Ejercicio 1

Sean $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, con U un abierto de $\mathbb{R}^n$. Si existen las derivadas parciales de f y como funciones son continuas en U entonces f es diferenciable.

Demostración.
jknvjkdfnvjkdnjkdnv dnvjknvjdfnvjdfn

jnvjknvjdnvnvkdnvnv
jknvjkdnvjdnvkjdnvjdnvkdmvlkcmvlv nbbkdnk kdfddj i o fjdj kdf df gd jg kks kdfks kfdfnkfvdfnvnd kl vvd vlfk vdf vdfvnklmvk nvk v v nvlknvksn v vvdfnvdfvksnvkvnjkfioeh tuught vn n oa vnj nod
Sean $\overline{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ fijo y $f = (f_1, f_2, \ldots, f_n)$.
P.d. $\displaystyle\lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x}) \left(x_j - y_j\right) \right|}{\|\overline{x} - \overline{y}\|} = 0 \quad \forall\, i = 1, 2, \ldots, m.$
Para todo $i = 1, 2, \ldots, m$ y $j = 1, 2, \ldots, n$ sea $g : \mathbb{R} \to \mathbb{R}$ dada por $g_j(x) = f_i\left(x_1, \ldots, x_{j-1}, x, y_{j+1}, \ldots, y_n\right)$, donde $\overline{x} = (x_1, x_2, \ldots, x_n) \in U$. Luego, para cada $i = 1, 2, \ldots, m$, se tiene que

$$\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) &= f_i\left(x_1, x_2, \ldots, x_n\right) - f_i\left(x_1, x_2, \ldots, y_n\right) + f_i\left(x_1, x_2, \ldots, y_n\right) - \\
&\qquad f_i\left(x_1, x_2, \ldots, y_{n-1}, y_n\right) + f_i\left(x_1, x_2, \ldots, y_{n-1}, y_n\right) - \cdots - \\
&\qquad f_i\left(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n\right) + f_i\left(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n\right) - \cdots - \\
&\qquad f_i\left(x_1, y_2, \ldots, y_n\right) + f_i\left(x_1, y_2, \ldots, y_n\right) - f_i\left(y_1, y_2, \ldots, y_n\right) \\
&= g_n(x_n) - g_n(y_n) + g_{n-1}(x_{n-1}) - g_{n-1}(y_{n-1}) + g_{n-2}(x_{n-1}) - \cdots - \\
&\qquad g_j(y_j) + g_{j-1}(x_{j-1}) - \cdots - g_2(y_2) + g_1(x_1) - g_1(y_1)
\end{aligned}$$

Ya que f_i es diferenciable en U, se tiene que, para todo $j = 1, 2, \ldots, n$, g_j también lo es. Así, por el Teorema del Valor Medio, para cada $j = 1, 2, \ldots, n$ existe $a_j \in \left[\text{mín}\left\{x_j, y_j\right\}, \text{máx}\left\{x_j, y_j\right\}\right]$ tal que $g_j(x_j) - g_j(y_j) = g_j(a_j)\left(x_j - y_j\right)$. De este modo,

$$\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) &= g_n'(a_n)\left(x_n - y_n\right) + g_{n-1}'(a_{n-1})\left(x_{n-1} - y_{n-1}\right) + \cdots + g_j'(a_j)\left(x_j - y_j\right) + \cdots + g_1'(a_1)\left(x_1 - y_1\right) \\
&= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\left(x_1 - y_1\right) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\left(x_2 - y_2\right) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\left(x_j - y_j\right) + \cdots + \frac{\partial f_i}{\partial x_n}(\overline{z_n})\left(x_n - y_n\right)
\end{aligned}$$

donde $\overline{z_j} = (x_1, \ldots, x_{j-1}, a_j, y_{j+1}, \ldots, y_n)$. Posteriormente,

