# Boosted Second Price Auctions:

## Revenue Optimization for Heterogeneous Bidders

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Due to its simplicity and desirable incentive properties, the second price auction has been the prevalent auction format used by advertising exchanges. However, even with the optimized choice of reserve prices, this auction is not revenue optimal when the bidders are heterogeneous and their valuation distributions differ significantly. In order to optimize the revenue of advertising exchanges, we propose an auction format called the *boosted second price auction*, which assigns a boost value to each bidder. The auction favors bidders with higher boost values and allocates the item to the bidder with the highest boosted bid.

We propose a data-driven approach to optimize boost values using the previous bids of the bidders. Our analysis of auction data from a large online advertising exchange shows that our algorithm can improve revenue by up to 6%. Furthermore, we observe that the data-driven algorithm assigns higher boosts to advertisers with more stable bidding behavior. We show how this connects to Myerson's optimal mechanism design framework for heterogeneous bidders and propose a boosted second price auction, where bid distributions with lower inverse hazard rates receive a higher boost. We also establish conditions that guarantee that these boosted auctions will increase revenue over the second price auctions and obtain a high fraction of the optimal revenue.

*Key words*: boosted second price auctions, heterogeneity, inverse hazard rate, data-driven, online advertising

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## 1. Introduction

The design of revenue-maximizing selling mechanisms is crucial for many marketplaces, including online advertising markets. The seminal work of Roger Myerson showed that when all the bidders are homogenous, second price (SP) auctions, which allocate items to bidders with the highest submitted bids, are revenue optimal (Myerson 1981). However, in the case of heterogeneous bidders, the auctioneer can benefit from not allocating the item to the bidder with the highest submitted bid. To maximize revenue, submitted bids should be transformed into *virtual bids*, and the bidder with the highest non-negative virtual bid should receive the item. In practice, this transformation is difficult to implement, as it requires a rather precise estimate of the probability density function of the bidders' willingness to pay (valuation). This difficulty highlights the need for a simple and easy-to-implement mechanism that takes into account the heterogeneity of bidders.

We propose the boosted second price (BSP) auction for heterogeneous bidders. In this auction, as in the optimal mechanism, the item may not be allocated to the bidder with the highest submitted bid. Instead, submitted bids are transformed into "boosted bids," and the bidder with the highest boosted bid wins the item. The bidder's boosted bid is that bidder's boost factor multiplied by his submitted bid. BSP auctions are truthful and bidders never pay more than their submitted bids. The intuition is that by assigning different boosts to different bidders, auctioneers can differentiate between bidders in heterogeneous environments to extract more revenue from them.

Now the question becomes how to design such an auction. To answer this question, let us start with the following observation regarding the heterogeneity of bidders/advertisers in online advertising markets. Analyzing bids submitted to a large advertising exchange, we can roughly order bidders based on their bidding and targeting behaviors. We have summarized the results of these analyses in Figures 1a and 1b.[1] These figures respectively illustrate the participation rate of bidders versus the average and standard deviation of their submitted bids, and the data suggest that in an extreme case, bidders can be divided into two groups: *brand* and *retargeting*. Brand bidders participate more often in auctions but submit low average bids and exhibit low variation in their submitted bids. Typically, these bidders have broad targeting criteria and are mostly concerned with displaying their ads to as many Internet users as possible. In other words, they are interested in creating awareness of a product or social event. In contrast, retargeting bidders have more restrictive targeting criteria and participate less often in auctions. In addition, the average and standard deviation of their submitted bids are rather high. These bidders usually submit low bids unless they are retargeting users who have previously visited their websites.

We note that due to the high variation in the bids submitted by retargeting bidders, the second price auctions—even with personalized reserve prices—may not be able to extract high revenue from these bidders. By setting a high reserve, these auctions can yield more revenue from retargeting bidders when these bidders submit high bids. However, the high reserve excludes retargeting bidders from auctions when their bids are low. This can negatively impact revenue, considering that the probability of retargeting a specific Internet user may be low, and that, as a result, the bids submitted by these bidders are often low.

To gain insight into on how boosts can increase revenue in such environments, let us consider the following example.[2] Assume that there are two bidders. The valuation of the first bidder (brand

[1] These figures were generated based on the bids submitted for an ad slot with high traffic volume. The dataset was anonymized.

[2] See Example 1 for more details.

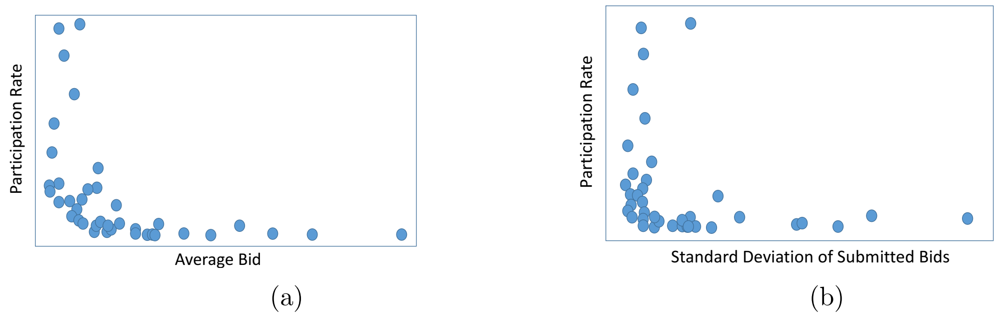



**Figure 1 (1a) Participation rate of bidders versus their average bids and (1b) participation rate of bidders versus the standard deviation of their submitted bids. Each node represents one bidder. The bidder participation rate is the probability that the bidder clears his reserve price. We do not disclose the values of the x and y axes in these figures because doing so could reveal sensitive dataset information.**

advertiser) is always 1, and the valuation of the second bidder (retargeting advertiser) is drawn from $F(x) = 1 - \frac{1}{x}$ for any $x \geq 1$. It is easy to see that the optimal personalized reserve price[3] for each bidder is equal to 1, which yields a revenue of 1 for the auction. Now consider a BSP auction mechanism with a (multiplicative) boost of $\beta_1 = 2$ for the first bidder and no boost for the second bidder ($\beta_2 = 1$) and a reserve price of 1 for each bidder. Let $b_i$, $i = 1, 2$ be the bid of bidder $i$. We consider the following cases:

- If the second bid, $b_2$, is in $[1, 2]$, the first bidder, who has the highest boosted bid, equal to $2 = b_1 \times \beta_1 = 1 \times 2$, receives the item at price 1, which equals his reserve price. More specifically, the payment of the boosted bidder is equal to the maximum of the bidder's reserve and the boosted bid of the other bidder divided by his own boost (i.e., $(b_2 \times \beta_2)/\beta_1$); in this case, the maximum will always be equal to 1.
- If the second bid is greater than 2, the second bidder receives the item at a price equal to 2, which is the maximum of his own reserve and the boosted bid of the first bidder divided by the boost of the second bidder, which is equal to $(b_1 \times \beta_1)/\beta_2 = (1 \times 2)/1 = 2$; see the description of the BSP auction in Section 2.

Therefore, the expected revenue of the BSP auction would be equal to 1.5, which is a 50% improvement over the SP auction with personalized reserve.

Even though the above example is quite stylized, we will show that the insight extends to practical settings.

[3] By the optimal personalized reserve price, we mean the monopoly price $r$ that solves $r = \arg\max_x x(1 - F(x))$. For $F(x) = 1 - \frac{1}{x}$, the solution of the aforementioned optimization problem is not unique. However, one can perturb the distribution slightly so that $r = 1$ becomes the unique solution of the optimization problem.

### 1.1. Contribution and Organization of the Paper

In Section 2, we formally present the BSP auction. In Section 3, we investigate the problem of optimizing boosts for the bidders. Unfortunately, as we show in Theorem 1, the problem of choosing the optimal boost values is NP-complete. However, we propose a data-driven approach that, given a history of the bids, iteratively optimizes the boost for each bidder.

In Section 4, we conduct counterfactual simulations using data from a large advertising exchange. We observe that our algorithm converges very quickly and outperforms SP auctions. Namely, it obtains up to about 30% more revenue than the SP auction with no reserve, and up to about 6% more than the SP auction with optimized personalized reserve prices. In order to understand how the bidding behavior of the bidders impacts their boosts, we do a regression analysis of the boosts assigned by our algorithm across different auctions. Our analysis shows that boost values are negatively correlated with the coefficient of variation and standard deviation of the bids of a bidder, confirming our intuition that bidders with more stable bidding patterns would receive higher boost values.

In Section 5, we connect our empirical observations with the theory of optimal auction design. Myerson (1981) shows that the optimal auction allocates the item to the bidder with the highest (non-negative) virtual value, which is defined as the value of the bidder, $v_i$, minus his inverse hazard rate (IHR) at the submitted bid, $\frac{1-F_i(v_i)}{f_i(v_i)}$, where $F_i$ denotes the valuation distribution of bidder $i$. Note that one can think of $-\frac{1-F_i(v_i)}{f_i(v_i)}$ as the boost value of bidder $i$, but $-\frac{1-F_i(v_i)}{f_i(v_i)}$ depends on the bid itself. Building on this observation, we look at our auction data to understand the patterns for the IHR, $\frac{1-F_i(\cdot)}{f_i(\cdot)}$, of bid distributions across different bidders.

Roughly speaking, we observe that (advertising exchange) advertisers who are more stable and submit lower average bids tend to have lower IHRs. Further, based on our empirical analysis, favoring more stable bidders with lower average valuations could increase the revenue of auctions. Hence, building on these observations, we propose a boosting guideline, called *conservative boosting*, that assigns higher boost values to bidders with lower IHRs. We prove that conservative boosting obtains more than half of the optimal revenue when bidders are "heterogeneous," and surpasses the SP auction. We also establish (tight) parametric bounds on the maximum revenue losses of the BSP and SP auctions compared to that of the optimal mechanism.

### 1.2. Related Work

Our work is related to the literature on revenue-maximizing mechanisms and auctions. One of the main challenges in designing revenue-maximizing mechanisms is heterogeneity of the buyers/bidders. When buyers are homogenous, the optimal mechanism allocates the item to the buyer with the highest submitted bid as long as the highest submitted bid is greater than a certain reserve.

This implies that the auctioneer can earn the maximum revenue by running either first-price (FP) or SP auctions (Myerson 1981).

With heterogeneous buyers, the revenue equivalence between FP and SP auctions does not hold anymore. More specifically, the expected revenue of the FP auction can be either lower or higher than the expected revenue of the SP auction (Maskin and Riley 2000). When buyers are heterogeneous, the auctioneer might find it beneficial not to allocate the item to the buyer with the highest submitted bid. The allocation rule of the optimal mechanism depends on common knowledge regarding the probability density functions of buyers' valuations and could be very sensitive to estimation errors in these functions (Myerson 1981).

Due to these issues, simpler mechanisms such as SP and FP auctions have been used in practice. With heterogeneous buyers, SP auctions with individually optimized reserve prices can receive only 50% of the optimal revenue in the worst case (Hartline and Roughgarden 2009). This result holds when the distribution of valuations is regular. Alaei et al. (2015) also show that when buyers are heterogeneous and regular, there exists a single reserve such that the SP auction with this reserve gets at least $\frac{1}{e}$ fraction of the optimal revenue. Our BSP auction improves upon the SP auction by taking advantage of buyer heterogeneity. We show that BSP auctions with conservative boosting yield more revenue than the SP auction not only on average but also in the worst case. In addition, the parameters of the BSP auction can be learned effectively using our data-driven approach. Thus, this auction format can embrace the heterogeneity among bidders without knowing the probability density of bidders' valaution.

Let us now discuss FP auctions. For homogenous bidders, at the equilibrium, bidders shade their valuations (Riley and Samuelson 1981). However, with heterogeneous buyers, pure Nash equilibrium may not always exist (Lebrun 1996). Since the FP auction is not truthful, another practical downside of the FP auction is that buyers may not follow the equilibrium and shade their bids in the ways that we might expect. Furthermore, via analyzing the submitted bids, it is very challenging to pin down the bidding strategies and shading rules employed by buyers, especially when the buyers are heterogeneous. This makes FP auctions less attractive in online advertising markets.

In online advertising markets, targeting is one of main reasons for heterogeneity among bidders; see Korula et al. (2016) for a survey.[4] In online advertising, buyers use cookie-matching technologies to target their ads towards the best set of Internet users, who have the potential to be more

[4] Heterogeneity is not only important in online advertising markets; it can also play an important role in the structure of revenue optimal selling mechanisms in other marketplaces, such as retail, procurement, and business-to-business markets (see, for example, Golrezaei et al. 2017, Duenyas et al. 2013, and Pilehvar et al. 2016).

interested in their products and services. This level of targeting, in addition to the diverse preferences of advertisers, can lead to a very heterogeneous environment (Levin and Milgrom 2010, Chen and Stallaert 2014).

Arnosti et al. (2016) study the adverse selection in online ad markets for the impressions that are sold via auctions vs. guaranteed-delivery contracts, where the valuations of the buyers are correlated via a common value component. They show that to address the adverse selection, the platform should sometimes allocate the impression to the guaranteed-delivery contracts, even when the bids from the auction are higher. This is similar to assigning higher boosts to those advertisers. In our private-value setting, we do not encounter the adverse selection problem. Nevertheless, we show that assigning boosts, based on the bidding patterns of the advertisers, can increase revenue.

Another way to deal with heterogeneity is via controlling access to users' information. Motivated by this fact, several authors have investigated optimal information disclosure in these markets; see Bergemann and Pesendorfer 2007, Eső and Szentes 2007, Kakade et al. 2013, and Golrezaei and Nazerzadeh 2016. In the present paper, however, we assume that all buyers can access information with which to target Internet users. This is usually the case when there are several advertising markets that are in competition with each other, and this competition reduces the markets' abilities to limit buyer access to user information. When all buyers can implement targeting strategies, the resulting market can be very heterogeneous. In this work, we propose a new auction format that enables advertising exchanges to differentiate between buyers based on their targeting behaviors.

