EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the first semester 2015

14 October 2014

Part 1

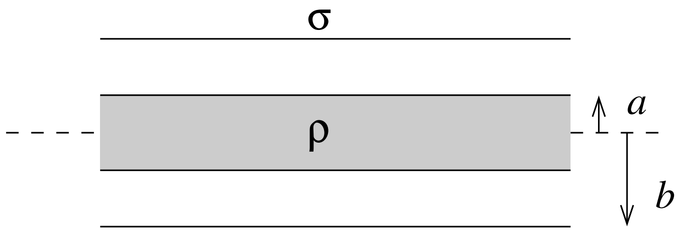
Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number (**EUFxxx**).
- This test contains questions on:
  electromagnetism, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 60 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding page of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- Do not write anything on the list of constants and formulae.
  Return it at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

Q1. a) A solid dielectric cylinder, of infinite length and radius a, has a positive and uniform bulk charge density ρ. A cylindrical shell, also dielectric, of radius $b > a$, with its axis coinciding with that of the solid cylinder, has a negative and uniform surface charge density σ, such that the total charge of the cylinder plus the shell, in a given length, vanishes, so that $\sigma = -\rho a^2/2b$. Determine the electric field $\vec{E}(r)$ in the regions $r < a$, $a < r < b$ and $b < r$ where r is the distance from the axis of the cylinder.

b) Next, suppose that the cylinder plus the shell moves together to the right with velocity $\vec{v}$. The motion gives rise to an electric current $I = \pi a^2 \rho v$ in the massive cilinder, to the right and uniformly distributed on the cross section, such that the current density is given by $\vec{J} = \rho\vec{v}$. Similarly, the shell in motion gives rise to a current of the same intensity I, but in the inverse direction (to the left). Determine the magnetic induction $\vec{B}$ in the regions $r < a$, $a < r < b$ and $b < r$.



Q2. The electric field of a monochromatic plane wave in vacuum is given by

$$\vec{E}(z,t) = (E_1\,\hat{x} + E_2\,\hat{y})e^{i(kz-\omega t)}.$$

where $\hat{x}$ and $\hat{y}$ are the Cartesian versors in directions x and y, respectively, and E_1 and E_2 are constant.

a) Find the magnetic induction $\vec{B}(z,t)$.

b) Shown that the electric field and the magnetic induction are orthogonal to each other.

c) Find the Poynting vector of the wave.

Q3. Consider a gas of diatomic molecules with frequency of oscillation, ω, and moment of inertia, I. At room temperature, the energies of molecular vibration are much larger than k_BT. Therefore, the majority of the molecules are found in the vibrational state of lowest energy. On the other hand, the characteristic energy of the rotational states is much lower than k_BT. The rotational-vibrational energy $E(n,\ell)$ of the state of a diatomic molecule is characterized by the quantum number n, for the vibrational energy, and by the quantum number ℓ, for the rotational energy.

a) Write $E(n,\ell)$ for $n = 0$ and any ℓ.

b) Supppose that a molecule suffers a transition from an initial state with $n = 0$ and any ℓ to an excited state with $n = 1$. Determine the two allowed total energies for the molecule after the transition, bearing in mind that the selection rules impose $\Delta\ell = \pm 1$. Determine the difference in energy between these two allowed states and the initial state, as well as the respective transition frequencies.

c) Consider the state of the molecule in which $n = 0$ and any ℓ. Knowing that the degeneracy of the state is $2\ell + 1$, determine the population of the rotational-vibrational state, $N(E)$, as a function of E, from the Boltzmann distribution.

d) For $n = 0$, the state $\ell = 0$ is not the most populated state at room temperature. For small values of ℓ, the population of the state increases slightly with relation to $\ell = 0$ due to the increase in the density of states. For large values of ℓ, the population decreases due to the Boltzmann factor. Determine the value of ℓ for which the population is a maximum.

