Definitions

Trapezoidal rule. Let f be an integrable function on $[a,b]$, let n be a positive integer, and let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be partition of $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$.

The **n-th trapezoidal estimate** of $\int f(x)\,dx$, denoted T_n, is defined as

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)) \approx \int_a^b f(x)\,dx.$$

The trapezoidal estimate is the result of approximating the graph of f with the polygon passing through the points $P_0 = (x_0, f(x_0)), P_1 = (x_1, f(x_1)), \ldots, P_n = (x_n, f(x_n))$.

Simpson's rule. Let f be an integrable function on $[a,b]$, let $n = 2r$ be an even positive integer, and let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be partition of $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$.

The **n-th Simpson's rule estimate** of $\int f(x)\,dx$, denoted S_n, is defined as

$$S_n = \frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) \approx \int_a^b f(x)\,dx.$$

The Simpson's rule estimate is the result of approximating the graph of f over each of the r subintervals $[x_{2(k-1)}, x_{2k}]$ with the unique "parabolic arc"[1] passing through $P_{2(k-1)} = (x_{2(k-1)}, f(x_{2(k-1)}))$, $P_{2k-1} = (x_{2k-1}, f(x_{2k-1})), P_{2k} = (x_{2k}, f(x_{2k}))$.

Theory

Error estimates. Let f be an integrable function on $[a,b]$, let n be a positive integer, and let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

be partition of $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$.

1. Let RS_n be either the right or left Riemann sum for this partition. Suppose $|f'(x)| \leq M$ for all x in $[a,b]$. Then
$$\left|\int_a^b f(x)\,dx - RS_n\right| \leq \frac{M(b-a)^2}{2n}.$$
2. Let T_n be the n-th trapezoidal estimate of $\int_a^b f(x)\,dx$. Suppose $|f''(x)| \leq N$ for all x in $[a,b]$. Then
$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{N(b-a)^3}{12n^2}.$$
3. Suppose n is even, and let S_n be the n-th Simpson's rule estimate of $\int_a^b f(x)\,dx$. Suppose $|f^{(4)}(x)| \leq K$ for all x in $[a,b]$. Then
$$\left|\int_a^b f(x)\,dx - S_n\right| \leq \frac{K(b-a)^5}{180n^4}.$$

[1] If the three points happen to be colinear, then the "parabolic arc" will actually be a line.

Examples

1. Let $f(x) = \frac{1}{x}$. Recall that we have by definition $\ln 4 = \int_1^4 f(x)\,dx$. Compute (a) the $n = 6$ trapezoidal estimate of I, and (b) the $n = 6$ Simpson's rule estimate of I.

2. Let $f(x) = \frac{4}{x^2+1}$, and let $I = \int_0^1 f(x)\,dx$. Observe that $I = 4(\arctan(1) - \arctan 0) = \pi$.

   Compute (a) the $n = 6$ trapezoidal estimate of I, and (b) the $n = 6$ Simpson's rule estimate of I.

3. Compute bounds for the errors in (a) the $n = 10$ trapezoidal estimate of $\ln 4$ and (b) the $n = 10$ Simpson's rule estimate of $\ln 4$.

4. Compute bounds for the errors in (a) the $n = 10$ trapezoidal estimate of $\pi = \int_0^1 4/(x^2+1)\,dx$ and (b) the $n = 10$ Simpson's rule estimate of $\pi = \int_0^1 4/(x^2+1)\,dx$.

   Hint. Letting $f(x) = 4/(x^2+1)$, we have

$$\begin{aligned} f''(x) &= \frac{8(3x^2-1)}{(x^2+1)^3} \\ f^{(4)}(x) &= \frac{96(5x^4-10x^2+1)}{(x^2+1)^5}. \end{aligned}$$

5. Find (a) an n such that the n-th trapezoidal estimate of $\pi = \int_0^1 4/(x^2+1)\,dx$ is within 10^{-9} of the actual value, and (b) an n such that the n-th Simpson's rule estimate of $\pi = \int_0^1 4/(x^2+1)\,dx$ is within 10^{-9} of the actual value.