Homework 2 solutions

Question 1

$$\left\{ \begin{array}{l} x' = \frac{1}{\sqrt{1-\beta^2}}(x - \beta ct) \\ ct' = \frac{1}{\sqrt{1-\beta^2}}(ct - \beta t) \end{array} \right. \tag{1}$$

Taylor-expansion

$$x' = x - ct\beta \tag{2}$$

$$ct' = x - t\beta \tag{3}$$

Question 2

Prove that every orthogonal transformation $A \in \mathbb{R}^2$ with $det(A) = 1$ can be parameterized with ϑ, so that

$$A(\vartheta) = \begin{bmatrix} C_\vartheta & S_\vartheta \\ -S_\vartheta & C_\vartheta \end{bmatrix} \tag{4}$$

Proof. We have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5}$$

with

$$A^T A = \mathbb{1} \tag{6}$$

and

$$det(A) = 1 \tag{7}$$

so that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

therefore

$$\begin{cases} a^2 + c^2 = b^2 + d^2 = 1 \\ ad - cb = 1 \\ ab + cd = 0 \end{cases} \tag{9}$$

substitute

$$a = S_\vartheta \quad b = S_\varphi \quad c = C_\vartheta \quad d = C_\varphi \tag{10}$$

we get

$$S_\vartheta^2 + C_\vartheta^2 = S_\varphi^2 + C_\varphi^2 = 1 \tag{11}$$

$$S_\vartheta C_\varphi - S_\varphi C_\vartheta = S_{\vartheta-\varphi} = 1 \tag{12}$$

$$S_\vartheta S_\varphi + C_\vartheta C_\varphi = C_{\vartheta-\varphi} = 0 \tag{13}$$

therefore

$$\vartheta - \varphi = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \tag{14}$$

$$S_\varphi = C_\vartheta, \quad C_\varphi = -S_\vartheta \tag{15}$$

$$A = \begin{bmatrix} S_\vartheta & C_\vartheta \\ C_\vartheta & -S_\vartheta \end{bmatrix}$$

□

Question 3

$$\vec{F}_g = -mg\hat{z} \tag{16}$$

$$\vec{F}_R = -\alpha\vec{v}(t) \tag{17}$$

$$(\vec{F}_g + \vec{F}_R)_z = (\vec{F})_z = m\ddot{z} = -mg - \alpha\dot{z} \tag{18}$$

$$m\ddot{z} = -mg - \alpha\dot{z} \tag{19}$$

$$m\frac{d^2}{dt^2}z = -mg - \alpha\frac{d}{dt}z \tag{20}$$

Substitute $u = \dot{z}$

$$m\frac{du}{dt} = -mg - \alpha u \tag{21}$$

$$\frac{1}{g + \frac{\alpha}{m}u}du = -dt \tag{22}$$

$$\int \frac{1}{g + \frac{\alpha}{m}u}du = -\int dt \tag{23}$$

$$\frac{m}{\alpha}\ln(g + \frac{\alpha}{m}u) = -t + C_0 \tag{24}$$

with initial condition $\vec{v}(0) = \vec{v}_0$, we have $u(0) = \vec{v}_z$, therefore

$$C_0 = \frac{m}{\alpha}\ln(g + \frac{\alpha}{m}\vec{v}_z) \tag{25}$$

then we have

$$\frac{m}{\alpha}\ln(g + \frac{\alpha}{m}u) = -t + \frac{m}{\alpha}\ln(g + \frac{\alpha}{m}\vec{v}_z) \tag{26}$$

$$(g + \frac{\alpha}{m}u)^{\frac{m}{\alpha}} = e^{-t}(g + \frac{\alpha}{m}\vec{v}_z)^{\frac{m}{\alpha}} \tag{27}$$

$$g + \frac{\alpha}{m}\frac{dz}{dt} = e^{-\frac{\alpha}{m}t}(g + \frac{\alpha}{m}\vec{v}_z) \tag{28}$$

set $\frac{\alpha}{m} = k$

$$gdt + kdz = e^{-kt}(g + k\vec{v}_z)dt \tag{29}$$

$$\int kdz = \int e^{-kt}(g + k\vec{v}_z)dt - \int gdt \tag{30}$$

$$kz = -\frac{1}{k}e^{-kt}(g + k\vec{v}_z) - gt + C_1 \tag{31}$$

with initial condition $z(0) = 0$

$$C_1 = \frac{1}{k}(g + k\vec{v}_z) \tag{32}$$

therefore

$$z = \frac{1}{k^2}(g + k\vec{v}_z) - \frac{1}{k^2}e^{-kt}(g + k\vec{v}_z) - \frac{1}{k}gt \tag{33}$$

$$= \frac{1}{k^2}[(g + k\vec{v}_z)(1 - e^{-kt}) - kgt] \tag{34}$$

with $k = \frac{\alpha}{m}$