EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the first semester 2016

14 October 2015

Part 1

Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  electromagnetism, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding page of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- Do not write anything on the list of constants and formulae.
  Return it at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

Q1. Defining the Hertz vector $\vec{Z}$ by the expressions:

$$\vec{\nabla} \cdot \vec{Z} = -\phi; \qquad \vec{A} = \mu_0 \epsilon_0 \frac{\partial \vec{Z}}{\partial t}, \tag{1}$$

where ϕ e $\vec{A}$ are, respectively, the scalar and vector potentials.

a) Show that these potentials satisfy the Lorentz gauge:

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0; \tag{2}$$

b) Shows that for a medium without sources, ($\rho = 0$, $\vec{J} = 0$) and with $\mu = \mu_0$ the vector $\vec{Z}$ satisfies the following expressions:

$$\nabla^2 \vec{Z} - \frac{1}{c^2} \frac{\partial^2 \vec{Z}}{\partial t^2} = -\frac{\vec{P}}{\epsilon_0}; \qquad \vec{B} = \frac{1}{c^2} \vec{\nabla} \times \frac{\partial \vec{Z}}{\partial t}; \qquad \vec{E} = \vec{\nabla} \times \vec{\nabla} \times \vec{Z} - \frac{\vec{P}}{\epsilon_0}, \tag{3}$$

where $\vec{P}$ is the polarization vector.

Q2. Consider a very thin hollow disc with inner radius r_1 and outer radius r_2 lying over the xy plane and with its axis centered at $z = 0$ (as illustrated in figure 1).

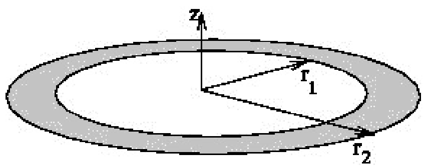

Figura 1: Hollow disk.

The ring has a superficial charge density given by:

$$\sigma(r) = \frac{\sigma_0}{r}, \tag{4}$$

where $r = \sqrt{x^2 + y^2}$.

a) Find the electric field $\vec{E}(x = y = 0, z)$ on the axis z;

b) Suppose now that the ring starts to rotate with angular velocity ω_0. Find the current density $\vec{J}_s = \sigma \vec{v}$, where $\vec{v}$ is the linear velocity;

c) Find the magnetic field $\vec{H}(x = y = 0, z)$ on the axis z, produced by the current density $\vec{J}_s$.

Q3. A positive pion π^+ can decay according to the reaction $\pi^+ \to \mu^+ + \nu_\mu$, i.e., it can decay into a positive muon μ^+ accompanied by a muonic neutrino ν_μ. Neglecting the mass m_ν of the neutrino and considering the pion initially at rest in an inertial frame S, determine, in terms of the mass of the pion (m_π) and of the muon (m_μ):

a) The modulus of the muon linear momentum.
b) The total energy of the muon.
c) The muon velocity.
d) The average distance one muon runs (in the vacuum) before also decaying. Use the symbol τ for the muon mean life-time measured in its own referential.

Q4. Consider a non-relativistic particle of mass m performing a simple harmonic movement with frequency ν.
a) Determine, in terms of ν, the allowed energy levels E for this particle, from the Bohr-Sommerfeld quantization rule $\oint p_q dq = nh$.
b) Consider a system containing a large number of these particles in thermal equilibrium. From the allowed energy levels for each particle, determined in the previous item, calculate the total average energy $\langle E \rangle$, where $P(E_n) = Ae^{-E_n/kT}$ is the distribution function.

Q5. Consider a Carnot engine operated with an ideal paramagnet whose equation of state is given by the Curie law

$$M = D\frac{H}{T},$$

with M the magnetization, H the magnetic field, T the temperature and D a constant. The internal energy change is given in terms of variation of the entropy and the magnetization by $dU = T\,dS + H\,dM$ (the term HdM is analogous to the term $-PdV$ for the ideal gas) and also by $dU = C_M dT$, with C_M constant.

a) Determine the relationship that links the initial values of magnetization and temperature M_i, T_i to the final values M_f, T_f in an adiabatic transformation, in terms of C_M and D.

b) Represent the cycle composed of two adiabatic and two isothermal transformations in a diagram H-M. The isotherms correspond respectively to a higher temperature, T_Q, and a lower, T_F. Indicate the four states in the vertices of the diagram as (M_1,H_1) (beginning of the cycle, in the highest magnetization and at the temperature T_Q) (M_2,H_2), (M_3,H_3), (M_4,H_4).

c) Calculate the total work done in the cycle, as a function of M_1, M_2, T_Q, T_F and of the constant D.

d) Get the efficiency of the cycle, given by the ratio between the total work done and absorbed heat (at the temperature T_Q).