Targeting can also cause irregularity in the distribution of valuations. This is so because advertisers' bidding behavior can be significantly different depending on whether or not an Internet user is a good match for them. When the distribution of buyers' valuations is irregular, the optimal mechanism may need to run a lottery; that is, with irregular distributions, the optimal mechanism may not be able to make a deterministic allocation. Motivated by that, Celis et al. (2014) advocate the use of a randomized auction format that they call *BIN-TAC*, short for "buy it now or take a chance," which improves revenue when the valuation distributions are homogeneous but irregular. In the present work, in contrast, we focus on the impact of heterogeneity caused by targeting. However, some of our results still hold even if the distribution of valuations is irregular. We would like to point out that unlike the BIN-TAC mechanism, the allocation rule of the BSP auction is deterministic.

Many authors have used theory to advocate and study new schemes; see Lejeune and Turner (2015), Golrezaei and Nazerzadeh (2016), Bergemann et al. (2017), Allouah and Besbes (2017), Conitzer et al. (2017), and Balseiro and Gur (2017) for some recent work. To advocate a new auction format in the online advertising market, we may need more than theoretical results. Here, in the current work, we validate our new auction format theoretically, via simulations, as well as empirically. We note that there are only few papers that promote a selling mechanism based on theoretical results and empirical studies. One of these papers is by Chu et al. (2011), who argue that bundle-size pricing yields a good approximation of the optimal mixed bundling pricing scheme for a monopolist selling multiple goods. Another paper is by Paes Leme et al. (2016), who compare two ways of applying monopoly prices in SP auctions. There are also few papers that are mainly focused on empirical studies; see, for example, Ostrovsky and Schwarz (2011) and Athey and Nekipelov (2010) for empirical analysis in search advertising and Goldfarb and Tucker (2011b) and Goldfarb and Tucker (2011a) for empirical work on privacy and targeting in online advertising.

## 2. Boosted Second Price Auction

In this section, we present the BSP auction. We assume that there are $n$ bidders, indexed by $i \in [n] = \{1, 2, \ldots, n\}$, who are interested in a good. The BSP auction assigns two parameters to each bidder: reserve price $r_i$ and boost $\beta_i$. For each bidder $i$, the auctioneer computes a boosted bid, which is a function of the submitted bid of bidder $i$, $b_i$, and the assigned boost value $\beta_i$. In this paper, we focus on multiplicative boosts, in which the boosted bid of bidder $i$ is given by $b_i\beta_i$. The boosts can be also applied additively. However, to ease the exposition, we only consider one way of applying the boost values.

We denote the BSP auction by BSP$(\mathbf{r}, \boldsymbol{\beta})$, where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is the vector of the reserve prices and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ is the vector of the boost values. We now present the BSP auction.

**Boosted Second Price Auction (BSP$(\mathbf{r}, \boldsymbol{\beta})$)**

- First, each bidder $i$ submits his bid $b_i$.
- Define $S$ as a set of bidders whose bids exceed their reserve prices, i.e., $S = \{i : b_i \geq r_i\}$.
- If set $S$ is empty, the good is not allocated. Otherwise, the good is allocated to bidder $i^*$ with the highest boosted bid, i.e., $i^* = \arg\max_{i \in S}\{b_i\beta_i\}$, and he pays $\max\{r_{i^*}, \max_{i \in S, i \neq i^*}\{b_i\beta_i\}/\beta_{i^*}\}$. For other bidders, the payement is zero.

In the BSP auction, the winner is one of the bidders who clears his reserve price and has the highest boosted bid. The winner then pays the maximum of his reserve price and the second highest boosted bid divided by his own boost value. Note that the payment of the winner, which is the minimum bid that he needs to submit to win the auction, does not exceed the submitted bid of the winner. With standard arguments, since the allocation probability is monotone in the submitted bid, and the payment of the winner is not a function of his submitted bid, we obtain the following result.

Proposition 1 **(BSP Auction is Truthful)**. *In BSP auctions, a weakly dominant strategy of each bidder $i$ is to bid truthfully, i.e., $b_i$ is equal to the true value of the bidder, denoted by $v_i$.*

We now discuss the role of reserve prices in more detail. In the BSP auction, we first discard bidders who do not meet their reserve prices, and then we determine a winner from the rest of the bidders. This way of enforcing reserve prices is called "eager" by Paes Leme et al. (2016). Another way of enforcing reserve prices is the approach known as "lazy," in which the winner is determined first and then the reserve prices are applied. Paes Leme et al. (2016) show that eager SP auctions outperform lazy auctions in terms of revenue when the bidders are either independent or symmetric. As such, we assume that reserve prices are applied eagerly in both the BSP and the SP auctions. Note that the SP auction with the vector of reserve prices $\mathbf{r}$ is equivalent to BSP auctions with boosts $\beta_i = \beta$ for $i \in [n]$ and the same vector of reserve prices. Thus, boost values in the BSP auctions allow the seller to further differentiate between bidders.

The fact that boosted bids are linear in submitted bids can simplify the problem of learning parameters of BSP auctions. In particular, the difference in revenue between the BSP auction when boosts are optimized using historical data and the BSP auction when boosts are optimized by having access to the true distribution of valuations is in the order of $U\sqrt{\frac{n \log n}{T}}$, where $U$ is the maximum revenue in an auction, $T$ is the number of previous auctions, and $n$ is the number of bidders; see Balcan et al. (2017) for details. Motivated by this, in the next section, we study the problem of learning the optimal boost values using historical data. In addition, in Section 6, we discuss the consequence of using historical data in learning the parameters of the BSP auction.

## 3. Data-driven Approach to Design BSP Auctions

In this section, we present a data-driven approach to optimize the boost values in BSP auctions. Let us define the problem formally. Assume that there are $n$ bidders indexed by $i \in [n]$. Suppose that we have access to the bids submitted by each of these $n$ bidders over the course of $T$ auctions. Let $b_i^t$ be the bid submitted by bidder $i$ in auction $t \in [T]$. We denote the bids submitted by bidders in auction $t$, i.e., $(b_1^t, b_2^t, \ldots, b_n^t)$, by $\mathbf{b}^t$. Given these submitted bids and reserve prices $\mathbf{r}$, we solve the following optimization problem:

$$\max_{\{\beta_i : i \in [n], \beta_i \in \mathcal{B}\}} \sum_{t=1}^{T} \mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, \boldsymbol{\beta}) \;=\; \max_{\{\beta_i : i \in [n], \beta_i \in \mathcal{B}\}} \sum_{t=1}^{T} \max\left\{ \frac{b_{(2)}^t \beta_{(2)}^t}{\beta_{(1)}^t}, r_{(1)} \right\}, \tag{Data-Driven BSP}$$

where $\mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, \boldsymbol{\beta})$ is the revenue of the BSP auction in period $t$ when the submitted bids, reserve prices, and assigned boost values are respectively $\mathbf{b}^t$, $\mathbf{r}$, and $\boldsymbol{\beta}$. More precisely, $\mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, \boldsymbol{\beta})$ is the second-highest boosted bid divided by the boost value of the winner, i.e., the bidder with the highest boosted bid. Here, $b_{(i)}^t$, $r_{(i)}$, and $\beta_{(i)}$ are, respectively, the bid, reserve price, and boost value associated with the $i^{th}$ highest boosted bid in auction $t$. In addition, $\mathcal{B}$ is the feasible region for $\beta_i$, $i \in [n]$. We note that in Problem (Data-Driven BSP), we do not optimize over the reserve prices; we determine the optimal boost values given the reserve prices $\mathbf{r}$.

Theorem 1. *Problem (Data-Driven BSP) is NP-complete.*

*Proof of Theorem 1* To show the result, we consider an instance of this problem in which the feasible region, $\mathcal{B}$, is $\{1, \beta_H\}$ and the reserve prices are set to zero. We then reduce this instance of the problem to an edge bipartization problem, which is NP-complete (Guo et al. 2006). The edge bipartization problem is the problem of deleting as few edges as possible to make a graph bipartite.

Let $H$ and $L$ be constants such that $H > L$. Given a graph $G = (V, E)$, we map the edge bipartization problem on this graph to the following instance of Problem (Data-Driven BSP) with $|V|$ bidders and $2|E|$ auctions. Each vertex is a bidder, and each edge $e = (u, v)$ represents two auctions. In the first auction, the bids of bidders $u$ and $v$ are respectively $H$ and $L$, and the bids of other bidders are zero. In the second auction, the bids of bidders $u$ and $v$ are respectively $L$ and $H$, and the bids of other bidders are zero. Furthermore, we set $\beta_H$ to $\frac{H}{L}$.

In the following, we describe the structure of the optimal solution of Problem (Data-Driven BSP) for the described instance. Consider an edge $e = (u, v)$. If bidders $u$ and $v$ have the same boosts, then the expected revenue from the auctions associated with this edge will be $2L$. Now, if one of the bidders has a boost of $\frac{H}{L}$ and the other one has a boost of 1, the expected revenue from the auctions associated with this edge is $H + \frac{L^2}{H}$, which is always greater than $2L$. Now let us assign the boosts to the bidders arbitrarily. Let $S_H$ be the set of bidders/vertices with a boost of $\frac{H}{L}$, and let $S_L$ be the set of bidders/vertices with a boost of 1. Assume that there is no edge within set $S_L$ and that there is no edge within set $S_H$. That is, every edge connects a vertex in $S_L$ to a vertex in $S_H$. Then, graph $G$ is a bipartite graph whose partition has the parts $S_H$ and $S_L$. In that case, the total revenue is $|E| \times (H + \frac{L^2}{H})$, which is the maximum possible revenue. When there are $k$ edges within $S_L$ and $S_H$, i.e., there are $k$ edges that do not connect $S_H$ to $S_L$, then we have $k$ edges $(u, v)$ such that the boosts of $u$ and $v$ are the same. In that case, the revenue is $(|E| - k) \times (H + \frac{L^2}{H}) + k \times 2L$. Since $H + \frac{L^2}{H} < 2L$, we prefer to minimize the number of edges that do not connect $S_H$ and $S_L$. In other words, the best solution of Problem (Data-Driven BSP) is obtained by finding as few edges as possible to make graph $G$ bipartite, which is an NP-complete problem. $\square$

Due to this negative result, we will turn our attention to a special case of Problem (Data-Driven BSP), in which the feasible region $\mathcal{B} = \mathbb{R}^+$. We point out that this problem is still challenging to solve, as its objective function is discontinuous and non-convex in boost values and can have multiple local maxima. Nonetheless, we present an iterative algorithm, called *BSP Alternating Minimizer* (BSP-AM). In this algorithm, we successively optimize one of the boost values while fixing all other boost values (c.f. Beck (2017)). A similar approach has shown promise in other applications; see, for example, Nesterov (2012). As we will show in Theorem 2, the algorithm has the following good properties: (i) it is simple and easy to implement because each iteration of the algorithm can be solved effectively, and (ii) it converges to a coordinate maximum. A vector $\mathbf{x} \in \mathbb{R}^n$ is a coordinate-wise maximum of a function $G : \mathbb{R}^n \to \mathbb{R}$ if, for any $y \in \mathbb{R}$ and for any $i \in [n]$, $G(\mathbf{x} + y e_i) \leq G(\mathbf{x})$, where $e_i \in \mathbb{R}^n$ is the unit vector that is zero everywhere except in its $i$-th coordinate.

We now present the algorithm.

BSP Alternating Minimizer (BSP-AM)

Choose $\beta_i = 1$, $i \in [n]$, as our initial values.
Until convergence is reached or for some fixed number of iterations:
For $i \in [n]$, update $\beta_i$ to

$$\beta_i \;=\; \arg\max_{y \geq 0} \;\; \mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, (y, \boldsymbol{\beta}_{-i})). \tag{1}$$

In the algorithm, $\boldsymbol{\beta}_{-i}$ is the boost value of all the bidders except bidder $i$. Observe that at any step of the algorithm, the updated boost values increase the revenue of the BSP auction. Thus, when the algorithm terminates, it is guaranteed that the BSP auction outperforms the SP auction with the same vector of reserve prices.

In the following, we show that algorithm BSP-AM converges to a coordinate maximum. Our empirical study demonstrates that the algorithm converges after two or three iterations. We further show that the optimization problem in (1) can be solved effectively. To solve problem (1), we evaluate its objective function for $O(T)$ points and return the best one. We need a few definitions before presenting the result. Let $S^t = \{i : b_i^t \geq r_i\}$ be the set of bidders who clear their reserve price in auction $t \in [T]$. In addition, define $\tau_i = \{t \in [T] : b_i^t \geq r_i, |S^t| \geq 2\}$ as the set of all the auctions in which both $b_i^t \geq r_i$ and the number of bids that clear their reserve price is at least two.[5] Let $h_{-i}^t$ and $s_{-i}^t$, $t \in \tau_i$, be respectively the highest and second-highest[6] boosted bids of the bidders in set $S^t \backslash \{i\}$. Define $r_{-i}^*$ and $\beta_{-i}^*$ respectively as the reserve price and boost value associated with $h_{-i}^t$. Finally, let set

$$A_i^t \;=\; \left\{ \frac{h_{-i}^t}{b_i^t}, \frac{s_{-i}^t}{b_i^t}, \frac{h_{-i}^t}{r_i}, \frac{r_{-i}^* \beta_{-i}^*}{b_i^t} \right\}.$$

Theorem 2 **(BSP-AM Algorithm)**. *To compute the optimal solution of problem (1) in the BSP-AM algorithm, it suffices to evaluate its objective at the following points $\bigcup_{t \in \tau_i} A_i^t$ and return the best one. In addition, the BSP-AM algorithm converges to a coordinate maximum.*

[5] When there is only one bidder in an auction, assigning boosts does not change the allocation and payment of the auction.

