Homework 0 solutions

Question 1

(a)

$$\hat{q}_k = \frac{1}{|\frac{\partial \vec{r}}{\partial q_k}|}\frac{\partial \vec{r}}{\partial q_k} \tag{1}$$

The relation between two curvilinear coordinates and Cartesian coordinates are given by

$$\begin{aligned}
&\text{cylindrical coordinates } (\rho, \varphi, z)\\
&x = \rho\cos(\varphi)\\
&y = \rho\sin(\varphi)\\
&z = z
\end{aligned}$$

$$\begin{aligned}
&\text{spherical coordinates } (r, \vartheta, \varphi)\\
&x = r\sin(\vartheta)\cos(\varphi)\\
&y = r\sin(\vartheta)\sin(\varphi)\\
&z = r\cos(\vartheta)
\end{aligned}$$

We do a transformation using equation (1) to spherical coordinates:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin(\vartheta)\cos(\varphi) \\ r\sin(\vartheta)\sin(\varphi) \\ r\cos(\vartheta) \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \vartheta} = \begin{bmatrix} r\cos(\vartheta)\cos(\varphi) \\ r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -r\sin(\vartheta)\sin(\varphi) \\ r\sin(\vartheta)\cos(\varphi) \\ 0 \end{bmatrix}$$

with

$$\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial r}\right| &= \sqrt{\sin^2(\vartheta)\cos^2(\varphi) + \sin^2(\vartheta)\sin^2(\varphi) + \cos^2(\vartheta)} = 1\\
\left|\frac{\partial \vec{r}}{\partial \vartheta}\right| &= \sqrt{r\cos^2(\vartheta)\cos^2(\varphi) + r^2\cos^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)} = r\\
\left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{r^2\sin^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)\cos^2(\varphi)} = r\sin(\vartheta)
\end{aligned}$$

we get the unit vectors:

$$\hat{r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \hat{\vartheta} = \begin{bmatrix} \cos(\vartheta)\cos(\varphi) \\ \cos(\vartheta)\sin(\varphi) \\ -\sin(\vartheta) \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \tag{2}$$

with transformation matrix:

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \sin(\vartheta)\sin(\varphi) & \cos(\vartheta) \\ \cos(\vartheta)\cos(\varphi) & \cos(\vartheta)\sin(\varphi) & -\sin(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{3}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \tag{4}$$

We now verify the orthogonality of our new unit vectors using cross product:

$$
\begin{aligned}
\hat{r} \times \hat{\vartheta} &= \begin{bmatrix} \sin(\vartheta)\sin(\varphi)(-\sin(\vartheta)) - \cos(\vartheta)\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\cos(\vartheta)\cos(\varphi) - \sin(\vartheta)\cos(\varphi)(-\sin(\vartheta)) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} \\
&= \begin{bmatrix} -(\sin^2(\vartheta)\sin(\varphi) + \cos^2(\vartheta)\sin(\varphi)) \\ \cos^2(\vartheta)\cos(\varphi) + \sin^2(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} = \hat{\varphi}
\end{aligned}
$$

correct, but I wanna do a double check:

$$
\hat{\vartheta} \times \hat{\varphi} = \begin{bmatrix} \cos(\vartheta)\sin(\varphi)0 - (-\sin(\vartheta))\cos(\varphi) \\ (-\sin(\vartheta))(-\sin(\varphi)) - \cos(\vartheta)\cos(\varphi)0 \\ \cos(\vartheta)\cos(\varphi)\cos(\varphi) - \cos(\vartheta)\sin(\varphi)(-\sin(\varphi)) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} = \hat{r}
$$

