Homework 2 solutions

Question 3

(a)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{1}$$

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} = \cdots \tag{2}$$

$$\begin{align}
[\hat{L}_1, \hat{L}_2] &= [x_2p_3 - x_3p_2, x_3p_1 - x_1p_3] \tag{3}\\
[A+B, C+D] &= [A+B, C] + [A+B, D] \tag{4}\\
&= [A,C] + [B,C] + [A,D] + [B,D] \tag{5}\\
\Rightarrow [L_1, L_2] &= \underset{\text{I}}{[x_2p_3, x_3p_1]} - \underset{\text{II}}{[x_3p_2, x_3p_1]} - \underset{\text{III}}{[x_2p_3, x_1p_3]} + \underset{\text{IV}}{[x_3p_2, x_1p_3]} \tag{6}
\end{align}$$

$$\begin{align}
[A, BC] &= B[A,C] + [A,B]C \tag{7}\\
[AB, C] &= A[B,C] + [A,C]B \tag{8}
\end{align}$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \ [p_i, x_j] = -i\hbar\delta_{ij} \tag{9}$$

$$[x_i, x_j] = [p_i, p_j] = 0 \tag{10}$$

$$\begin{align}
\text{I} &= [x_2p_3, x_3]p_1 + x_3[x_2p_2, p_1] \tag{11}\\
&= [x_2, x_3]p_3p_1 + x_2[p_3, x_3]p_1 + x_3[x_2, p_1]p_3 + x_2x_3[p_3, p_1] \tag{12}\\
&= -i\hbar x_2p_1 \tag{13}\\
\text{II} &= [x_3p_2, x_3]p_1 + x_3[x_3p_2, p_1] \tag{14}\\
&= x_3[p_2, x_3]p_1 + +[x_3, x_3]p_1p_2 + x_3x_3[p_2, p_1] + x_3[x_3, p_1]p_2 \tag{15}\\
&= 0 \tag{16}\\
\text{III} &= [x_2p_3, x_1]p_3 + x_1[x_2p_3, p_3] = 0 \tag{17}\\
\text{IV} &= [x_3p_2, x_1]p_3 + x_1[x_3p_2, p_3] \tag{18}\\
&= [x_3, x_1]p_2p_3 + x_3[p_2, x_1]p_3 + x_1[x_3, p_3]p_2 + x_1x_3[p_2, p_3] \tag{19}\\
&= i\hbar x_1p_2 \tag{20}
\end{align}$$

$$[L_1, L_2] = i\hbar(x_1p_2 - x_2p_1) = i\hbar L_3 \tag{21}$$

$$[L_i, L_j] = \varepsilon_{ijk} i\hbar L_k \tag{22}$$

(b)

$$[\hat{\vec{L}}_1^2, \hat{L}_1] \tag{23}$$

$$\begin{aligned}
\hat{\vec{L}}^2 &= \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 && (24)\\
&= [L_1^2 + L_2^2 + L_3^2, L_1] = 0 && (25)\\
&= [L_2^2, L_1] + [L_3^2, L_1] && (26)\\
&= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3 && (27)\\
&= -i\hbar L_2 L_3 - i\hbar L_3 L_2 + i\hbar L_3 L_2 + i\hbar L_2 L_3 && (28)\\
&= -i\hbar(L_2 L_3 + L_3 L_2) + i\hbar(L_3 L_2 + L_2 L_3) = 0 && (29)
\end{aligned}$$

(c)

$$\begin{aligned}
\vec{L} \times \vec{L} &= \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} \times \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} && (30)\\
&= \begin{bmatrix} (x_3 p_1 - x_1 p_3)(x_1 p_2 - x_2 p_1) - (x_1 p_2 - x_2 p_1)(x_3 p_1 - x_1 p_3) \\ (x_1 p_2 - x_2 p_1)(x_2 p_3 - x_3 p_2) - (x_2 p_3 - x_3 p_2)(x_1 p_2 - x_2 p_1) \\ (x_2 p_3 - x_3 p_2)(x_3 p_1 - x_1 p_3) - (x_3 p_1 - x_1 p_3)(x_2 p_3 - x_3 p_2) \end{bmatrix} && (31)\\
&= \begin{bmatrix} [L_2, L_3] \\ [L_3, L_1] \\ [L_1, L_2] \end{bmatrix} = 0 && (32)
\end{aligned}$$

$$(\vec{L} \times \vec{L})_l = \sum_{m,n} \varepsilon_{lmn} L_m L_n = \varepsilon_{lmn}(L_m L_n - L_n L_m) = \varepsilon_{lmn}[L_m, L_n] = i\hbar L_l \tag{33}$$

(d)

$$\begin{aligned}
[\frac{p^2}{2m}, x_i] &= \frac{1}{2m}[p^2, x_i] && (34)\\
&= \frac{1}{2m}[p_1^2 + p_2^2 + p_3^2, x_i] && (35)\\
&= \frac{1}{2m}[p_i^2, x_i] && (36)\\
&= \frac{1}{2m}(p_i[p_i, x_i] + [p_i, x_i]p_i) && (37)\\
&= \frac{1}{2m}(p_i(-i\hbar) - i\hbar p_i) && (38)\\
&= \frac{-i\hbar}{m} p_i && (39)
\end{aligned}$$