[6] When there are only two bidders in set $S^t$, $t \in \tau_i$, then we set $s_{-i}^t$, to zero.

The proof of Theorem 2 is given in Section 7.1. As is clear in the proof of the theorem, $\mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, (y, \boldsymbol{\beta}_{-i}))$ can be discontinuous at points $y = \frac{h^t_{-i}}{b^t_i}, \frac{s^t_{-i}}{b^t_i}$ that are in set $A^t_i$, $t \in \tau_i$, $i \in [n]$. Because of that, in Theorem 2, we need to compute $\lim_{y \to (\frac{h^t_{-i}}{b^t_i})^+} \mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, (y, \boldsymbol{\beta}_{-i}))$ and $\lim_{y \to (\frac{h^t_{-i}}{b^t_i})^-} \mathrm{Rev}(\mathbf{b}^t, \mathbf{r}, (y, \boldsymbol{\beta}_{-i}))$ and, among $\{(\frac{h^t_{-i}}{b^t_i})^+, (\frac{h^t_{-i}}{b^t_i})^-\}$, choose the one that leads to higher revenue. A similar procedure should be used for point $y = \frac{s^t_{-i}}{b^t_i}$.

Next, we proceed to use/evaluate our algorithm with a real auction dataset.

## 4. Empirical Analysis of Data-driven BSP Auctions

Here, we seek to understand how much more revenue the BSP auction can earn, relative to the SP auction. To answer this question, we use bids submitted to a large advertising exchange over the course of one day. The bids in this dataset were submitted to one of the ad slots with the highest traffic volume. The data are provided at the level of an impression, which consists of a set of bidders who participate in an auction run for the impression. For each auction, we have access to the bids submitted by all bidders.

We divide our dataset randomly into training and test datasets that have roughly equal size. We use our training dataset to compute the reserve prices and boost values. We then evaluate the computed parameters on the test dataset.

**Boost values:** Here, we only optimize the boost values of the top bidders with the highest spending using the BSP-AM algorithm.[7] We note that the revenue gain of the BSP auction (with respect to the SP auction) stays roughly the same when we optimize the boost values of more than 8 top bidders. Thus, we consider the top $k = 2, 4, 6$, and 8 bidders and we optimize the boost values for these bidders, called *selected bidders*. After optimizing the boost value of a selected bidder, i.e., after solving problem (1) in the BSP-AM algorithm, we normalize the boost values of all the bidders, including the non-selected bidders, such that the minimum boost value is one; that is, without loss of generality, for any $i \in [n]$, we set $\beta_i$ to $\beta_i / \min_{i' \in [n]}\{\beta_{i'}\}$. Thus, the (normalized) boost values of the bidders can potentially change after updating the boost value of a selected bidder. Then, the algorithm stops at iteration $m$ when either $m$ reaches 200 or $\mathsf{Err} = \sum_{i \in [n]} (\beta^m_i - \beta^{m-1}_i)^2 \leq 0.01$. Here, $\beta^m_i$ is the boost value of bidder $i$ at the end of iteration $m$.[8]

**Reserve prices:** We set the reserve prices in two ways. In the first way, the reserve prices of all the bidders are set to zero in both SP and BSP auctions. In the second way, we set the reserves

[7] Optimizing the boost values of these bidders does not necessarily mean that we favor them in the allocation rule.

[8] In each iteration, we update the boost value of each selected bidder once.

to the monopoly prices, where the monopoly price for bidder $i$ solves the following optimization problem:

$$\bar{r}_i = \arg\max_r \Big\{ r \times \Pr[\text{submitted bid of bidder } i \text{ is greater than } r] \Big\}. \tag{2}$$

We use the training dataset to estimate the probability that bids submitted by bidder $i$ are greater than a certain reserve $r$. We point out that $\bar{r}_i$ is an optimal reserve price for bidder $i$ when either he is the only bidder who attends the SP auction or the bidders are all homogenous in terms of their valuation distribution.[9]

Table 1 summarizes the result of our analysis. The first and second columns represent the number of selected bidders and the reserve price, respectively, used in the BSP and SP auctions. As stated earlier, to perform a fair comparison, we use the same reserve prices in the SP and BSP auctions. The third and fourth columns depict the Err and the number of iterations when the algorithm stops. We observe that the algorithm converges after two or three iterations and that, when it stops, Err is very small. Finally, the last column of the table presents two numbers in the form of $(a, b)$, where $a$ is the average gain of the BSP auction (relative to the SP auction) and $b$ is the standard error of the average gain in the test dataset. To compute these numbers, we sample the test dataset 200 times with a rate of 5%.[10] We then evaluate the revenue gain of the BSP auction in each sampled dataset and take the average.

We now discuss the revenue gain of the BSP auction. When the monopoly reserve prices are used, the revenue gain of the BSP auction can be as low as 2.43% and as high as 6.75%. Overall, the revenue gain increases as the number of selected bidders increases. However, the highest jump in the revenue gain happens when we increase the number of selected bidders from 2 to 4. When increasing the number of selected bidders from 4 to 8, the jump in the revenue gain is non-significant (is less than 0.3%). When the reserve prices are zero, the revenue gain spans from 16.55% to 29.28%. Observe that the revenue gain under zero reserve is much larger than that under the monopoly reserve. This is the case because, with zero reserve, there is more room for improvement. Put differently, when the reserve prices are not set appropriately, then the revenue gain of the BSP auction is higher.

**Robustness:** Here, we investigate to what extent the revenue gain of the BSP auction is sensitive to the choice of the top bidders. With this aim, we allow the BSP auction to have some flexibility in choosing its selected bidders: the BSP auction randomly chooses its selected bidders from a

[9] Another way to calculate reserve prices is to optimize them jointly based on the set of bidders who participate in each auction. This approach requires more computation power. In fact, Paes Leme et al. (2016) show that optimizing reserve prices jointly is an NP-complete problem. Thus, here, we focus on monopoly prices. Recall that Hartline and Roughgarden (2009) show that SP auctions with monopoly prices receive at least 50% of the optimal revenue.

[10] We note that 5% of the test dataset includes the submitted bids in more than 9000 auctions.

| # of selected bidders | Reserve price | Err | # of iterations | Average revenue gain & its standard error in test (in %) |
|---|---|---|---|---|
| 2 | Monopoly | 4e-05 | 3 | (2.43, 0.01) |
| 4 | | 1e-04 | 2 | (6.48, 0.05) |
| 6 | | 5e-04 | 3 | (6.65, 0.04) |
| 8 | | 1e-03 | 3 | (6.75, 0.04) |
| 2 | Zero | 1e-04 | 2 | (16.55, 0.08) |
| 4 | | 1e-04 | 3 | (18.91, 0.11) |
| 6 | | 6e-03 | 2 | (28.93, 0.14) |
| 8 | | 1e-03 | 3 | (29.28, 0.10) |

**Table 1 BSP auction versus SP auction**

pool of the top 15 bidders with the highest spending and then employs the BSP-AM algorithm to optimize the boost values of the selected bidders. Here, we focus on the case where the number of selected bidders is 4. This is motivated by Table 1, where we show that by updating the boost values of the 4 top bidders, we can obtain a high fraction (more than 90%) of the revenue gain of the BSP auction with monopoly prices.[11] Finally, we restrict our attention to the BSP auction with monopoly reserve prices, as (i) in practice, reserve prices are set to extract more revenue from the bidders (Paes Leme et al. 2016, Ostrovsky and Schwarz 2011), and (ii) the revenue gain of the BSP auction is smaller when the reserve prices are set.

We observe that the average revenue gain of the BSP auction (relative to the SP auction) is 5.1%[12], where the average is taken over 300 samples. Here, each sample corresponds to a set of 4 bidders (with updated boost values) that has been randomly selected from the top 15 bidders. We note that the average revenue gain is consistent with the result presented in the table.

**Who is favored?** Next, we would like to provide additional insight into the optimized boost values. We are interested in learning what set of bidders is favored by receiving higher boost values in the BSP auction and how the boost values change. To do so, we use our 300 samples, and for each sample, we choose one of its selected bidders randomly. Let us call him bidder $i$. Then, we regress the log of the boost value of bidder $i$, i.e., $\log(\beta_i)$, as follows,

$$\log(\beta_i) \;=\; c_0 + c_1.CV + c_2.\mathrm{STD} + c_3.\mathrm{Scale} + c_4.\mathrm{STD}_{\max} + c_5.\mathrm{STD}_{\min}\,, \tag{3}$$

where CV and STD are the coefficient of variation and standard deviation, respectively, of the submitted bids of bidder $i$. Scale is defined as $\log(\frac{\beta_{\max}}{\beta_{\min}})$, where $\beta_{\max} = \max_{j\neq i, j\in \mathsf{Sel}}\{\beta_j\}$ and $\beta_{\min} = \min_{j\neq i, j\in \mathsf{Sel}}\{\beta_j\}$. Here, $\mathsf{Sel}$ is the set of selected bidders. We consider $\log(\frac{\beta_{\max}}{\beta_{\min}})$ to account for the

[11] By updating the boost values of the top 15 bidders, the BSP auction with the monopoly reserves outperforms the SP auction by 6.51%. We obtain a similar result for a higher number of selected bidders.

[12] The standard error of the average revenue gain is 0.25%.

normalization factor. Recall that the revenue of the BSP auction stays the same as we scale all the boost values by a constant factor. $\text{STD}_{\max}$ and $\text{STD}_{\min}$ are the standard deviations of the submitted bids of $\beta_{\max}$ and $\beta_{\min}$, respectively.

| | Coefficient ($c_i$'s) | t value | p value |
|---|---|---|---|
| Intercept | 0.98 | 5.46 | 1.03e-07 *** |
| CV | -0.16 | -7.01 | 1.64e-11 *** |
| STD | -1.07 | -5.57 | 5.70e-8 *** |
| Scale | 0.25 | 4.54 | 7.98e-6 *** |
| $\text{STD}_{\max}$ | 1.49 | 5.05 | 7.60e-7 *** |
| $\text{STD}_{\min}$ | -0.04 | -0.26 | 0.798 |

**Table 2 Result of the regressing the log of a boost value on his CV, STD, Scale, $\text{STD}_{\max}$, and $\text{STD}_{\min}$. Here, $R^2$ and the adjusted $R^2$ are, respectively, $0.29$ and $0.28$. Significance codes: 0 '***', 0.001 '**' , 0.01 '*'.**

The result of the regression is provided in Table 2. All the considered parameters in the regression are significant, except $\text{STD}_{\min}$. We point out that as the volatility (CV and STD) in the submitted bids of bidders increases, the boost values decrease. Thus, the BSP auction favors the bidders with more stable bidding behavior. As another observation, when the standard deviation of a selected bidder with the highest boost value increases, the boost value of the bidder goes up. Put differently, a bidder is favored more when there is higher variation in the submitted bids of the other bidders.

Next, we study the BSP auction in a standard model. This enables us to strengthen our intuition for the BSP auction.

## 5. BSP Auctions in a Classic Bayesian Model

In this section, we connect our insights from our empirical analysis to the optimal auction design framework and present a procedure to determine boost values in a way that is provably revenue improving compared with the SP auction. We consider the following standard model. Assume that there are $n$ bidders who participate in an auction for a single good valued at zero by the seller. In the context of online advertising, the seller (auctioneer) is an advertising exchange, bidders are advertisers, and the good is an advertisement opportunity (impression). Bidders are risk neutral. The valuation of a bidder $i$, which is denoted by $v_i$, is drawn from distribution $F_i : [\underline{v}_i, \bar{v}_i] \to [0,1]$. The valuations of bidders are private and are independent of each other. However, distributions $F_1, F_2, \ldots, F_n$ are public information. We assume that the distribution $F_i$, $i \in [n]$ has density $f_i : [\underline{v}_i, \bar{v}_i] \to \mathbb{R}^+$.

We denote the IHR associated with distribution $F_i$ by $\alpha_i : [\underline{v}_i, \bar{v}_i] \to \mathbb{R}^+$ and define it as $\alpha_i(x) = \frac{1-F_i(x)}{f_i(x)}$. Furthermore, we denote the virtual value associated with distribution $F_i$ by $\phi_i : [\underline{v}_i, \bar{v}_i] \to \mathbb{R}$, and define it as $\phi_i(x) = x - \alpha_i(x)$.

Next, we present and discuss the optimal mechanism in this setting when the valuation distribution $F_i$, $i \in [n]$, is regular.[13] The discussion can help us provide insight into the BSP auction.

**Optimal Mechanism:** When bidders are symmetric, that is, $F_i = F$ for any $i \in [n]$, the optimal mechanism is an SP auction with a reserve price $r$ where $r$ solves $r = \phi^{-1}(0)$ and $\phi(x) = x - \frac{1-F(x)}{f(x)}$. In this case, the optimal mechanism depends on the distribution of valuations only via the reserve price $r$. This is so because, in the SP auction, the good is allocated to the bidder with the highest submitted bid.

When the bidders are not symmetric, SP auctions cannot earn the maximum revenue. In the optimal mechanism, for each bidder $i$ whose submitted bid $b_i$ is greater than his reserve price $r_i$, a virtual bid $\phi_i(b_i) = b_i - \frac{1-F_i(b_i)}{f_i(b_i)}$ is calculated, where $r_i = \phi_i^{-1}(0)$. Then, the good is allocated to bidder $i^*$ with the highest non-negative virtual bid, and bidder $i^*$ pays $\max\{r_{i^*}, \phi_{i^*}^{-1}(\phi_j(b_j))\}$, where $j = \arg\max_{i \neq i^*}\{\phi_i(b_i)\}$ is the bidder with the second-highest virtual bid (Myerson 1981). [14]

We note that in the optimal mechanism, bids are first transformed into virtual bids and then bidders compete on this basis. That is, the bidder with the highest virtual bid wins the auction, and his payment is the minimum bid with which he can win the auction.[15] In addition, observe that in the optimal mechanism, if all bidders who clear their reserve prices submit the same bid, the one with the lowest IHR will get the item. This is the case because the virtual bid/value of a bidder is his bid/value minus his IHR at the submitted bid. Inspired by this observation, in the following, we empirically investigate the IHR of the bidders and its relationship with the boost values. This investigation helps us pin down a property of the IHR of the bidders. We will use this property later to provide a boosting rule that helps the seller extract a high fraction of the optimal revenue.

