

Examen. Límite, continuidad y derivada.

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1. (a) Afirmación: $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$.

Demostración.

Sean $(x,y) \in \mathbb{R}^2 \setminus \{0\}$, $\epsilon > 0$ y $\delta = \epsilon$. Se tiene que

$$0 \leq (|x| + |y|)^2 = |x|^2 + 2|xy| + |y|^2$$

$$\implies 2|xy| \leq |x|^2 + |y|^2$$

$$\implies \frac{|xy|}{x^2 + y^2} \leq \frac{1}{2}$$

$$\implies \frac{|x| \cdot |xy|}{x^2 + y^2} \leq \frac{|x|}{2} \quad (1)$$

Luego, si $0 < \|(x,y) - (0,0)\| < \delta$ entonces

$$\sqrt{x^2 + y^2} < \delta = \epsilon$$

$$\implies |x| \leq \sqrt{x^2 + y^2} < \epsilon$$

$$\implies \frac{|x|}{2} \leq |x| < \epsilon$$

$$\implies \frac{|x| \cdot |xy|}{x^2 + y^2} \leq \frac{|x|}{2} < \epsilon \quad (\text{por (1)})$$

$$\implies \frac{|x^2 y|}{x^2 + y^2} < \epsilon$$

$$\implies \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| < \epsilon$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

- (b) Afirmación: $\lim_{(x,y) \rightarrow (0,0)} g(x,y)$ no existe.

Demostración.

Sean $(x,y) \in \mathbb{R}^2 \setminus \{0\}$, $\epsilon > 0$ y $\delta = \sqrt{\epsilon}$. Si $y = 0$ y $0 < \|(x,y) - (0,0)\| < \delta$ entonces

$$\|(x,0)\| = |x| < \delta = \sqrt{\epsilon}$$

$$\implies |x| < \sqrt{\epsilon}$$

$$\implies |x|^2 < \epsilon$$

$$\implies |(x^2 + 1) - 1| < \epsilon$$

$$\therefore \lim_{(x,0) \rightarrow (0,0)} g(x,y) = 1$$

Ahora, sea $\delta_1 = \sqrt{\epsilon - 1}$. Si $x = y$ y $0 < \|(x,y) - (0,0)\| < \delta_1$ entonces

$$\|(x,x)\| = |x|\sqrt{2} < \delta_1 = \sqrt{\epsilon - 1}$$

$$\implies |x| < |x|\sqrt{2} < \sqrt{\epsilon - 1}$$

$$\implies |x| < \sqrt{\epsilon - 1}$$

$$\implies |x|^2 < \epsilon - 1$$

$$\implies |x^2| + 1 < \epsilon$$

$$\implies |x^2 + 1 - 0| < \epsilon$$

$$\therefore \lim_{(x,x) \rightarrow (0,0)} g(x,y) = 0$$

Como los límites obtenidos difieren entonces $\lim_{(x,y) \rightarrow (0,0)} g(x,y)$ no existe.

2. Demostración.

Sean $(x, y) \in \mathbb{R}^2$, $\epsilon > 0$ y $\delta = \sqrt{\epsilon}$. Como $|f(x, y)| \leq x^2 + y^2$ se cumple que

$$0 \leq |f(0, 0)| \leq 0^2 + 0^2 = 0$$

Por lo que $f(0, 0) = 0$.

Luego, si $0 < \|(x, y) - (0, 0)\| < \delta$ entonces

$$\|(x, y)\| < \sqrt{\epsilon}$$

$$\implies \sqrt{x^2 + y^2} < \sqrt{\epsilon}$$

$$\implies x^2 + y^2 < \epsilon$$

$$\implies |f(x, y)| \leq x^2 + y^2 < \epsilon$$

$$\implies |f(x, y)| < \epsilon$$

$$\implies |f(x, y) - 0| < \epsilon$$

$$\implies |f(x, y) - f(0, 0)| < \epsilon$$

Por lo tanto, f es continua en $\bar{0}$.

