

**Proposición.** Sean  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ , con  $U$  un abierto de  $\mathbb{R}^n$ . Si existen las derivadas parciales de  $f$  y como funciones son continuas en  $U$  entonces  $f$  es diferenciable.

**Demostración.**

Sean  $\bar{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  fijo y  $f = (f_1, f_2, \dots, f_n)$ .

$$\text{P.d. } \lim_{\bar{x} \rightarrow \bar{y}} \frac{\left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x}) (x_j - y_j) \right|}{\|\bar{x} - \bar{y}\|} = 0 \quad \forall i = 1, 2, \dots, m.$$

Para todo  $i = 1, 2, \dots, m$  y  $j = 1, 2, \dots, n$  sea  $g : \mathbb{R} \rightarrow \mathbb{R}$  dada por  $g_j(x) = f_i(x_1, \dots, x_{j-1}, x, y_{j+1}, \dots, y_n)$ , donde  $\bar{x} = (x_1, x_2, \dots, x_n) \in U$ . Luego, para cada  $i = 1, 2, \dots, m$ , se tiene que

$$\begin{aligned} f_i(\bar{x}) - f_i(\bar{y}) &= f_i(x_1, x_2, \dots, x_n) - f_i(x_1, x_2, \dots, y_n) + f_i(x_1, x_2, \dots, y_n) - \\ &\quad f_i(x_1, x_2, \dots, y_{n-1}, y_n) + f_i(x_1, x_2, \dots, y_{n-1}, y_n) - \dots - \\ &\quad f_i(x_1, \dots, x_{j-1}, y_j, y_{j+1}, \dots, y_n) + f_i(x_1, \dots, x_{j-1}, y_j, y_{j+1}, \dots, y_n) - \dots - \\ &\quad f_i(x_1, y_2, \dots, y_n) + f_i(x_1, y_2, \dots, y_n) - f_i(y_1, y_2, \dots, y_n) \\ &= g_n(x_n) - g_n(y_n) + g_{n-1}(x_{n-1}) - g_{n-1}(y_{n-1}) + g_{n-2}(x_{n-1}) - \dots - \\ &\quad g_j(y_j) + g_{j-1}(x_{j-1}) - \dots - g_2(y_2) + g_1(x_1) - g_1(y_1) \end{aligned}$$

Ya que  $f_i$  es diferenciable en  $U$ , se tiene que, para todo  $j = 1, 2, \dots, n$ ,  $g_j$  también lo es. Así, por el Teorema del Valor Medio, para cada  $j = 1, 2, \dots, n$  existe  $a_j \in [\min\{x_j, y_j\}, \max\{x_j, y_j\}]$  tal que  $g_j(x_j) - g_j(y_j) = g_j(a_j)(x_j - y_j)$ . De este modo,

$$\begin{aligned} f_i(\bar{x}) - f_i(\bar{y}) &= g'_n(a_n)(x_n - y_n) + g'_{n-1}(a_{n-1})(x_{n-1} - y_{n-1}) + \dots + g'_j(a_j)(x_j - y_j) + \dots + g'_1(a_1)(x_1 - y_1) \\ &= \frac{\partial f_i}{\partial x_1}(\bar{z}_1)(x_1 - y_1) + \frac{\partial f_i}{\partial x_2}(\bar{z}_2)(x_2 - y_2) + \dots + \frac{\partial f_i}{\partial x_j}(\bar{z}_j)(x_j - y_j) + \dots + \frac{\partial f_i}{\partial x_n}(\bar{z}_n)(x_n - y_n) \end{aligned}$$

donde  $\bar{z}_j = (x_1, \dots, x_{j-1}, a_j, y_{j+1}, \dots, y_n)$ . Posteriormente,

