Exercises Regarding **Grp** and **Ab**

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1. Let $\varphi : G \to H$ be a morphism in a category C with products. Explain why there is a unique morphism $(\varphi \times \varphi) : G \times G \to H \times H$ compatible in the evident way with the natural projections.

   Solution. Considering the universal property for $G \times G$, it provides us with the two natural projection morphisms: $\pi_G^1 : G \times G \to G$ and $\pi_G^2 : G \times G \to G$ mapping onto the first and second components, respectively. Let π_H^1, π_H^2 be the respective natural projections for $H \times H$. We have the two morphisms $\varphi \circ \pi_G^i : G \times G \to H$, for each i, yielding the following diagram.

$$\begin{tikzcd}
 & H \times H \arrow[dl, "\pi_H^1"'] \arrow[dr, "\pi_H^2"] & \\
H & & H \\
 & G \times G \arrow[ul, "\varphi\circ\pi_G^1"] \arrow[ur, "\varphi\circ\pi_G^2"'] \arrow[uu, dashed, "f"] &
\end{tikzcd}$$

   The universal property for $H \times H$ guarantees the existence of a morphism $f : G \times G \to H \times H$ for which $\pi_H^2 \circ f = \varphi \circ \pi_G^2$ and $\pi_H^1 \circ f = \varphi \circ \pi_G^1$. If we think about how such a morphism $\varphi \times \varphi$ should interact with the natural projections, we want the following relations equations to hold.

$$\pi_H^1(\varphi \times \varphi)(a, b) = \pi_H^1(\varphi(a), \varphi(b)) = \varphi(a)$$
$$\pi_H^2(\varphi \times \varphi)(a, b) = \pi_H^2(\varphi(a), \varphi(b)) = \varphi(b)$$

   Observe that $\varphi(a)$ is also obtained by mapping $(a, b) \mapsto a \mapsto \varphi(a)$, which is just $\varphi(a) = \varphi(\pi_G^1(a, b))$. Similarly, $\varphi(b) = \varphi(\pi_G^2(a, b))$. That is, a morphism obtained as above is the best candidate for $\varphi \times \varphi$. Since such a morphism is unique, it's safe to say that this is the morphism we're looking for.

$\square$

2. Let $\varphi : G \to H$, $\psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G$, $H \times H$, $K \times K$ as in the previous exercise. Prove that
$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$

Solution. Similar to last time, $\psi \circ \varphi \circ \pi_G^1$ and $\psi \circ \varphi \circ \pi_G^2$ are morphisms from $G \times G$ to K. The universal property for $K \times K$ gives us a unique morphism, which we called $(\psi \circ \phi) \times (\psi \circ \phi) : G \times G \to K \times K$. To show equality, it suffices to prove that the morpism $(\psi \times \psi) \circ (\varphi \times \varphi)$ satisfies the universal property for $K \times K$. From the diagram for $H \times H$

$$\begin{tikzcd}
& H \times H \arrow[dl, "\pi_H^1"'] \arrow[dr, "\pi_H^2"] & \\
H & & H \\
& G \times G \arrow[ul, "\varphi\pi_G^1"] \arrow[ur, "\varphi\pi_G^2"'] \arrow[uu, dashed, "\varphi\times\varphi"] &
\end{tikzcd}$$

we have the equations
$$\pi_H^i(\varphi \times \varphi) = \varphi\pi_G^i \tag{1}$$

for i = 1, 2. Similarly, from the diagram for $K \times K$,

$$\begin{tikzcd}
& K \times K \arrow[dl, "\pi_K^1"'] \arrow[dr, "\pi_K^2"] & \\
K & & K \\
& H \times H \arrow[ul, "\psi\pi_H^1"] \arrow[ur, "\psi\pi_H^2"'] \arrow[uu, dashed, "\psi\times\psi"] &
\end{tikzcd}$$

we have the equations
$$\pi_K^i(\psi \times \psi) = \psi\pi_H^i \tag{2}$$

for i = 1, 2. Let us consider how the universal property for $K \times K$ from $G \times G$ should look, then we will use the above equations to show equality.

$$
\begin{tikzcd}
& K \times K \arrow[dl, "\pi_K^1"'] \arrow[dr, "\pi_K^2"] & \\
K & & K \\
& G \times G \arrow[ul, "(\psi\varphi)\pi_G^1"] \arrow[ur, "(\psi\varphi)\pi_G^2"'] \arrow[uu, dashed, "(\psi\varphi)\times(\psi\varphi)"] &
\end{tikzcd}
$$

Thus, we need to prove $\pi_K^i(\psi \times \psi)(\varphi \times \varphi) = (\psi\varphi)\pi_G^i$. Applying (2) immediately yields

$$
\begin{aligned}
[\pi_K^i(\psi \times \psi)](\varphi \times \varphi) &= [\psi \pi_H^i](\varphi \times \varphi) \\
&= \psi[\pi_H^i(\varphi \times \varphi)] \\
&= \psi(\varphi \pi_G^i) \qquad\qquad \text{(by equation (1)).}
\end{aligned}
$$

By the universal property of products, the morphism from $G \times G$ to $K \times K$ is unique, which proves equality. $\square$

3. Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in **Ab**.

