# Polytropic Stars

Stars are self gravitating globes of gas in kept in hydrostatic equilibrium by internal pressure support. The hydrostatic equilibrium condition, as mentioned earlier, is:

$$\frac{\partial P}{\partial x^\alpha} = \left(\rho + \frac{P}{c^2}\right) F_\alpha \tag{1}$$

Using a metric that is applicable to the inside of the spherically symmetric stellar matter distribution, one obtains from here the condition

$$\frac{dP}{dr} = -\frac{G}{r^2(1 - 2GM(r)/rc^2)}\left(M(r) + \frac{4\pi r^3 P(r)}{c^2}\right)\left(\rho(r) + \frac{P(r)}{c^2}\right) \tag{2}$$

where

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{3}$$

These equations are called the *Tolman-Oppenheimer-Volkoff* equations, or in short, TOV equations.

In the non-relativistic limit, the conditions $P \ll \rho c^2$, and $r \gg 2GM(r)/c^2$, yield the simpler hydrostatic equilibrium equations

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{4}$$

with

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{5}$$

These are valid when the particles providing the mass are non-relativistic. Instead of $M(r)$, these two equations can also be written in terms of the gravitational potential $\Phi(r)$:

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\rho \tag{6}$$

$$\nabla^2\Phi = 4\pi G\rho \tag{7}$$

The first equation gives the force balance and the second is the Poisson's equation determining the potential $\Phi$ from the mass distribution. For the spherically symmetric case of stars the Poisson's equation reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho \tag{8}$$

The pressure support in a star could have several origins:

1. Thermal Pressure of the hot gas, $P = \rho kT/\mu m_p$. Stars supported by thermal pressure constitute the multitude of luminous, shining stars that we are familiar with. Our Galaxy contains about $10^{11}$ such stars.
2. Degeneracy pressure of electrons, $P \propto \rho^{5/3}$ (non-relativistic) and $P \propto \rho^{4/3}$ (relativistic). Stars supported primarily by the electron degeneracy pressure are called *White Dwarfs*.
3. Degeneracy pressure of neutrons and repulsive strong interactions above nuclear density. Stars supported in this manner are called *Neutron Stars*
4. A giant bag of up, down and strange quarks, the pressure coming from their degeneracy pressure and a confinement pressure (called Bag pressure). Such stars are called *Quark Stars* or *Strange Stars*.

Normal Stars and White Dwarfs are in the regime where non-relativistic version of the hydrostatic equilibrium equation can be used. The energy per particle in these stars does not exceed the rest mass of the main mass providers (protons and neutrons), and the radii of these stars are much larger than their Schwarzschild radii.

We will start our description of stars with white dwarfs, since their equation of state has pressure as a function of density alone (such equations are called *barotropic*) which makes the solution of hydrostatic equilibrium a relatively simple task.

In either of the two regimes (non-relativistic and relativistic) the pressure of the degenerate electron gas has a power-law dependence on density: $P_e \propto \rho^\gamma$, with $\gamma$ taking on values $5/3$ and $4/3$ respectively. Such equations of state are called "polytropic", and are parametrized by a polytropic index $n$ which is defined to be $n = 1/(\gamma - 1)$.

Let us see how to solve the hydrostatic equilibrium for a general polytropic equation of state. Let the equation of state be written as

$$P = K\rho^\gamma \equiv K\rho^{1+\frac{1}{n}} \tag{9}$$

Using this in eq. 6 we can write

$$\frac{d\Phi}{dr} = -\gamma K \rho^{\gamma-2} \frac{d\rho}{dr} \tag{10}$$

For values of $\gamma$ other than 1 eq. 10 has a solution

$$\rho = \left( \frac{-\Phi}{(n+1)K} \right)^n \tag{11}$$

where $\Phi$ has been set to zero at the stellar surface ($\rho = 0$). This can now be introduced into the Poisson's equation (eq. 8) to yield a differential equation for $\Phi$:

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G \left( \frac{-\Phi}{(n+1)K} \right)^n \tag{12}$$

Defining dimensionless variables $z, w$ by

$$z = Ar, \quad A^2 = \frac{4\pi G}{(n+1)^n K^n} (-\Phi_c)^{n-1} = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$$

$$w = \frac{\Phi}{\Phi_c} = \left( \frac{\rho}{\rho_c} \right)^{1/n}$$

(where subscript c represents values at the centre of the star), we can write eq. 12 as

$$\frac{d^2 w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + w^n = 0$$

or

$$\frac{1}{z^2}\frac{d}{dz}\left( z^2 \frac{dw}{dz} \right) + w^n = 0 \tag{13}$$

which is called the Lane-Emden equation. For real stars the solutions must be finite at the centre, $z = 0$. This implies that $dw/dz$ must vanish at the centre. The central boundary conditions for solving the equation are then $w = 1$ (by definition) and $w' = 0$ at $z = 0$.

$$\rho(r) = \rho_c w^n, \quad \rho_c = \left( \frac{-\Phi_c}{(n+1)K} \right)^n \tag{14}$$

In general the Lane-Emden equation needs to be solved numerically. Only for $n = 0, 1$and 5 do analytic solutions for $w$ exist.

