1 Homework 2 solutions

Question 3

(a)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} = \cdots$$

$$\begin{aligned}
[\hat{L}_1, \hat{L}_2] &= [x_2p_3 - x_3p_2, x_3p_1 - x_1p_3] \\
[A+B, C+D] &= [A+B, C] + [A+B, D] \\
&= [A, C] + [B, C] + [A+D] + [B+D] \\
\Rightarrow [L_1, L_2] &= \underbrace{[x_2p_3, x_3p_1]}_{I} - [x_3p_2, x_3p_1] - [x_2p_3, x_1p_2] - \underbrace{[x_3p_2, x_1p_3]}_{IV}
\end{aligned}$$

$$\begin{aligned}
I &= [x_2p_3, x_3]p_1 + x_3[x_2p_2, p_1] \\
&= [x_2, x_3]p_3p_1 + x_2[p_3, x_3]p_1 + x_3[x_2, p_1]p_3 + x_2x_3[p_3, p_1] \\
&= -i\hbar x_2p_1 \\
IV &= i\hbar x_1p_2
\end{aligned}$$

$$[L_1, L_2] = i\hbar(x_1p_2 - x_2p_1) = i\hbar L_3$$

$$[L_i, L_j] = \varepsilon_{ijk} i\hbar L_k$$

(b)

$$[\hat{\vec{L}}_1^2, \hat{L}_1]$$

$$\begin{aligned}
\hat{\vec{L}}^2 &= \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 \\
&= [L_1^2 + L_2^2 + L_3^2, L_1] = 0 \\
&= [L_2^2, L_1] + [L_3^2, L_1] \\
&= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3 \\
&= -i\hbar L_2L_3 - i\hbar L_3L_2 + i\hbar L_3L_2 + i\hbar L_2L_3 \\
&= -i\hbar(L_2L_3 + L_3L_2) + i\hbar(L_3L_2 + L_2L_3) = 0
\end{aligned}$$

(c)

$$\vec{L}\times\vec{L} = \begin{bmatrix} x_2p_3 - x_3p_2 \\ x_3p_1 - x_1p_3 \\ x_1p_2 - x_2p_1 \end{bmatrix} \times \begin{bmatrix} x_2p_3 - x_3p_2 \\ x_3p_1 - x_1p_3 \\ x_1p_2 - x_2p_1 \end{bmatrix}$$
$$= \begin{bmatrix} (x_3p_1 - x_1p_3)(x_1p_2 - x_2p_1) - (x_1p_2 - x_2p_1)(x_3p_1 - x_1p_3) \\ (x_1p_2 - x_2p_1)(x_2p_3 - x_3p_2) - (x_2p_3 - x_3p_2)(x_1p_2 - x_2p_1) \\ (x_2p_3 - x_3p_2)(x_3p_1 - x_1p_3) - (x_3p_1 - x_1p_3)(x_2p_3 - x_3p_2) \end{bmatrix}$$
$$= \begin{bmatrix} [L_2, L_3] \\ [L_3, L_1] \\ [L_1, L_2] \end{bmatrix} = 0$$

$$(\vec{L}\times\vec{L})_l = \sum_{m,n} \varepsilon_{lmn} L_m L_n = \varepsilon_{lmn}(L_m L_n - L_n L_m) = \varepsilon_{lmn}[L_m, L_n] = i\hbar L_l$$

(d)

$$\begin{aligned} [\frac{p^2}{2m}, x_i] &= \frac{1}{2m}[p^2, x_i] \\ &= \frac{1}{2m}[p_1^2 +_2^2 + p_3^2, x_i] \\ &= \frac{1}{2m}[p_i^2, x_i] \\ &= \frac{1}{2m}(p_i[p_i, x_i] + [p_i, x_i]p_i) \\ &= \frac{1}{2m}(p_i(-i\hbar) - i\hbar p_i) \\ &= \frac{-i\hbar}{m} p_i \end{aligned}$$

$$f(x) = \sum_n a_n x^n$$
$$f(\hat{x}) = \sum_n a_n \hat{x}^n$$

therefore,

$$\begin{aligned} [f(\hat{x}), p_1] &= [\sum_n a_n \hat{x}^n, p_1] \\ &= \sum_n a_n [x_1^n, p_1] \end{aligned}$$

$$\begin{aligned} [x_1, p_1] &= i\hbar \\ [x_1^2, p_1] &= 2i\hbar x_1 \\ [x_1^n, p_1] &= ni\hbar x_1^{n-1} \\ [x_1^{n+1}, p_1] &= [x_1 \cdot x_1^n, p_1] = x_1[x_1^n, p_1] + [x_1, p_1]x_1^n = ni\hbar x_1^n + i\hbar x_1^n = (n+1)i\hbar x^n \end{aligned}$$

therefore,

$$
\begin{aligned}
&= \sum_n a_n n i\hbar x_1^{n-1} \\
f(\hat{x}_1) &= \sum_n a_n \hat{x}_1^n
\end{aligned}
$$

and

$$
\begin{aligned}
[\hat{x}_1, f(\hat{x}_1)] &= i\hbar f'(\hat{p}_1) \\
[f(\hat{x}_1), \hat{p}_1] &= i\hbar f'(\hat{x}_1) \\
[V(\hat{\vec{r}}), \hat{p}_i] &= i\hbar \frac{\partial V}{\partial \hat{x}_i}
\end{aligned}
$$

Question 4

(a)

$$\psi(x,0) = Ae^{ik_0x}e^{\frac{-x^2}{2b^2}}$$

$$\rho(x,0) = |\psi(x,0)|^2$$

is the density of the wave function

$$\rho(x,0)dx$$

is the probability to find the particle in $(x, x+dx)$
therefore,

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \rho(x,0)dx \\
&= A^2 \int e^{-\frac{x^2}{2b^2}} dx \\
(u = \frac{x}{b}) &= A^2 b \int e^{-u^2} du \\
&= A^2 b\sqrt{\pi} \\
\Rightarrow A &= \frac{1}{\sqrt{b\sqrt{\pi}}}
\end{aligned}$$

(b)

$$\begin{aligned}
\psi(x,0) &= \frac{1}{2\pi}\int dk e^{ikx}\tilde{\psi}(k) \\
\tilde{\psi}(k) &= \int dx e^{-ikx}\psi(x,0)
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}(k) &= A\int e^{-ikx-ik_0x}e^{-\frac{x^2}{2b^2}} \\
\tilde{\psi}(k) &= \sqrt{2b\sqrt{\pi}}e^{\frac{-b^2}{2}(k-k_0)^2}
\end{aligned}$$

(c) Set potential of e in (b) equal to -1

(d)

$$\psi(x,t) = \frac{\sqrt{b}}{\pi^{\frac{1}{4}}}\frac{e^{\frac{-b^2k_0^2}{2}}}{\sqrt{b^2+\frac{i\hbar t}{m}}}e^{\frac{(b^2k_0+ix)^2}{2(b^2+\frac{i\hbar t}{m})}}$$