Definitions

One-to-one. A function f is one-to-one on the set X if $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$. We express this with logical notation as

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2),$$

or equivalently, using the contrapositive,

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

Monotonic functions. Let f be a a real-valued function defined on the set X.

- The function f is **increasing on** X if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 < x_2$. Using logical notation:
$$x_1 < x_2 \implies f(x_1) < f(x_2).$$
- The function f is **decreasing on** X if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 < x_2$. Using logical notation:
$$x_1 < x_2 \implies f(x_1) > f(x_2).$$
- The function f is **monotonic on** X if f is increasing on X or f is decreasing on X.

Inverse function. Suppose f is one-to-one on the set X, and let Y be the range of f. The **inverse function of** f is the function f^{-1} with domain Y defined by the following rule:

- Given $b \in Y$ there is a unique element $a \in X$ such that $f(a) = b$.
- We define $f^{-1}(b) = a$.

Theory

Horizontal line test. Let f be a real-valued function defined on X, and let $\mathcal{C}$ be the graph of f over X. The function f is one-to-one on X if and only if for all $c \in \mathbb{R}$ the horizontal line $y = c$ intersects $\mathcal{C}$ in *at most* one point.

Monotonic functions are one-to-one. If f is monotonic on X then f is invertible on X.

Inverse function compendium. Let f be one-to-one on its domain X, and let Y be the range of f. Let f^{-1} be the inverse of f.

1. $f(a) = b$ if and only if $f^{-1}(b) = a$.
2. The domain of f^{-1} is Y, the range of f; the range of f^{-1} is X, the domain of f.
3. We have
$$f^{-1}(f(a)) = a \text{ for all } a \in X$$
$$f(f^{-1}(b)) = b \text{ for all } b \in Y.$$
4. The point $P = (x, y)$ is on the graph of f if and only if the point $Q = (y, x)$ is on the graph of f^{-1}.
5. The graph of f^{-1} is the reflection of the graph of f through the line $y = x$.

Derivative formula for inverses. Assume f is one-to-one and differentiable on the interval I, and that $f'(x) \neq 0$ for all $x \in I$. Let J be the range of f. Then:

1. The inverse function f^{-1} is differentiable on J.

2. We have
$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$
for all $b \in J$. Alternatively, letting a be the unique element of D such that $f(a) = b$, we have
$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Examples

1. Let $f(x) = x^2 + 1$.
   (a) Show that f is not one-to-one on $(-\infty, \infty)$.
   (b) Show that f is one-to-one on $(-\infty, 0]$.
   (c) Compute a formula for the inverse of f on the domain $(-\infty, 0]$.

2. Let $f(x) = x^5 + x^3 + 3x - 5$.
   (a) Show that f is one-to-one.
   (b) Plot three points on the graph of f^{-1}.
   (c) Compute $(f^{-1})'(-5)$ and $(f^{-1})'(-8)$.