EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the second semester of 2017

April 04, 2017

Part 1

Instructions

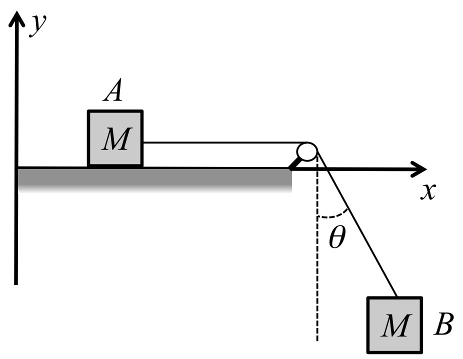
- **Do not write your name on the test.**
  It should be identified only by your candidate number.
- This test contains questions on:
  classical mechanics, quantum mechanics, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding sheet of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- **Do not write anything on the list of constants and formulae.**
  Return both this test and the list of constants and formulae at the end of the test. The latter will be used in the test tomorrow.

Have a good exam!

Q1. Gravity's acceleration g can be measured with reasonable precision by means of a simple pendulum, which consists of a body of mass m attached to a wire of negligible mass and length l.

(a) Find the expression for the period of the pendulum in terms of its parameters.
(b) A group of students went to the lab in order to obtain an accurate value of the local acceleration of gravity. They built a simple pendulum with a metallic mass attached to the lab's ceiling through a thin wire. The metallic mass had a spherical shape of radius $r = 8.00 \pm 0.05$ cm and mass $m = 10.0 \pm 0.1$ kg and was linked to the wire so that at rest the center of mass of the sphere was 4.00 ± 0.02 m from the ceiling. The wire's mass was 7.4 ± 0.2 g. The period of oscillation was measured for different initial lateral displacements between 5.0 ± 0.1 and 10.0 ± 0.1 cm. The students determined that, for this range of lateral displacements, the period did not depend on the initial position, within the experimental uncertainty. The pendulum was found to perform 10 complete oscillations in 40.0 ± 0.5 s. Determine the value of g they found, together with the experimental uncertainty. Use, if necessary, $\pi^2 = 9.86960$.

Q2. Two bodies of identical mass M are linked by a uniform inextensible string of length l. The body A slides without friction on a uniform table and the body B hangs from the side of the table, the string passing through a pulley of negligible radius, as shown in the figure.



(a) Find the common acceleration of the bodies if the angle θ is **kept constant and equal to zero** and the mass of the string is negligible.
(b) Consider now the more general motion in which the angle θ is allowed to vary. Assume that θ is always less than $\pi/2$ and that the body B never touches the table. Write the system's Lagrangian and the equations of motion (**do not try to solve the equations**). Show that we recover the result of item (a) if we set $\theta = 0$.
(c) Assume now that θ is again kept constant and equal to zero, but the string has a **non-negligible** mass m. Write the system's Lagrangian and the equations of motion. It is not necessary to solve the equations.

Q3. A particle of rest mass m and total relativistic energy equal to double its rest energy, collides head-on with an identical particle (same rest mass m), initially at rest. After the collision, a single particle is formed with rest mass M (totally inelastic collision). For each item below, express your answer **in terms of** c **and** m.

(a) Find the speed v of the incident particle before the collision.
(b) Using the conservation of energy-momentum,

(i) determine the speed V of the final resulting particle in terms of the speed v of the incident particle. Use the result of item (a) to obtain the numerical value of V/c.
(ii) determine the mass M of the final resulting particle.

(c) Find the kinetic energy of the resulting particle.

Q4. Consider the one-dimensional **time-independent** Schrödinger equation for energies E within the interval $[0,U_0]$ ($U_0 > 0$) for a particle of mass m in a square well potential given by

$$V(x) = \begin{cases} U_0, & \text{if } x < 0, \\ 0, & \text{if } 0 < x < L, \\ U_0, & \text{if } x > L. \end{cases}$$

(a) Is the energy spectrum E discrete or continuous? Why?
(b) Write the general form of the wave function in each of the 3 regions of the potential.
(c) Write the equation whose solution gives the energy spectrum. It is not necessary to solve this equation.
(d) What happens to the energy spectrum in the limit $L \to 0$?

Q5. The Stirling engine is a heat engine whose cycle consists of two isothermal and two isochoric (isovolumetric) processes. One mole of an ideal monatomic gas ($C_V = 3R/2$) is subject to a Stirling cycle consisting of the following processes: (1) isothermal compression to 1/3 of the initial volume V_0 at temperature T_0; (2) constant-volume heating to double the initial temperature T_0; (3) isothermal expansion up to the initial volume V_0; (4) constant-volume cooling down to the initial temperature T_0.

