Ejercicio 1

Sean $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, con U un abierto de $\mathbb{R}^n$. Si existen las derivadas parciales de f y como funciones son continuas en U entonces f es diferenciable.

Demostración.

Sean $\overline{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ fijo y $f = (f_1, f_2, \dots, f_n)$.

P.d. $$\lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) \right|}{\|\overline{x} - \overline{y}\|} = 0 \quad \forall\, i = 1, 2, \dots, m.$$

Para todo $i = 1, 2, \dots, m$ y $j = 1, 2, \dots, n$, sea $g : \mathbb{R} \to \mathbb{R}$ dada por $g_j(x) = f_i\left(x_1, \dots, x_{j-1}, x, y_{j+1}, \dots, y_n\right)$, donde $\overline{x} = (x_1, x_2, \dots, x_n) \in U$. Luego, para cada $i = 1, 2, \dots, m$, se tiene que

$$\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) &= f_i\left(x_1, x_2, \dots, x_n\right) - f_i\left(x_1, x_2, \dots, y_n\right) + f_i\left(x_1, x_2, \dots, y_n\right) - \\
&\quad f_i\left(x_1, x_2, \dots, y_{n-1}, y_n\right) + f_i\left(x_1, x_2, \dots, y_{n-1}, y_n\right) - \cdots - \\
&\quad f_i\left(x_1, \dots, x_{j-1}, y_j, y_{j+1}, \dots, y_n\right) + f_i\left(x_1, \dots, x_{j-1}, y_j, y_{j+1}, \dots, y_n\right) - \cdots - \\
&\quad f_i\left(x_1, y_2, \dots, y_n\right) + f_i\left(x_1, y_2, \dots, y_n\right) - f_i\left(y_1, y_2, \dots, y_n\right) \\
&= g_n(x_n) - g_n(y_n) + g_{n-1}(x_{n-1}) - g_{n-1}(y_{n-1}) + g_{n-2}(x_{n-1}) - \cdots - \\
&\quad g_j(y_j) + g_{j-1}(x_{j-1}) - \cdots - g_2(y_2) + g_1(x_1) - g_1(y_1)
\end{aligned}$$

Ya que f_i es diferenciable en U, se tiene que, para todo $j = 1, 2, \dots, n$, g_j también lo es. Así, por el Teorema del Valor Medio, para cada $j = 1, 2, \dots, n$ existe $a_j \in \left[\text{mín}\left\{x_j, y_j\right\}, \text{máx}\left\{x_j, y_j\right\}\right]$ tal que $g_j(x_j) - g_j(y_j) = g_j(a_j)\left(x_j - y_j\right)$. De este modo,

$$\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) &= g_n'(a_n)\left(x_n - y_n\right) + g_{n-1}'(a_{n-1})\left(x_{n-1} - y_{n-1}\right) + \cdots + g_j'(a_j)\left(x_j - y_j\right) + \cdots + \\
&\quad g_1'(a_1)\left(x_1 - y_1\right) \\
&= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\left(x_1 - y_1\right) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\left(x_2 - y_2\right) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\left(x_j - y_j\right) + \cdots + \\
&\quad \frac{\partial f_i}{\partial x_n}(\overline{z_n})\left(x_n - y_n\right)
\end{aligned}$$

donde $\overline{z_j} = (x_1, \dots, x_{j-1}, a_j, y_{j+1}, \dots, y_n)$. Posteriormente,

$$\begin{aligned}
f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) &= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\left(x_1 - y_1\right) - \frac{\partial f_i}{\partial x_1}(\overline{x})\left(x_1 - y_1\right) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\left(x_2 - y_2\right) - \\
&\quad \frac{\partial f_i}{\partial x_2}(\overline{x})\left(x_2 - y_2\right) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\left(x_j - y_j\right) - \\
&\quad \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) + \cdots + \frac{\partial f_i}{\partial x_n}(\overline{z_n})\left(x_n - y_n\right) - \\
&\quad \frac{\partial f_i}{\partial x_n}(\overline{x})\left(x_n - y_n\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right)(x_1 - y_1) + \\
&\quad \left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right)(x_2 - y_2) + \cdots + \\
&\quad \left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right)(x_j - y_j) + \cdots + \\
&\quad \left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right)(x_n - y_n)
\end{aligned}
$$

