UNIVERSIDAD AUTÓNOMA DEL ESTADO DE MÉXICO
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS
Cálculo Diferencial Vectorial
Profesores: Dr. Félix Capulín Pérez
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Tarea: Polinomio de Taylor

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En cada uno de los ejercicios, determinar la fórmula de Taylor de segundo orden para la función dada alrededor del punto x_0, y_0.

1. $f(x, y) = (x + y)^2$, donde $x_0 = 0$, $y_0 = 0$.

   Solución.

   $f(0,0) = (0+0)^2 = 0$

   $\dfrac{\partial f}{\partial x}(0,0) = 2(x+y)|_{(0,0)} = 0$

   $\dfrac{\partial f}{\partial y}(0,0) = 2(x+y)|_{(0,0)} = 0$

   $\dfrac{\partial^2 f}{\partial x^2}(0,0) = 2$

   $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 2$

   $\dfrac{\partial^2 f}{\partial y^2}(0,0) = 2$

   Así, el Polinomio de Taylor es

$$
\begin{aligned}
f((h_1, h_2) - (x_0, y_0)) &= f(h_1, h_2) \\
&= f(0,0) + h_1 \frac{\partial f}{\partial x}(0,0) + h_2 \frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2 \frac{\partial^2 f}{\partial x^2}(0,0) + \right. \\
&\quad \left. h_1 h_2 \frac{\partial^2 f}{\partial y \partial x}(0,0) + h_2 h_1 \frac{\partial^2 f}{\partial x \partial y}(0,0) + h_2^2 \frac{\partial^2 f}{\partial y^2}(0,0)\right) \\
&= \frac{1}{2}\left(2h_1^2 + 2h_1h_2 + 2h_2h_1 + 2h_2^2\right) \\
&= h_1^2 + 2h_1h_2 + h_2^2
\end{aligned}
$$

   Notemos que $f(x, y) = (x + y)^2 = x^2 + 2xy + y^2$. Por lo tanto, es igual a su Polinomio de Taylor.

2. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, donde $x_0 = 0$, $y_0 = 0$.

   Solución.

   $f(0,0) = \dfrac{1}{0^2 + 0^2 + 1} = 1$

$$\frac{\partial f}{\partial x}(0,0) = -\frac{2x}{(x^2+y^2+1)^2}\bigg|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = -\frac{2y}{(x^2+y^2+1)^2}\bigg|_{(0,0)} = 0$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(0,0) &= \frac{(x^2+y^2+1)^2(-2)+(2x)(2x)(2(x^2+y^2+1))}{(x^2+y^2+1)^4}\bigg|_{(0,0)} \\
&= \frac{-2(x^2+y^2+1)^2+8x^2(x^2+y^2+1)}{(x^2+y^2+1)^4}\bigg|_{(0,0)} \\
&= \frac{-2(x^2+y^2+1)+8x^2}{(x^2+y^2+1)^3}\bigg|_{(0,0)} \\
&= \frac{6x^2-2y^2-2}{(x^2+y^2+1)^3}\bigg|_{(0,0)} \\
&= -2
\end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{8xy(x^2+y^2+1)}{(x^2+y^2+1)^4}\bigg|_{(0,0)} = 0$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2}(0,0) &= -\frac{(x^2+y^2+1)^2(-2)+(2y)(2y)(2(x^2+y^2+1))}{(x^2+y^2+1)^4}\bigg|_{(0,0)} \\
&= -\frac{-2(x^2+y^2+1)^2+8y^2(x^2+y^2+1)}{(x^2+y^2+1)^4}\bigg|_{(0,0)} \\
&= -\frac{-2(x^2+y^2+1)+8y^2}{(x^2+y^2+1)^3}\bigg|_{(0,0)} \\
&= -\frac{-2x^2+6y^2-2}{(x^2+y^2+1)^3}\bigg|_{(0,0)} \\
&= -2
\end{aligned}$$

Así, el Polinomio de Taylor es

$$\begin{aligned}
f((h_1,h_2)-(x_0,y_0)) &= f(h_1,h_2) \\
&= f(0,0) + h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2\frac{\partial^2 f}{\partial x^2}(0,0) + \right. \\
&\quad \left. h_1h_2\frac{\partial^2 f}{\partial y\partial x}(0,0) + h_2h_1\frac{\partial^2 f}{\partial x\partial y}(0,0) + h_2^2\frac{\partial^2 f}{\partial y^2}(0,0)\right) \\
&= 1 + \frac{1}{2}\left(-2h_1^2 - 2h_2^2\right) \\
&= 1 - h_1^2 - h_2^2
\end{aligned}$$

3. $f(x,y) = e^{x+y}$, donde $x_0 = 0$, $y_0 = 0$.