$$\begin{aligned}
[V(x), p_i]\psi &= (V(-i\hbar\partial_{x_i}) - (-i\hbar\partial_{x_i})V)\psi && (40)\\
&= V(-i\hbar)\partial_{x_i}\psi - (-i\hbar)\partial_{x_i}(V\psi) && (41)\\
&= V(-i\hbar)\partial_{x_i}\psi - V(-i\hbar)\partial_{x_i}\psi - (-i\hbar)(\partial_{x_i}V)\psi && (42)\\
&= \underbrace{i\hbar(\partial_{x_i}V)}_{[V(x), p_i]}\psi && (43)
\end{aligned}$$

$$[L_1, x_1] = [x_2 p_3 - x_3 p_2, x_1] = 0 \tag{44}$$

$$\begin{aligned}
[L_1, x_2] &= [x_2 p_3 - x_3 p_2, x_2] &(45)\\
&= [x_2 p_3, x_2] - [x_3 p_2, x_2] &(46)\\
&= x_2[p_3, x_2] + [x_2, x_2]p_3 - (x_3[p_2, x_2] + [x_3, x_2]p_2) &(47)\\
&= -x_3[p_2, x_2] &(48)\\
&= i\hbar x_3 &(49)
\end{aligned}$$

Question 4

(a)

$$\psi(x,0) = Ae^{ik_0x}e^{\frac{-x^2}{2b^2}} \tag{50}$$

$$\rho(x,0) = |\psi(x,0)|^2 = \psi^*\psi \tag{51}$$

is the probability density of the wave function

$$\rho(x,0)dx \tag{52}$$

is the probability to find the particle in $(x, x+dx)$
therefore,

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \rho(x,0)dx && (53)\\
&= A^2 \int e^{-\frac{x^2}{b^2}}dx && (54)\\
(u = \frac{x}{b}) &= A^2 b \int e^{-u^2}du && (55)\\
&= A^2 b\sqrt{\pi} && (56)\\
\Rightarrow A &= \frac{1}{\sqrt{b}\sqrt[4]{\pi}} && (57)
\end{aligned}$$

(b)

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx e^{-ikx}\psi(x,0) \tag{58}$$

$$\begin{aligned}
\tilde{\psi}(k) &= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}\int_{-\infty}^{\infty} dx e^{-ikx}e^{ik_0x}e^{\frac{-x^2}{2b^2}} && (59)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}\int_{-\infty}^{\infty} dx \exp\left[-i(k-k_0)x - \frac{x^2}{2b^2}\right] && (60)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}\int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2b^2}\Big(x^2 + 2b^2 i(k-k_0)x - b^4(k-k_0)^2\Big) - \frac{b^2}{2}(k-k_0)^2\right] && (61)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}\int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2b^2}\Big(x + b^2 i(k-k_0)\Big)^2 - \frac{b^2}{2}(k-k_0)^2\right] && (62)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2}\int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2b^2}\Big(x + b^2 i(k-k_0)\Big)^2\right] && (63)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2}\int_{-\infty}^{\infty} dx' \exp\left[-\frac{x'^2}{2b^2}\right] && (64)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2}\sqrt{2}b\int_{-\infty}^{\infty} dx'' \exp\left[-x''^2\right] && (65)\\
&= \frac{1}{\sqrt{b}\sqrt[4]{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2}\sqrt{2b\pi} && (66)\\
&= \sqrt{2b\sqrt{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2} && (67)
\end{aligned}$$

(c)

$$(\tilde{\psi}(k))^2 = e^{-b^2(k-k_0)^2} 2b\sqrt{\pi} \tag{68}$$

$$\Rightarrow e^{-1} = e^{-b^2(k-k_0)^2} \tag{69}$$

$$-1 = -b^2(k-k_0)^2 \tag{70}$$

$$\frac{1}{b} = |k-k_0| = \frac{1}{2}\Delta k \tag{71}$$

(d)

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[i\Big(kx - \frac{\hbar k^2}{2m}t\Big)\right]\tilde{\psi}(k) \tag{72}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[i\Big(kx - \frac{\hbar k^2}{2m}t\Big)\right]\sqrt{2b\sqrt{\pi}}e^{-\frac{b^2}{2}(k-k_0)^2} \tag{73}$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[i\Big(kx - \frac{\hbar k^2}{2m}t\Big) - \frac{b^2}{2}(k-k_0)^2\right] \tag{74}$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[ikx - \frac{\hbar k^2}{2m}it - \frac{b^2}{2}k^2 + b^2kk_0 - \frac{b^2}{2}k_0^2\right] \tag{75}$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi}e^{-\frac{b^2}{2}k_0^2}\int_{-\infty}^{\infty} dk \exp\left[ikx - \frac{\hbar k^2}{2m}it - \frac{b^2}{2}k^2 + b^2kk_0\right] \tag{76}$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi}e^{-\frac{b^2}{2}k_0^2}\int_{-\infty}^{\infty} dk \exp\left[k(ix + b^2k_0) + k^2\Big(-\frac{\hbar}{2m}it - \frac{b^2}{2}\Big)\right] \tag{77}$$

$$= \frac{\sqrt{2b\sqrt{\pi}\pi}}{2\pi}e^{-\frac{b^2}{2}k_0^2}\frac{1}{\sqrt{\frac{\hbar}{2m}it + \frac{b^2}{2}}} \exp\left[\frac{(ix+b^2k_0)^2}{4(\frac{\hbar}{2m}it + \frac{b^2}{2})}\right] \tag{78}$$

$$= \frac{\sqrt{b}}{\sqrt{\pi}\sqrt{\frac{\hbar}{m}it + b^2}} \exp\left[\frac{(ix+b^2k_0)^2}{2(\frac{\hbar}{m}it + b^2)} - \frac{b^2}{2}k_0^2\right] \tag{79}$$

(e)