(a) Sketch the above cycle in a P x V (pressure vs volume) diagram.
(b) Determine the variation of the internal energy of the gas in processes 1 and 2 in terms of R and T_0.
(c) Determine the work done by the gas in process 3 in terms of R and T_0.
(d) Determine the engine's thermal efficiency.

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April 05, 2017

Part 2

Instructions

- **Do not write your name on the test.**
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- This test contains questions on:
  electromagnetism, quantum mechanics, modern physics, and statistical mechanics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
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Q6. Two thin circular loops lie on the xy plane of a coordinate system with their centers at the origin. One loop has radius b and a linear electric charge distribution $-\lambda < 0$ and the other has radius $2b$ and a linear electric charge distribution $2\lambda > 0$.

(a) Find the electric potential $V(z)$ at the point $P = (0,0,z)$.
(b) Find the electric field vector $\mathbf{E}(z)$ at the point $P = (0,0,z)$.
(c) Write the equation for Newton's second law for a particle of charge $q > 0$ and mass m, constrained to move along the z axis and subject to the electric field of item (b). No other force besides the electric force acts on the particle.
(d) Find the frequency of small oscillations of the particle of item (c) in the neighborhood of $z = 0$. Hint: linearize the force around $z = 0$.

Q7. A plane monochromatic circularly-polarized electromagnetic wave propagates in vacuum, described **in complex notation** by the electric field $\tilde{\mathbf{E}}(\mathbf{r},t) = E_0 e^{i(kz-\omega t)}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$, where $\omega = ck$ is the angular frequency, c is the speed of light in vacuum, k is the wave number, E_0 is a real amplitude and $i = \sqrt{-1}$.

(a) Find the *real* (physical) electric field $\mathbf{E}(\mathbf{r},t)$.
(b) Find the *real* (physical) magnetic field $\mathbf{B}(\mathbf{r},t)$ using Maxwell's equations. If you want, you can use $\nabla \to i\mathbf{k}$.
(c) Find the linear momentum density of the electromagnetic wave $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$.
(d) Find the angular momentum density of the electromagnetic wave $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g}$. Hint: use cylindrical coordinates $\mathbf{r} = \rho\hat{\rho} + z\hat{z}$.

Q8. An electron with kinetic energy $E_{\text{kin}} = 22$ eV collides with a hydrogen atom that is initially in its ground state. Only part of the incident electron's energy is transferred to the atom, which undergoes a transition to an excited state with quantum number n. At a time Δt after the collision, the atom decays back to its ground state, emitting a photon with energy 10.2 eV.

(a) Using a non-relativistic approximation, determine the de Broglie wavelength λ of the incident electron.
(b) Determine the quantum number n of the excited state of the hydrogen atom.
(c) Obtain the uncertainty in the energy of the emitted photon if $\Delta t = 10^{-8}$ s.
(d) Justify the use of the non-relativistic approximation employed in item (a).

Q9. Consider a quantum system whose Hilbert space is spanned by an **orthonormal basis** of 3 states $|1\rangle$, $|2\rangle$ e $|3\rangle$. A generic state of the system can then be represented in that basis as a column vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where x, y and z are complex coefficients. The system's Hamiltonian, on the other hand, can be represented in the same basis as a complex square matrix

$$H = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & M_{23} \\ 0 & - & E_3 \end{pmatrix}.$$

(a) What is the value of the missing matrix element of H? What is the value of the imaginary part of E_3?
(b) A certain observable A acts on the basis states as follows

$$\begin{aligned} A|1\rangle &= 2|1\rangle, \\ A|2\rangle &= 2|2\rangle, \\ A|3\rangle &= |3\rangle. \end{aligned}$$

Write the matrix that represents A in that basis. Can this observable be measured simultaneously with the energy? Justify.
(c) What are the eigenenergies of the system?
(d) Assume that $E_1 = 1$, $E_2 = E_3 = 3$ and $M_{23} = 1$ and that the system is prepared at time $t = 0$ in the state $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Find the state for $t > 0$.

Q10. Consider a system composed of N localized magnetic ions of spin 1 in contact with a thermal reservoir at temperature T. The system can be approximately described by the Hamiltonian

$$H = D \sum_{i=1}^{N} \sigma_i^2 - h \sum_{i=1}^{N} \sigma_i,$$

where σ_i is the (dimensionless) z-component of the i-th spin, which can take the values $0, +1$ and -1, $h > 0$ is an external magnetic field and $D > 0$ is an anisotropy term.

(a) Find the partition function of the system.
(b) Find the Helmholtz free energy per ion as a function of the temperature.
(c) Find the internal energy per ion as a function of the temperature.
(d) Assume now that $h = 0$. Find the specific heat as a function of the temperature.