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FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS
Cálculo Diferencial Vectorial
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Tarea: Multiplicadores de Lagrange

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1. $\nabla(\mathbf{f}+\mathbf{g}) = \nabla\mathbf{f}+\nabla\mathbf{g}$

   Solución.

$$\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial(f+g)}{\partial x_1}, \frac{\partial(f+g)}{\partial x_2}, \dots, \frac{\partial(f+g)}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1}+\frac{\partial g}{\partial x_1}, \frac{\partial f}{\partial x_2}+\frac{\partial g}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}+\frac{\partial g}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) + \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}\right) \\
&= \nabla f + \nabla g
\end{aligned}$$

   Por lo tanto, $\nabla(f+g) = \nabla f + \nabla g$.

2. $\nabla(\mathbf{cf}) = \mathbf{c}\nabla\mathbf{f}$

   Solución.

$$\begin{aligned}
\nabla(cf) &= \left(\frac{\partial(cf)}{\partial x_1}, \frac{\partial(cf)}{\partial x_2}, \dots, \frac{\partial(cf)}{\partial x_n}\right) \\
&= \left(c\frac{\partial f}{\partial x_1}, c\frac{\partial f}{\partial x_2}, \dots, c\frac{\partial f}{\partial x_n}\right) \\
&= c\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) \\
&= c\nabla f
\end{aligned}$$

   Por lo tanto, $\nabla(cf) = c\nabla f$.

3. $\nabla(\mathbf{fg}) = \mathbf{f}\nabla\mathbf{g}+\mathbf{g}\nabla\mathbf{f}$

   Solución.

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial(fg)}{\partial x_1}, \frac{\partial(fg)}{\partial x_2}, \dots, \frac{\partial(fg)}{\partial x_n}\right) \\
&= \left(f\frac{\partial g}{\partial x_1}+g\frac{\partial f}{\partial x_1}, f\frac{\partial g}{\partial x_2}+g\frac{\partial f}{\partial x_2}, \dots, f\frac{\partial g}{\partial x_n}+g\frac{\partial f}{\partial x_n}\right) \\
&= f\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}\right) + g\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) \\
&= f\nabla g + g\nabla f
\end{aligned}$$

   Por lo tanto, $\nabla(fg) = f\nabla g + g\nabla f$.

4. $\nabla\left(\frac{\mathbf{f}}{\mathbf{g}}\right) = \frac{\mathbf{g}\ \nabla\mathbf{f} - \mathbf{f}\ \nabla\mathbf{g}}{\mathbf{g^2}}$

   Solución.

   Calculando $\nabla\left(\frac{1}{g}\right)$:

$$\begin{aligned}
\nabla\left(\frac{1}{g}\right) &= \left(\frac{\partial}{\partial x_1}\left(\frac{1}{g}\right), \frac{\partial}{\partial x_2}\left(\frac{1}{g}\right), \dots, \frac{\partial}{\partial x_n}\left(\frac{1}{g}\right)\right) \\
&= \left(-\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_1}, -\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_2}, \dots, -\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_n}\right) \\
&= -\frac{1}{g^2}\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}\right) \\
&= -\frac{1}{g^2}\nabla g
\end{aligned}$$

   Luego, por el inciso 3, se tiene que

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \nabla\left(f\cdot\frac{1}{g}\right) \\
&= f\nabla\left(\frac{1}{g}\right) + \frac{1}{g}\nabla f \\
&= f\left(-\frac{1}{g^2}\nabla g\right) + \frac{1}{g}\nabla f \\
&= \frac{\nabla f}{g} - \frac{f\nabla g}{g^2} \\
&= \frac{g\ \nabla f - f\ \nabla g}{g^2}
\end{aligned}$$

   Por lo tanto, $\nabla\left(\frac{f}{g}\right) = \frac{g\ \nabla f - f\ \nabla g}{g^2}$.

5. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

   Solución.

$$\begin{aligned}
\operatorname{div}(F+G) &= \frac{\partial(f_1+g_1)}{\partial x_1} + \frac{\partial(f_2+g_2)}{\partial x_2} + \dots + \frac{\partial(f_n+g_n)}{\partial x_n} \\
&= \frac{\partial f_1}{\partial x_1} + \frac{\partial g_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial g_2}{\partial x_2} + \dots + \frac{\partial f_n}{\partial x_n} + \frac{\partial g_n}{\partial x_n} \\
&= \operatorname{div}F + \operatorname{div}G
\end{aligned}$$

   Por lo tanto, $\operatorname{div}(F+G) = \operatorname{div}F + \operatorname{div}G$

6. $\operatorname{rot}(\mathbf{F}+\mathbf{G}) = \operatorname{rot}\mathbf{F} + \operatorname{rot}\mathbf{G}$

