Joint Entrance Examination for Postgraduate Courses in Physics

EUF

First Semester 2013

Part 1 — 16 October 2012

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST.** It should be identified **only by your candidate number (EUFxxx)**.
- This test is the **first part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Electromagnetism, Modern Physics, Thermodynamics and Statistical Mechanics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted in the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for marking. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, ou Q2, or . . .) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will not be marked.**
  Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked SCRATCH at the end of the answer booklet. **DO NOT DETACH THEM**. The scratch sheets will be discarded and **solutions written on them will be ignored**.
- Do **NOT** write **ANYTHING** on the list of Constants and Formulae provided; **RETURN IT** at the end of the test, as it will be used in the test tomorrow.

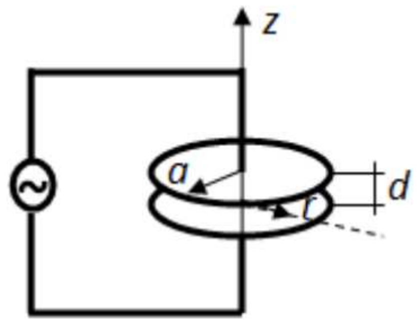
Have a good exam!

Q1. Consider a solid sphere, uniformly charged, with charge Q and radius R.

(a) Determine the electric field vector $\vec{E}$ at a point a distance r from the center of the sphere, for $r > R$ and for $r \leq R$.
(b) Obtain the force $d\vec{F}$ acting on a volume element dV of the sphere, located at position $\vec{r}$.
(c) Determine now, by direct integration, the total force $\vec{F}$ acting on the upper hemisphere of the sphere.

Q2. A parallel plate capacitor is formed by two circular discs of radius a, separated by a distance $d \ll a$, in vacuum. The plane discs are connected to an alternating current generator of frequency ω, which produces a uniform charge on the plate of the capacitor, given by $q(t) = q_0 \sin(\omega t)$. Edge effects are neglected. Assuming low frequencies, so that $(\omega a/c) \ll 1$ (where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light), the electric field $\vec{E}$ between the plates can be considered uniform. Consider a system of cylindrical coordinates (r,θ,z) with the z axis passing through the center of the plates, as shown in the figure.

(a) Find the expression for the electric field $\vec{E}$ between the plates.
(b) Calculate the magnetic field $\vec{B}$ as a function of the radius r in the region between the capacitor plates.
(c) Compute the Poynting vector $\vec{S} = (\vec{E} \times \vec{B})/\mu_0$.
(d) Using the approximation of low frequencies, show that the conservation of energy is satisfied, as expressed by the condition $\nabla \cdot \vec{S} + \partial u/\partial t = 0$, where $u = \frac{1}{2}(\epsilon_0 \vec{E}^2 + \vec{B}^2/\mu_0)$ is the energy density of the electromagnetic field.



Q3. A particle of mass m contained in a one-dimensional potential well has a wave function given by:

$$\psi(x) = \begin{cases} 0 & \text{for } x < -L/2 \\ A\cos\dfrac{3\pi x}{L} & \text{for } -L/2 < x < L/2 \\ 0 & \text{for } x > L/2 \end{cases}$$

(a) Calculate the normalization constant A.
(b) Calculate the probability of finding a particle in the range from $-L/4 < x < L/4$.
(c) Using the solution of the time independent Schrödinger equation for this particle in that potential well, find the energy corresponding to the wave function in terms of m, L, and h.
(d) Calculate the wavelength of the photon emitted by this particle in the transition to the ground state, in terms of m, L, and h.

