Lorentz Transformations and Gamma Factor

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Abstract

Quantum Observables

1 Lorentz Transformations

Define an event to have spacetime coordinates (t, x, y, z) in system S and (t', x', y', z') in a reference frame moving at a velocity v with respect to that frame S. Then the Lorentz transformation specifies that these coordinates are related in the following way

$$t' = \gamma\ (t - vx)\,, \tag{1.1}$$
$$x' = \gamma\ (x - vt)\,, \tag{1.2}$$
$$y' = y\,, \tag{1.3}$$
$$z' = z, \tag{1.4}$$

where γ is the Lorentz factor given by the following expression

$$\gamma = \frac{1}{\sqrt{1 - v^2}}\,. \tag{1.5}$$

The inverse Lorentz transformations are given by

$$t = \gamma\ (t' + vx')\,, \tag{1.6}$$
$$x = \gamma\ (x' + vt')\,, \tag{1.7}$$
$$y = y'\,, \tag{1.8}$$
$$z = z', \tag{1.9}$$

2 Relativity of Simultaneity

In the special theory of relativity there is no privileged observer since space is not absolute and it is in the same footing as time coordinate. If an observer in reference frame S watches two events happening in simultaneously another observer in an moving inertial refernce frame S' may not agree with the simultaneity of the two events. This is known as the lack of absolute simultaneity. This is readily seen by the Lorentz transformations.

Considering two events $E_a = (t_a, x_a)$ and $E_b = (t_b, x_b)$ perceived by an observer A in reference frame S. The two events are simultaneous for this observer if $t_a = t_b$ and they do not happen in the same place $x_a \neq x_b$. For simplicity consider $t_a = x_a = 0$, then for another observer in S' that moves with speed v with respect to S the two events happen in spacetime coordinates

$$t'_a = \gamma\,(t_a - vx_a) = 0\,, \tag{2.1}$$

$$x'_a = \gamma\,(x_a - vt_a) = 0\,, \tag{2.2}$$

$$t'_b = \gamma\,(t_b - vx_b) = -\gamma v x_b\,, \tag{2.3}$$

$$x'_b = \gamma\,(x_b - vt_b) = \gamma x_b\,. \tag{2.4}$$

For the observer in S' event (t_b, x_b) happens before event (t_a, x_a).

3 Time Dilation

In relativity time dilation is the difference in the perception of elapsed time as measured by two clocks in different reference frames. It is either due to a relative velocity between the frames in special relativity or due to a difference in gravitational potential between their locations which is known as gravitational time dilation in the framework of general relativity.

Time dilation can be inferred from the observed constancy of the speed of light in all reference frames dictated by the second postulate of special relativity. It can be deduced from a simple thought experiment. Consider a vertical clock consisting of two mirrors A and B between which a light pulse is bouncing. The separation of the mirrors is L and the clock ticks once each time the light pulse hits either of the mirrors. In the frame in which the clock is at rest the light pulse traces out a path of length $2L$ and the period of the clock is $2L$ divided by the speed of light c

$$\Delta t = \frac{2L}{c}\,. \tag{3.1}$$

The schematics of the thought experiment is shown in the figure bellow. The figure on the left is how the observer in the rest refence frame S perceives the emitting light pulse and the figure on the right is the perception from the point of view of an observer in the moving inertial reference frame S'.

From the frame of reference of a moving observer traveling at the relative speed v relative to the resting frame of the clock on the figure on the right, the light pulse is seen as tracing out a longer angled path. The total time for the light pulse to trace its path as perceived by observer in S' is given by:

$$\Delta t' = \frac{2D}{c}\,, \tag{3.2}$$

the length of the half path $2D$ can be calculated with a simple application of Pithagoras theorem

