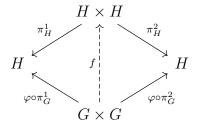
## Exercises Regarding **Grp** and **Ab**

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April 24, 2019

1. Let  $\varphi: G \to H$  be a morphism in a category C with products. Explain why there is a unique morphism  $(\varphi \times \varphi): G \times G \to H \times H$  compatible in the evident way with the natural projections.

Solution. Considering the universal property for  $G \times G$ , it provides us with the two natural projection morphisms:  $\pi_G^1: G \times G \to G$  and  $\pi_G^2: G \times G \to G$  mapping onto the first and second components, respectively. Let  $\pi_H^1, \pi_H^2$  be the respective natural projections for  $H \times H$ . We have the two morphisms  $\varphi \circ \pi_G^i: G \times G \to H$ , for each i, yielding the following diagram.



The universal property for  $H \times H$  guarantees the existence of a morphism  $f: G \times G \to H \times H$  for which  $\pi_H^2 \circ f = \varphi \circ \pi_G^2$  and  $\pi_H^1 \circ f = \varphi \circ \pi_G^1$ . If we think about how such a morphism  $\varphi \times \varphi$  should interact with the natural projections, we want the following relations equations to hold.

$$\pi_H^1(\varphi \times \varphi)(a,b) = \pi_H^1(\varphi(a),\varphi(b)) = \varphi(a)$$

$$\pi_H^2(\varphi \times \varphi)(a,b) = \pi_H^2(\varphi(a),\varphi(b)) = \varphi(b)$$

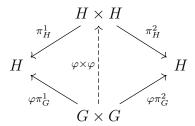
Observe that  $\varphi(a)$  is also obtained by mapping  $(a,b) \mapsto a \mapsto \varphi(a)$ , which is just  $\varphi(a) = \varphi(\pi_G^1(a,b))$ . Similarly,  $\varphi(b) = \varphi(\pi_G^2(a,b))$ . That is, a morphism obtained as

above is the best candidate for  $\varphi \times \varphi$ . Since such a morphism is unique, it's safe to say that this is the morphism we're looking for.

2. Let  $\varphi: G \to H$ ,  $\psi: H \to K$  be morphisms in a category with products, and consider morphisms between the products  $G \times G$ ,  $H \times H$ ,  $K \times K$  as in the previous exercise. Prove that

$$(\psi\varphi)\times(\psi\varphi)=(\psi\times\psi)(\varphi\times\varphi).$$

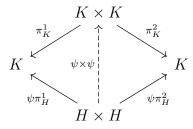
Solution. Similar to last time,  $\psi \circ \varphi \circ \pi_G^1$  and  $\psi \circ \varphi \circ \pi_G^2$  are morphisms from  $G \times G$  to K. The universal property for  $K \times K$  gives us a unique morphism, which we called  $(\psi \circ \phi) \times (\psi \circ \phi) : G \times G \to K \times K$ . To show equality, it suffices to prove that the morpism  $(\psi \times \psi) \circ (\varphi \times \varphi)$  satisfies the universal property for  $K \times K$ . From the diagram for  $H \times H$ 



we have the equations

$$\pi_H^i(\varphi \times \varphi) = \varphi \pi_G^i \tag{1}$$

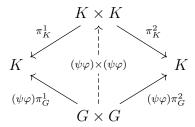
for i = 1, 2. Similarly, from the diagram for  $K \times K$ ,



we have the equations

$$\pi_K^i(\psi \times \psi) = \psi \pi_H^i \tag{2}$$

for i = 1, 2. Let us consider how the universal property for  $K \times K$  from  $G \times G$  should look, then we will use the above equations to show equality.



Thus, we need to prove  $\pi_K^i(\psi \times \psi)(\varphi \times \varphi) = (\psi \varphi)\pi_G^i$ . Applying (2) immediately yields

$$\begin{split} [\pi_K^i(\psi\times\psi)](\varphi\times\varphi) &= [\psi\pi_H^i](\varphi\times\varphi) \\ &= \psi[\pi_H^i(\varphi\times\varphi)] \\ &= \psi(\varphi\pi_G^i) & \text{(by equation (1))}. \end{split}$$

By the universal property of products, the morphism from  $G \times G$  to  $K \times K$  is unique, which proves equality.

3. Show that if G, H are abelian groups, then  $G \times H$  satisfies the universal property for coproducts in  $\mathbf{Ab}$ .

