EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the second semester of 2019

April 02, 2019

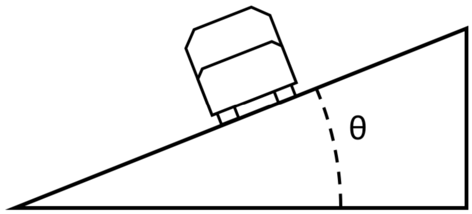
Part 1

This test contains questions on classical mechanics, modern physics, quantum mechanics, and thermodynamics. All questions have the same weight.

Useful information for the solution of this test can be found in the accompanying formula sheet.

Have a good exam!

Q1. A car moves on a circular track inclined by an angle θ with respect to the horizontal. The figure below shows the plane normal to the car's movement. The acceleration due to gravity is g.



(a) Show schematically all the forces acting on the car if friction is negligible.
(b) Still neglecting friction, find the speed of the car if it follows a circular path of radius R.
(c) **From this item on, assume that there is friction between the car tyres and the track and the coefficient of static friction is μ.** If the car moves with the highest possible speed without sliding, show schematically all the forces that act on the car.
(d) What is the highest possible speed of the car if it is able to make the curve of radius R without sliding?
(e) Now assume that $\sin\theta > \mu\cos\theta$ and that the car moves much more slowly. Show schematically all the forces that act on the car in this case. Find the lowest possible speed of the car if it is able to make the curve of radius R without sliding.

Q2. Consider a pendulum composed of a small body of mass m suspended by a massless spring of elastic constant k. The relaxed length of the spring (i.e., without any suspended mass) is l. Assume that the pendulum's motion is constrained to a fixed vertical plane. Use generalized coordinates such that the spring length is $l + r(t)$ and that the angle it forms with the vertical is $\theta(t)$.

(a) Write the system's kinetic energy in terms of r, θ, and their time derivatives.
(b) Write the system's potential energy in terms of r, θ, and their time derivatives.
(c) Write the system's Lagrangian.
(d) Write the Euler-Lagrange equations for r and θ.
(e) Now consider the strictly vertical motion of the pendulum, i.e., $\theta(t) = 0$ for any t. Find the general solution (in terms of two arbitrary constants) of the Euler-Lagrange equation for r.

Q3. Cosmic rays that impinge on the Earth's atmosphere give rise to a cascade of particles with various energies, among them muons. Muons are unstable and decay spontaneously according to the law $N(t) = N_0 e^{-t/\tau}$, where $N(t)$ and N_0 are the number of muons at instants t and $t = 0$, respectively, and τ is the muon's lifetime, whose value is $\tau = 2.0$ μs, when measured in its proper reference frame. A speed-selective muon detector D_1 is mounted on the top of a mountain, 2.94×10^3 m above sea level. The detector is tuned to detect particles with speed $v = 0.98c$. In a given time interval, 1.5×10^3 muons are detected. In another similar detector D_2, mounted at sea level, the same measurement is performed (same speed and time interval) and a number of muons is obtained that is orders of magnitude greater than the value expected according to non-relativistic physics.

(a) Find the number of muons that would be expected in the detector D_2 according to non-relativistic physics.
(b) Explain *qualitatively* why the number of muons observed in the detector D_2 is greater than

the non-relativistic expectation.
(c) Calculate the muon count in the detector D_2: (i) from the point of view of an observer in the reference frame attached to the detector and (ii) from the point of view of an observer in the muons' proper reference frame.
(d) Consider a reference frame S in which both the muons and the detector D_2 move approaching each other with equal speeds. From the point of view of an observer in the reference frame S, the muon count observed in the detector D_2 is smaller than, greater than or equal to the one from item (c)? Justify your answer.

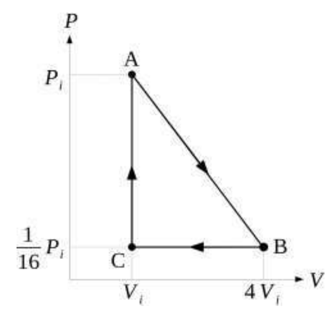
Q4. The wave function that describes the one-dimensional quantum dynamics of a particle of mass m as a function of time t in the presence of a confining potential is

$$\Psi(x,t) = C\left(x^2 - \frac{\hbar}{4am}\right)e^{-a\left[(mx^2/\hbar)+5it\right]},$$

where C and a are real positive constants with appropriate dimensions.

