Homework 2 solutions

Question 3

(a)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , \ \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{1}$$

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} = \cdots \tag{2}$$

$$[\hat{L}_1, \hat{L}_2] = [x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3] \tag{3}$$

$$[A + B, C + D] = [A + B, C] + [A + B, D] \tag{4}$$

$$= [A, C] + [B, C] + [A, D] + [B, D] \tag{5}$$

$$\Rightarrow [L_1, L_2] = \underset{\text{I}}{[x_2 p_3, x_3 p_1]} - \underset{\text{II}}{[x_3 p_2, x_3 p_1]} - \underset{\text{III}}{[x_2 p_3, x_1 p_3]} + \underset{\text{IV}}{[x_3 p_2, x_1 p_3]} \tag{6}$$

$$[A, BC] = B[A, C] + [A, B]C \tag{7}$$

$$[AB, C] = A[B, C] + [A, C]B \tag{8}$$

$$[x_i, p_j] = i\hbar \delta_{ij}, \ [p_i, x_j] = -i\hbar \delta_{ij} \tag{9}$$

$$[x_i, x_j] = [p_i, p_j] = 0 \tag{10}$$

$$\text{I} = [x_2 p_3, x_3] p_1 + x_3 [x_2 p_2, p_1] \tag{11}$$

$$= [x_2, x_3] p_3 p_1 + x_2 [p_3, x_3] p_1 + x_3 [x_2, p_1] p_3 + x_2 x_3 [p_3, p_1] \tag{12}$$

$$= -i\hbar x_2 p_1 \tag{13}$$

$$\text{II} = [x_3 p_2, x_3] p_1 + x_3 [x_3 p_2, p_1] \tag{14}$$

$$= x_3 [p_2, x_3] p_1 + +[x_3, x_3] p_1 p_2 + x_3 x_3 [p_2, p_1] + x_3 [x_3, p_1] p_2 \tag{15}$$

$$= 0 \tag{16}$$

$$\text{III} = [x_2 p_3, x_1] p_3 + x_1 [x_2 p_3, p_3] = 0 \tag{17}$$

$$\text{IV} = [x_3 p_2, x_1] p_3 + x_1 [x_3 p_2, p_3] \tag{18}$$

$$= [x_3, x_1] p_2 p_3 + x_3 [p_2, x_1] p_3 + x_1 [x_3, p_3] p_2 + x_1 x_3 [p_2, p_3] \tag{19}$$

$$= i\hbar x_1 p_2 \tag{20}$$

$$[L_1, L_2] = i\hbar (x_1 p_2 - x_2 p_1) = i\hbar L_3 \tag{21}$$

$$[L_i, L_j] = \varepsilon_{ijk} i\hbar L_k \tag{22}$$

(b)

$$[\hat{\vec{L}}_1^2, \hat{L}_1] \tag{23}$$

$$\hat{\vec{L}}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 \tag{24}$$
$$= [L_1^2 + L_2^2 + L_3^2, L_1] = 0 \tag{25}$$
$$= [L_2^2, L_1] + [L_3^2, L_1] \tag{26}$$
$$= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3 \tag{27}$$
$$= -i\hbar L_2 L_3 - i\hbar L_3 L_2 + i\hbar L_3 L_2 + i\hbar L_2 L_3 \tag{28}$$
$$= -i\hbar(L_2 L_3 + L_3 L_2) + i\hbar(L_3 L_2 + L_2 L_3) = 0 \tag{29}$$

(c)

$$\vec{L} \times \vec{L} = \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} \times \begin{bmatrix} x_2 p_3 - x_3 p_2 \\ x_3 p_1 - x_1 p_3 \\ x_1 p_2 - x_2 p_1 \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} (x_3 p_1 - x_1 p_3)(x_1 p_2 - x_2 p_1) - (x_1 p_2 - x_2 p_1)(x_3 p_1 - x_1 p_3) \\ (x_1 p_2 - x_2 p_1)(x_2 p_3 - x_3 p_2) - (x_2 p_3 - x_3 p_2)(x_1 p_2 - x_2 p_1) \\ (x_2 p_3 - x_3 p_2)(x_3 p_1 - x_1 p_3) - (x_3 p_1 - x_1 p_3)(x_2 p_3 - x_3 p_2) \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} [L_2, L_3] \\ [L_3, L_1] \\ [L_1, L_2] \end{bmatrix} = 0 \tag{32}$$

$$(\vec{L} \times \vec{L})_l = \sum_{m,n} \varepsilon_{lmn} L_m L_n = \varepsilon_{lmn}(L_m L_n - L_n L_m) = \varepsilon_{lmn}[L_m, L_n] = i\hbar L_l \tag{33}$$

(d)

$$[\frac{p^2}{2m}, x_i] = \frac{1}{2m}[p^2, x_i] \tag{34}$$

$$= \frac{1}{2m}[p_1^2 + p_2^2 + p_3^2, x_i] \tag{35}$$

$$= \frac{1}{2m}[p_i^2, x_i] \tag{36}$$

$$= \frac{1}{2m}(p_i[p_i, x_i] + [p_i, x_i]p_i) \tag{37}$$

$$= \frac{1}{2m}(p_i(-i\hbar) - i\hbar p_i) \tag{38}$$

$$= \frac{-i\hbar}{m}p_i \tag{39}$$

$$[V(x), p_i] = ? \tag{40}$$

$$[L_1, x_1] = [x_2 p_3 - x_3 p_2, x_1] = 0 \tag{41}$$

2

$$[L_1, x_2] = [x_2 p_3 - x_3 p_2, x_2] \tag{42}$$

$$= [x_2 p_3, x_2] - [x_3 p_2, x_2] \tag{43}$$

$$= x_2[p_3, x_2] + [x_2, x_2]p_3 - (x_3[p_2, x_2] + [x_3, x_2]p_2) \tag{44}$$

$$= -x_3[p_2, x_2] \tag{45}$$

$$= i\hbar x_3 \tag{46}$$

Question 4

(a)

$$\psi(x,0) = Ae^{ik_0x}e^{\frac{-x^2}{2b^2}} \tag{47}$$

$$\rho(x,0) = |\psi(x,0)|^2 \tag{48}$$

is the probability density of the wave function

$$\rho(x,0)dx \tag{49}$$

is the probability to find the particle in $(x, x+dx)$

therefore,

$$1 = \int_{-\infty}^{\infty} \rho(x,0)dx \tag{50}$$

$$= A^2 \int e^{-\frac{x^2}{2b^2}}dx \tag{51}$$

$$(u = \frac{x}{b}) = A^2 b \int e^{-u^2}du \tag{52}$$

$$= A^2 b\sqrt{\pi} \tag{53}$$

$$\Rightarrow A = \frac{1}{\sqrt{b}\sqrt[4]{\pi}} \tag{54}$$

(b)

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx e^{-ikx}\psi(x,0) \tag{55}$$

$$\tilde{\psi}(k) = \frac{1}{\sqrt{b}\sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx e^{-ikx}e^{ik_0x}e^{\frac{-x^2}{2b^2}} \tag{56}$$

$$= \frac{1}{\sqrt{b}\sqrt[4]{\pi}} \int_{-\infty}^{\infty} dx \exp\left[i(k_0-k)x - \frac{x^2}{2b^2}\right] \tag{57}$$

$$= \tag{58}$$

(c)

$$\frac{1}{e} = e^{-1} \tag{59}$$

(d)

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[i\left(kx - \frac{\hbar k^2}{2m}t\right)\right]\tilde{\psi}(k) \tag{60}$$

(e)

4