EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the second semester 2015

14 April 2015

Part 1

Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  electromagnetism, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding page of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- Do not write anything on the list of constants and formulae.
  Return it at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

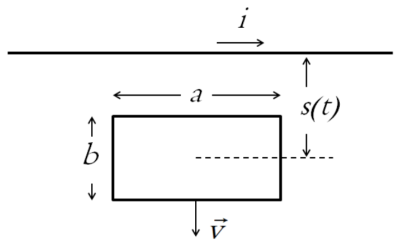
Q1. A rectangular conducting loop (length a, width b and resistance R) is located in the vicinity of an infinitely long straight wire that is carrying a current i to the right, as shown in the figure. The loop moves away from the wire with a constant speed $\vec{v}$, so that the distance of the center of the coil to the wire is given by $s(t) = s_0 + vt$. Calculate:

a) the module of the magnetic field produced by the current at a point distant r from the wire. Indicate the direction of the field in the region enclosed by the loop;

b) the magnetic flux in the region delimited by the loop for a given $s(t)$;

c) the electromotive force induced in the loop at the distance $s(t)$;

d) the current induced in the loop, i_{ind}. Indicate the direction of that current.



Q2. A conductive medium has electric conductivity σ, magnetic permeability μ_0 and electric permittivity $\epsilon = K\epsilon_0$, where K is the real dielectric constant. The wave equation for the electric field in this medium is given by $\nabla^2\vec{E} - K\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \frac{\sigma}{\epsilon_0}\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0$, with $\frac{1}{c^2} = \mu_0\epsilon_0$.

a) Show that the monochromatic plane wave function $\vec{E}(z,t) = \vec{E}_0 e^{i(\omega t - \tilde{q}z)}$ is a solution of the differential equation above. Find the relationship between the complex wavenumber, $\tilde{q}$, and the angular frequency, ω, so that $\vec{E}(z,t)$ is the solution. Show also that $\tilde{q}$ becomes real in the case of an insulating medium.

b) Find the complex dielectric constant, $\tilde{K}$, using the relationship between the wave number and the dielectric constant, $\tilde{q^2} = \tilde{K}\frac{\omega^2}{c^2}$. Check that the real part of $\tilde{K}$ equals K, as expected, and write the imaginary part of $\tilde{K}$.

c) Make the approximation for low frequencies in the complex dielectric constant expression of item (b) and calculate the complex refractive index, $\tilde{n} = \sqrt{\tilde{K}}$. Show that the real and imaginary parts of $\tilde{n}$ are the same in this case.

d) The penetration depth of the wave in a conductive medium, δ , is given by the inverse of the imaginary part of the wavenumber, q_i, or $\delta = 1/q_i$. Remember that $\tilde{q} = \tilde{n}\frac{\omega}{c}$ and calculate the penetration depth for silver (Ag) in the region of microwave ($f = \frac{\omega}{2\pi} = 10$ GHz), for which the approximation on item (c) is valid. The conductivity of the silver is in this frequency range $\sigma_{Ag} = 3 \times 10^{+7}(\Omega m)^{-1}$. Simplify the numerical calculation to obtain the order of magnitude of δ_{Ag} (1 m, 10 cm, 1 cm ...).

Q3. Consider two photons propagating along the x-axis in opposite directions. The energies of the photons are 5 MeV and 2 MeV, respectively.

a) Calculate the relative velocity between the photons.

b) What is the value of the total energy of the system?

c) What is the total momentum of the system?

d) Calculate the rest energy of the system.

Q4. A X-ray photon with wavelength $\lambda = 10^{-10}$ m, is backscattered in a Compton experiment, i.e., the scattering angle is 180^o respecting to the incidence axis.

a) Calculate the frequency of the backscattered photon.

b) What is the direction of the momentum of the electron ejected in the scattering, respecting to that of the incident foton?

c) What is the module of the velocity of the electron ejected in the scattering?

Q5. A cylindrical container of circular cross section A and fixed base was positioned vertically on a flat surface and filled with an ideal gas. On its (open) upper end, a movable circular piston of mass M was perfectly adjusted. Suppose that the piston remains horizontally oriented and only slides up and down, without friction, in contact with the inner wall of the cylinder. Assume that the ratio γ between the specific heats of the gas at constant pressure and at constant volume is known.

a) Compute the equilibrium pressure for the gas in the container, given the atmospheric pressure p_0.

b) Write down the expression for the variation of the pressure p in terms of the variation of the volume V due to a small displacement of the piston. Suppose that, for small displacements of the piston, the states of the gas are described by a quasi-static adiabatic process.

c) Determine the additional force exerted on the piston when it has a displacement dx with respect to its equilibrium position.

d) Obtain the angular frequency for small oscillations of the piston from the equilibrium position, in terms of V, A, M, p_0 and γ.

