Definitions

Average value of a function. Let f be integrable on $[a,b]$. The **average value of f over** $[a,b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Difference-evaluation notation. Let g be a real-valued function, and let a,b be elements of the domain of g. The notation $\Big[g(x)\Big]_a^b$ is defined as follows:

$$\Big[g(x)\Big]_a^b = g(b) - g(a).$$

It is worthwile recording some simple identities involving this notation:

$$\Big[f(x) \pm g(x)\Big]_a^b = \Big[f(x)\Big]_a^b \pm \Big[g(x)\Big]_a^b$$
$$\Big[cg(x)\Big]_a^b = c\Big[g(x)\Big]_a^b$$

We will often abbreviate the notation $\Big[g(x)\Big]_a^b$ to $g(x)\Big]_a^b$.

Theory

Fundamental theorem of calculus (I). Let f be continuous on an open interval I containing a. Let $F(x)$ be the function defined on I as

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F(x)$ is differentiable on I and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

for all $x \in I$.

Corollary. If f is continuous on the open interval I, then f has an antiderivative on I.

Fundamental theorem of calculus (II). Let f be continuous on the interval $[a,b]$. If $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

FTC II: rate of change version. Suppose g is differentiable on $[a,b]$. The derivative function g' computes the (instantaneous) rate of change of g with respect to x. By FTC II, we have:

$$\int_a^b g'(x)\,dx = g(b) - g(a).$$

In other words, the integral of the *rate of change* of a function over $[a,b]$ is the *net change* of that function from a to b.

Procedures

Antiderivative method of computing integrals. Suppose $f(x)$ is continuous on $[a, b]$. The antiderivative method for computing $\int_a^b f(x)\,dx$ proceeds as follows:

1. Find an antiderivative of f: i.e., find F such that $F'(x) = f(x)$ for all $x \in [a, b]$.
2. By the fundamental theorem of calculus (II) we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Examples

1. Use the fundamental theorem of calculus to compute the following definite integrals.

   (a) $\displaystyle\int_a^b 1 - x^3\,dx$

   (b) $\displaystyle\int_0^{10} \frac{1}{\sqrt{2t+1}}\,dt$

   (c) $\displaystyle\int_{3\pi/4}^{\pi} \sec^2 u\,du.$

2. Let $f(x) = \sin(x/3)$, and let $\mathcal{C}$ be the graph of f.

   For each region $\mathcal{R}$ compute the area of $\mathcal{R}$ and the signed area of $\mathcal{R}$.

   Include a diagram of $\mathcal{C}$ and $\mathcal{R}$. Make sure your answer is consistent with your graph. If your answer happens to be 0, use the diagram to explain why.

   (a) $\mathcal{R}$ is the region between $\mathcal{C}$ and the x-axis, from $x = -\pi$ to $x = \pi$.

   (b) $\mathcal{R}$ is the region between $\mathcal{C}$ and the x-axis, from $x = -\pi$ to $x = \pi/2$.

   (c) $\mathcal{R}$ is the region between $\mathcal{C}$ and the x-axis, from $x = 0$ to $x = 6\pi$.

3. Let $F(x) = \int_1^x \frac{1}{t^2}\,dt$. Make a table of values of $F(x)$ for $x = 1, 2, 3, 4, 5$. Explain graphically what $F(b)$ is for any $b \geq 1$.

4. For each $F(x)$ defined below, use the fundamental theorem of calculus (along with some other useful pieces of theory) to compute $F'(x) = \dfrac{d}{dx}F(x)$.

   (a) $F(x) = \displaystyle\int_x^5 \sqrt{t+1}\,dt$

   (b) $F(x) = \displaystyle\int_{-2}^{\sin x} \cos(u^2)\,du$

   (c) $F(x) = \displaystyle\int_{4x}^{\sqrt{x^2+1}} \sin(s^2)\,ds$