$$
\begin{aligned}
\implies \left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right)\right| &= \left|\left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right)(x_1 - y_1) + \right. \\
&\quad \left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right)(x_2 - y_2) + \cdots + \\
&\quad \left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right)(x_j - y_j) + \cdots + \\
&\quad \left.\left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right)(x_n - y_n)\right| \\
&\leq \left|\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right| |x_1 - y_1| + \\
&\quad \left|\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right| |x_2 - y_2| + \cdots + \\
&\quad \left|\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right| |x_j - y_j| + \cdots + \\
&\quad \left|\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right| |x_n - y_n| \\
&\leq \left|\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right| \|\overline{x} - \overline{y}\| + \\
&\quad \left|\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right| \|\overline{x} - \overline{y}\| + \cdots + \\
&\quad \left|\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right| \|\overline{x} - \overline{y}\| + \cdots + \\
&\quad \left|\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right| \|\overline{x} - \overline{y}\| \\
&= \left(\left|\frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1})\right| + \left|\frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2})\right| + \cdots + \right. \\
&\quad \left.\left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j})\right| + \cdots + \left|\frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n})\right|\right) \|\overline{x} - \overline{y}\|
\end{aligned}
$$

$$\implies \frac{\left|f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right)\right|}{\|\overline{x} - \overline{y}\|} \leq \left|\frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1})\right| + \left|\frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2})\right| + \cdots + \left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j})\right| + \cdots + \left|\frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n})\right|$$

Después, como las derivadas parciales son continuas en U se da que $\lim\limits_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{y})\right| = 0$, y puesto que $\overline{z_j} = (x_1, \ldots, x_{j-1}, a_j, y_{j+1}, \ldots, y_n)$ con $a_j \in \left[\text{mín}\left\{x_j, y_j\right\}, \text{máx}\left\{x_j, y_j\right\}\right]$, se tiene que $\overline{z_j}$ tiende a $\overline{y}$ conforme $\overline{x}$ tiende a $\overline{y}$, para todo $j = 1, 2, \ldots, n$. De esta manera,

$$\begin{aligned}
0 \leq \lim_{\overline{x}\to\overline{y}} \frac{\left|f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right)\right|}{\|\overline{x} - \overline{y}\|} &\leq \lim_{\overline{x}\to\overline{y}} \Bigg(\left|\frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1})\right| + \left|\frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2})\right| + \cdots + \left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j})\right| + \cdots + \left|\frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n})\right| \Bigg) \\
&= \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1})\right| + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2})\right| + \cdots + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j})\right| + \cdots + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n})\right| \\
&= \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{y})\right| + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{y})\right| + \cdots + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{y})\right| + \cdots + \lim_{\overline{x}\to\overline{y}} \left|\frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{y})\right| \\
&= 0
\end{aligned}$$

De esta manera, $\lim\limits_{\overline{x}\to\overline{y}} \dfrac{\left|f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right)\right|}{\|\overline{x} - \overline{y}\|} = 0 \quad \forall\, i = 1, \ldots, m.$

Por último,

$$0 \leq \lim_{\overline{x} \to \overline{y}} \frac{\|f(\overline{x}) - f(\overline{y}) - Df(\overline{y})(\overline{x} - \overline{y})\|}{\|\overline{x} - \overline{y}\|} \leq \sum_{i=1}^{m} \lim_{\overline{x} \to \overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\left(x_j - y_j\right) \right|}{\|\overline{x} - \overline{y}\|} = 0$$

Ya que $\overline{y}$ fue arbitrario, se concluye que f es diferenciable en U. ■

Ejercicio 2

Sean $A, B \subseteq \mathbb{R}$ no vacíos. Si $a \leq b$ para todo $a \in A$ y para todo $b \in B$, entonces A está acotada superiormente y B está acotado inferiormente, y además, $\sup(A) \leq \inf(B)$.

Demostración.

