4. Demuestra o da un contraejemplo. Para $i = 1, 2, 3$, sea G_i una gráfica.

a) $G_1 + G_2 = G_2 + G_1$

Demostración.

Ya que

$$\begin{aligned} V(G_1 + G_2) &= V(G_1) \cup V(G_2) \\ &= V(G_2) \cup V(G_1) \\ &= V(G_2 + G_1) \end{aligned}$$

y

$$\begin{aligned} A(G_1 + G_2) &= A(G_1) \cup A(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\} \\ &= A(G_2) \cup A(G_1) \cup \{vu \mid v \in V(G_2), u \in V(G_1)\} \\ &= A(G_2 + G_1) \end{aligned}$$

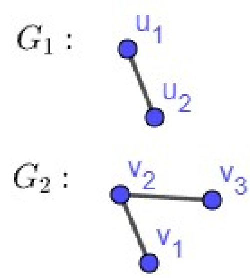
se tiene que $G_1 + G_2 = G_2 + G_1$ ■

b) $G_1 \times G_2 = G_2 \times G_1$

Afirmación: $G_1 \times G_2 \neq G_2 \times G_1$

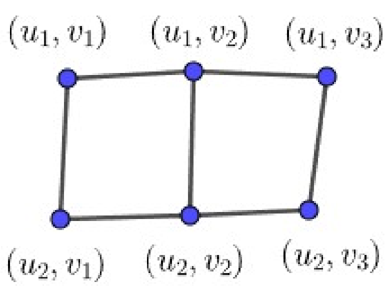
Sean $G_1 = (V, A)$ y $G_2 = (V, A)$ gráficas con $V(G_1) = \{u_1, u_2\}$, $A(G_1) = \{u_1u_2\}$, $V(G_2) = \{v_1, v_2, v_3\}$ y $A(G_2) = \{v_1v_2, v_2v_3\}$.

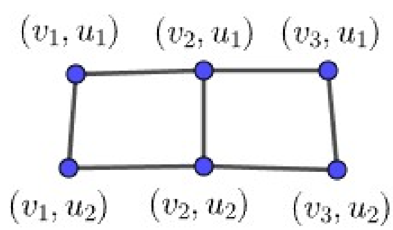
Como



$$\begin{aligned} V(G_1 \times G_2) &= V(G_1) \times V(G_2) \\ &= \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_2, v_1), (u_2, v_2), (u_2, v_3)\} \\ &\neq \{(v_1, u_1), (v_1, u_2), (v_2, u_1), (v_2, u_2), (v_3, u_1), (v_3, u_2)\} \\ &= V(G_2) \times V(G_1) \\ &= V(G_2 \times G_1) \end{aligned}$$

se tiene que $G_1 \times G_2 \neq G_2 \times G_1$





c) $(G_1 + G_2) + G_3 = G_1 + (G_2 + G_3)$

Demostración.

$$\begin{aligned} V((G_1 + G_2) + G_3) &= V(G_1 + G_2) \cup V(G_3) \\ &= [V(G_1) \cup V(G_2)] \cup V(G_3) \\ &= V(G_1) \cup [V(G_2) \cup V(G_3)] \\ &= V(G_1) \cup V(G_2 + G_3) \\ &= V(G_1 + (G_2 + G_3)) \end{aligned}$$

Ahora, sea $uv \in A((G_1 + G_2) + G_3)$. P.d. $uv \in A(G_1 + (G_2 + G_3))$.

Ya que

$$\begin{aligned}A\left(\left(G_1+G_2\right)+G_3\right) &= A\left(G_1+G_2\right)\cup A\left(G_3\right)\cup\left\{ab \mid a\in V\left(G_1+G_2\right), b\in V\left(G_3\right)\right\}\\ &= A\left(G_1\right)\cup A\left(G_2\right)\cup\left\{ab \mid a\in V\left(G_1\right), b\in V\left(G_2\right)\right\}\cup A\left(G_3\right)\cup\\ &\quad \left\{ab \mid a\in V\left(G_1+G_2\right), b\in V\left(G_3\right)\right\}\end{aligned}$$

se dan los siguientes casos:

- Si $uv\in A(G_1)$ entonces $uv\in A(G_1)\cup A(G_2+G_3)\cup\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.
- Si $uv\in A(G_2)$ entonces $uv\in A(G_1)\cup A(G_2)\cup A(G_3)\cup\{ab \mid a\in V(G_2), b\in V(G_3)\}\cup\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.
- Si $uv\in A(G_3)$ entonces $uv\in A(G_1)\cup A(G_2)\cup A(G_3)\cup\{ab \mid a\in V(G_2), b\in V(G_3)\}\cup\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.
- Si $uv\in\{ab \mid a\in V(G_1), b\in V(G_2)\}$ entonces $uv\in\{ab \mid a\in V(G_1), b\in V(G_2)\cup V(G_3)\}=\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.
- Si $uv\in\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}=\{ab \mid a\in V(G_1)\cup V(G_2), b\in V(G_3)\}$ entonces se tienen los siguientes casos:
  - Si $u\in V(G_1)$ entonces $uv\in\{ab \mid a\in V(G_1), b\in V(G_3)\}$. Así, $uv\in\{ab \mid a\in V(G_1), b\in V(G_2)\cup V(G_3)\}=\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$. De esta manera, $uv\in A(G_1)\cup A(G_2+G_3)\cup\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.
  - Si $u\in V(G_2)$ entonces $uv\in\{ab \mid a\in V(G_2), b\in V(G_3)\}$. De esta forma, $uv\in A(G_1)\cup A(G_2)\cup A(G_3)\cup\{ab \mid a\in V(G_2), b\in V(G_3)\}\cup\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}$, es decir, $uv\in A(G_1+(G_2+G_3))$.

