

$$35. \ y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}; \quad y(0) = \frac{1}{2}, \quad y'(0) = \frac{5}{2}, \quad y''(0) = -\frac{9}{2}$$

Solución.

Ecuación homogénea asociada: $y''' - 2y'' + y' = 0$

Polinomio característico: $m^3 - 2m^2 + m = 0$

$$\implies m(m^2 - 2m + 1) = 0$$

$$\implies m(m - 1)^2 = 0$$

$$\implies m_1 = 0, \ m_2 = 1 \text{ es una raíz de multiplicidad } 2.$$

Soluciones l.i.: $y_1(x) = e^{0x} = 1$, $y_2(x) = e^x$ y $y_3(x) = xe^x$.

Funcion complementaria: $y_c(x) = c_1 + c_2e^x + c_3xe^x$.

Sea $y_p(x) = Ax + Bx^2e^x + Ce^{5x}$

$$\implies y'_p(x) = A + 2Bxe^x + Bx^2e^x + 5Ce^{5x}$$

$$\implies y''_p(x) = 2Be^x + 4Bxe^x + Bx^2e^x + 25Ce^{5x}$$

$$\implies y'''_p(x) = 6Be^x + 6Bxe^x + Bx^2e^x + 125Ce^{5x}$$

Sustituyendo en la E.D.

$$\begin{aligned} y_p(x)''' - 2y_p(x)'' + y_p(x)' &= 6Be^x + 6Bxe^x + Bx^2e^x + 125Ce^{5x} \\ &\quad - 2(2Be^x + 4Bxe^x + Bx^2e^x + 25Ce^{5x}) \\ &\quad + A + 2Bxe^x + Bx^2e^x + 5Ce^{5x} \\ &= A + 2Be^x + 80Ce^{5x} \end{aligned}$$

$$\implies 2 - 24e^x + 40e^{5x} = A + 2Be^x + 80Ce^{5x}$$

De esto, se tiene que

$$\begin{aligned} A &= 2 \\ -24 &= 2B \\ 40 &= 80C \end{aligned}$$

por lo que

$$\begin{aligned} A &= 2 \\ B &= -12 \\ C &= \frac{1}{2} \end{aligned}$$

Así, $y_p(x) = 2x - 12x^2e^x + \frac{1}{2}e^{5x}$.

De esta manera, la solución general es:

$$y(x) = c_1 + c_2 e^x + c_3 x e^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}$$

Luego,

$$y'(x) = c_2 e^x + c_3 e^x + c_3 x e^x + 2 - 24x e^x - 12x^2 e^x + \frac{5}{2} e^{5x}$$

$$y''(x) = c_2 e^x + 2c_3 e^x + c_3 x e^x - 24e^x - 48x e^x - 12x^2 e^x + \frac{25}{2} e^{5x}$$

Pero

$$\begin{aligned} -\frac{9}{2} = y''(0) &= c_2 + 2c_3 - 24 + \frac{25}{2} = c_2 + 2c_3 - \frac{23}{2} \\ \implies 7 &= c_2 + 2c_3 \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{5}{2} = y'(0) &= c_2 + c_3 + 2 + \frac{5}{2} = c_2 + c_3 + \frac{9}{2} \\ \implies 2 &= -c_2 - c_3 \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{1}{2} = y(0) &= c_1 + c_2 + \frac{1}{2} \\ \implies 0 &= c_1 + c_2 \end{aligned} \tag{3}$$

Sumando (1) y (2): $9 = c_3$

Sustituyendo c_3 en (1): $7 = c_2 + 2(9) = c_2 + 18 \implies c_2 = -11$

Sustituyendo c_2 en (3): $0 = c_1 - 11 \implies c_1 = 11$

Por lo tanto,

$$y(x) = 11 - 11e^x + 9xe^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}$$