Homework 4 solutions

Question 1

(a) Conservation of momentum in *lab frame*

$$p_1 + p_2 = p_1' + p_2' = P_l \tag{1}$$

Conservation of energy in *lab frame*

$$\frac{1}{2}p_1 v_1 + \frac{1}{2}p_2 v_2 = \frac{1}{2}p_1' v_1' + \frac{1}{2}p_2' v_2' + Q = E \tag{2}$$

$$\frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 = \frac{1}{2m_1}p_1'^2 + \frac{1}{2m_2}p_2'^2 + Q = E \tag{3}$$

Conservation of momentum in *zero momentum frame*

$$p_a + p_b = p_a' + p_b' = 0 \tag{4}$$
$$p_a = -p_b \tag{5}$$
$$p_a' = -p_b' \tag{6}$$

Conservation of energy in *zero momentum frame*

$$\frac{1}{2}p_a v_a + \frac{1}{2}p_b v_b = \frac{1}{2}p_a' v_a' + \frac{1}{2}p_b' v_b' + Q = E \tag{7}$$

Position of two particles in *lab frame* are

$$r_1, r_2 \tag{8}$$

Position of two particles in *zero momentum frame* are

$$r_a = r_1 - r_s \tag{9}$$
$$r_b = r_2 - r_s \tag{10}$$

where r_s is the position of center of mass. It is given by

$$r_s = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \tag{11}$$

therefore we get

$$r_a = r_1 - \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{(m_1 + m_2)r_1 - m_1 r_1 - m_2 r_2}{m_1 + m_2} = \frac{m_2(r_1 - r_2)}{m_1 + m_2} \tag{12}$$
$$r_b = r_2 - \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{(m_1 + m_2)r_2 - m_1 r_1 - m_2 r_2}{m_1 + m_2} = \frac{m_1(r_2 - r_1)}{m_1 + m_2} \tag{13}$$

similarly, the velocity of center of mass is

$$r_s = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \tag{14}$$

and the velocity of two particles refers to *zero momentum frame* are

$$v_a = \frac{m_2(v_1 - v_2)}{m_1 + m_2} \tag{15}$$
$$v_b = \frac{m_1(v_2 - v_1)}{m_1 + m_2} \tag{16}$$

then the momentum refers to *zero momentum frame* are

$$p_a = m_1 v_a = \frac{m_1 m_2 (v_1 - v_2)}{m_1 + m_2} \tag{17}$$

$$p_b = m_2 v_b = \frac{m_1 m_2 (v_2 - v_1)}{m_1 + m_2} \tag{18}$$

with reduced mass

$$\mu = \frac{m_1 + m_2}{m_1 m_2} \tag{19}$$

we get

$$p_a = \frac{v_1 - v_2}{\mu} \tag{20}$$

$$p_b = \frac{v_2 - v_1}{\mu} \tag{21}$$

now we take equation (7)

$$\frac{1}{2} p_a v_a + \frac{1}{2} p_b v_b = \frac{1}{2} p'_a v'_a + \frac{1}{2} p'_b v'_b + Q = E \tag{22}$$

$$\frac{p_a^2}{m_1} + \frac{p_b^2}{m_2} = \frac{p_a'^2}{m_1} + \frac{p_b'^2}{m_2} + 2Q \tag{23}$$

$$\frac{1}{\mu} p_a^2 = \frac{1}{\mu} p_a'^2 + 2Q \tag{24}$$

$$\frac{1}{\mu} p_a'^2 = \frac{1}{\mu} p_a^2 - 2Q \tag{25}$$

$$p_a'^2 = p_a^2 - 2\mu Q = \frac{v_1^2 - 2 v_1 v_2 + v_2^2}{\mu^2} - 2\mu Q \tag{26}$$