

1 Homework 1 solutions

Question 1

(a)

$$\hat{q}_k = \frac{1}{\left| \frac{\partial \vec{r}}{\partial q_k} \right|} \frac{\partial \vec{r}}{\partial q_k} \quad (1)$$

The relation between two curvilinear coordinates and Cartesian coordinates are given by

cylindrical coordinates (ρ, φ, z)

$$x = \rho \cos(\varphi)$$

$$y = \rho \sin(\varphi)$$

$$z = z$$

spherical coordinates (r, ϑ, φ)

$$x = r \sin(\vartheta) \cos(\varphi)$$

$$y = r \sin(\vartheta) \sin(\varphi)$$

$$z = r \cos(\vartheta)$$

We do a transformation using equation (1) to spherical coordinates:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin(\vartheta) \cos(\varphi) \\ r \sin(\vartheta) \sin(\varphi) \\ r \cos(\vartheta) \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \vartheta} = \begin{bmatrix} r \cos(\vartheta) \cos(\varphi) \\ r \cos(\vartheta) \sin(\varphi) \\ -r \sin(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -r \sin(\vartheta) \sin(\varphi) \\ r \sin(\vartheta) \cos(\varphi) \\ 0 \end{bmatrix}$$

with

$$\begin{aligned} \left| \frac{\partial \vec{r}}{\partial r} \right| &= \sqrt{\sin^2(\vartheta) \cos^2(\varphi) + \sin^2(\vartheta) \sin^2(\varphi) + \cos^2(\vartheta)} = 1 \\ \left| \frac{\partial \vec{r}}{\partial \vartheta} \right| &= \sqrt{r^2 \cos^2(\vartheta) \cos^2(\varphi) + r^2 \cos^2(\vartheta) \sin^2(\varphi) + r^2 \sin^2(\vartheta)} = r \\ \left| \frac{\partial \vec{r}}{\partial \varphi} \right| &= \sqrt{r^2 \sin^2(\vartheta) \sin^2(\varphi) + r^2 \sin^2(\vartheta) \cos^2(\varphi)} = r \sin(\vartheta) \end{aligned}$$

we get the unit vectors:

$$\hat{r} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \hat{\vartheta} = \begin{bmatrix} \cos(\vartheta) \cos(\varphi) \\ \cos(\vartheta) \sin(\varphi) \\ -\sin(\vartheta) \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \quad (2)$$

with transformation matrix:

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \\ \cos(\vartheta) \cos(\varphi) & \cos(\vartheta) \sin(\varphi) & -\sin(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (3)$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \cos(\vartheta) \cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \quad (4)$$

We now verify the orthogonality of our new unit vectors using cross product:

$$\begin{aligned}\hat{r} \times \hat{\vartheta} &= \begin{bmatrix} \sin(\vartheta) \sin(\varphi)(-\sin(\vartheta)) - \cos(\vartheta) \cos(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \cos(\vartheta) \cos(\varphi) - \sin(\vartheta) \cos(\varphi)(-\sin(\vartheta)) \\ \sin(\vartheta) \cos(\varphi) \cos(\vartheta) \sin(\varphi) - \sin(\vartheta) \sin(\varphi) \cos(\vartheta) \cos(\varphi) \end{bmatrix} \\ &= \begin{bmatrix} -(\sin^2(\vartheta) \sin(\varphi) + \cos^2(\vartheta) \sin(\varphi)) \\ \cos^2(\vartheta) \cos(\varphi) + \sin^2(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \cos(\varphi) \cos(\vartheta) \sin(\varphi) - \sin(\vartheta) \sin(\varphi) \cos(\vartheta) \cos(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} = \hat{\varphi}\end{aligned}$$

correct, but I wanna do a double check:

$$\hat{\vartheta} \times \hat{\varphi} = \begin{bmatrix} \cos(\vartheta) \sin(\varphi)0 - (-\sin(\vartheta)) \cos(\varphi) \\ (-\sin(\vartheta))(-\sin(\varphi)) - \cos(\vartheta) \cos(\varphi)0 \\ \cos(\vartheta) \cos(\varphi) \cos(\varphi) - \cos(\vartheta) \sin(\varphi)(-\sin(\varphi)) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} = \hat{r}$$