We start with revisiting our empirical study. As an example, we consider two subsets of four bidders. The four bidders in each subset are among the top 15 bidders and will be used as selected bidders in the BSP auction. In Figure 2a, we present the (estimated) IHR of the submitted bids of the bidders and their boost values. To estimate the IHR, we first fit the distribution of the submitted bids with a lognormal distribution. The lognormal distribution is a good fit for these bidders in the sense that the mean squared error between the empirical and the estimated (lognormal)

[13] A distribution $F_i$ is regular when $\alpha_i(\cdot)$ is increasing.

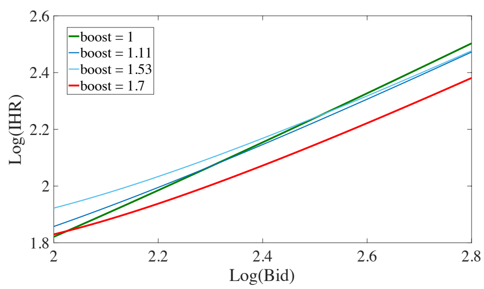
[14] In Section 6, we discuss the main difficulties in implementing the optimal auction.

[15] This implies that, in the optimal mechanism, the seller must calculate/estimate the virtual bids of bidders, where the virtual bid depends on the probability distributions, $F_i(\cdot)$'s, and the probability density functions, $f_i(\cdot)$'s. In fact, the estimation of virtual bids is very sensitive to estimation errors in the probability density functions. This sensitivity with respect to the assumed structure of the environment makes this mechanism less attractive. We note that the BSP auction does not suffer from this drawback because it has few parameters per bidder, and as we show in our empirical study, the parameters can be efficiently learned using historical data.

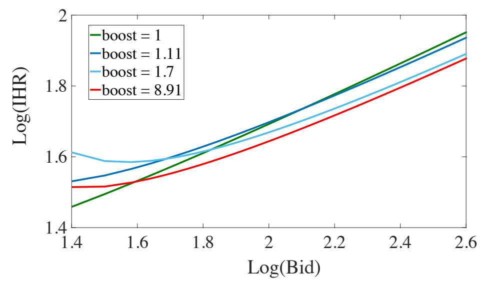
distributions is non-significant.[16] Here, we assume that bidders submit true values when they participate in the auctions. This is a reasonable assumption because in this advertising exchange, SP auctions are run, and these auctions are truthful mechanisms.[17]

Figure 2 shows that bidders with higher IHR values tend to get lower boost values. We will formalize this intuition in Section 5.1. As another observation, after some (bid) value, roughly speaking, one can find a consistent ranking for the bidders based on their IHR. This important property, which is summarized below, is also observed for other subsets of bidders.

**Property 1** *There exists $\rho > 0$ and the ordering $(h_1, h_2, \ldots, h_n)$ such that $\alpha_{h_1}(x) \leq \alpha_{h_2}(x) \leq \ldots \leq \alpha_{h_n}(x)$ for any $x \geq \rho$.*




(a) Subset 1




(b) Subset 2

**Figure 2** **Log of the (estimated) IHR versus log of bids when a subset of (four) top bidders is selected in the BSP auction. For confidentiality, the x and y values in this figure are scaled.**

The above property is related to hazard rate ordering (Shaked and Shanthikumar 1994). The random variable $X$ is said to be larger than the random variable $Y$ in the hazard rate ordering (denoted by $X \geq_h Y$) if $\frac{f_X(v)}{1-F_X(v)} \leq \frac{f_Y(v)}{1-F_Y(v)}$ for any $v \in \mathbb{R}$ where $f_Z$ and $F_Z$, $Z \in \{X, Y\}$ are respectively the probability density and probability distribution of random variable $Z$ and where $\frac{f_Z(v)}{1-F_Z(v)} = \frac{1}{\alpha_Z(v)}$ is the hazard rate of the random variable $Z$. Thus, when $X \geq_h Y$, we have $\alpha_X(v) \geq \alpha_Y(v)$. Then, when $\rho = \min_{i\in[n]}\{\underline{v}_i\}$, Property 1 implies that $v_{h_1} \leq_h v_{h_2} \leq_h \ldots \leq_h v_{h_n}$. Note that if $v_i \leq_h v_j$, then $\frac{1-F_j(v)}{1-F_i(v)}$ is increasing in $v \in [\min\{\underline{v}_i, \underline{v}_j\}, \max\{\bar{v}_i, \bar{v}_j\}]$. From this, it follows that when $v_i \leq_h v_j$, we have $\underline{v}_i \leq \underline{v}_j$ and $\bar{v}_i \leq \bar{v}_j$, where $\bar{v}_i$ and $\underline{v}_i$ are respectively the maximum and minimum values that variable $v_i$ takes. In addition, by definition, given that $\bar{v}_i \leq \underline{v}_j$, we have $v_i \leq_h v_j$.

Hazard rate ordering is weaker than *likelihood ordering* and stronger than *stochastic ordering* (Shaked and Shanthikumar 1994). The random variable $X$ is said to be larger than the random

[16] The mean squared error between the empirical and estimated distributions, denoted by $F$ and $\hat{F}$, respectively, is given by $\frac{1}{1000}\sum_{i\in[1000]}(F(x_i) - \hat{F}(x_i))^2$ where $\{x_i, i \in [1000]\}$ is the set of 1000 equally spaced points that cover the entire range of empirical distribution $F$. For example, for the bidder, shown in Figure 2a, with boost values of $1, 1.11, 1.53$, and $1.7$, the mean squared error is 0.0005, 0.0003, 0.0013, and 0.0015, respectively.

[17] Here, we assume that bidders are not budget constrained.

variable $Y$ in likelihood ordering (denoted by $X \geq_l Y$) if $\frac{f_X(v)}{f_Y(v)} \leq \frac{f_X(v')}{f_Y(v')}$ for any $v < v'$, $v, v' \in \mathbb{R}$. Note that if $X \geq_l Y$ (and assuming that $F_X \neq F_Y$), then it follows that $f_X$ and $f_Y$ cross exactly once. In addition, given that $X \geq_l Y$, we have $X \geq_h Y$. The random variable $X$ is said to be larger than the random variable $Y$ in stochastic ordering (denoted by $X \geq_s Y$) if $F_X(v) \leq F_Y(v)$ for any $v \in \mathbb{R}$. Then, given that $X \geq_h Y$, we have $X \geq_s Y$.

In Appendix 8.1, we further discuss Property 1 by providing several examples. We show that random variables that have higher IHR values tend to have a higher average and variance.

### 5.1. Insights into Boost Values

In this section, we theoretically justify why bidders with a lower IHR should be favored in the BSP auction. With this aim, we present a boosting rule called *conservative boosting* that assigns higher boost values to bidders with a lower IHR. We show that the BSP auction with conservative boost values yields more revenue than the SP auction and earns at least a constant fraction of the optimal revenue and that this fraction is always greater than $\frac{1}{2}$ if we can assign different boost values to different bidders. This verifies that the BSP auction even outperforms the SP auction in terms of its worst-case performance. Note that Hartline and Roughgarden (2009) show that the SP auction with monopoly prices obtains at least 50% of the optimal revenue.

Throughout this section, we assume that Property 1 holds. This is motivated by our empirical analysis where we show that top bidders can be ordered based on their IHR; see Figure 2. Further, roughly speaking, this property implies that there is a bidder spectrum ranging from brand to retargeting. To see why note that higher IHR implies higher average and variance, and as we observed in Figures 1a and 1b, the average and standard deviation of submitted bids of retargeting bidders are higher than those of brand bidders.

We now present the conservative rule. In this boosting rule, boosts are chosen such that the BSP auction makes fewer misallocation mistakes than the SP auction. That is, this boosting rule guarantees that when the BSP auction misallocates the good, the SP auction also misallocates the good. We say that the good is misallocated in an auction if the winner of the auction is not the same as the winner of the optimal mechanism.

We now present our boosting rule. *We note that the main purposes of this boosting rule are to provide theoretical support for our insights from the empirical results and to show that under a standard model, one can design the BSP auction in a manner such that it strictly outperforms the SP auction.*

For simplicity, we assume that $r_i = \bar{r}_i$, $i \in [n]$.[18] Let $l_{i,j}(\frac{\beta_i}{\beta_j}) = \max\{\bar{r}_i \frac{\beta_i}{\beta_j}, \bar{r}_j\}$ and $u_{i,j}(\frac{\beta_i}{\beta_j}) = \min\{\bar{v}_j, \bar{v}_i \frac{\beta_i}{\beta_j}\}$. Assume that bidders can be ordered based on their IHR such that bidder $j = 2, \ldots, n$

[18] Most of the results in this section still hold when $r_i \neq \bar{r}_i$.

has a higher IHR than bidder $j-1$; see Property 1 for details. Then, with conservative boosting, for any $i<j$, we must satisfy the following equation:[19]

$$\beta_i \times \frac{1}{x}\phi_i^{-1}(\phi_j(x)) \ \leq \ \beta_j \ \leq \ \beta_i, \quad x \in \left[ l_{i,j}(\frac{\beta_i}{\beta_j}), u_{i,j}(\frac{\beta_i}{\beta_j}) \right], \tag{4}$$

where $\phi_i^{-1}(\cdot)$ is the inverse function of $\phi_i(\cdot)$. [20] Note that the support for $x$ in the above inequality depends on $\frac{\beta_i}{\beta_j}$. Thus, at first glance, it may not be obvious how to determine a range for $\frac{\beta_i}{\beta_j}$ using the above inequality. In Appendix 8.2, we provide a simple algorithm that uses the above inequality to find upper bounds for $\frac{\beta_i}{\beta_j}$. We then present a sequential procedure that uses these upper bounds to return the maximum allowable boost value for each bidder. In this procedure, we determine the boost (maximum boost value) of bidder $i$ after determining the boost values of bidders $i+1, i+2, \ldots, n$. That is, the boost value of a bidder does not depend on the boost values of other bidders whose IHRs are less than his.

Next, we provide insights into Eq. (4). Consider $x \in \left[ l_{i,j}(\frac{\beta_i}{\beta_j}), u_{i,j}(\frac{\beta_i}{\beta_j}) \right]$, and let us transfer $x$ to the domain of bidder $i$'s valuation as follows: $x \mapsto \phi_i^{-1}(\phi_j(x))$. We note that the virtual value of bidder $j$ at point $x$ and the virtual value of bidder $i$ at point $\phi_i^{-1}(\phi_j(x))$ are both equal to $\phi_j(x)$. Thus, the optimal mechanism, which compares the virtual values of bidders for allocating the good, is indifferent to the allocation of the good to bidder $i$ with valuation $\phi_i^{-1}(\phi_j(x))$ or to bidder $j$ with valuation $x$. Now Eq. (4) ensures that the boost value of bidder $i$, relative to that of bidder $j$, is not set too high, such that the BSP auction prefers bidder $i$ to bidder $j$.

There is an alternative way to think about conservative boosting. Let $K$ be the winner of the optimal mechanism, and let $J$ be the winner of the BSP auction. The lower bound in Eq. (4) ensures that whenever the BSP auction allocates the good to bidder $J>K$ rather than bidder $K$, the SP auction also misallocates the good. Similarly, the upper bound guarantees that whenever the BSP auction allocates the good to bidder $J<K$ rather than bidder $K$, the SP auction does not properly allocate the good. For more details, see Lemma 2 in the appendix.

We note that the winner of the SP auction, denoted by $I$, is always greater than or equal to $K$. This implies that the winner of the BSP auction with conservative boosting is also greater than or equal to $K$. As a result, bidders with higher IHRs have a greater chance of winning in the BSP auction than in the optimal mechanism. To further clarify, assume that there are only two bidders in the auction. When the bidder with the highest IHR wins in the optimal mechanism, the winners of the SP auction and the BSP auction with conservative boosting are the same bidder. In contrast, when the bidder with the lowest IHR wins in the optimal auction, the BSP and SP auctions may

[19] Without loss of generality, one can set $\beta_n$ to one.

[20] When this equation cannot be satisfied, we set $\beta_j$ equal to $\beta_i$. Furthermore, if $\phi_i^{-1}(\cdot)$ is not unique, we choose the largest one.

not choose the same bidders as their winners. However, in the BSP auction, this bidder wins the good with a higher probability (relative to the SP auction). That is, the BSP allocation rule is closer to that of the optimal mechanism.

The following theorem shows that the BSP auction with conservative boosting outperforms the SP auction in terms of the obtained revenue.

THEOREM 3 **(BSP Auction with Conservative Boosting)**. *Suppose that Property 1 and Eq. (4) hold. Then, for any vector of reserve prices $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ such that $r_{(2)} \geq \rho$, we have $R_{bsp}(\boldsymbol{r}, \boldsymbol{\beta}) \geq R_{sp}(\boldsymbol{r})$, where $r_{(2)}$ is the second smallest reserve among $(r_1, r_2, \ldots, r_n)$, and $R_{bsp}(\boldsymbol{r}, \boldsymbol{\beta})$ is the revenue of the BSP auction with boosts $\boldsymbol{\beta}$ and reserve prices $\boldsymbol{r}$, and $R_{sp}(\boldsymbol{r})$ is the revenue of the SP auction with reserve prices $\boldsymbol{r}$. In addition, for heterogeneous bidders—when the allocation rule of the BSP and SP auctions differs for a set of valuations with positive probability mass—the revenue of the BSP auction is strictly greater than that of the SP auction.*

Next, we compare the revenue of the BSP auction with that of the optimal mechanism.

