Joint Entrance Examination for Postgraduate Courses in Physics

EUF

Second Semester/2012

Part 1 – 24 April 2012

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST. It should be identified only by your candidate number (EUFxxx).**
- This test is the **first part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Electromagnetism, Modern Physics, Thermodynamics and Statistical Mechanics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted in the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for marking. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, Q2, or . . .) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will not be marked.**
  Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked SCRATCH at the end of the answer booklet. **DO NOT DETACH THEM.** The scratch sheets will be discarded and **solutions written on them will be ignored.**
- Do **NOT** write **ANYTHING** on the List of Constants and Formulae provided; **RETURN IT** at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

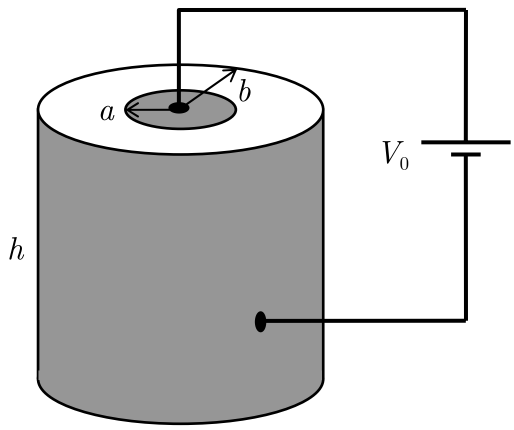
Q1. A hollow cylinder of height h, external radius b and internal radius a is made of a homogeneous material with electrical conductivity σ and electric permittivity ε. Another material with high electrical conductivity fills the central hole and surrounds the outer surface, serving as electric contacts, as illustrated in the figure below. Assume $h >> b$, so that edge effects can be neglected. A battery maintains a potential difference V_0 between the contacts (take $V = 0$ on the outer surface).

(a) Show that, in the steady state ($\frac{\partial \rho}{\partial t} = 0$), the charge density ρ within the material of the cylinder vanishes.

(b) Show that, in this case, the electric potential satisfies the Laplace equation and obtain the electric field $\vec{E}(\vec{r})$ within the material of the cylinder.

(c) Calculate the total free charge accumulated on the inner surface (radius a) and the capacitance between the two contacts.

(d) Calculate the electrical resistance between these two contacts.



Q2. A very long conducting cylinder of radius a carries a current I along its z axis. The current density at a point within the cylinder varies according to:

$$\vec{J}(r, \varphi, z) = \hat{z}\,\frac{J_0}{r}\sin\left(\frac{\pi r}{a}\right),$$

where r is the radial distance from the axis to that point.

(a) Determine the constant J_0 in terms of I and a.

(b) Calculate the magnetic field $\vec{B}$ outside the conducting cylinder ($r > a$) and express your result in terms of I and a.

(c) Calculate the magnetic field $\vec{B}$ inside the conducting cylinder ($r < a$) and express your result in terms of I and a.

(d) Sketch a qualitative graph of the magnitude of the magnetic field, $B(r)$, indicating its behavior at $r = 0$ and $r = a$.

Q3. (a) Use the de Broglie relation for the wavelength associated with a particle and obtain the angular momentum quantization relation for an electron in orbital motion in an atom, within the Bohr model ($L = n\hbar$, with n=1, 2, 3, ...).

(a) Use the expression above to show that the energies of the electronic states of a hydrogen atom are given by

$$E_n = -\frac{m_e e^4}{8\epsilon_0^2 h^2 n^2} ,$$

where e and m_e are the charge and mass of the electron, respectively.

(b) Calculate the ionization energy of a doubly ionized lithium atom ($Z = 3$), knowing that the ionization energy of hydrogen is 13.6 eV.

(c) In spectroscopy, the Balmer series is associated with a set of transitions in which the electron in the hydrogen atom decays from an excited state to a final state with $n_f = 2$. In this series, the line known as H_β is attributed to a transition from a state with $n_i = 4$. Estimate the wavelength of the H_β line and identify in which region of the electromagnetic spectrum it is situated.

Q4. Regarding an experiment on the photoelectric effect, Figure 1 below illustrates a graph of the photoelectric current as a function of the potential difference V between the electron collector and a sodium target. Curves (a) and (b) correspond to different intensities of the incident light and V_0 the so called "stopping potential" or "limit potential". Moreover, Figure 2 displays measurements of the stopping potential as a function of the incident light frequency. Using these graphs:

(a) estimate the value of Planck's constant in eVs, indicating your procedure;

(b) estimate the value of the "work function" for sodium;

(c) estimate the value of the kinetic energy of the fastest photoelectron emitted when light of frequency 10^{15} Hz shines on the sodium target;

(d) cite a feature of the photoelectric effect than can be explained classically and another one that cannot be explained within the wave theory of electromagnetic radiation.

