Homework 6 solutions

Question 1

Proof.

$$\Theta_{kl} = \sum_{\nu=1}^{N} m_\nu (r_\nu'^2 \delta_{kl} - x'_{\nu k} x'_{\nu l}) = \sum_{\nu=1}^{N} m_\nu (r_\nu'^2 \delta_{lk} - x'_{\nu l} x'_{\nu k}) = \Theta_{lk} \tag{1}$$

□

Question 2 A homogeneous ellipsoid is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2}$$

with mass density ρ

$$\Theta_{kl} = \int dV \rho (r^2 \delta_{kl} - x_k x_l) \tag{3}$$

therefore

$$\Theta_{xx} = \int dV \rho (y^2 + z^2) \tag{4}$$

$$\Theta_{yy} = \int dV \rho (x^2 + z^2) \tag{5}$$

$$\Theta_{zz} = \int dV \rho (x^2 + y^2) \tag{6}$$

In spherical coordinates the ellipsoid can be written as

$$\frac{r^2 C_\vartheta^2 S_\varphi^2}{a^2} + \frac{r^2 S_\vartheta^2 S_\varphi^2}{b^2} + \frac{r^2 C_\varphi^2}{c^2} = 1 \tag{7}$$

define

$$x' = ax = ar^2 C_\vartheta S_\varphi \tag{8}$$

$$y' = by = br^2 S_\vartheta S_\varphi \tag{9}$$

$$z' = cz = cr^2 C_\varphi \tag{10}$$

therefore the infinitesimal volume element dV is to be multiplied with a coefficient abc resulting $dV' = abcdV = abcr^2 S_\vartheta dr d\varphi d\vartheta$. Then integrals from equation 4, 5 and 6 are to calculated.

$$\Theta_{xx} = 8\rho abc \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr (r^2 S_\vartheta^2 S_\varphi^2 + r^2 C_\varphi^2) r^2 S_\vartheta \tag{11}$$

$$= 8\rho abc \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr (r^4 S_\vartheta^3 S_\varphi^2 + r^4 S_\vartheta C_\varphi^2) \tag{12}$$

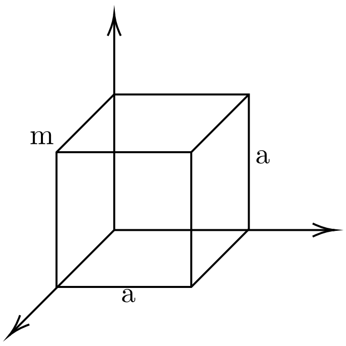
$$= \frac{8\rho abc}{5} \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\frac{\pi}{2}} d\varphi (S_\vartheta^3 S_\varphi^2 + S_\vartheta C_\varphi^2) \tag{13}$$

$$= \lightning \tag{14}$$

Question 3

(a)

$$\Theta_{kl} = \sum_{\nu=1}^{N} m_\nu (r_\nu'^2 \delta_{kl} - x'_{\nu k} x'_{\nu l}) \tag{15}$$



We have 8 particles in total

$$\begin{aligned}
&P_1(0,0,0)\\
&P_2(a,0,0)\\
&P_3(0,a,0)\\
&P_4(0,0,a)\\
&P_5(a,a,0)\\
&P_6(a,0,a)\\
&P_7(0,a,a)\\
&P_8(a,a,a)
\end{aligned}$$

the inertia tensor is therefore given by

$$\begin{aligned}
\Theta_{xx} &= m \sum_{\nu=1}^{N} (r_\nu'^2 - x^2) && (16)\\
&= m(0 + 0 + a^2 + a^2 + a^2 + a^2 + 2a^2 + 2a^2) && (17)\\
&= 8a^2 m = \Theta_{yy} = \Theta_{zz} && (18)\\
\Theta_{xy} &= m \sum_{\nu=1}^{N} (-xy) && (19)\\
&= m(-0 - 0 - 0 - 0 - a^2 - 0 - 0 - a^2) && (20)\\
&= -2a^2 m = \Theta_{xz} = \Theta_{zy} && (21)
\end{aligned}$$

$$\overleftrightarrow{\Theta} = a^2 m \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix} \tag{22}$$

(b) Consider a situation that four particles in the top layer have a mass of m, while particles in the bottom layer have $2m$.

$$
\begin{aligned}
&P_1(0,0,0), 2m\\
&P_2(a,0,0), 2m\\
&P_3(0,a,0), 2m\\
&P_4(0,0,a), m\\
&P_5(a,a,0), 2m\\
&P_6(a,0,a), m\\
&P_7(0,a,a), m\\
&P_8(a,a,a), m
\end{aligned}
$$

the inertia tensor is therefore given by

$$
\begin{aligned}
\Theta_{xx} &= m\sum_{\nu=1}^{N}(r_\nu'^2 - x^2) && (23)\\
&= 2m(0+0+a^2+a^2) + m(a^2+a^2+2a^2+2a^2) && (24)\\
&= 10a^2m = \Theta_{yy} && (25)\\
\Theta_{zz} &= m\sum_{\nu=1}^{N}(r_\nu'^2 - z^2) && (26)\\
&= 2m(0+a^2+a^2+2a^2) + m(0+a^2+a^2+2a^2) && (27)\\
&= 12a^2m && (28)\\
\Theta_{xy} &= m\sum_{\nu=1}^{N}(-xy) && (29)\\
&= 2m(-0-0-0a^2) + m(-0-0-0-a^2) && (30)\\
&= -3a^2m && (31)\\
\Theta_{xz} &= m\sum_{\nu=1}^{N}(-xz) && (32)\\
&= 2m(-0-0-0-0) + m(-0-a^2-0-a^2) && (33)\\
&= -2a^2m = \Theta_{yz} && (34)
\end{aligned}
$$

$$
\overleftrightarrow{\Theta} = a^2m\begin{pmatrix} 10 & -3 & -2\\ -3 & 10 & -2\\ -2 & -2 & 12 \end{pmatrix} \tag{35}
$$