3. Demostración.

Como f es continua en \bar{x}_0 se tiene que $\forall \epsilon > 0$, en particular para $\epsilon = f(\bar{x}_0)$, existe $\delta > 0$ tal que si $\bar{y} \in B_\delta(\bar{x}_0)$ entonces $f(\bar{y}) \in B_\epsilon(f(\bar{x}_0))$. Así, $|f(\bar{x}_0) - f(\bar{y})| < \epsilon = f(\bar{x}_0)$ y de esto

$$-f(\bar{x}_0) < f(\bar{x}_0) - f(\bar{y}) < f(\bar{x}_0)$$

$$\implies f(\bar{x}_0) - f(\bar{y}) < f(\bar{x}_0)$$

$$\implies -f(\bar{y}) < 0$$

$$\therefore f(\bar{y}) > 0.$$

4. Demostración.

Sea $\bar{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ un punto fijo. La derivada de f en \bar{x}_0 es

$$Df(\bar{x}_0) = \left(\frac{\partial f}{\partial x}(\bar{x}_0), \frac{\partial f}{\partial y}(\bar{x}_0) \right) = (1, 1)$$

Luego,

$$\begin{aligned} \lim_{\bar{x} \rightarrow \bar{x}_0} \frac{|f(\bar{x}) - (1(x - x_0) + 1(y - y_0) + f(\bar{x}_0))|}{\|\bar{x} - \bar{x}_0\|} &= \lim_{\bar{x} \rightarrow \bar{x}_0} \frac{|2 + x + y - (x - x_0 + y - y_0 + 2 + x_0 + y_0)|}{\|\bar{x} - \bar{x}_0\|} \\ &= \lim_{\bar{x} \rightarrow \bar{x}_0} \frac{|2 + x + y - (x + y + 2)|}{\|\bar{x} - \bar{x}_0\|} \\ &= \lim_{\bar{x} \rightarrow \bar{x}_0} \frac{|0|}{\|\bar{x} - \bar{x}_0\|} \\ &= 0 \end{aligned}$$

Por lo tanto, f es diferenciable en cualquier punto.

5. Obteniendo las derivadas parciales

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\
&= \left(\frac{(u^2 - v^2)(2u) - (u^2 + v^2)(2u)}{(u^2 - v^2)^2} \right) (-e^{-x-y}) + \left(\frac{(u^2 - v^2)(2v) - (u^2 + v^2)(-2v)}{(u^2 - v^2)^2} \right) (ye^{xy}) \\
&= \left(\frac{2u^3 - 2uv^2 - 2u^3 - 2uv^2}{(u^2 - v^2)^2} \right) (-e^{-x-y}) + \left(\frac{2u^2v - 2v^3 + 2u^2v + 2v^3}{(u^2 - v^2)^2} \right) (ye^{xy}) \\
&= \left(\frac{4uv^2}{(u^2 - v^2)^2} \right) e^{-x-y} + \left(\frac{4u^2v}{(u^2 - v^2)^2} \right) ye^{xy} \\
&= \frac{4uv(v e^{-x-y} + u y e^{xy})}{(u^2 - v^2)^2} \\
&= \frac{4e^{-x-y} e^{xy} (e^{xy} e^{-x-y} + e^{-x-y} y e^{xy})}{((e^{-x-y})^2 - (e^{xy})^2)^2} \\
&= \frac{4e^{-x-y} e^{xy} [e^{xy} e^{-x-y} (1 + y)]}{(e^{-2(x+y)} - e^{2xy})^2} \\
&= \frac{4e^{-2(x+y)} e^{2xy} (1 + y)}{(e^{-2(x+y)} - e^{2xy})^2} \\
&= \frac{4e^{2(xy-x-y)} (1 + y)}{(e^{-2(x+y)} - e^{2xy})^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial h}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \\
&= \left(\frac{(u^2 - v^2)(2u) - (u^2 + v^2)(2u)}{(u^2 - v^2)^2} \right) (-e^{-x-y}) + \left(\frac{(u^2 - v^2)(2v) - (u^2 + v^2)(-2v)}{(u^2 - v^2)^2} \right) (xe^{xy}) \\
&= \frac{4uv(v e^{-x-y} + u x e^{xy})}{(u^2 - v^2)^2} \\
&= \frac{4e^{-x-y} e^{xy} (e^{xy} e^{-x-y} + e^{-x-y} x e^{xy})}{((e^{-x-y})^2 - (e^{xy})^2)^2} \\
&= \frac{4e^{-x-y} e^{xy} [e^{xy} e^{-x-y} (1 + x)]}{(e^{-2(x+y)} - e^{2xy})^2} \\
&= \frac{4e^{-2(x+y)} e^{2xy} (1 + x)}{(e^{-2(x+y)} - e^{2xy})^2}
\end{aligned}$$