Now we do the same transformation to cylindrical coordinates:

$$
\vec{r} = \begin{bmatrix} \rho\cos(\varphi) \\ \rho\sin(\varphi) \\ z \end{bmatrix}
$$

$$
\frac{\partial \vec{r}}{\partial \rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

with

$$
\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial \rho}\right| &= \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1 \\
\left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{\rho^2\sin^2(\varphi) + \rho^2\cos^2(\varphi)} = \rho \\
\left|\frac{\partial \vec{r}}{\partial z}\right| &= 1
\end{aligned}
$$

therefore:

$$
\hat{\rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}
$$

with transformation matrix:

$$
\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{6}
$$

$$
\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \tag{7}
$$

(b) The position vector $\vec{r}$ in combination of basis vectors from cartesian coordinates and factors of curvilinear coordinates is given by

$$(\textit{sperical})\ \vec{r} = r\sin(\vartheta)\cos(\varphi)\hat{x} + r\sin(\vartheta)\sin(\varphi)\hat{y} + r\cos(\vartheta)\hat{z} \tag{8}$$
$$(\textit{cylindrical})\ \vec{r} = \rho\cos(\varphi)\hat{x} + \rho\sin(\varphi)\hat{y} + z\hat{z} \tag{9}$$

our goal is to obtain an expression of position vector, such that it is explicitly expressed by curvilinear coordinates, like in the following equation:

$$\vec{r} = \vec{r}(q_1, q_2, q_3) = \sum_{k=1}^{3} c_k(q_1, q_2, q_3)\hat{q}_k(q_1, q_2, q_3)$$

Now we replace unit vectors of cartesian coordinates with which from curvilinear coordinates using transformation matrices.

For spherical coordinates, we use equation (4):

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi)\\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi)\\ \cos(\vartheta) & -\sin(\vartheta) & 0\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\vartheta}\\ \hat{\varphi}\end{bmatrix}$$

we insert position vector form (8),

$$\begin{aligned}
&\begin{bmatrix}r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta)\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}\\
=&\begin{bmatrix}r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta)\end{bmatrix}\begin{bmatrix}\sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi)\\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi)\\ \cos(\vartheta) & -\sin(\vartheta) & 0\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\vartheta}\\ \hat{\varphi}\end{bmatrix}\\
=&\begin{bmatrix}r\sin^2(\vartheta)\cos^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi) & -r\sin(\vartheta)\cos(\varphi)\sin(\varphi)\\ r\sin^2(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\sin(\varphi)\cos(\varphi)\\ r\cos^2(\vartheta) & -r\cos(\vartheta)\sin(\vartheta) & 0\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\vartheta}\\ \hat{\varphi}\end{bmatrix}\\
=&\begin{bmatrix}r\sin^2(\vartheta)\cos^2(\varphi)\hat{r} + r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi)\hat{\vartheta} - r\sin(\vartheta)\cos(\varphi)\sin(\varphi)\hat{\varphi}\\ r\sin^2(\vartheta)\sin^2(\varphi)\hat{r} + r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi)\hat{\vartheta} + r\sin(\vartheta)\sin(\varphi)\cos(\varphi)\hat{\varphi}\\ r\cos^2(\vartheta)\hat{r} - r\cos(\vartheta)\sin(\vartheta)\hat{\vartheta}\end{bmatrix}\\
=&(r\sin^2(\vartheta)\cos^2(\varphi) + r\sin^2(\vartheta)\sin^2(\varphi) + r\cos^2(\vartheta))\hat{r}\\
&+ (r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi) + r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi) - r\cos(\vartheta)\sin(\vartheta))\hat{\vartheta}\\
&+ (-r\sin(\vartheta)\cos(\varphi)\sin(\varphi) + r\sin(\vartheta)\sin(\varphi))\cos(\varphi)\hat{\varphi}\\
=&r\hat{r}
\end{aligned}$$

