Definitions

Antiderivative. Let f be a real-valued function defined on an interval I. A function F is called an antiderivative of f if $F'(x) = f(x)$ for all $x \in I$.

Indefinite integral. Let f be a real-valued function defined on an interval I and suppose f has an antiderivative. The **indefinite integral** of f with respect to x is the notation

$$\int f \; dx$$

and is used to denote the general antiderivative of f. Thus if F is a particular antiderivative, then we write

$$\int f \; dx = F(x) + C$$

to express the fact that the general antiderivative of f is of the form $F(x)+C$ for some $C \in \mathbb{R}$. The symbol $\int$ is called the **integral symbol**, the function f is called the **integrand** of the integral, and x is called the **variable of integration**.

Theory

General antiderivative theorem. Let f be a real-valued function defined on an interval I and suppose F is an antiderivative of f.

1. Given any $C \in \mathbb{R}$, the function $F(x) + C$ is an antiderivative of f.
2. If G is an antiderivative of f, then there is a $C \in \mathbb{R}$ such that

   $$G(x) = F(x) + C$$

   for all $x \in I$.
3. Thus (1) and (2) imply that the **general antiderivative** of f on I can be expressed as $F(x) + C$, where C is any real number.

Antiderivative formulas. The following antiderivative (or indefinite integral) formulas follow directly from a corresponding derivative formula.

$$\int 0\, dx = C \qquad \int x^r\, dx = \frac{x^{r+1}}{r+1} + C, r \neq -1$$

$$\int \cos kx\, dx = \frac{1}{k}\sin kx + C \qquad \int \sin kx\, dx = -\frac{1}{k}\cos kx + C$$

$$\int \sec^2 kx\, dx = \frac{1}{k}\tan kx + C \qquad \int \csc^2 kx\, dx = -\frac{1}{k}\cot kx + C$$

$$\int \sec kx \tan kx\, dx = \frac{1}{k}\sec kx + C \qquad \int \csc x \cot x\, dx = -\frac{1}{k}\csc kx + C$$

Antiderivative rules. Let f and g be real-valued functions defined on an interval I. Suppose F is an antiderivative of f and G is an antiderivative of g.

1. Given any constant $a \in \mathbb{R}$, the function aF is an antiderivative of af, and hence

   $$\int af\, dx = aF(x) + C.$$

2. The function $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$, and hence

   $$\int f \pm g\, dx = F(x) \pm G(x) + C.$$

Examples

1. Find an antiderivative for the given function.

   (a) $f(x) = x^7$

   (b) $f(x) = \dfrac{1}{\sqrt{x}}$

   (c) $f(x) = 2\sin x - x^{2/3}$

2. Find an antiderivative for the given function.

   (a) $f(x) = \sec^2 5x$

   (b) $f(x) = 2x\cos(x^2)$

   (c) $f(x) = \cos(x^2)$

3. At time $t = 0$ minutes a tank containing 100 gallons of water begins leaking. After t minutes the rate at which the water leaves the tank is given by

$$r(t) = \frac{1}{\sqrt{2t+1}}.$$

   Let $f(t)$ be the amount of water in the tank after t minutes. Find a formula for $f(t)$.

4. Consider the differential equation

$$f''(x) = -\frac{2}{3}\cos(2x) + x. \qquad (*)$$

   (a) Find the general formula for a function $f(x)$ satisfying $(*)$.

   (b) Find the unique function $f(x)$ satisfying $(*)$ and the initial conditions

$$f(0) = 0, f'(0) = -1.$$