   Solución.

$$
\begin{aligned}
\operatorname{rot}(F+G) &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1+g_1 & f_2+g_2 & f_3+g_3 \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2+g_2 & f_3+g_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1+g_1 & f_3+g_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1+g_1 & f_2+g_2 \end{vmatrix} \widehat{k} \\
&= \left(\frac{\partial(f_3+g_3)}{\partial y} - \frac{\partial(f_2+g_2)}{\partial z} \right) \widehat{i} - \left(\frac{\partial(f_3+g_3)}{\partial x} - \frac{\partial(f_1+g_1)}{\partial z} \right) \widehat{j} + \\
&\quad \left(\frac{\partial(f_2+g_2)}{\partial x} - \frac{\partial(f_1+g_1)}{\partial y} \right) \widehat{k} \\
&= \left(\frac{\partial f_3}{\partial y} + \frac{\partial g_3}{\partial y} - \frac{\partial f_2}{\partial z} - \frac{\partial g_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial f_3}{\partial x} + \frac{\partial g_3}{\partial x} - \frac{\partial f_1}{\partial z} - \frac{\partial g_1}{\partial z} \right) \widehat{j} + \\
&\quad \left(\frac{\partial f_2}{\partial x} + \frac{\partial g_2}{\partial x} - \frac{\partial f_1}{\partial y} - \frac{\partial g_1}{\partial y} \right) \widehat{k} \\
&= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \widehat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \widehat{k} + \\
&\quad \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial g_3}{\partial x} - \frac{\partial g_1}{\partial z} \right) \widehat{j} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right) \widehat{k} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix} \widehat{k} + \\
&\quad \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ g_2 & g_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ g_1 & g_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ g_1 & g_2 \end{vmatrix} \widehat{k} \\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix} \\
&= \operatorname{rot} F + \operatorname{rot} G
\end{aligned}
$$

Por lo tanto, $\operatorname{rot}(F+G) = \operatorname{rot} F + \operatorname{rot} G$.

7. $\operatorname{div}(\mathbf{fF}) = \mathbf{f}\operatorname{div}\mathbf{F} + \nabla\mathbf{fF}$

Solución.

$$
\begin{aligned}
\operatorname{div}(fF) &= \frac{\partial(fg_1)}{\partial x_1} + \frac{\partial(fg_2)}{\partial x_2} + \cdots + \frac{\partial(fg_n)}{\partial x_n} \\
&= f\frac{\partial g_1}{\partial x_1} + g_1\frac{\partial f}{\partial x_1} + f\frac{\partial g_2}{\partial x_2} + g_2\frac{\partial f}{\partial x_2} + \cdots + f\frac{\partial g_n}{\partial x_n} + g_n\frac{\partial f}{\partial x_n}
\end{aligned}
$$

$$
\begin{aligned}
&= f\left(\frac{\partial g_1}{\partial x_1}+\frac{\partial g_2}{\partial x_2}+\cdots+\frac{\partial g_n}{\partial x_n}\right)+g_1\frac{\partial f}{\partial x_1}+g_2\frac{\partial f}{\partial x_2}+\cdots+g_n\frac{\partial f}{\partial x_n}\\
&= f\,\operatorname{div} F+\nabla f F
\end{aligned}
$$

Por lo tanto, $\operatorname{div}(fF) = f\,\operatorname{div} F+\nabla f F$.

8. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\ \operatorname{rot}\mathbf{F}-\mathbf{F}\cdot\ \operatorname{rot}\mathbf{G}$

Solución.

