JIRSS (2016)
Vol. 15, No. 1, pp 59-70
DOI:10.7508/jirss.2016.01.004

Distribution of Order Statistics for Exchangeable Random Variables

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Abstract. Let $T_1, \ldots, T_n$ be exchangeable random variables and suppose that $T_{i:n}$ represents the ith order statistic among T_i's, $i = 1, \ldots, n$. In this paper some expressions for the joint distribution of $(T_{1:n}, \ldots, T_{n:n})$, marginal distribution of $T_{i:n}$ and the joint distribution of $(T_{r:n}, T_{k:n})$, $1 \leq r \leq k \leq n$ in terms of the joint distribution (or joint reliability) function of T_i's are provided. Using these and when $\{T_1, \ldots, T_n\}$ is a sequence of lifetimes, some expressions for the mean residual life functions of a $n-k+1$-out-of-n system, $H_n^k(t) = E(T_{k:n} - t|T_{1:n} > t)$ and $M_n^{r,k}(t) = E(T_{k:n} - t|T_{r:n} > t)$, $1\leq r \leq k \leq n$ in terms of the joint survival function of T_i's are given. Also, some examples are provided.

Keywords. DFR, Exchangeable random variables, IFR, Mean residual life function, Order statistics, $(n-k+1)$-out-of-n system.

MSC: Primary xx; Secondary xx.

1 Introduction

Order statistics play an important role in probability and statistical inference, particularly in reliability theory and survival analysis. For the basic theory of order statistics and description of their role in statistics and applications, see for example David and Nagaraja (2003) and Arnold et al. (2008). Main motivation for the present work is the emphasize on the important role and the application of order statistics in reliability

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theory to use the obtained results for order statistics in system reliability theory, particularly when the random variables are exchangeable. We then use it to compute the mean residual life function, which is very important in reliability and survival analysis. In reliability analysis, the assumption of dependence of lifetimes of components of the system is more realistic than assumption of independence. For example, system components may be affected by a common shock, see e.g. Barlow and Proschan (1975). A kind of dependence is exchangeability which has attracted the interest of many authors in recent years, see for example Navarro et al. (2005), Navarro et al. (2007), Zhang (2010), Eryilmaz and Tank (2012), Eryilmaz (2012) and Yilmaz (2012). Arellano-Valle and Gentone (2007) studied the distribution of linear combinations of order statistics of arbitrary dependent random variables. Bairamov and Parsi (2011) combined two independent samples consisting of exchangeable random variables and then obtained some results on the distributions of order statistics of the mixed sample. Eryilmaz (2013) obtained an expression for the sums of marginal distributions of the order statistics, in terms of the common marginal distribution of the exchangeable random variables.

The random variables $T_1, \ldots, T_n$ are said to be exchangeable if

$$P(T_1 \leq t_1, \ldots, T_n \leq t_n) = P(T_{\pi(1)} \leq t_1, \ldots, T_{\pi(n)} \leq t_n)$$

, where $\pi = (\pi(1), \ldots, \pi(n))$ is an arbitrary permutation of $\{1, \ldots, n\}$, i.e., the joint distribution of $T_1, \ldots, T_n$ is symmetric in $t_1, \ldots, t_n$. Note that T_i's are identically distributed. It is well-known that, when $T_1, \ldots, T_n$ are independent and have a common distribution function F, survival function $\bar{F} = 1 - F$ and density function $f = F'$, then

$$f_{(T_{1:n},\ldots,T_{n:n})}(t_1, \ldots, t_n) = n! f_{(T_1,\ldots,T_n)}(t_1, \ldots, t_n) = n! \prod_{i=1}^{n} f(t_i),$$

$$\bar{F}_{i:n}(t) = P(T_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t),$$

$$f_{i:n}(t) = \bar{F}'_{i:n}(t) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(t) \bar{F}^{n-i}(t) f(t)$$

and

$$\bar{F}_{r,k:n}(t,s) = P(T_{r:n} > t, T_{k:n} > s) = \sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \sum_{j=0}^{k-i-1} \binom{n-i}{j} [\bar{F}(t) - \bar{F}(s)]^j \bar{F}^{n-i-j}(s)$$

for $s > t$ and $1 \leq r \leq k \leq n$, see for example David and Nagaraja (2003).

When $T_1, \ldots, T_n$ are independent but not identically distributed (INID case), some expressions for the above formulas are given by Balakrishnan (2007). Also the results for the copula of the order statistics are obtained by Navarro and Spizzichino (2010).

