Procedures

Reverse substitution technique (indefinite integral). To compute $\int f(x)\,dx$ using reverse substitution, proceed as follows:

1. Choose a 1-1, differentiable substitution function g with differentiable inverse and assemble the two equations
$$x = g(t)$$
$$dx = g'(t)\,dt$$
2. Compute
$$\int f(g(t))g'(t)\,dt = F(t) + C.$$
3. We conclude that
$$\int f(x)\,dx = F(g^{-1}(x)) + C.$$
Alternatively, we compute an antiderivative for $f(x)$ by expressing the function $F(t)$ from (2) as a function of x using $x = g(t)$ and $g^{-1}(x) = t$.

Reverse substitution technique (definite integral). To compute $\int_a^b f(x)\,dx$ using reverse substitution substitution, proceed as follows:

1. Choose a 1-1, differentiable substitution function g with differentiable inverse and assemble the two equations
$$x = g(t)$$
$$dx = g'(t)\,dt$$
2. Then we have
$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=g^{-1}(a)}^{t=g^{-1}(b)} f(g(t))g'(t)\,dt.$$

Comment. What is the difference between our original (forward) substitution and reverse subsitution?

- Forward substitution allows us to find an antiderivative of $f(u(x))u'(x)$ from an antiderivative of $f(x)$: namely,
$$F(x) \text{ is an antiderivative of } f(x) \implies F(u(x)) \text{ is an antiderivative of } f(u(x))u'(x).$$
- Reverse substitution allows us to find an antiderivative of $f(x)$ from an antiderivative of $f(g(t))g'(t)$: namely,
$$F(t) \text{ is an antiderivative of } f(g(t))g'(t) \implies F(g^{-1}(x)) \text{ is an antiderivative of } f(x).$$

Trigonometric substitution. The table below indicates potentially helpful (reverse) substitutions for functions f containing particular forms of expressions.

$$f(x) \text{ contains } \sqrt{a^2 - x^2} \implies \text{try } \begin{array}{c} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{array}, -\pi/2 \leq \theta \leq \pi/2$$

$$f(x) \text{ contains } x^2 + a^2 \implies \text{try } \begin{array}{c} x = a\tan\theta \\ dx = a\sec^2\theta\, d\theta \end{array}, -\pi/2 < \theta < \pi/2$$

$$f(x) \text{ contains } \sqrt{x^2 - a^2} \implies \text{try } \begin{array}{c} x = a\sec\theta \\ dx = a\sec\theta\tan\theta\, d\theta \end{array}, 0 < \theta < \pi/2 \text{ or } \pi/2 < \theta < \pi$$

Examples

1. Derive the area formula for a circle of radius r using calculus.
2. Find an antiderivative of $\sqrt{1-x^2}$.
3. Compute the following integrals.
   (a) $\displaystyle\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{\sqrt{x^2-1}}{x}\, dx$
   (b) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}$
   (c) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\, dx$, $x \leq -1$.
   Note. This is the indefinite integral version of (a). To finish the computation you need to use the arcsec function, which is defined as the inverse of sec with restricted domain $[-1, \pi/2) \cup (\pi/2, 1]$. We don't officially cover arcsec in this course, but this exercise is good practice nonetheless.