EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the first semester of 2018

October 03, 2017

Part 1

This test contains questions on classical mechanics, quantum mechanics, modern physics, and thermodynamics. All questions have the same weight.

Have a good exam!

Q1. A particle of mass m moves on a vertical plane (xz plane, where x is the horizontal direction and z is the vertical direction) under the action of the gravitational force $\mathbf{F}_g = m\mathbf{g} = -mg\hat{\mathbf{z}}$, where g is the acceleration of gravity. At the initial instant $t = 0$, the particle is at the origin with velocity $\mathbf{v}_0 = (v_0 \cos\theta)\hat{\mathbf{x}} + (v_0 \sin\theta)\hat{\mathbf{z}}$, where $v_0 > 0$ and $0 < \theta < \pi/4$.
(a) Write the equations of motion for the components x and z of the particle's position.
(b) Obtain the components $v_x(t)$ and $v_z(t)$ of the particle's velocity as functions of time.
(c) Obtain the components $x(t)$ and $z(t)$ of the particle's position as functions of time.
(d) Obtain the particle's angular momentum **vector** $\mathbf{L}(t)$ with respect to the origin as a function of time.
(e) Obtain the torque **vector** $\mathbf{N}(t)$ due to the gravitational force $\mathbf{F}_g$ with respect to the origin as a function of time. Find the relation between $\mathbf{L}(t)$ and $\mathbf{N}(t)$.

Q2. A particle of mass m moves in two dimensions (xy plane) under the action of two conservative forces whose potentials are

$$U_1(y) = \lambda y \qquad \text{and} \qquad U_2(r) = \frac{1}{2}kr^2,$$

where λ and k are positive constants and $r = \sqrt{x^2 + y^2}$ is the distance from the particle to the origin of the reference frame.

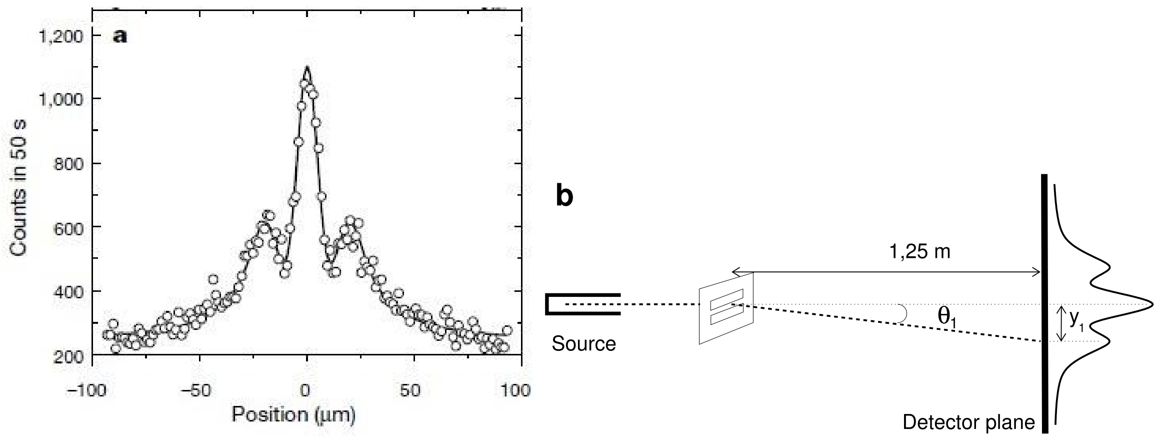
(a) Find the force **vector** $\mathbf{F}_2$ derived from the potential $U_2(r)$.
(b) Write the particle's Lagrangian using polar coordinates in the plane r and θ, and obtain the corresponding equations of motion.
(c) Find the Hamiltonian of the system. Remember that the Hamiltonian must be written in terms of the coordinates r and θ and their canonically conjugate momenta.
(d) Find the angular momentum **vector** $\mathbf{L}$ using polar coordinates r and θ. Determine under what condition the angular momentum is conserved.