In Section 2 some expressions for the right sides of the above formulas are given when $T_1, \ldots, T_n$ are exchangeable random variables. In Section 3, some explicit formulas for the mean residual life (MRL) functions $H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t)$ and $M_n^{r,k}(T) = E(T_{k:n} - t | T_{r:n} > t)$ of a $n-k+1$-out-of-n system with exchangeable components in terms of the joint survival function of T_i's, $\bar{\mathbf{F}}(t_1, \ldots, t_n) = P(T_1 > t_1, \ldots, T_n > t_n)$ are provided, $1 \le r \le k \le n$. Finally in Section 4, concluding remarks and a suggestion for future works are given.

2 Joint and marginal distribution functions

Let $(T_1, \ldots, T_n)$ be an exchangeable random vector and suppose $T_{1:n}, \ldots,$ $T_{n:n}$ are corresponding order statistics. We note that the joint density function $f_{(T_1,\ldots,T_n)}(t_1, \ldots, t_n)$ is symmetric in $t_1, \ldots, t_n$. Therefore we can write

$$f_{(T_{1:n},\ldots,T_{n:n})}(x_1, \ldots, x_n) = n! f_{(T_1,\ldots,T_n)}(x_1, \ldots, x_n), \quad x_1 < x_2 < \cdots < x_n. \tag{2.1}$$

We now consider the survival function of $T_{i:n}$. As T_i's are exchangeable random variables, we can write

$$\bar{F}_{i:n}(t) = P(T_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} P(T_1 \le t, \ldots, T_j \le t, T_{j+1} > t, \ldots, T_n > t).$$

The above equation can be written in terms of the joint survival function of T_i's which is given as follows.

Comment 1. We have

$$\begin{aligned}
\bar{F}_{i:n}(t) &= \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} P(T_{1:j} > t) \\
&= \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \bar{\mathbf{F}}(\underbrace{t, \ldots, t}_{j}, \underbrace{-\infty, \ldots, -\infty}_{n-j}) \\
&= 1 - \sum_{j=i}^{n} (-1)^{j-i} \binom{j-1}{i-1} \binom{n}{j} \mathbf{F}(\underbrace{t, \ldots, t}_{j}, \underbrace{\infty, \ldots, \infty}_{n-j}).
\end{aligned} \tag{2.2}$$

The proof of the above equation follows from Equation (3.4.2) in David and Nagaraja (2003, Page 46).

Equation (2.2) shows that the survival (or distribution) function of $T_{i:n}$ can be written as a linear combination of the joint (or survival) function of $T_1, \ldots, T_n$. This kind of representation is called generalized (or negative) mixtures (see, e.g., Navarro et al. (2007)). By taking derivative from both sides of the (2.2) with respect to t, density function of $T_{i:n}$ is obtained.

We now consider $\bar{F}_{r,k:n}$ the joint distribution function of $(T_{r:n}, T_{k:n})$, $1 \le r < k \le n$. For $r = 1$ and $s > t$ we have

$$\bar{F}_{1,k:n}(t,s) = \sum_{j=0}^{k-1} \binom{n}{j} P(t < T_1 \le s, \ldots, t < T_j \le s, T_{j+1} > s, \ldots, T_n > s).$$

Comment 2. For $s > t$ and $1 \le k \le n$,

$$\begin{aligned} \bar{F}_{1,k:n}(t,s) &= P(T_{1:n} > t, T_{k:n} > s) \\ &= \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{n}{j} \binom{n-j-1}{n-k} \bar{\mathbf{F}}(\underbrace{t, \ldots, t}_{j}, \underbrace{s, \ldots, s}_{n-j}) \end{aligned} \tag{2.3}$$

The proof follows from Equation (3.4.3) in (David and Nagaraja , 2003, Page 46).

Equation (2.3) shows that the joint reliability of $(T_{1:n}, T_{k:n})$ can be written as a linear combination of the joint survival function of T_i's, $\bar{\mathbf{F}}(x_1, \ldots, x_n) = P(T_1 > x_1, \ldots, T_n > x_n)$.

