

Quantum Observables

O.L. Santos-Pereira

e-mail: osvaldo.pereira@sciencedata.ai

Abstract

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1 Time Evolution

The Hamiltonian generates the time evolution of quantum states. If $|\psi(t)\rangle$ is the state of the system at time t , then

$$H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle. \quad (1.1)$$

This equation is the Schrödinger equation. It takes the same form as the Hamilton-Jacobi equation, which is one of the reasons H is also called the Hamiltonian. Given the state at some initial time ($t = 0$), we can solve it to obtain the state at any subsequent time. In particular, if H is independent of time, then

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(t=0)\rangle. \quad (1.2)$$

The exponential operator on the right hand side of the Schrodinger equation is usually defined by the corresponding power series in H .

$$U = e^{-iHt/\hbar}, \quad (1.3)$$

is a unitary operator. It is the time evolution operator, or propagator, of a closed quantum system. If the Hamiltonian is time-independent, $U(t)$ form a one parameter unitary group, this gives rise to the physical principle of detailed balance.

In the more general formalism of Dirac, the Hamiltonian is typically implemented as an operator on a Hilbert space in the following way: The eigenkets $|a\rangle$ (eigenvectors) of H , provide an orthonormal basis for the Hilbert space. The spectrum of allowed energy levels of the system is given by the set of eigenvalues, denoted E_a , solving the eigenvalue equation

$$H|a\rangle = E_a|a\rangle, \quad (1.4)$$

Since H is a Hermitian operator, the energy is always a real number. From a mathematically rigorous point of view, care must be taken with the above assumptions. Operators on infinite-dimensional Hilbert spaces need not have eigenvalues. The set of eigenvalues does not necessarily coincide with the spectrum of an operator. However, all routine quantum mechanical calculations can be done using the physical formulation.

2 Three Level System - Example

Considering a quantum system in a Hilbert space, span by an orthonormal basis of three eigenvectors $|1\rangle, |2\rangle, |3\rangle$, that are degenerate eigenvector of an observable W with the same eigenvalues ω . The matrix form for the observable W is given by the following expression

$$\langle i|W|j\rangle = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \omega I. \quad (2.1)$$

Let us consider the Hamiltonian H_0 for an isolated system given by

$$H_0 = \Omega|1\rangle\langle 1| + \Omega|2\rangle\langle 2| + \Omega|3\rangle\langle 3|, \quad (2.2)$$

now let us consider the following perturbation A given by

$$A = \Omega|1\rangle\langle 3| + \Omega|3\rangle\langle 1|, \quad (2.3)$$

and another external perturbation B given by

$$B = \Delta|3\rangle\langle 2| + \Delta|2\rangle\langle 3|, \quad (2.4)$$

and considering the full perturbed Hamiltonian system given by the sum of the isolated system H_0 and the two perturbations A and B

$$H = H_0 + A + B. \quad (2.5)$$

It is straightforward to see that the Hamiltonian H for the system acts as the following in the eigenvectors

$$H|1\rangle = \Omega|1\rangle + \Omega|3\rangle, \quad (2.6)$$

$$H|2\rangle = \Omega|2\rangle + \Delta|3\rangle, \quad (2.7)$$

$$H|3\rangle = \Omega|1\rangle + \Delta|2\rangle + \Omega|3\rangle, \quad (2.8)$$

where Ω and Δ are real constants with dimensions of energy. The matrix form for the perturbed Hamiltonian in that basis is given by

$$\langle i|H|j\rangle = \begin{pmatrix} \Omega & 0 & \Omega \\ 0 & \Omega & \Delta \\ \Omega & \Delta & \Omega \end{pmatrix}. \quad (2.9)$$

It is straightforward to show that the observable $W = \omega I$ commutes with H , since it is proportional to the identity matrix, and this means that it is possible to measure both

H and W simultaneously. To find the eigenvectors and eigenvalues for the perturbed Hamiltonian, one has to solve the secular equation $\det |H - \lambda I| = 0$, in matrix form

$$\begin{vmatrix} \Omega - \lambda & 0 & \Omega \\ 0 & \Omega - \lambda & \Delta \\ \Omega & \Delta & \Omega - \lambda \end{vmatrix} = 0. \quad (2.10)$$

The resulting characteristic polynomial for this eigenvalue equation is the following

$$(\Omega - \lambda)^3 - (\Omega - \lambda)(\Omega^2 + \Delta^2) = 0. \quad (2.11)$$

Solving the polynomial equation for λ , one arrives at the following equations for the eigenvalues for the perturbed Hamiltonian, giving rise to now to a set of non degenerate eigenvalues due to the two external perturbations

$$\lambda_1 = \Omega, \quad (2.12)$$

$$\lambda_2 = \Omega + \sqrt{\Omega^2 + \Delta^2}, \quad (2.13)$$

$$\lambda_3 = \Omega - \sqrt{\Omega^2 + \Delta^2}. \quad (2.14)$$

Considering the eigenvector for the Hamiltonian to be $|\psi_i\rangle = (a_i, b_i, c_i)$, where a_i, b_i, c_i are complex numbers to be determined, it is straightforward to solve the matrix equation $(H - \lambda I)|\psi_i\rangle = 0$. In matrix form, this equation is

$$\begin{pmatrix} \Omega - \lambda_i & 0 & \Omega \\ 0 & \Omega - \lambda_i & \Delta \\ \Omega & \Delta & \Omega - \lambda_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.15)$$

Solving for a_i, b_i, c_i the result for the eigenvectors $|\lambda_i\rangle$ for the Hamiltonian are

$$|\lambda_1\rangle = \frac{\Delta}{\sqrt{\Omega^2 + \Delta^2}}|1\rangle - \frac{\Omega}{\sqrt{\Omega^2 + \Delta^2}}|2\rangle, \quad (2.16)$$

$$\sqrt{2}|\lambda_2\rangle = -\frac{\Omega}{\sqrt{\Omega^2 + \Delta^2}}|1\rangle - \frac{\Delta}{\sqrt{\Omega^2 + \Delta^2}}|2\rangle + |3\rangle, \quad (2.17)$$

$$\sqrt{2}|\lambda_3\rangle = \frac{\Omega}{\sqrt{\Omega^2 + \Delta^2}}|1\rangle + \frac{\Delta}{\sqrt{\Omega^2 + \Delta^2}}|2\rangle + |3\rangle. \quad (2.18)$$

Supposing that the initial state $|\psi_0\rangle$, at the instant $t = 0$ is the eigenvector $|\lambda_1\rangle$, then for an instant $t > 0$, the state $|\psi_t\rangle$ can be found by applying the time evolution operator $\exp(-iHt/\hbar)$ to the initial state

$$|\psi_t\rangle = e^{-iHt/\hbar}|\lambda_1\rangle = e^{-i\Omega t/\hbar} \frac{\Delta|1\rangle - \Omega|2\rangle}{\sqrt{\Omega^2 + \Delta^2}}. \quad (2.19)$$

References

- [1] Cohen-Tannoudji, Quantum Mechanics, vol.2.
- [2] David J. Griffiths, Quantum Mechanics.