And for cylindrical coordinates:

$$\begin{aligned}
\begin{bmatrix} \rho\cos(\varphi) & & \\ & \rho\sin(\varphi) & \\ & & z \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} &= \begin{bmatrix} \rho\cos(\varphi) & & \\ & \rho\sin(\varphi) & \\ & & z \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} \rho\cos^2(\varphi) & -\rho\sin(\varphi)\cos(\varphi) & 0 \\ \rho\sin^2(\varphi) & \rho\sin(\varphi)\cos(\varphi) & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} \rho\cos^2(\varphi)\hat{\rho} - \rho\sin(\varphi)\cos(\varphi)\hat{\varphi} \\ \rho\sin^2(\varphi)\hat{\rho} + \rho\sin(\varphi)\cos(\varphi)\hat{\varphi} \\ z\hat{z} \end{bmatrix} \\
&= (\rho\cos^2(\varphi) + \rho\sin^2(\varphi))\hat{\rho} + (-\rho\sin(\varphi)\cos(\varphi) + \rho\sin(\varphi)\cos(\varphi))\hat{\varphi} + z\hat{z} \\
&= \rho\hat{\rho} + z\hat{z}
\end{aligned}$$

(c) The (square of) line element is given by

$$dl := |d\vec{r}|^2$$

with $d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$
In spherical coordinates, it is

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \vartheta} d\vartheta + \frac{\partial \vec{r}}{\partial \varphi} d\varphi \\ &= \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} dr + \begin{bmatrix} r\cos(\vartheta)\cos(\varphi) \\ r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta) \end{bmatrix} d\vartheta + \begin{bmatrix} -r\sin(\vartheta)\sin(\varphi) \\ r\sin(\vartheta)\cos(\varphi) \\ 0 \end{bmatrix} d\varphi \end{aligned}$$

therefore,

$$\begin{aligned} |d\vec{r}|^2 &= \begin{bmatrix} \sin(\vartheta)^2\cos(\varphi)^2 \\ \sin(\vartheta)^2\sin(\varphi)^2 \\ \cos(\vartheta)^2 \end{bmatrix} dr^2 + \begin{bmatrix} r^2\cos(\vartheta)^2\cos(\varphi)^2 \\ r^2\cos(\vartheta)^2\sin(\varphi)^2 \\ r^2\sin(\vartheta)^2 \end{bmatrix} d\vartheta^2 + \begin{bmatrix} r^2\sin(\vartheta)^2\sin(\varphi)^2 \\ r^2\sin(\vartheta)^2\cos(\varphi)^2 \\ 0 \end{bmatrix} d\varphi^2 \\ &= dr^2 + r^2 d\vartheta^2 + r^2\sin^2(\theta) d\varphi^2 \end{aligned}$$

In cylindrical coordinates, it is

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{\rho}}{\partial \rho} dr + \frac{\partial \vec{r}}{\partial \varphi} d\varphi + \frac{\partial \vec{r}}{\partial z} dz \\ &= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix} dr + \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix} d\vartheta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\varphi \end{aligned}$$

therefore

$$\begin{aligned} |d\vec{r}|^2 &= \begin{bmatrix} \cos^2(\varphi) \\ \sin^2(\varphi) \\ 0 \end{bmatrix} d\rho^2 + \begin{bmatrix} \rho^2\sin^2(\varphi) \\ \rho^2\cos^2(\varphi) \\ 0 \end{bmatrix} d\varphi^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dz^2 \\ &= d\rho^2 + \rho^2 d\varphi^2 + dz^2 \end{aligned}$$

(d) The position vector in spherical coordinates is

$$\vec{r} = r\hat{r}$$

the speed $\vec{v}(t)$ is the derivative of position vector $\vec{r}$ with respect to time t

$$\begin{aligned}\vec{v}(t) = \dot{\vec{r}} &= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}\\ &= \dot{r}(t)\begin{bmatrix}\sin(\vartheta(t))\cos(\varphi(t))\\ \sin(\vartheta(t))\sin(\varphi(t))\\ \cos(\vartheta(t))\end{bmatrix} + r(t)\underbrace{\frac{d}{dt}\begin{bmatrix}\sin(\vartheta(t))\cos(\varphi(t))\\ \sin(\vartheta(t))\sin(\varphi(t))\\ \cos(\vartheta(t))\end{bmatrix}}_{\text{🀐}}\end{aligned}$$

