1 Homework 1 solutions

Question 1

(a)

$$\hat{q}_k = \frac{1}{|\frac{\partial \vec{r}}{\partial q_k}|}\frac{\partial \vec{r}}{\partial q_k} \tag{1}$$

The relationship between two curvilinear coordinates and Cartesian coordinates are given by

$$\begin{aligned}
&\text{cylinder coordinates } (\rho, \varphi, z)\\
&x = \rho\cos(\varphi)\\
&y = \rho\sin(\varphi)\\
&z = z
\end{aligned}$$

$$\begin{aligned}
&\text{spherical coordinates } (r, \vartheta, \varphi)\\
&x = r\sin(\vartheta)\cos(\varphi)\\
&y = r\sin(\vartheta)\sin(\varphi)\\
&z = r\cos(\vartheta)
\end{aligned}$$

We do a transformation using equation (1) to spherical coordinates:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin(\vartheta)\cos(\varphi) \\ r\sin(\vartheta)\sin(\varphi) \\ r\cos(\vartheta) \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \vartheta} = \begin{bmatrix} r\cos(\vartheta)\cos(\varphi) \\ r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -r\sin(\vartheta)\sin(\varphi) \\ r\sin(\vartheta)\cos(\varphi) \\ 0 \end{bmatrix}$$

with

$$\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial r}\right| &= \sqrt{\sin^2(\vartheta)\cos^2(\varphi) + \sin^2(\vartheta)\sin^2(\varphi) + \cos^2(\vartheta)} = 1\\
\left|\frac{\partial \vec{r}}{\partial \vartheta}\right| &= \sqrt{r\cos^2(\vartheta)\cos^2(\varphi) + r^2\cos^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)} = r\\
\left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{r^2\sin^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)\cos^2(\varphi)} = r\sin(\vartheta)
\end{aligned}$$

we get the unit vectors:

$$\hat{r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \hat{\vartheta} = \begin{bmatrix} \cos(\vartheta)\cos(\varphi) \\ \cos(\vartheta)\sin(\varphi) \\ -\sin(\vartheta) \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \tag{2}$$

with transformation matrix:

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \sin(\vartheta)\sin(\varphi) & \cos(\vartheta) \\ \cos(\vartheta)\cos(\varphi) & \cos(\vartheta)\sin(\varphi) & -\sin(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{3}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \tag{4}$$

We now verify the orthogonality of our new unit vectors using cross product:

$$\begin{aligned}
\hat{r} \times \hat{\vartheta} &= \begin{bmatrix} \sin(\vartheta)\sin(\varphi)(-\sin(\vartheta)) - \cos(\vartheta)\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\cos(\vartheta)\cos(\varphi) - \sin(\vartheta)\cos(\varphi)(-\sin(\vartheta)) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} \\
&= \begin{bmatrix} -(\sin^2(\vartheta)\sin(\varphi) + \cos^2(\vartheta)\sin(\varphi)) \\ \cos^2(\vartheta)\cos(\varphi) + \sin^2(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} = \hat{\varphi}
\end{aligned}$$

correct, but I wanna do a double check:

$$\hat{\vartheta} \times \hat{\varphi} = \begin{bmatrix} \cos(\vartheta)\sin(\varphi)0 - (-\sin(\vartheta))\cos(\varphi) \\ (-\sin(\vartheta))(-\sin(\varphi)) - \cos(\vartheta)\cos(\varphi)0 \\ \cos(\vartheta)\cos(\varphi)\cos(\varphi) - \cos(\vartheta)\sin(\varphi)(-\sin(\varphi)) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} = \hat{r}$$

Now we do the same transformation to cylinder coordinates:

$$\vec{r} = \begin{bmatrix} \rho\cos(\varphi) \\ \rho\sin(\varphi) \\ z \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with

$$\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial \rho}\right| &= \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1 \\
\left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{\rho^2\sin^2(\varphi) + \rho^2\cos^2(\varphi)} = \rho \\
\left|\frac{\partial \vec{r}}{\partial z}\right| &= 1
\end{aligned}$$

therefore:

$$\hat{\rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with transformation matrix:

$$\begin{aligned}
\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix} &= \begin{bmatrix} \cos\varphi & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} &= \begin{bmatrix} \cos\varphi & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix}
\end{aligned}$$

(b) 1