Homework 3 solutions

Question 3

(a) Wave function given by

$$\psi(x,t) = \phi(x)e^{-iEt/\hbar} \tag{1}$$

insert into Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi \tag{2}$$

we get

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\phi(x)e^{-iEt/\hbar} &= \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\phi(x)e^{-iEt/\hbar} && (3)\\
E\phi(x)e^{-iEt/\hbar} &= \Big(-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + V\phi(x)\Big)e^{-iEt/\hbar} && (4)\\
E\phi(x) &= -\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + V\phi(x) && (5)\\
V &= E + \frac{1}{\phi(x)}\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} && (6)
\end{aligned}$$

(b) For gven function

$$\phi(x) = Ne^{-x^2/2\sigma^2} \tag{7}$$

the potential is therefore

$$\begin{aligned}
V(x) &= E + \frac{1}{Ne^{-x^2/2\sigma^2}}\frac{\hbar^2}{2m}\frac{\partial^2 Ne^{-x^2/2\sigma^2}}{\partial x^2} && (8)\\
&= E + \frac{1}{Ne^{-x^2/2\sigma^2}}\frac{\hbar^2}{2m}\frac{x^2-\sigma^2}{\sigma^4}Ne^{-x^2/2\sigma^2} && (9)\\
&= E + \frac{\hbar^2}{2m}\frac{x^2-\sigma^2}{\sigma^4} && (10)
\end{aligned}$$

(c) For gven function

$$\phi(x) = Ne^{-x/2l} \tag{11}$$

the potential is therefore

$$\begin{aligned}
V(x) &= E + \frac{1}{Ne^{-x/2l}}\frac{\hbar^2}{2m}\frac{\partial^2 Ne^{-x/2l}}{\partial x^2} && (12)\\
&= E + \frac{1}{Ne^{-x^2/2\sigma^2}}\frac{\hbar^2}{2m}\frac{1}{4l^2}Ne^{-x^2/2\sigma^2} && (13)\\
&= E + \frac{\hbar^2}{8ml^2} && (14)\\
& && (15)
\end{aligned}$$

Question 4

(a) To show whether a wave function ψ is normalized, we have to find out if the following intergal of probability amplitude $\rho = |\psi|^2$ is equal to one:

$$\int_{-\infty}^{\infty} \rho dx \tag{16}$$

with wave function given by

$$\psi(x) = \frac{1}{\sqrt{L}}\theta\left(\frac{L}{2} - |x|\right) \tag{17}$$

therefore we have

$$\rho(x) = |\psi(x)|^2 = \frac{1}{L}\theta\left(\frac{L}{2} - |x|\right) \tag{18}$$

since $(\theta(\xi))^2 = \theta(\xi)$, and thus the integral is

$$\begin{aligned} & \frac{1}{L}\int_{-\infty}^{\infty} \theta\left(\frac{L}{2} - |x|\right) dx && (19) \\ = & \frac{1}{L}L && (20) \\ = & 1 && (21) \end{aligned}$$

(b)

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx = \int_{-\infty}^{\infty} x|\psi|^2 dx \tag{22}$$

with wave function given by

$$\psi(x) = \frac{1}{\sqrt{L}}\theta\left(\frac{L}{2} - |x|\right) \tag{23}$$

we get

$$\langle x \rangle = \frac{1}{L}\int_{-\infty}^{\infty} x\theta\left(\frac{L}{2} - |x|\right) dx = 0 \tag{24}$$

similaraly we have

$$\langle x^2 \rangle = \frac{1}{L}\int_{-\infty}^{\infty} x^2\theta\left(\frac{L}{2} - |x|\right) dx = \frac{L^2}{12} \tag{25}$$

and

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle} = \sqrt{\frac{x^2}{12}} = \frac{\sqrt{3}}{6}L \tag{26}$$

(c)

$$
\begin{aligned}
\tilde{\psi}(x) &= \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)dx & (27)\\
&= \int_{-\infty}^{\infty} e^{-ipx/\hbar}\frac{1}{\sqrt{L}}\theta\left(\frac{L}{2}-|x|\right)dx & (28)\\
&= \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} e^{-ipx/\hbar}dx & (29)\\
&= \frac{1}{\sqrt{L}}\left[\frac{e^{-ipx/\hbar}}{-ip/\hbar}\right]_{-L/2}^{L/2} & (30)\\
&= \frac{1}{\sqrt{L}}\left[\frac{e^{-ipL/2\hbar}}{-ip/\hbar}-\frac{e^{ipL/2\hbar}}{-ip/\hbar}\right] & (31)\\
&= \frac{\hbar}{-ip\sqrt{L}}(e^{-ipL/2\hbar}-e^{ipL/2\hbar}) & (32)\\
&= \frac{2\hbar}{p\sqrt{L}}\sin\left(\frac{pL}{2\hbar}\right) & (33)
\end{aligned}
$$

(d) The expectation value of momentum is given by

$$
\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty}\tilde{\psi}^*(p)p\tilde{\psi}(p)dp & (34)\\
&= \frac{4\hbar^2}{L}\int_{-\infty}^{\infty}\frac{1}{p}\sin^2\left(\frac{pL}{2\hbar}\right)dp & (35)\\
&= 0 & (36)
\end{aligned}
$$

(e)

$$
\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty}\tilde{\psi}^*(p)p^2\tilde{\psi}(p)dp & (37)\\
&= \frac{4\hbar^2}{L}\int_{-\infty}^{\infty}\sin^2\left(\frac{pL}{2\hbar}\right)dp & (38)\\
&= \infty & (39)
\end{aligned}
$$

therefore

$$\Delta p = \infty \tag{40}$$