UNIVERSIDAD AUTÓNOMA DEL ESTADO DE MÉXICO
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS
Cálculo Diferencial Vectorial
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Tarea: Multiplicadores de Lagrange

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1. $\nabla(\mathbf{f}+\mathbf{g}) = \nabla\mathbf{f} + \nabla\mathbf{g}$

   Solución.

$$\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial(f+g)}{\partial x_1}, \frac{\partial(f+g)}{\partial x_2}, \dots, \frac{\partial(f+g)}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_1}, \frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} + \frac{\partial g}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) + \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}\right) \\
&= \nabla f + \nabla g
\end{aligned}$$

   Por lo tanto, $\nabla(f+g) = \nabla f + \nabla g$.

2. $\nabla(\mathbf{cf}) = \mathbf{c}\nabla\mathbf{f}$

   Solución.

$$\begin{aligned}
\nabla(cf) &= \left(\frac{\partial(cf)}{\partial x_1}, \frac{\partial(cf)}{\partial x_2}, \dots, \frac{\partial(cf)}{\partial x_n}\right) \\
&= \left(c\frac{\partial f}{\partial x_1}, c\frac{\partial f}{\partial x_2}, \dots, c\frac{\partial f}{\partial x_n}\right) \\
&= c\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) \\
&= c\nabla f
\end{aligned}$$

   Por lo tanto, $\nabla(cf) = c\nabla f$.

3. $\nabla(\mathbf{fg}) = \mathbf{f}\nabla\mathbf{g} + \mathbf{g}\nabla\mathbf{f}$

   Solución.

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial(fg)}{\partial x_1}, \frac{\partial(fg)}{\partial x_2}, \dots, \frac{\partial(fg)}{\partial x_n}\right) \\
&= \left(f\frac{\partial g}{\partial x_1} + g\frac{\partial f}{\partial x_1}, f\frac{\partial g}{\partial x_2} + g\frac{\partial f}{\partial x_2}, \dots, f\frac{\partial g}{\partial x_n} + g\frac{\partial f}{\partial x_n}\right) \\
&= f\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}\right) + g\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right) \\
&= f\nabla g + g\nabla f
\end{aligned}$$

   Por lo tanto, $\nabla(fg) = f\nabla g + g\nabla f$.

4. $\nabla\left(\dfrac{\mathbf{f}}{\mathbf{g}}\right) = \dfrac{\mathbf{g}\ \nabla\mathbf{f} - \mathbf{f}\ \nabla\mathbf{g}}{\mathbf{g^2}}$

Solución.

Calculando $\nabla\left(\dfrac{1}{g}\right)$:

$$\begin{aligned}
\nabla\left(\frac{1}{g}\right) &= \left(\frac{\partial}{\partial x_1}\left(\frac{1}{g}\right), \frac{\partial}{\partial x_2}\left(\frac{1}{g}\right), \ldots, \frac{\partial}{\partial x_n}\left(\frac{1}{g}\right)\right) \\
&= \left(-\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_1}, -\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_2}, \ldots, -\frac{1}{g^2}\cdot\frac{\partial g}{\partial x_n}\right) \\
&= -\frac{1}{g^2}\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_n}\right) \\
&= -\frac{1}{g^2}\nabla g
\end{aligned}$$

Luego, por el inciso 3, se tiene que

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \nabla\left(f\cdot\frac{1}{g}\right) \\
&= f\nabla\left(\frac{1}{g}\right) + \frac{1}{g}\nabla f \\
&= f\left(-\frac{1}{g^2}\nabla g\right) + \frac{1}{g}\nabla f \\
&= \frac{\nabla f}{g} - \frac{f\nabla g}{g^2} \\
&= \frac{g\ \nabla f - f\ \nabla g}{g^2}
\end{aligned}$$

Por lo tanto, $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\ \nabla f - f\ \nabla g}{g^2}$.

5. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

Solución.

$$\begin{aligned}
\operatorname{div}(F+G) &= \frac{\partial(f_1+g_1)}{\partial x_1} + \frac{\partial(f_2+g_2)}{\partial x_2} + \cdots + \frac{\partial(f_n+g_n)}{\partial x_n} \\
&= \frac{\partial f_1}{\partial x_1} + \frac{\partial g_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial g_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n} + \frac{\partial g_n}{\partial x_n} \\
&= \operatorname{div}F + \operatorname{div}G
\end{aligned}$$

Por lo tanto, $\operatorname{div}(F+G) = \operatorname{div}F + \operatorname{div}G$

6. $\operatorname{rot}(\mathbf{F}+\mathbf{G}) = \operatorname{rot}\mathbf{F} + \operatorname{rot}\mathbf{G}$

Solución.

