1 Homework 1 solutions

Question 1

(a)

$$\hat{q}_k = \frac{1}{\left|\frac{\partial \vec{r}}{\partial q_k}\right|} \frac{\partial \vec{r}}{\partial q_k} \tag{1}$$

The relation between two curvilinear coordinates and Cartesian coordinates are given by

$$\text{cylindrical coordinates } (\rho, \varphi, z)$$
$$x = \rho \cos(\varphi)$$
$$y = \rho \sin(\varphi)$$
$$z = z$$

$$\text{spherical coordinates } (r, \vartheta, \varphi)$$
$$x = r \sin(\vartheta) \cos(\varphi)$$
$$y = r \sin(\vartheta) \sin(\varphi)$$
$$z = r \cos(\vartheta)$$

We do a transformation using equation (1) to spherical coordinates:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin(\vartheta) \cos(\varphi) \\ r \sin(\vartheta) \sin(\varphi) \\ r \cos(\vartheta) \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \vartheta} = \begin{bmatrix} r \cos(\vartheta) \cos(\varphi) \\ r \cos(\vartheta) \sin(\varphi) \\ -r \sin(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -r \sin(\vartheta) \sin(\varphi) \\ r \sin(\vartheta) \cos(\varphi) \\ 0 \end{bmatrix}$$

with

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2(\vartheta) \cos^2(\varphi) + \sin^2(\vartheta) \sin^2(\varphi) + \cos^2(\vartheta)} = 1$$

$$\left|\frac{\partial \vec{r}}{\partial \vartheta}\right| = \sqrt{r \cos^2(\vartheta) \cos^2(\varphi) + r^2 \cos^2(\vartheta) \sin^2(\varphi) + r^2 \sin^2(\vartheta)} = r$$

$$\left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \sqrt{r^2 \sin^2(\vartheta) \sin^2(\varphi) + r^2 \sin^2(\vartheta) \cos^2(\varphi)} = r \sin(\vartheta)$$

we get the unit vectors:

$$\hat{r} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \hat{\vartheta} = \begin{bmatrix} \cos(\vartheta) \cos(\varphi) \\ \cos(\vartheta) \sin(\varphi) \\ -\sin(\vartheta) \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \tag{2}$$

with transformation matrix:

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \\ \cos(\vartheta) \cos(\varphi) & \cos(\vartheta) \sin(\varphi) & -\sin(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{3}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \cos(\vartheta) \cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \tag{4}$$

We now verify the orthogonality of our new unit vectors using cross product:

$$\hat{r} \times \hat{\vartheta} = \begin{bmatrix} \sin(\vartheta)\sin(\varphi)(-\sin(\vartheta)) - \cos(\vartheta)\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\cos(\vartheta)\cos(\varphi) - \sin(\vartheta)\cos(\varphi)(-\sin(\vartheta)) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} -(\sin^2(\vartheta)\sin(\varphi) + \cos^2(\vartheta)\sin(\varphi)) \\ \cos^2(\vartheta)\cos(\varphi) + \sin^2(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} = \hat{\varphi}$$

correct, but I wanna do a double check:

$$\hat{\vartheta} \times \hat{\varphi} = \begin{bmatrix} \cos(\vartheta)\sin(\varphi)0 - (-\sin(\vartheta))\cos(\varphi) \\ (-\sin(\vartheta))(-\sin(\varphi)) - \cos(\vartheta)\cos(\varphi)0 \\ \cos(\vartheta)\cos(\varphi)\cos(\varphi) - \cos(\vartheta)\sin(\varphi)(-\sin(\varphi)) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} = \hat{r}$$

Now we do the same transformation to cylindrical coordinates:

$$\vec{r} = \begin{bmatrix} \rho\cos(\varphi) \\ \rho\sin(\varphi) \\ z \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix} , \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix} , \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with

$$\left| \frac{\partial \vec{r}}{\partial \rho} \right| = \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1$$

$$\left| \frac{\partial \vec{r}}{\partial \varphi} \right| = \sqrt{\rho^2\sin^2(\varphi) + \rho^2\cos^2(\varphi)} = \rho$$

$$\left| \frac{\partial \vec{r}}{\partial z} \right| = 1$$

therefore:

$$\hat{\rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix} , \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} , \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

with transformation matrix:

$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{6}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \tag{7}$$

(b) The position vector $\vec{r}$ in combination of basis vectors from cartesian coordinates and factors of curvilinear coordinates is given by

$$(sperical) \ \vec{r} = r\sin(\vartheta)\cos(\varphi)\hat{x} + r\sin(\vartheta)\sin(\varphi)\hat{y} + r\cos(\vartheta)\hat{z} \tag{8}$$

$$(cylindrical) \ \vec{r} = \rho\cos(\varphi)\hat{x} + \rho\sin(\varphi)\hat{y} + z\hat{z} \tag{9}$$

our goal is to obtain an expression of position vector, such that it is explicitly expressed by curvilinear coordinates, like in the following equation:

$$\vec{r} = \vec{r}(q_1, q_2, q_3) = \sum_{k=1}^{3} c_k(q_1, q_2, q_3)\hat{q}_k(q_1, q_2, q_3)$$

Now we replace unit vectors of cartesian coordinates with which from curvilinear coordinates using transformation matrices.