(a) Through dimensional analysis, find the physical dimension of the constant C.
(b) Is the particle in an energy eigenstate? If yes, what is the corresponding energy eigenvalue? Justify your answers.
(c) Find the standard deviations of the position x and linear momentum p of the particle if the expectation values of x^2 and p^2 for this state are $\langle \hat{x}^2 \rangle = C^2\sqrt{\frac{\pi}{2}\frac{5}{32}}\left(\frac{\hbar}{am}\right)^{7/2}$ and $\langle p^2 \rangle = 40\sqrt{\frac{2}{\pi}\frac{\hbar^2}{C^2}}\left(\frac{am}{\hbar}\right)^{7/2}$. Are the standard deviations consistent with the uncertainty principle? Justify your answer.
(d) Find the potencial energy function of the particle.

Q5. A sample of 1.0 mole of an ideal monatomic gas (molar heat capacities $C_V = 3R/2$ and $C_p = 5R/2$) undergoes the cyclical thermodynamic process shown in the pressure vs volume diagram below. The cycle is performed in the clockwise direction. The pressures and volumes at points A, B and C are shown in the figure and all transformations undergone by the gas are reversible, following the straight continuous lines AB, BC and CA. Give your answers in terms of the universal gas constant R and the gas pressure P_i and volume V_i at point A.



(a) Find the gas temperatures T_A, T_B and T_C at the points A, B and C.
(b) Find the heat exchanged between the environment and the gas in each cycle, Q_{cycle}.
(c) Find the heat exchanged between the environment and the gas between points A and B, Q_{AB}.
(d) Find the variation of the gas entropy between points A and B, $\Delta S_{AB} = S_B - S_A$.

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For the second semester of 2019

April 03, 2019

Part 2

This test contains questions on electromagnetism, modern physics, quantum mechanics, and statistical mechanics. All questions have the same weight.

Useful information for the solution of this test can be found in the accompanying formula sheet.

Have a good exam!

Q6. A parallel-plate capacitor has its conducting plates normal to the z direction. One of the plates, at $z = 0$, has an electric potential $V = 0$, whereas the other one, at $z = d$, has an electric potential $V = V_0$, where V_0 is a constant. In the space between the plates, filled with a dielectric with electric permittivity ϵ, the free electric charge density is given by

$$\rho_F(z) = \rho_0 \, e^{-\alpha z},$$

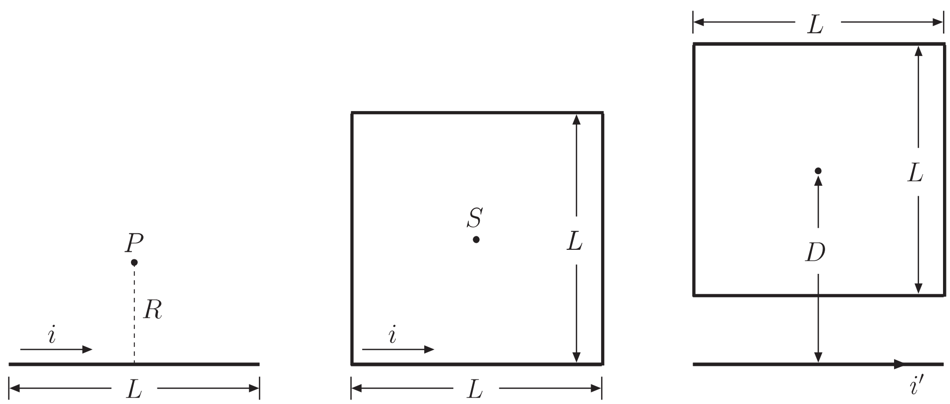
where ρ_0 and α are constants. Neglect border effects.

(a) Show that the potential between the plates has the form

$$V(z) = A + Bz - \frac{\rho_0}{\epsilon\alpha^2} \, e^{-\alpha z},$$

where A and B are constants.
(b) Find the constants A and B from the boundary conditions.
(c) Find the electric field **vector** between the plates.

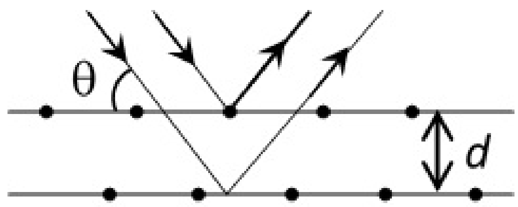
Q7. (a) A straight segment of wire of length L carries a current i. Find the magnitude of the magnetic field B generated by the wire segment at the point P, located a distance R from the segment and equidistant from its ends, as shown in Figure (a).



