Proposición. Sean $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, con U un abierto de $\mathbb{R}^n$. Si existen las derivadas parciales de f y como funciones son continuas en U entonces f es diferenciable.

Demostración.

Sean $\overline{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ fijo y $f = (f_1, f_2, \ldots, f_n)$.

P.d. $$\lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right|}{\|\overline{x} - \overline{y}\|} = 0 \quad \forall\, i = 1, 2, \ldots, m.$$

Para todo $i = 1, 2, \ldots, m$ y $j = 1, 2, \ldots, n$ sea $g : \mathbb{R} \to \mathbb{R}$ dada por $g_j(x) = f_i\left(x_1, \ldots, x_{j-1}, x, y_{j+1}, \ldots, y_n\right)$, donde $\overline{x} = (x_1, x_2, \ldots, x_n) \in U$. Luego, para cada $i = 1, 2, \ldots, m$, se tiene que

$$\begin{aligned} f_i(\overline{x}) - f_i(\overline{y}) &= f_i\left(x_1, x_2, \ldots, x_n\right) - f_i\left(x_1, x_2, \ldots, y_n\right) + f_i\left(x_1, x_2, \ldots, y_n\right) - \\ &\quad f_i\left(x_1, x_2, \ldots, y_{n-1}, y_n\right) + f_i\left(x_1, x_2, \ldots, y_{n-1}, y_n\right) - \cdots - \\ &\quad f_i\left(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n\right) + f_i\left(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n\right) - \cdots - \\ &\quad f_i\left(x_1, y_2, \ldots, y_n\right) + f_i\left(x_1, y_2, \ldots, y_n\right) - f_i\left(y_1, y_2, \ldots, y_n\right) \\ &= g_n(x_n) - g_n(y_n) + g_{n-1}(x_{n-1}) - g_{n-1}(y_{n-1}) + g_{n-2}(x_{n-1}) - \cdots - \\ &\quad g_j(y_j) + g_{j-1}(x_{j-1}) - \cdots - g_2(y_2) + g_1(x_1) - g_1(y_1) \end{aligned}$$

Ya que f_i es diferenciable en U, se tiene que, para todo $j = 1, 2, \ldots, n$, g_j también lo es. Así, por el Teorema del Valor Medio, para cada $j = 1, 2, \ldots, n$ existe $a_j \in \left[\min\left\{x_j, y_j\right\}, \max\left\{x_j, y_j\right\}\right]$ tal que $g_j(x_j) - g_j(y_j) = g_j(a_j)\,(x_j - y_j)$. De este modo,

$$\begin{aligned} f_i(\overline{x}) - f_i(\overline{y}) &= g_n'(a_n)\,(x_n - y_n) + g_{n-1}'(a_{n-1})\,(x_{n-1} - y_{n-1}) + \cdots + g_j'(a_j)\,(x_j - y_j) + \cdots + g_1'(a_1)\,(x_1 - y_1) \\ &= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\,(x_1 - y_1) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\,(x_2 - y_2) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\,(x_j - y_j) + \cdots + \frac{\partial f_i}{\partial x_n}(\overline{z_n})\,(x_n - y_n) \end{aligned}$$

donde $\overline{z_j} = (x_1, \ldots, x_{j-1}, a_j, y_{j+1}, \ldots, y_n)$. Posteriormente,

$$\begin{aligned} f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) &= \frac{\partial f_i}{\partial x_1}(\overline{z_1})\,(x_1 - y_1) - \frac{\partial f_i}{\partial x_1}(\overline{x})\,(x_1 - y_1) + \frac{\partial f_i}{\partial x_2}(\overline{z_2})\,(x_2 - y_2) - \\ &\quad \frac{\partial f_i}{\partial x_2}(\overline{x})\,(x_2 - y_2) + \cdots + \frac{\partial f_i}{\partial x_j}(\overline{z_j})\,(x_j - y_j) - \\ &\quad \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) + \cdots + \frac{\partial f_i}{\partial x_n}(\overline{z_n})\,(x_n - y_n) - \frac{\partial f_i}{\partial x_n}(\overline{x})\,(x_n - y_n) \\ &= \left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x})\right)(x_1 - y_1) + \\ &\quad \left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x})\right)(x_2 - y_2) + \cdots + \\ &\quad \left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x})\right)(x_j - y_j) + \cdots + \\ &\quad \left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x})\right)(x_n - y_n) \end{aligned}$$

