Quantum Observables

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Abstract

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1 Time Evolution

The Hamiltonian generates the time evolution of quantum states. If $|\psi(t)\rangle$ is the state of the system at time t, then

$$H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle. \tag{1.1}$$

This equation is the Schrödinger equation. It takes the same form as the Hamilton-Jacobi equation, which is one of the reasons H is also called the Hamiltonian. Given the state at some initial time ($t = 0$), we can solve it to obtain the state at any subsequent time. In particular, if H is independent of time, then

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(t=0)\rangle. \tag{1.2}$$

The exponential operator on the right hand side of the Schrodinger equation is usually defined by the corresponding power series in H.

$$U = e^{-iHt/\hbar}, \tag{1.3}$$

is a unitary operator. It is the time evolution operator, or propagator, of a closed quantum system. If the Hamiltonian is time-independent, $U(t)$ form a one parameter unitary group, this gives rise to the physical principle of detailed balance.

In the more general formalism of Dirac, the Hamiltonian is typically implemented as an operator on a Hilbert space in the following way: The eigenkets $|a\rangle$ (eigenvectors) of H, provide an orthonormal basis for the Hilbert space. The spectrum of allowed energy levels of the system is given by the set of eigenvalues, denoted E_a, solving the eigenvalue equation

$$H|a\rangle = E_a|a\rangle, \tag{1.4}$$

Since H is a Hermitian operator, the energy is always a real number. From a mathematically rigorous point of view, care must be taken with the above assumptions. Operators on infinite-dimensional Hilbert spaces need not have eigenvalues . The set of eigenvalues does not necessarily coincide with the spectrum of an operator. However, all routine quantum mechanical calculations can be done using the physical formulation.

2 Three Level System - Example

Considering a quantum system in a Hilbert space, span by an orthonormal basis of three eigenvectors $|1\rangle, |2\rangle, |3\rangle$, that are degenerate eigenvector of an observable W with the same eigenvalues ω. The Hamiltonian H for the system in those states acts as the following

$$H|1\rangle = \Omega|1\rangle + \Omega|3\rangle, \tag{2.1}$$
$$H|2\rangle = \Omega|2\rangle + \Delta|3\rangle, \tag{2.2}$$
$$H|3\rangle = \Omega|1\rangle + \Delta|2\rangle + \Omega|3\rangle, \tag{2.3}$$

where Ω and Δ are real constants with dimensions of energy. The matrix for the Hamiltonian for that basis is

$$\langle i|H|j\rangle = \begin{pmatrix} \Omega & 0 & \Omega \\ 0 & \Omega & \Delta \\ \Omega & \Delta & \Omega \end{pmatrix}. \tag{2.4}$$

The matrix for the observable W is,

$$\langle i|W|j\rangle = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \omega I. \tag{2.5}$$

It is straightforward to show that the observable $W = \omega I$ commutes with H, since it is proportional to the identity matrix. That means it is possible to measure both H and W simultaneously. To find the eigenvectors, and eigenvalues for the Hamiltonian one has to solve the secular equation $\det|H - \lambda I| = 0$, in matrix form

$$\begin{vmatrix} \Omega - \lambda & 0 & \Omega \\ 0 & \omega - \lambda & \Delta \\ \Omega & \Delta & \Omega - \lambda \end{vmatrix} = 0. \tag{2.6}$$

The resulting polynomial is

$$(\Omega - \lambda)^3 - (\Omega - \lambda)(\Omega^2 + \Delta^2) = 0. \tag{2.7}$$

Solving the polynomial equation for λ,

$$\lambda_1 = \Omega, \tag{2.8}$$
$$\lambda_2 = \Omega + \sqrt{\Omega^2 + \Delta^2}, \tag{2.9}$$
$$\lambda_3 = \Omega - \sqrt{\Omega^2 + \Delta^2}. \tag{2.10}$$

Considering the eigenvector for the Hamiltonian to be $|\psi_i\rangle = (a_i, b_i, c_i)$, where a_i, b_i, c_i are complex numbers to be determined, it is straightforward to solve the matrix equation

$(H - \lambda I)|\psi_i\rangle = 0$. In matrix form, this equation is

$$\begin{pmatrix} \Omega - \lambda_i & 0 & \Omega \\ 0 & \omega - \lambda_i & \Delta \\ \Omega & \Delta & \Omega - \lambda_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.11}$$

Solving for a_i, b_i, c_i and after normalization, the result for the eigenvectors $|\lambda_i\rangle$ for the Hamiltonian are

$$|\lambda_1\rangle = \frac{\Delta|1\rangle - \Omega|2\rangle}{\sqrt{\Omega^2 + \Delta^2}}, \tag{2.12}$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \left(\frac{\Omega|1\rangle + \Delta|2\rangle}{\sqrt{\Omega^2 + \Delta^2}} + |3\rangle \right), \tag{2.13}$$

$$|\lambda_3\rangle = \frac{1}{\sqrt{2}} \left(\frac{\Omega|1\rangle + \Delta|2\rangle}{\sqrt{\Omega^2 + \Delta^2}} - |3\rangle \right). \tag{2.14}$$

Supposing that the initial state $|\psi_0\rangle$, at the instant $t = 0$ is the eigenvector $|\lambda_1\rangle$, then for an instant $t > 0$, the state $|\psi_t\rangle$ can be found by applying the time evolution operator $\exp(-iHt/\hbar)$ to the initial state

$$|\psi_t\rangle = e^{-iHt/\hbar}|\lambda_1\rangle = e^{-i\Omega t/\hbar}\frac{\Delta|1\rangle - \Omega|2\rangle}{\sqrt{\Omega^2 + \Delta^2}}. \tag{2.15}$$

References

[1] Cohen-Tannoudji, Quantum Mechanics, vol.2.

[2] David J. Griffiths, Quantum Mechanics.