$$\text{Así, } Dh(x, y) = \left(\frac{4e^{2(xy-x-y)} (1 + y)}{(e^{-2(x+y)} - e^{2xy})^2}, \frac{4e^{-2(x+y)} e^{2xy} (1 + x)}{(e^{-2(x+y)} - e^{2xy})^2} \right).$$

Por otro lado, sea $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ con $w(x, y) = (u(x, y), v(x, y))$ entonces $h(x, y) = (f \circ w)(x, y)$. Luego,

$$Dw(x, y) = \begin{pmatrix} -e^{-x-y} & -e^{-x-y} \\ ye^{xy} & xe^{xy} \end{pmatrix}$$

$$\begin{aligned}
Df(u, v) &= \left(\frac{(u^2 - v^2)(2u) - (u^2 + v^2)(2u)}{(u^2 - v^2)^2}, \frac{(u^2 - v^2)(2v) - (u^2 + v^2)(-2v)}{(u^2 - v^2)^2} \right) \\
&= \left(\frac{2u^3 - 2uv^2 - 2u^3 - 2uv^2}{(u^2 - v^2)^2}, \frac{2u^2v - 2v^3 + 2u^2v + 2v^3}{(u^2 - v^2)^2} \right) \\
&= \left(-\frac{4uv^2}{(u^2 - v^2)^2}, \frac{4u^2v}{(u^2 - v^2)^2} \right)
\end{aligned}$$

Después, por regla de la cadena

$$\begin{aligned}
Dh(x, y) &= Df(w(x, y)) \cdot Dw(x, y) \\
&= \left(-\frac{4e^{-x-y}(e^{xy})^2}{((e^{-x-y})^2 - (e^{xy})^2)^2} \quad \frac{4(e^{-x-y})^2 e^{xy}}{((e^{-x-y})^2 - (e^{xy})^2)^2} \right) \cdot \begin{pmatrix} -e^{-x-y} & -e^{-x-y} \\ ye^{xy} & xe^{xy} \end{pmatrix} \\
&= \left(-\frac{4e^{2xy-x-y}}{(e^{-2(x+y)} - e^{2xy})^2} \quad \frac{4e^{xy-2x-2y}}{(e^{-2(x+y)} - e^{2xy})^2} \right) \cdot \begin{pmatrix} -e^{-x-y} & -e^{-x-y} \\ ye^{xy} & xe^{xy} \end{pmatrix} \\
&= \left(-\frac{4e^{2xy-x-y}(-e^{-x-y})}{(e^{-2(x+y)} - e^{2xy})^2} + \frac{4e^{xy-2x-2y}(ye^{xy})}{(e^{-2(x+y)} - e^{2xy})^2} \quad -\frac{4e^{2xy-x-y}(-e^{-x-y})}{(e^{-2(x+y)} - e^{2xy})^2} + \frac{4e^{xy-2x-2y}(xe^{xy})}{(e^{-2(x+y)} - e^{2xy})^2} \right) \\
&= \left(\frac{4e^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} + \frac{4ye^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} \quad \frac{4e^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} + \frac{4xe^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} \right) \\
&= \left(\frac{4e^{2(xy-x-y)}(1+y)}{(e^{-2(x+y)} - e^{2xy})^2} \quad \frac{4e^{2(xy-x-y)}(1+x)}{(e^{-2(x+y)} - e^{2xy})^2} \right)
\end{aligned}$$

Por lo tanto, se cumple la regla de la cadena.