$$
\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) &= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\,(x_1 - y_1) - \frac{\partial f_i}{\partial x_1}(\overline{x})\,(x_1 - y_1) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\,(x_2 - y_2) - \\
&\quad \frac{\partial f_i}{\partial x_2}(\overline{x})\,(x_2 - y_2) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\left(x_j - y_j\right) - \\
&\quad \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) + \cdots + \frac{\partial f_i}{\partial x_n}(\overline{z_n})\,(x_n - y_n) - \frac{\partial f_i}{\partial x_n}(\overline{x})\,(x_n - y_n) \\
&= \left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right)(x_1 - y_1) + \\
&\quad \left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right)(x_2 - y_2) + \cdots + \\
&\quad \left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right)\left(x_j - y_j\right) + \cdots + \\
&\quad \left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right)(x_n - y_n)
\end{aligned}
$$

$$
\begin{aligned}
\implies \left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x}) \left(x_j - y_j\right) \right| &= \left| \left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right) (x_1 - y_1) + \right. \\
&\qquad \left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right) (x_2 - y_2) + \cdots + \\
&\qquad \left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right) \left(x_j - y_j\right) + \cdots + \\
&\qquad \left. \left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right) (x_n - y_n) \right| \\
&\leq \left| \frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right| |x_1 - y_1| + \\
&\qquad \left| \frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right| |x_2 - y_2| + \cdots + \\
&\qquad \left| \frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right| \left|x_j - y_j\right| + \cdots + \\
&\qquad \left| \frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right| |x_n - y_n| \\
&\leq \left| \frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right| \|\overline{x} - \overline{y}\| + \\
&\qquad \left| \frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right| \|\overline{x} - \overline{y}\| + \cdots + \\
&\qquad \left| \frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right| \|\overline{x} - \overline{y}\| + \cdots + \\
&\qquad \left| \frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right| \|\overline{x} - \overline{y}\| \\
&= \left(\left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots + \right. \\
&\qquad \left. \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right| \right) \|\overline{x} - \overline{y}\|
\end{aligned}
$$

$$
\begin{aligned}
\implies \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x}) \left(x_j - y_j\right) \right|}{\|\overline{x} - \overline{y}\|} &\leq \left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots + \\
&\qquad \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right|
\end{aligned}
$$

Después, como las derivadas parciales son continuas en U se da que $\lim\limits_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_j}(\overline{x})-\frac{\partial f_i}{\partial x_j}(\overline{y})\right| = 0$, y puesto que $\overline{z_j} = (x_1,\ldots,x_{j-1},a_j,y_{j+1},\ldots,y_n)$ con $a_j \in \left[\text{mín}\left\{x_j,y_j\right\},\text{máx}\left\{x_j,y_j\right\}\right]$, se tiene que $\overline{z_j}$ tiende a $\overline{y}$ conforme $\overline{x}$ tiende a $\overline{y}$, para todo $j=1,2,\ldots,n$. De esta manera,