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Part 2

Instructions

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  classical mechanics, quantum mechanics, and statistical mechanics.
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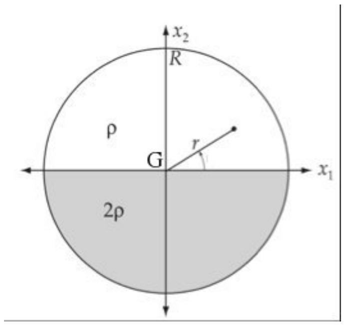
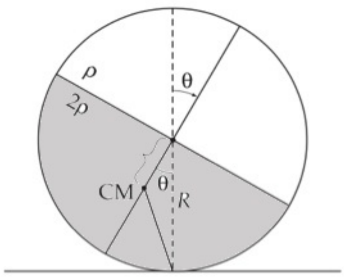
Q6. A disk of radius R is composed by two halves, each with mass densities 1ρ and 2ρ respectively.



a) What is the moment of inertia of the disk, with respect to the axis perpendicular to the disk plane passing through its geometric center G?
b) Determine the coordinates x_1 and x_2 of the center of mass of the disk.
c) What is the moment of inertia of the disk, with respect to the axis perpendicular to the disk plane passing through its center of mass?
d) Consider the movement of the disk in a straight line on a horizontal plane perpendicular to the disk plane, without sliding. Find $\lambda(\theta)$ implicitly defined by

$$v(t) = \lambda(\theta)\, R\, \frac{d\theta}{dt},$$

where θ is the angle between the vertical axis and the straight line passing through the geometric center and the center of mass (see figure), $v(t)$ is the module of the center of mass velocity, and $\frac{d\theta}{dt}$ is the module of disk rotation velocity.



Q7. Consider an object of mass M that moves under the action of a Coulomb-like central force, modified by a force proportional to the inverse of r^3,

$$F(r) = -\frac{k}{r^2} - \frac{q}{r^3},$$

where r is the radial coordinate, and k and q are positive constants.

Let the total energy of the system be described by

$$E = \frac{M}{2}\dot{r}^2 + \frac{M}{2}r^2\dot{\theta}^2 - \frac{k}{r} - \frac{q}{2r^2},$$

and the angular momentum of the system given by $L = M\, r^2\, \dot{\theta}$.

a) For the case where the object describes a circular orbit (equilibrium), find the orbit radius as a function of the parameters k, q, M e L of the system.

b) For the same conditions of item a), find the total energy E as a function of the parameters k, q and L, of the system.

c) By identifying the effective potential for the radial movement as

$$V_{ef}(r) = \frac{L^2}{2mr^2} - \frac{k}{r} - \frac{q}{2r^2},$$

check under which conditions on the constants q, L and M, the radial coordinate of the circular orbit obeys a stable equilibrium configuration.

d) In the case the radial coordinate of the particle move out of the (stable) equilibrium condition and oscillate approximately harmonically (around the radius of the circular orbits), find the relationship between the radial oscillation period and the period of revolution (angular movement) in terms of the constants q, M and L.

Q8. Consider a system composed by a pair $\mathbf{A}$ and $\mathbf{B}$ of spins 1/2 described by the state

$$|\psi\rangle = \alpha|\mathbf{A}_+\rangle \otimes |\mathbf{B}_-\rangle + \beta|\mathbf{A}_-\rangle \otimes |\mathbf{B}_+\rangle + \gamma|\mathbf{A}_-\rangle \otimes |\mathbf{B}_-\rangle + \delta|\mathbf{A}_+\rangle \otimes |\mathbf{B}_+\rangle$$

(with $\alpha,\beta,\gamma,\delta \in \mathbb{C}$) in the Hilbert space $\mathcal{H}_{\mathbf{A}} \otimes \mathcal{H}_{\mathbf{B}}$, where the state $|\mathbf{A}_\pm\rangle$ satisfies $\langle \mathbf{A}_\pm|\mathbf{A}_\pm\rangle = 1$, $\langle \mathbf{A}_\pm|\mathbf{A}_\mp\rangle = 0$ and

$$\hat{S}^{\mathbf{A}}_z|\mathbf{A}_\pm\rangle = \pm\frac{\hbar}{2}|\mathbf{A}_\pm\rangle, \quad \hat{S}^{\mathbf{A}}_\mp|\mathbf{A}_\pm\rangle = \hbar|\mathbf{A}_\mp\rangle, \quad \hat{S}^{\mathbf{A}}_\pm|\mathbf{A}_\pm\rangle = 0.$$

