EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the second semester 2014

23 April 2014

Part 1

Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  electromagnetism, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 60 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding page of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- Do not write anything on the list of constants and formulae.
  Return it at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

Q1. A capacitor is made up of a spherical conductor of radius R_1, concentric with a spherical shell of radius R_2 and negligible thickness, with $R_1 < R_2$. The inner conductor has charge $+Q$ and the outer conductor has charge $-Q$.

(a) Calculate the electric field and the energy density as a function of r, where r is the radial distance from the center of the spherical surfaces, for any r.

(b) Determine the capacitance C of the capacitor.

(c) Calculate the energy of the electric field stored in a spherical shell of radius r, thickness dr and volume $4\pi r^2 dr$, located between the conductors. Integrate the expression obtained to find the total energy stored between the conductors. Give your answer in terms of the charge Q and the capacitance C.

Q2. Two identical coils, each one consisting of a ring of radius R and negligible thickness, are assembled with their axes coincident with the z-axis, as seen in figure below. Their centers are separated by a distance d, with the middle point P coincident with the origin of the z-axis. Each coil carries an electric current of intensity I. Both currents are counterclockwise.

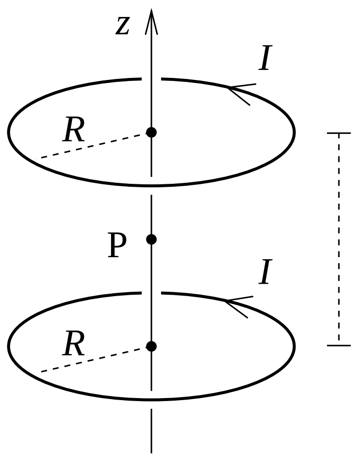
(a) Use the Biot-Savart law to determine the magnetic field of a single coil along its symmetry axis.

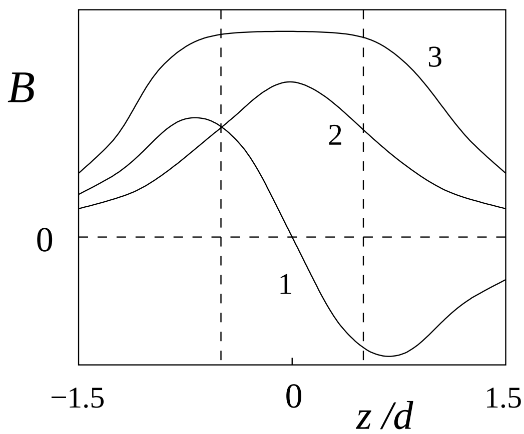
(b) From the results of the previous item, find the magnetic field $B(z)$ of the two coils along the z-axis.

(c) Assuming that the spacing d is equal to the radius R of the coils, show that, at the point P, the following equalities are valid: $dB/dz = 0$ and $d^2B/dz^2 = 0$.

(d) Considering the plots of B (in arbitrary units) versus z, shown below, which curve describes the magnetic field along the z-axis in the configuration of item (b)? Justify.

(e) Assuming that the current of the top coil is reversed, determine the new value of the magnetic field at the point P.





Q3. The Planck radiation law allows us to obtain the following energy density of the black body espectrum of a cavity at temperature T:

$$\rho(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{e^{h\nu/kT} - 1}.$$

(a) Express the energy density as a function of the wave length $\lambda = c/\nu$ instead of the frequency ν.

(b) Show that for large wave lengths and high temperatures, the previous result reduces to the classical Rayleigh-Jeans law.

(c) Obtain the Stefan-Boltzmann law from the Planck radiation law. Notice that the radiance $R(\lambda)$, which is the energy flux per unit area through a small hole in the cavity, is given by $R(\lambda) = c\rho(\lambda)/4$.

Q4. Consider a head on relativistic collision, completely inelastic, of two particles that move along the x-axis. Both particles have mass m. Before the collision, an observer A, in an inertial reference frame, notices that they move with constant velocities but in opposite directions, that is, particle 1 moves with velocity v and particle 2 moves with velocity $-v$. According to another observer B, however, particle 1 is at rest before the collision.

(a) Determine the velocity v'_x of particle 2 as measured by observer B before the collision.

(b) Find the velocities v_A and v'_B of the resultant particle after the collision, as measured by the observers A and B, respectively.

(c) Use relativistic mass-energy conservation to calculate the mass M of the resultant particle after the collision.

Q5. The pressure p of a gas behaves, as a function of the temperature T and the molar volume v, according to the following equation of state

$$p = \frac{RT}{v} - \frac{a}{v^2},$$

where a is a positive constant and R is the universal gas constant.

(a) Use the identity

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_v - p$$

to determine the molar energy u as a function of v.

(b) Assuming that $c_v = (\partial u/\partial T)_v$ is a constant equal to c, find u as a function of T and v.

(c) In a free gas expansion, does the temperature increase or decrease? Take into account that, in a free expansion, u remains invariant and v increases.

(d) Demonstrate the identity of item (a).

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Part 2

Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  classical mechanics, quantum mechanics, and statistical mechanics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 60 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
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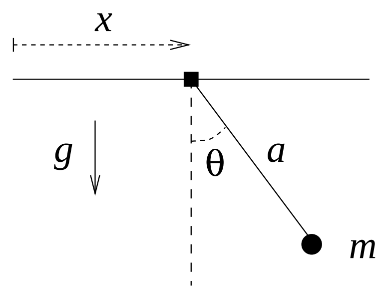
Have a good exam!