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Part 2

Instructions

- **Do not write your name on the test.**
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- This test contains questions on:
  classical mechanics, quantum mechanics, and statistical mechanics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
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Have a good exam!

Q6. A particle of mass m is subject to a conservative central force whose potential energy is given by $U(r) = k\left(r^2 - a^2\right)\mathrm{e}^{-b\,r^2}$, where r is the spherical radial coordinate, and k, a and b are real and positive constants.

a) Determine the units of the constant k, a and b in the SI (International System of Units).

b) Sketch a graph of the function $U(r)$, determining its maxima and minima as a function of the given parameters.

c) Determine the particle energy ranges E for which (i) the particle is in bound orbits and (ii) in unbound orbits. (iii) Determine the conditions, if any, for the existence of orbits with constant radius.

d) Determine the force acting on the particle, the equilibrium conditions, if any, and if they do exist, determine the frequency of particle oscillation in radial motion close to the point(s) of stable equilibrium.

Q7. A particle of mass m is constrained to a spherical surface of fixed radius a, and no external force acts on it.

a) Determine the Lagrangian of the particle using appropriate coordinates in three-dimensional space ($\mathbb{R}^3$) and establish the equation of the constraints.

b) Using the method of Lagrange multipliers, find the equations of motion and determine the constraint force, ie, determine the Lagrange multiplier and interpret the result.

c) Establish the constants of motion for the particle .

d) Assuming now that the radius of the sphere varies in time with the function $a(t) = a_0\,(1 + \cos\omega t)$, where a_0 and ω are constant, determine the constants of motion for the particle.

Q8. Consider a free particle of mass m confined to a circle of perimeter L.

a) Write the corresponding Schroedinger equation.

b) Calculate the *normalized* wave function $\psi = \psi(t,x)$, where x is the position of the particle ($0 \leq x < L$), assuming that the particle has well-defined values of momentum and energy: p and E, respectively.

c) Assuming that the particle is in an energy eigenstate, what are the two smallest (non-zero) corresponding eigenvalues?

d) Consider a particle in an energy eigenstate with the lowest non-zero energy value. Write down its wave function in order that it has a probability density to be found between x and $x + \delta x$ equal to $(2/L)[\cos(2\pi x/L)]^2$. (Remember that $(\cos x)^2 = (\cos 2x + 1)/2$).

Q9. Consider a system composed by a pair **A** and **B** of spins 1/2 described by the state

$$|\psi\rangle = \alpha\left(|z_+^{\mathbf{A}}\rangle \otimes |z_-^{\mathbf{B}}\rangle - |z_-^{\mathbf{A}}\rangle \otimes |z_+^{\mathbf{B}}\rangle\right)$$

where

$$\begin{aligned} \hat{S}_x|x_\pm^{\mathbf{A}}\rangle &= \pm\frac{\hbar}{2}|x_\pm^{\mathbf{A}}\rangle, \quad \langle x_\pm^{\mathbf{A}}|x_\pm^{\mathbf{A}}\rangle = 1, && (1)\\ \hat{S}_y|y_\pm^{\mathbf{A}}\rangle &= \pm\frac{\hbar}{2}|y_\pm^{\mathbf{A}}\rangle, \quad \langle y_\pm^{\mathbf{A}}|y_\pm^{\mathbf{A}}\rangle = 1, && (2)\\ \hat{S}_z|z_\pm^{\mathbf{A}}\rangle &= \pm\frac{\hbar}{2}|z_\pm^{\mathbf{A}}\rangle, \quad \langle z_\pm^{\mathbf{A}}|z_\pm^{\mathbf{A}}\rangle = 1, && (3) \end{aligned}$$

(and analogously for **B**) and where we write the spin operators as

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

in the base of the eigenstates of $\hat{S}_z$:

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

Answer the questions:

a) What is the value of $\alpha \in \mathbb{R}$ for which $|\psi\rangle$ is normalized?

b) What is the probability of measuring in the z-direction: $-\hbar/2$ for spin **A** and $+\hbar/2$ for spin **B**?

c) What is the probability of measuring in the x-direction: $+\hbar/2$ for spin **A** and $-\hbar/2$ for spin **B**?

d) What is the probability of measuring in the z-direction $-\hbar/2$ for spin **A** and in the x-direction $+\hbar/2$ for spin **B**?

Q10. Consider a system consisting of a large number N of distinguishable molecules, which do not interact among themselves. Each molecule has two possible energy states: 0 and $\epsilon > 0$.

a) Obtain the entropy density S/N of the system as a function of the average energy per molecule E/N, of ϵ and the Boltzmann constant k_B.

b) Considering the system in thermal equilibrium at the inverse temperature $\beta = 1/k_B T$, compute E/N.

c) What is the maximum value for E/N in the above case? Compare it to the maximum value of this quantity if it was possible that all elements of the system were in the state of maximum energy.