Homework 1 solutions

Question 1

An elliptic curve is given by

$$\begin{cases} x = aC_t \\ y = bS_t \\ z = 0 \end{cases} \quad \text{or} \quad \vec{r}(t) = \begin{pmatrix} aC_t \\ bS_t \\ 0 \end{pmatrix} \tag{1}$$

we are going to find the TNB-frame of this curve.

First we determine the infinitesimal arc length ds

$$ds = \Big|\frac{\vec{r}(t)}{dt}\Big| dt = \sqrt{(-aS_t)^2 + (bC_t)^2}dt = \sqrt{a^2+b^2}dt = v(t)dt \tag{2}$$

where $v(t) = \sqrt{a^2+b^2}$ is a constant.

According to Frenet–Serret formulas, the tangential unit vector is given by

$$\mathbf{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \begin{pmatrix} \frac{-aS_t}{\sqrt{a^2+b^2}} \\ \frac{bC_t}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix} \tag{3}$$

and the normal unit vector is

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}\frac{dt}{ds}}{|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}|} \tag{4}$$

where

$$\frac{d\mathbf{T}}{dt} = \begin{pmatrix} \frac{-aC_t}{\sqrt{a^2+b^2}} \\ \frac{-bS_t}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix} \tag{5}$$

therefore

$$\frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \begin{pmatrix} \frac{-aC_t}{a^2+b^2} \\ \frac{-bS_t}{a^2+b^2} \\ 0 \end{pmatrix}, \quad |\frac{d\mathbf{T}}{dt}\frac{dt}{ds}| = \sqrt{\frac{1}{a^2+b^2}} \tag{6}$$

$$\mathbf{N} = \begin{pmatrix} \frac{-aC_t}{\sqrt{a^2+b^2}} \\ \frac{-bS_t}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix} \tag{7}$$

the binormal unit vector is given by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{pmatrix} 0 \\ 0 \\ \frac{abS_{2t}}{a^2+b^2} \end{pmatrix} \tag{8}$$

The velocity and acceleration vector is given by

$$\vec{v}(t) = \begin{pmatrix} -aS_t \\ bC_t \\ 0 \end{pmatrix}, \quad \vec{a}(t) = \begin{pmatrix} -aC_t \\ -bS_t \\ 0 \end{pmatrix} = a(t)\mathbf{T} + \frac{v^2(t)}{R(t)}\mathbf{N} \tag{9}$$

where $a(t) = 0$ due to constant velocity
Therefore

$$\begin{pmatrix} -aC_t \\ -bS_t \\ 0 \end{pmatrix} = \frac{a^2 + b^2}{R} \begin{pmatrix} \frac{-aC_t}{\sqrt{a^2+b^2}} \\ \frac{-bS_t}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix} \tag{10}$$

$$R = \sqrt{a^2 + b^2} \tag{11}$$

Question 2

A helix is given by

$$\vec{r}(t) = \begin{pmatrix} \rho C_{\omega t} \\ \rho S_{\omega t} \\ \frac{1}{2}at^2 \end{pmatrix} \tag{12}$$

$$ds = \left|\frac{\vec{r}(t)}{dt}\right| dt = \sqrt{\rho^2\omega^2 + a^2t^2}dt \tag{13}$$

$$\mathbf{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \begin{pmatrix} \rho C_{\omega t} \\ \rho S_{\omega t} \\ \frac{1}{2}at^2 \end{pmatrix} \frac{1}{\sqrt{\rho^2\omega^2 + a^2t^2}} \tag{14}$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}\frac{dt}{ds}}{|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}|} \tag{15}$$

$$\frac{d\mathbf{T}}{dt} = \begin{pmatrix} \rho(\omega S_{\omega t}(\rho^2\omega^2 + a^2t^2) + a^2tC_{\omega t}) \\ \rho(\omega C_{\omega t}(\rho^2\omega^2 + a^2t^2\omega) - a^2tS_{\omega t}) \\ \frac{1}{2}a^3t^3 + a\rho^2t\omega^2 \end{pmatrix} \frac{1}{(\rho^2\omega^2 + a^2t^2)^{3/2}} \tag{16}$$

$$\frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \begin{pmatrix} \rho(\omega S_{\omega t}(\rho^2\omega^2 + a^2t^2) + a^2tC_{\omega t}) \\ \rho(\omega C_{\omega t}(\rho^2\omega^2 + a^2t^2\omega) - a^2tS_{\omega t}) \\ \frac{1}{2}a^3t^3 + a\rho^2t\omega^2 \end{pmatrix} \frac{1}{(\rho^2\omega^2 + a^2t^2)^2} \tag{17}$$

$$|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}| = \sqrt{\left(\frac{a^3t^3}{2} + a\rho^2t\omega^2\right)^2 + \rho^2(\omega C_{\omega t}(a^2t^2\omega + \rho^2\omega^2) - a^2tS_{\omega t})^2 + \rho^2(\omega S_{\omega t}(a^2t^2 + \rho^2\omega^2) + a^2tC_{\omega t})^2} \tag{18}$$

