

Homework 3 solutions

Question 3

(a) Wave function given by

$$\psi(x, t) = \phi(x)e^{-iEt/\hbar} \quad (1)$$

insert into Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi \quad (2)$$

we get

$$i\hbar \frac{\partial}{\partial t} \phi(x)e^{-iEt/\hbar} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \phi(x)e^{-iEt/\hbar} \quad (3)$$

$$E\phi(x)e^{-iEt/\hbar} = \left(-\frac{\hbar^2}{2m} \frac{\partial^2 \phi(x)}{\partial x^2} + V\phi(x) \right) e^{-iEt/\hbar} \quad (4)$$

$$E\phi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi(x)}{\partial x^2} + V\phi(x) \quad (5)$$

$$V = E + \frac{1}{\phi(x)} \frac{\hbar^2}{2m} \frac{\partial^2 \phi(x)}{\partial x^2} \quad (6)$$

(b) For given function

$$\phi(x) = Ne^{-x^2/2\sigma^2} \quad (7)$$

the potential is therefore

$$V(x) = E + \frac{1}{Ne^{-x^2/2\sigma^2}} \frac{\hbar^2}{2m} \frac{\partial^2 Ne^{-x^2/2\sigma^2}}{\partial x^2} \quad (8)$$

$$= E + \frac{1}{Ne^{-x^2/2\sigma^2}} \frac{\hbar^2}{2m} \frac{x^2 - \sigma^2}{\sigma^4} Ne^{-x^2/2\sigma^2} \quad (9)$$

$$= E + \frac{\hbar^2}{2m} \frac{x^2 - \sigma^2}{\sigma^4} \quad (10)$$

(c) For given function

$$\phi(x) = Ne^{-x/2l} \quad (11)$$

the potential is therefore

$$V(x) = E + \frac{1}{Ne^{-x/2l}} \frac{\hbar^2}{2m} \frac{\partial^2 Ne^{-x/2l}}{\partial x^2} \quad (12)$$

$$= E + \frac{1}{Ne^{-x/2l}} \frac{\hbar^2}{2m} \frac{1}{4l^2} Ne^{-x/2l} \quad (13)$$

$$= E + \frac{\hbar^2}{8ml^2} \quad (14)$$

$$(15)$$

Question 4

- (a) To show whether a wave function ψ is normalized, we have to find out if the following integral of probability amplitude $\rho = |\psi|^2$ is equal to one:

$$\int_{-\infty}^{\infty} \rho dx \quad (16)$$

with wave function given by

$$\psi(x) = \frac{1}{\sqrt{L}} \theta\left(\frac{L}{2} - |x|\right) \quad (17)$$

therefore we have

$$\rho(x) = |\psi(x)|^2 = \frac{1}{L} \theta\left(\frac{L}{2} - |x|\right) \quad (18)$$

since $(\theta(\xi))^2 = \theta(\xi)$, and thus the integral is

$$\frac{1}{L} \int_{-\infty}^{\infty} \theta\left(\frac{L}{2} - |x|\right) dx \quad (19)$$

$$= \frac{1}{L} L \quad (20)$$

$$= 1 \quad (21)$$

(b)

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx = \int_{-\infty}^{\infty} x |\psi|^2 dx \quad (22)$$

with wave function given by

$$\psi(x) = \frac{1}{\sqrt{L}} \theta\left(\frac{L}{2} - |x|\right) \quad (23)$$

we get

$$\langle x \rangle = \frac{1}{L} \int_{-\infty}^{\infty} x \theta\left(\frac{L}{2} - |x|\right) dx = 0 \quad (24)$$

similarly we have

$$\langle x^2 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} x^2 \theta\left(\frac{L}{2} - |x|\right) dx = \frac{L^2}{12} \quad (25)$$

and

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{x^2}{12}} = \frac{\sqrt{3}}{6} L \quad (26)$$

(c)

$$\tilde{\psi}(x) = \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) dx \quad (27)$$

$$= \int_{-\infty}^{\infty} e^{-ipx/\hbar} \frac{1}{\sqrt{L}} \theta\left(\frac{L}{2} - |x|\right) dx \quad (28)$$

$$= \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} e^{-ipx/\hbar} dx \quad (29)$$

$$= \frac{1}{\sqrt{L}} \left[\frac{e^{-ipx/\hbar}}{-ip/\hbar} \right]_{-L/2}^{L/2} \quad (30)$$

$$= \frac{1}{\sqrt{L}} \left[\frac{e^{-ipL/2\hbar}}{-ipL/2\hbar} - \frac{-e^{ipL/2\hbar}}{-ipL/2\hbar} \right] \quad (31)$$

$$(32)$$