Definitions

Exponential function. The **exponential function**, denoted exp, is defined as the inverse of the natural logarithm function. In other words, letting $f(x) = \ln x$, we have $f^{-1}(x) = \exp(x)$. We also write e^x for $\exp(x)$.

Exponential function with base a. Let a be a fixed positive number. The **exponential function with base** a, denoted $f(x) = a^x$, is the function with domain all real numbers defined as

$$a^x = e^{x \ln a}.$$

Logarithmic function with base a. Let a be a fixed positive number. The **logarathmic function with base** a, denoted $f(x) = \log_a(x)$ is defined as the inverse function of $g(x) = a^x$.

Theory

Properties of the exponential function. The following properties hold:

1. The exponential function is differentiable (hence also continuous) on all of $\mathbb{R}$ and satisfies
$$\frac{d}{dx}e^x = e^x.$$
Equivalently, we have
$$\int e^x \, dx = e^x + C.$$
2. The exponential function is increasing and hence one-to-one. Its graph is always concave up.
3. We have
$$\lim_{x\to\infty} e^x = \infty$$
$$\lim_{x\to-\infty} e^x = 0$$
4. The domain of exp is $\mathbb{R} = (-\infty, \infty)$; the range of exp is $(0, \infty)$.
5. $e^0 = 1$.
6. We have
$$e^{x+y} = e^x e^y \qquad e^{x-y} = e^x/e^y \qquad e^{xy} = (e^x)^y$$
for all $x, y \in \mathbb{R}$.
7. We have
$$\ln(e^x) = x, \text{ for all } x; \qquad e^{\ln x} = x, \text{ for all } x \in (0, \infty).$$

Logarithmic and exponential compendium. The table below summarizes the important properties of our various families of logarithmic and exponential functions.

$f(x)$	$\ln x$	$\log_a x,\ a>1$	$\log_a x,\ 0<a<1$	e^x	$a^x,\ a>1$	$a^x,\ 0<a<1$
Domain	$(0,\infty)$			$(-\infty,\infty)$		
Range	$(-\infty,\infty)$			$(0,\infty)$		
Monotonicity	Increasing		Decreasing	Increasing		Decreasing
Limit as $x\to\infty$	∞		$-\infty$	∞		0
Limit as $x\to 0^+$	$-\infty$		∞	$*$		
Limit as $x\to-\infty$	$*$			0		∞
Inverse	e^x	a^x		$\ln x$	$\log_a x$	
Relation to base-e	$\ln x=\log_e x$	$\log_a x=\frac{\ln x}{\ln a}$		$*$	$a^x=e^{x\ln a}$	
Algebra	$\log_a(xy)=\log_a x+\log_a y$ $\log_a(x^y)=y\log_a x$ $\log_a(a^x)=x$			$a^{x+y}=a^xa^y$ $a^{xy}=(a^x)^y$ $a^{\log_a x}=x$		

Derivative/antiderivative compendium. We collect here the new derivative formulas obtained via logarithms and exponential functions, along with their equivalent antiderivative formulas.

$$\begin{aligned}
\frac{d}{dx}\ln|x|=\frac{1}{x} &\iff \int\frac{1}{x}\,dx=\ln|x|+C\\
\frac{d}{dx}\ln|\cos x|=-\tan x &\iff \int\tan x\,dx=-\ln|\cos x|+C=\ln|\sec x|+C\\
\frac{d}{dx}\ln|\sin x|=\cot x &\iff \int\cot x\,dx=\ln|\sin x|+C\\
\frac{d}{dx}\ln|\sec x+\tan x|=\sec x &\iff \int\sec x\,dx=\ln|\sec x+\tan x|+C\\
\frac{d}{dx}\ln|\csc x+\cot x|=-\csc x &\iff \int\csc x\,dx=-\ln|\csc x+\cot x|+C\\
\frac{d}{dx}e^x=e^x &\iff \int e^x\,dx=e^x+C\\
\frac{d}{dx}a^x=(\ln a)a^x &\iff \int a^x\,dx=\frac{1}{\ln a}a^x+C\\
\frac{d}{dx}\log_a|x|=\frac{1}{(\ln a)\,x} &\iff \int\frac{1}{(\ln a)\,x}\,dx=\log_a|x|+C
\end{aligned}$$

Examples

1. Find all t satisfying $2^{-t^2}=\dfrac{1}{16}$. Simplify your answer as much as possible.
2. Compute $f'(x)$ for each of the following functions.
   (a) $f(x)=\ln(\sin x)e^{\cos x}$
   (b) $f(x)=\log_3(2^x+3^{x^2})$
3. Compute the following definite indefinite integrals.
   (a) $\displaystyle\int(e^t)^2\sin(e^{2t})\,dt$

(b) $\int_0^{\pi} \sin(2^x) 2^{\cos(2^x)+x} \, dx$