25. $y'' - (x+1)y' - y = 0$

Solución.

La E.D. no tiene puntos singulares.

Considerando al punto ordinario $x_0 = -1$ entonces

$$y(x) = \sum_{n=0}^{\infty} c_n (x+1)^n$$

$$\Longrightarrow y'(x) = \sum_{n=1}^{\infty} n c_n (x+1)^{n-1}$$

$$\Longrightarrow y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x+1)^{n-2}$$

Sustituyendo en la E.D.

$$\sum_{n=2}^{\infty} n(n-1)c_n(x+1)^{n-2} - (x+1)\sum_{n=1}^{\infty} nc_n(x+1)^{n-1} - \sum_{n=0}^{\infty} c_n(x+1)^n = 0$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1)c_n(x+1)^{n-2} - \sum_{n=1}^{\infty} nc_n(x+1)^{n} - \sum_{n=0}^{\infty} c_n(x+1)^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}(x+1)^{n} - \sum_{n=1}^{\infty} nc_n(x+1)^{n} - \sum_{n=0}^{\infty} c_n(x+1)^n = 0$$

$$\Longrightarrow 2c_2 - c_0 + \sum_{n=1}^{\infty} (n+2)(n+1)c_{n+2}(x+1)^{n} - \sum_{n=1}^{\infty} nc_n(x+1)^{n} - \sum_{n=1}^{\infty} c_n(x+1)^n = 0$$

$$\Longrightarrow 2c_2 - c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n](x+1)^n = 0$$

$$\Longrightarrow 2c_2 - c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - (n+1)c_n](x+1)^n = 0$$

$$\Longrightarrow 2c_2 - c_0 + \sum_{n=1}^{\infty} \{(n+1)[(n+2)c_{n+2} - c_n]\}(x+1)^n = 0$$

$$\Longrightarrow 2c_2 - c_0 = 0 \quad \text{y} \quad (n+1)[(n+2)c_{n+2} - c_n] = 0 \text{ con } n = 1, 2, 3, \ldots$$

$$\Longrightarrow 2c_2 - c_0 = 0 \quad \text{y} \quad (n+2)c_{n+2} - c_n = 0 \text{ con } n = 1, 2, 3, \ldots$$

$$\Longrightarrow c_2 = \frac{c_0}{2} \quad \text{y} \quad c_{n+2} = \frac{c_n}{n+2} \text{ con } n = 1, 2, 3, \ldots$$

Obteniendo algunos términos:

$$c_3 = \frac{c_1}{3}$$

$c_4 = \frac{c_2}{4} = \frac{c_0}{2 \cdot 4}$
$c_5 = \frac{c_3}{5} = \frac{c_1}{3 \cdot 5}$
$c_6 = \frac{c_4}{6} = \frac{c_0}{2 \cdot 4 \cdot 6}$
$c_7 = \frac{c_5}{7} = \frac{c_1}{3 \cdot 5 \cdot 7}$
Así, la solución general es:

$$\begin{aligned}
y(x) &= c_0 + c_1(x+1) + \left(\frac{c_0}{2}\right)(x+1)^2 + \left(\frac{c_1}{3}\right)(x+1)^3 + \left(\frac{c_0}{2 \cdot 4}\right)(x+1)^4 \\
&\quad + \left(\frac{c_1}{3 \cdot 5}\right)(x+1)^5 + \left(\frac{c_0}{2 \cdot 4 \cdot 6}\right)(x+1)^6 + \left(\frac{c_1}{3 \cdot 5 \cdot 7}\right)(x+1)^7 + \cdots \\
&= c_0\left(1 + \frac{(x+1)^2}{2} + \frac{(x+1)^4}{2 \cdot 4} + \frac{(x+1)^6}{2 \cdot 4 \cdot 6} + \cdots\right) \\
&\quad + c_1\left((x+1) + \frac{(x+1)^3}{3} + \frac{(x+1)^5}{3 \cdot 5} + \frac{(x+1)^7}{3 \cdot 5 \cdot 7} + \cdots\right) \\
&= c_0\left(1 + \frac{(x+1)^2}{2!} + \frac{3(x+1)^4}{4!} + \frac{3 \cdot 5(x+1)^6}{6!} + \cdots\right) \\
&\quad + c_1\left((x+1) + \frac{2(x+1)^3}{3!} + \frac{2 \cdot 4(x+1)^5}{5!} + \frac{2 \cdot 4 \cdot 6(x+1)^7}{7!} + \cdots\right) \\
&= c_0\left(1 + \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{n} 2j-1}{(2n)!}(x+1)^{2n}\right) + c_1\left((x+1) + \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{n} 2j}{(2n+1)!}(x+1)^{2n+1}\right)
\end{aligned}$$

De esta manera, $y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{n} 2j-1}{(2n)!}(x+1)^{2n}$ y $y_2(x) = (x+1) + \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{n} 2j}{(2n+1)!}(x+1)^{2n+1}$ son soluciones linealmente independientes.