$$
\begin{aligned}
\operatorname{rot}(F+G) &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1+g_1 & f_2+g_2 & f_3+g_3 \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2+g_2 & f_3+g_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1+g_1 & f_3+g_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1+g_1 & f_2+g_2 \end{vmatrix} \widehat{k} \\
&= \left(\frac{\partial (f_3+g_3)}{\partial y} - \frac{\partial (f_2+g_2)}{\partial z} \right) \widehat{i} - \left(\frac{\partial (f_3+g_3)}{\partial x} - \frac{\partial (f_1+g_1)}{\partial z} \right) \widehat{j} + \\
&\quad \left(\frac{\partial (f_2+g_2)}{\partial x} - \frac{\partial (f_1+g_1)}{\partial y} \right) \widehat{k} \\
&= \left(\frac{\partial f_3}{\partial y} + \frac{\partial g_3}{\partial y} - \frac{\partial f_2}{\partial z} - \frac{\partial g_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial f_3}{\partial x} + \frac{\partial g_3}{\partial x} - \frac{\partial f_1}{\partial z} - \frac{\partial g_1}{\partial z} \right) \widehat{j} + \\
&\quad \left(\frac{\partial f_2}{\partial x} + \frac{\partial g_2}{\partial x} - \frac{\partial f_1}{\partial y} - \frac{\partial g_1}{\partial y} \right) \widehat{k} \\
&= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \widehat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \widehat{k} + \\
&\quad \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z} \right) \widehat{i} - \left(\frac{\partial g_3}{\partial x} - \frac{\partial g_1}{\partial z} \right) \widehat{j} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right) \widehat{k} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix} \widehat{k} + \\
&\quad \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ g_2 & g_3 \end{vmatrix} \widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ g_1 & g_3 \end{vmatrix} \widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ g_1 & g_2 \end{vmatrix} \widehat{k} \\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix} \\
&= \operatorname{rot} F + \operatorname{rot} G
\end{aligned}
$$

Por lo tanto, $\operatorname{rot}(F+G) = \operatorname{rot} F + \operatorname{rot} G$.

7. $\operatorname{div}(\mathbf{fF}) = \mathbf{f} \operatorname{div} \mathbf{F} + \nabla \mathbf{fF}$

   Solución.

$$
\begin{aligned}
\operatorname{div}(fF) &= \frac{\partial (f g_1)}{\partial x_1} + \frac{\partial (f g_2)}{\partial x_2} + \cdots + \frac{\partial (f g_n)}{\partial x_n} \\
&= f \frac{\partial g_1}{\partial x_1} + g_1 \frac{\partial f}{\partial x_1} + f \frac{\partial g_2}{\partial x_2} + g_2 \frac{\partial f}{\partial x_2} + \cdots + f \frac{\partial g_n}{\partial x_n} + g_n \frac{\partial f}{\partial x_n}
\end{aligned}
$$

$$
\begin{aligned}
&= f\left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \cdots + \frac{\partial g_n}{\partial x_n}\right) + g_1\frac{\partial f}{\partial x_1} + g_2\frac{\partial f}{\partial x_2} + \cdots + g_n\frac{\partial f}{\partial x_n}\\
&= f\,\operatorname{div} F + \nabla f F
\end{aligned}
$$

Por lo tanto, $\operatorname{div}(fF) = f\,\operatorname{div} F + \nabla f F$.

11. $\operatorname{rot}(fF) = f\operatorname{rot} F + \nabla f \times F$

Solución.

$$
\begin{aligned}
\operatorname{rot}(fF) &= \nabla \times fF\\
&= \nabla \times (ff_1, ff_2, ff_3)\\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ ff_1 & ff_2 & ff_3 \end{vmatrix}\\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ ff_2 & ff_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z}\\ ff_1 & ff_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\\ ff_1 & ff_2 \end{vmatrix}\widehat{k}\\
&= \left(\frac{\partial(ff_3)}{\partial y} - \frac{\partial(ff_2)}{\partial z}\right)\widehat{i} - \left(\frac{\partial(ff_3)}{\partial x} - \frac{\partial(ff_1)}{\partial z}\right)\widehat{j} + \left(\frac{\partial(ff_2)}{\partial x} - \frac{\partial(ff_1)}{\partial y}\right)\widehat{k}\\
&= \left(f\cdot\frac{\partial f_3}{\partial y} + f_3\cdot\frac{\partial f}{\partial y} - f\cdot\frac{\partial f_2}{\partial z} - f_2\cdot\frac{\partial f}{\partial z}\right)\widehat{i}\\
&\quad - \left(f\cdot\frac{\partial f_3}{\partial x} + f_3\cdot\frac{\partial f}{\partial x} - f\cdot\frac{\partial f_1}{\partial z} - f_1\cdot\frac{\partial f}{\partial z}\right)\widehat{j}\\
&\quad + \left(f\cdot\frac{\partial f_2}{\partial x} + f_2\cdot\frac{\partial f}{\partial x} - f\cdot\frac{\partial f_1}{\partial y} - f_1\cdot\frac{\partial f}{\partial y}\right)\widehat{k}\\
&= f\left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\widehat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\widehat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\widehat{k}\right] +\\
&\quad \left(f_3\cdot\frac{\partial f}{\partial y} - f_2\cdot\frac{\partial f}{\partial z}\right)\widehat{i} - \left(f_3\cdot\frac{\partial f}{\partial x} - f_1\cdot\frac{\partial f}{\partial z}\right)\widehat{j} + \left(f_2\cdot\frac{\partial f}{\partial x} - f_1\cdot\frac{\partial f}{\partial y}\right)\widehat{k}\\
&= f\left(\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f_2 & f_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z}\\ f_1 & f_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\\ f_1 & f_2 \end{vmatrix}\widehat{k}\right) +\\
&\quad \begin{vmatrix} \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\\ f_2 & f_3 \end{vmatrix}\widehat{i} - \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z}\\ f_1 & f_3 \end{vmatrix}\widehat{j} + \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ f_1 & f_2 \end{vmatrix}\widehat{k}\\
&= f\begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\\ f_1 & f_2 & f_3 \end{vmatrix}\\
&= f\operatorname{rot} F + \nabla f \times F
\end{aligned}
$$