Q6. A simple pendulum consists of a particle of mass m suspended by an inextensible thread of length a and neglible mass. Its point of suspension is connected to a support that moves horizontally without friction as shown in the figure. Assume that the support is very small and that the pendulum moves only in the vertical plane. Using x and θ as generalized coordinates, where x is the horizontal position of the support and θ is the angular displacement of the pendulum, as seen in the figure, the movement of the system is described by the Lagrangian:



$$\mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{m}{2}(a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta) + mga\cos\theta.$$

(a) Obtain the equation of motion for the coordinate θ.

(b) Assuming that the angular displacements are small and that the support is subject to a forced harmonic movement of frequency ω, that is, described by $x(t) = x_0 \cos\omega t$, find the general solution $\theta(t)$ of the equation of motion for the coordinate θ.

(c) In the previous item, find the ressonance frequency ω_R.

(d) Write the general solution for $\theta(t)$, when the initial conditions are $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ and the support moves with frequency $\omega < \omega_R$.

Q7. A tritium atom can be described *classically* as a nucleus of electrical charge $+e$, consisting of one proton and two neutrons, encircled by an orbital electron of charge $-e$, which follows a circular orbit of radius r_0. In a process known as beta decay, the tritium nucleus transforms itself into a helium nucleus, consisting of two protons and one neutron, with the emission of a pair of particles that quickly escapes from the atomic system. As a consequence of this process, the helium atom becomes one fold ionized, and the orbital electron suddenly passes to a new situation, orbiting now around a nucleus with charge $+2e$.

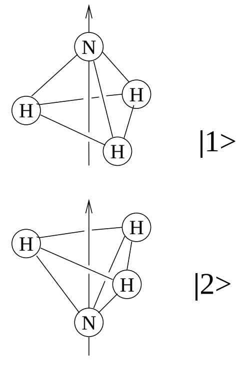
(a) Assuming that the pair of particles that escapes from the atom has a total linear moment of absolute value equal to p, obtain the recoil velocity of the helium atom of mass M.

(b) Obtain the energy E_b of the orbital electron before the beta decay.

(c) Calculate the energy E_a of the orbital electron after the beta decay and obtain the ratio $\rho = E_b/E_a$.

(d) Determine the total angular momentum of the electron as a functioin of r_0 and the mass m of the electron. Calculate the greatest and the smallest distance between the electron and the nucleus in the new orbit in terms of r_0.

Q8. Consider two states $|1\rangle$ and $|2\rangle$ of the ammonia molecule as illustrated in the figure. Assume that they are orthonormalized, $\langle i|j\rangle = \delta_{ij}$ and that these two are the only states accessible to the system, so that we may describe it by using a basis comprised by $|1\rangle$ and $|2\rangle$. In this basis, the Hamiltonian H of the system is given by



$$H = \begin{pmatrix} E_0 & -E_1 \\ -E_1 & E_0 \end{pmatrix}$$

(a) If, initially, the system is in state $|1\rangle$, will it remain in state $|1\rangle$ at a later time? And if it is in state $|2\rangle$, will it remain in state $|2\rangle$?

(b) Find the eigenvalues E_I and E_{II} and the respective eigenvectors $|I\rangle$ and $|II\rangle$ of H, expressing them in terms of $|1\rangle$ and $|2\rangle$.

(c) Based on the above result, we can predict at least one possible emission frequency of the electromagnetic radiation for an ammonia molecule. What is this frequency?

(d) Find the probability of measuring the energy E_I in the following state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|1\rangle - \frac{2}{\sqrt{5}}|2\rangle.$$

Q9. A quantum particle of mass m is subject to the potential

$$V = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2).$$

(a) Obtain the energy levels of this particle. That is, determine the eigenvalues of

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi.$$

(b) Consider the ground state and the two first excited levels. Set up a table showing for each of these three levels, the value of the energy, the degeneracy and the respective states in terms of the quantum numbers.

(c) Using

$$\nabla^2\psi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\psi\right]$$

and keeping in mind that $L^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}$, write a differential equation of the item (a) for the radial part of the wave function (it is not necessary to solve it). Identify in this equation the effective potencial $V_{\text{eff}}(r)$.

(d) Solve the differential equation of the previous item for the case in which $\ell = 0$ and determine the corresponding eigenvalue. To this end, assume a solution of the type $e^{-\alpha r^2}$ and determine α.

Q10. Consider a classic monoatomic gas comprising N noninteracting atoms of mass m enclosed in a vessel of volume V, at temperature T. The Hamiltonian corresponding to an atom is given by $\mathcal{H} = (p_x^2 + p_y^2 + p_z^2)/2m$.

(a) Show that the atomic canonical partition function is $\zeta = V/\lambda^3$, where $\lambda = h/\sqrt{2\pi m k_B T}$ is the thermal de Broglie wave length.

(b) Using ζ of the previous item, find the partition function Z of the system and the Helmholtz free energy F. Obtain, also, the free energy per atom $f = F/N$ in the thermodynamic limit $N \to \infty$, $V \to \infty$, $v = V/N$ fixed.

(c) Find the internal energy U and the pressure p of the gas.

(d) Calculate the entropy per atom in the thermodynamic limit.