$$\Longrightarrow \left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right| = \left| \left(\frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right)(x_1 - y_1) + \right.$$
$$\left(\frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right)(x_2 - y_2) + \cdots +$$
$$\left(\frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right)(x_j - y_j) + \cdots +$$
$$\left. \left(\frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right)(x_n - y_n) \right|$$
$$\leq \left| \frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right| |x_1 - y_1| +$$
$$\left| \frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right| |x_2 - y_2| + \cdots +$$
$$\left| \frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right| |x_j - y_j| + \cdots +$$
$$\left| \frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right| |x_n - y_n|$$
$$\leq \left| \frac{\partial f_i}{\partial x_1}(\overline{z_1}) - \frac{\partial f_i}{\partial x_1}(\overline{x}) \right| \|\overline{x} - \overline{y}\| +$$
$$\left| \frac{\partial f_i}{\partial x_2}(\overline{z_2}) - \frac{\partial f_i}{\partial x_2}(\overline{x}) \right| \|\overline{x} - \overline{y}\| + \cdots +$$
$$\left| \frac{\partial f_i}{\partial x_j}(\overline{z_j}) - \frac{\partial f_i}{\partial x_j}(\overline{x}) \right| \|\overline{x} - \overline{y}\| + \cdots +$$
$$\left| \frac{\partial f_i}{\partial x_n}(\overline{z_n}) - \frac{\partial f_i}{\partial x_n}(\overline{x}) \right| \|\overline{x} - \overline{y}\|$$
$$= \left(\left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots + \right.$$
$$\left. \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right| \right) \|\overline{x} - \overline{y}\|$$
$$\Longrightarrow \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right|}{\|\overline{x} - \overline{y}\|} \leq \left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots +$$
$$\left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right|$$

Después, como las derivadas parciales son continuas en U se da que $\lim\limits_{\overline{x} \to \overline{y}} \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{y}) \right| = 0$, y puesto que $\overline{z_j} = (x_1, \ldots, x_{j-1}, a_j, y_{j+1}, \ldots, y_n)$ con $a_j \in \left[\text{mín}\{x_j, y_j\}, \text{máx}\{x_j, y_j\}\right]$, se tiene que $\overline{z_j}$ tiende a $\overline{y}$ conforme $\overline{x}$ tiende a $\overline{y}$, para todo $j = 1, 2, \ldots, n$. De esta manera,

$$
\begin{aligned}
0 \leq \lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right|}{\|\overline{x}-\overline{y}\|} &\leq \lim_{\overline{x}\to\overline{y}} \left(\left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots + \right.\\
&\quad \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \\
&\quad \left. \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right| \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{z_1}) \right| + \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{z_2}) \right| + \cdots + \\
&\quad \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{z_j}) \right| + \cdots + \\
&\quad \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{z_n}) \right| \\
&= \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_1}(\overline{x}) - \frac{\partial f_i}{\partial x_1}(\overline{y}) \right| + \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_2}(\overline{x}) - \frac{\partial f_i}{\partial x_2}(\overline{y}) \right| + \cdots + \\
&\quad \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_j}(\overline{x}) - \frac{\partial f_i}{\partial x_j}(\overline{y}) \right| + \cdots + \\
&\quad \lim_{\overline{x}\to\overline{y}} \left| \frac{\partial f_i}{\partial x_n}(\overline{x}) - \frac{\partial f_i}{\partial x_n}(\overline{y}) \right| \\
&= 0
\end{aligned}
$$

De esta manera, $\displaystyle\lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right|}{\|\overline{x}-\overline{y}\|} = 0 \quad \forall\, i = 1, \ldots, m.$

Por último,

$$
0 \leq \lim_{\overline{x}\to\overline{y}} \frac{\|f(\overline{x}) - f(\overline{y}) - Df(\overline{y})\,(\overline{x}-\overline{y})\|}{\|\overline{x}-\overline{y}\|} \leq \sum_{i=1}^{m} \lim_{\overline{x}\to\overline{y}} \frac{\left| f_i(\overline{x}) - f_i(\overline{y}) - \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\overline{x})\,(x_j - y_j) \right|}{\|\overline{x}-\overline{y}\|} = 0
$$

Ya que $\overline{y}$ fue arbitrario, se concluye que f es diferenciable en U. ■

Proposición. Sean $A, B \subseteq \mathbb{R}$ no vacíos. Si $a \leq b$ para todo $a \in A$ y para todo $b \in B$, entonces A está acotada superiormente y B está acotado inferiormente, y además, $\sup(A) \leq \inf(B)$.

Demostración.

Sea $b \in B$, ya que $a \leq b$ para todo $a \in A$, se tiene que A está acotado superiormente. De igual forma, sea $a \in A$ como $a \leq b$ para todo $b \in B$, se da que B está acotado inferiormente. Además, dado que A y B son no vacíos, se obtiene que el supremo y el ínfimo de A y B existen, respectivamente. Como todo elemento de B es cota superior de A se tiene que $\sup(A)$ es una cota inferior de B. Por lo tanto, $\sup(A) \leq \inf(B)$, pues $\inf(B)$ es la mayor cota inferior de B.