Definitions

Theory

Trigonometric identities. The following identities hold for all $\theta, \phi \in \mathbb{R}$.

1. $\cos\theta\cos\phi = \dfrac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$
2. $\sin\theta\sin\phi = \dfrac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$
3. $\sin\theta\cos\phi = \dfrac{\sin(\theta-\phi)+\sin(\theta+\phi)}{2}$
4. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
5. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Procedures

Comment. The basic strategy for computing integrals of functions of the form $\sin^m x\cos^n x$ or $\tan^m x\sec^n x$ is to use one of the four substitutions

$$\begin{array}{cccc} u=\sin x & u=\cos x & u=\tan x & u=\sec x \\ du=\cos x\,dx & du=-\sin x\,dx & du=\sec^2 x\,dx & du=\sec x\tan x\,dx, \end{array}$$

"peel off" what is necessary for du, and express the rest of the integrand as a polynomial in u using the trigonometric identities.

$$\sin^2 x+\cos^2 x=1 \qquad \sec^2 x=\tan^2+1.$$

Integrating $\sin^m x\cos^n x$. Let m and n be nonnegative integers. When computing

$$\int \sin^m x\cos^n x\,dx$$

the following strategies often help.

1. If $m=2k+1$ is odd, write

$$\int \sin^m x\cos^n x\,dx = \int (1-\cos^2 x)^k\cos^n x\sin x\,dx$$

and use the substitution $u=\sin x, du=\cos x\,dx$.

2. If $n=2k+1$ is odd, write

$$\int \sin^m x\cos^n x\,dx = \int \sin^m x(1-\sin^2 x)^k\cos x\,dx$$

and use the substitution $u=\cos x, du=-\sin x\,dx$.

3. If m and n are both even use $\sin^2 x=\dfrac{1-\cos 2x}{2}$ and $\cos^2 x=\dfrac{1+\cos 2x}{2}$ to reduce to a lower power of $\cos 2x$.

Integrating $\tan^m x \sec^n x$**.** Let m and n be nonnegative integers. When computing

$$\int \tan^m x \sec^n x \, dx$$

the following strategies often help.

1. If $m = 2k + 1$ is odd and $n \geq 1$, write

$$\int \tan^m x \sec^n x \, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

   and use the substitution $u = \sec x, du = \sec x \tan x \, dx$.

2. If $n = 2k$ is even, write

$$\int \tan^m x \sec^n x \, dx = \int (\tan^2 x + 1)^{k-1} \tan^m x \sec^2 x \, dx$$

   and use the substitution $u = \tan x, du = \sec^2 x \, dx$.

3. If m is even and n is odd, express everything in terms of $\sec x$ and possibly use integration by parts.

Examples

Compute the following indefinite integrals.

1. $\displaystyle\int \sin^3 x \cos^2 x \, dx$

2. $\displaystyle\int \sin^2 x \cos^4 x \, dx$

3. $\displaystyle\int \sec^4 x \, dx$

4. $\displaystyle\int \tan^5 x \sec^7 x \, dx$

5. $\displaystyle\int \sec^3 x \, dx$

6. $\displaystyle\int \tan^5 x \, dx$