   Solución.

   $f(0,0) = e^{0+0} = 1$

$$\frac{\partial f}{\partial x}(0,0) = e^{x+y}|_{(0,0)} = 1$$
$$\frac{\partial f}{\partial y}(0,0) = e^{x+y}|_{(0,0)} = 1$$
$$\frac{\partial^2 f}{\partial x^2}(0,0) = e^{x+y}|_{(0,0)} = 1$$
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = e^{x+y}|_{(0,0)} = 1$$
$$\frac{\partial^2 f}{\partial y^2}(0,0) = e^{x+y}|_{(0,0)} = 1$$

Así, el Polinomio de Taylor es

$$\begin{aligned} f((h_1,h_2)-(x_0,y_0)) &= f(h_1,h_2) \\ &= f(0,0) + h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2\frac{\partial^2 f}{\partial x^2}(0,0) + \right. \\ &\left. h_1h_2\frac{\partial^2 f}{\partial y\partial x}(0,0) + h_2h_1\frac{\partial^2 f}{\partial x\partial y}(0,0) + h_2^2\frac{\partial^2 f}{\partial y^2}(0,0)\right) \\ &= 1 + h_1 + h_2 + \frac{1}{2}\left(h_1^2 + 2h_1h_2 + h_2^2\right) \\ &= 1 + h_1 + h_2 + h_1h_2 + \frac{h_1^2+h_2^2}{2} \end{aligned}$$

4. $f(x,y) = e^{-x^2-y^2}\cos(xy)$, donde $x_0 = 0$, $y_0 = 0$.

Solución.

$f(0,0) = e^{-0^2-0^2}\cos(0\cdot 0) = 1$

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \left.\left[-2xe^{-x^2-y^2}\cos(xy) - ye^{-x^2-y^2}\operatorname{sen}(xy)\right]\right|_{(0,0)} \\ &= \left. e^{-x^2-y^2}\left(-2x\cos(xy) - y\operatorname{sen}(xy)\right)\right|_{(0,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \left.\left[-2ye^{-x^2-y^2}\cos(xy) - xe^{-x^2-y^2}\operatorname{sen}(xy)\right]\right|_{(0,0)} \\ &= \left. e^{-x^2-y^2}\left(-2y\cos(xy) - x\operatorname{sen}(xy)\right)\right|_{(0,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(0,0) &= \left[-2xe^{-x^2-y^2}\left(-2x\cos(xy) - y\operatorname{sen}(xy)\right) + \right. \\ &\left.\left. e^{-x^2-y^2}\left(-2\cos(xy) + 2xy\operatorname{sen}(xy) - y^2\cos(xy)\right)\right]\right|_{(0,0)} \\ &= -2 \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x}(0,0) &= \left[-2ye^{-x^2-y^2}\left(-2x\cos(xy) - y\,\mathrm{sen}(xy)\right) + \right.\\
&\qquad \left.\left. e^{-x^2-y^2}\left(2x^2\,\mathrm{sen}(xy) - \mathrm{sen}(xy) - xy\cos(xy)\right)\right]\right|_{(0,0)}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2}(0,0) &= \left[-2ye^{-x^2-y^2}\left(-2y\cos(xy) - x\,\mathrm{sen}(xy)\right) + \right.\\
&\qquad \left.\left. e^{-x^2-y^2}\left(-2\cos(xy) + 2xy\,\mathrm{sen}(xy) - x^2\cos(xy)\right)\right]\right|_{(0,0)}\\
&= -2
\end{aligned}
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Así, el Polinomio de Taylor es

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\begin{aligned}
f((h_1,h_2)-(x_0,y_0)) &= f(h_1,h_2)\\
&= f(0,0) + h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2\frac{\partial^2 f}{\partial x^2}(0,0) + \right.\\
&\qquad \left. h_1h_2\frac{\partial^2 f}{\partial y\partial x}(0,0) + h_2h_1\frac{\partial^2 f}{\partial x\partial y}(0,0) + h_2^2\frac{\partial^2 f}{\partial y^2}(0,0)\right)\\
&= 1 + \frac{1}{2}\left(-2h_1^2 - 2h_2^2\right)\\
&= 1 - h_1^2 - h_2^2
\end{aligned}
$$

5. $f(x,y) = \mathrm{sen}(xy) + \cos(xy)$, donde $x_0 = 0$, $y_0 = 0$.