THEOREM 4 **(Worst Case Performance of BSP Auction)**. *Suppose that Property 1 holds, $\phi_i(\cdot)$ is non-decreasing for $i \in [n]$, and the second smallest monopoly reserve price is greater than $\rho$. Then, if Eq. (4) is satisfied or $J \geq K$ for any realization of valuations, we have*

$$R_{opt} \leq R_{bsp}(\bar{\boldsymbol{r}}, \boldsymbol{\beta}) \left(1 + \max_{j \in [n-1]} \left\{\frac{\beta_{j+1}}{\beta_j}\right\}\right),$$

*where $R_{opt}$ is the optimal revenue, $K$ is the winner of the optimal mechanism, and $J$ is the winner of the BSP auction.*

Theorem 4 compares the optimal mechanism and the BSP auction. Before discussing this theorem, we note that, in Appendix 8.3, we compare these mechanisms in a different way: we present parametric upper bounds on the revenue losses of BSP (with conservative boosting) and SP auctions, where the revenue loss is computed against the optimal revenue. These upper bounds highlight when BSP auctions perform better. Specifically, we show that the revenue loss of the BSP auction is small when the estimation errors of the virtual bids of each pair of bidders are close to each other. Here, we consider the boosted bid of a bidder to be a linear estimation of his virtual bid. We establish a similar bound for the SP auction and show that under conservative boosting, the maximum revenue loss of the BSP auction is less than that of the SP auction. We note that the bounds on the maximum revenue losses of the BSP and SP auctions are tight.

We now discuss Theorem 4. The theorem shows that the BSP auction earns at least

$$\frac{1}{\left(1 + \max_{j \in [n-1]} \left\{\frac{\beta_{j+1}}{\beta_j}\right\}\right)} \geq \frac{1}{2}$$

fraction of the optimal revenue when Eq. (4) holds or $J \geq K$. As mentioned earlier, when Eq. (4) holds, we have $J \geq K$. However, in Theorem 4, we explicitly mention the condition of $J \geq K$ because, in some cases, it is easier to verify this condition. Observe that the worst case performance of the BSP auction is governed by a pair of neighboring bidders, i.e., $(j, j+1)$, whose IHR ordering is closest. Furthermore, as bidders become more heterogeneous in such a way that the seller can assign them more distinctive boost values, the BSP auction performs better. This result shows that the BSP auction performs better than the SP auction in the worst case. Note that Hartline and Roughgarden (2009) find that the SP auction receives at least 50% of the optimal revenue. These authors show that this result is tight in an example in which there are two bidders and the valuation of one of the bidders is drawn from the equal-revenue distribution. In the following, we revisit this example to show that the bound presented in Theorem 4 is tight.

Example 1 (The bound is Tight). Assume that there are two bidders. The valuation of the first bidder is always 1, and the valuation of the second bidder is drawn from $F_2(x) = 1 - \frac{1}{x}$ for any $x \geq 1$. Note that for distribution $F_2$, the virtual value is $\phi_2(x) = 0$. Then, the monopoly price for both bidders is 1.[21]

The revenue of the SP auction with monopoly prices is $\Pr[v_2 > 1] \times 1 + \Pr[v_2 \leq 1] \times 1 = 1$. Now, consider the BSP auction with monopoly prices that assigns boost $\beta_1$ to bidder 1. Then, the BSP auction allocates the good to the second bidder when his valuation is higher than $\beta_1$, and allocates the good to the first bidder otherwise. This auction earns the expected revenue of $\beta_1 \times \frac{1}{\beta_1} + (1 - \frac{1}{\beta_1}) = 2 - \frac{1}{\beta_1}$. Note that as $\beta_1$ goes to infinity, the BSP auction yields revenue of 2, which is the optimal revenue. In addition, for this example, we have $J \geq K$ for any $\beta_1 \geq 1$. This verifies that the bound in Theorem 4 is tight. $\square$

We next evaluate the BSP auction with conservative boosting using our dataset.

### 5.2. Evaluating Conservative Boosting Empirically

In this section, we evaluate the conservative boosting rule using our dataset; see the description of the dataset in Section 4. We further compare conservative boosting, which is inspired by theory, with our data-driven approach. Overall, we show that the data-driven boosting surpasses the theory-driven boosting, as (i) it relies on fewer assumptions, and as a result, is more robust, and (ii) by considering all the submitted bids in its optimization problems, it can better capture the patterns and potential correlation in the bids submitted to auctions. Further, not surprisingly, we observe that the gap between theory-driven and data-driven approaches gets wider when the assumptions under which the theoretical results hold are more violated.

[21] The distribution of the second bidder can be perturbed so that his monopoly price of 1 is uniquely optimal.

Similar to Section 4, we only optimize the boost values of a subset of the top bidders. As before, we call these bidders "selected bidders." In particular, for our selected bidders, we consider the two subsets of buyers with size 4, which are shown in Figure 2. We then determine the boost values of these bidders via conservative boosting. As stated earlier, for these bidders, the lognormal distribution is a good fit for their bid distributions. This allows us to apply the conservative boosting rule to these bidders. We note that unlike our data-driven approach, the conservative boosting rule needs to have access to a good and reliable estimation of the probability density of submitted bids. Such estimation may not always be available. Nonetheless, here, to assess the conservative boosting rule, we focus on selected bidders for whom we have a good estimate of bid distributions.

We use our training dataset to fit the bid distribution of selected bidders with (truncated) lognormal distributions. [22] We then use the estimated bid distributions to estimate the IHR of the selected bidders. Using the estimated IHR, we order the selected bidders such that bidder $i$ has a higher IHR than bidder $i-1$. We then compute the upper bound on $\frac{\beta_i}{\beta_j}$ for any $j > i,\ i,j \in \mathsf{Sel}$, where $\mathsf{Sel}$ is the set of selected bidders. Specifically, we apply the procedure in Appendix 8.2.

After determining the boost values of the selected bidders, to perform a fair comparison with the data-driven approach, we further choose/optimize the boost value of the unselected buyers using our training dataset.[23] For reserve prices, we use the monopoly reserve prices, computed using our training dataset. Having learned the boost values and reserve prices from the training dataset, we next proceed to compare conservative boosting with the data-driven approach using our test dataset. [24]

We observe that the BSP auctions with data-driven and theory-driven boosting rules outperform the SP auction. However, in terms of the revenue gain over the SP auction, the data-driven approach outperforms conservative boosting by 42% and 6% when the set of selected bidders is equal to subset 1 or 2, respectively. The better performance of conservative boosting under subset 2 is due to the fact that bidders in this subset can be better ordered based on their IHR; see Figures 2a and 2b.[25]

[22] The truncated lognormal distribution is a probability distribution of a lognormally distributed random variable whose value is bounded below. Suppose $X \sim \text{Logn}(\mu, \sigma)$. Then, $X$ conditioned on $X < L$ has a truncated lognormal distribution. We denote the distribution of truncated $X$ with $\text{Logn}(\mu, \sigma, L)$. When we estimate the bid distribution of a bidder, we set $L$ to his maximum submitted bid.

[23] Recall that all the unselected buyers get the same boost value. Thus, optimizing the boost value of unselected buyers is a one-dimensional optimization problem and can be solved effectively. We point out that in our data-driven approach, even if we optimize the boost value of the selected bidders, we incorporate the submitted bids of unselected bidders in our optimization problems.

[24] To do so, we sample the test dataset 200 times with a rate of 5%. We then compare the two boosting rules in each sampled dataset and take the average over the 200 sampled datasets.

[25] That is, under subset 2, Property 1 holds for smaller value of $\rho$ and by Theorem 3, conservative boosting performs well when $\rho$ is small enough. Nevertheless, one can still apply conservative boosting even if $\rho$ is not small.

## 6. Discussion and Conclusion

In this work, we presented a simple and effective auction format called the BSP auction that embraces the heterogeneity of bidders. We combined theory and empirics to show that BSP auctions improve revenue relative to widely used SP auctions by favoring bidders with more stable bidding behavior. Our empirical studies also showed that by learning the parameters of the BSP auction using historical data, the BSP auction beats the SP auction by up to 29%.

In the following, we discuss some important aspects of the BSP auction and our data-driven approach.

**Parsimonious Auction Format:** The BSP auction is parsimonious auction format in the sense that it requires at most two parameters per bidder, and as shown in our empirical studies, these parameters can be learned effectively. This highlights the fact the BSP auction is easy-to-implement, a property that the optimal mechanism lacks. Implementing the optimal auction requires calculating virtual bid, $b - \frac{1-F_i(b)}{f_i(b)}$, for bidders, where the virtual bids are very sensitive to the estimation errors in density functions since these functions appear in the denominator. Further, estimated density functions may not be regular (Celis et al. 2014). With non-regular distributions, ironing is necessary and allocation would be randomized, which is not practically plausible. To sum up, comparing with BSP, which only needs to learn two parameters per bidder, the optimal mechanism needs accurate estimations of the probability density “function” per bidder.

**Data-driven Approach:** We have shown that our data-driven approach that learns/tunes the parameters of the BSP auction using historical data performs well. Generally, learning from historical data has proven to be an effective tool to optimize the parameters of other types of auctions, including SP auctions (Dhangwatnotai et al. 2015, Balcan et al. 2017, Paes Leme et al. 2016, Ostrovsky and Schwarz 2011). However, this technique can have a drawback: it can incentivize the bidders to change their bidding behavior to game the learning algorithm of the seller. For instance, in the SP auctions, the bidder may have the incentive to shade their bids in order to enjoy lower reserve prices in the future. In fact, the problem of mitigating the negative impact of strategic behavior of bidders even in the SP auction is still an open problem. Empirical work that studied reserve price optimization in the SP auctions have not considered this problem (Paes Leme et al. 2016, Ostrovsky and Schwarz 2011). Further, only recently, some theoretical papers have made some progress in this direction under some assumptions; see, for example, Golrezaei et al. (2018), Kanoria and Nazerzadeh (2017), Mahdian et al. (2017), Amin et al. (2013, 2014). It would be interesting to study how strategic bidders change their behavior to game the learning algorithm of the BSP auctions and how this behavior impacts the seller. Since the BSP auction favors bidders with more stable bidding behavior, the strategic bidders may have the incentive to stabilize their bids. In contrast to shading, which always reduces revenue, in some cases, stabilizing bids can help the seller gain more revenue. We demonstrate this point in the following example.

Assume that there are two bidders whose valuations are drawn from the uniform distribution in the range of $[0, 1]$. These bidders participate in a BSP auction with no reserve and with optimized boost values. Suppose that bidder one is strategic and aims at stabilizing his bidding strategy. In particular, he (linearly) transforms his bids to the range of $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, where $\delta \in [0, \frac{1}{2}]$. Meanwhile, the second bidder is truthful and submits his true value. Note that the optimal boost values, as well as the utility of bidders, depend on the value of $\delta$. Then, the strategic bidder chooses the utility-maximizing value for $\delta$, which is around 0.28. By doing this, the utility of the strategic bidder goes up by approximately 10% relative to his utility under truthful bidding. More interestingly, the seller's revenue also improves by around 10%.

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# Appendix

## 7. Proofs of Statements

### 7.1. Proof of Theorem 2

We first show that problem (1) can be solved effectively. We then show that the BSP-AM algorithm converges to a coordinate maximum.

Observe that at any auction $t \notin \tau_i$, either bidder $i$ does not clear his reserve price, or only one bidder clears his reserve price. In this case, changing the boost value of bidder $i$ cannot impact the revenue of the BSP auction, i.e., the objective function of problem (1). Thus, in the following, we will focus on all the auctions in set $\tau_i$. The revenue of the BSP auction in these auctions as a function of boost of bidder $i$, $y$, can be written as

$$\sum_{t\in\tau_i}\Bigg[\mathbb{I}\{b_i^t y \ \geq\ h_{-i}^t\}\max\left\{\frac{h_{-i}^t}{y}, r_i\right\} + \mathbb{I}\{h_{-i}^t \ >\ b_i^t y\ \geq\ s_{-i}^t\}\max\left\{\frac{b_i^t y}{\beta_{-i}^*}, r_{-i}^*\right\}$$
$$+\,\mathbb{I}\{s_{-i}^t\ >\ b_i^t y\}\max\left\{\frac{s_{-i}^t}{\beta_{-i}^*}, r_{-i}^*\right\}\Bigg], \tag{5}$$

where $h_{-i}^t$ and $s_{-i}^t$ are respectively the highest and second highest boosted bid of the bidders in set $S^t\backslash\{i\}$. Recall that $S^t$ is the set of bidders that clear their reserve prices. Also, $\mathbb{I}\{X\}$ is one when event $X$ happens and zero otherwise. Note that in the first term, bidder $i$ wins the auction and he pays $\max\left\{\frac{h_{-i}^t}{y}, r_i\right\}$. In the second and third terms, he does not win. In the second term, bidder $i$ has the second highest boost bid and because of this, his boost value influences the revenue of the BSP auction. Whereas, in the third term, the boost value of bidder $i$ does not have any impact on the revenue of the BSP auction.

In the following, we argue that the revenue of the BSP auction, given in (5), is piecewise convex in $y$; that is we can divide the interval $[0,\infty)$ into $M+1$ disjoint subintervals such that (5) is differentiable and convex in $y$ in any subinterval. Here, $M$ is the cardinality of set $\bigcup_{t\in\tau_i} A_i^t$. In particular, let $\bigcup_{t\in\tau_i} A_i^t = \{a_1, a_2, \ldots, a_M\}$ where $a_1 < a_2 < \ldots < a_M$. Then, the first and last subintervals are $(0, a_1)$ and $(a_M, \infty)$, respectively, and $i$-th subinterval is $(a_{i-1}, a_i)$. The fact that that (5) is convex in each subinterval implies that to solve problem (1), it suffices to evaluate its objective at $a_1, \ldots, a_M$, and return the best one.

