Assignment 2

1) Prove that if T is a tree with a vertex of degree K, then T has at least K leaves.

Proof:

Suppose T is a tree.
Let v be a vertex in T with degree K.
Thus V has K neighbors.
Since T is a tree each neighbor of v is either a leaf or has degree at least 2.
If all neighbors of v are leaves then T has at least K leaves.
If a neighbor, u, has degree at least 2, then by removing edge uv we have that the component containing u is still a tree and contains at least one leaf.
Therefore if T is a tree with a vertex of degree K, then T has at least K leaves.

QED

2/ Prove that an edge e is a bridge in G if and only if e belongs to every spanning tree of G

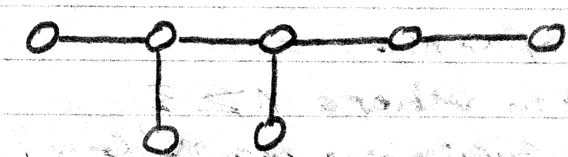
Proof:

Suppose e is a bridge in G.
Assume G is a connected graph.
Therefore removing e will increase the number of components from 1 to 2.
Now consider every spanning tree of G.
Any spanning tree is a connected graph which contains all vertices of G.
Therefore, if e was removed from G we would gain 2 components.
Therefore e must belong to every spanning tree of G to keep each component connected.
Suppose now, e is not a bridge.
Therefore removing e will not disconnect G.
Thus, some other path from the endpoints of e exists.
Therefore, e does not belong to every spanning tree.
Therefore, e is a bridge in G if and only if e belongs to every spanning tree of G.

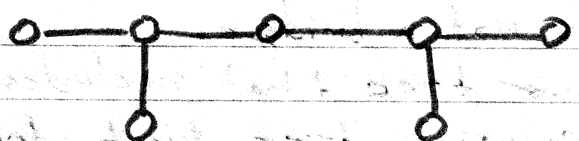
QED

3) Prove that there exists a pair of non-isomorphic graphs with the same degree sequence.

Proof:



Degree Sequence: 1111 233



Degree Sequence: 1111 233

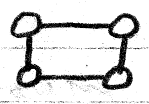
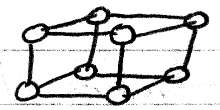
4) Prove that there is only one positive integer K such that no graph contains exactly K spanning trees.

Proof:

K_1 and P_2 have only 1 spanning tree.
C_n has n spanning trees where $n \geq 3$
Toward a contradiction, suppose a graph G has 2 spanning trees.
Let u and v be two verticies in G
Assume edge uv is not a bridge.
Let T be the spanning tree that includes uv and T' be the spanning tree that does not include uv.
Since uv is not a bridge there exists some path from u to v in T'.
However, this implies the path from u to v in T' plus the original edge uv would form a cycle
But C_n has n spanning trees where $n \geq 3$ ↯
Therefore, their is only one positive integer K (K=2) such that no graph contains exactly K spanning trees.

QED

5/ How many automorphisms of Q_3 exist

$Q_2 =$  $Q_3 =$ 

Q_2 has 8 automorphisms:

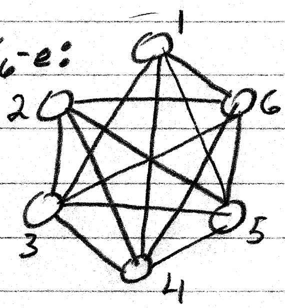
itself
itself rotated 90° 3 times
itself mirrored
itself mirrored then rotated 90° 3 times

Therefore, each face has 8 automorphisms and Q_3 has 6 faces, thus:

Q_3 has $6 \times 8 = 48$ automorphisms

6) Let $G = K_6 - e$.
How many automorphisms of G exist

K_6-e:



Vertices 1 and 2 must stay in the same locations therefore vertices 3, 4, 5 and 6 can be moved around in any permutation thus 4!
Therefore G has $2 \cdot 4! = 48$ automorphisms.
3, 4, 5 and 6 have 4! permutations for each 1 and 2 combo which is 2 possibilities