with

$$\text{🀐} = \begin{bmatrix}\dot{\vartheta}(t)\cos(\vartheta(t))\cos(\varphi(t)) - \sin(\vartheta(t))\dot{\varphi}(t)\sin(\varphi(t))\\ \dot{\vartheta}(t)\cos(\vartheta(t))\sin(\varphi(t)) + \sin(\vartheta(t))\dot{\varphi}(t)\cos(\varphi(t))\\ -\dot{\vartheta}(t)\sin(\vartheta(t))\end{bmatrix}$$

therefore

$$\vec{v}(t) = \dot{r}(t)\underbrace{\begin{bmatrix}\sin(\vartheta(t))\cos(\varphi(t))\\ \sin(\vartheta(t))\sin(\varphi(t))\\ \cos(\vartheta(t))\end{bmatrix}}_{\hat{r}(t)} + r(t)\underbrace{\begin{bmatrix}\dot{\vartheta}(t)\cos(\vartheta(t))\cos(\varphi(t)) - \sin(\vartheta(t))\dot{\varphi}(t)\sin(\varphi(t))\\ \dot{\vartheta}(t)\cos(\vartheta(t))\sin(\varphi(t)) + \sin(\vartheta(t))\dot{\varphi}(t)\cos(\varphi(t))\\ -\dot{\vartheta}(t)\sin(\vartheta(t))\end{bmatrix}}_{\dot{\hat{r}}(t)} \tag{10}$$

the acceleration a is given by

$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t) = \ddot{r}(t)\hat{r}(t) + \dot{r}(t)\dot{\hat{r}}(t) + \dot{r}(t)\dot{\hat{r}}(t) + r(t)\ddot{\hat{r}}(t) \tag{11}$$

with

$$\begin{aligned}\ddot{\hat{r}}(t) &= \begin{bmatrix}\ddot{\vartheta}C_\vartheta C_\varphi + \dot{\vartheta}(\dot{\vartheta}S_\vartheta C_\varphi - \dot{\varphi}C_\vartheta S_\varphi) - \ddot{\varphi}S_\vartheta S_\varphi - \dot{\varphi}(\dot{\vartheta}C_\vartheta S_\varphi + \dot{\varphi}S_\vartheta C_\varphi)\\ \ddot{\vartheta}C_\vartheta S_\varphi + \dot{\vartheta}(-\dot{\vartheta}S_\vartheta S_\varphi + \dot{\varphi}C_\vartheta C_\varphi) + \ddot{\varphi}S_\vartheta C_\varphi + \dot{\varphi}(\dot{\vartheta}C_\vartheta C_\varphi - \dot{\varphi}S_\vartheta S_\varphi)\\ -\ddot{\vartheta}S_\vartheta - (\dot{\vartheta})^2 C_\vartheta\end{bmatrix}\\ \dot{\hat{r}}(t) &= \begin{bmatrix}\dot{\vartheta}(t)\cos(\vartheta(t))\cos(\varphi(t)) - \sin(\vartheta(t))\dot{\varphi}(t)\sin(\varphi(t))\\ \dot{\vartheta}(t)\cos(\vartheta(t))\sin(\varphi(t)) + \sin(\vartheta(t))\dot{\varphi}(t)\cos(\varphi(t))\\ -\dot{\vartheta}(t)\sin(\vartheta(t))\end{bmatrix}\\ \hat{r}(t) &= \begin{bmatrix}\sin(\vartheta(t))\cos(\varphi(t))\\ \sin(\vartheta(t))\sin(\varphi(t))\\ \cos(\vartheta(t))\end{bmatrix}\end{aligned}$$

Question 2

(a) The trajectory of the mass point is given by $\vec{r}(t)$

Cross product of two vectors gives a "directionalized(?) area"(a vector) with magnitude equals to area of the parallelogram that spanned by two vectors.