Now we do the same transformation to cylindrical coordinates:

$$\begin{aligned}\vec{r} &= \begin{bmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ z \end{bmatrix} \\ \frac{\partial \vec{r}}{\partial \rho} &= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -\rho \sin(\varphi) \\ \rho \cos(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

with

$$\begin{aligned}\left| \frac{\partial \vec{r}}{\partial \rho} \right| &= \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1 \\ \left| \frac{\partial \vec{r}}{\partial \varphi} \right| &= \sqrt{\rho^2 \sin^2(\varphi) + \rho^2 \cos^2(\varphi)} = \rho \\ \left| \frac{\partial \vec{r}}{\partial z} \right| &= 1\end{aligned}$$

therefore:

$$\hat{\rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (5)$$

with transformation matrix:

$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (6)$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \quad (7)$$

- (b) The position vector \vec{r} in combination of basis vectors from cartesian coordinates and factors of curvilinear coordinates is given by

$$(spherical) \quad \vec{r} = r \sin(\vartheta) \cos(\varphi) \hat{x} + r \sin(\vartheta) \sin(\varphi) \hat{y} + r \cos(\vartheta) \hat{z} \quad (8)$$

$$(cylindrical) \quad \vec{r} = \rho \cos(\varphi) \hat{x} + \rho \sin(\varphi) \hat{y} + z \hat{z} \quad (9)$$

our goal is to obtain an expression of position vector, such that it is explicitly expressed by curvilinear coordinates, like in the following equation:

$$\vec{r} = \vec{r}(q_1, q_2, q_3) = \sum_{k=1}^3 c_k(q_1, q_2, q_3) \hat{q}_k(q_1, q_2, q_3)$$

Now we replace unit vectors of cartesian coordinates with which from curvilinear coordinates using transformation matrices.

For spherical coordinates, we use equation (4):

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \cos(\vartheta) \cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}$$

we insert position vector form (8),

$$\begin{aligned} & \begin{bmatrix} r \sin(\vartheta) \cos(\varphi) & & \\ & r \sin(\vartheta) \sin(\varphi) & \\ & & r \cos(\vartheta) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\ &= \begin{bmatrix} r \sin(\vartheta) \cos(\varphi) & & \\ & r \sin(\vartheta) \sin(\varphi) & \\ & & r \cos(\vartheta) \end{bmatrix} \begin{bmatrix} \sin(\vartheta) \cos(\varphi) & \cos(\vartheta) \cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta) \sin(\varphi) & \cos(\vartheta) \sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \\ &= \begin{bmatrix} r \sin^2(\vartheta) \cos^2(\varphi) & r \sin(\vartheta) \cos(\vartheta) \cos^2(\varphi) & -r \sin(\vartheta) \cos(\varphi) \sin(\varphi) \\ r \sin^2(\vartheta) \sin^2(\varphi) & r \sin(\vartheta) \cos(\vartheta) \sin^2(\varphi) & r \sin(\vartheta) \sin(\varphi) \cos(\varphi) \\ r \cos^2(\vartheta) & -r \cos(\vartheta) \sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \\ &= \begin{bmatrix} r \sin^2(\vartheta) \cos^2(\varphi) \hat{r} + r \sin(\vartheta) \cos(\vartheta) \cos^2(\varphi) \hat{\vartheta} - r \sin(\vartheta) \cos(\varphi) \sin(\varphi) \hat{\varphi} \\ r \sin^2(\vartheta) \sin^2(\varphi) \hat{r} + r \sin(\vartheta) \cos(\vartheta) \sin^2(\varphi) \hat{\vartheta} + r \sin(\vartheta) \sin(\varphi) \cos(\varphi) \hat{\varphi} \\ r \cos^2(\vartheta) \hat{r} - r \cos(\vartheta) \sin(\vartheta) \hat{\vartheta} \end{bmatrix} \\ &= (r \sin^2(\vartheta) \cos^2(\varphi) + r \sin^2(\vartheta) \sin^2(\varphi) + r \cos^2(\vartheta)) \hat{r} \\ &\quad + (r \sin(\vartheta) \cos(\vartheta) \cos^2(\varphi) + r \sin(\vartheta) \cos(\vartheta) \sin^2(\varphi) - r \cos(\vartheta) \sin(\vartheta)) \hat{\vartheta} \\ &\quad + (-r \sin(\vartheta) \cos(\varphi) \sin(\varphi) + r \sin(\vartheta) \sin(\varphi) \cos(\varphi)) \hat{\varphi} \\ &= r \hat{r} \end{aligned}$$