$$
\begin{aligned}
\operatorname{div}(F\times G) &= \operatorname{div}\begin{vmatrix}\widehat{i} & \widehat{j} & \widehat{k}\\ f_1 & f_2 & f_3\\ g_1 & g_2 & g_3\end{vmatrix}\\
&= \operatorname{div}\left(\begin{vmatrix} f_2 & f_3\\ g_2 & g_3\end{vmatrix}\widehat{i}-\begin{vmatrix} f_1 & f_3\\ g_1 & g_3\end{vmatrix}\widehat{j}+\begin{vmatrix} f_1 & f_2\\ g_1 & g_2\end{vmatrix}\widehat{k}\right)\\
&= \operatorname{div}\left[(f_2g_3-f_3g_2)\widehat{i}-(f_1g_3-f_3g_1)\widehat{j}+(f_1g_2-f_2g_1)\widehat{k}\right]\\
&= \operatorname{div}\,(f_2g_3-f_3g_2, f_3g_1-f_1g_3, f_1g_2-f_2g_1)\\
&= \frac{\partial(f_2g_3-f_3g_2)}{\partial x}+\frac{\partial(f_3g_1-f_1g_3)}{\partial y}+\frac{\partial(f_1g_2-f_2g_1)}{\partial z}\\
&= f_2\cdot\frac{\partial g_3}{\partial x}+g_3\cdot\frac{\partial f_2}{\partial x}-f_3\cdot\frac{\partial g_2}{\partial x}-g_2\cdot\frac{\partial f_3}{\partial x}+f_3\cdot\frac{\partial g_1}{\partial y}+g_1\cdot\frac{\partial f_3}{\partial y}-\\
&\quad f_1\cdot\frac{\partial g_3}{\partial y}-g_3\cdot\frac{\partial f_1}{\partial y}+f_1\cdot\frac{\partial g_2}{\partial z}+g_2\cdot\frac{\partial f_1}{\partial z}-f_2\cdot\frac{\partial g_1}{\partial z}-g_1\cdot\frac{\partial f_2}{\partial z}\\
&= g_1\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+g_2\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+g_3\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)-\\
&\quad f_1\left(\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z}\right)-f_2\left(\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial x}\right)-f_3\left(\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y}\right)\\
&= (g_1,g_2,g_3)\cdot\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z},\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x},\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)-\\
&\quad (f_1,f_2,f_3)\cdot\left(\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z},\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial x},\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y}\right)\\
&= (g_1,g_2,g_3)\cdot\left[\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\widehat{i}-\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)\widehat{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\widehat{k}\right]-\\
&\quad (f_1,f_2,f_3)\cdot\left[\left(\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z}\right)\widehat{i}-\left(\frac{\partial g_3}{\partial x}-\frac{\partial g_1}{\partial z}\right)\widehat{j}+\left(\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y}\right)\widehat{k}\right]\\
&= (g_1,g_2,g_3)\cdot\left(\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_2 & f_3\end{vmatrix}\widehat{i}-\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ f_1 & f_3\end{vmatrix}\widehat{j}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ f_1 & f_2\end{vmatrix}\widehat{k}\right)-\\
&\quad (f_1,f_2,f_3)\cdot\left(\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ g_2 & g_3\end{vmatrix}\widehat{i}-\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ g_1 & g_3\end{vmatrix}\widehat{j}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ g_1 & g_2\end{vmatrix}\widehat{k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (g_1, g_2, g_3) \cdot \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} - (f_1, f_2, f_3) \cdot \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix} \\
&= G \cdot \operatorname{rot} F - F \cdot \operatorname{rot} G
\end{aligned}
$$

Por lo tanto, $\operatorname{div}(F \times G) = G \cdot \operatorname{rot} F - F \cdot \operatorname{rot} G$.

9. $\operatorname{div}(\operatorname{rot} \mathbf{F}) = \mathbf{0}$

Solución.

$$
\begin{aligned}
\operatorname{div}(\operatorname{rot} F) &= \operatorname{div} \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\
&= \operatorname{div} \left(\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix} \widehat{k} \right) \\
&= \operatorname{div} \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \widehat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \widehat{k} \right] \\
&= \operatorname{div} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\
&= \frac{\partial \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z} \right)}{\partial x} + \frac{\partial \left(\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x} \right)}{\partial y} + \frac{\partial \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y} \right)}{\partial z} \\
&= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} \\
&= 0
\end{aligned}
$$

Por lo tanto, $\operatorname{div}(\operatorname{rot} F) = 0$

10. $\operatorname{rot}(\nabla \mathbf{f}) = \overline{\mathbf{0}}$

Solución.

$$
\begin{aligned}
\operatorname{rot}(\nabla f) &= \operatorname{rot} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{vmatrix} \widehat{k}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\widehat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\widehat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\widehat{k} \\
&= 0\widehat{i} - 0\widehat{j} + 0\widehat{k} \\
&= \overline{0}
\end{aligned}
$$

Por lo tanto, $\operatorname{rot}(\nabla f) = \overline{0}$.

11. $\operatorname{rot}(\mathbf{fF}) = \mathbf{f}\operatorname{rot}\mathbf{F} + \nabla\mathbf{f} \times \mathbf{F}$

Solución.

