EUF

Joint Entrance Examination for Postgraduate Courses in Physics

For the first semester of 2017

October 04, 2016

Part 1

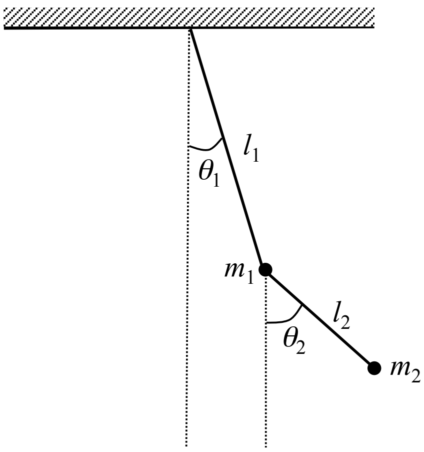
Instructions

- **Do not write your name on the test.**
  It should be identified only by your candidate number.
- This test contains questions on:
  classical mechanics, quantum mechanics, modern physics, and thermodynamics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
- The use of calculators or other electronic instruments is not permitted during the exam.
- **Answer each question on the corresponding sheet of the answer booklet.**
  The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. Remember to write the number of the question (Qx) and your candidate number on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked **scratch** at the end of the answer booklet. Do not detach them. The scratch sheets will be discarded and solutions written on them will be ignored.
- **Do not write anything on the list of constants and formulae.**
  Return both this test and the list of constants and formulae at the end of the test. The latter will be used in the test tomorrow.

Have a good exam!

Q1. A body of mass m falls from rest along a straight line inside a fluid. The acceleration of gravity $\vec{g}$ can be considered as constant. The body is also subject to a drag force proportional to the velocity: $\vec{F}_r = -km\vec{v}$, where k is a constant. Neglect the buoyant force of the fluid.
(a) Obtain the body's speed as a function of time.
(b) What is the body's terminal speed (speed in the limit $t \to \infty$)?
(c) Find $z(t)$, the body's position as a function of time (consider $z(0) = 0$).
(d) Find $z(v)$, the body's position as a function of its speed.

Q2. The planar double pendulum consists of two particles of masses m_1 and m_2 and two massless rigid rods of lengths l_1 and l_2, which swing, under the action of gravity $\vec{g}$, on a fixed vertical plane, as depicted in the figure below. Considering $\vec{g}$ as constant and adopting as generalized coordinates the angles θ_1 and θ_2 in the figure, obtain:



(a) The kinetic energy of the system.
(b) The potential energy of the system.
(c) The Lagrangian of the system.
(d) The equations of motion for θ_1 and θ_2.

Q3. Consider the quantum non-relativistic dynamics of a particle of mass m in a three-dimensional isotropic harmonic potential of angular frequency ω given by

$$V(x,y,z) = \frac{m\omega^2}{2}(x^2 + y^2 + z^2).$$

(a) Write the eigenstates $|n_1,n_2,n_3\rangle$ of the total Hamiltonian $\hat{H}$ in terms of the eigenstates of one-dimensional harmonic oscillators $|n_i\rangle$ $(i = 1,2,3)$ and also the eigenenergies of $\hat{H}$.
(b) One of the eigenenergies of the system is $\frac{7}{2}\hbar\omega$. What is its degeneracy?
(c) The observable $\hat{H}$ is measured when the system is in the following state (assume the eigenstates $|n_1,n_2,n_3\rangle$ are normalized)

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0,0,1\rangle + \frac{1}{2}|0,1,0\rangle + \frac{1}{2}|0,1,1\rangle.$$

What are the possible outcomes and what are their probabilities?
(d) Suppose the outcome of the measurement in item (c) was $\frac{5}{2}\hbar\omega$. Consider $t = 0$ the instant immediately after the measurement. Find the state of the system $|\psi(t)\rangle$ for $t > 0$.

Q4. The energy density $u(T)$ of electromagnetic radiation in equilibrium at temperature T can be expressed (starting from thermodynamic arguments) as

$$u(T) = \int_0^\infty \nu^3 f\left(\frac{\nu}{T}\right)\, d\nu, \tag{1}$$

where ν is the radiation frequency.

