12. $f(t) = e^{-2t-5}$

Solución.

$$\begin{aligned}
F(s) &= \mathscr{L}\left\{e^{-2t-5}\right\} \\
&= \int_0^\infty e^{-st}e^{-2t-5}\,\mathrm{d}t \\
&= \lim_{r\to\infty}\int_0^r e^{-st}e^{-2t-5}\,\mathrm{d}t \\
&= \lim_{r\to\infty} e^{-5}\int_0^r e^{-t(s+2)}\,\mathrm{d}t \\
&= e^{-5}\lim_{r\to\infty}\left[-\left.\frac{e^{-t(s+2)}}{s+2}\right|_0^r\right]\mathrm{d}t \\
&= e^{-5}\lim_{r\to\infty}\left[-\frac{e^{-r(s+2)}}{s+2}+\frac{1}{s+2}\right] \\
&= \frac{e^-5}{s+2} \quad \text{con } s > -2
\end{aligned}$$

32. $f(t) = \cos(5t) + \operatorname{sen}(2t)$

Solución.

$$\begin{aligned}
F(s) &= \mathscr{L}\left\{\cos(5t) + \operatorname{sen}(2t)\right\} \\
&= \mathscr{L}\left\{\cos(5t)\right\} + \mathscr{L}\left\{\operatorname{sen}(2t)\right\} \\
&= \frac{s}{s^2+25} + \frac{2}{s^2+4}
\end{aligned}$$

41. Una definición de la función gamma está dada por la integral impropia

$\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t, \alpha > 0$

a) Demuestre que $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$.

b) Demuestre que $\mathscr{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$

Demostración.

a)

$$\begin{aligned}
\Gamma(\alpha+1) &= \int_0^\infty t^{\alpha+1-1}e^{-t}\,\mathrm{d}t \\
&= \int_0^\infty t^{\alpha}e^{-t}\,\mathrm{d}t \\
&= \lim_{r\to\infty}\left[\int_0^r t^{\alpha}e^{-t}\,\mathrm{d}t\right]
\end{aligned}$$

Sean $u = t^\alpha$ y $\mathrm{d}v = e^{-t}\mathrm{d}t$ entonces $\mathrm{d}u = \alpha t^{\alpha-1}\,\mathrm{d}t$ y $v = -e^{-t}$. Así,

$$\begin{aligned}
\Gamma(\alpha+1) &= \lim_{r\to\infty}\left[-t^\alpha e^{-t}\Big|_0^r + \int_0^r \alpha\, t^{\alpha-1}e^{-t}\,\mathrm{d}t\right]\\
&= \lim_{r\to\infty}\left[-r^\alpha e^{-r} + \alpha\int_0^r t^{\alpha-1}e^{-t}\,\mathrm{d}t\right]\\
&= \alpha\int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t\\
&= \alpha\,\Gamma(\alpha)
\end{aligned}$$

$\therefore \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

b) $\mathscr{L}\{t^\alpha\} = \displaystyle\int_0^\infty e^{-st}t^\alpha\,\mathrm{d}t$

Sea $u = st$ entonces $\mathrm{d}u = s\,\mathrm{d}t$ y $u = \dfrac{u}{s}$. Aplicando cambio de variables:

$$\begin{aligned}
\mathscr{L}\{t^\alpha\} &= \int_{s(0)}^{s(\infty)}\left(\frac{u}{s}\right)^\alpha \frac{e^{-u}}{s}\,\mathrm{d}u\\
&= \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,\mathrm{d}u \qquad (\text{con } s > 0)\\
&= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \qquad (\text{con } \alpha > -1)
\end{aligned}$$

$\therefore \mathscr{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

28. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4-9}\right\}$

$$\frac{1}{s^4-9} = \frac{1}{(s^2-3)(s^2+3)}$$

Sea $\dfrac{1}{(s^2-3)(s^2+3)} = \dfrac{As+B}{s^2-3} + \dfrac{Cs+D}{s^2+3}$

$$\begin{aligned}
\implies 1 &= (As+B)(s^2+3) + (Cs+D)(s^2-3)\\
&= As^3 + 3As + Bs^2 + 3B + Cs^3 - 3Cs + Ds^2 - 3D\\
&= s^3(A+C) + s^2(B+D) + s(3A-3C) + 3B - 3D
\end{aligned}$$

De aqui, se obtiene el siguiente sistema de ecuaciones

$$A + C = 0 \tag{1}$$

$$B + D = 0 \tag{2}$$

$$3A - 3C = 0 \tag{3}$$

$$3B - 3D = 1 \tag{4}$$

De (1): $A = -C$. Sustituyendo esto en (3) se tiene que $-3C - 3C = 0 \Longrightarrow -6C = 0 \Longrightarrow C = 0$. De esta manera, $A = -0 = 0$

Luego, de (2): $B = -D$. Sustituyendo esto en (4) $-3D - 3D = 1 \Longrightarrow -6D = 1 \Longrightarrow D = -\dfrac{1}{6}$. De esta forma, $B = \dfrac{1}{6}$.

Así $\dfrac{1}{(s^2-3)(s^2+3)} = \dfrac{\frac{1}{6}}{s^2-3} - \dfrac{\frac{1}{6}}{s^2+3}$.

Después,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s^4-9}\right\} &= \mathscr{L}^{-1}\left\{\frac{\frac{1}{6}}{s^2-3} - \frac{\frac{1}{6}}{s^2+3}\right\} \\
&= \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s^2-3}\right\} - \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s^2+3}\right\} \\
&= \frac{1}{6\sqrt{3}}\mathscr{L}^{-1}\left\{\frac{\sqrt{3}}{s^2-3}\right\} - \frac{1}{6\sqrt{3}}\mathscr{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+3}\right\} \\
&= \frac{\operatorname{senh}(t\sqrt{3})}{6\sqrt{3}} - \frac{\operatorname{sen}(t\sqrt{3})}{6\sqrt{3}}
\end{aligned}$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^4-9}\right\} = \frac{\operatorname{senh}(t\sqrt{3})}{6\sqrt{3}} - \frac{\operatorname{sen}(t\sqrt{3})}{6\sqrt{3}}$$