Q4. The decay of the muon obeys the following differential equation

$$\frac{dN(t)}{dt} = -RN(t)$$

where $N(t)$ is the number of muons present at time t e $dN(t)/dt$ represents the rate of decay of the muons at the same instant of time t. The proportionality constant R is called the decay constant. The mean lifetime of the muon is $\overline{t} = 2\mu$s, i.e. in this time interval, $N(\overline{t})/N(0) = (1/e) \approx (1/2.73)$. If the speed of muons in the direction of the Earth's surface is equal to $0.998c$:

(a) In the muon reference frame, what is the value of R for the muon decay?
(b) Without considering relativistic corrections, estimate how many muons would be detected at sea level, if 10^8 muons were detected at an altitude of 9 km.
(c) Consider now the relativistic correction and repeat the estimate of item (b).

Q5. An ideal gas of N monatomic molecules of mass m is in thermal equilibrium at an absolute temperature T. The gas is contained in a cubic box of edge L, with two sides parallel to the Earth's surface. Consider the effect of the gravitational field on the molecules. The acceleration of gravity is g. Determine:

(a) the partition function of a gas molecule;
(b) the average kinetic energy of a gas molecule;
(c) the average potential energy of a gas molecule;
(d) the average potential energy of a molecule of gas for $mgL/k_BT \ll 1$. Do the calculation to second order of mgL/k_BT.

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EUF

First Semester 2013

Part 2 — 17 October 2012

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST.** It should be identified **only by your candidate number (EUFxxx)**.
- This test is the **second part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Classical Mechanics, Quantum Mechanics, Thermodynamics and Statistical Mechanics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted in the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for marking. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, ou Q2, or ...) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will not be marked.**

  Use separate extra sheets for each question. Do not detach the extra sheets.
- If you need spare paper for rough notes or calculations, use the sheets marked SCRATCH at the end of the answer booklet. **DO NOT DETACH THEM**. The scratch sheets will be discarded and **solutions written on them will be ignored**.
- It is **NOT** necessary to return the list of Constant and Formulae.

Have a good exam!

Q6. An acrobat of mass m is initially at rest at the end of a horizontal, homogeneous bar, of mass $M = 3m$ and length D. The bar rotates around a vertical axis passing through its center. The acrobat then begins to walk along the bar, toward the axis of rotation, with constant speed. Consider the initial period of rotation of the system to be equal to T_0.

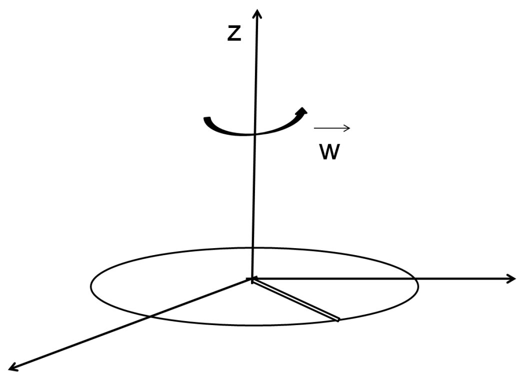
(a) Determine the torque of forces acting on the acrobat with respect to the center of the bar.
(b) Determine the angular momentum of the system when the acrobat reaches the center of the bar. Determine the period of rotation of the system in this situation.
(c) Determine the energy in the initial and final positions of the system. In this movement, the energy of the system changed?

Consider the acrobat as a point mass.

Given: $I_{CM}(bar) = \dfrac{1}{12} M D^2$

Q7. A particle of mass m lies within a hollow cylinder, which is smooth, narrow, and long, lying in a horizontal plane which rotates with constant angular velocity ω. The fixed axis of rotation passes through one end of the cylinder, as shown in the figure.

(a) Write the Lagrangian for the particle.
(b) Obtain Lagrange's equations for the motion of the particle.
(c) Determine the particle's motion, assuming that, initially, it is released from the center of rotation with velocity $\vec{v}_0$.
(d) Obtain the Hamiltonian function (H) for the motion of this particle and Hamilton's equations of motion.
(e) Among the physical quantities H and E (energy), which are conserved? Explain your answer.



