1 Homework 1 solutions

Question 1

(a)

$$\hat{q}_k = \frac{1}{|\frac{\partial \vec{r}}{\partial q_k}|}\frac{\partial \vec{r}}{\partial q_k} \tag{1}$$

The relation between two curvilinear coordinates and Cartesian coordinates are given by

$$\begin{aligned}&\text{cylindrical coordinates } (\rho, \varphi, z)\\ &x = \rho\cos(\varphi)\\ &y = \rho\sin(\varphi)\\ &z = z\end{aligned}$$

$$\begin{aligned}&\text{spherical coordinates } (r, \vartheta, \varphi)\\ &x = r\sin(\vartheta)\cos(\varphi)\\ &y = r\sin(\vartheta)\sin(\varphi)\\ &z = r\cos(\vartheta)\end{aligned}$$

We do a transformation using equation (1) to spherical coordinates:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin(\vartheta)\cos(\varphi) \\ r\sin(\vartheta)\sin(\varphi) \\ r\cos(\vartheta) \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \vartheta} = \begin{bmatrix} r\cos(\vartheta)\cos(\varphi) \\ r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta) \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -r\sin(\vartheta)\sin(\varphi) \\ r\sin(\vartheta)\cos(\varphi) \\ 0 \end{bmatrix}$$

with

$$\begin{aligned}\left|\frac{\partial \vec{r}}{\partial r}\right| &= \sqrt{\sin^2(\vartheta)\cos^2(\varphi) + \sin^2(\vartheta)\sin^2(\varphi) + \cos^2(\vartheta)} = 1\\ \left|\frac{\partial \vec{r}}{\partial \vartheta}\right| &= \sqrt{r\cos^2(\vartheta)\cos^2(\varphi) + r^2\cos^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)} = r\\ \left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{r^2\sin^2(\vartheta)\sin^2(\varphi) + r^2\sin^2(\vartheta)\cos^2(\varphi)} = r\sin(\vartheta)\end{aligned}$$

we get the unit vectors:

$$\hat{r} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix}, \quad \hat{\vartheta} = \begin{bmatrix} \cos(\vartheta)\cos(\varphi) \\ \cos(\vartheta)\sin(\varphi) \\ -\sin(\vartheta) \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} \tag{2}$$

with transformation matrix:

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \sin(\vartheta)\sin(\varphi) & \cos(\vartheta) \\ \cos(\vartheta)\cos(\varphi) & \cos(\vartheta)\sin(\varphi) & -\sin(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{3}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \tag{4}$$

We now verify the orthogonality of our new unit vectors using cross product:

$$
\begin{aligned}
\hat{r} \times \hat{\vartheta} &= \begin{bmatrix} \sin(\vartheta)\sin(\varphi)(-\sin(\vartheta)) - \cos(\vartheta)\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\cos(\vartheta)\cos(\varphi) - \sin(\vartheta)\cos(\varphi)(-\sin(\vartheta)) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} \\
&= \begin{bmatrix} -(\sin^2(\vartheta)\sin(\varphi) + \cos^2(\vartheta)\sin(\varphi)) \\ \cos^2(\vartheta)\cos(\varphi) + \sin^2(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\cos(\varphi)\cos(\vartheta)\sin(\varphi) - \sin(\vartheta)\sin(\varphi)\cos(\vartheta)\cos(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix} = \hat{\varphi}
\end{aligned}
$$

correct, but I wanna do a double check:

$$
\hat{\vartheta} \times \hat{\varphi} = \begin{bmatrix} \cos(\vartheta)\sin(\varphi)0 - (-\sin(\vartheta))\cos(\varphi) \\ (-\sin(\vartheta))(-\sin(\varphi)) - \cos(\vartheta)\cos(\varphi)0 \\ \cos(\vartheta)\cos(\varphi)\cos(\varphi) - \cos(\vartheta)\sin(\varphi)(-\sin(\varphi)) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{bmatrix} = \hat{r}
$$

Now we do the same transformation to cylindrical coordinates:

$$
\vec{r} = \begin{bmatrix} \rho\cos(\varphi) \\ \rho\sin(\varphi) \\ z \end{bmatrix}
$$

$$
\frac{\partial \vec{r}}{\partial \rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{bmatrix} -\rho\sin(\varphi) \\ \rho\cos(\varphi) \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

with

$$
\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial \rho}\right| &= \sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1 \\
\left|\frac{\partial \vec{r}}{\partial \varphi}\right| &= \sqrt{\rho^2\sin^2(\varphi) + \rho^2\cos^2(\varphi)} = \rho \\
\left|\frac{\partial \vec{r}}{\partial z}\right| &= 1
\end{aligned}
$$

therefore:

$$
\hat{\rho} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{bmatrix}, \quad \hat{\varphi} = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}
$$

with transformation matrix:

$$
\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{6}
$$

$$
\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ z \end{bmatrix} \tag{7}
$$

(b) The position vector $\vec{r}$ in combination of basis vectors from cartesian coordinates and the factors from curvilinear coordinates is given by

$$(sperical)\vec{r} = r\sin(\vartheta)\cos(\varphi)\hat{x} + r\sin(\vartheta)\sin(\varphi)\hat{y} + r\cos(\vartheta)\hat{z} \tag{8}$$

$$(cylindrical)\vec{r} = \rho\cos(\varphi)\hat{x} + \rho\sin(\varphi)\hat{y} + z\hat{z} \tag{9}$$

our goal is to obtain an expression of position vector, such that it is explicitly expressed by curvilinear coordinates, like the following equation:

$$\vec{r} = \vec{r}(q_1, q_2, q_3) = \sum_{k=1}^{3} c_k(q_1, q_2, q_3)\hat{q}_k(q_1, q_2, q_3)$$

Now we replace the unit vectors of cartesian coordinates with which from curvilinear coordinates using transformation matrices.

For spherical coordinates, we use equation (4):

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}$$

we insert position vector form (8),

$$\begin{aligned}
&\begin{bmatrix} r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\
=&\begin{bmatrix} r\sin(\vartheta)\cos(\varphi) & & \\ & r\sin(\vartheta)\sin(\varphi) & \\ & & r\cos(\vartheta) \end{bmatrix} \begin{bmatrix} \sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\vartheta)\sin(\varphi) & \cos(\vartheta)\sin(\varphi) & \cos(\varphi) \\ \cos(\vartheta) & -\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix} \\
=&\begin{bmatrix} r\sin^2(\vartheta)\cos^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\cos^2(\varphi) & -r\sin(\vartheta)\cos(\varphi)\sin(\varphi) \\ r\sin^2(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\sin^2(\varphi) & r\sin(\vartheta)\cos(\vartheta)\sin(\varphi) \\ r\cos^2(\vartheta) & -r\cos(\vartheta)\sin(\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\vartheta} \\ \hat{\varphi} \end{bmatrix}
\end{aligned}$$