$$\begin{aligned} f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - y_j) &= \frac{\partial f_i}{\partial x_1}(\bar{z}_1)(x_1 - y_1) - \frac{\partial f_i}{\partial x_1}(\bar{x})(x_1 - y_1) + \frac{\partial f_i}{\partial x_2}(\bar{z}_2)(x_2 - y_2) - \\ &\quad \frac{\partial f_i}{\partial x_2}(\bar{x})(x_2 - y_2) + \dots + \frac{\partial f_i}{\partial x_j}(\bar{z}_j)(x_j - y_j) - \\ &\quad \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - y_j) + \dots + \frac{\partial f_i}{\partial x_n}(\bar{z}_n)(x_n - y_n) - \frac{\partial f_i}{\partial x_n}(\bar{x})(x_n - y_n) \\ &= \left( \frac{\partial f_i}{\partial x_1}(\bar{z}_1) - \frac{\partial f_i}{\partial x_1}(\bar{x}) \right) (x_1 - y_1) + \\ &\quad \left( \frac{\partial f_i}{\partial x_2}(\bar{z}_2) - \frac{\partial f_i}{\partial x_2}(\bar{x}) \right) (x_2 - y_2) + \dots + \\ &\quad \left( \frac{\partial f_i}{\partial x_j}(\bar{z}_j) - \frac{\partial f_i}{\partial x_j}(\bar{x}) \right) (x_j - y_j) + \dots + \\ &\quad \left( \frac{\partial f_i}{\partial x_n}(\bar{z}_n) - \frac{\partial f_i}{\partial x_n}(\bar{x}) \right) (x_n - y_n) \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - y_j) \right| = \left| \left( \frac{\partial f_i}{\partial x_1}(\bar{z}_1) - \frac{\partial f_i}{\partial x_1}(\bar{x}) \right) (x_1 - y_1) + \right. \\
& \quad \left( \frac{\partial f_i}{\partial x_2}(\bar{z}_2) - \frac{\partial f_i}{\partial x_2}(\bar{x}) \right) (x_2 - y_2) + \cdots + \\
& \quad \left( \frac{\partial f_i}{\partial x_j}(\bar{z}_j) - \frac{\partial f_i}{\partial x_j}(\bar{x}) \right) (x_j - y_j) + \cdots + \\
& \quad \left. \left( \frac{\partial f_i}{\partial x_n}(\bar{z}_n) - \frac{\partial f_i}{\partial x_n}(\bar{x}) \right) (x_n - y_n) \right| \\
& \leq \left| \frac{\partial f_i}{\partial x_1}(\bar{z}_1) - \frac{\partial f_i}{\partial x_1}(\bar{x}) \right| |x_1 - y_1| + \\
& \quad \left| \frac{\partial f_i}{\partial x_2}(\bar{z}_2) - \frac{\partial f_i}{\partial x_2}(\bar{x}) \right| |x_2 - y_2| + \cdots + \\
& \quad \left| \frac{\partial f_i}{\partial x_j}(\bar{z}_j) - \frac{\partial f_i}{\partial x_j}(\bar{x}) \right| |x_j - y_j| + \cdots + \\
& \quad \left| \frac{\partial f_i}{\partial x_n}(\bar{z}_n) - \frac{\partial f_i}{\partial x_n}(\bar{x}) \right| |x_n - y_n| \\
& \leq \left| \frac{\partial f_i}{\partial x_1}(\bar{z}_1) - \frac{\partial f_i}{\partial x_1}(\bar{x}) \right| \|\bar{x} - \bar{y}\| + \\
& \quad \left| \frac{\partial f_i}{\partial x_2}(\bar{z}_2) - \frac{\partial f_i}{\partial x_2}(\bar{x}) \right| \|\bar{x} - \bar{y}\| + \cdots + \\
& \quad \left| \frac{\partial f_i}{\partial x_j}(\bar{z}_j) - \frac{\partial f_i}{\partial x_j}(\bar{x}) \right| \|\bar{x} - \bar{y}\| + \cdots + \\
& \quad \left| \frac{\partial f_i}{\partial x_n}(\bar{z}_n) - \frac{\partial f_i}{\partial x_n}(\bar{x}) \right| \|\bar{x} - \bar{y}\| \\
& = \left( \left| \frac{\partial f_i}{\partial x_1}(\bar{x}) - \frac{\partial f_i}{\partial x_1}(\bar{z}_1) \right| + \left| \frac{\partial f_i}{\partial x_2}(\bar{x}) - \frac{\partial f_i}{\partial x_2}(\bar{z}_2) \right| + \cdots + \right. \\
& \quad \left. \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{z}_j) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\bar{x}) - \frac{\partial f_i}{\partial x_n}(\bar{z}_n) \right| \right) \|\bar{x} - \bar{y}\| \\
& \Rightarrow \frac{\left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x})(x_j - y_j) \right|}{\|\bar{x} - \bar{y}\|} \leq \left| \frac{\partial f_i}{\partial x_1}(\bar{x}) - \frac{\partial f_i}{\partial x_1}(\bar{z}_1) \right| + \left| \frac{\partial f_i}{\partial x_2}(\bar{x}) - \frac{\partial f_i}{\partial x_2}(\bar{z}_2) \right| + \cdots + \\
& \quad \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{z}_j) \right| + \cdots + \left| \frac{\partial f_i}{\partial x_n}(\bar{x}) - \frac{\partial f_i}{\partial x_n}(\bar{z}_n) \right|
\end{aligned}$$