It is easy to observe that the expression in (5) is continuous and differentiable in any aforementioned subinterval. Thus, in the following, we show that in any subinterval, (5) is convex in $y$. The derivative of (5) w.r.t. $y$ is given by

$$\sum_{t\in\tau_i}\Bigg[-\,\mathbb{I}\left\{\frac{h_{-i}^t}{r_i}\ \geq\ y\ \geq\ \frac{h_{-i}^t}{b_i^t}\right\}\cdot\frac{h_{-i}^t}{y^2} + \mathbb{I}\left\{\frac{h_{-i}^t}{b_i^t}\ >\ y\ \geq\ \max\{\frac{s_{-i}^t}{b_i^t}, \frac{r_{-i}^*\beta_{-i}^*}{b_i^t}\}\right\}\cdot\frac{b_i^t}{\beta_{-i}^*}.$$

Taking another derivative w.r.t. $y$ leads to

$$2\sum_{t\in\tau_i}\mathbb{I}\left\{\frac{h_{-i}^t}{r_i}\geq y \;\geq\; \frac{h_{-i}^t}{b_i^t}\right\}\cdot\frac{h_{-i}^t}{y^3}\;\geq\; 0.$$

The above equation completes the first part of the proof.

We now show the second part of the proof. Observe the objective function of Problem (Data-Driven BSP) is bounded, as it is less than $\sum_{t\in[T]}\max_{i\in S^t}\{b_i^t\}$. This is the case, because the BSP auction does not charge the winner more than his submitted bid. Then, considering the fact that in any iteration, the objective increases, we can conclude that the BSP-AM algorithm converges.

So far, we established that the algorithm converges. Next, we argue that the limit point of the algorithm is a coordinate maximum. Suppose contrary to our claim this is not the case and the limit point, denoted by $\boldsymbol{\beta}^*$ is not a coordinate maximum. This implies that at $\boldsymbol{\beta}^*$, there should exist $i\in[n]$ such that the objective function increases by changing the boost of bidder $i$. This, in turn, contradicts that fact that $\boldsymbol{\beta}^*$ is the limit point of the algorithm.

### 7.2. Proof of Theorem 3

Let $I$ and $J$ and $K$ be the winner of the SP, BSP auctions, and optimal mechanism, respectively. We will show that for any vector of submitted bids, $\phi_I(v_I)\leq\phi_J(v_J)$. This will verify that the revenue of the BSP auction is greater than the revenue of the SP auction because $R_{bsp}(\mathbf{r},\boldsymbol{\beta})=\mathbb{E}[\phi_J(v_J)]$ and $R_{sp}(\mathbf{r},\boldsymbol{\beta})=\mathbb{E}[\phi_I(v_I)]$. Throughout the proof, we assume that bidders bid truthfully.

We start with some definitions. Consider the BSP auction with a vector of boosts $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$ and a vector of reserve prices $\mathbf{r}=(r_1,r_2,\ldots,r_n)$. Let $A_{bsp}^{k\to j}(v_k,v_j,\mathbf{r},\boldsymbol{\beta})$ be an event under which

- bidder $k$ with bid/valuation $v_k$ has a higher virtual value than bidder $j$, i.e., $\phi_k(v_k)\geq\phi_j(v_j)$,
- the boosted bid of bidder $k$ is less than the boosted bid of bidder $j$, i.e., $v_k\beta_k\leq v_j\beta_j$, and
- both bidders clear their reserve prices, i.e., $v_k\geq r_k$ and $v_j\geq r_j$.

We note that under event $A_{bsp}^{k\to j}(v_k,v_j,\mathbf{r},\boldsymbol{\beta})$, the winner of the BSP auction might not be the same as the winner of the optimal mechanism. This is the case because while bidder $k$ has a higher virtual value than bidder $j$, his boosted bid is less than that of bidder $j$.

Similarly, let $A_{sp}^{k\to j}(v_k,v_j,\mathbf{r})$ be the events that in the SP auction with reserve prices $\mathbf{r}$, bidder $k$ with bid/valuation $v_k$ has a higher virtual value than bidder $j$, i.e., $\phi_k(v_k)\geq\phi_j(v_j)$, bidder $k$ submits a lower bid than bidder $j$, i.e., $v_k\leq v_j$, and both bidders clear their reserve prices, i.e., $v_k\geq r_k$ and $v_j\geq r_j$.

We make use of the following lemma. The lemma guarantees that every time event $A_{bsp}^{k\to j}(v_k,v_j,\mathbf{r},\boldsymbol{\beta})$ happens, event $A_{sp}^{k\to j}(v_k,v_j,\mathbf{r})$ will happen too. This ensures that the BSP auction makes fewer misallocation mistakes. Later, we show that this property will lead to more revenue for the BSP auction.

Lemma 1. *Suppose boosts $\boldsymbol{\beta}$, and reserve prices $\boldsymbol{r}$ fulfill the conditions in Theorem 3. Given that $\mathbb{I}\{A_{sp}^{k\to j}(v_k, v_j, \boldsymbol{r})\} = 0$, then $\mathbb{I}\{A_{bsp}^{k\to j}(v_k, v_j, \boldsymbol{r}, \boldsymbol{\beta})\} = 0$.*

Lemma 1 is the key technical part in proving the result. The proof of this lemma is provided at the end of this section.

We are now ready to show the result by induction. In our base case, we show that $\phi_I(v_I) \le \phi_J(v_J)$ when there are two bidders in the auctions, i.e., $n = 2$. Then, we show that if $\phi_I(v_I) \le \phi_J(v_J)$ in auctions with $n-1$ bidders, we have $\phi_I(v_I) \le \phi_J(v_J)$ in auctions with $n$ bidders.

We first show the base case of the induction. Consider the following scenarios. Scenario 1: $J \neq K$. Under this scenario, the winner of the BSP, $J$, is not the same as the winner of the optimal mechanism, $K$. Scenario 2- $J = K$.

Consider scenario 1. By Lemma 1, if $J \neq K$, we have $I \neq K$ and as a result $I = J$ and $\phi_I(v_I) = \phi_J(v_J)$. Now, consider the second scenario where $J = K$. Then, since $I$ might not the same as $K$, we have $\phi_J(v_J) = \phi_K(v_K) \ge \phi_I(v_I)$.

Now assume that $\phi_J(v_J) \ge \phi_I(v_I)$ in any auctions with $n-1$ bidders. We will show that the same result holds in auctions with $n$ bidders. Consider BSP and SP auctions with $n$ bidders and let $I$ and $J$ be the winner of the SP and BSP auctions, respectively, when all $n$ bidders participate. Clearly, if $I = J$, we get the result. Thus, let us assume that $I \neq J$. We consider the following cases.

- $J \neq K$ and $I = K$: We show that it is not possible to have $J \neq K$ and $I = K$. Suppose, contrary to our claim, we have $I = K$ and $J \neq K$. Since $J \neq K$, we have $\mathbb{I}\{A_{bsp}^{K\to J}(v_K, v_J, \mathbf{r}, \boldsymbol{\beta})\} = 1$. Then, by Lemma 1, it should be the case that $\mathbb{I}\{A_{sp}^{K\to J}(v_K, v_J, \mathbf{r})\} = 1$. This implies that $I$ cannot be equal to $K$.
- $J = K$: In this case, we have $\phi_J(v_J) = \phi_K(v_K) \ge \phi_I(v_I)$, as the winner of the optimal mechanism has the highest virtual value.
- $I, J \neq K$: Here, we will use our induction assumption. To show the result, we construct another environment with all the bidders expect bidder $K$. In this new environment, the bids of all bidders is the same as bids of bidders in the original setting. Also, the winner of the BSP auction in this environment is still bidder $J$, and the winner of the SP auction in this environment is bidder $I$. By the induction assumption, we know in any environment with $n-1$ bidders, we have $\phi_I(v_I) \le \phi_J(v_J)$. This completes the induction.

*Proof of Lemma 1* We divide the proof into two parts. In the first part, $k > j$ and in the second part, $k < j$.

**First part:** We show that for any $k > j$, when $\mathbb{I}\{A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 1$, then $\mathbb{I}\{A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})\} = 1$. Recall that under event $A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$, we have $v_k\beta_k \le v_j\beta_j$, and $\phi_k(v_k) \ge \phi_j(v_j)$, and under event $A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})$, we have $v_k \le v_j$, $\phi_k(v_k) \ge \phi_j(v_j)$. We consider the following cases:

- $v_k \le v_j$: First note that by our assumption $\beta_j - \beta_k \ge 0$ when $k > j$. Then, if $v_k \le v_j$, both events $A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})$ and $A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$ happen when $\phi_k(v_k) \ge \phi_j(v_j)$, $v_k \ge r_k$, and $v_j \ge r_j$.
- $v_k > v_j$: In this case, $\mathbb{I}\{A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})\} = 0$. In the following, we will show that $\mathbb{I}\{A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 0$. Contrary to our claim, let us assume that $\mathbb{I}\{A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 1$. Then, we have

$$\phi_k(v_k) > \phi_j(v_j) > \phi_j(\frac{\beta_k v_k}{\beta_j}) \ ,$$

where the second inequality holds because $\phi_j(\cdot)$ is increasing and $v_j\beta_j > v_k\beta_k$. But, $\phi_k(v_k)$ cannot be greater than $\phi_j(\frac{\beta_k v_k}{\beta_j})$ because by our assumption for any $j < k$,

$$\max_{x\in[\max\{r_j\frac{\beta_j}{\beta_k}, r_k\}, \min\{\bar{v}_k, \bar{v}_j\frac{\beta_j}{\beta_k}\}]} \left\{\frac{1}{x}\phi_j^{-1}(\phi_k(x))\right\} \le \frac{\beta_k}{\beta_j}.$$

**Second Part:** Next, we will show that for any $k < j$, given that $\mathbb{I}\{A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 1$, we have $\mathbb{I}\{A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})\} = 1$. Note that under event $A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$, we have $v_j \ge v_k\frac{\beta_k}{\beta_j}$. Then, considering the fact that $\frac{\beta_k}{\beta_j} \ge 1$, having $v_j \ge v_k\frac{\beta_k}{\beta_j}$ implies that $v_j \ge v_k$. Thus, we can conclude that when event $A_{bsp}^{k\to j}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$ happens, then event $A_{sp}^{k\to j}(v_k, v_j, \mathbf{r})$ will happen too. $\square$

### 7.3. Proof of Theorem 4

Let $J$ be the winner of the BSP auction and $K$ be the winner of the optimal mechanism. We can write the revenue of the optimal mechanism as follows.

$$R_{opt} \ = \ \mathbb{E}[\phi_K(v_K)] \ = \ R_{opt}^{=} + R_{opt}^{\neq} \ ,$$

where $R_{opt}^{=} = \mathbb{E}[\phi_K(v_K)|K = J]\Pr[K = J]$ and $R_{opt}^{\neq} = \mathbb{E}[\phi_K(v_K)|K \neq J]\Pr[K \neq J]$. By definition,

$$\begin{aligned} R_{opt}^{=} \ &= \ \mathbb{E}[\phi_K(v_K)|K = J]\Pr[K = J] \ = \ \mathbb{E}[\phi_J(v_J)|K = J]\Pr[K = J] \\ &\le \ \mathbb{E}[\phi_J(v_J)|K = J]\Pr[K = J] + \mathbb{E}[\phi_J(v_J)|K \neq J]\Pr[K \neq J] \ = \ R_{bsp}(\mathbf{r}, \boldsymbol{\beta}), \end{aligned}$$

where the inequality holds because $\phi_J(v_J) \ge 0$. This follows from the fact that reserve price $r_J = \bar{r}_J$. Then, to show the result, it suffices to verify that

$$R_{opt}^{\neq} \ \le \ \max_{k<j}\left\{\frac{\beta_j}{\beta_k}\right\} R_{bsp}(\mathbf{r}, \boldsymbol{\beta}).$$

To do so, we make use of the following lemma.

Lemma 2. *Suppose that boost values $\boldsymbol{\beta}$ and reserve prices $\mathbf{r}$ fulfill the conditions in Theorem 3, then we have $J \ge K$, where $J$ is the winner of the BSP auction and $K$ is the winner of the optimal mechanism.*

The proof of Lemma 2 is given at the end of this section. By Lemma 2, we have

$$\begin{aligned}
R^{\neq}_{opt} &= \mathbb{E}[\phi_K(v_K)|K<J]\,\Pr[K<J] \\
&\leq \mathbb{E}[v_K \frac{\beta_K}{\beta_J}\frac{\beta_J}{\beta_K}|K<J]\,\Pr[K<J] \\
&\leq \mathbb{E}[(\text{payment in BSP})\frac{\beta_J}{\beta_K}|K<J]\,\Pr[K<J] \\
&\leq \max_{k<j}\left\{\frac{\beta_j}{\beta_k}\right\} \times R_{bsp}(\mathbf{r},\boldsymbol{\beta}) \;\leq \max_{j=1,2,\dots,n-1}\left\{\frac{\beta_{j+1}}{\beta_j}\right\} \times R_{bsp}(\mathbf{r},\boldsymbol{\beta}),
\end{aligned}$$

where the first inequality holds because $\phi_K(v_K) \leq v_K$ and the second inequality holds because the payment in the BSP auction is the second highest boosted bid divided by the boost of the winner. Finally, the last inequality follows because $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$.

*Proof of Lemma 2* To prove the result, we show that $\mathbb{I}\{A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 0$ when $k > j$; see the definition of event $A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$ in Section 7.2. Under event $A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$, the optimal mechanism and BSP auction have different preferences over bidders $k$ and $j$: While the BSP prefers bidder $j$, the optimal mechanism prefers bidder $k$.

In Lemma 1, we show that when $k > j$ and $v_k \geq v_j$, we have $\mathbb{I}\{A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 0$. Thus, to complete the proof, we will show that when $k > j$ and $v_k < v_j$, we have $\mathbb{I}\{A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 0$.