And for cylindrical coordinates:

$$\begin{aligned}
\begin{bmatrix} \rho \cos(\varphi) & & \\ & \rho \sin(\varphi) & \\ & & z \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} &= \begin{bmatrix} \rho \cos(\varphi) & & \\ & \rho \sin(\varphi) & \\ & & z \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} \rho \cos^2(\varphi) & -\rho \sin(\varphi) \cos(\varphi) & 0 \\ \rho \sin^2(\varphi) & \rho \sin(\varphi) \cos(\varphi) & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} \rho \cos^2(\varphi) \hat{\rho} - \rho \sin(\varphi) \cos(\varphi) \hat{\varphi} \\ \rho \sin^2(\varphi) \hat{\rho} + \rho \sin(\varphi) \cos(\varphi) \hat{\varphi} \\ z \hat{z} \end{bmatrix} \\
&= (\rho \cos^2(\varphi) + \rho \sin^2(\varphi)) \hat{\rho} + (-\rho \sin(\varphi) \cos(\varphi) + \rho \sin(\varphi) \cos(\varphi)) \hat{\varphi} + z \hat{z} \\
&= \rho \hat{\rho} + z \hat{z}
\end{aligned}$$

(c) The (square of) line element is given by

$$dl := |d\vec{r}|^2$$

with $d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$

In spherical coordinates, it is

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \vartheta} d\vartheta + \frac{\partial \vec{r}}{\partial \varphi} d\varphi \\ &= \begin{bmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} dr + \begin{bmatrix} r \cos(\vartheta) \cos(\varphi) \\ r \cos(\vartheta) \sin(\varphi) \\ -r \sin(\vartheta) \end{bmatrix} d\vartheta + \begin{bmatrix} -r \sin(\vartheta) \sin(\varphi) \\ r \sin(\vartheta) \cos(\varphi) \\ 0 \end{bmatrix} d\varphi \end{aligned}$$

therefore,

$$\begin{aligned} |d\vec{r}|^2 &= \begin{bmatrix} \sin(\vartheta)^2 \cos(\varphi)^2 \\ \sin(\vartheta)^2 \sin(\varphi)^2 \\ \cos(\vartheta)^2 \end{bmatrix} dr^2 + \begin{bmatrix} r^2 \cos(\vartheta)^2 \cos(\varphi)^2 \\ r^2 \cos(\vartheta)^2 \sin(\varphi)^2 \\ r^2 \sin(\vartheta)^2 \end{bmatrix} d\vartheta^2 + \begin{bmatrix} r^2 \sin(\vartheta)^2 \sin(\varphi)^2 \\ r^2 \sin(\vartheta)^2 \cos(\varphi)^2 \\ 0 \end{bmatrix} d\varphi^2 \\ &= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2(\vartheta) d\varphi^2 \end{aligned}$$

In cylindrical coordinates, it is

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \varphi} d\varphi + \frac{\partial \vec{r}}{\partial z} dz \\ &= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix} d\rho + \begin{bmatrix} -\rho \sin(\varphi) \\ \rho \cos(\varphi) \\ 0 \end{bmatrix} d\varphi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dz \end{aligned}$$

therefore

$$\begin{aligned} |d\vec{r}|^2 &= \begin{bmatrix} \cos^2(\varphi) \\ \sin^2(\varphi) \\ 0 \end{bmatrix} d\rho^2 + \begin{bmatrix} \rho^2 \sin^2(\varphi) \\ \rho^2 \cos^2(\varphi) \\ 0 \end{bmatrix} d\varphi^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dz^2 \\ &= d\rho^2 + \rho^2 d\varphi^2 + dz^2 \end{aligned}$$