In our case, the circular sector A can be approximated to a triangle with a infinitesimal arc element $d\vec{r}$, therefore we have

$$dA = \frac{1}{2}|\vec{r}(t) \times d\vec{r}(t)|$$

$d\vec{r}(t)$ is the path of mass point along the the orbit $\vec{r}(t)$ in time dt

$$\begin{aligned} dA &= \frac{1}{2}|\vec{r}(t) \times \vec{v}(t)dt| \\ &= \frac{1}{2}|\vec{r}(t) \times \vec{v}(t)|dt \\ &= \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)|dt \end{aligned}$$

therefore

$$f(t) = \frac{dA}{dt} = \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)|$$

(b) We are going to prove that

$$\frac{d}{dt}f(t) = 0$$

$$\begin{aligned} df(t) &= f(t+dt) - f(t) \\ &= \frac{1}{2}|\vec{r}(t+dt) \times \dot{\vec{r}}(t+dt)| - \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)| \\ &= \frac{1}{2}|(\vec{r}(t) + \dot{\vec{r}}(t)dt) \times (\dot{\vec{r}}(t) + \ddot{\vec{r}}(t)dt)| - \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)| \\ &= \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t) + \vec{r}(t) \times \ddot{\vec{r}}(t)dt + \underbrace{\dot{\vec{r}}(t)\ddot{\vec{r}}(t)dt^2}_{dt^2=0}| - \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)| \\ &= \frac{1}{2}(|\vec{r}(t) \times \dot{\vec{r}}(t) + \underbrace{\vec{r}(t) \times \ddot{\vec{r}}(t)dt}_{=0}| - |\vec{r}(t) \times \dot{\vec{r}}(t)|) \\ &= \frac{1}{2}(|\vec{r}(t) \times \dot{\vec{r}}(t)| - |\vec{r}(t) \times \dot{\vec{r}}(t)|) \\ &= 0 \\ \Rightarrow \frac{d}{dt}f(t) &= 0 \end{aligned}$$

(c) It is given that

$$\vec{r}(t) = \rho\hat{\rho} + z\hat{z}$$

and

$$\dot{\vec{r}}(t) = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{z}\hat{z}$$

with $z(t) = 0$

$$\dot{\vec{r}}(t) = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} = \dot{\rho}\begin{pmatrix} C_\varphi \\ S_\varphi \\ 0 \end{pmatrix} + \rho\begin{pmatrix} -\dot{\varphi}S_\varphi \\ \dot{\varphi}C_\varphi \\ 0 \end{pmatrix} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi}$$

therefore

$$\begin{aligned} f(t) &= \frac{1}{2}|\vec{r}(t) \times \dot{\vec{r}}(t)| \\ &= \frac{1}{2}|(\rho\hat{\rho} + z\hat{z}) \times (\dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi})| \\ &= \frac{1}{2}|\rho\hat{\rho} \times \rho\dot{\varphi}\hat{\varphi} + z\hat{z} \times \dot{\rho}\hat{\rho} + z\hat{z} \times \rho\dot{\varphi}\hat{\varphi}| \\ &= \frac{1}{2}| - \rho^2\dot{\varphi}\hat{z} + z\dot{\rho}\hat{\varphi} - z\rho\dot{\varphi}\hat{\rho}| \\ &= \frac{1}{2}\sqrt{\rho^4\dot{\varphi}^2 + z^2\dot{\rho}^2 + z^2\rho^2\dot{\varphi}^2} \\ (z(t) = 0) &= \frac{1}{2}\rho^2\dot{\varphi} \end{aligned}$$

Question 3

(a) We are going to prove

$$(R_1R_2)^T(R_1R_2) = \mathbb{1}$$

Proof.

