EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the second semester of 2016

April 05, 2016

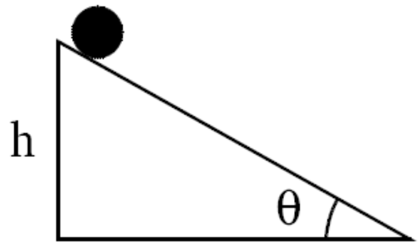
Part 1

Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  classical mechanics, modern physics, quantum mechanics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding sheet of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- **Do not write anything on the list of constants and formulae.**
  Return both this test and the list of constants and formulae at the end of the test. The latter will be used in the test tomorrow.

Have a good exam!

Q1. A solid bronze sphere of mass m and radius r rolls without slipping down an inclined plane after being released from rest from a height h. The sphere's moment of inertia with respect to an axis through its center is $I = 2mr^2/5$ and the acceleration of gravity is g. The inclined plane makes an angle θ with the horizontal, as shown in the figure.



(a) Is there friction between the sphere and the plane? How did you reach this conclusion?
(b) Is there conservation of mechanical energy? Justify your answer taking into consideration your answer to item (a).
(c) Using energy arguments, find the speed with which the sphere arrives at the bottom of the plane.
(d) Now obtain the speed at the bottom of the plane found in item (c) using dynamical arguments (i. e., through Newton's second law).

Q2. A mass m is attached to one end of a rigid rod of negligible mass and length l. The other end of the rod is fixed to a point in space and the mass-rod system moves in a vertical plane in a region where the acceleration of gravity is g.

(a) Write the system's Lagrangian.
(b) Obtain the equation of motion that describes the system.
(c) Determine the points of equilibrium of this system and classify them according to their stability, justifying your answers.
(d) Find the frequency of small oscillations around the point of stable equilibrium.

Q3. In relativistic Compton scattering, a photon with energy-momentum $(E_0, \vec{p}_0)$ collides with an electron of mass m initially at rest. The scattered photon is observed to emerge in a direction that makes an angle θ with the direction of incidence, with energy-momentum $(E, \vec{p})$.
(a) If $\vec{p}_e$ is the momentum of the scattered electron, write the equations for the conservation of energy-momentum.
(b) Obtain the relation

$$\frac{1}{E} - \frac{1}{E_0} = \frac{1}{mc^2}\,(1 - \cos\theta)\,.$$

(c) If the wavelength of the incident photon is λ_0, find the wavelength of the scattered photon when $\theta = \pi/2$.
(d) Under the same conditions as the previous item, what is the kinetic energy of the scattered electron? Give your answer in terms of λ_0, $\lambda_C \equiv h/(mc)$, and universal constants.

Q4. Consider the one-dimensional quantum dynamics of a particle with mass m under the influence of a harmonic potential. Its Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

where ω is the oscillator's angular frequency and

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}.$$

The eigenstates $|n\rangle$ $(n = 0,1,\ldots)$ of the Hamiltonian are non-degenerate, are eigenstates of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, and satisfy the relations

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

(a) Calculate the matrix elements of the operators $\hat{x}$ and $\hat{p}$ in the basis of eigenstates of the Hamiltonian.
(b) Calculate the expectation value of the operator $\hat{x}^2$ for any eigenstate $|n\rangle$.
(c) Calculate the ratio between the average total energy and the average potential energy of the system for any eigenstate $|n\rangle$.
(d) Use the equation of motion of operators in the Heisenberg picture

$$i\hbar\frac{d\hat{O}_H(t)}{dt} = \left[\hat{O}_H(t),\hat{H}\right]\,,$$

where $\hat{O}_H(t) = e^{i\hat{H}t/\hbar}\hat{O}e^{-i\hat{H}t/\hbar}$, to obtain the time evolution of the operator $\hat{a}_H(t)$.

Q5. A thermal engine operates with a monatomic ideal gas in a cycle which starts with an adiabatic expansion from a state A with volume V_0 to a state B whose volume is rV_0 (with $r > 1$). This is followed by an isothermal contraction from B to state C, which has the same volume as A. Finally, the cycle is completed by an isovolumetric compression from C back to A.
(a) Sketch the engine cycle on a $P - V$ diagram.
(b) Calculate (i) the total work done by the gas and (ii) the heat injected into the gas, both during a complete cycle. Write your answer in terms of r, $\gamma \equiv c_P/c_V$ and the extreme temperatures $T_{\max}$ e $T_{\min}$, namely, the maximum and the minimum temperatures between which the engine operates. Remember that $c_P - c_V = R$.
(c) Find the efficiency of the cycle.
(d) Write the efficiency of the cycle in terms of $T_{\max}$ and $T_{\min}$ only (if you have not already done so in item (c)). Consider the case $T_{\max} = 2T_{\min} > 0$. Find the ratio between the efficiency of this engine and that of a Carnot engine. Which one has the largest efficiency? Is your answer consistent with what you expect from the second law of thermodynamics? Justify.

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Part 2

Instructions

- **Do not write your name on the test.**
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  classical mechanics, quantum mechanics, and statistical mechanics.
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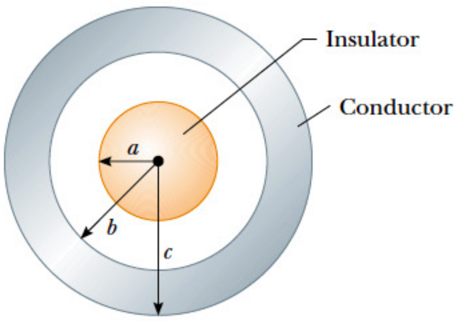
Q6. An insulating sphere of radius a has a uniform charge density ρ and total charge Q. A hollow conducting uncharged sphere, whose inner and outer radii are b and c, respectively, is concentric with the insulating sphere, as shown in the figure below.



