RMT

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1 On the eigenvalues distribution

Assume gaussian random matrix $X \sim \mathcal{X}_{n\times n}(0, \Sigma)$, $X \in \mathbb{R}^{n\times n}$, which means $X^V \sim \mathcal{X}_{n^2}(0, I_n \otimes \Sigma)$. Define the columns of X to be $X_1, \cdots, X_n$, $X_n \in \mathbb{R}^n$ and $X^V = (X_1^T, \cdots, X_n^T)^T \in \mathbb{R}^{n^2}$. And $\cdot \otimes \cdot$ means Kronecker product.

Assume the singular decompose of X is $X = A\Lambda B'$, where $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ and A, B are orthogonal matrix $A^T A = B^T B = I_n$. Thus $A^T X = \Lambda B^T$ and $A^T X X^T A = \Lambda B^T B \Lambda = \Lambda^2$.

For calculation simplicity, assume $\Sigma = I_n$, which we will justify later.

As $\Lambda^2 = diag(\lambda_1^2, \cdots, \lambda_n^2) = A^T X X^T A$, Λ^2 is the eigenvalue of XX^T. As $X \sim \mathcal{X}_{n\times n}(0, I_n)$, $XX^T \sim W(n, I_n)$ where W means Wishart distribution. To find the distribution of Λ^2, first state Weyl's Integration Formula[1].

Theorem 1.1 (Weyl's Integration Formula)**.** *If $X \in \mathbb{R}^{n\times n}$ is a real symmetric random matrix with density $g(\lambda_1', \cdots, \lambda_n')$ and g is exchangeable, the joint density of $(\lambda_1', \cdots, \lambda_n')$ is $Cg(\lambda_1', \cdots, \lambda_n')\prod_{i<j}|\lambda_i' - \lambda_j'|$.*

A direct corollary is

Corollary 1.2. *The joint density of $\Lambda^2 = (\lambda_1^2, \cdots, \lambda_n^2)$ is*

$$f_1(\Lambda^2) = \left[2^{n^2/2}\pi^{n(n-1)/2}\prod_{i=1}^{n}\Gamma((n+1-i)/2)\right]^{-1}\left[\prod_{i=1}^{n}\lambda_i^{-1/2}e^{-\lambda_i/2}\right]\left[\prod_{i<j}|\lambda_i - \lambda_j|\right].$$

As $f_1(\Lambda^2)$ is an exchangeable function, the order statistics $\Lambda^2_{ord} = (\lambda^2_{(1)}, \cdots, \lambda^2_{(n)})$ with $\lambda^2_{(1)} \geq \cdots \geq \lambda^2_{(n)}$ has density

$$f_1^{ord}(\Lambda^2_{ord}) = n! f_1(\Lambda^2_{ord}),\ \lambda^2_{(1)} \geq \cdots \geq \lambda^2_{(n)}.$$

Given the low rank p and the error is of magnitude

$$\beta_1 = \frac{\sum\limits_{i=1}^{p} \lambda_{(i)^2}}{\sum\limits_{i=1}^{n} \lambda_{(i)^2}}.$$

2 On the Fourier Concentration

Assume gaussian random matrix $X \sim \mathcal{X}_{n\times n}(0, \Sigma)$ and the Fourier transformation F given X, both $X,\ F \in \mathbb{R}^{n\times n}$. Let $\Theta = \{(u_i, v_i) : i \leq 2np\}$ be the index set we choose to learn parameters on F. The Fourier transformation is

$$F_{u,v} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} X_{i,j} \exp\left\{-j'2\pi(iu + jv)/n\right\},$$

further decompose

$$\exp\left\{-j'2\pi(iu+jv)/n\right\} = \cos\left(2\pi(iu+jv)/n\right) - j'\sin\left(2\pi(iu+jv)/n\right) = r_{ijuv} + j' im_{ijuv},$$

where j' is $\sqrt{-1}$ and $r_{ijuv},\ im_{ijuv}$ be real and imagine part respectively. Rearrange r_{ijuv} and im_{ijuv} to $R_{uv},\ Im_{uv}$ such that

$$F_{u,v} = X^{VT} R_{uv} + j' X^{VT} Im_{uv},$$

and we have

$$|F_{u,v}|^2 = X^{VT}\left[R_{uv}R_{uv}^T + Im_{uv}Im_{uv}^T\right] X^V.$$

Let $S_{uv} = R_{uv}R_{uv}^T + Im_{uv}Im_{uv}^T$ and the error of Fourier transformation is of magnitude

$$\beta_2 = \frac{\sum\limits_{(u,v)\in\Theta} X^{VT} S_{uv} X^V}{\sum\limits_{(u,v)} X^{VT} S_{uv} X^V} \overset{def.}{=} \frac{X^{VT} S_\Theta X^V}{X^{VT} S_0 X^V}$$

References

[1] Theodor Bröcker and Tammo Tom Dieck. *Representations of compact Lie groups*, volume 98. Springer Science & Business Media, 2013.