Topic 1 | Projectile Motion with Air Resistance

A Case Study in Computer Analysis

In our study of projectile motion, we assumed that air-resistance effects are negligibly small. But in fact air resistance (often called *air drag,* or simply *drag*) has a major effect on the motion of many objects, including tennis balls, bicycle riders, and airplanes. In Section 5.3 we considered how a fluid resistance force affected a body falling straight down. We'd now like to extend this analysis to a projectile moving in a plane.

It's not difficult to include the force of air resistance in the equations for a projectile, but solving them for the position and velocity as functions of time, or the shape of the path, can get quite complex. Fortunately, it is fairly easy to make quite precise numerical approximations to these solutions, using a computer. That's what this section is about.

When we omitted air drag, the only force acting on a projectile with mass m was its weight $\vec{w} = m\vec{g}$. The components of the projectile's acceleration were simply

$$a_x = 0 \quad a_y = -g$$

The $+x$-axis is horizontal, and the $+y$-axis is vertically upward. We must now include the air drag force acting on the projectile. At the speed of a tossed tennis ball or faster, the magnitude f of the air drag force is approximately proportional to the square of the projectile's speed relative to the air:

$$f = Dv^2 \tag{T1.1}$$

where $v^2 = v_x^2 + v_y^2$. We'll assume that the air is still, so $\vec{v}$ is the velocity of the projectile relative to the ground as well as to the air (Fig. T1.1a). The direction of $\vec{f}$ is opposite the direction of $\vec{v}$, so we can write $\vec{f} = -Dv\vec{v}$ and the components of $\vec{f}$ are

$$f_x = -Dvv_x \quad f_y = -Dvv_y$$

Note that each component is opposite in direction to the corresponding component of velocity and that $f = \sqrt{f_x^2 + f_y^2}$. The free-body diagram is shown in Fig. T1.1b. Newton's second law gives

$$\Sigma F_x = -Dvv_x = ma_x \quad \Sigma F_y = -mg - Dvv_y = ma_y$$

and the components of acceleration including the effects of both gravity and air drag are

$$a_x = -(D/m)vv_x \quad a_y = -g - (D/m)vv_y \tag{T1.2}$$

The constant D depends on the density ρ of air, the silhouette area A of the body (its area as seen from the front), and a dimensionless constant C called the **drag coefficient** that depends on the shape of the body. Typical values of C for baseballs, tennis balls, and the like are in the range from 0.2 to 1.0. In terms of these quantities, D is given by

$$D = \frac{\rho C A}{2} \tag{T1.3}$$

(a)

(b)
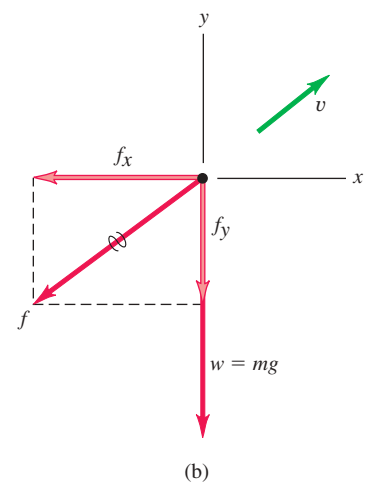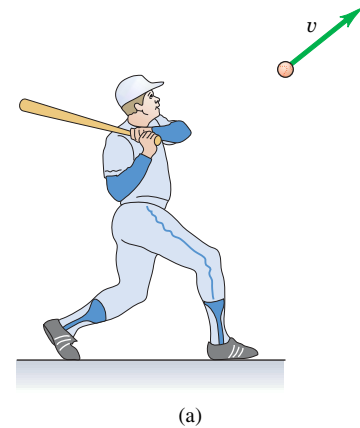
T1.1 (a) A projectile in flight with velocity $\vec{v}$ relative to the ground. (b) Free-body diagram for the projectile. In still air, the air drag force $\vec{f}$ is always directed opposite $\vec{v}$.

Now comes the basic idea of our numerical calculation. The acceleration components a_x and a_y are constantly changing as the velocity components change. But over a sufficiently short time interval Δt, we can regard the acceleration as essentially constant. If we know the coordinates and velocity components at some time t, we can find these quantities at a slightly later time $t + \Delta t$ using the formulas for constant acceleration. Here's how we do it. During a time interval Δt, the average x-component of acceleration is $a_x = \Delta v_x / \Delta t$ and the x-velocity v_x changes by an amount $\Delta v_x = a_x \Delta t$. Similarly, v_y changes by an amount $\Delta v_y = a_y \Delta t$. So the values of the x-velocity and y-velocity at the end of the interval are

$$v_x + \Delta v_x = v_x + a_x \Delta t \qquad v_y + \Delta v_y = v_y + a_y \Delta t \qquad \text{(T1.4)}$$

While this is happening, the projectile is moving, so the coordinates are also changing. The average x-velocity during the time interval Δt is the average of the value v_x (at the beginning of the interval) and $v_x + \Delta v_x$ (at the end of the interval), or $v_x + \Delta v_x / 2$. During Δt the coordinate x changes by an amount

$$\Delta x = (v_x + \Delta v_x / 2)\Delta t = v_x \Delta t + \frac{1}{2}a_x(\Delta t)^2$$

and similarly for y. Compare this to Eq. (T2.12) for motion with constant acceleration; it's exactly the same expression. So the coordinates of the projectile at the end of the interval are

$$x + \Delta x = x + v_x \Delta t + \frac{1}{2}a_x(\Delta t)^2 \qquad y + \Delta y = y + v_y \Delta t + \frac{1}{2}a_y(\Delta t)^2 \quad \text{(T1.5)}$$

We have to specify the starting conditions, that is, the initial values of x, y, v_x, and v_y. Then we can step through the calculation to find the position and velocity at the end of each interval in terms of their values at the beginning, and thus to find the values at the end of any number of intervals. It would be a lot of work to do all the calculations for 100 or more steps with a hand calculator, but the computer does it for us quickly and accurately.

