MATH-3420 - Assignment 2

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1. Which values of p and q generate the Primitive Pythagorean Triple given by $65^2 + 72^2 = 97^2$?

   Take $y = 72$, thus we have:

$$\begin{aligned} y &= 2pq \\ 72 &= 2pq \\ pq &= 36 \end{aligned}$$

   So the possible values of p and q are the factors of 36. Using $p = 4$ and $q = 9$, we have:

$$\begin{aligned} x &= q^2 - p^2 \\ x &= 9^2 - 4^2 \\ x &= 81 - 16 \\ x &= 65 \end{aligned}$$

   and:

$$\begin{aligned} z &= p^2 + q^2 \\ z &= 4^2 + 9^2 \\ z &= 16 + 81 \\ z &= 97 \end{aligned}$$

   Therefore the values $p = 4$ and $q = 9$ generate the Primitive Pythagorean Triple given by $65^2 + 72^2 = 97^2$.

2. (a) Find all Primitive Pythagorean Triples (x, y, z) such that $x = 15$.
Take $x = 15$, thus we have:

$$\begin{aligned} x &= q^2 - p^2 \\ 15 &= (q+p)(q-p) \end{aligned}$$

So the possible values of p and q are two numbers that when added or subtracted from one another give the factors of 15. The possible factors of 15 are 1 and 15, and 3 and 5.

For the factors 1 and 15, we have that $p = 7$ and $q = 8$ will satisfy this. Thus:

$$\begin{aligned} y &= 2pq \\ y &= 2(7)(8) \\ y &= 112 \end{aligned}$$

and:

$$\begin{aligned} z &= p^2 + q^2 \\ z &= 7^2 + 8^2 \\ z &= 49 + 64 \\ z &= 113 \end{aligned}$$

For the factors 3 and 5, we have that $p = 1$ and $q = 4$ will also satisfy our conditions. Thus:

$$\begin{aligned} y &= 2pq \\ y &= 2(1)(4) \\ y &= 8 \end{aligned}$$

and:

$$\begin{aligned} z &= p^2 + q^2 \\ z &= 1^2 + 4^2 \\ z &= 1 + 16 \\ z &= 17 \end{aligned}$$

Therefore, all Primitive Pythagorean Triples (x, y, z) such that $x = 15$ are, $(15, 112, 113)$ and $(15, 8, 17)$.

(b) Find all Pythagorean Triples (x, y, z) such that $x = 15$.

As per (a) we found $(15, 112, 113)$ and $(15, 8, 17)$ to be two of the Pythagorean Triples (x, y, z) such that $x = 15$.

Noticing that 15 can be written as $15 = 3*5$ we can use the Primitive Pythagorean Triples $(3, 4, 5)$ and $(5, 12, 13)$ to generate the remaining two Pythagorean Triples (x, y, z) such that $x = 15$.

Therefore we have:

$$\begin{aligned} (3,4,5) &\qquad (5,12,13) \\ 5(3,4,5) &\qquad 3(5,12,13) \\ (15,20,25) &\qquad (15,36,39) \end{aligned}$$

Thus the set of all Pythagorean Triples (x, y, z) such that $x = 15$ is $S = \{(15, 112, 113), (15, 8, 17), (15, 20, 25), (15, 36, 39)\}$

3. (a) Show that for any Pythagorean Triple (x, y, z), 4 divides y.

Proof. Suppose we have some Pythagorean Triple (x, y, z). We know then:

$$y = 2pq$$

We also know by definition that p and q must have opposite parity. Therefore, where $k \in \mathbb{Z}$, we have:

$$\begin{aligned} y &= 2(2k)(2k+1) \\ y &= 8k^2 + 4k \\ y &= 4(2k^2 + k) \end{aligned}$$

Therefore, we clearly see that $4|y$

$\square$

(b) Show that for any Pythagorean Triple (x, y, z), 12 divides xy.

Proof. Suppose we have some Pythagorean Triple (x, y, z).

As per (a) we have proven y to be divisible by 4. As per class we have also proven that one of x or y is divisible by 3. Since y is divisible by 4, we can assume x to be divisible by 3 Therefore since x is divisible by 3 and y is divisible 4, xy must be divisible by 12, since $(3)(4) = 12$.

$\square$

4. Solve the following linear congruence's, or explain why no solution exists:

   (a) $4x \equiv 7 \pmod{11}$
   First we find d to be:

   $$d = \gcd(a, n) = \gcd(4, 11) = 1$$

   Therefore, we can see a solution exists since $1|7$. Repeatedly adding 11 to 7 we gain:

   $$4x \equiv 40 \pmod{11}$$

   Thus with $\frac{n}{d} = \frac{11}{1} = 11$ the solution is:

   $$x \equiv 10 \pmod{11}$$

   (b) $3x \equiv 4 \pmod{12}$
   First we find d to be:

   $$d = \gcd(a, n) = \gcd(3, 12) = 3$$

   Therefore, we can see no solution exists since $3 \nmid 4$.

   (c) $6x \equiv 4 \pmod{8}$
   First we find d to be:

   $$d = \gcd(a, n) = \gcd(6, 8) = 2$$

   Therefore, we can see a solution exists since $2|4$. Repeatedly adding 8 to 4 we gain:

   $$6x \equiv 12 \pmod{8}$$

   Thus with $\frac{n}{d} = \frac{8}{2} = 4$ the solution is:

   $$x \equiv 2 \pmod{4}$$
   $$x \equiv 2, 6 \pmod{8}$$