Solution. The obvious choice for embeddings of G, H into  $G \times H$  works here; denote them by  $i_G, i_H$ . Let  $Z \in \mathbf{Ab}$  and let  $\varphi_G : G \to Z$ ,  $\varphi_H : H \to Z$  be morphisms. Define the map

$$f: G \times H \to Z$$
  
 $(g,h) \mapsto \varphi_G(g)\varphi_H(h).$ 

We see that f is a map satisfying the commutativity of the diagram for a product:

$$f(i_G(g)) = f(g, 1)$$

$$= \varphi_G(g)\varphi_H(1)$$

$$= \varphi_G(g),$$

hence  $fi_G = \varphi_G$  and similarly,  $fi_H = \varphi_H$ . Let us see whether f preserves structure.

For  $(g_i, h_i) \in G \times H$ , where i = 1, 2, we have

$$f((g_1, h_1)(g_2, h_2)) = f((g_1g_2, h_1h_2))$$

$$= \varphi_G(g_1g_2)\varphi_H(h_1h_2)$$

$$= \varphi_G(g_1)\varphi_G(g_2)\varphi_H(h_1)\varphi_H(h_2)$$

$$= \varphi_G(g_1)\varphi_H(h_1)\varphi_G(g_2)\varphi_H(h_2)$$

$$= f((g_1, h_1))f((g_2, h_2)).$$

Therefore, f is a group homomorphism. To show existence, let  $\psi: G \times H \to Z$  be another such morphism for which  $\psi i_G = \varphi_G$  and  $\psi i_H = \varphi_H$ . Observe

$$\psi(i_G(g)) = \psi(g, 1) = \varphi_G(g)$$
$$= \varphi_G(g)\varphi_H(1)$$
$$= f(g, 1),$$

hence  $\psi(g,1) = f(g,1)$ . Similarly,  $\psi(1,h) = f(1,h)$ . For an arbitrary  $(g,h) \in G \times H$ , (g,h) = (g,1)(1,h), so these together prove  $\psi(g,h) = f(g,h)$ , so f is indeed unique. We can see precisely where f would fail to be a group homomorphism by looking at the above equations. In order for there to be coproducts in  $\mathbf{Ab}$  it is necessary that Z is an abelian group, for each Z. Otherwise, we would be unable to commute the elements into a form where f preserves structure. This is exactly the reason that the regular cartesian product fails to be a coproduct in  $\mathbf{Grp}$ . Finding a counterexample would consist of taking the cartesian product of two abelian groups, then considering the map from the product into a non-abelian group.

4. Consider the product of the cyclic groups  $C_2, C_3$ . By the exercise above,  $C_2 \times C_3$  is a coproduct in **Ab**. Show that it is *not* a coproduct of  $C_2, C_3$  in **Grp**.

Solution. Let us consider the coproduct mapping into the symmetric group of order 3,  $S_3$ . Define  $\varphi: C_2 \to S_3$  by  $n \mapsto (12)^n$  and  $\psi: C_3 \to S_3$  by  $n \mapsto (123)^n$ . It is easy to see that these are embedding homomorphisms. By the universal property there is a homomorphism  $\varphi \times \psi: C_2 \times C_3 \to S_3$ . Following the remark at the end of the previous exercise, let us investigate the commutativity between  $C_2 \times C_3$  and  $S_3$  through  $\varphi \times \psi$ . As of now, the value of  $\varphi \times \psi$  is undetermined. It is easy to see that, since  $(\varphi \times \psi)(0, m) = \psi(m)$  and  $(\varphi \times \psi)(n, 0) = \varphi(n)$ , it follows that  $(\varphi \times \psi)((n, 0)(0, m))) = \varphi(n)$ 

 $(\varphi \times \psi)(n,0)(\varphi \times \psi)(0,m) = \varphi(n)\psi(m)$ , since  $(\varphi \times \psi)$  is a homomorphism. However, this is taxing on the commutativity between (12) and (123) in  $S_3$ :

$$\begin{split} \varphi(1)\psi(1) &= (\varphi \times \psi)(1,1) = (\varphi \times \psi)(0+1,1+0) \\ &= (\varphi \times \psi)[(0,1)+(1,0)] \\ &= (\varphi \times \psi)(0,1)(\varphi \times \psi)(1,0) \\ &= [\varphi(0)\psi(1)][\varphi(1)\psi(0)] \\ &= \psi(1)\varphi(1) \end{split}$$

We have just shown that  $\varphi(1)\psi(1) = \psi(1)\varphi(1)$ , or,

$$\varphi(1)\psi(1) = (12)(123) = (23) = \psi(1)\varphi(1) = (123)(12) = (13)$$

a contradiction. Therefore,  $C_2 \times C_3$  is not a coproduct. So products and coproducts differ in **Grp**.