Q4. Suppose that a photon collides with an electron that is initially at rest in a reference frame S, as in figure 1A. Most of the time, the photon is simply scattered from its original trajectory, but, occasionally, the event results in the annihilation of the photon and the creation of an electron-positron pair, in the presence of the original electron. Assume that the details of the interaction that produce the pair are such that the three resulting particles move to the right, as seen in figure 1B, with the same velocity u, that is, in a way that all are at rest in a reference frame S′, which is moving to the right with velocity u with respect to S.

a) Write down the energy and momentum conservation laws before and after the pair creation.

b) Using relativistic energy-momentum conservation, find, in the system S′, the photon energy for the creation of a pair of particles with energy equivalent to the rest energy of 2 electrons.

c) Use the relation $m_0^2c^4 = E^2 - p^2c^2$ to find the relation $u/c = pc/E$.

d) Determine from (c) the velocity u with which the three particles move in the frame S.

Figura 1: (A) Situation before the collision, in the frame S. (B) Situation after the collision in the frame S.

Q5. An ideal gas enclosed in a vessel, initially with volume, V_A, and pressure p_A (state A), undergoes an isobaric expansion until reaching a volume V_B (state B). The gas then undergoes an adiabatic expansion, until its pressure becomes p_C (state C), so that an isobaric contraction (until state D) followed by an adiabatic compression, will drive the gas again to the initial situation (state A). Assume that the ratio γ between the specific heat at constant pressure and constant volume is given.

a) Show the transformations described above in a $p - V$ diagram, indicating the states A, B, C and D.

(b) Calculate the heat exchanged in each part of the cycle, in terms of p_A, V_A, V_B, p_C and γ.

(c) Determine the efficiency of the cycle, that is, the ratio between the work performed by the gas and the heat absorbed by the gas.

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Part 2

Instructions

- **Do not write your name on the test.**
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  classical mechanics, quantum mechanics, and statistical mechanics.
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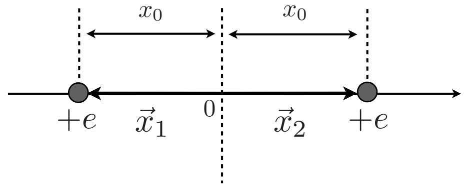
Q6. It is possible to set up traps capable of confining ions of mass m and charge q. In particular, the trap can restrict the motion of the ions to just one spatial direction, x. Thus, consider two calcium ions singly ionized (Ca^+), subject to an external harmonic confining potential, $U(x) = m\omega^2 x^2/2$. In addition, these ions interact through the Coulomb repulsion,



$$F_C = \frac{e^2}{(x_1 - x_2)^2}$$

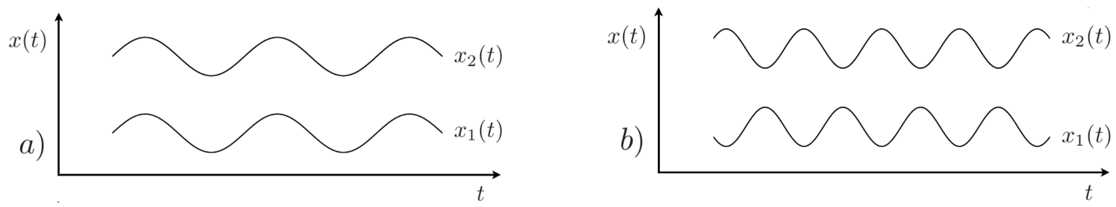
where x_1 and x_2 are the positions of the calcium ions and, for simplicity, we define: $e^2 = q^2/(4\pi\varepsilon_0)$. The figure above defines a convenient coordinate system and represents the ions at the equilibrium positions for which $-x_1 = x_2 = x_0$. The aim of this problem is the study of the normal modes of this one-dimensional chain comprising the two calcium ions.

a) Find the equilibrium position x_0 in terms of e, m and ω.

b) Write down Newton's equations for the motion of each ion and obtain the frequency of oscillation of the system when the separation of the ions is constant. This is the first normal mode of oscillation of the chain.

c) The second mode corresponds to an antisymmetric motion of the ions, in which case the center of mass is at rest at $x = 0$. Obtain the second mode in the limit of small oscillations. Obtain the ratio between the frequencies of the two normal modes of oscillation of the system.

d) Figures a) and b) below represent the normal mode of oscillation of this system of two ions. Identify the first and the second normal mode obtained, respectively, in items b and c above. Which one has the smaller energy?