(d) The position vector in spherical coordinates is

$$\vec{r} = r\hat{r}$$

the speed $\vec{v}(t)$ is the derivative of position vector \vec{r} with respect to time t

$$\begin{aligned}\vec{v}(t) &= \dot{\vec{r}} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \\ &= \dot{r}(t) \begin{bmatrix} \sin(\vartheta(t)) \cos(\varphi(t)) \\ \sin(\vartheta(t)) \sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix} + r(t) \underbrace{\frac{d}{dt} \begin{bmatrix} \sin(\vartheta(t)) \cos(\varphi(t)) \\ \sin(\vartheta(t)) \sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix}}_{\boxed{\text{A}}}\end{aligned}$$

with

$$\boxed{\text{A}} = \begin{bmatrix} \dot{\vartheta}(t) \cos(\vartheta(t)) \cos(\varphi(t)) - \sin(\vartheta(t)) \dot{\varphi}(t) \sin(\varphi(t)) \\ \dot{\vartheta}(t) \cos(\vartheta(t)) \sin(\varphi(t)) + \sin(\vartheta(t)) \dot{\varphi}(t) \cos(\varphi(t)) \\ -\dot{\vartheta}(t) \sin(\vartheta(t)) \end{bmatrix}$$

therefore

$$\vec{v}(t) = \dot{r}(t) \underbrace{\begin{bmatrix} \sin(\vartheta(t)) \cos(\varphi(t)) \\ \sin(\vartheta(t)) \sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix}}_{\hat{r}(t)} + r(t) \underbrace{\begin{bmatrix} \dot{\vartheta}(t) \cos(\vartheta(t)) \cos(\varphi(t)) - \sin(\vartheta(t)) \dot{\varphi}(t) \sin(\varphi(t)) \\ \dot{\vartheta}(t) \cos(\vartheta(t)) \sin(\varphi(t)) + \sin(\vartheta(t)) \dot{\varphi}(t) \cos(\varphi(t)) \\ -\dot{\vartheta}(t) \sin(\vartheta(t)) \end{bmatrix}}_{\dot{\hat{r}}(t)} \quad (10)$$

the acceleration a is given by

$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t) = \ddot{r}(t)\hat{r}(t) + \dot{r}(t)\dot{\hat{r}}(t) + \dot{r}(t)\dot{\hat{r}}(t) + r(t)\ddot{\hat{r}}(t) \quad (11)$$

with

$$\begin{aligned}\ddot{\vec{r}}(t) &= \begin{bmatrix} \ddot{\vartheta}C_{\vartheta}C_{\varphi} + \dot{\vartheta}(\dot{\vartheta}S_{\vartheta}C_{\varphi} - \dot{\varphi}C_{\vartheta}S_{\varphi}) - \ddot{\varphi}S_{\vartheta}S_{\varphi} - \dot{\varphi}(\dot{\vartheta}C_{\vartheta}S_{\varphi} + \dot{\varphi}S_{\vartheta}C_{\varphi}) \\ \ddot{\vartheta}C_{\vartheta}S_{\varphi} + \dot{\vartheta}(-\dot{\vartheta}S_{\vartheta}S_{\varphi} + \dot{\varphi}C_{\vartheta}C_{\varphi}) + \ddot{\varphi}S_{\vartheta}C_{\varphi} + \dot{\varphi}(\dot{\vartheta}C_{\vartheta}C_{\varphi} - \dot{\varphi}S_{\vartheta}S_{\varphi}) \\ -\dot{\vartheta}S_{\vartheta} - (\dot{\vartheta})^2C_{\vartheta} \end{bmatrix} \\ \dot{\hat{r}}(t) &= \begin{bmatrix} \dot{\vartheta}(t) \cos(\vartheta(t)) \cos(\varphi(t)) - \sin(\vartheta(t)) \dot{\varphi}(t) \sin(\varphi(t)) \\ \dot{\vartheta}(t) \cos(\vartheta(t)) \sin(\varphi(t)) + \sin(\vartheta(t)) \dot{\varphi}(t) \cos(\varphi(t)) \\ -\dot{\vartheta}(t) \sin(\vartheta(t)) \end{bmatrix} \\ \hat{r}(t) &= \begin{bmatrix} \sin(\vartheta(t)) \cos(\varphi(t)) \\ \sin(\vartheta(t)) \sin(\varphi(t)) \\ \cos(\vartheta(t)) \end{bmatrix}\end{aligned}$$