Assume that, contrary to our claim, $\mathbb{I}\{A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 1$ when $k > j$ and $v_k < v_j$. Then, given that $\mathbb{I}\{A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})\} = 1$, we have

$$\phi_j(v_j) < \phi_k(v_k) \leq \phi_k(v_j),$$

where the second inequality holds because $v_k < v_j$ and $\phi_k(\cdot)$ is increasing. But, the above equation cannot hold because for $k > j$, we have $\phi_j(v_j) \geq \phi_k(v_j)$. This implies that $A^{k\to j}_{bsp}(v_k, v_j, \mathbf{r}, \boldsymbol{\beta})$ does not happen when $k > j$ and $v_k < v_j$. $\square$

## 8. Extensions and Discussions

Here, we first discuss Property 1. Then, we present a simple procedure for computing the suggested upper bounds by conservative boosting. We also provide an algorithm that allows us to apply these upper bounds sequentially. Finally, we present parametric upper bounds on the revenue losses of BSP and SP auctions.

### 8.1. Property 1

Here, we provide several examples to shed light on Property 1.

Example 2. Suppose that the random variables $X$ and $Y$ are drawn from the normal distributions $N(\mu_1, \sigma)$ and $N(\mu_2, \sigma)$[26], respectively, where $\mu_1 \geq \mu_2$. Then, $X \geq_l Y$ and consequently $X \geq_h Y$.

[26] The distribution of a normal random variable with mean $\mu$ and standard deviation $\sigma$ is denoted by $N(\mu, \sigma)$.

Likelihood ordering is closed under increasing functions. Therefore, the above result can be extended to lognormal distributions.

EXAMPLE 3. Suppose that the random variables $X$ and $Y$ are drawn from the lognormal distributions $\text{Logn}(\mu_1, \sigma)$ and $\text{Logn}(\mu_2, \sigma)$, respectively, where $\mu_1 \geq \mu_2$. Then, $X \geq_l Y$ and consequently $X \geq_h Y$.

We would like to point out that the average and variance of random variable $X$ with distribution $\text{Logn}(\mu_1, \sigma)$ are respectively $e^{\mu_1 + \sigma^2/2}$ and $(e^{\sigma^2} - 1)e^{2\mu_1 + \sigma^2}$. This implies that the random variable $X \sim \text{Logn}(\mu_1, \sigma)$ that dominates the random variable $Y \sim \text{Logn}(\mu_2, \sigma)$ in hazard order has a higher average and variance.

EXAMPLE 4. Suppose that the random variable $X$ is drawn from the exponential distribution with rate $\lambda_1$, $\text{Exp}(\lambda_1)$, and the random variable $Y$ is drawn from $\text{Exp}(\lambda_2)$, where $\lambda_1 \leq \lambda_2$. Then, $X \geq_l Y$ and thus $X \geq_h Y$.

Note that the average and standard deviation of a random variable with the exponential distribution $\text{Exp}(\lambda)$ is $\lambda^{-1}$. Then, given that $\lambda_1 \leq \lambda_2$, the random variable $X \sim \text{Exp}(\lambda_1)$ that dominates the random variable $Y \sim \text{Exp}(\lambda_2)$ has a larger average and standard deviation than random variable $Y$. Roughly speaking, when a random variable dominates another random variable, the dominating random variable has a higher average and a greater variance. Then, Property 1 suggests that there is a spectrum of bidders ranging from brand bidders (those with the lowest IHRs) to retargeting bidders (those with the highest IHRs). This is the case because the average and standard deviation of the brand bidders are lower than those of the retargeting bidders.

Thus far, we have discussed Property 1 when $\rho = \min_{i=1,\ldots,n}\{\underline{v}_i\}$. Given that $\rho > \min_{i=1,\ldots,n}\{\underline{v}_i\}$, this assumption becomes relaxed because the ordering must be held for a smaller range. The following example clarifies this point.

EXAMPLE 5. Suppose that we draw the random variable $X$ from the lognormal distribution $\text{Logn}(\mu_1, \sigma_1)$ and the random variable $Y$ from the lognormal distribution $\text{Logn}(\mu_2, \sigma_2)$, where $\mu_1 \geq \mu_2$ and $\sigma_1 \geq \sigma_2$. Then, there exists $\rho$ such that $\alpha_X(v) \geq \alpha_Y(v)$ for any $v \geq \rho$.

*Proof of the result in Example 5* Consider the random variable $X \sim \text{Logn}(\mu_1, \sigma_1)$. Then, using the L'Hpital's rule, $\alpha_X(v) = \frac{1 - F_X(v)}{f_x(v)}$ goes to infinity at the rate $\frac{\sigma_1^2 v}{\sigma_1^2 + \ln(v) - \mu_1}$. Note that $\frac{\sigma_1^2 v}{\sigma_1^2 + \ln(v) - \mu_1}$ is increasing in $\sigma_1$ and $\mu_1$. Then, given that $Y \sim \text{Logn}(\mu_2, \sigma_2)$, $\mu_1 \geq \mu_2$, and $\sigma_1 \geq \sigma_2$, we have $\alpha_X(v) \geq \alpha_Y(v)$ for a large enough $v$. $\square$

Note that even if $X \sim \text{Logn}(\mu, \sigma_1)$ and $Y \sim \text{Logn}(\mu, \sigma_2)$ with $\sigma_1 \geq \sigma_2$, we do not have $X \geq_h Y$. Therefore, Property 1 is less restrictive than hazard rate ordering.

### 8.2. Upper Bounds on Boosts and a Sequential Procedure

This section has two parts. In the first part, we simplify Eq. (4) and present an algorithm to identify an upper bound on $\frac{\beta_i}{\beta_j}$, $i < j$. In the second part, we present a sequential procedure to determine the maximum allowable boost values. We show that BSP auctions with these boost values outperform SP auctions.

We start by characterizing an upper bound on $\frac{\beta_i}{\beta_j}$ using Eq. (4). Let us denote this upper bound by $\mu_{i,j}$. Eq. (4) is fulfilled if for any $i<j$, $j=2,\ldots,n$, we have

$$\beta_i \times \min\left\{\max_{x\in[\bar{r}_j, u_{i,j}(\frac{\beta_i}{\beta_j})]}\left\{\frac{1}{x}\phi_i^{-1}(\phi_j(x))\right\}, 1\right\} \;\leq\; \beta_j \;\leq\; \beta_i\ . \tag{6}$$

This is the case because $l_{i,j}(\frac{\beta_i}{\beta_j})\geq \bar{r}_j$. We note that for any $x\geq \max_{x\in[\underline{v}_i,\bar{v}_i]}\{\phi_i(x)\}=\phi_i(\bar{v}_i)$, we have $\phi_i(x)=x$. This implies that if $\phi_j(u_{i,j}(\frac{\beta_i}{\beta_j}))\geq\phi_i(\bar{v}_i)$, then the above inequality enforces $\beta_i$ to be equal to $\beta_j$. To avoid this, we must have $\bar{v}_i\frac{\beta_i}{\beta_j}\leq\bar{v}_j$ and $\phi_j(u_{i,j}(\frac{\beta_i}{\beta_j}))=\phi_j(\bar{v}_i\frac{\beta_i}{\beta_j})\leq\phi_i(\bar{v}_i)$. The former ensures that $u_{i,j}(\frac{\beta_i}{\beta_j})=\bar{v}_i\frac{\beta_i}{\beta_j}$ and the latter guarantees that $\beta_j$ is not forced to be equal to $\beta_i$. Recall that $u_{i,j}(\frac{\beta_i}{\beta_j})=\min\{\bar{v}_j,\bar{v}_i\frac{\beta_i}{\beta_j}\}$. This leads to the following:

$$\frac{\beta_i}{\beta_j} \;\leq\; \min\left\{\frac{\bar{v}_j}{\bar{v}_i}, \frac{1}{\bar{v}_i}\phi_j^{-1}(\phi_i(\bar{v}_i))\right\} \;=\; \frac{1}{\bar{v}_i}\phi_j^{-1}(\phi_i(\bar{v}_i)).$$

Define $\bar{\mu}_{i,j}=\frac{1}{\bar{v}_i}\phi_j^{-1}(\phi_i(\bar{v}_i))$. We can think of $\bar{\mu}_{i,j}$ as an initial upper bound on $\frac{\beta_i}{\beta_j}$. Then, Eq. (4) is satisfied if we have

$$\beta_i \times \max_{x\in[\bar{r}_j,\bar{v}_i\bar{\mu}_{i,j}]}\left\{\frac{1}{x}\phi_i^{-1}(\phi_j(x))\right\} \;\leq\; \beta_j \quad\Rightarrow\quad \frac{\beta_i}{\beta_j}\;\leq\; G(\bar{\mu}_{i,j})\ , \tag{7}$$

where $G(y) \;=\; \left(\max_{x\in[\bar{r}_j,\bar{v}_i y]}\left\{\frac{1}{x}\phi_i^{-1}(\phi_j(x))\right\}\right)^{-1}$. We note that the input $y$ appears in the range of the maximum. By Eq. (7), the new upper bound on $\frac{\beta_i}{\beta_j}$ is $G(\bar{\mu}_{i,j})$. If the new upper bound is greater than the old one, i.e., $\bar{\mu}_{i,j}$, then we can return $\bar{\mu}_{i,j}$ as an upper bound on $\frac{\beta_i}{\beta_j}$. If not, we can either report $G(\bar{\mu}_{i,j})$ as an upper bound or we can repeat this procedure multiple times to obtain a better bound; see the following algorithm.

<u>DETERMINING UPPER BOUNDS ON BOOST VALUES</u>

- Let $\bar{\mu}_{i,j}=\frac{1}{\bar{v}_i}\phi_j^{-1}(\phi_i(\bar{v}_i))$ where $j>i$.

Set $O \;=\; \bar{\mu}_{i,j}$ and $k=1$. Also, choose a value for $W\in\mathbb{Z}^+$.

```
—WHILE w ≤ W.
    Calculate G(O).
    IF G(O) ≥ O
        BREAK.
    ELSE
        O = G(O) and w = w + 1.
    END IF
—END WHILE
```

- $\mu_{i,j}=O$

In this algorithm, at any step, we have an old upper bound denoted by $O$. Initially, we set $O$ to $\bar{\mu}_{i,j}$. Then, at any step, we calculate $G(O)$. If $G(O)\geq O$, we report $O$ as an upper bound. When

$G(O) < O$, we can either report $G(O)$ as an upper bound or we can obtain a better bound by setting the old upper bound to $G(O)$ and repeating this procedure. In other words, if we stop, the upper bound is $G(O)$. But if we continue, the new upper bound is $G(G(O))$, which is larger than $G(O)$. This follows because $O > G(O)$ and $G(\cdot)$ is non-increasing. In the algorithm, parameter $W$ is the maximum number of times that we would like to repeat this procedure. Note that even if we set $W = 1$, the returned upper bound satisfies Eq. (7).

Next, we present a sequential procedure for applying the upper bounds suggested by conservative boosting. We then show that by following this procedure, Eq. (6) is satisfied and as a result, the BSP auction earns more revenue than the SP auction.

We need few notations to describe this sequential procedure. For any $i < j$, let $\hat{\mu}_{i,j} = \min_{k\in[i]}\{\mu_{k,j}\}$. We note that $\hat{\mu}_{i,j}$ satisfies the conservative boosting guideline as $\hat{\mu}_{i,j} \le \mu_{i,j}$. Furthermore, when $\mu_{i,j}$ is decreasing in $i$, we get $\hat{\mu}_{i,j} = \mu_{i,j}$. In our sequential procedure, we use $\hat{\mu}_{i,j}$ as an upper bound on $\frac{\beta_i}{\beta_j}$. One important feature of these upper bounds is that for any $j$, $\hat{\mu}_{i,j}$ is decreasing in $i$. This allows us to apply these upper bounds sequentially.

*Sequential Procedure:* This procedure has $n$ steps in which in each step, we determine the boost value of a bidder. We start with bidder $n$, i.e., the bidder with the highest IHR, and without loss of generality, we set $\beta_n$ to 1. Then, we move to bidder $n-1$, i.e., the bidder with the second highest IHR, and we set $\beta_{n-1} = \beta_n\hat{\mu}_{n-1,n}$. Generally, we choose the boost value of bidder $i$ after we determine the boost values of bidders $i+1, i+2, \ldots, n$. More precisely, we set the boost value of bidder $i$ to the following:

$$\beta_i = \min_{j=i+1,i+2,\ldots,n}\{\beta_j\hat{\mu}_{i,j}\}, \quad i \in [n-1]. \tag{8}$$

Observe that the boost value of bidder $i$ is only a function of the boost values of bidders $i+1, i+2, \ldots, n$ and does not depend on the boost values of bidders $1, 2, \ldots, i-1$.

As we show in the following proposition, by choosing boost values sequentially according to Eq. (8), we obtain a BSP auction that outperforms the SP auction.

PROPOSITION 2 **(Applying Conservative Boosting Sequentially)**. *Suppose that Property 1 holds. Then, if we choose boost values sequentially according to Eq. (8), the BSP auction earns more revenue than the SP auction.*

*Proof of Proposition 2* To show the result, we will verify that by choosing the boosts sequentially according to Eq. (8), our guideline, given in Eq. (6), is fulfilled. Recall that Eq. (6) is satisfied if for any $i < j$, we have $1 \le \frac{\beta_i}{\beta_j} \le \hat{\mu}_{i,j}$. For any $i < j$, the inequality $\frac{\beta_i}{\beta_j} \le \hat{\mu}_{i,j}$ is enforced when we determine the boost of bidder $i$. In the following, we show that even we do not explicitly enforce $\frac{\beta_i}{\beta_j} \ge 1$, this inequality is also satisfied.