For spherical coordinates, we use equation (4):

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}$$

we insert position vector form (8),

$$\begin{bmatrix} r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

$$= \begin{bmatrix} r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta) \end{bmatrix} \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} r\sin^2(\vartheta)\cos^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi) & -r\sin(\vartheta)\cos(\varphi)\sin(\varphi) \\ r\sin^2(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\sin(\varphi)\cos(\varphi) \\ r\cos^2(\vartheta) & -r\cos(\vartheta)\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} r\sin^2(\vartheta)\cos^2(\varphi)\hat{r} + r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi)\hat{\vartheta} - r\sin(\vartheta)\cos(\varphi)\sin(\varphi)\hat{\varphi} \\ r\sin^2(\vartheta)\sin^2(\varphi)\hat{r} + r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi)\hat{\vartheta} + r\sin(\vartheta)\sin(\varphi)\cos(\varphi)\hat{\varphi} \\ r\cos^2(\vartheta)\hat{r} - r\cos(\vartheta)\sin(\vartheta)\hat{\vartheta} \end{bmatrix}$$

$$= (r\sin^2(\vartheta)\cos^2(\varphi) + r\sin^2(\vartheta)\sin^2(\varphi) + r\cos^2(\vartheta))\hat{r}$$

$$+ (r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi) + r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi) - r\cos(\vartheta)\sin(\vartheta))\hat{\vartheta}$$

$$+ (-r\sin(\vartheta)\cos(\varphi)\sin(\varphi) + r\sin(\vartheta)\sin(\varphi))\cos(\varphi)\hat{\varphi}$$

$$= r\hat{r}$$

And for cylindrical coordinates:

$$
\begin{bmatrix} \rho\cos(\varphi) & & \\ & \rho\sin(\varphi) & \\ & & z \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \rho\cos(\varphi) & & \\ & \rho\sin(\varphi) & \\ & & z \end{bmatrix}\begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix}
$$

$$
= \begin{bmatrix} \rho\cos^2(\varphi) & -\rho\sin(\varphi)\cos(\varphi) & 0 \\ \rho\sin^2(\varphi) & \rho\sin(\varphi)\cos(\varphi) & 0 \\ 0 & 0 & z \end{bmatrix}\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix}
$$

$$
= \begin{bmatrix} \rho\cos^2(\varphi)\hat{\rho} - \rho\sin(\varphi)\cos(\varphi)\hat{\varphi} \\ \rho\sin^2(\varphi)\hat{\rho} + \rho\sin(\varphi)\cos(\varphi)\hat{\varphi} \\ z\hat{z} \end{bmatrix}
$$

$$
=(\rho\cos^2(\varphi) + \rho\sin^2(\varphi))\hat{\rho} + (-\rho\sin(\varphi)\cos(\varphi) + \rho\sin(\varphi)\cos(\varphi))\hat{\varphi} + z\hat{z}
$$

$$
=\rho\hat{\rho} + z\hat{z}
$$

(c) The (square of) line element is given by

$$dl := |d\vec{r}|^2$$

with $d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$

In spherical coordinates, it is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \vartheta} d\vartheta + \frac{\partial \vec{r}}{\partial \varphi} d\varphi$$

$$= \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} dr + \begin{bmatrix} r\cos(\vartheta)\cos(\varphi) \\ r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta) \end{bmatrix} d\vartheta + \begin{bmatrix} -r\sin(\vartheta)\sin(\varphi) \\ r\sin(\vartheta)\cos(\varphi) \\ 0 \end{bmatrix} d\varphi$$

therefore,

$$|d\vec{r}|^2 = \begin{bmatrix} \sin(\vartheta)^2\cos(\varphi)^2 \\ \sin(\vartheta)^2\sin(\varphi)^2 \\ \cos(\vartheta)^2 \end{bmatrix} dr^2 + \begin{bmatrix} r^2\cos(\vartheta)^2\cos(\varphi)^2 \\ r^2\cos(\vartheta)^2\sin(\varphi)^2 \\ r^2\sin(\vartheta)^2 \end{bmatrix} d\vartheta^2 + \begin{bmatrix} r^2\sin(\vartheta)^2\sin(\varphi)^2 \\ r^2\sin(\vartheta)^2\cos(\varphi)^2 \\ 0 \end{bmatrix} d\varphi^2$$

$$= dr^2 + r^2 d\vartheta^2 + r^2\sin^2(\theta)d\varphi^2$$

In cylindrical coordinates, it is

$$d\vec{r} = \frac{\partial \vec{\rho}}{\partial \rho} dr + \frac{\partial \vec{r}}{\partial \varphi} d\varphi + \frac{\partial \vec{r}}{\partial z} dz$$

$$= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix} dr + \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix} d\vartheta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\varphi$$

therefore

$$|d\vec{r}|^2 = \begin{bmatrix} \cos^2(\varphi) \\ \sin^2(\varphi) \\ 0 \end{bmatrix} d\rho^2 + \begin{bmatrix} \rho^2\sin^2(\varphi) \\ \rho^2\cos^2(\varphi) \\ 0 \end{bmatrix} d\varphi^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dz^2$$

$$= d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

(d) The position vector in spherical coordinates is

$$\vec{r} = r\hat{r}$$

the speed $\vec{v}(t)$ is the derivative of position vector $\vec{r}$ with respect to time t

$$\dot{\vec{r}} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}(t)\begin{bmatrix} \sin(\vartheta(t))\cos(\varphi(t)) \\ \sin(\vartheta(t))\sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix} + r(t)\underbrace{\frac{d}{dt}\begin{bmatrix} \sin(\vartheta(t))\cos(\varphi(t)) \\ \sin(\vartheta(t))\sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix}}_{🀄}$$

with

$$🀄 = \begin{bmatrix} \dot{\vartheta}(t)\cos(\vartheta(t))\cos(\varphi(t)) - \sin(\vartheta(t))\dot{\varphi}(t)\sin(\varphi(t)) \\ \\ \\ \end{bmatrix}$$