Note that using the joint and marginal reliability functions, the joint distribution function of $(T_{1:n}, T_{k:n})$ is obtained as

$$F_{1,k:n}(t,s) = P(T_{1:n} \le t, T_{k:n} \le s) = 1 - \bar{F}_{T_{1:n}}(t) - \bar{F}_{T_{k:n}}(s) + \bar{F}_{1,k:n}(t,s).$$

We now consider the joint reliability function of $(T_{r:n}, T_{k:n})$ when $1 < r < k \le n$. Using the corresponding properties for order statistics and exchangeability assumption of T_i's we can write

$$P(T_{r:n} > t, T_{k:n} > s) = \sum_{i=0}^{r-1} \binom{n}{i} \sum_{j=0}^{k-i-1} \binom{n-i}{j} \times$$

$$P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s). \tag{2.4}$$

Lemma 2.1. *For* $s > t$, $P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s)$

$$\begin{aligned}
&= \bar{\mathbf{F}}(\underbrace{-\infty, \ldots, -\infty}_{i}, \underbrace{t, \ldots, t}_{j}, \underbrace{s, \ldots, s}_{n-i-j}) \\
&- \sum_{l=1}^{j} (-1)^{l+1} \binom{j}{l} \bar{\mathbf{F}}(\underbrace{-\infty, \ldots, -\infty}_{i}, \underbrace{t, \ldots, t}_{j-l}, \underbrace{s, \ldots, s}_{n-i-j+l}) \\
&- \sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \bar{\mathbf{F}}(\underbrace{-\infty, \ldots, -\infty}_{i-l}, \underbrace{t, \ldots, t}_{l+j}, \underbrace{s, \ldots, s}_{n-i-j}) \\
&+ \sum_{l_1=1}^{i} \sum_{l_2=1}^{j} (-1)^{l_1+l_2} \binom{i}{l_1} \binom{j}{l_2} \bar{\mathbf{F}}(\underbrace{-\infty, \ldots, -\infty}_{i-l_1}, \underbrace{t, \ldots, t}_{j+l_1-l_2}, \underbrace{s, \ldots, s}_{n-i-j+l_2}).
\end{aligned} \tag{2.5}$$

We assume that any summation in Equation (2.5) is equal to zero if $i = 0$ or $j = 0$. Note that when $i = 0$, the given probability in Lemma 2.1 is $P(t < T_1 \le s, \ldots, t < T_j \le s, T_{j+1} > s \ldots, T_n > s)$. Similarly for $j = 0$, it is $P(T_1 \le t, \ldots, T_i \le t, T_{i+1} > s, \ldots, T_n > s)$.

Proof. We can write

$$\begin{aligned}
P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s) \\
= P(A \cap B' \cap C'),
\end{aligned}$$

where the events A, B and C are defined as

$$\begin{aligned}
A &= (T_{i+1} > t, \ldots, T_{i+j} > t, T_{i+j+1} > s, \ldots, T_n > s), \\
B &= \cup_{l=i+1}^{i+j} (T_l > s), \\
C &= \cup_{l=1}^{i} (T_l > t).
\end{aligned}$$

Using the principle of inclusion-exclusion and noting that $P(A \cap B' \cap C') = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$, the result follows. If we replace Equation (2.5) in Equation (2.4), joint reliability function of $(T_{r:n}, T_{k:n})$ is again obtained as a linear combination of the joint survival function of T_i's.

□

Comment 3. Similar expressions for the joint distribution (reliability) of two order statistics (or systems) were obtained by Navarro and Balakrishnan (2010).

3 MRL Function of a $n-k+1$-out-of-n System with Exchangeable Components

The mean residual life (MRL) and the failure rate functions are very important in reliability analysis. In this section, we assume that the exchangeable and nonnegative random variables $T_1, \ldots, T_n$ represent the lifetimes of n components which are connected in a $n-k+1$-out-of-n system. It is well-known that the lifetime of this system is $T_{k:n}$. From the results given in the previous section, we shall obtain the MRL function of the system, in terms of the joint survival function of T_i's, $\bar{\mathbf{F}}(x_1, \ldots, x_n)$. Particularly, we consider two MRL functions

$$H_n^k(t) = E(T_{k:n} - t|T_{1:n} > t) \text{ and } M_n^{r,k}(t) = E(T_{k:n} - t|T_{r:n} > t),\ 1 \leq r \leq k \leq n,$$

in which $H_n^k(t)$ measures MRL of the system when all components of the system are working at or before time t whereas in $M_n^{r,k}(t)$ the number of those components is at least $n-r+1$.

The following lemma gives an expression for $H_n^k(t)$.

Lemma 3.1. *We have*

$$H_n^k(t) = E(T_{k:n} - t|T_{1:n} > t) = \sum_{j=n-k+1}^{n} (-1)^{j-n+k-1} \binom{n}{j} \binom{j-1}{n-k} H_{j,n}^1(t), \tag{3.1}$$

where

$$H_{j,n}^1(t) = E(T_{1:j} - t|T_{1:n} > t) = \int_0^\infty \frac{\bar{\mathbf{F}}(\overbrace{t+x, \ldots, t+x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} dx.$$

Proof. We know that

$$H_n^k(t) = \int_0^\infty P(T_{k:n} - t > x|T_{1:n} > t)dx = \int_0^\infty \frac{P(T_{k:n} > t+x, T_{1:n} > t)}{\bar{\mathbf{F}}(t, \ldots, t)} dx.$$

Now, the proof follows from Equation (2.3). □

Comment 4. Asadi and Bayramoglu (2006), studied the MRL function of a k-out-of-n system in IID cases. Navarro and Hernandez (2008), obtained some results on MRL functions of finite mixtures and systems. In Remark 5 of their paper the expression (3.1) is extended to the general case. From this remark we can obtain an expression for the conditional reliability and the Equation (3.1) also follows from this expression.