The surface of the polytropic star is defined by the value $z = z_n$, for which $\rho = 0$ and thus $w = 0$. For $n < 5$ the stellar radius is finite.

Table 1: Numerical values for polytropic models of different $n$

| $n$ | $z_n$ | $(-z^2 dw/dz)_{z=z_n}$ |
|---|---|---|
| 0 | 2.4494 | 4.8988 |
| 1 | 3.14159 | 3.14159 |
| 1.5 | 3.65375 | 2.71406 |
| 2 | 4.35287 | 2.41105 |
| 3 | 6.89685 | 2.01824 |
| 4 | 14.97155 | 1.79723 |
| 4.5 | 31.8365 | 1.73780 |
| 5 | $\infty$ | 1.73205 |

The mass $M(r)$ included within a radius $r$ can be found from

$$M(r) = \int_0^r 4\pi\rho r^2 dr = 4\pi\rho_c \int_0^r w^n r^2 dr = 4\pi\rho_c \frac{r^3}{z^3} \int_0^z w^n z^2 dz$$

From Lane-Emden equation it is seen that the last integrand is a total derivative and can therefore be integrated to yield

$$M(r) = 4\pi\rho_c r^3 \left(-\frac{1}{z}\frac{dw}{dz}\right)$$

and for the case of the surface, we get the total mass of the confuguration:

$$M = 4\pi\rho_c R^3 \left(-\frac{1}{z}\frac{dw}{dz}\right)_{z=z_n} \tag{15}$$

where $R = z_n/A$ is the radius of the configuration. The value of $z_n$ and $(-z^2 dw/dz)_{z=z_n}$, obtained numerically, are listed in table 1.

White dwarf models can be constructed starting from a chosen $\rho_c$. From the definition of $A$, we find the radius

$$R = \frac{z_n}{A} = z_n \left(\frac{(n+1)K}{4\pi G}\right)^{1/2} \rho_c^{\frac{1-n}{2n}} \tag{16}$$

The mass

$$M \propto \rho_c R^3 \propto \rho_c^{\frac{3-n}{2n}}$$

and hence

$$R \propto M^{\frac{1-n}{3-n}}$$

The white dwarf equation of state corresponds to $n = 3/2$ in the non-relativistic limit and $n = 3$ in the relativistic limit:

$$P_e = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e}\left(\frac{\rho}{\mu_e m_p}\right)^{5/3} \quad \text{(non-relativistic)} \tag{17}$$

$$P_e = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\left(\frac{\rho}{\mu_e m_p}\right)^{4/3} \quad \text{(relativistic)} \tag{18}$$

In the non-relativistic limit the radius of the white dwarf decreases with mass: $R \propto M^{-1/3}$. In the relativistic limit, however, $M \propto R^{(3-n)/(1-n)}$ becomes independent of radius or central density and reaches an unique limiting value for a given K:

$$M = 4\pi\left(-\frac{w'}{z}\right)_{z_3} z_3^3\left(\frac{K}{\pi G}\right)^{3/2} \tag{19}$$

Using $K$ for the degenerate electron gas (above) we find the limiting mass of a white dwarf:

$$M_{\mathrm{Ch}} = \frac{5.836}{\mu_e^2}\, M_\odot \tag{20}$$

which is known as the Chandrasekhar limit. For a typical white dwarf composition $\mu_e = 2$ and the limiting mass works out to be 1.46 $M_\odot$. Masses larger than this cannot be supported by electron degeneracy pressure against self gravity. Such configurations, if devoid of thermal pressure support, must then collapse to higher densities.

We have seen here an example of application of polytropic solutions to stars with barotropic equations of state. In normal stars, however, pressure is a function of two variables, density and temperature (for a fixed composition), so polytropic relations are not applicable in general. However, there are several special situations which impose a pre-determined relation between density and temperature that allows one to write pressure as a function of density alone, and in some of these situations polytropic equations give a good description of the star or parts of the star. We will encounter such situations when we discuss normal stars in detail.