Solución.

$f(0,0) = \mathrm{sen}(0\cdot 0) + \cos(0\cdot 0) = 1$

$$\frac{\partial f}{\partial x}(0,0) = \left.\left[y\cos(xy) - y\,\mathrm{sen}(xy)\right]\right|_{(0,0)} = \left. y\left(\cos(xy) - \mathrm{sen}(xy)\right)\right|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \left.\left[x\cos(xy) - x\,\mathrm{sen}(xy)\right]\right|_{(0,0)} = \left. x\left(\cos(xy) - \mathrm{sen}(xy)\right)\right|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \left. y\left(-y\,\mathrm{sen}(xy) - y\cos(xy)\right)\right|_{(0,0)} = \left. -y^2\left(\mathrm{sen}(xy) + \cos(xy)\right)\right|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y\partial x}(0,0) = \left.\cos(xy) - \mathrm{sen}(xy) + y\left(-x\,\mathrm{sen}(xy) - x\cos(xy)\right)\right|_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = \left. x\left(-x\,\mathrm{sen}(xy) - x\cos(xy)\right)\right|_{(0,0)} = \left. -x^2\left(\mathrm{sen}(xy) + \cos(xy)\right)\right|_{(0,0)} = 0$$

Así, el Polinomio de Taylor es

$$
\begin{aligned}
f((h_1,h_2)-(x_0,y_0)) &= f(h_1,h_2)\\
&= f(0,0) + h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2\frac{\partial^2 f}{\partial x^2}(0,0) + \right.\\
&\qquad \left. h_1h_2\frac{\partial^2 f}{\partial y\partial x}(0,0) + h_2h_1\frac{\partial^2 f}{\partial x\partial y}(0,0) + h_2^2\frac{\partial^2 f}{\partial y^2}(0,0)\right)\\
&= 1 + h_1h_2
\end{aligned}
$$

6. $f(x,y) = e^{(x-1)^2}\cos(y)$, donde $x_0 = 1$, $y_0 = 0$.

Solución.

$f(1,0) = e^{(1-1)^2}\cos(0) = 1$

$\dfrac{\partial f}{\partial x}(1,0) = 2(x-1)e^{(x-1)^2}\cos(y)\Big|_{(1,0)} = 0$

$\dfrac{\partial f}{\partial y}(1,0) = -e^{(x-1)^2}\operatorname{sen}(y)\Big|_{(1,0)} = 0$

$\dfrac{\partial^2 f}{\partial x^2}(1,0) = 2\cos(y)\left[e^{(x-1)^2} + 2(x-1)^2 e^{(x-1)^2}\right]\Big|_{(1,0)} = 2$

$\dfrac{\partial^2 f}{\partial y \partial x}(1,0) = -2(x-1)e^{(x-1)^2}\operatorname{sen}(y)\Big|_{(1,0)} = 0$

$\dfrac{\partial^2 f}{\partial y^2}(1,0) = -e^{(x-1)^2}\cos(y)\Big|_{(1,0)} = -1$

Así, el Polinomio de Taylor es

$$
\begin{aligned}
f((h_1,h_2)-(x_0,y_0)) &= f(h_1-1,h_2) \\
&= f(1,0) + (h_1-1)\frac{\partial f}{\partial x}(1,0) + h_2\frac{\partial f}{\partial y}(1,0) + \\
&\quad \frac{1}{2}\left((h_1-1)^2\frac{\partial^2 f}{\partial x^2}(1,0) + (h_1-1)h_2\frac{\partial^2 f}{\partial y\partial x}(1,0) + \right. \\
&\quad \left. h_2(h_1-1)\frac{\partial^2 f}{\partial x\partial y}(1,0) + h_2^2\frac{\partial^2 f}{\partial y^2}(1,0)\right) \\
&= 1 + \frac{1}{2}\left(2(h_1-1)^2 - h_2^2\right) \\
&= 1 + (h_1-1)^2 - \frac{h_2^2}{2}
\end{aligned}
$$