Joint Entrance Examination for Postgraduate Courses in Physics

EUF

For the first semester 2014

Part 1 — 15 October 2013

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST.** It should be identified **only by your candidate number (EUFxxx)**.
- This test is the **first part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Electromagnetism, Modern Physics, and Thermodynamics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted during the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, or Q2, or . . .) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.**
- If you need spare paper for rough notes or calculations, use the sheets marked SCRATCH at the end of the answer booklet. **DO NOT DETACH THEM**. The scratch sheets will be discarded and **solutions written on them will be ignored**.
- Do **NOT** write **ANYTHING** on the list of Constants and Formulae; **RETURN IT** at the end of the test, as it will be used in the test tomorrow.

Have a good exam!

Q1. Consider an arbitrary macroscopic conductor whose surface is closed and smooth. Using Gauss's law and considering that the curl of the electrostatic field is zero:

(a) Calculate the electric field inside the conductor;

(b) Determine the normal component of the electric field at the external surface of the conductor in terms of the surface charge density;

(c) Determine the tangential component of the electric field at the surface of the conductor.

Q2. Consider a set of solutions of electromagnetic plane waves in vacuum, whose fields (electric and magnetic) are described by the real part of the functions: $\vec{u}(\vec{x},t) = \vec{A}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, where $\vec{k}$ is the wave vector, which determines the direction of wave propagation, and ω is the angular frequency, which is related to the wave vector through $\omega = v|\vec{k}|$, where $v = 1/\sqrt{\epsilon\mu}$ is the velocity of the wave propagation.

(a) Show that the divergence of $\vec{u}(\vec{x},t)$ satisfies: $\vec{\nabla}\cdot\vec{u} = i\vec{k}\cdot\vec{u}$;

(b) Show that the curl of $\vec{u}(\vec{x},t)$ satisfies: $\vec{\nabla}\times\vec{u} = i\vec{k}\times\vec{u}$;

(c) Show that the waves are transversal and that the vectors $\vec{E}$, $\vec{B}$ and $\vec{k}$ are mutually perpendicular.

Q3. In 1913, Niels Bohr introduced his atomic model by adapting the Rutherford model to the quantization ideas that were arising at that the time. In honor of this event, address the following items in terms of fundamental quantities.

(a) Use the quantization rule for the angular momentum, $L = \hbar n$, of an electron around an atom of atomic number Z to find an expression for the radii of the allowed orbits.

(b) According to the Bohr model, the transition between different orbits is accompanied by the emission/absorption of a photon. Determine the energy of the photon emitted as the result of the transition between the first excited state and the ground state of a hydrogen atom.

(c) Consider an electron trapped in a one-dimensional infinite square well of width a. Determine an expression for the electronic energy levels using the Bohr-Sommerfeld quantization rule $\oint p dx = hn$.

(d) Find the width a of the well in terms of the Bohr radius, so that the energy of a photon emitted on account of the transition between the first excited state and the ground state is equal to the energy obtained in part (b).

Q4. γ-rays produced by pair annihilation show a considerable Compton scattering. Consider that a photon with energy m_0c^2 is produced by the annihilation of an electron and a positron, where m_0 is the rest mass of the electron and c is the light velocity. Suppose that the photon is scattered by a free electron and the scattering angle is θ.

(a) Find the maximum possible kinetic energy of the back scattered electron.

(b) If the scattering angle is $\theta = 120°$, determine the photon energy and the kinetic energy of the electron after the scattering.

(c) If $\theta = 120°$, what is the direction of motion of the electron after the scattering, relative to the direction of the incident photon?

Q5. One mole of a simple ideal gas is enclosed in a vessel with initial volume v_0 and initial pressure p_0. The ideal gas is described by the equations $pv = RT$ and $u = cRT$, where p is the pressure, v is the molar volume, T is the absolute temperature, u is the molar energy, R and c are constant. The gas expands from the initial state to a state corresponding to a final volume $2v_0$ through a given process. Determine the work W performed by the gas and the heat Q received by the gas for each one of the processes listed below. The final answers should be given only in terms of (v_0, p_0) and c.

(a) Free expansion. Determine also the variation in temperature ΔT.