$$D = \sqrt{\left(\frac{1}{2}v\Delta t'\right)^2 + L^2}\,. \tag{3.3}$$

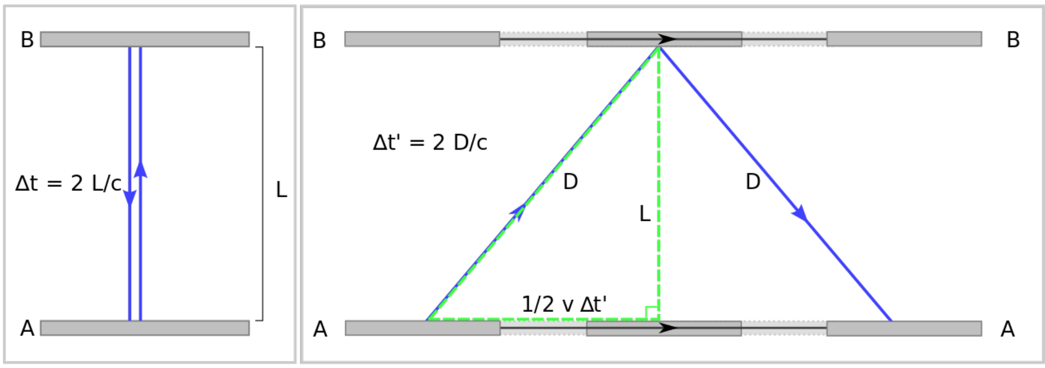

Figure 1: Schematics of the thought experiment that deduces time time dilation. Figure on the left is form the perspective of the observer at rest in the reference frame S. Figure on the right is from the perspective of observer in moving refence frame S'

Eliminating D and L from these equations results in

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.4}$$

which expresses the fact that the moving observer's period of the clock $\Delta t'$ is longer than the period Δt in the rest frame S. Another way of deducing this result is using the Lorentz transformation. Suppose the observer in S perceives the ticks of its clock in a period Δt and in the same location $\Delta x = 0$, then from the Lorentz transformation

$$\Delta t' = \gamma \Delta t = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.5}$$

4 Lengh Contraction

Length contraction is the phenomenon that a moving object's length is measured to be shorter than its proper length, which is the length as measured in the object's own rest frame. It only happens in the direction in which the particle is traveling. To deduce the result for the length contraction suppose that an observer in a rest frame S measures the length of a rod as $L = x_2 - x_1$ where x_2 and x_1 are the space coordinates of both ends of the rod measured at times respectively t_2 and t_1 and $\Delta t = t_2 - t_1$. According to the Lorentz transformation for an observer in a moving inertial reference frame S' the lenght of the rod $L' = x'_2 - x'_1$ and the time interval $\Delta t'$ are given by the following expressions

$$L' = x'_2 - x'_1 = \gamma(L - v\Delta t)\,, \tag{4.1}$$

$$\Delta t' = t'_2 - t'_1 = \gamma(\Delta t - vL)\,, \tag{4.2}$$

and the measures in the frame S' must be made at the same time, that is $\Delta t' = 0$, then for this frame of reference the time period Δt that happened in the frame S is given by

$$\Delta t = vL\,, \tag{4.3}$$

and inserting this result in equation (4.1)

$$L' = \gamma(1 - v^2)L \tag{4.4}$$

but $1 - v^2 = \gamma^{-2}$ and we finally arrive at the formula for the length contraction due to the relative movement between two inertial reference frames

$$L' = \frac{L}{\gamma}\,. \tag{4.5}$$

5 Exercises

5.1 Signal Emiting Observation with Telescope

Two clocks located at the origins of the K and K' systems, which have a relative speed v, are synchronized when the origins coincide. After a time t, an observer at the origin of the K system observes the K' clock by means of a telescope. What does the K clock read?

solution:

The signal that arrives at the origin of K was send from K' at a time τ and the distance from K is $v\tau$. It takes a time $v\tau/c$ for the light signal to reach K. The total elapsed time from the emited signal and the observation at the origin is

$$t = \tau + \frac{v\tau}{c}\,, \tag{5.1}$$

and solving for τ

$$\tau = \frac{t}{1 + \frac{v}{c}} \tag{5.2}$$

and the distance the K' frame traveled is $v\tau$, from Lorentz transformation

$$\tau' = \gamma\left(\tau - \frac{v}{c^2}v\tau\right) = \gamma\tau\left(1 - \frac{v^2}{c^2}\right) = \frac{\tau}{\gamma}\,. \tag{5.3}$$

and inserting the expression from equation (5.2) in equation (5.3)

$$\tau' = \sqrt{\frac{c-v}{c+v}}t\,. \tag{5.4}$$

5.2 Synchronized clocks

Synchronized clocks are stationed at regular intervals, a million km apart, along a straight line. When the clock next to you reads T hours

(a) What time do you see on the nth clock down the line?

(b) What time do you observe on that clock?

solution:

Supposing the clocks are regularly separated by a distance Δx, so the light from the nth clock will take a time $\Delta t = n\Delta x/c$ to reach me at the origin. So the time I see on the nth clock is just

$$t = T - \frac{n\Delta x}{c}\,, \tag{5.5}$$

and since all clocks are synchronize they all read the time T and that is the time observed.

