Examen. Límite, continuidad y derivada.

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1. (a) Afirmación: $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

   Demostración.

   Sean $(x,y) \in \mathbb{R}^2 \setminus \{\overline{0}\}$, $\epsilon > 0$ y $\delta = \epsilon$. Se tiene que

   $0 \leq (|x| + |y|)^2 = |x|^2 - 2|xy| + |y|^2$

   $\implies 2|xy| \leq |x|^2 + |y|^2$

   $\implies \dfrac{|xy|}{x^2+y^2} \leq \dfrac{1}{2}$

   $$\implies \frac{|x| \cdot |xy|}{x^2+y^2} \leq \frac{|x|}{2} \tag{1}$$

   Luego, si $0 < \|(x,y) - (0,0)\| < \delta$ entonces

   $\sqrt{x^2+y^2} < \delta = \epsilon$

   $\implies |x| \leq \sqrt{x^2+y^2} < \epsilon$

   $\implies \dfrac{|x|}{2} \leq |x| < \epsilon$

   $\implies \dfrac{|x| \cdot |xy|}{x^2+y^2} \leq \dfrac{|x|}{2} < \epsilon$ (por (1))

   $\implies \dfrac{|x^2y|}{x^2+y^2} < \epsilon$

   $\implies \left| \dfrac{x^2y}{x^2+y^2} - 0 \right| < \epsilon$

   $\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$

   (b) Afirmación: $\lim_{(x,y)\to(0,0)} g(x,y)$ no existe.

   Demostración.

   Sean $(x,y) \in \mathbb{R}^2 \setminus \{\overline{0}\}$, $\epsilon > 0$ y $\delta = \sqrt{\epsilon}$. Si $y = 0$ y $0 < \|(x,y) - (0,0)\| < \delta$ entonces

   $\|(x,0)\| = |x| < \delta = \sqrt{\epsilon}$

   $\implies |x| < \sqrt{\epsilon}$

   $\implies |x|^2 < \epsilon$

   $\implies \left|(x^2+1) - 1\right| < \epsilon$

   $\therefore \lim_{(x,0)\to(0,0)} g(x,y) = 1$

   Ahora, sea $\delta_1 = \sqrt{\epsilon - 1}$. Si $x = y$ y $0 < \|(x,y) - (0,0)\| < \delta_1$ entonces

   $\|(x,x)\| = |x|\sqrt{2} < \delta_1 = \sqrt{\epsilon - 1}$

   $\implies |x| < |x|\sqrt{2} < \sqrt{\epsilon - 1}$

   $\implies |x| < \sqrt{\epsilon - 1}$

   $\implies |x|^2 < \epsilon - 1$

   $\implies |x^2| + 1 < \epsilon$

   $\implies \left|x^2 + 1 - 0\right| < \epsilon$

   $\therefore \lim_{(x,x)\to(0,0)} g(x,y) = 0$

   Como los límites obtenidos difieren entonces $\lim_{(x,y)\to(0,0)} g(x,y)$ no existe.

2. **Demostración.**

   Sean $(x, y) \in \mathbb{R}^2$, $\epsilon > 0$ y $\delta = \sqrt{\epsilon}$. Como $|f(x, y)| \leq x^2 + y^2$ se cumple que

   $0 \leq |f(0, 0)| \leq 0^2 + 0^2 = 0$

   Por lo que $f(0, 0) = 0$.

   Luego, si $0 < \|(x, y) - (0, 0)\| < \delta$ entonces

   $\|(x, y)\| < \sqrt{\epsilon}$

   $\implies \sqrt{x^2 + y^2} < \sqrt{\epsilon}$

   $\implies x^2 + y^2 < \epsilon$

   $\implies |f(x, y)| \leq x^2 + y^2 < \epsilon$

   $\implies |f(x, y)| < \epsilon$

   $\implies |f(x, y) - 0| < \epsilon$

   $\implies |f(x, y) - f(0, 0)| < \epsilon$

   Por lo tanto, f es continua en $\overline{0}$.