Después, como las derivadas parciales son continuas en  $U$  se da que  $\lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{y}) \right| = 0$ , y puesto que  $\bar{z}_j = (x_1, \dots, x_{j-1}, a_j, y_{j+1}, \dots, y_n)$  con  $a_j \in [\min\{x_j, y_j\}, \max\{x_j, y_j\}]$ , se tiene que  $\bar{z}_j$  tiende a  $\bar{y}$  conforme  $\bar{x}$  tiende a  $\bar{y}$ , para todo  $j = 1, 2, \dots, n$ . De esta manera,

$$\begin{aligned}
0 &\leq \lim_{\bar{x} \rightarrow \bar{y}} \frac{\left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x}) (x_j - y_j) \right|}{\|\bar{x} - \bar{y}\|} \leq \lim_{\bar{x} \rightarrow \bar{y}} \left( \left| \frac{\partial f_i}{\partial x_1}(\bar{x}) - \frac{\partial f_i}{\partial x_1}(\bar{z}_1) \right| + \left| \frac{\partial f_i}{\partial x_2}(\bar{x}) - \frac{\partial f_i}{\partial x_2}(\bar{z}_2) \right| + \cdots + \right. \\
&\quad \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{z}_j) \right| + \cdots + \\
&\quad \left. \left| \frac{\partial f_i}{\partial x_n}(\bar{x}) - \frac{\partial f_i}{\partial x_n}(\bar{z}_n) \right| \right) \\
&= \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_1}(\bar{x}) - \frac{\partial f_i}{\partial x_1}(\bar{z}_1) \right| + \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_2}(\bar{x}) - \frac{\partial f_i}{\partial x_2}(\bar{z}_2) \right| + \cdots + \\
&\quad \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{z}_j) \right| + \cdots + \\
&\quad \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_n}(\bar{x}) - \frac{\partial f_i}{\partial x_n}(\bar{z}_n) \right| \\
&= \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_1}(\bar{x}) - \frac{\partial f_i}{\partial x_1}(\bar{y}) \right| + \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_2}(\bar{x}) - \frac{\partial f_i}{\partial x_2}(\bar{y}) \right| + \cdots + \\
&\quad \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{y}) \right| + \cdots + \\
&\quad \lim_{\bar{x} \rightarrow \bar{y}} \left| \frac{\partial f_i}{\partial x_n}(\bar{x}) - \frac{\partial f_i}{\partial x_n}(\bar{y}) \right| \\
&= 0
\end{aligned}$$

De esta manera,  $\lim_{\bar{x} \rightarrow \bar{y}} \frac{\left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x}) (x_j - y_j) \right|}{\|\bar{x} - \bar{y}\|} = 0 \quad \forall i = 1, \dots, m.$

Por último,

$$0 \leq \lim_{\bar{x} \rightarrow \bar{y}} \frac{\|f(\bar{x}) - f(\bar{y}) - Df(\bar{y})(\bar{x} - \bar{y})\|}{\|\bar{x} - \bar{y}\|} \leq \sum_{i=1}^m \lim_{\bar{x} \rightarrow \bar{y}} \frac{\left| f_i(\bar{x}) - f_i(\bar{y}) - \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\bar{x}) (x_j - y_j) \right|}{\|\bar{x} - \bar{y}\|} = 0$$

Ya que  $\bar{y}$  fue arbitrario, se concluye que  $f$  es diferenciable en  $U$ . ■