And similarly for $|\mathbf{B}_\pm\rangle$. By recalling that

$$\hat{S}_z \equiv \hat{S}^{\mathbf{A}}_z \otimes \hat{I}^{\mathbf{B}} + \hat{I}^{\mathbf{A}} \otimes \hat{S}^{\mathbf{B}}_z$$

and also

$$\hat{S}_x \equiv \hat{S}^{\mathbf{A}}_x \otimes \hat{I}^{\mathbf{B}} + \hat{I}^{\mathbf{A}} \otimes \hat{S}^{\mathbf{B}}_x, \quad \hat{S}_y \equiv \hat{S}^{\mathbf{A}}_y \otimes \hat{I}^{\mathbf{B}} + \hat{I}^{\mathbf{A}} \otimes \hat{S}^{\mathbf{B}}_y$$

with $I^{\mathbf{A}}$,$I^{\mathbf{B}}$ being the identity operators acting in the corresponding Hilbert spaces, answer:

a) What is the dimension of the Hilbert space $\mathcal{H}_{\mathbf{A}} \otimes \mathcal{H}_{\mathbf{B}}$ of the pair of spins $\mathbf{A}$ and $\mathbf{B}$?

b) Consider the state $|\psi\rangle$ with $\alpha = \beta = \gamma = 0$. What is the most general value of $\delta \in \mathbb{C}$ that normalizes $|\psi\rangle$?

c) Consider the state $|\psi\rangle$ with $\alpha = -\beta = 1/\sqrt{2}$ and $\gamma = \delta = 0$. What is the expectation value of the operator $\hat{S}_z$ for this state?

d) Consider the state $|\psi\rangle$ with $\alpha = \beta = 1/\sqrt{2}$ and $\gamma = \delta = 0$. Determine whether $|\psi\rangle$ is an eigenstate of the spin operator $\hat{S}^2 \equiv \hat{S}^2_x + \hat{S}^2_y + \hat{S}^2_z$. In the case it is an eigenstate, what is the corresponding eigenvalue? (Hint: remember that $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ and that $\left[\hat{S}_x,\hat{S}_y\right] = i\hbar\hat{S}_z$.)

Q9. Consider a harmonic oscillator with frequency ω , mass m and Hamiltonian

$$\hat{H} = (1/2 + \hat{n})\hbar\omega, \tag{5}$$

where $\hat{n} \equiv \hat{a}^\dagger \hat{a}$ with $\hat{n}|n\rangle = n|n\rangle$ and we recall that the lowering and raising operators satisfy

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Assuming that at the time $t = 0$ the oscillator is in a coherent state $|z\rangle$ defined by

$$\hat{a}|z\rangle = z|z\rangle,$$

respond:

a) What is the value of $\langle z|\hat{n}|z\rangle$ for $z = \frac{1}{2}\exp(i\pi/4)$, assuming that $|z\rangle$ is normalized?

b) Assuming that at $t = 0$ the oscillator is in the ground state $|0\rangle$, determine the state form at time $t = 1/10$ s for $\omega = 5\pi$ s^{-1}.

c) What is the value of c_n (as a function of n and z) so that the coherent state $|z\rangle = \sum_{n=0}^{+\infty} c_n|n\rangle$ (expanded in the eigenstates $|n\rangle$ of the number operator $\hat{n}$) be normalized? (Remember that $e^x = \sum_{n=0}^{+\infty} x^n/n!$)

d) Use the result of the previous item and calculate the numerical value of $|\langle z'|z\rangle|^2$ for $z = 1/2\exp(i\pi/4)$ and $z' = 1/4\exp(i\pi/4)$.

Q10. Consider a system of N non-interacting spins 1/2, with magnetic dipole moment of magnitude μ, in the presence of a uniform magnetic field B.

a) Write down the Hamiltonian of the system.

b) Considering the system in thermal equilibrium at the inverse temperature $\beta = 1/k_BT$, calculate the partition function $Z(\beta,B)$.

c) Calculate the magnetization M as a function of T and B.

d) Get the expression for M in the high temperature limit and weak magnetic field.