$$
\begin{aligned}
\operatorname{rot}(fF) &= \nabla \times fF \\
&= \nabla \times (ff_1, ff_2, ff_3) \\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ff_1 & ff_2 & ff_3 \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ff_2 & ff_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ ff_1 & ff_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ ff_1 & ff_2 \end{vmatrix}\widehat{k} \\
&= \left(\frac{\partial(ff_3)}{\partial y} - \frac{\partial(ff_2)}{\partial z}\right)\widehat{i} - \left(\frac{\partial(ff_3)}{\partial x} - \frac{\partial(ff_1)}{\partial z}\right)\widehat{j} + \left(\frac{\partial(ff_2)}{\partial x} - \frac{\partial(ff_1)}{\partial y}\right)\widehat{k} \\
&= \left(f \cdot \frac{\partial f_3}{\partial y} + f_3 \cdot \frac{\partial f}{\partial y} - f \cdot \frac{\partial f_2}{\partial z} - f_2 \cdot \frac{\partial f}{\partial z}\right)\widehat{i} \\
&\quad - \left(f \cdot \frac{\partial f_3}{\partial x} + f_3 \cdot \frac{\partial f}{\partial x} - f \cdot \frac{\partial f_1}{\partial z} - f_1 \cdot \frac{\partial f}{\partial z}\right)\widehat{j} \\
&\quad + \left(f \cdot \frac{\partial f_2}{\partial x} + f_2 \cdot \frac{\partial f}{\partial x} - f \cdot \frac{\partial f_1}{\partial y} - f_1 \cdot \frac{\partial f}{\partial y}\right)\widehat{k} \\
&= f\left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\widehat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\widehat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\widehat{k}\right] + \\
&\quad \left(f_3 \cdot \frac{\partial f}{\partial y} - f_2 \cdot \frac{\partial f}{\partial z}\right)\widehat{i} - \left(f_3 \cdot \frac{\partial f}{\partial x} - f_1 \cdot \frac{\partial f}{\partial z}\right)\widehat{j} + \left(f_2 \cdot \frac{\partial f}{\partial x} - f_1 \cdot \frac{\partial f}{\partial y}\right)\widehat{k} \\
&= f\left(\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix}\widehat{k}\right) + \\
&\quad \begin{vmatrix} \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ f_2 & f_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \\ f_1 & f_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ f_1 & f_2 \end{vmatrix}\widehat{k} \\
&= f\begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\
&= f\operatorname{rot}F + \nabla f \times F
\end{aligned}
$$

Por lo tanto, $\operatorname{rot}(fF) = f \operatorname{rot} F + \nabla f \times F$.

12. $\nabla^2(\mathbf{fg}) = \mathbf{f}\nabla^2\mathbf{g} + \mathbf{g}\nabla^2\mathbf{f} + \mathbf{2}(\nabla\mathbf{f} \cdot \nabla\mathbf{g})$

Solución.

$$\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2(fg)}{\partial {x_1}^2} + \frac{\partial^2(fg)}{\partial {x_2}^2} + \cdots + \frac{\partial^2(fg)}{\partial {x_n}^2} \\
&= f \cdot \frac{\partial^2 g}{\partial {x_1}^2} + 2 \cdot \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_1} + g\frac{\partial^2 f}{\partial {x_1}^2} + f \cdot \frac{\partial^2 g}{\partial {x_2}^2} + 2 \cdot \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_2} + \\
&\quad g\frac{\partial^2 f}{\partial {x_2}^2} + \ldots + f \cdot \frac{\partial^2 g}{\partial {x_n}^2} + 2 \cdot \frac{\partial f}{\partial x_n} \cdot \frac{\partial g}{\partial x_n} + g\frac{\partial^2 f}{\partial {x_n}^2} \\
&= f \cdot \frac{\partial^2 g}{\partial {x_1}^2} + f \cdot \frac{\partial^2 g}{\partial {x_2}^2} + \ldots + f \cdot \frac{\partial^2 g}{\partial {x_n}^2} + g\frac{\partial^2 f}{\partial {x_1}^2} + g\frac{\partial^2 f}{\partial {x_2}^2} + \ldots + \\
&\quad g\frac{\partial^2 f}{\partial {x_n}^2} + 2 \cdot \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_1} + 2 \cdot \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_2} + \ldots + 2 \cdot \frac{\partial f}{\partial x_n} \cdot \frac{\partial g}{\partial x_n} \\
&= f\left(\frac{\partial^2 g}{\partial {x_1}^2} + \frac{\partial^2 g}{\partial {x_2}^2} + \ldots + \frac{\partial^2 g}{\partial {x_n}^2}\right) + g\left(\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} + \ldots + \frac{\partial^2 f}{\partial {x_n}^2}\right) + \\
&\quad 2\left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_2} + \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial g}{\partial x_n}\right) \\
&= f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)
\end{aligned}$$

Por lo tanto, $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.

13. $\operatorname{div}(\nabla\mathbf{f} \times \nabla\mathbf{g}) = \mathbf{0}$

Solución.

Por el inciso 8, se sabe que

$\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{rot}(\nabla f) - \nabla f \cdot \operatorname{rot}(\nabla g)$

Y por el inciso 10 se tiene que

$\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \overline{0} - \nabla f \cdot \overline{0}$

Por lo tanto, $\operatorname{div}(\nabla f \times \nabla g) = 0$.