Homework 2 solutions

Question 3

(a)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{1}$$

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} = \cdots \tag{2}$$

$$[\hat{L}_1, \hat{L}_2] = [x_2p_3 - x_3p_2, x_3p_1 - x_1p_3] \tag{3}$$

$$[A+B, C+D] = [A+B, C] + [A+B, D] \tag{4}$$

$$= [A, C] + [B, C] + [A, D] + [B, D] \tag{5}$$

$$\Rightarrow [L_1, L_2] = \underset{\text{I}}{[x_2p_3, x_3p_1]} - \underset{\text{II}}{[x_3p_2, x_3p_1]} - \underset{\text{III}}{[x_2p_3, x_1p_3]} + \underset{\text{IV}}{[x_3p_2, x_1p_3]} \tag{6}$$

$$[A, BC] = B[A, C] + [A, B]C \tag{7}$$

$$[AB, C] = A[B, C] + [A, C]B \tag{8}$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \ [p_i, x_j] = -i\hbar\delta_{ij} \tag{9}$$

$$[x_i, x_j] = [p_i, p_j] = 0 \tag{10}$$

$$\text{I} = [x_2p_3, x_3]p_1 + x_3[x_2p_2, p_1] \tag{11}$$

$$= [x_2, x_3]p_3p_1 + x_2[p_3, x_3]p_1 + x_3[x_2, p_1]p_3 + x_2x_3[p_3, p_1] \tag{12}$$

$$= -i\hbar x_2p_1 \tag{13}$$

$$\text{II} = [x_3p_2, x_3]p_1 + x_3[x_3p_2, p_1] \tag{14}$$

$$= x_3[p_2, x_3]p_1 + +[x_3, x_3]p_1p_2 + x_3x_3[p_2, p_1] + x_3[x_3, p_1]p_2 \tag{15}$$

$$= 0 \tag{16}$$

$$\text{III} = [x_2p_3, x_1]p_3 + x_1[x_2p_3, p_3] = 0 \tag{17}$$

$$\text{IV} = [x_3p_2, x_1]p_3 + x_1[x_3p_2, p_3] \tag{18}$$

$$= [x_3, x_1]p_2p_3 + x_3[p_2, x_1]p_3 + x_1[x_3, p_3]p_2 + x_1x_3[p_2, p_3] \tag{19}$$

$$= i\hbar x_1p_2 \tag{20}$$

$$[L_1, L_2] = i\hbar(x_1p_2 - x_2p_1) = i\hbar L_3 \tag{21}$$

$$[L_i, L_j] = \varepsilon_{ijk} i\hbar L_k \tag{22}$$

(b)

$$[\hat{\vec{L}}_1^2, \hat{L}_1] \tag{23}$$

$$\begin{aligned}
\hat{\vec{L}}^2 &= \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 && (24)\\
&= [L_1^2 + L_2^2 + L_3^2, L_1] = 0 && (25)\\
&= [L_2^2, L_1] + [L_3^2, L_1] && (26)\\
&= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3 && (27)\\
&= -i\hbar L_2 L_3 - i\hbar L_3 L_2 + i\hbar L_3 L_2 + i\hbar L_2 L_3 && (28)\\
&= -i\hbar(L_2 L_3 + L_3 L_2) + i\hbar(L_3 L_2 + L_2 L_3) = 0 && (29)
\end{aligned}$$

(c)

$$\vec{L}\times\vec{L} = \begin{bmatrix} x_2p_3 - x_3p_2 \\ x_3p_1 - x_1p_3 \\ x_1p_2 - x_2p_1 \end{bmatrix} \times \begin{bmatrix} x_2p_3 - x_3p_2 \\ x_3p_1 - x_1p_3 \\ x_1p_2 - x_2p_1 \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} (x_3p_1 - x_1p_3)(x_1p_2 - x_2p_1) - (x_1p_2 - x_2p_1)(x_3p_1 - x_1p_3) \\ (x_1p_2 - x_2p_1)(x_2p_3 - x_3p_2) - (x_2p_3 - x_3p_2)(x_1p_2 - x_2p_1) \\ (x_2p_3 - x_3p_2)(x_3p_1 - x_1p_3) - (x_3p_1 - x_1p_3)(x_2p_3 - x_3p_2) \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} [L_2, L_3] \\ [L_3, L_1] \\ [L_1, L_2] \end{bmatrix} = 0 \tag{32}$$

$$(\vec{L}\times\vec{L})_l = \sum_{m,n} \varepsilon_{lmn} L_m L_n = \varepsilon_{lmn}(L_m L_n - L_n L_m) = \varepsilon_{lmn}[L_m, L_n] = i\hbar L_l \tag{33}$$