Q8. Consider an harmonic oscillator (in one dimension, x) of mass m and frequency ω. At the instant $t = 0$, the oscillator state is $|\psi(0)\rangle = \sum_n c_n|\phi_n\rangle$ where the $|\phi_n\rangle$ are the stationary states,i.e., $H|\phi_n\rangle = E_n|\phi_n\rangle$, with H the Hamiltonian and $E_n = (n+1/2)\hbar\omega$ with $n = 0,1,2,\ldots$

(a) Considering the states $|\phi_n\rangle$ to be normalized, determine the condition which the coefficients c_n must satisfy in order that the state $|\psi(0)\rangle$ be also normalized. Calculate the probability $\mathcal{P}$ that a measurement of the energy of the oscillator made at a later instant, t, will give a result larger than $2\hbar\omega$.

(b) If only c_0 and c_1 differ from zero, give an expression for the expectation value of the energy, $\langle H\rangle$, in terms of c_0 and c_1. With the condition $\langle H\rangle = \hbar\omega$, compute $|c_0|^2$ and $|c_1|^2$.

(c) The state vector $|\psi(0)\rangle$ is defined except for a global phase factor, allowing a choice of c_0 real and positive. With this choice, writing $c_1 = |c_1|e^{i\theta_1}$ and keeping the condition $\langle H\rangle = \hbar\omega$, determine θ_1 such that $\langle X\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$.

Observation: Remember that the effect of the position operator X on the stationary states of the harmonic oscillator is

$$X|\phi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,|\phi_{n+1}\rangle + \sqrt{n}\,|\phi_{n-1}\rangle\right]$$

with the understanding that the second term above is zero for $n = 0$.

(d) With $|\psi(0)\rangle$ determined as in the previous item, write $|\psi(t)\rangle$ for $t > 0$ and calculate $\langle X\rangle(t)$.

Q9. Let $\vec{L}$,$\vec{R}$ and $\vec{P}$ be operators of angular momentum, position and linear momentum, respectively.

(a) Using the commutation relations $[L_i,L_j] = i\hbar\sum_k \epsilon_{ijk}L_k$, show that

$$\vec{L}\times\vec{L} = i\hbar\vec{L}$$

(b) With the definition $\vec{L} = \vec{R}\times\vec{P}$ and using $[R_i,P_j] = i\hbar\delta_{ij}$, show that

$$[L_i,R_j] = i\hbar\sum_k \epsilon_{ijk}R_k$$

(c) Assuming that the operators $\vec{R}$ and $\vec{P}$ are Hermitian, i.e. $R_i^\dagger = R_i$ e $P_i^\dagger = P_i$, show that the components of the angular momentum operator $\vec{L} = \vec{R}\times\vec{P}$ are also Hermitian operators.

Observation: In the above expressions, ϵ_{ijk} is the completely antisymmetric tensor, i.e.,

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two or all three indices are equal;} \\ +1 & \text{if } ijk \text{ is 123, 231 or 312;} \\ -1 & \text{in all other cases.} \end{cases}$$

If needed, use the identity $\sum_{i,j}\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$.

Q10. The radiation in a resonant cavity can be viewed as a gas of photons whose pressure on the walls of a cavity volume V is given by

$$P = \frac{aT^4}{3},$$

where a is a constant. The internal energy of the gas is given by $U = aT^4V$. Consider that, initially, the temperature of the cavity is T_0 and its volume is V_0.

(a) Determine the work done in an isothermal process in which the cavity volume is doubled. Provide the answer in terms of T_0, V_0 and the constant a.

(b) Determine the heat supplied in an isothermal process in which the cavity volume is doubled. Provide the answer in terms of T_0, V_0 and the constant a.

(c) Using the relation:

$$đQ = \left(\frac{\partial U}{\partial T}\right)_V dT + \left[\left(\frac{\partial U}{\partial V}\right)_T + P\right] dV,$$

determine the equation that describes an adiabatic process in terms of variables V and T.

(d) Determine the work done in an adiabatic process in which the cavity volume is doubled. Express the result in terms of T_0, V_0 and the constant a.