Definitions

Theory

Chain rule (antiderivative version). Let u be a differentiable function on its domain, and suppose f is continuous on the range of u. Suppose $F(x)$ is an antiderivative of $f(x)$. Then $F(u(x))$ is an antiderivative of $f(u(x))u'(x)$: i.e.,

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C.$$

Alternatively, letting $u = u(x)$ we have

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du.$$

Comment. Before seeing how to correctly use the chain rule (antiderivative version) to compute indefinite integrals, it is worthwhile noting a tempting, but *incorrect* method: namely, if $F(x)$ is an antiderivative of $f(x)$ it is not in general true that $F(u(x))$ is an antiderivative of $f(u(x))$. Indeed, the chain rule tells us that $F(u(x))$ is an antiderivative of $f(u(x))u'(x)$.

Procedures

Substitution technique (indefinite integrals). We wish to compute $\displaystyle\int f(x)\,dx$.

1. Pick a differentiable substitution function $u = u(x)$. Set

$$u = u(x) \tag{1}$$
$$du = u'(x)\,dx \tag{2}$$

2. Algebraically manipulate equations (1) and (2) to find a function g such that

$$f(x)\,dx = g(u)\,du.$$

   By the chain rule (antiderivative form) we have

$$\int f(x)\,dx = \int g(u)\,du.$$

3. If possible, find an antiderivative G of g. Then $F(x) = G(u(x))$ is an antiderivative of $f(x)$: i.e.,

$$\int f(x)\,dx = G(u(x)) + C$$

Comment. There is no such thing as a *correct* or *incorrect* substitution, and you are encouraged to be creative with your choice of substitution $u(x)$. Instead think of a substitution as either *helpful* or *not helpful* (or possibly *somewhat helpful*). The success of a particular choice of $u(x)$ depends on two factors:

1. Can you algebraically find a function g such that $f(x) = g(u(x))u'(x)$?
2. Having found a suitable g, can you find an antiderivative G of g?

Substitution technique (definite integrals). We wish to compute the definite integral $\int_a^b f(x)\,dx$ using a substitution $u = u(x)$. We can proceed in two different ways.

1. **Two-step method**. *First* find an antiderivative $F(x)$ of $f(x)$ using the substitution method for indefinite integrals, then use the FTC to compute $\int_a^b f(x)\,dx = F(b) - F(a)$.
2. **Streamlined method**. Find the g such that $f(x)\,dx = g(u)\,du$ (as with indefinite integral substitution) then convert the original definite integral into a new definite integral with respect to u by also *changing the limits of integration*:

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=u(a)}^{u=u(b)} g(u)\,du.$$

Examples

1. **More or less obvious substitutions**. Use the substitution technique to compute the following indefinite integrals.
   (a) $\int x^2\sqrt{x^3+1}\,dx$
   (b) $\int -\sin t\sqrt{\cos t}\,dt$
   (c) $\int \frac{\sin(\sqrt{u})}{\sqrt{u}}\,du$
2. **Less obvious substitutions**. Use the substitution technique to compute the following indefinite integrals.
   (a) $\int \frac{x}{\sqrt{x+1}}\,dx$
   (b) $\int (1+\sqrt{t})^{100}\,dt$
3. **Substitution with definite integrals**. Use the substitution technique to compute the following definite integrals. You may use either the two-step or streamlined method.
   (a) $\int_\pi^{2\pi} \cos^2(x)\sin x\,dx$
   (b) $\int_1^2 \sqrt{s^8+s^6}\,ds$