

# Quantum Observables

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## Abstract

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## 1 Time Evolution

The Hamiltonian generates the time evolution of quantum states. If  $|\psi(t)\rangle$  is the state of the system at time  $t$ , then

$$H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle. \quad (1.1)$$

This equation is the Schrödinger equation. It takes the same form as the Hamilton-Jacobi equation, which is one of the reasons  $H$  is also called the Hamiltonian. Given the state at some initial time ( $t = 0$ ), we can solve it to obtain the state at any subsequent time. In particular, if  $H$  is independent of time, then

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(t=0)\rangle. \quad (1.2)$$

The exponential operator on the right hand side of the Schrodinger equation is usually defined by the corresponding power series in  $H$ .

$$U = e^{-iHt/\hbar}, \quad (1.3)$$

is a unitary operator. It is the time evolution operator, or propagator, of a closed quantum system. If the Hamiltonian is time-independent,  $U(t)$  form a one parameter unitary group, this gives rise to the physical principle of detailed balance.

In the more general formalism of Dirac, the Hamiltonian is typically implemented as an operator on a Hilbert space in the following way: The eigenkets  $|a\rangle$  (eigenvectors) of  $H$ , provide an orthonormal basis for the Hilbert space. The spectrum of allowed energy levels of the system is given by the set of eigenvalues, denoted  $E_a$ , solving the eigenvalue equation

$$H|a\rangle = E_a|a\rangle, \quad (1.4)$$

Since  $H$  is a Hermitian operator, the energy is always a real number. From a mathematically rigorous point of view, care must be taken with the above assumptions. Operators on infinite-dimensional Hilbert spaces need not have eigenvalues. The set of eigenvalues does not necessarily coincide with the spectrum of an operator. However, all routine quantum mechanical calculations can be done using the physical formulation.

## 2 Three Level System - Example

Considering a quantum system in a Hilbert space, span by an orthonormal basis of three eigenvectors  $|1\rangle, |2\rangle, |3\rangle$ , that are degenerate eigenvector of an observable  $W$  with the same eigenvalues  $\omega$ . The Hamiltonian  $H$  for the system in those states acts as the following

$$H|1\rangle = \Omega|1\rangle + \Omega|3\rangle, \quad (2.1)$$

$$H|2\rangle = \Omega|2\rangle + \Delta|3\rangle, \quad (2.2)$$

$$H|3\rangle = \Omega|1\rangle + \Delta|2\rangle + \Omega|3\rangle, \quad (2.3)$$

where  $\Omega$  and  $\Delta$  are real constants with dimensions of energy. The matrix for the Hamiltonian for that basis is

$$\langle i|H|j\rangle = \begin{pmatrix} \Omega & 0 & \Omega \\ 0 & \Omega & \Delta \\ \Omega & \Delta & \Omega \end{pmatrix}. \quad (2.4)$$

The matrix for the observable  $W$  is,

$$\langle i|W|j\rangle = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \omega I. \quad (2.5)$$

It is straightforward to show that the observable  $W = \omega I$  commutes with  $H$ , since it is proportional to the identity matrix. That means it is possible to measure both  $H$  and  $W$  simultaneously. To find the eigenvectors, and eigenvalues for the Hamiltonian one has to solve the secular equation  $\det|H - \lambda I| = 0$ , in matrix form

$$\begin{vmatrix} \Omega - \lambda & 0 & \Omega \\ 0 & \omega - \lambda & \Delta \\ \Omega & \Delta & \Omega - \lambda \end{vmatrix} = 0. \quad (2.6)$$

The resulting polynomial is

$$(\Omega - \lambda)^3 - (\Omega - \lambda)(\Omega^2 + \Delta^2) = 0. \quad (2.7)$$

Solving the polynomial equation for  $\lambda$ ,

$$\lambda_1 = \Omega, \quad (2.8)$$

$$\lambda_2 = \Omega + \sqrt{\Omega^2 + \Delta^2}, \quad (2.9)$$

$$\lambda_3 = \Omega - \sqrt{\Omega^2 + \Delta^2}. \quad (2.10)$$

Considering the eigenvector for the Hamiltonian to be  $|\psi_i\rangle = (a_i, b_i, c_i)$ , where  $a_i, b_i, c_i$  are complex numbers to be determined, it is straightforward to solve the matrix equation

$(H - \lambda I)|\psi_i\rangle = 0$ . In matrix form, this equation is

$$\begin{pmatrix} \Omega - \lambda_i & 0 & \Omega \\ 0 & \omega - \lambda_i & \Delta \\ \Omega & \Delta & \Omega - \lambda_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.11)$$

Solving for  $a_i, b_i, c_i$  and after normalization, the result for the eigenvectors  $|\lambda_i\rangle$  for the Hamiltonian are

$$|\lambda_1\rangle = \frac{\Delta|1\rangle - \Omega|2\rangle}{\sqrt{\Omega^2 + \Delta^2}}, \quad (2.12)$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \left( \frac{\Omega|1\rangle + \Delta|2\rangle}{\sqrt{\Omega^2 + \Delta^2}} + |3\rangle \right), \quad (2.13)$$

$$|\lambda_3\rangle = \frac{1}{\sqrt{2}} \left( \frac{\Omega|1\rangle + \Delta|2\rangle}{\sqrt{\Omega^2 + \Delta^2}} - |3\rangle \right). \quad (2.14)$$

Supposing that the initial state  $|\psi_0\rangle$ , at the instant  $t = 0$  is the eigenvector  $|\lambda_1\rangle$ , then for an instant  $t > 0$ , the state  $|\psi_t\rangle$  can be found by applying the time evolution operator  $\exp(-iHt/\hbar)$  to the initial state

$$|\psi_t\rangle = e^{-iHt/\hbar}|\lambda_1\rangle = e^{-i\Omega t/\hbar} \frac{\Delta|1\rangle - \Omega|2\rangle}{\sqrt{\Omega^2 + \Delta^2}}. \quad (2.15)$$

## References

- [1] Cohen-Tannoudji, Quantum Mechanics, vol.2.
- [2] David J. Griffiths, Quantum Mechanics.