Definitions

Natural logarithm. The **natural logarithm function** is defined as

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

where x is an element of $(0, \infty)$.

Euler's number. **Euler's number**, denoted e, is the unique number satisfying $\ln e = 1$. In other words, e is the number satisfying

$$1 = \int_1^e \frac{1}{t}\, dt.$$

Theory

Properties of the natural logarithm. The following properties hold:

1. The natural logarithm is differentiable (hence also continuous) on $(0, \infty)$ and satisfies
$$\frac{d}{dx} \ln x = \frac{1}{x}.$$
for all x in $(0, \infty)$.

2. The natural logarithm is increasing on $(0, \infty)$ and hence one-to-one. The graph of ln is always concave down.

3. We have
$$\lim_{x\to\infty} \ln x = \infty$$
$$\lim_{x\to 0^+} \ln x = -\infty$$

4. The domain of ln is $(0, \infty)$; the range of ln is $(-\infty, \infty)$.

5. $\ln 1 = 0$.

6. We have
$$\begin{aligned} \ln(ab) &= \ln a + \ln b, \text{ for all } a, b \in (0, \infty). \\ \ln(a/b) &= \ln a - \ln b, \text{ for all } a, b \in (0, \infty),\ b \neq 0. \\ \ln a^r &= r \ln a, \text{ for all } a \in (0, \infty) \text{ and } r \text{ rational.} \end{aligned}$$

Corollary. The function $f(x) = \ln|x|$ is an antiderivative of $1/x$ on its entire domain $D = (-\infty, 0) \cup (0, \infty)$: i.e., we have

$$\int \frac{1}{x}\, dx = \ln|x|\, dx + C.$$

Further trigonometric antiderivative formulas. The following antiderivative formulas hold:

1. $\displaystyle\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C$

2. $\int \cot x \, dx = \ln |\sin x| + C$

3. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

4. $\int \csc x \, dx = -\ln |\csc x + \cot x| + C$