(d)

$$[\frac{p^2}{2m}, x_i] = \frac{1}{2m}[p^2, x_i] \tag{34}$$

$$= \frac{1}{2m}[p_1^2 +_2^2 + p_3^2, x_i] \tag{35}$$

$$= \frac{1}{2m}[p_i^2, x_i] \tag{36}$$

$$= \frac{1}{2m}(p_i[p_i, x_i] + [p_i, x_i]p_i) \tag{37}$$

$$= \frac{1}{2m}(p_i(-i\hbar) - i\hbar p_i) \tag{38}$$

$$= \frac{-i\hbar}{m} p_i \tag{39}$$

$$f(x) = \sum_n a_n x^n \tag{40}$$

$$f(\hat{x}) = \sum_n a_n \hat{x}^n \tag{41}$$

therefore,

$$[f(\hat{x}), p_1] = [\sum_n a_n \hat{x}^n, p_1] \tag{42}$$

$$= \sum_n a_n [x_1^n, p_1] \tag{43}$$

$$[x_1, p_1] = i\hbar \tag{44}$$

$$[x_1^2, p_1] = 2i\hbar x_1 \tag{45}$$

$$[x_1^n, p_1] = ni\hbar x_1^{n-1} \tag{46}$$

$$[x_1^{n+1}, p_1] = [x_1 \cdot x_1^n, p_1] = x_1[x_1^n, p_1] + [x_1, p_1]x_1^n = ni\hbar x_1^n + i\hbar x_1^n = (n+1)i\hbar x^n \tag{47}$$

therefore,

$$= \sum_n a_n n i\hbar x_1^{n-1} \tag{48}$$

$$f(\hat{x}_1) = \sum_n a_n \hat{x}_1^n \tag{49}$$

and

$$[\hat{x}_1, f(\hat{x}_1)] = i\hbar f'(\hat{p}_1) \tag{50}$$

$$[f(\hat{x}_1), \hat{p}_1] = i\hbar f'(\hat{x}_1) \tag{51}$$

$$[V(\hat{\vec{r}}), \hat{p}_i] = i\hbar \frac{\partial V}{\partial \hat{x}_i} \tag{52}$$

Question 4

(a)

$$\psi(x,0) = Ae^{ik_0x}e^{\frac{-x^2}{2b^2}} \tag{53}$$

$$\rho(x,0) = |\psi(x,0)|^2 \tag{54}$$

is the density of the wave function

$$\rho(x,0)dx \tag{55}$$

is the probability to find the particle in $(x, x+dx)$

therefore,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \rho(x,0)dx && (56)\\ &= A^2 \int e^{-\frac{x^2}{2b^2}} dx && (57)\\ (u = \frac{x}{b}) &= A^2 b \int e^{-u^2} du && (58)\\ &= A^2 b\sqrt{\pi} && (59)\\ \Rightarrow A &= \frac{1}{\sqrt{b\sqrt{\pi}}} && (60) \end{aligned}$$

(b)

$$\begin{aligned} \psi(x,0) &= \frac{1}{2\pi} \int dk e^{ikx} \tilde{\psi}(k) && (61)\\ \tilde{\psi}(k) &= \int dx e^{-ikx} \psi(x,0) && (62) \end{aligned}$$

$$\begin{aligned} \tilde{\psi}(k) &= A \int e^{-ikx - ik_0x} e^{-\frac{x^2}{2b^2}} && (63)\\ \tilde{\psi}(k) &= \sqrt{2b\sqrt{\pi}} e^{\frac{-b^2}{2}(k-k_0)^2} && (64) \end{aligned}$$

(c) Set potential of e in (b) equal to -1

(d)

$$\psi(x,t) = \frac{\sqrt{b}}{\pi^{\frac{1}{4}}} \frac{e^{\frac{-b^2k_0^2}{2}}}{\sqrt{b^2 + \frac{i\hbar t}{m}}} e^{\frac{(b^2k_0 + ix)^2}{2(b^2 + \frac{i\hbar t}{m})}} \tag{65}$$