Remark 1. We note that from Comment 1 and in view of Lemma 2.1, the MRL function

$$M_n^{r,k}(t) = E(T_{k:n} - t | T_{r:n} > t) = \int_0^\infty \frac{P(T_{k:n} > t + x, T_{r:n} > t)}{P(T_{r:n} > t)} dx$$

can also be written in terms of the joint survival function of T_i's but the expression becomes lengthy.

We now give some examples for determining $H_n^k(t)$.

Example 3.1. Suppose that the joint distribution of $T_1, \ldots, T_n$ is Marshal and Olkin's multivariate exponential with the survival function

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = exp\left[-\sum_{i=1}^{n} \lambda_i x_i - \sum_{i_1 < i_2} \lambda_{i_1, i_2} \max(x_{i_1}, x_{i_2}) - \cdots - \lambda_{12\ldots n} \max(x_1, \ldots, x_n) \right].$$

For the special case $\lambda_1 = \cdots = \lambda_n = \lambda_{12} = \cdots = \lambda_{12\ldots n} = \lambda$, $\bar{\mathbf{F}}(x_1, \ldots, x_n)$ is exchangeable. It can be shown that

$$\frac{\bar{\mathbf{F}}(\overbrace{t + x, \ldots, t + x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} = exp\{-(2^n - 2^{n-j})\lambda x\}$$

and hence from Equation (3.1) we have

$$H_n^k(t) = \lambda^{-1} \sum_{i=n-k+1}^{n} \frac{(-1)^{i+k-n-1}}{2^n - 2^{n-i}} \binom{n}{i} \binom{i-1}{n-k} = \lambda^{-1} \sum_{i=0}^{k-1} \frac{(-1)^{k-i-1}}{2^n - 2^i} \binom{n}{i} \binom{n-i-1}{n-k},$$

which is a positive constant. Another special case corresponds to $\lambda_1 = \cdots = \lambda_n = \lambda > 0$, and other λ's equal to 0 (i.e., the IID case). In this case,

$$\frac{\bar{\mathbf{F}}(\overbrace{t + x, \ldots, t + x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} = exp(-\lambda j x)$$

and therefore

$$H_n^k(t) = \lambda^{-1} \sum_{i=n-k+1}^{n} \frac{(-1)^{i+k-n-1}}{i} \binom{n}{i} \binom{i-1}{n-k} = \lambda^{-1} \sum_{i=0}^{k-1} \frac{(-1)^{k-i-1}}{n-i} \binom{n}{i} \binom{n-i-1}{n-k},$$

which is again a positive constant.

Example 3.2. Assume that $T_1, \ldots, T_n$ are distributed as Mardia's multivariate Pareto distribution with the joint survival function

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = \left[\theta^{-1} \sum_{i=1}^{n} x_i - n + 1\right]^{-a}, x_i > \theta > 0, a > 1.$$

We can show that

$$\begin{aligned} E(T_{1:j} - t | T_{1:n} > t) &= \int_0^\infty \frac{\bar{\mathbf{F}}(\overbrace{t + x, \ldots, t + x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} dx \\ &= \frac{nt - n\theta + \theta}{j(a-1)} \end{aligned}$$

and hence, from Equation (3.1), for $t > \theta$,

$$\begin{aligned} H_n^k(t) &= \sum_{j=n-k+1}^{n} (-1)^{j+k-n-1} \binom{n}{j} \binom{j-1}{n-k} \frac{nt - n\theta + \theta}{j(a-1)} \\ &= \left(\sum_{j=n-k+1}^{n} c_j(k, n) \right) \frac{nt - n\theta + \theta}{a - 1}, \end{aligned}$$

where $c_j(k, n) = (-1)^{j+k-n-1} \binom{n}{j} \binom{j-1}{n-k} / j$. Note that $\sum_{j=n-k+1}^{n} c_j(k, n) = (a-1)/\theta H_n^k(\theta) \geq 0$ and therefore $H_n^k(t)$ is a linearly increasing function of t.