Question 3

(a) Prove the following identity

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{19}$$

Proof.

$$\varepsilon_{ijk}\varepsilon_{lmk} = \begin{cases} 1 & \text{for} \quad i = l, j = m \\ -1 & \text{for} \quad i = m, j = l \end{cases} \tag{20}$$

where $i \neq j \neq k$ and $l \neq m \neq k$

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \begin{cases} \delta_{ll}\delta_{mm} - \delta_{lm}\delta_{ml} = 1 - 0 = 1 & \text{for} \quad i = l, j = m \\ \delta_{ml}\delta_{lm} - \delta_{mm}\delta_{ll} = 0 - 1 = -1 & \text{for} \quad i = m, j = l \end{cases} \tag{21}$$

where $i \neq j$ and $l \neq m$

Consider the third case, when $i = j = l = m$, we have

$$\varepsilon_{ijk}\varepsilon_{lmk} = 0 \tag{22}$$

and

$$\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = 1 - 1 = 0 \tag{23}$$

□

(b) Prove the following identity

$$(\vec{a} \times \vec{b})_i = \varepsilon_{ijk} a_j b_k \tag{24}$$

Proof.

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_i \\ a_j \\ a_k \end{pmatrix} \times \begin{pmatrix} b_i \\ b_j \\ b_k \end{pmatrix} = \begin{pmatrix} a_j b_k - a_k b_j \\ a_k b_i - a_i b_k \\ a_i b_j - a_j b_i \end{pmatrix} = \begin{pmatrix} (\vec{a} \times \vec{b})_i \\ (\vec{a} \times \vec{b})_j \\ (\vec{a} \times \vec{b})_k \end{pmatrix} \tag{25}$$

therefore

$$(\vec{a} \times \vec{b})_i = a_j b_k - a_k b_j \tag{26}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for} \quad j \neq k \neq i \\ -1 & \text{for} \quad j, k \text{ inverse}(\varepsilon_{ikj}) \\ 0 & \text{for} \quad (j = k) \vee (j = i) \vee (k = i) \end{cases} \tag{27}$$

therefore

$$\varepsilon_{ijk} a_j b_k = a_j b_k - a_k b_j$$

□

Question 4

(a) A field is given by

$$\vec{F} = \left(\frac{-y}{r^2}, \frac{x}{r^2}, 0 \right) \cdot \beta \tag{28}$$

To verify if the field is conservative, we apply $\nabla\times$

$$\nabla \times \vec{F} = \beta \nabla \times \left(\frac{-y}{r^2}, \frac{x}{r^2}, 0 \right) = \beta \nabla \times \vec{F}' \tag{29}$$

since

$$(\nabla \times \vec{F})_i = \epsilon_{ijk} \partial_j F_k \tag{30}$$

we get

$$\nabla \times \vec{F}' = \begin{pmatrix} \partial_y \vec{F}'_z - \partial_z \vec{F}'_y \\ \partial_z \vec{F}'_x - \partial_x \vec{F}'_z \\ \partial_y \vec{F}'_x - \partial_x \vec{F}'_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} \end{pmatrix} = 0 \tag{31}$$

the field is curl-free.

(b) A circle path is given by

$$\vec{r} = \begin{pmatrix} C_t \\ S_t \end{pmatrix} \tag{32}$$

calculate the line integral

$$\oint_C \vec{F} \cdot d\vec{l} \tag{33}$$

infinitesimal line element

$$dl = \left| \frac{d\vec{r}}{dt} \right| dt = dt \tag{34}$$

therefore

$$\mathbf{T} = \frac{d\vec{r}}{dt} = \begin{pmatrix} -S_t \\ C_t \end{pmatrix} \tag{35}$$

$$d\vec{l} = dt\mathbf{T} = \begin{pmatrix} -S_t dt \\ C_t dt \end{pmatrix} \tag{36}$$

therefore

$$\oint_C \vec{F} \cdot d\vec{l} = \int_0^{2\pi} \vec{F}(r(t)) \cdot \mathbf{T} dt = \beta \int_0^{2\pi} \begin{pmatrix} -S_t \\ C_t \end{pmatrix} \cdot \begin{pmatrix} -S_t \\ C_t \end{pmatrix} dt = \beta \int_0^{2\pi} dt = 2\pi\beta \tag{37}$$