Solution. The obvious choice for embeddings of G, H into $G \times H$ works here; denote them by i_G, i_H. Let $Z \in \mathbf{Ab}$ and let $\varphi_G : G \to Z$, $\varphi_H : H \to Z$ be morphisms. Define the map

$$
\begin{aligned}
f : G \times H &\to Z \\
(g, h) &\mapsto \varphi_G(g)\varphi_H(h).
\end{aligned}
$$

We see that f is a map satisfying the commutativity of the diagram for a product:

$$
\begin{aligned}
f(i_G(g)) &= f(g, 1) \\
&= \varphi_G(g)\varphi_H(1) \\
&= \varphi_G(g),
\end{aligned}
$$

hence $fi_G = \varphi_G$ and similarly, $fi_H = \varphi_H$. Let us see whether f preserves structure.

For $(g_i, h_i) \in G \times H$, where $i = 1, 2$, we have

$$\begin{aligned} f((g_1, h_1)(g_2, h_2)) &= f((g_1g_2, h_1h_2)) \\ &= \varphi_G(g_1g_2)\varphi_H(h_1h_2) \\ &= \varphi_G(g_1)\varphi_G(g_2)\varphi_H(h_1)\varphi_H(h_2) \\ &= \varphi_G(g_1)\varphi_H(h_1)\varphi_G(g_2)\varphi_H(h_2) \\ &= f((g_1, h_1))f((g_2, h_2)). \end{aligned}$$

Therefore, f is a group homomorphism. To show existence, let $\psi : G \times H \to Z$ be another such morphism for which $\psi i_G = \varphi_G$ and $\psi i_H = \varphi_H$. Observe

$$\begin{aligned} \psi(i_G(g)) = \psi(g, 1) &= \varphi_G(g) \\ &= \varphi_G(g)\varphi_H(1) \\ &= f(g, 1), \end{aligned}$$

hence $\psi(g, 1) = f(g, 1)$. Similarly, $\psi(1, h) = f(1, h)$. For an arbitrary $(g, h) \in G \times H$, $(g, h) = (g, 1)(1, h)$, so these together prove $\psi(g, h) = f(g, h)$, so f is indeed unique. We can see precisely where f would fail to be a group homomorphism by looking at the above equations. In order for there to be coproducts in **Ab** it is necessary that Z is an abelian group, for each Z. Otherwise, we would be unable to commute the elements into a form where f preserves structure. This is exactly the reason that the regular cartesian product fails to be a coproduct in **Grp**. Finding a counterexample would consist of taking the cartesian product of two abelian groups, then considering the map from the product into a non-abelian group. $\square$

4. Consider the product of the cyclic groups C_2, C_3. By the exercise above, $C_2 \times C_3$ is a coproduct in **Ab**. Show that it is *not* a coproduct of C_2, C_3 in **Grp**.

   Solution. Let us consider the coproduct mapping into the symmetric group of order 3, S_3. Define $\varphi : C_2 \to S_3$ by $n \mapsto (12)^n$ and $\psi : C_3 \to S_3$ by $n \mapsto (123)^n$. It is easy to see that these are embedding homomorphisms. By the universal property there is a homomorphism $\varphi \times \psi : C_2 \times C_3 \to S_3$. Following the remark at the end of the previous exercise, let us investigate the commutativity between $C_2 \times C_3$ and S_3 through $\varphi \times \psi$. As of now, the value of $\varphi \times \psi$ is undetermined. It is easy to see that, since $(\varphi \times \psi)(0, m) = \psi(m)$ and $(\varphi \times \psi)(n, 0) = \varphi(n)$, it follows that $(\varphi \times \psi)((n, 0)(0, m))) =$

$(\varphi \times \psi)(n,0)(\varphi \times \psi)(0,m) = \varphi(n)\psi(m)$, since $(\varphi \times \psi)$ is a homomorphism. However, this is taxing on the commutativity between (12) and (123) in S_3:

$$\begin{aligned}
\varphi(1)\psi(1) = (\varphi \times \psi)(1,1) &= (\varphi \times \psi)(0+1, 1+0) \\
&= (\varphi \times \psi)[(0,1) + (1,0)] \\
&= (\varphi \times \psi)(0,1)(\varphi \times \psi)(1,0) \\
&= [\varphi(0)\psi(1)][\varphi(1)\psi(0)] \\
&= \psi(1)\varphi(1)
\end{aligned}$$

We have just shown that $\varphi(1)\psi(1) = \psi(1)\varphi(1)$, or,

$$\varphi(1)\psi(1) = (12)(123) = (23) = \psi(1)\varphi(1) = (123)(12) = (13)$$

a contradiction. Therefore, $C_2 \times C_3$ is not a coproduct. So products and coproducts differ in **Grp**. $\square$