3. **Demostración.**

   Como f es continua en $\overline{x}_0$ se tiene que $\forall\, \epsilon > 0$, en particular para $\epsilon = f(\overline{x}_0)$, existe $\delta > 0$ tal que si $\overline{y} \in B_\delta(\overline{x}_0)$ entonces $f(\overline{y}) \in B_\epsilon(f(\overline{x}_0))$. Así, $|f(\overline{x}_0) - f(\overline{y})| < \epsilon = f(\overline{x}_0)$ y de esto

   $-f(\overline{x}_0 < f(\overline{x}_0) - f(\overline{y}) < f(\overline{x}_0)$

   $\implies f(\overline{x}_0) - f(\overline{y}) < f(\overline{x}_0)$

   $\implies -f(\overline{y}) < 0$

   $\therefore f(\overline{y}) > 0$.

4. **Demostración.**

   Sea $\overline{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ un punto fijo. La derivada de f en $\overline{x}_0$ es

   $Df(\overline{x}_0) = \left(\frac{\partial f}{\partial x}(\overline{x}_0), \frac{\partial f}{\partial y}(\overline{x}_0)\right) = (1, 1)$

   Luego,

$$\begin{aligned}
\lim_{\overline{x}_0 \to \overline{x}_0} \frac{|f(\overline{x}) - (1(x - x_0) + 1(y - y_0) + f(\overline{x}_0))|}{\|\overline{x} - \overline{x}_0\|} &= \lim_{\overline{x} \to \overline{x}_0} \frac{|2 + x + y - (x - x_0 + y - y_0 + 2 + x_0 + y_0)|}{\|\overline{x} - \overline{x}_0\|} \\
&= \lim_{\overline{x} \to \overline{x}_0} \frac{|2 + x + y - (x + y + 2)|}{\|\overline{x} - \overline{x}_0\|} \\
&= \lim_{\overline{x} \to \overline{x}_0} \frac{|0|}{\|\overline{x} - \overline{x}_0\|} \\
&= 0
\end{aligned}$$

   Por lo tanto, f es diferenciable en cualquier punto.

5. Obteniendo las derivadas parciales

$$
\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\\
&= \left(\frac{(u^2-v^2)(2u)-(u^2+v^2)(2u)}{(u^2-v^2)^2}\right)(-e^{-x-y})+\left(\frac{(u^2-v^2)(2v)-(u^2+v^2)(-2v)}{(u^2-v^2)^2}\right)(ye^{xy})\\
&= \left(\frac{2u^3-2uv^2-2u^3-2uv^2}{(u^2-v^2)^2}\right)(-e^{-x-y})+\left(\frac{2u^2v-2v^3+2u^2v+2v^3}{(u^2-v^2)^2}\right)(ye^{xy})\\
&= \left(\frac{4uv^2}{(u^2-v^2)^2}\right)e^{-x-y}+\left(\frac{4u^2v}{(u^2-v^2)^2}\right)ye^{xy}\\
&= \frac{4uv(ve^{-x-y}+uye^{xy})}{(u^2-v^2)^2}\\
&= \frac{4e^{-x-y}e^{xy}(e^{xy}e^{-x-y}+e^{-x-y}ye^{xy})}{((e^{-x-y})^2-(e^{xy})^2)^2}\\
&= \frac{4e^{-x-y}e^{xy}[e^{xy}e^{-x-y}(1+y)]}{(e^{-2(x+y)}-e^{2xy})^2}\\
&= \frac{4e^{-2(x+y)}e^{2xy}(1+y)}{(e^{-2(x+y)}-e^{2xy})^2}\\
&= \frac{4e^{2(xy-x-y)}(1+y)}{(e^{-2(x+y)}-e^{2xy})^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial h}{\partial y} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\\
&= \left(\frac{(u^2-v^2)(2u)-(u^2+v^2)(2u)}{(u^2-v^2)^2}\right)(-e^{-x-y})+\left(\frac{(u^2-v^2)(2v)-(u^2+v^2)(-2v)}{(u^2-v^2)^2}\right)(xe^{xy})\\
&= \frac{4uv(ve^{-x-y}+uxe^{xy})}{(u^2-v^2)^2}\\
&= \frac{4e^{-x-y}e^{xy}(e^{xy}e^{-x-y}+e^{-x-y}xe^{xy})}{((e^{-x-y})^2-(e^{xy})^2)^2}\\
&= \frac{4e^{-x-y}e^{xy}[e^{xy}e^{-x-y}(1+x)]}{(e^{-2(x+y)}-e^{2xy})^2}\\
&= \frac{4e^{-2(x+y)}e^{2xy}(1+x)]}{(e^{-2(x+y)}-e^{2xy})^2}
\end{aligned}
$$