To do so, we show that $\beta_i \geq \beta_{i+1}$ for any $i = 1, 2, \ldots, n-1$. By Eq. (8), $\beta_i \geq \beta_{i+1}$ if

$$\min_{j=i+1,i+2,\ldots,n} \{\beta_j \hat{\mu}_{i,j}\} \;\geq\; \beta_{i+1} \;=\; \min_{k=i+2,i+3,\ldots,n} \{\beta_k \hat{\mu}_{i+1,k}\} \;.$$

This holds if we have

$$\beta_j \hat{\mu}_{i,j} \;\geq\; \beta_{i+1} \;=\; \min_{k=i+2,i+3,\ldots,n} \{\beta_k \hat{\mu}_{i+1,k}\} \qquad j = i+1, i+2, \ldots, n \;. \tag{9}$$

When $j = i+1$, Eq. (9) is satisfied, as $\beta_{i+1}\hat{\mu}_{i,i+1} \geq \beta_{i+1}$; that is, $\hat{\mu}_{i,i+1} \geq 1$. For any $j \geq i+2$, Eq. (9) holds if $\beta_j \hat{\mu}_{i,j}$ is greater than one of these terms: $\beta_k \hat{\mu}_{i+1,k}$ for $k = i+2, i+3, \ldots, n$. Then considering the fact that $\hat{\mu}(i,j)$ is decreasing in $i$, we have $\beta_j \hat{\mu}_{i,j} \geq \beta_j \hat{\mu}_{i+1,j}$. This confirms that Eq. (9) is fulfilled for any $j \geq i+2$. $\square$

### 8.3. Upper Bounds on the Revenue Loss of the BSP

Here, we present parametric bounds on the revenue losses of BSP and SP auctions. The bounds are easy to compute because each bound term concerns only one pair of bidders. Furthermore, the bounds provide insights into BSP auctions.

The maximum revenue loss is governed by pairs of bidders, $(j,k)$'s, such that (B)SP auctions sort them differently than does the optimal mechanism. That is, while the (boosted) bid of bidder $j$ is greater than the (boosted) bid of bidder $k$, the virtual bid of bidder $j$ is less than that of bidder $k$. In this case, the allocation rule of the optimal mechanism may not be the same as the allocation rule of (B)SP auctions. Here, we determine the maximum revenue loss by specifying the most expensive misallocation that can be made by (B)SP auctions.

In this section, for simplicity, we assume that $r_i = \bar{r}_i$ for $i \in [n]$. We need few definitions to present the result. Recall that

$$A_{bsp}^{k\to j}(v_k, v_j, \bar{\mathbf{r}}, \boldsymbol{\beta}) = \mathbb{I}\big\{\phi_j(v_j) \leq \phi_k(v_k), \quad \beta_j v_j \geq \beta_k v_k, \quad v_j \in [\bar{r}_j, \bar{v}_j], \text{ and } v_k \in [\bar{r}_k, \bar{v}_k]\big\}$$

is an event under which bidder $k$ has a higher virtual value than bidder $j$, while bidder $k$ has a lower boosted bid than bidder $j$. In this event, both bidders clear their reserve prices. In the following, for ease of notation, we denote $A_{bsp}^{k\to j}(v_k, v_j, \bar{\mathbf{r}}, \boldsymbol{\beta})$ with $A_{bsp}^{k\to j}$. In event $A_{bsp}^{k\to j}$, the optimal mechanism prefers bidder $k$ to bidder $j$, but the BSP auction prefers bidder $j$ to bidder $k$. This is so because in event $A_{bsp}^{k\to j}$, bidder $j$ has a higher boosted bid than bidder $k$. Let

$$\mathcal{L}_{bsp} = \max_{(j,k),k\neq j} \left\{ \mathbb{E}\Big[ (\phi_k(v_k) - \phi_j(v_j)) - (\frac{\beta_k v_k}{\beta_j} - v_j) | A_{bsp}^{k\to j} \Big] \times \Pr\left[A_{bsp}^{k\to j}\right] \right\}.$$

As it becomes clear later, $\mathcal{L}_{bsp}$ is the maximum loss that BSP auction incurs by misallocating the good. Let us parse the term inside the expectation, i.e., $(\phi_k(v_k) - \phi_j(v_j)) - (\frac{\beta_k v_k}{\beta_j} - v_j)$. First of all, this term is small if the virtual value and boosted bid of bidder $j$ are close to those of bidder $k$. There is another way to think about this term. Consider the BSP auction with two bidders $j$ and $k$. Assign boost $\hat{\beta}_j = 1$ to bidder $j$ and boost $\hat{\beta}_k = \frac{\beta_k}{\beta_j}$ to bidder $k$. Obviously, the revenue of the BSP auction with these boosts is equal to that of the BSP auction with boosts $\beta_j$ and $\beta_k$. In this new BSP auction, regard the boosted bids of bidders $j$ and $k$ as proxies for their virtual bids and denote the boosted bid of bidder $j$ by $\hat{\phi}_j(v_j)$ and the boosted bid of bidder $k$ by $\hat{\phi}_k(v_k)$; that is: $\hat{\phi}_j(v_j) = v_j$ and $\hat{\phi}_k(v_k) = \frac{\beta_k v_k}{\beta_j}$. Then, $\mathcal{L}_{bsp}$ can be written as follows: $\mathcal{L}_{bsp} = \max_{(j,k),k\neq j}\Big\{\mathbb{E}\Big[\Big(\phi_k(v_k)-\hat{\phi}_k(v_k)\Big)-\Big(\phi_j(v_j)-\hat{\phi}_j(v_j)\Big)\,|A_{bsp}^{k\to j}\Big]\times\Pr\left[A_{bsp}^{k\to j}\right]\Big\}$. This shows that if our error in estimating the virtual value of bidder $j$, i.e., $\Big(\phi_j(v_j)-\hat{\phi}_j(v_j)\Big)$, is close to that of bidder $k$, i.e., $\Big(\phi_k(v_k)-\hat{\phi}_k(v_k)\Big)$, the BSP auction performs well. That is, it is not important how big the estimation errors of $\phi_j(v_j)$ and $\phi_k(v_k)$ are. What is important is their relative estimation errors.

Similarly, we let

$$\mathcal{L}_{sp} = \max_{(j,k),j\neq k}\left\{\mathbb{E}\left[(-\alpha_k(v_k)+\alpha_j(v_j))\middle|A_{sp}^{k\to j}(v_k,v_j,\bar{\mathbf{r}})\right]\times\Pr[A_{sp}^{k\to j}(v_k,v_j,\bar{\mathbf{r}})]\right\},$$

where event $A_{sp}^{k\to j}(v_k,v_j,\bar{\mathbf{r}})$ is given by

$$A_{sp}^{k\to j}(v_k,v_j,\bar{\mathbf{r}}) = \mathbb{I}\big\{(v_j-v_k)\in[0,\alpha_j(v_j)-\alpha_k(v_k)],\ v_j\geq\bar{r}_j,\ \text{and } v_k\geq\bar{r}_k\big\},$$

Note that $\mathcal{L}_{sp}$ is equal to $\mathcal{L}_{bsp}$ when $\beta_i=\beta\neq 0$ for any $i\in[n]$. We are now ready to present our results.

Theorem 5 **(Maximum Revenue Loss)**. *Given that $\phi_i(\cdot)$ is an increasing function for any $i\in[n]$, the revenue of the BSP auction with the monopoly reserve prices $\bar{\boldsymbol{r}}$ and boosts $\boldsymbol{\beta}$ satisfies the following inequality: $R_{bsp}(\bar{\boldsymbol{r}},\boldsymbol{\beta}) \;\geq\; R_{opt}-\mathcal{L}_{bsp}$. Furthermore, for the SP auction, we have $R_{sp}(\bar{\boldsymbol{r}}) \;\geq\; R_{opt}-\mathcal{L}_{sp}$. Finally, under the conditions stated in Theorem 3, we have $\mathcal{L}_{sp}\geq\mathcal{L}_{bsp}$.*

We point out that the bounds for the BSP and SP auctions, i.e., $R_{bsp}(\bar{\mathbf{r}},\boldsymbol{\beta}) \;\geq\; R_{opt}-\mathcal{L}_{bsp}$ and $R_{sp}(\bar{\mathbf{r}}) \;\geq\; R_{opt}-\mathcal{L}_{sp}$, are tight when the BSP and SP auctions are revenue optimal. To see why, note that when the BSP and SP auctions are optimal, events $A_{bsp}^{k\to j}$ and $A_{sp}^{k\to j}$ do not occur, and as a result, $\mathcal{L}_{bsp}$ and $\mathcal{L}_{sp}$ are zero.

By Theorem 5, the upper bound of the revenue loss of the BSP auction with conservative boosting is less than that of the SP auction. Then, by the definitions of $\mathcal{L}_{bsp}$ and $\mathcal{L}_{sp}$, we can conclude that the boosted bids in BSP auctions better approximate the virtual bids, compared with the bids in the SP auctions.

**8.3.1. Proof of Theorem 5** We first prove the upper bounds on the revenue loss of the BSP and SP auctions. We then show that the upper bounds on the revenue loss of the BSP auction with conservative boosting is less than that of the SP auction.

Throughout the proof, to simplify the notations, we denote $R_{bsp}(\bar{\mathbf{r}}, \boldsymbol{\beta})$ by $R_{bsp}$. Let $J$ be the winner of the BSP auction and let $K$ be the winner of the optimal mechanism. By the revenue equivalence, the revenue of the optimal mechanism can be written as

$$R_{opt} \ = \ \mathbb{E}[\phi_K(v_K)] \ = \ R_{opt}^{=} + R_{opt}^{\neq} \ ,$$

where $R_{opt}^{=} = \mathbb{E}[\phi_K(v_K)|J=K]\Pr[J=K]$ and $R_{opt}^{\neq} = \mathbb{E}[\phi_K(v_K)|J\neq K]\Pr[J\neq K]$. Similarly, we can define $R_{bsp}^{=} = \mathbb{E}[\phi_J(v_J)|J=K]\Pr[J=K]$ and $R_{bsp}^{\neq} = \mathbb{E}[\phi_J(v_J)|J\neq K]\Pr[J\neq K]$. Note that $R_{opt}^{=} = R_{bsp}^{=}$. Thus, to verify our claim, in the following, we show that

$$R_{bsp}^{\neq} \ \geq R_{opt}^{\neq} - \mathbb{E}\left[-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J)\big|A_{bsp}^{K\to J}\right]\Pr[A_{bsp}^{K\to J}]\,.$$

By definition,

$$\begin{aligned}
R_{bsp}^{\neq} \ &= \ \mathbb{E}[v_J - \alpha_J(v_J)|J\neq K]\Pr[J\neq K] \\
&\geq \ \mathbb{E}[v_K\frac{\beta_K}{\beta_J} - \alpha_J(v_J)|J\neq K]\Pr[J\neq K] \\
&= \ \mathbb{E}[v_K - \alpha_K(v_K) + \alpha_K(v_K) + v_K\left(\frac{\beta_K}{\beta_J}-1\right) - \alpha_J(v_J)|J\neq K]\Pr[J\neq K] \\
&= \ R_{opt}^{\neq} - \mathbb{E}[-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J)|J\neq K]\Pr[J\neq K]\,,
\end{aligned}$$

where the inequality holds because $v_K\beta_K \leq v_J\beta_J$. In the following, we show that $-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J) \geq 0$. This allows us to get a lower bound on $R_{bsp}^{\neq}$ by replacing event $J\neq K$ with event $A_{bsp}^{K\to J}(v_K, v_J, \bar{\mathbf{r}}, \boldsymbol{\beta})$ in the above equation. Note that when $J\neq K$, event $A_{bsp}^{K\to J}(v_K, v_J, \bar{\mathbf{r}}, \boldsymbol{\beta})$ happens. That is, we get

$$\begin{aligned}
R_{bsp}^{\neq} \ &\geq R_{opt}^{\neq} - \mathbb{E}\left[-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J)\big|A_{bsp}^{K\to J}\right]\Pr[A_{bsp}^{K\to J}] \\
&\geq R_{opt}^{\neq} - \max_{k\neq j, k,j\in[n]}\left\{\mathbb{E}[-\alpha_k(v_k) - v_k\left(\frac{\beta_k}{\beta_j}-1\right) + \alpha_j(v_j)|A_{bsp}^{K\to j}] \times \Pr[A_{bsp}^{k\to j}]\right\}
\end{aligned}$$

The above equation leads to the desired result.

Now let us show that $-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J) \geq 0$. Since $K$ and $J$ are the winner of the optimal mechanism and BSP auction, respectively, we have $v_K - \alpha_K(v_K) \geq v_J - \alpha_J(v_J)$ and $v_K\beta_K \leq v_J\beta_J$. This implies that

$$-\alpha_K(v_K) - v_K\left(\frac{\beta_K}{\beta_J}-1\right) + \alpha_J(v_J) \ \geq \ -v_K + v_J - v_K\left(\frac{\beta_K}{\beta_J}-1\right) \ = \ v_J - v_K\left(\frac{\beta_K}{\beta_J}\right) \ \geq \ 0$$

Similarly, one can show that $R_{sp} \geq R_{opt} - \mathcal{L}_{sp}$.

Next, we show that $\mathcal{L}_{sp} \geq \mathcal{L}_{bsp}$ if the BSP auctions follow the conservative boosting procedure. By Lemma 1, when $\mathbb{I}\{A_{sp}^{k\to j}\} = 0$, then $\mathbb{I}\{A_{bsp}^{k\to j}\} = 0$. Then, by definition of $\mathcal{L}_{sp}$ and $\mathcal{L}_{bsp}$, the upper bound $\mathcal{L}_{bsp}$ is smaller than $\mathcal{L}_{sp}$ if $\frac{\beta_k}{\beta_j} \geq 1$ when both events $A_{sp}^{k\to j}$ and $A_{bsp}^{k\to j}$ happen. To show this we use Lemma 2 where we show that under $A_{bsp}^{k\to j}$, we should have $j > k$. This implies that $\frac{\beta_k}{\beta_j} \geq 1$ and completes the proof.