Sea $b \in B$, ya que $a \leq b$ para todo $a \in A$, se tiene que A está acotado superiormente. De igual forma, sea $a \in A$ como $a \leq b$ para todo $b \in B$, se da que B está acotado inferiormente. Además, dado que A y B son no vacíos, se obtiene que el supremo y el ínfimo de A y B existen, respectivamente. Como todo elemento de B es cota superior de A se tiene que $\sup(A)$ es una cota inferior de B. Por lo tanto, $\sup(A) \leq \inf(B)$, pues $\inf(B)$ es la mayor cota inferior de B.

Ejercicio 3

Si $f \in \mathcal{R}(\alpha)$ en $[a, b]$ y $a < c < b$ entonces $f \in \mathcal{R}(\alpha)$ en $[a, c]$ y en $[c, b]$ y

$$\int_a^b f \, \mathrm{d}\alpha = \int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha$$

Demostración.

Afirmación. $f \in \mathcal{R}(\alpha)$ en $[a, c]$ y en $[c, b]$.

Sea $\varepsilon > 0$. Ya que $f \in \mathcal{R}(\alpha)$ en $[a, b]$ existe $P_\varepsilon = \{a = x_0, x_1, \ldots, x_n = b\} \in \gamma_{[a,b]}$ tal que $U(f, P_\varepsilon, \alpha) - L(f, P_\varepsilon, \alpha) < \varepsilon$. Luego, para cada $j = 1, \ldots, n$ sean $M_j = \sup\left\{f(x) \,\middle|\, x \in \left[x_{j-1}, x_j\right]\right\}$ y $m_j = \inf\left\{f(x) \,\middle|\, x \in \left[x_{j-1}, x_j\right]\right\}$. Ya que $a < c < b$, existe $i = 1, \ldots, n$ tal que $x_{i-1} < c \leq x_i$, por lo que se definen $M'_c = \sup\{f(x) \mid x \in [x_{i-1}, c]\}$, $M''_c = \sup\{f(x) \mid x \in [c, x_i]\}$, $m'_c = \inf\{f(x) \mid x \in [x_{i-1}, c]\}$ y $m''_c = \inf\{f(x) \mid x \in [c, x_i]\}$.

Después, considerando las particiones $P'_\varepsilon = \{a = x_0, x_1, \ldots, x_{i-1}, c\} \in \gamma_{[a,c]}$ y $P''_\varepsilon = \{c, x_i, x_{i+1}, \ldots, x_n\} \in \gamma_{[c,b]}$, se tiene que

$$\mathcal{U}\left(f, P'_\varepsilon, \alpha\right) = \sum_{j=1}^{i-1} M_j \Delta\alpha_j + M'_c\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right],$$

$$\mathcal{U}\left(f, P''_\varepsilon, \alpha\right) = M''_c\left[\alpha(x_i) - \alpha\left(c\right)\right] + \sum_{j=i+1}^{n} M_j \Delta\alpha_j,$$

$$\mathcal{L}\left(f, P'_\varepsilon, \alpha\right) = \sum_{j=1}^{i-1} m_j \Delta\alpha_j + m'_c\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right] \qquad \text{y}$$

$$\mathcal{L}\left(f, P''_\varepsilon, \alpha\right) = m''_c\left[\alpha(x_i) - \alpha\left(c\right)\right] + \sum_{j=i+1}^{n} m_j \Delta\alpha_j.$$

Puesto que $[x_{i-1}, c] \cup [c, x_i] = [x_{i-1}, x_i]$ se da que $M'_c \leq M_i, M''_c \leq M_i, m_i \leq m'_c$ y $m_i \leq m''_c$. Así,

$$\begin{aligned}
\mathcal{U}\left(f, P'_\varepsilon, \alpha\right) + \mathcal{U}\left(f, P''_\varepsilon, \alpha\right) &= \sum_{j=1}^{i-1} M_j \Delta\alpha_j + M'_c\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right] + M''_c\left[\alpha(x_i) - \alpha\left(c\right)\right] + \\
&\quad \sum_{j=i+1}^{n} M_j \Delta\alpha_j \\
&\leq \sum_{j=1}^{i-1} M_j \Delta\alpha_j + M_i\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right] + M_i\left[\alpha(x_i) - \alpha\left(c\right)\right] + \\
&\quad \sum_{j=i+1}^{n} M_j \Delta\alpha_j \\
&= \sum_{j=1}^{n} M_j \Delta\alpha_j \\
&= U(f, P_\varepsilon, \alpha)
\end{aligned}$$