Exactly how you implement this plan depends on whether you use a computer language such as C++, BASIC, FORTRAN, or Pascal or use a spreadsheet or a numerical-analysis software package such as MathCad. Here's a sketch of a general algorithm using a programming language.

Step 1: Identify the parameters of the problem, m, A, C, and ρ, and evaluate D.
Step 2: Choose the time interval Δt and the initial values of x, y, v_x, v_y and t. You may want to express the initial velocity components in terms of the magnitude v_0 and direction α_0 of the initial velocity.
Step 3: Choose the maximum number of intervals N or the maximum time $t_{max} = N\Delta t$ for which you want to get the numerical solution.
Step 4: Loop (or iterate) Steps 5 through 9 while $n < N$ or $t < t_{max}$:
  Step 5: Calculate the acceleration components.

$$a_x = -(D/m)vv_x \qquad a_y = -g - (D/m)vv_y$$

  Step 6: Print or plot x, y, v_x, v_y, a_x, and a_y.
  Step 7: Calculate the new velocity components from Eq. (T1.4):

$$v_x + \Delta v_x = v_x + a_x \Delta t \qquad v_y + \Delta v_y = v_y + a_y \Delta t$$

  Step 8: Calculate the new coordinates from Eq. (T1.5):

$$x + \Delta x = x + v_x \Delta t + \frac{1}{2}a_x(\Delta t)^2 \qquad y + \Delta y = y + v_y \Delta t + \frac{1}{2}a_y(\Delta t)^2$$

Step 9: Increment the time by Δt:

$$t = t + \Delta t$$

Step 10: Stop.

If you are using a spreadsheet, the following notation may be useful. Let $x(n)$, $y(n)$, $v_x(n)$, and $v_y(n)$ be the values at the end of the nth interval. These values appear across the nth line of the spreadsheet. The initial values are $x(1)$, $y(1)$, $v_x(1)$, and $v_y(1)$. Then Eqs. (T1.4) and (T1.5) become

$$v_x(n + 1) = v_x(n) + a_x(n)\Delta t \qquad x(n + 1) = x(n) + v_x(n)\Delta t + \frac{1}{2}a_x(n)(\Delta t)^2 \quad \text{(T1.6)}$$

$$v_y(n + 1) = v_y(n) + a_y(n)\Delta t \qquad y(n + 1) = y(n) + v_y(n)\Delta t + \frac{1}{2}a_y(n)(\Delta t)^2 \quad \text{(T1.7)}$$

In Eqs. (T1.6) and (T1.7) we compute $a_x(n)$ and $a_y(n)$ by substituting the values $v_x(n)$ and $v_y(n)$ into Eq. (T1.2).

Figure T1.2 shows the trajectory of a baseball with and without air drag. The radius of the baseball is $r = 0.0366$ m, and $A = \pi r^2$. The mass is $m = 0.145$ kg, and we have chosen the drag coefficient to be $C = 0.5$ (appropriate for a batted ball or pitched fastball) and the density of air to be $\rho = 1.2$ kg/m^3 (appropriate for a ballpark at sea level). In this example the baseball was given an initial velocity of 50 m/s at an angle of 35° above the $+x$-axis. You can see that both the range of the baseball and the maximum height reached are substantially less than the zero-drag calculation would lead you to believe. Calculating a baseball's trajectory and ignoring air drag is quite unrealistic. Air drag is an important part of the game of baseball!
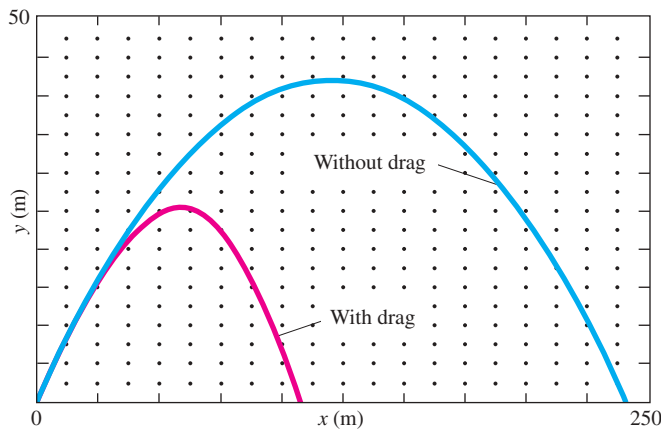
T1.2 Computer-generated trajectories of a baseball with and without drag. Note that different scales are used on the horizontal and vertical axes.