(a) (b) (c)

(b) Use the result of item (a) and find B when $R \ll L$. Compare the result with the magnitude of the magnetic field generated by a very long wire.
(c) Consider now a square loop of side L carrying a current i, as shown in Figure (b). Find the magnitude of the magnetic field B at the point S at the center of the loop.
(d) A square loop of wire of side L and no current is placed close to an infinitely long wire that carries a current i'. The distance from the long wire to the center of the wire loop is D, as shown in Figure (c). Find the magnetic flux generated by the wire and threading the square loop.

Q8. A neutron beam forming a matter wave of wavelength λ is incident on a crystal making an angle $\theta \in [0,\pi/2]$ with a set of crystalline planes. The distance between two adjacent crystalline planes is d, as shown in the Figure. The rest energy of a neutron is $E_n^0 = 940$ MeV.



(a) Find the expression relating λ, θ and d and describing the Bragg reflection.
(b) A certain crystal has a set of crystalline planes separated by $d = 1.1$ Å. If neutrons with kinetic energy $K = 1.9$ eV are used, at how many different values of θ is the corresponding Bragg reflection observed?
(c) Explain how a crystal can be used as a filter to select neutron speeds from a beam with a wide speed distribution.
(d) The relations $E = h\nu$ and $p = h/\lambda$ are also valid in the relativistic regime, where E is the total relativistic energy of the particle, p is its linear momentum and λ and ν are the wavelength and frequency of the associated matter wave, respectively. Find the **phase velocity** of a matter wave associated with relativistic neutrons in terms of E_n^0 and λ. Is the result greater than, smaller than or equal to the speed of light in vacuum? Find also the **group velocity** of the matter wave, compare it with the phase velocity, and comment on the results.

Q9. Consider a one-dimensional quantum harmonic oscillator of mass m and angular frequency ω, governed by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

where $\hat{x}$ and $\hat{p}_x$ are the canonically conjugate position and linear momentum operators of the particle, respectively. The energy eigenstates are denoted by $|n\rangle$ ($n = 0,1,\ldots$), with corresponding eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega$. This problem can also be formulated in terms of non-Hermitian operators $\hat{a}$ and $\hat{a}^\dagger$, where $\hat{a}^\dagger$ is the Hermitian adjoint of $\hat{a}$ and

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}_x.$$

The action of these operators on the energy eigenstates satisfies the relations

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

(a) Find the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ and re-write the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$. Why is the operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$ called *number operator*?
Consider now a **two-dimensional harmonic oscillator**, whose Hamiltonian is given by

$$\hat{H}_2 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega_x^2\hat{x}^2 + \frac{1}{2}m\omega_y^2\hat{y}^2$$

(b) Give arguments that justify the fact that the energy eigenstates of the two-dimensional problem can be written as $|n_x,n_y\rangle \equiv |n_x\rangle \otimes |n_y\rangle$, with $|n_x\rangle$ and $|n_y\rangle$ eigenstates of one-dimensional

harmonic oscillators with angular frequencies ω_x and ω_y, respectively. Find the energy eigenvalues $E_{n_x n_y}$ of the two-dimensional problem.
(c) Assume the state of the particle at instant $t = 0$ is given by

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{5}}\Big(|n_x = 2, n_y = 0\rangle + 2|n_x = 1, n_y = 1\rangle\Big).$$

Write the state $|\psi(t)\rangle$ for a generic time $t > 0$. If a measurement of the total energy of the system is made at an instant $t' > 0$, what is the probability that the state right after the measurement gives the expectation value of $\hat{p}_x^2$ as $\langle p_x^2\rangle = 5m\hbar\omega_x/2$?
(d) For the isotropic potential case ($\omega_x = \omega_y \equiv \omega$), find the degree of degeneracy of the n-th excited state.

Q10. A system of N distinguishable non-interacting particles is governed by the Hamiltonian

$$H = \sum_{i=1}^{N} \varepsilon_i. \tag{1}$$

The energy of each particle ε_i can assume only two values: $\varepsilon_i = 0$ or $\varepsilon_i = \Delta > 0$. Thus, each microstate of the system is described by the set of values $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$.

(a) If the system has total energy E, which is an integer multiple of Δ, the total number of possible microstates is

$$\Omega(E,N) = \frac{N!}{(E/\Delta)!(N - E/\Delta)!}. \tag{2}$$

Making use of the basic postulate of statistical mechanics, obtain the probability of finding the system in a specific microstate.
(b) Find the entropy per particle of the system $s = S/N$ as a function of the energy per particle $u = E/N$ in the regime $N \gg 1$. Use the approximation $\ln N! \approx N \ln N - N$, valid for $N \gg 1$.
(c) Find the temperature of the system as a function of u. Is there a range of values of u for which the temperature is negative?
(d) Find the specific heat of the system. Is there any regime in which the specific heat is negative?