$$\begin{aligned}
0 \leq \lim_{\overline{x}\to\overline{y}} \frac{\left|f_i(\overline{x})-f_i(\overline{y})-\sum\limits_{j=1}^{n}\frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j-y_j\right)\right|}{\|\overline{x}-\overline{y}\|} &\leq \lim_{\overline{x}\to\overline{y}}\left(\left|\frac{\partial f_i}{\partial x_1}(\overline{x})-\frac{\partial f_i}{\partial x_1}(\overline{z_1})\right|+\left|\frac{\partial f_i}{\partial x_2}(\overline{x})-\frac{\partial f_i}{\partial x_2}(\overline{z_2})\right|+\cdots+\right.\\
&\quad \left|\frac{\partial f_i}{\partial x_j}(\overline{x})-\frac{\partial f_i}{\partial x_j}(\overline{z_j})\right|+\cdots+\\
&\quad \left.\left|\frac{\partial f_i}{\partial x_n}(\overline{x})-\frac{\partial f_i}{\partial x_n}(\overline{z_n})\right|\right)\\
&= \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_1}(\overline{x})-\frac{\partial f_i}{\partial x_1}(\overline{z_1})\right|+\lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_2}(\overline{x})-\frac{\partial f_i}{\partial x_2}(\overline{z_2})\right|+\cdots+\\
&\quad \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_j}(\overline{x})-\frac{\partial f_i}{\partial x_j}(\overline{z_j})\right|+\cdots+\\
&\quad \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_n}(\overline{x})-\frac{\partial f_i}{\partial x_n}(\overline{z_n})\right|\\
&= \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_1}(\overline{x})-\frac{\partial f_i}{\partial x_1}(\overline{y})\right|+\lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_2}(\overline{x})-\frac{\partial f_i}{\partial x_2}(\overline{y})\right|+\cdots+\\
&\quad \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_j}(\overline{x})-\frac{\partial f_i}{\partial x_j}(\overline{y})\right|+\cdots+\\
&\quad \lim_{\overline{x}\to\overline{y}}\left|\frac{\partial f_i}{\partial x_n}(\overline{x})-\frac{\partial f_i}{\partial x_n}(\overline{y})\right|\\
&= 0
\end{aligned}$$

De esta manera, $\lim\limits_{\overline{x}\to\overline{y}} \dfrac{\left|f_i(\overline{x})-f_i(\overline{y})-\sum\limits_{j=1}^{n}\frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j-y_j\right)\right|}{\|\overline{x}-\overline{y}\|} = 0 \quad \forall\, i=1,\ldots,m.$

Por último,

$$0 \leq \lim_{\overline{x} \to \overline{y}} \frac{\| f(\overline{x}) - f(\overline{y}) - Df(\overline{y})(\overline{x} - \overline{y}) \|}{\| \overline{x} - \overline{y} \|} \leq \sum_{i=1}^{m} \lim_{\overline{x} \to \overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x}) \left(x_j - y_j \right) \right|}{\| \overline{x} - \overline{y} \|} = 0$$

Ya que $\overline{y}$ fue arbitrario, se concluye que f es diferenciable en U. ■

Proposición. Sean $A, B \subseteq \mathbb{R}$ no vacíos. Si $a \leq b$ para todo $a \in A$ y para todo $b \in B$, entonces A está acotada superiormente y B está acotado inferiormente, y además, $\sup(A) \leq \text{ínf}(B)$.

Demostración.

Sea $b \in B$, ya que $a \leq b$ para todo $a \in A$, se tiene que A está acotado superiormente. De igual forma, sea $a \in A$ como $a \leq b$ para todo $b \in B$, se da que B está acotado inferiormente. Además, dado que A y B son no vacíos, se obtiene que el supremo y el ínfimo de A y B existen, respectivamente. Como todo elemento de B es cota superior de A se tiene que $\sup(A)$ es una cota inferior de B. Por lo tanto, $\sup(A) \leq \text{ínf}(B)$, pues $\text{ínf}(B)$ es la mayor cota inferior de B.

c) Si $f \in \mathcal{R}(\alpha)$ en $[a, b]$ y $a < c < b$ entonces $f \in \mathcal{R}(\alpha)$ en $[a, c]$ y en $[c, b]$ y

$$\int_a^b f \, \mathrm{d}\alpha = \int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha$$

Demostración.

Sea $\varepsilon > 0$. Ya que $f \in \mathcal{R}(\alpha)$ en $[a, b]$ existe $P_\varepsilon = \{a = x_0, x_1, \ldots, x_n = b\} \in \gamma_{[a,b]}$ tal que $U(f, P_\varepsilon, \alpha) - L(f, P_\varepsilon, \alpha) < \varepsilon$. Luego, sean $P'_\varepsilon = \{a = x_0, x_1, \ldots, x_{i-1}, c\} \in \gamma_{[a,c]}$ y $P''_\varepsilon = \{a = c, x_i, x_{i+1}, \ldots, x_n\} \in \gamma_{[c,b]}$, se tiene que