(b) Quasi-static isentropic expansion. Also, find the final pressure p_f, using the fact that, in this process for an ideal gas, $pv^\gamma =$ constant, where $\gamma = (c+1)/c$.

(c) Quasi-static isobaric expansion.

(d) Quasi-static isothermal expansion.

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Part 2 — 16 October 2013

Instructions:

- **DO NOT WRITE YOUR NAME ON THE TEST.** It should be identified **only by your candidate number (EUFxxx)**.
- This test is the **first part** of the joint entrance exam for Postgraduate Physics.

  It contains questions on: Classical Mechanics, Quantum Mechanics, and Statistical Mechanics. All questions have the same weight.
- The duration of this test is **4 hours**. Candidates must remain in the exam room for a minimum of **90 minutes**.
- The use of **calculators** or other electronic instruments is **NOT** permitted during the exam.
- **ANSWER EACH QUESTION ON THE CORRESPONDING PAGE OF THE ANSWER BOOKLET.** The sheets with answers will be reorganized for correction. If you need more answer space, use the extra sheets in the answer booklet. **Remember to write the number of the question (Q1, or Q2, or . . .) and your candidate number (EUFxxx) on each extra sheet. Extra sheets without this information will be discarded. Use separate extra sheets for each question. Do not detach the extra sheets.**
- If you need spare paper for rough notes or calculations, use the sheets marked SCRATCH at the end of the answer booklet. **DO NOT DETACH THEM**. The scratch sheets will be discarded and **solutions written on them will be ignored**.
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Have a good exam!

Q6. Two particles, A and B, of masses m and M ($m \neq M$), respectively, are connected to the ends of an inextensible thread of length ℓ and of negligible mass which passes through a hole in a horizontal table, as shown in the figure below. The particle A moves without friction over the table while the other moves vertically under the combined action of gravity, acceleration $\vec{g}$, and the traction of the thread (disregard the friction between the thread and the hole).

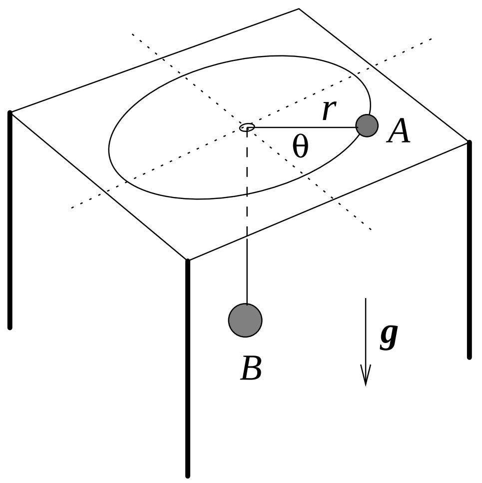
(a) Assuming that the initial position of A is $r = r_0$, which initial velocity should be given to the particle so that B remains at rest below the surface of the table?

(b) Obtain the equations of motion, assuming that the Lagrangian that describes an arbitrary movement of this system is given by

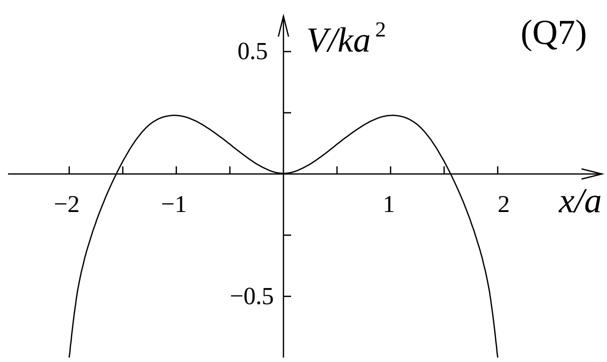
$$\mathcal{L} = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mg(r-\ell).$$

(c) Determine the conserved quantities and give the meaning of each of them

(d) If B is slightly and vertically displaced from its position, there will occur small oscillations in the system. Obtain the period of these oscillations in terms of the equilibrium radius r_{eq} and the other quantities that characterize the system (m, M and g).



(Q6)



(Q7)

Q7. A particle of mass m is subject to the one-dimensional potential

$$V(x) = \frac{1}{2}kx^2 - \frac{k}{4a^2}x^4,$$

shown in the figure above, where k and a are positive constants.

(a) Determine the force $F(x)$, obtain the equilibrium points, and describe their nature.

