

Homework 3 solutions

Question 3

(a)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad (1)$$

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} = \dots \quad (2)$$

$$[\hat{L}_1, \hat{L}_2] = [x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3] \quad (3)$$

$$[A + B, C + D] = [A + B, C] + [A + B, D] \quad (4)$$

$$= [A, C] + [B, C] + [A, D] + [B, D] \quad (5)$$

$$\Rightarrow [L_1, L_2] = \underbrace{[x_2 p_3, x_3 p_1]}_I - \underbrace{[x_3 p_2, x_3 p_1]}_{II} - \underbrace{[x_2 p_3, x_1 p_3]}_{III} + \underbrace{[x_3 p_2, x_1 p_3]}_{IV} \quad (6)$$

$$[A, BC] = B[A, C] + [A, B]C \quad (7)$$

$$[AB, C] = A[B, C] + [A, C]B \quad (8)$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \quad [p_i, x_j] = -i\hbar\delta_{ij} \quad (9)$$

$$[x_i, x_j] = [p_i, p_j] = 0 \quad (10)$$

$$I = [x_2 p_3, x_3] p_1 + x_3 [x_2 p_2, p_1] \quad (11)$$

$$= [x_2, x_3] p_3 p_1 + x_2 [p_3, x_3] p_1 + x_3 [x_2, p_1] p_3 + x_2 x_3 [p_3, p_1] \quad (12)$$

$$= -i\hbar x_2 p_1 \quad (13)$$

$$II = [x_3 p_2, x_3] p_1 + x_3 [x_3 p_2, p_1] \quad (14)$$

$$= x_3 [p_2, x_3] p_1 + [x_3, x_3] p_1 p_2 + x_3 x_3 [p_2, p_1] + x_3 [x_3, p_1] p_2 \quad (15)$$

$$= 0 \quad (16)$$

$$III = [x_2 p_3, x_1] p_3 + x_1 [x_2 p_3, p_3] = 0 \quad (17)$$

$$IV = [x_3 p_2, x_1] p_3 + x_1 [x_3 p_2, p_3] \quad (18)$$

$$= [x_3, x_1] p_2 p_3 + x_3 [p_2, x_1] p_3 + x_1 [x_3, p_3] p_2 + x_1 x_3 [p_2, p_3] \quad (19)$$

$$= i\hbar x_1 p_2 \quad (20)$$

$$[L_1, L_2] = i\hbar(x_1 p_2 - x_2 p_1) = i\hbar L_3 \quad (21)$$

$$[L_i, L_j] = \varepsilon_{ijk} i\hbar L_k \quad (22)$$

(b)

$$[\hat{\vec{L}}_1^2, \hat{L}_1] \quad (23)$$

$$\hat{\vec{L}}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 \quad (24)$$

$$= [L_1^2 + L_2^2 + L_3^2, L_1] = 0 \quad (25)$$

$$= [L_2^2, L_1] + [L_3^2, L_1] \quad (26)$$

$$= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3 \quad (27)$$

$$= -i\hbar L_2 L_3 - i\hbar L_3 L_2 + i\hbar L_3 L_2 + i\hbar L_2 L_3 \quad (28)$$

$$= -i\hbar(L_2 L_3 + L_3 L_2) + i\hbar(L_3 L_2 + L_2 L_3) = 0 \quad (29)$$

(c)

$$\vec{L} \times \vec{L} = \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} \times \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} \quad (30)$$

$$= \begin{bmatrix} (x_3 p_1 - x_1 p_3)(x_1 p_2 - x_2 p_1) - (x_1 p_2 - x_2 p_1)(x_3 p_1 - x_1 p_3) \\ (x_1 p_2 - x_2 p_1)(x_2 p_3 - x_3 p_2) - (x_2 p_3 - x_3 p_2)(x_1 p_2 - x_2 p_1) \\ (x_2 p_3 - x_3 p_2)(x_3 p_1 - x_1 p_3) - (x_3 p_1 - x_1 p_3)(x_2 p_3 - x_3 p_2) \end{bmatrix} \quad (31)$$

$$= \begin{bmatrix} [L_2, L_3] \\ [L_3, L_1] \\ [L_1, L_2] \end{bmatrix} = 0 \quad (32)$$

$$(\vec{L} \times \vec{L})_l = \sum_{m,n} \varepsilon_{lmn} L_m L_n = \varepsilon_{lmn} (L_m L_n - L_n L_m) = \varepsilon_{lmn} [L_m, L_n] = i\hbar L_l \quad (33)$$

(d)

$$[\frac{p^2}{2m}, x_i] = \frac{1}{2m} [p^2, x_i] \quad (34)$$

$$= \frac{1}{2m} [p_1^2 + p_2^2 + p_3^2, x_i] \quad (35)$$

$$= \frac{1}{2m} [p_i^2, x_i] \quad (36)$$

$$= \frac{1}{2m} (p_i [p_i, x_i] + [p_i, x_i] p_i) \quad (37)$$

$$= \frac{1}{2m} (p_i (-i\hbar) - i\hbar p_i) \quad (38)$$

$$= \frac{-i\hbar}{m} p_i \quad (39)$$

$$[V(x), p_i] \psi = (V(-i\hbar \partial_{x_i}) - (-i\hbar \partial_{x_i}) V) \psi \quad (40)$$

$$= V(-i\hbar) \partial_{x_i} \psi - (-i\hbar) \partial_{x_i} (V \psi) \quad (41)$$

$$= V(-i\hbar) \partial_{x_i} \psi - V(-i\hbar) \partial_{x_i} \psi - (-i\hbar) (\partial_{x_i} V) \psi \quad (42)$$