Q3. Consider the quantum dynamics of a beam of particles of mass m that move strictly along the x axis of a reference frame, in the positive x direction. They are subjected to a step potential ($V_0 > 0$)

$$V(x) = \begin{cases} 0, & \text{if } x < 0, \\ V_0, & \text{if } x > 0. \end{cases}$$

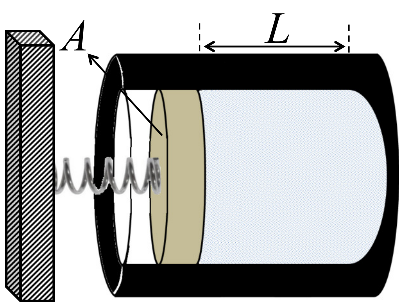
(a) If the total energy E of each particle is such that $E > V_0$, find the general form of the solution of the time-independent Schrödinger equation in the two regions of the potential.
(b) According to classical mechanics, when $E > V_0$ all the particles cross the potential step. According to quantum mechanics, however, some particles are reflected. What percentage of particles is reflected by the potential step if $E > V_0$?
(c) Now assume that $E < V_0$. What is the general form of the solution of the time-independent Schrödinger equation for $x > 0$?
(d) Find the reflection probability if $E = V_0/2$.

Q4. Figure **a** below [extracted from the article *Wave-particle duality of C_{60} molecules* by M. Arndt *et al.*, Nature **401**, 680 (1999)] shows the diffraction pattern obtained when a beam of fullerene molecules (C_{60}) goes through a diffraction grating. It shows the molecular count at the detector versus the vertical position y (in μm) measured from the intersection of the beam direction and the detector plane, as shown schematically in figure **b**. The detector plane was 1.25 m away from the diffraction grating.



(a) From the graph, estimate the position y_1 of the first peak beside the central diffraction maximum and find the corresponding angle θ_1.
(b) Upon seeing the diffraction pattern, a student assumed that it was an interference pattern from an electromagnetic wave going through a double slit. She was told that the distance between the slits was 100 nm. Based on these assumptions and using the angle of the first peak obtained in item (a), find the wavelength λ of the incident wave.
(c) If each C_{60} molecule has a speed of 220 m/s, calculate the absolute value of the linear momentum of the molecule. The molar mass of carbon is 12 g/mol.
(d) Using the result of item (c), calculate the de Broglie wavelength of a C_{60} molecule in the beam.

Q5. A cylindrical region of cross section $A = 5 \times 10^{-3}$ m^2 and whose length is initially $L = 25$ cm is occupied by 5×10^{-2} mol of an ideal monatomic gas ($c_V = 3R/2 = 12.5\ \mathrm{J\,mol^{-1}\,K^{-1}}$). One of the bases of the cylinder is a frictionless movable piston which is attached to one end of a spring, initially in its relaxed state. The other end of the spring is attached to a fixed wall, as shown in the figure. The spring's elastic constant is 400 N/cm. The gas is in equilibrium at a temperature of 300 K and an external pressure of 1 atm = 1×10^5 N/m^2. An amount of heat Q is then added quasi-statically to the gas, which expands by pushing the piston and compressing the spring by 2 cm.



(a) Find the final pressure in the gas.
(b) Find the final temperature of the gas.
(c) Find the work done by the gas.
(d) Find the amount of heat Q added to the gas.

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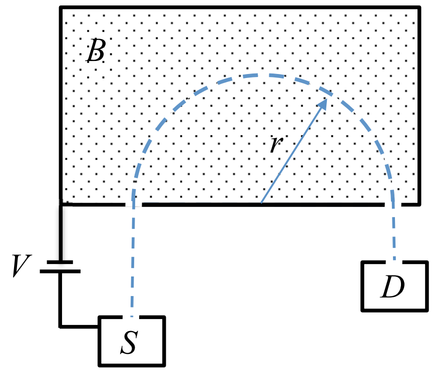
Part 2

This test contains questions on electromagnetism, quantum mechanics, modern physics, and statistical mechanics. All questions have the same weight.

Have a good exam!

Q6. A simple mass spectrometer can be built as shown schematically in the figure. A beam of positive ions of mass m and charge q (dashed line) leaves a source S, is accelerated by the electric potential difference V and goes into a chamber through an entrance slit. Inside the chamber there is a uniform magnetic field B coming out of the paper plane in the normal direction. The beam then moves along a semicircular path of radius r, comes out through an exit slit and is detected by a detector D if the ion mass is compatible with the path. Thus, it is possible to select the ions according to their mass, by adjusting the value of B. The entrance and exit slits are identical. The whole system is in vacuum. Neglect the gravitational interaction of the beam.