(b) Calculate the period of small oscillations occurring around the point of stable equilibrium.

(c) Assume that the particle is at rest at the point $x = 0$ and receives, instantaneously, an impulse that gives the particle a speed v in the direction of positive x. Address the following cases: $0 < v \leq a\sqrt{k/2m}$ and $v > a\sqrt{k/2m}$.

(d) Sketch the phase diagram of the system ($\dot{x}$ versus x for a constant energy) for the several types of motion. Indicate clearly the curve corresponding to the transition from the periodic to the non-periodic motion as well as the corresponding values of the energy.

Q8. Consider the problem of a particle of mass m whose motion along the x-axis is restricted to the interval $0 \leq x \leq a$, that is, it is confined in a box with walls placed at positions $x = 0$ and $x = a$.

(a) Determine the wave function and the energy of the ground state.

(b) Suppose that the particle is described by the following wave function:

$$\psi(x) = A\left[\sin\frac{\pi x}{a} - 3i\sin\frac{2\pi x}{a}\right],$$

where A is a normalization constant. Determine A and calculate the probability of finding the result $2\pi^2\hbar^2/ma^2$ in the measurement of the energy.

(c) Suppose now that the particle is in the ground state. What is the probability distribution of the linear momentum of the particle in this state?

(d) Considering, again, that the particle is in the ground state, suppose that the walls are removed instantaneously, so that the particle becomes free ($\hat{\mathcal{H}} = \hat{p}^2/2m$). What is the energy of this free particle?

Q9. Consider a particle of spin 1/2, whose magnetic moment is $\vec{M} = \gamma\vec{S}$, where γ is a constant. It is possible to describe the quantum state of this particle using the space spanned by the eigenvectors $|+\rangle$ and $|-\rangle$ of the operator $\hat{S}_z$, which measures the projection of the spin in the z direction,

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle, \qquad \hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle.$$

The particle is subject to a uniform magnetic field $\vec{B} = B\hat{y}$, parallel to the y axis, so that the Hamiltonian is given by

$$\hat{\mathcal{H}} = -\vec{M}\cdot\vec{B} = -\gamma B\hat{S}_y.$$

Initially, at time $t = 0$, the particle is in the state $|\psi(0)\rangle = |+\rangle$.

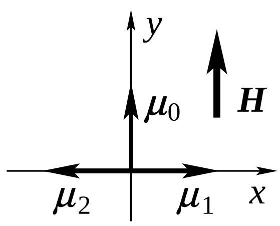
(a) What are the possible values of the projection of the spin on the y axis?

(b) Find the eigenvectors of $\hat{S}_y$.

(c) Obtain $|\psi(t)\rangle$ for $t > 0$ in terms of $|+\rangle$ and $|-\rangle$ defined above.

(d) Obtain the average values of the observables S_x, S_y e S_z as functions of time.

Q10. A certain magnetic material comprises N noninteracting magnetic atoms, whose magnetic moments $\boldsymbol{\mu}$ can be in three possible directions, according to the figure: $\boldsymbol{\mu}_0 = \mu\hat{\boldsymbol{y}}$, $\boldsymbol{\mu}_1 = \mu\hat{\boldsymbol{x}}$ and $\boldsymbol{\mu}_2 = -\mu\hat{\boldsymbol{x}}$. The system is in thermal equilibrium at temperature T in the presence of a magnetic field parallel to the y direction, $\boldsymbol{H} = H\hat{\boldsymbol{y}}$, so that the levels of energy of an atom are $\epsilon_0 = -\mu H$, $\epsilon_1 = 0$ and $\epsilon_2 = 0$.



(a) Obtain the canonical partition function z of an atom , the canonical partition function Z of the system and the Helmholtz free energy f per atom.

(b) Determine the average energy $u = \langle\epsilon_n\rangle$ and the entropy s per atom.

(c) Obtain the magnetization per atom $\boldsymbol{m} = m_x\hat{\boldsymbol{x}} + m_y\hat{\boldsymbol{y}} = \langle\boldsymbol{\mu}_n\rangle$.

(d) Verify that the isothermal susceptibility $\chi_T = (\partial m_y/\partial H)_T$ at zero field obeys the Curie law, $\chi_T \propto 1/T$.