(a) **Using only Eq.** (1), find $u(T)$ up to a constant factor (that is independent of T). What is the dimension of this factor?
(b) In 1900, Planck discovered that

$$\nu^3 f\left(\frac{\nu}{T}\right) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\mathrm{e}^{(h\nu/k_B T)} - 1},$$

where h is Planck's constant (which relates the quantum of energy to the frequency), c is the speed of light in vacuum and k_B is Boltzmann's constant.

(i) Discuss, without deriving it, the physical significance of the factor $\frac{8\pi\nu^2}{c^3}d\nu$.
(ii) Find the behavior of the energy distribution in the limit in which the photon energy is much smaller than the thermal energy $k_B T$.
(iii) What is the meaning of the result of item (ii) in the context of classical physics?

(c) The constants c, $\hbar = h/(2\pi)$ and the gravitational constant G may be used to define absolute units of time (t_P), distance (l_P) and mass (m_P). Find these quantities in terms of products of powers of $\hbar$, c and G. Find also *Planck's temperature* T_P. Estimate the order of magnitude of t_P, l_P, m_P and T_P in the international system of units.

Q5. A student wants to measure the specific heat c_x of an unknown substance x. For that purpose, he can use a calorimeter, a device that ideally does not exchange heat with the environment. The calorimeter's heat capacity K is known. The calorimeter is initially at room temperature T_{amb}. The experimental protocol consists of putting a known mass of water m_{H_2O} at room temperature T_{amb} in the calorimeter, adding a known mass of substance x, m_x, initially at a temperature $T_x > T_{amb}$, and measuring the final equilibrium temperature T_{eq}. If necessary, use $\sqrt{2} \cong 1{,}4$ and $\sqrt{26} \cong 5{,}1$.
(a) Write the equation required for determining c_x from the data provided.
(b) Considering $K = 30.0$ cal/°C, the specific heat of water $c_{H_2O} = 1.0$ cal/(g °C) and using $m_{H_2O} = 50.0$ g, $m_x = 200$ g, $T_x = 37.8$ °C, $T_{amb} = 25.0 \pm 0.1$ °C, the student found the final equilibrium temperature of the system to be $T_{eq} = 27.8 \pm 0.1$ °C. Find c_x. Express your result with the uncertainty in the value of c_x.

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For the first semester of 2017

October 05, 2016

Part 2

Instructions

- **Do not write your name on the test.**
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- This test contains questions on:
  electromagnetism, quantum mechanics, modern physics, and statistical mechanics.
  All questions have the same weight.
- The duration of this test is **4 hours**.
  Candidates must remain in the exam room for a minimum of 90 minutes.
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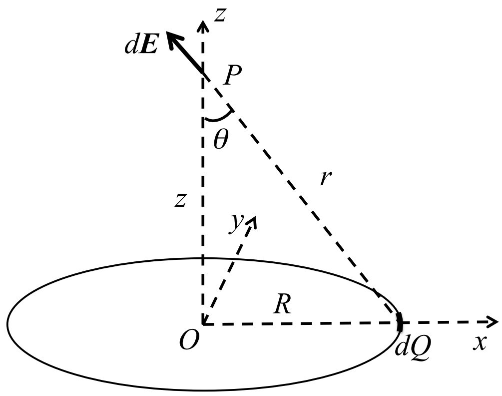
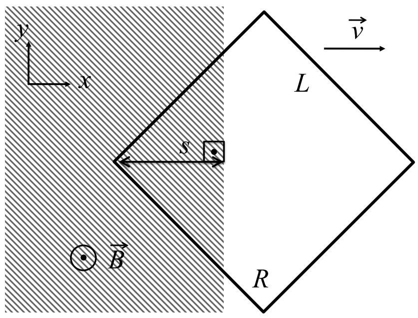
Q6. A thin ring of radius R and total charge $Q > 0$ uniformly distributed throughout its perimeter is fixed on the xy plane of a coordinate system with its center at the origin O. Let P be a point with coordinates (0,0,z) (see the figure below).