Remark 2. It is shown in IID case that the MRL function $H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t)$ is an increasing (a decreasing) function of t when the common distribution function of component lifetimes, $F(t) = P(T \leq t)$, has decreasing (increasing) failure rate, that is, the distribution F has IFR (DFR) property (see for example Asadi and Goliforushani (2008)). We see in Example 3.1 that the common distribution of T_i's is Exponential distribution with parameter λ, which has constant failure rate $r_F(t) = f(t)/\bar{F}(t) = \lambda$ and its MRL function $H_n^k(t)$ is also a positive constant. This property was already mentioned in Remark 5 of Navarro and Hernandez (2008). In Example 3.2, the common distribution of T_i's is $F(t) = 1 - (t/\theta)^{-a}$ and its failure rate function is $r_F(t) = f(t)/\bar{F}(t) = a/t$ which is a DFR distribution. Moreover, its MRL function $H_n^k(t)$ is increasing in t. Although this property in Examples 3.1 and 3.2 is similar to that of the IID case mentioned above, this does not hold in general for exchangeable random variables. See the following example.

Example 3.3. Assume that the dependence between the T_i's is modeled by FGM copula which is given by $\mathbf{F}(x_1,\ldots,x_n)=\prod_{i=1}^n x_i\{1+\theta\prod_{i=1}^n(1-x_i)\}$ for $-1\le\theta\le 1$ and $0\le x_i\le 1$, $i=1,\ldots,n$. Now suppose $n=2$, $\theta=1$ and $k=1$. From Equation (3.1),

$$H_2^1(t)=E(T_{1:2}-t|T_{1:2}>t)=\int_0^\infty\frac{\bar{\mathbf{F}}(t+x,t+x)}{\bar{\mathbf{F}}(t,t)}dx=\int_t^1\frac{\bar{\mathbf{F}}(x,x)}{\bar{\mathbf{F}}(t,t)}dx.$$

Note that

$$\bar{\mathbf{F}}(x_1,x_2)=1-x_1-x_2+\mathbf{F}(x_1,x_2)=1-x_1-x_2+2x_1x_2-x_1^2x_2-x_1x_2^2+x_1^2x_2^2.$$

After simple calculation we obtain

$$H_2^1(t)=\frac{t^4/5-t^3/20+37t^2/60-23t/60+37/60}{1-t+t^2-t^3}.$$

We see that the common distribution of T_i's, $F(t)=t$, $0<t<1$, is an IFR distribution but it is easy to verify that $H_2^1(t)$ is not decreasing in t, for example $d/dtH_2^1(t)>0$ for $1/3<t<1$.

It is obvious for a series system that if $\bar{\mathbf{F}}(t+x,\ldots,t+x)/\bar{\mathbf{F}}(t,\ldots,t)$ is decreasing (increasing) in t, for all $x\ge 0$, then the MRL function of the system, $H_n^1(t)=E(T_{1:n}-t|T_{1:n}>t)=\int_0^\infty\bar{\mathbf{F}}(t+x,\ldots,t+x)/\bar{\mathbf{F}}(t,\ldots,t)dx$, is also decreasing (increasing) in t. A sufficient condition for this is that the joint distribution of T_i's be a multivariate IFR (DFR) distribution. We recall that a joint distribution $\mathbf{F}(x_1,\ldots,x_n)$ is said to be a multivariate IFR (DFR) distribution if $\bar{\mathbf{F}}(t_1+x,t_2+x,\ldots,t_n+x)/\bar{\mathbf{F}}(t_1,t_2,\ldots,t_n)$ decreases (increases) in $t_1,t_2,\ldots,t_n$ for all $x\ge 0$ (see for example Barlow and Proschan (1975)).

4 Concluding Remarks

In this paper, we considered a sequence of exchangeable random variables $T_1,\ldots,T_n$. Let $T_{1:n}<\cdots<T_{n:n}$ denote the ordered values of T_i's. In Section 2, we obtained some expressions for the distribution of $T_{i:n}$ and the joint distribution of $(T_{r:n},T_{k:n})$, $1\le r<k\le n$. Using these two mean residual life (MRL) functions $H_n^k(t)=E(T_{k:n}-t|T_{1:n}>t)$ and $M_n^{r,k}(t)=E(T_{k:n}-t|T_{r:n}>t)$, of a $n-k+1$-out-of-n system are considered in Section 3. The present work might be useful to obtain some more properties of the reliability and the MRL functions of a general coherent system with exchangeable components.

Acknowledgment

The author would like to thank the referee and the Associate Editor for careful reading and giving valuable and constructive comments, which led to a significantly improved version of the paper.

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