Figure 1

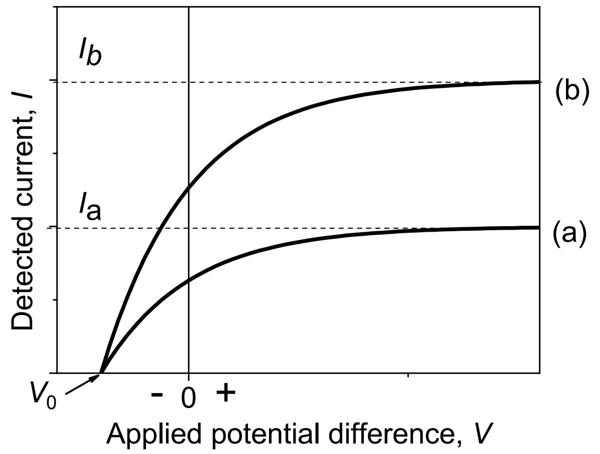
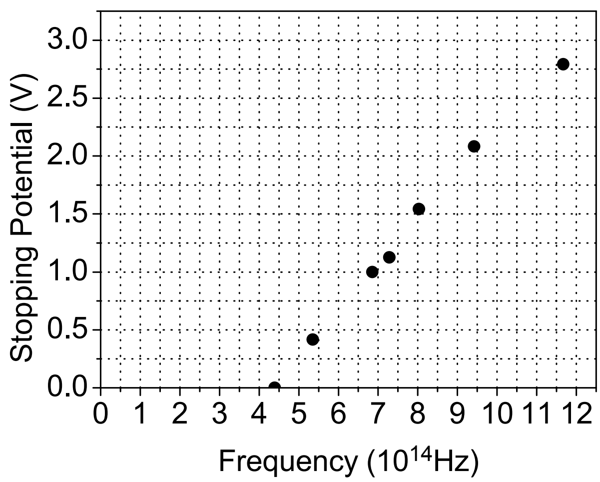


Figure 2



Q5. Two identical bodies of constant (finite) thermal capacity C_P are at temperatures T_1 and T_2, respectively, where $T_2 > T_1$. Assume that, throughout the processes described below, the bodies remain at constant pressure and do not undergo phase transitions.

(a) If the bodies are put in thermal contact, but insulated from the rest of the Universe, determine the equilibrium temperature.

(b) Determine the change in entropy of the system for the process described in (a).

Now consider the bodies to be used as hot and cold reservoirs for a small thermal engine, which will work until the two bodies reach thermal equilibrium.

(c) Assuming that this process is reversible, determine the final equilibrium temperature.

(d) Calculate the total work done by the engine during the process described in (c).

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EUF

Second Semester/2012

Part 2 – 25 April 2012

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST. It should be identified only by your candidate number (EUFxxx).**
- This test is the **second part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Classical Mechanics, Quantum Mechanics, Thermodynamics and Statistical Mechanics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted in the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for marking. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, Q2, or . . .) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will not be marked.**
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Have a good exam!

Q6. A celestial body of mass m approaches the Sun (mass $M >> m$) along a hyperbolic trajectory and when it is at a distance r_0 from it, its velocity is v_0 and makes an angle of 30° with the vector direction towards the Sun.

(a) Calculate the angular momentum L and the energy E of this celestial body.

(b) Determine the distance r_p of its closest approach to the Sun, expressing your result in terms of L and E.

(c) When the celestial body reaches the distance of closest approach, r_p , it collides with a small asteroid such that its mass does not change, but thereafter it describes a circular orbit of radius r_p in the same plane as the previous trajectory. Calculate the new energy and angular momentum of the celestial body after the collision, expressing your result in terms of r_p.

Q7. A ball of mass $m = 450$ g is tied to a spring whose potential energy as a function of its elongation x is shown on the figure below (solid line). Express your answers in SI units.

(a) Determine the spring's elastic constant, for small displacements.