(a) Summing up the contributions of all charge elements of the ring, find the electric field vector $\vec{E}(z)$ (magnitude and direction) at the point P.
(b) Proceed analogously to item (a) and find the electric potential $V(z)$ at the point P.
(c) A point particle of charge $-q < 0$ and mass m starts at rest from a point with coordinates (0,0,z_0), very far from the origin (i. e., $z_0 \gg R$) and moves along the z axis. What is its speed when it reaches the ring center? Neglect the effects of electromagnetic radiation generated by the particle in its journey towards the ring center.

Q7. A rigid square loop of wire of side L has total electric resistance R. The loop is on the xy plane of a coordinate system and moves with velocity $\vec{v}$ away from a region where there is a uniform magnetic field $\vec{B}$ (shaded area of the figure below) pointing out of the page (in the positive z direction). Consider the instant when the left-hand vertex of the loop is a distance s inside the shaded region ($0 < s < \sqrt{2}L/2$).



(a) Find the magnetic flux through the loop as a function of s.
(b) Find the magnitude and the circulation direction of the electric current induced in the loop.
(c) Find the total magnetic force (magnitude and direction) on the loop when the induced current is flowing in it. What additional force other than the magnetic force must be applied to the loop so that it moves with constant velocity under the sole action of these two forces?

Q8. Consider the quantum non-relativistic dynamics of an electron (mass m and charge $-e$) moving along the x axis in a one-dimensional harmonic oscillator potential with angular frequency ω. The electron is also subject to an electric field $\vec{E} = E\hat{x}$ along the same axis.

(a) Write the total Hamiltonian of the system.
(b) Let $|n\rangle$ ($n = 0,1,2,\ldots$) be the eigenstates of the harmonic oscillator. Let us now consider the effect of the electric field as a small perturbation that changes only slightly the eigenenergies and eigenstates of the harmonic oscillator. Let $\hat{V}_E$ be the term of the Hamiltonian due to the electric field. In this case, the correction to the energy of the first excited state to *linear order in E* is given by the average value of $\hat{V}_E$ in $|1\rangle$. Calculate this correction.
(c) The correction to the first excited eigenstate to linear order in E is given by

$$|\delta\psi_1^{(1)}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{eE}{\hbar\omega}\left(|0\rangle - \sqrt{2}\,|2\rangle\right). \qquad (2)$$

The correction to the energy of the first excited state to *quadratic order in E* is given by the matrix element of $\hat{V}_E$ between $|1\rangle$ and the state in Eq. (2). Calculate this correction.
(d) Calculate the exact eigenenergies of the system (harmonic oscillator plus electric field) by means of a coordinate transformation. Compare the exact result with the perturbative calculation of items (b) and (c). Do they agree? Why?

Q9. An extra-solar planet is cT light-years distant from the Earth (c is the speed of light and T is the time in years it takes light to reach the planet from the Earth). An expedition is planned to send astronauts to the planet in such a way that they should age $3T/4$ years in the outward journey. The journey is made almost completely at constant speed. Neglect, therefore, the small portions with accelerated motion.

(a) What should the astronauts' constant speed relative to the Earth be on the outward journey?
(b) According to the astronauts, what will the traveled distance be on the outward journey?
(c) Each year (according to the spaceship's clock) the astronauts send a light pulse back to the Earth. What is the period of arrival of the pulses at the Earth?
(d) In the middle of the outward journey, a couple of astronauts decides to the return to Earth in a space module. According to the astronauts that remain in the original spaceship, the module travels at speed $5c/6$. Find the *total time* (measured from the Earth) the couple of astronauts will have stayed away from our planet.

Q10. A system of N one-dimensional, localized and independent quantum oscillators is in equilibrium with a thermal reservoir at temperature T. The energies of each oscillator are given by

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \qquad\qquad n = 1,3,5,7,\ldots$$

Note that the possible values of n are the odd positive integers only.

(a) Find an expression for the internal energy per oscillator u as a function of temperature T. What is the expression for u in the classical limit ($\hbar\omega_0 \ll k_BT$)? Sketch a plot of u versus T.
(b) Find an expression for the entropy per oscillator s as a function of temperature T. What is the expression for s in the classical limit? Sketch a plot of s versus T.