(a) Find the ions' speed as they come out of the exit slit in terms of V, m and q.
(b) Find m as a function of V, B, r and q.
(c) The mass resolution of the apparatus is limited by size of the exit slit, which determines the error in the radius r. Consider a situation in which $r = 10$ cm, $V = 4.0 \times 10^3$ V, $B = 1.00$ T and each ion has one electron less than the corresponding neutral atom. If the slits' size is 100 μm, what is the mass resolution of the apparatus?
(d) Is it possible to use the apparatus to distinguish between the two carbon isotopes ^{12}C and ^{14}C? Justify your answer.

Q7. Consider the propagation of electromagnetic waves in a linear, homogeneous and isotropic medium with electrical conductivity σ, electrical permittivity ϵ and magnetic permeability μ, in the absence of sources of free electric charges (ρ_F=0).

(a) Write Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in the medium, in terms of σ, ϵ and μ.
(b) Find the differential equation satisfied by the electric field $\mathbf{E}$ alone.
(c) Consider a plane wave solution $\mathbf{E}\,(x,t) = \mathbf{E}_0 e^{i(kx-\omega t)}$ and obtain the relation between k and ω in terms of σ, ϵ and μ.
(d) Regarding the result of item (c), interpret physically the difference between the cases $\sigma \neq 0$ and $\sigma = 0$.

Q8. Consider a quantum system whose Hilbert space is spanned by an **orthonormal basis** of 3 states, $|1\rangle$, $|2\rangle$ and $|3\rangle$. The system's Hamiltonian can be represented in this basis through the square matrix

$$H_0 = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}.$$

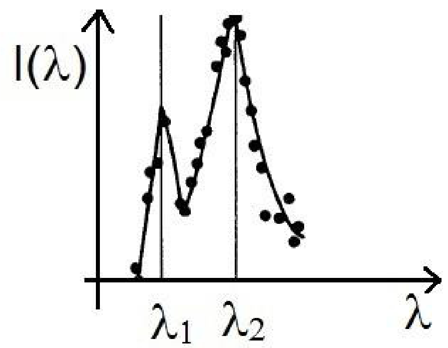
(a) When the system is prepared through a certain experimental protocol P_1, measurements of its energy return the value E_0 with probability 1. What is the state prepared by P_1?

(b) When the system is prepared through a different experimental protocol P_2, measurements of its energy return the value E also with probability 1. What are all the possible states prepared by P_2?
(c) At the instant $t = 0$, an external perturbation is turned on and the Hamiltonian becomes, in the same basis,

$$H = H_0 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & W \\ 0 & W & 0 \end{pmatrix}.$$

What are the energy eigenvalues and eigenvectors in the presence of the external perturbation?
(d) Before $t = 0$, the system had been prepared in the state $|2\rangle$. What is the state of the system after turning on the external perturbation, i.e., for $t > 0$?

Q9. The figure below shows the intensity of scattered X-rays $I(\lambda)$ with wavelength λ by a graphite target in the famous 1923 experiment by Compton. The X-rays are detected at an angle θ fixed with respect to the direction of incidence on the target. Some of the photons experience elastic scattering (without losing energy) and others undergo Compton scattering. As a result, it can be seen that $I(\lambda)$ shows two peaks at wavelengths λ_1 and $\lambda_2 > \lambda_1$.



(a) What is the wavelength of the incident X-rays and of the X-rays that undergo Compton scattering? Justify your answer.

Consider now an event of Compton scattering in which the energy of the incident photon is 23 keV and the scattering angle is $\theta = 60^\circ$.

(b) Find the wavelength of the incident photon.
(c) Find the wavelength of the scattered photon.
(d) Find the kinetic energy of the electron after the scattering event.

Q10. Consider a gas of $N \gg 1$ **classical** point particles of mass m in a cubic box ($0 \le x \le L$, $0 \le y \le L$ and $0 \le z \le L$) in the vicinity of Earth's surface. The particles are subject to the gravitational potential $V(z) = mgz$, where g is the acceleration of gravity and z is the particle's height measured from the surface of the Earth. The gas is in equilibrium at a temperature T.

(a) Write the Hamiltonian of a single particle in the box as a function of its coordinates and the components of its linear momentum p_x, p_y and p_z.
(b) Find the partition function of the system of N **classical** particles.
(c) Assume that $mgL \ll k_BT$ and find the partition function in this regime. You can use the following approximation: $e^{-x} \approx 1 - x$, if $x \ll 1$.
(d) Obtain the pressure in the gas in the regime of item (c) as a function of N, T and $V = L^3$.