5.3 Traveling Star

A star is traveling with speed v at an angle θ to the line of sight. What is its apparent speed across the sky?

solution:

Suppose the light signal passes two points in space, from point a to point b with speed v at an angle θ from the line of sight for an observer in Earth. The light signal from the star leaves point a at a time τ_a and leaves point b at time τ_b. The distance from point a to Earth is x_a and from point b to Earth is x_b. The two light signals reach the observer in Earth at respective times

$$t_a = \tau_a + \frac{x_a}{c}\,, \tag{5.6}$$

$$t_b = \tau_b + \frac{x_b}{c}\,. \tag{5.7}$$

The star travels with speed v and it traces out a path from point a to point b given by the following expression

$$x_b - x_a = -v\left(\tau_b - \tau_a\right)\cos\theta\,, \tag{5.8}$$

and the total time period elapsed from the emission of the light signals to reach Earth is

$$\Delta t = t_b - t_a = \tau_b - \tau_a + \frac{x_b - x_a}{c}\,, \tag{5.9}$$

inserting the relation found for $x_b - x_a$ in the above equation and naming $\Delta\tau = \tau_b - \tau_a$

$$\Delta t = \Delta\tau\left(1 - \frac{v}{c}\cos\theta\right)\,, \tag{5.10}$$

from the perspective of the observer in Earth the star travels the apparent distance

$$\Delta s = v\Delta\tau\sin\theta = \frac{v\Delta t\sin\theta}{1 - \frac{v}{c}\cos\theta}\,, \tag{5.11}$$

and the apparent speed across the sky is

$$u = \frac{\Delta s}{\Delta t} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} \tag{5.12}$$

one can calculate the angle for maximum speed simply by differentiating u in relation to θ and setting it to zero then solving for $\cos\theta$ which reads

$$\cos\theta = \frac{v}{c} \tag{5.13}$$

and at this maximum angle the maximum speed is given by

$$u = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}} . \tag{5.14}$$

Notice that even if $v < c$ the speed u can be very high, since as v approaches c the speed $u \to \infty$.

5.4 Muon Life Time

A muon has a proper life time of τ and it is observed to travel a distance Δx in a laboratory. What is its speed?

solution:

If we simply calculate the speed as $\Delta\tau/d$ this would result in a speed greater than the speed of light. This happens since we have to take into account time dilation for the muon life time considering it moving in a inertial moving reference frame. From the perspective of the observer in the laboratory (rest frame) the time elapsed is

$$\Delta t = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} , \tag{5.15}$$

the speed measured by the observer in the laboratory is

$$v = \frac{\Delta x}{\Delta t} = \sqrt{1-\frac{v^2}{c^2}}\frac{\Delta x}{\Delta\tau} , \tag{5.16}$$

solving for v and considering $u = \Delta x/\Delta t$ we arrive at the following equation

$$v = \frac{u}{\sqrt{1+\frac{u^2}{c^2}}} , \tag{5.17}$$

5.5 Racing Spaceships

Spaceship A has proper length L_A and spaceship B has proper length L_B when they are at rest. They begin a race and a third observer C in a rest frame observes they have the same length. If the spaceship A has speed v_A what is the speed for the spaceship B?

solution:

The observer at rest notices two moving inertial reference frames A and B with respectively Lorentz factors γ_A and γ_B. From the length contraction, observer C perceives the length for the spaceship A as L_A/γ_A and the length of the spaceship B as L_B/γ_B and he notices that they are the same $L_A/\gamma_A = L_B/\gamma_B$ and we have the following equation

$$L_A^2\left(1-\frac{v_A^2}{c^2}\right) = L_B\left(1-\frac{v_B^2}{c^2}\right)\,, \tag{5.18}$$

and the speed of the spaceship B is given by

$$v_B = \sqrt{1-\frac{L_A^2}{L_B^2}\left(1-\frac{v_A^2}{c^2}\right)}c\,, \tag{5.19}$$

5.6 Racing Spaceships

A rod with proper length L moves making an angle θ with respect to the ground when an observer in a rest frame measures the speed of the rod as v. Calculate the angle θ that the observer at the rest frame observes.

solution:

The orthogonal projection of the proper length is $L\sin\theta$ and the parallel (to the movement) projection is $L\cos\theta$. We know that the Lorentz length contraction only takes place with lengths that are parallel to the movement of the frame, while the orthogonal does not suffer length contraction. So the tangent of the angle θ' observed in the rest frame is

$$\tan\theta' = \frac{L\sin\theta}{\frac{1}{\gamma}L\cos\theta} = \gamma\tan\theta \tag{5.20}$$

References

[Shannon, 1948] A Mathematical Theory of Communication.