Definitions

Volume of solid via cross sections. Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a solid region in 3-space.

For each $x_0 \in \mathbb{R}$ let $\mathcal{S}_{x_0}$ be the **cross section** of $\mathcal{S}$ consisting of all points of $\mathcal{S}$ whose x-coordinate is equal to x_0, and let $A(x_0)$ be the area of S_{x_0}.

Assume $A(x)$ is integrable on the interval $[a, b]$. We define the **volume** V of $\mathcal{S}$ between $x = a$ and $x = b$ as the integral of $A(x)$ from $x = a$ to $x = b$: i.e.,

$$V = \int_a^b A(x)\,dx.$$

The volume of $\mathcal{S}$ between $y = c$ and $y = d$, or $z = e$ and $z = f$ is defined similarly.

Solid of revolution. Given a planar region $\mathcal{R}$ and a line L in that plane, the **solid of revolution** with **axis of revolution** L is the solid region $\mathcal{S}$ obtained by rotating $\mathcal{R}$ about L.

Theory

Procedures

Volume via cross sections. To compute the volume of a solid region $\mathcal{S}$ via x-cross sections, proceed as follows:

1. Sketch $\mathcal{S}$ along with a typical cross section $\mathcal{S}_x$.
2. Derive a formula for $A(x)$ in terms of x.
3. Determine the appropriate limits of integration: $x = a$ and $x = b$.
4. Compute $\displaystyle\int_a^b A(x)\,dx$.

Volumes of solids of revolution. The cross section method can be applied to the *special case* of solids of revolution. The two cases below are typical, and the given procedures can be modified appropriately if a vertical axis is replaced with a horizontal one.

Cylinder (or disk) method. Suppose $f(x)$ is integrable on $[a, b]$ and that $f(x) \geq c$ for all $x \in [a, b]$. Let $\mathcal{R}$ be the region between the graph of $f(x)$ and the line $y = c$ from $x = a$ to $x = b$, and let $\mathcal{S}$ be the solid obtained by revolving $\mathcal{R}$ about the horizontal axis $y = c$.

- For each $x \in [a, b]$, S_x is a disc of radius $f(x) - c$ and area $A(x) = \pi(f(x) - c)^2$.
- The volume of $\mathcal{S}$ from $x = a$ to $x = b$ is thus

$$V = \int_a^b \pi(f(x) - c)^2\,dx.$$

Annulus (or washer) method. Suppose $p(y)$ and $q(y)$ are integrable on $[a, b]$ and that $p(y) \geq q(y) > c$ for all $y \in [a, b]$. Let $\mathcal{R}$ be the region between the graph of $p(y)$ and $q(y)$ over the interval $[a, b]$ in the y-axis, and let $\mathcal{S}$ be the solid obtained by revolving $\mathcal{R}$ about the vertical axis $x = c$.

- For each $y \in [a, b]$, S_y is an **annulus** of inner radius $q(y) - c$ and outer radius $p(y) - c$. The area of this annulus is $\pi((p(y) - c)^2 - (q(y) - c)^2)$.
- The volume of $\mathcal{S}$ from $y = a$ to $y = b$ is thus

$$V = \int_a^b \pi((p(y) - c)^2 - (q(y) - c)^2)\,dy.$$

Examples

1. Use the volume via cross sections method to compute the volume of a sphere of radius r.
2. Use the volume via cross sections method to compute the volume of a right circular cone of height h and base of radius r.
3. Let $\mathcal{R}$ be the region between the graph of $y = -\frac{3}{25}x^2 + 5$ and the x-axis from $x = 0$ to $x = 5$, and let $\mathcal{S}$ be the solid obtained by revolving $\mathcal{R}$ about the x-axis. Sketch $\mathcal{S}$ and compute its volume.
4. Let $\mathcal{R}$ be the region enclosed by the line $y + 2x = 2$ and the parabola $y^2 + x = 4$, and let $\mathcal{S}$ be the solid obtained revolving $\mathcal{R}$ about the y-axis. Sketch $\mathcal{R}$ and compute the volume of $\mathcal{S}$. Can you sketch, or at least describe $\mathcal{S}$?