## Definitions

**Integrable function.** Let $f$ be a function defined on the interval $[a, b]$.
We say the **definite integral of $f$ over $[a, b]$ exists** if there is a real number $J$ such that for *any* sequence of partitions $P_n$ of $[a, b]$ and *any* choice of Riemann sums $S_n$ corresponding to the partitions $P_n$, if the maximum width of a subinterval in $P_n$ approaches 0, then

$$\lim_{n\to\infty} S_n = J.$$

In plain English, the definite integral of $f$ exists if any sequence of Riemann sums corresponding to a finer and finer partition of $[a, b]$ approaches the same value $J$ in the limit.

In this case we say $f$ **is integrable over** $[a, b]$ and call $J$ the **definite integral of $f$ over** $[a, b]$, denoted

$$\int_a^b f(x)\,dx = J.$$

**Area and signed area of regions defined by functions.** Let $f$ be integrable over the interval $[a, b]$, let $\mathcal{C}$ be the graph of $f$, and let $\mathcal{R}$ be the region between $\mathcal{C}$ and the $x$-axis from $x = a$ to $x = b$.

- We define the **area** (or **total area**) of $\mathcal{R}$ to be the integral of $|f|$ over $[a, b]$: i.e.,

$$\text{area of } \mathcal{R} = \int_a^b |f(x)|\,dx.$$

- We define the **signed area** of $\mathcal{R}$ to be the integral of $f$ over $[a, b]$: i.e.,

$$\text{signed area of } \mathcal{R} = \int_a^b f(x)\,dx.$$

**Comment.** Let $f$ be integrable over the interval $[a, b]$, let $\mathcal{C}$ be the graph of $f$, and let $\mathcal{R}$ be the region between $\mathcal{C}$ and the $x$-axis from $x = a$ to $x = b$.

1. The area of $\mathcal{R}$ is always nonnegative, since $|f(x)| \geq 0$ for all $x \in [a, b]$.
2. If $f(x) \geq 0$ for all $x \in [a, b]$, then $f = |f|$ over $[a, b]$, and hence

$$\text{area of } \mathcal{R} = \int_a^b f(x)\,dx$$

   in this case.
3. Suppose $[a, b]$ can be partitioned into finitely many intervals over which $f$ is either always nonnegative ($\geq 0$) or always nonpositive ($\leq 0$). Then

$$\text{signed area of } \mathcal{R} = \int_a^b f(x)\,dx = (\text{area of regions where } f \geq 0) - (\text{area of regions where } f \leq 0).$$

## Procedures

**Direct computation of definite integral.** Suppose $f$ is integrable on the interval $[a, b]$. Since $\int_a^b f(x)\,dx$ can be computed using any sequence of Riemann sums, we may compute it as a limit of right Riemann sums $R_n$ corresponding to partitions of $[a, b]$ into $n$ equal subintervals. For such partitions the length of each subinterval is $\Delta x = (b-a)/n$, and the right endpoint of the $k$-th subinterval is $x_k = a + k(b-a)/n$. We conclude:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) \frac{(b-a)}{n}.$$

## Theory

**Integrable functions theorem.** Let $f$ be defined on the interval $[a, b]$. If $f$ is continuous everywhere on $[a, b]$, or if $f$ has at most finitely many jump discontinuities on $[a, b]$, then $f$ is integrable over $[a, b]$.

**Properties of definite integrals.** Let $f$ and $g$ be integrable over $[a, b]$.

1. $\int_a^a f(x)\,dx = 0$. (By definition)
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. (By definition)
3. **Sum and difference**. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. **Constant multiple**. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for any $c \in \mathbb{R}$.
5. **Additive**. For any $c \in \mathbb{R}$ we have

   $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

   as long as all of the integrals involved are defined.
6. **Max-min inequality**. If $f$ has a minumum value $\min f$ on $[a, b]$ and a maximum value $\max f$ on $[a, b]$, then

   $$(\min f)(b-a) \le \int_a^b f(x)\,dx \le (\max f)(b-a)$$

7. **Domination**. If $f(x) \le g(x)$ for all $x \in [a, b]$, then

   $$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

## Examples

1. Fix positive constants $m$ and $b$, and define $f(x) = mx + b$.

   (a) Fix a positive constant $a$. Compute $\int_0^a f(x)\,dx$ directly as a limit of right Riemann sums.

   (b) Graph $f(x)$ on $[0, a]$ and explain how your answer in (a) is consistent with known area formulas.

2. Fix a positive constant $b$. Compute $\displaystyle\int_0^b f(x)\,dx$ directly as a limit of right Riemann sums.

3. Let $f(x) = 1 - x^3$. Fix constants $a$ and $b$ with $0 < a < b$. Use your result in Example 2 and various integral properties (including the additivite property) to derive a formula for $\displaystyle\int_a^b f(x)\,dx$ in terms of $a$ and $b$.

4. Let $f(x) = 1 - x^3$. Fix a constant $b$ with $b > 1$, let $f(x) = 1 - x^3$, and let $\mathcal{R}$ be the region between the graph of $f$ and the $x$-axis from $x = 0$ to $x = b$.

   (a) Graph $f(x)$ on $[0, b]$. Your graph should reflect the assumption that $b > 1$.

   (b) Describe precisely how the signed area of $\mathcal{R}$ is a difference of areas of two distinct regions.

   (c) Compute the area of $\mathcal{R}$.