(a) Find the magnitude of the electric field in the regions:

(i) $r < a$; (ii) $a < r < b$; (iii) $b < r < c$ and (iv) $r > c$.

(b) Find the induced charge per unit area on the inner and outer surfaces of the conductor.

(c) Sketch the graph of the magnitude of the electric field E as a function of r. Identify in your graph each one of the regions listed in item (a).

Q7. Consider Maxwell's equations in differential form and solve each item below.

(a) Derive, showing every step, the wave equations in vector form for both the electric and the magnetic fields in vacuum. Remember:

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V} \quad \text{and} \quad \nabla\cdot(\nabla\times\mathbf{V}) = 0$$

(b) Write the wave equation for a generic scalar function $f(\vec{r},t)$ and, by comparing with the expressions obtained in (a), obtain the speed of propagation for both fields.

(c) A possible solution of the equations obtained in (a) is a linearly polarized plane wave. Consider an electromagnetic field solution of this kind, which propagates in the $\hat{z}$ direction. If ω is the angular frequency, k is the wave number, E_0 and B_0 are the amplitudes of the electric and magnetic fields, respectively, write explicitly the magnitude and direction of $\vec{E}$ and $\vec{B}$ as functions of position and time.

(d) Consider now Maxwell's equations in the presence of charges and currents and derive the equation that relates the electric charge and electric current densities (continuity equation). What conservation law is expressed mathematically by this equation?

Q8. Consider a spin-1/2 particle under the influence of a uniform magnetic field $\vec{B} = B\hat{z}$. The Hamiltonian for this problem is given by

$$\hat{H} = -\gamma B\hat{S}_z,$$

where γ is a constant. Let the states $|+\rangle$ and $|-\rangle$ be such that $\hat{S}_z\,|\pm\rangle = \pm(\hbar/2)\,|\pm\rangle$.

(a) What are the eigenvalues of the Hamiltonian?

(b) At time $t = 0$ the state of the particle is given by $|\psi\,(0)\rangle = [|+\rangle - |-\rangle]\,/\sqrt{2}$. Calculate the state of the particle at any time $t > 0$.

(c) Calculate the average value of the operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ for any given time $t \geq 0$. Remember that $\hat{S}_x\,|\pm\rangle = (\hbar/2)\,|\mp\rangle$ e $\hat{S}_y\,|\pm\rangle = \pm i(\hbar/2)\,|\mp\rangle$.

(d) What is smallest value of $t > 0$ for which the particle returns to its initial state?

Q9. When two events in space-time are separated in space by the vector $\Delta x\ \hat{x} + \Delta y\ \hat{y} + \Delta z\ \hat{z}$ and in time by Δt, the invariant interval between them, whose value is independent of the inertial reference frame, is defined as

$$\Delta s^2 \equiv \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2.$$

(a) Events (1) and (2) occur in **different** positions (x_1,y_1,z_1) and (x_2,y_2,z_2), respectively, in an inertial frame (S) and are such that the invariant interval is positive. Is there an inertial frame in which these events occur at the same point in space? Justify.

(b) Under the same conditions as in (a), could event (2) have been caused by event (1)? Justify your answer analyzing the propagation of a signal from (1) to (2) with velocity $\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$.

(c) A clock is at rest in an inertial frame (S′), which moves with velocity $\vec{V}$ with respect to (S).

(i) What is the sign of the invariant interval between events defined by two successive positions of the "clock hands" (neglect the spatial dimensions of the clock)?

(ii) Obtain the relation between the proper time interval $\Delta t'$ (measured in S′) and the time interval Δt measured in (S).

(d) The spatial separation between a source F and a particle detector D is $L\hat{x}$, in the lab frame (S). Consider events E_F and E_D, of production and detection of a particle, respectively. The particle moves from F to D with constant velocity $\vec{V} = V_0\hat{x}$ in the lab frame.

(i) What are the separations in space Δx and time Δt between E_F and E_D in the lab frame?

(ii) Let L' be the distance between F and D in the reference frame of the particle. What are the separations in space $\Delta x'$ and time $\Delta t'$ between E_F and E_D in the reference frame of the particle?

(iii) Find the relation between L' and L.

Q10. In a model for a solid crystal it is assumed that the N atoms are equivalent to $3N$ independent, one-dimensional classical harmonic oscillators of mass m, each one oscillating with the same angular frequency ω around its equilibrium position. At a distance x from that position, the potential energy is given by $U = m\omega^2 x^2/2$. From a few experimental data it is possible to estimate, in terms of the inter-atomic distance at low temperatures d, the root mean square displacement of the atoms when melting occurs. The answers to the items below allow us to obtain this estimate. Assume the crystal is in thermal equilibrium at absolute temperature T.

(a) The number of states in a cell in phase space (x,p) is given by $(dxdp)/h$, where h is Planck's constant. Obtain the partition function of the harmonic oscillator, $Z(T,\omega)$.

(b) Calculate the average number of oscillators whose position lies between x and $x + dx$.

(c) Find an expression for the average potential energy $\langle U \rangle$ per one-dimensional harmonic oscillator. Compare your result with the value expected from the equipartition theorem.

(d) Let x_0^2 be the mean square displacement around the equilibrium position when the crystal melts and let f be equal to x_0/d. Using $\langle U \rangle = m\omega^2 x_0^2/2$, obtain an estimate for f for an element whose atomic mass is $m = 1.0 \times 10^{-25}$ kg, melting temperature is $T_F = 1400$ K, $d = (10/3)$ Å $= (10/3) \times 10^{-10}$ m, and the frequency is such that $\hbar\omega/k_B = 300$ K.