y

$$\begin{aligned}
-\mathcal{L}\left(f, P'_\varepsilon, \alpha\right) - \mathcal{L}\left(f, P''_\varepsilon, \alpha\right) &= -\sum_{j=1}^{i-1} m_j \Delta\alpha_j - m'_c\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right] - m''_c\left[\alpha(x_i) - \alpha\left(c\right)\right] \\
&\quad - \sum_{j=i+1}^{n} m_j \Delta\alpha_j \\
&\leq -\sum_{j=1}^{i-1} m_j \Delta\alpha_j - m_i\left[\alpha(c) - \alpha\left(x_{i-1}\right)\right] - m_i\left[\alpha(x_i) - \alpha\left(c\right)\right] - \\
&\quad \sum_{j=i+1}^{n} m_j \Delta\alpha_j \\
&= \sum_{j=1}^{n} m_j \Delta\alpha_j \\
&= -\mathcal{L}(f, P_\varepsilon, \alpha)
\end{aligned}$$

De esta manera,

$$\begin{aligned}
&\mathcal{U}\left(f,P'_\varepsilon,\alpha\right)+\mathcal{U}\left(f,P''_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P'_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P''_\varepsilon,\alpha\right)\leq U(f,P_\varepsilon,\alpha)-\mathcal{L}(f,P_\varepsilon,\alpha)<\varepsilon\\
&\Longrightarrow \mathcal{U}\left(f,P'_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P'_\varepsilon,\alpha\right)+\mathcal{U}\left(f,P''_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P''_\varepsilon,\alpha\right)<\varepsilon\\
&\Longrightarrow \mathcal{U}\left(f,P'_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P'_\varepsilon,\alpha\right)<\varepsilon \quad \text{y} \quad \mathcal{U}\left(f,P''_\varepsilon,\alpha\right)-\mathcal{L}\left(f,P''_\varepsilon,\alpha\right)<\varepsilon
\end{aligned}$$

Por lo que $f\in\mathcal{R}(\alpha)$ en $[a,c]$ y en $[c,b]$.

Ahora, sea $P=\{a=x_0,x_1,\dots,x_n=b\}\in\gamma_{[a,b]}$, procediendo como antes, se da que existen $P'=\{a=x_0,x_1,\dots,x_{i-1},c\}\in\gamma_{[a,c]}$ y $P''=\{c,x_i,x_{i+1},\dots,x_n=b\}\in\gamma_{[c,b]}$ tales que $\mathcal{U}(f,P',\alpha)+\mathcal{U}(f,P'',\alpha)\leq\mathcal{U}(f,P,\alpha)$ y $\mathcal{L}(f,P',\alpha)+\mathcal{L}(f,P'',\alpha)\geq\mathcal{L}(f,P,\alpha)$. Posteriormente, se tiene que

$$\begin{aligned}
&\mathcal{L}(f,P',\alpha)\leq\int_a^c f\ \mathrm{d}\alpha\leq\mathcal{U}(f,P',\alpha) \quad \text{y} \quad \mathcal{L}(f,P'',\alpha)\leq\int_c^b f\ \mathrm{d}\alpha\leq\mathcal{U}(f,P'',\alpha)\\
&\Longrightarrow \mathcal{L}(f,P',\alpha)+\mathcal{L}(f,P'',\alpha)\leq\int_a^c f\ \mathrm{d}\alpha+\int_c^b f\ \mathrm{d}\alpha\leq\mathcal{U}(f,P',\alpha)+\mathcal{U}(f,P'',\alpha)\\
&\Longrightarrow \mathcal{L}(f,P,\alpha)\leq\int_a^c f\ \mathrm{d}\alpha+\int_c^b f\ \mathrm{d}\alpha\leq\mathcal{U}(f,P,\alpha)
\end{aligned}$$

Como P fue arbitraria, se obtiene que $\displaystyle\int_a^b f\ \mathrm{d}\alpha=\int_a^c f\ \mathrm{d}\alpha+\int_c^b f\ \mathrm{d}\alpha$.