$$= \underbrace{i\hbar (\partial_{x_i} V)}_{[V(x), p_i]} \psi \quad (43)$$

$$[L_1, x_1] = [x_2 p_3 - x_3 p_2, x_1] = 0 \quad (44)$$

$$[L_1, x_2] = [x_2 p_3 - x_3 p_2, x_2] \quad (45)$$

$$= [x_2 p_3, x_2] - [x_3 p_2, x_2] \quad (46)$$

$$= x_2 [p_3, x_2] + [x_2, x_2] p_3 - (x_3 [p_2, x_2] + [x_3, x_2] p_2) \quad (47)$$

$$= -x_3 [p_2, x_2] \quad (48)$$

$$= i\hbar x_3 \quad (49)$$

Question 4

(a)

$$\psi(x, 0) = Ae^{ik_0x} e^{\frac{-x^2}{2b^2}} \quad (50)$$

$$\rho(x, 0) = |\psi(x, 0)|^2 = \psi^* \psi \quad (51)$$

is the probability density of the wave function

$$\rho(x, 0)dx \quad (52)$$

is the probability to find the particle in $(x, x + dx)$

therefore,

$$1 = \int_{-\infty}^{\infty} \rho(x, 0)dx \quad (53)$$

$$= A^2 \int e^{-\frac{x^2}{b^2}} dx \quad (54)$$

$$(u = \frac{x}{b}) = A^2 b \int e^{-u^2} du \quad (55)$$

$$= A^2 b \sqrt{\pi} \quad (56)$$

$$\Rightarrow A = \frac{1}{\sqrt{b} \sqrt[4]{\pi}} \quad (57)$$

(b)

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} \psi(x, 0) \quad (58)$$

$$\tilde{\psi}(k) = \frac{1}{\sqrt{b} \sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} e^{ik_0x} e^{\frac{-x^2}{2b^2}} \quad (59)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx \exp \left[-i(k - k_0)x - \frac{x^2}{2b^2} \right] \quad (60)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx \exp \left[-\frac{1}{2b^2} \left(x^2 + 2b^2 i(k - k_0)x - b^4(k - k_0)^2 \right) - \frac{b^2}{2}(k - k_0)^2 \right] \quad (61)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx \exp \left[-\frac{1}{2b^2} \left(x + b^2 i(k - k_0) \right)^2 - \frac{b^2}{2}(k - k_0)^2 \right] \quad (62)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} e^{-\frac{b^2}{2}(k - k_0)^2} \int_{-\infty}^{\infty} dx \exp \left[-\frac{1}{2b^2} \left(x + b^2 i(k - k_0) \right)^2 \right] \quad (63)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} e^{-\frac{b^2}{2}(k - k_0)^2} \int_{-\infty}^{\infty} dx' \exp \left[-\frac{x'^2}{2b^2} \right] \quad (64)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} e^{-\frac{b^2}{2}(k - k_0)^2} \sqrt{2b} \int_{-\infty}^{\infty} dx'' \exp[-x''^2] \quad (65)$$

$$= \frac{1}{\sqrt{b} \sqrt[4]{\pi}} e^{-\frac{b^2}{2}(k - k_0)^2} \sqrt{2b\pi} \quad (66)$$

$$= \sqrt{2b\sqrt{\pi}} e^{-\frac{b^2}{2}(k - k_0)^2} \quad (67)$$

(c)

$$(\tilde{\psi}(k))^2 = e^{-b^2(k-k_0)^2} 2b\sqrt{\pi} \quad (68)$$

$$\Rightarrow e^{-1} = e^{-b^2(k-k_0)^2} \quad (69)$$

$$-1 = -b^2(k-k_0)^2 \quad (70)$$

$$\frac{1}{b} = |k-k_0| = \frac{1}{2}\Delta k \quad (71)$$

(d)

$$\psi(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp \left[i \left(kx - \frac{\hbar k^2}{2m} t \right) \right] \tilde{\psi}(k) \quad (72)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp \left[i \left(kx - \frac{\hbar k^2}{2m} t \right) \right] \sqrt{2b\sqrt{\pi}} e^{-\frac{b^2}{2}(k-k_0)^2} \quad (73)$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi} \int_{-\infty}^{\infty} dk \exp \left[i \left(kx - \frac{\hbar k^2}{2m} t \right) - \frac{b^2}{2}(k-k_0)^2 \right] \quad (74)$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi} \int_{-\infty}^{\infty} dk \exp \left[ikx - \frac{\hbar k^2}{2m} it - \frac{b^2}{2} k^2 + b^2 k k_0 - \frac{b^2}{2} k_0^2 \right] \quad (75)$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi} e^{-\frac{b^2}{2} k_0^2} \int_{-\infty}^{\infty} dk \exp \left[ikx - \frac{\hbar k^2}{2m} it - \frac{b^2}{2} k^2 + b^2 k k_0 \right] \quad (76)$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi} e^{-\frac{b^2}{2} k_0^2} \int_{-\infty}^{\infty} dk \exp \left[k(ix + b^2 k_0) + k^2 \left(-\frac{\hbar}{2m} it - \frac{b^2}{2} \right) \right] \quad (77)$$

$$= \frac{\sqrt{2b\sqrt{\pi}}}{2\pi} e^{-\frac{b^2}{2} k_0^2} \frac{1}{\sqrt{\frac{\hbar}{2m} it + \frac{b^2}{2}}} \exp \left[\frac{(ix + b^2 k_0)^2}{4(\frac{\hbar}{2m} it + \frac{b^2}{2})} \right] \quad (78)$$

$$= \frac{\sqrt{b}}{\sqrt{\pi} \sqrt{\frac{\hbar}{m} it + b^2}} \exp \left[\frac{(ix + b^2 k_0)^2}{2(\frac{\hbar}{m} it + b^2)} - \frac{b^2}{2} k_0^2 \right] \quad (79)$$

(e)