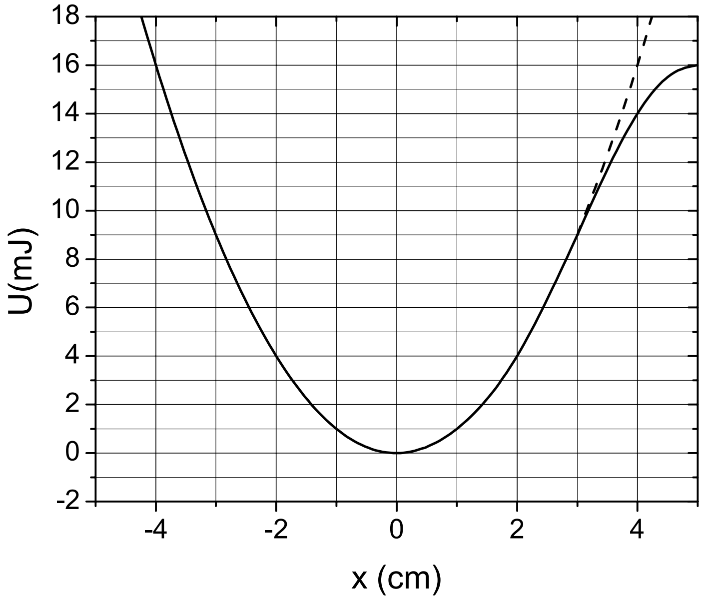
(b) Sketch a graph of the force acting on the ball as a function of the spring elongation.

Knowing that the ball's motion is one-dimensional and the spring's maximum elongation is 3 cm:

(c) determine its maximum speed;

(d) determine the ball's kinetic energy in this motion for spring elongation $\mathsf{x} = -2$ cm.

(e) Now determine the position ($\mathsf{x} < 0$) where the ball should be released from rest to reach the point $\mathsf{x} = 5$ cm with zero velocity.



Q8. Consider the one-dimensional quantum problem of a particle of mass m subject to the potential

$$V(x) = \begin{cases} +\infty & , x < 0 \\ 0 & , 0 < x < a \\ +\infty & , x > a \end{cases} .$$

(a) Write the time-independent Schrödinger equation for this problem.

(b) Solve this equation, finding all the independent acceptable solutions. That is: determine all possible values for the energy, E_n, and the corresponding normalized wavefunctions, $\psi_n(x)$.

Suppose now that the actual total potential has the form $V_{\text{total}}(x) = V(x) + W(x)$, where $W(x)$ is a small correction given by

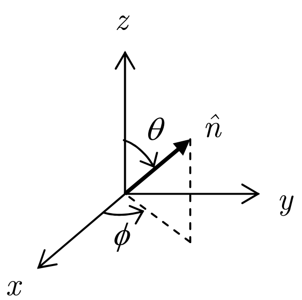
$$W(x) = \begin{cases} 0 & , x < 0 \\ W_0 \sin(\pi x/a) & , 0 < x < a \\ 0 & , x > a \end{cases}$$

(c) Using first order perturbation theory, calculate the correction to the ground state energy obtained in (b).

Q9. For a spin $\frac{1}{2}$ particle the spin operator is given by $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. Let $\hat{n}$ be the unit vector in the direction (θ, ϕ), as illustrated in the figure below.



(a) Calculating the scalar product, show explicitly that the operator which represents the spin component in that direction, $S_n = \hat{n}\cdot\vec{S}$, is given by

$$S_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} .$$

(b) Show that the only values that can be obtained in a measurement of S_n are $+\hbar/2$ and $-\hbar/2$, for any direction $\hat{n}$.

(c) Obtain the normalized column vector which represents the state in which a measurement of S_n necessarily gives the value $+\hbar/2$. Simplify the final answer expressing the θ dependence in terms of $sin(\theta/2)$ and $cos(\theta/2)$.

(d) Suppose now that $\theta = 60°$ and $\phi = 45°$. If the particle has been prepared in a state such that the z-component of the spin, S_z, has a well defined value of $+\hbar/2$, what is the probability of obtaining this same value in a measurement of S_n? *Give a numerical answer.*

Q10. Consider a gas composed of N ultra-relativistic particles (such that their energy ε may be written as $\varepsilon = c\,p$, where p is their linear momentum) confined in a container of volume V and at temperature T. Suppose the particles are indistinguishable and non-interacting, and that their thermal energy is sufficiently high to neglect quantum effects.

(a) Show that the partition function of this gas is $Z = \dfrac{(8\pi V)^N}{N!(hc/k_B T)^{3N}}$, where h is Planck's constant, c is the speed of light in vacuum and k_B is Boltzmann's constant.

(b) Determine the pressure of this gas.

(c) Calculate the entropy of this gas.

(d) Determine the internal energy of this gas.