Por lo anterior, $A((G_1+G_2)+G_3)\subseteq A(G_1+(G_2+G_3))$.

Por otro lado, sea $uv\in A(G_1+(G_2+G_3))$. P.d. $uv\in A((G_1+G_2)+G_3)$.

Como

$$\begin{aligned}A\left(G_1+\left(G_2+G_3\right)\right) &= A\left(G_1\right)\cup A\left(G_2+G_3\right)\cup\left\{ab \mid a\in V\left(G_1\right), b\in V\left(G_2+G_3\right)\right\}\\ &= A\left(G_1\right)\cup A\left(G_2\right)\cup A\left(G_3\right)\cup\left\{ab \mid a\in V\left(G_2\right), b\in V\left(G_3\right)\right\}\cup\\ &\quad \left\{ab \mid a\in V\left(G_1\right), b\in V\left(G_2+G_3\right)\right\}\end{aligned}$$

se dan los siguientes casos:

- Si $uv\in A(G_1)$ entonces $uv\in A(G_1)\cup A(G_2)\cup\{ab \mid a\in V(G_1), b\in V(G_2)\}\cup A(G_3)\cup\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$, es decir, $uv\in A((G_1+G_2)+G_3)$.
- Si $uv\in A(G_2)$ entonces $uv\in A(G_1)\cup A(G_2)\cup\{ab \mid a\in V(G_1), b\in V(G_2)\}\cup A(G_3)\cup\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$, es decir, $uv\in A((G_1+G_2)+G_3)$.
- Si $uv\in A(G_3)$ entonces $uv\in A(G_1+G_2)\cup A(G_3)\cup\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$, es decir, $uv\in A((G_1+G_2)+G_3)$.
- Si $uv\in\{ab \mid a\in V(G_2), b\in V(G_3)\}$ entonces $uv\in\{ab \mid a\in V(G_1)\cup V(G_2), b\in V(G_3)\}=\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$. Así, $uv\in A(G_1+G_2)\cup A(G_3)\cup\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$, es decir, $uv\in A((G_1+G_2)+G_3)$.
- Si $uv\in\{ab \mid a\in V(G_1), b\in V(G_2+G_3)\}=\{ab \mid a\in V(G_1), b\in V(G_2)\cup V(G_3)\}$ entonces se tienen los siguientes casos:
  - Si $v\in V(G_2)$ entonces $uv\in\{ab \mid a\in V(G_1), b\in V(G_2)\}$. Así, $uv\in A(G_1)\cup A(G_2)\cup\{ab \mid a\in V(G_1), b\in V(G_2)\}\cup A(G_3)\cup\{ab \mid a\in V(G_1+G_2), b\in V(G_3)\}$. De esta manera, $uv\in A((G_1+G_2)+G_3)$.

- Si $v \in V(G_3)$ entonces $uv \in \{ab \mid a \in V(G_1), b \in V(G_3)\} \subseteq \{ab \mid a \in V(G_1) \cup V(G_2), b \in V(G_3)\}$. Así, $uv \in \{ab \mid a \in V(G_1 + G_2), b \in V(G_3)\}$. De esta manera, $uv \in A(G_1 + G_2) \cup A(G_3) \cup \{ab \mid a \in V(G_1 + G_2), b \in V(G_3)\}$, es decir, $uv \in A((G_1 + G_2) + G_3)$.

Por lo anterior, $A((G_1 + G_2) + G_3) \supseteq A(G_1 + (G_2 + G_3))$.

Por lo tanto, $A((G_1 + G_2) + G_3) = A(G_1 + (G_2 + G_3))$. En conclusión, $(G_1 + G_2) + G_3 = G_1 + (G_2 + G_3)$.

d) $(G_1 \times G_2) \times G_3 = G_1 \times (G_2 \times G_3)$

Afirmación: $(G_1 \times G_2) \times G_3 \neq G_1 \times (G_2 \times G_3)$.

Sean $G_1 = (V, A)$, $G_2 = (V, A)$ y $G_3 = (V, A)$ gráficas con $V(G_1) = \{u_1\}$, $A(G_1) = \emptyset$, $V(G_2) = \{v_1, v_2\}$, $A(G_2) = \{v_1 v_2\}$, $V(G_3) = \{w_1, w_2, w_3\}$ y $A(G_3) = \{w_1 w_2, w_2 w_3\}$.

Como

$$\begin{aligned}
V((G_1 \times G_2) \times G_3) &= V(G_1 \times G_2) \times V(G_3) \\
&= [V(G_1) \times V(G_2)] \times V(G_3) \\
&= \{((u_1, v_1), w_1), ((u_1, v_1), w_2), ((u_1, v_1), w_3), ((u_1, v_2), w_1), ((u_1, v_2), w_2), \\
&\quad ((u_1, v_2), w_3)\} \\
&\neq \{(u_1, (v_1, w_1)), (u_1, (v_1, w_2)), (u_1, (v_1, w_3)), (u_1, (v_2, w_1)), (u_1, (v_2, w_2)), \\
&\quad (u_1, (v_2, w_3))\} \\
&= V(G_1) \times [V(G_2) \times V(G_3)] \\
&= V(G_1) \times V(G_2 \times G_3) \\
&= V(G_1 \times (G_2 \times G_3))
\end{aligned}$$

se tiene que $(G_1 \times G_2) \times G_3 \neq G_1 \times (G_2 \times G_3)$