Así, $Dh(x,y)=\left(\frac{4e^{2(xy-x-y)}(1+y)}{(e^{-2(x+y)}-e^{2xy})^2},\ \frac{4e^{-2(x+y)}e^{2xy}(1+x)}{(e^{-2(x+y)}-e^{2xy})^2}\right)$.

Por otro lado, sea $w:\mathbb{R}^2\to\mathbb{R}^2$ con $w(x,y)=(u(x,y),v(x,y))$ entonces $h(x,y)=(f\circ w)(x,y)$. Luego,

$$
Dw(x,y)=\begin{pmatrix}-e^{-x-y} & -e^{-x-y}\\ ye^{xy} & xe^{xy}\end{pmatrix}
$$

$$
\begin{aligned}
Df(u,v) &= \begin{pmatrix}\frac{(u^2-v^2)(2u)-(u^2+v^2)(2u)}{(u^2-v^2)^2} & \frac{(u^2-v^2)(2v)-(u^2+v^2)(-2v)}{(u^2-v^2)^2}\end{pmatrix}\\
&= \begin{pmatrix}\frac{2u^3-2uv^2-2u^3-2uv^2}{(u^2-v^2)^2} & \frac{2u^2v-2v^3+2u^2v+2v^3}{(u^2-v^2)^2}\end{pmatrix}\\
&= \begin{pmatrix}-\frac{4uv^2}{(u^2-v^2)^2} & \frac{4u^2v}{(u^2-v^2)^2}\end{pmatrix}
\end{aligned}
$$

Después, por regla de la cadena

$$
\begin{aligned}
Dh(x,y) &= Df(w(x,y)) \cdot Dw(x,y) \\
&= \begin{pmatrix} -\dfrac{4e^{-x-y}(e^{xy})^2}{((e^{-x-y})^2 - (e^{xy})^2)^2} & \dfrac{4(e^{-x-y})^2 e^{xy}}{((e^{-x-y})^2 - (e^{xy})^2)^2} \end{pmatrix} \cdot \begin{pmatrix} -e^{-x-y} & -e^{-x-y} \\ ye^{xy} & xe^{xy} \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{4e^{2xy-x-y}}{(e^{-2(x+y)} - e^{2xy})^2} & \dfrac{4e^{xy-2x-2y}}{(e^{-2(x+y)} - e^{2xy})^2} \end{pmatrix} \cdot \begin{pmatrix} -e^{-x-y} & -e^{-x-y} \\ ye^{xy} & xe^{xy} \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{4e^{2xy-x-y}(-e^{-x-y})}{(e^{-2(x+y)} - e^{2xy})^2} + \dfrac{4e^{xy-2x-2y}(ye^{xy})}{(e^{-2(x+y)} - e^{2xy})^2} & -\dfrac{4e^{2xy-x-y}(-e^{-x-y})}{(e^{-2(x+y)} - e^{2xy})^2} + \dfrac{4e^{xy-2x-2y}(xe^{xy})}{(e^{-2(x+y)} - e^{2xy})^2} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{4e^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} + \dfrac{4ye^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} & \dfrac{4e^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} + \dfrac{4xe^{2(xy-x-y)}}{(e^{-2(x+y)} - e^{2xy})^2} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{4e^{2(xy-x-y)}(1+y)}{(e^{-2(x+y)} - e^{2xy})^2} & \dfrac{4e^{2(xy-x-y)}(1+x)}{(e^{-2(x+y)} - e^{2xy})^2} \end{pmatrix}
\end{aligned}
$$

Por lo tanto, se cumple la regla de la cadena.