$$\begin{aligned}(R_1R_2)^T(R_1R_2) &= R_2^TR_1^TR_1R_2\\ &= R_2^T\mathbb{1}R_2\\ &= R_2^TR_2\\ &= \mathbb{1}\end{aligned}$$

□

(b) We are going to prove

$$(R\vec{a})\cdot(R\vec{b}) = \vec{a}\cdot\vec{b}$$

with R orthogonal

Proof.

$$\begin{aligned}(R\vec{a})\cdot(R\vec{b}) &= (R\vec{a})^T(R\vec{b})\\ &= \vec{a}^TR^TR\vec{b}\\ &= \vec{a}^T\mathbb{1}\vec{b}\\ \vec{a}^T\vec{b} &= \vec{a}\cdot\vec{b}\end{aligned}$$

□

(c) We define two rotation matrices that rotate two different angles

$$R(\varphi_1) = \begin{pmatrix} C_{\varphi_1} & S_{\varphi_1} & 0\\ -S_{\varphi_1} & C_{\varphi_1} & 0\\ 0 & 0 & 1\end{pmatrix},\ R(\varphi_2) = \begin{pmatrix} C_{\varphi_2} & S_{\varphi_2} & 0\\ -S_{\varphi_2} & C_{\varphi_2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

therefore

$$\begin{aligned}R(\varphi_1)R(\varphi_2) &= \begin{pmatrix} C_{\varphi_1} & S_{\varphi_1} & 0\\ -S_{\varphi_1} & C_{\varphi_1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} C_{\varphi_2} & S_{\varphi_2} & 0\\ -S_{\varphi_2} & C_{\varphi_2} & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} C_{\varphi_1}C_{\varphi_2} - S_{\varphi_1}S_{\varphi_2} & S_{\varphi_1}C_{\varphi_2} + C_{\varphi_1}S_{\varphi_2} & 0\\ -S_{\varphi_1}C_{\varphi_2} - C_{\varphi_1}S_{\varphi_2} & C_{\varphi_1}C_{\varphi_2} - S_{\varphi_1}S_{\varphi_2} & 0\\ 0 & 0 & 1\end{pmatrix}\\ (R(\varphi_1)R(\varphi_2))^T &= \begin{pmatrix} C_{\varphi_1}C_{\varphi_2} - S_{\varphi_1}S_{\varphi_2} & -S_{\varphi_1}C_{\varphi_2} - C_{\varphi_1}S_{\varphi_2} & 0\\ S_{\varphi_1}C_{\varphi_2} + C_{\varphi_1}S_{\varphi_2} & C_{\varphi_1}C_{\varphi_2} - S_{\varphi_1}S_{\varphi_2} & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned}$$

$$(R(\varphi_1)R(\varphi_2))^T(R(\varphi_1)R(\varphi_2)) = \mathbb{1}$$

We define two vectors

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

with dot product defined

$$\vec{a}^T \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

We apply a orthogonal transformation (rotation matrix $R(\varphi)$) to each vector

$$\begin{aligned}
(R(\varphi)\vec{a})^T (R(\varphi)\vec{b}) &= \left(\begin{pmatrix} C_\varphi & S_\varphi & 0 \\ -S_\varphi & C_\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right)^T \begin{pmatrix} C_\varphi & S_\varphi & 0 \\ -S_\varphi & C_\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
&= \begin{pmatrix} C_\varphi a_1 + S_\varphi a_2 \\ -S_\varphi a_1 + C_\varphi a_2 \\ a_3 \end{pmatrix}^T \begin{pmatrix} C_\varphi b_1 + S_\varphi b_2 \\ -S_\varphi b_1 + C_\varphi b_2 \\ b_3 \end{pmatrix} \\
&= (C_\varphi a_1 + S_\varphi a_2)(C_\varphi b_1 + S_\varphi b_2) + (-S_\varphi a_1 + C_\varphi a_2)(-S_\varphi b_1 + C_\varphi b_2) + a_3 b_3 \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a}^T \vec{b}
\end{aligned}$$