Q7. An artificial satellite of mass m is in an elliptic orbit around the Earth. Assume that the Earth is a sphere of uniform density with radius R and mass M, and denote by G the universal gravitational constant. Consider known d and D, the distances between the Earth's center and the points of smallest and greatest separation, respectively. A particle of mass m_0 smaller than m, collides head on and completely inelastically with the satellite at the point of smallest separation from the Earth. At the instant of collision, the satellite and the particle have the same speed but opposite velocities.

a) Obtain the velocity v_S of the satellite-particle system *just after* the collision in terms of v_p, the velocity at the point of smallest separation.

b) Express the angular momentum of the satellite at the points of minimum and maximum separation in terms of v_p and of v_a (the velocity at greatest separation), respectively, before the collision.

c) Find the velocity v_p, before the collision, in terms of M, d, D and G.

d) Find the energy E_S and the angular momentum L_S of the satellite-particle system, after the collision, in terms of m_0 and the quantities that characterize the motion of the satellite before the collision.

Q8. Let the spin state of an electron be given by

$$|\psi\rangle = \alpha \left(|z_+\rangle - \frac{\sqrt{2}}{2} |z_-\rangle \right)$$

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Bearing in mind that the spin operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ can be written in terms of the Pauli matrices as $\hat{\mathbf{S}} = \hbar\, \vec{\sigma}/2$ (see the list of formulae), where

$$\begin{aligned} \hat{S}_x|x_+\rangle &= +\frac{\hbar}{2}|x_+\rangle, & \hat{S}_x|x_-\rangle &= -\frac{\hbar}{2}|x_-\rangle, \\ \hat{S}_y|y_+\rangle &= +\frac{\hbar}{2}|y_+\rangle, & \hat{S}_y|y_-\rangle &= -\frac{\hbar}{2}|y_-\rangle, \\ \hat{S}_z|z_+\rangle &= +\frac{\hbar}{2}|z_+\rangle, & \hat{S}_z|z_-\rangle &= -\frac{\hbar}{2}|z_-\rangle, \end{aligned}$$

a) What is the value of $\alpha \in \mathbb{R}$ such that $|\psi\rangle$ is normalized?

b) What is the probability of measuring $-\hbar/2$ for the spin in the direction z?

c) What is the probability of measuring $+\hbar/2$ for the spin in the direction x?

d) What is the expectation value of the spin in the plane $y = 0$ in a direction of 45^0 between the axes x and z?

Q9. Let $\hat{A}$ be an operator associated with a physical observable A of a system satisfying $[\hat{A},\hat{H}] \neq 0$, where $\hat{H}$ is a time independent Hamiltonian operator. Now, let ϕ_+, ϕ_- be the normalized eigenvectors and a_+, a_- $(a_+ \neq a_-)$ the corresponding eigenvalues of $\hat{A}$:

$$\hat{A}\phi_+ = a_+\phi_+, \qquad \hat{A}\phi_- = a_-\phi_-,$$

with

$$\phi_+ = \frac{u_+ + u_-}{\sqrt{2}}, \qquad \phi_- = \frac{u_+ - u_-}{\sqrt{2}}$$

where

$$\hat{H}u_+ = E_+u_+, \qquad \hat{H}u_- = E_-u_-$$

a) Calculate the expectation value of $\hat{A}$ in state ϕ_+.

b) Calculate the projection of $\hat{H}u_+$ onto state u_-.

c) Assuming that the system is initially in an arbitrary state $\psi(0)$, write down the state $\psi(t)$ at a later time as a function of $\hat{H}$.

d) Calculate the expectation value of the observable A at time $t = \hbar\pi/[3(E_+ - E_-)]$ assuming that the system is initially in state $\psi(0) = \phi_+$ and $E_+ \neq E_-$.

Q10. Consider N three-dimensional non-interacting classical harmonic oscillators, of mass m and angular frequency ω, in contact with a heat reservoir at temperature T.

a) Write down the Hamiltonian of the system and find the canonical partition function.

b) Obtain the average value of the energy per oscillator. What is the heat capacity of the system?