Definitions

First-order differential equation. A **first-order differential equation in the unknown** $f(x)$ is an equation that can be written in the form

$$f'(x) = F(x, f(x)) \tag{$*$}$$

where $F(x, f(x))$ denotes an arbitrary expression involving x and $f(x)$. Equivalently, a first-order differential equation is an equation that can be written in the form

$$G(x, f'(x), f(x)) = 0,$$

where $G(x, f(x), f'(x))$ denotes an arbitrary expression involving x, $f(x)$, and $f'(x)$.

A **solution** to a differential equation is any function $f(x)$ that satisfies this equation. The **general solution** to a differential equation is a formula, possibly containing undetermined constants, describing all solutions to the differential equation.

Separable first-order differential equation. A **separable differential equation in the unknown** $f(x)$ is a differential equation that can be written in the form

$$f'(x) = g(x)h(f(x)), \text{ or equivalently, } \frac{dy}{dx} = g(x)h(y),$$

where $y = f(x)$.

Exponential growth and decay. Suppose the function $f(x)$ satisfies the equation

$$f'(x) = kf(x),$$

where k is a fixed constant.

If $k > 0$ then $f(x)$ is said to undergo **exponential growth**.

If $k < 0$ then $f(x)$ is said to undergo **exponential decay**.

Procedures

Separation of variables (prime form). To solve a separable differential equation of the form

$$f'(x) = g(x)h(f(x))$$

proceed as follows:

1. *Separation.* Write the equation as

$$\frac{f'(x)}{h(f(x))} = g(x).$$

and take take the indefinite integral of both sides.

$$\int \frac{f'(x)}{h(f(x))}\, dx = \int g(x)\, dx.$$

2. *Substitution.* Use the substitution $u = f(x)$ to rewrite this equality as

$$\int \frac{1}{h(u)}\, du = \int g(x)\, dx,$$

and attempt to find an antiderivative $F(u)$ of $1/h(u)$ and an antiderivative $G(x)$ for $g(x)$.

3. *Algebra.* Attempt to solve the resulting general equation

$$F(u) = G(x) + C$$

for $u = f(x)$.

Separation of variables (algebraic form). To solve a separable differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

proceed as follows:

1. *Separation.* Write the equation as

$$\frac{1}{h(y)}\, dy = g(x)\, dx$$

and take take the indefinite integral of both sides.

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx.$$

2. *Integration.* Attempt to find an antiderivative $F(y)$ of $1/h(y)$ and an antiderivative $G(x)$ for $g(x)$.

3. *Algebra.* Attempt to solve the resulting general equation

$$F(y) = G(x) + C$$

for y in terms of x.

Examples

1. Suppose a hot object cools in a room kept at constant temperature of T_0 (in celcius). Newton's law of cooling states that the rate at which the object cools (with respect to time) is proportional to the *difference* between its current temperature and the room temperature T_0.
   (a) Write a differential equation that describes Newton's law of cooling in this setting.
   (b) Find the general solution to this differential equation.
   (c) Find a the particular solution to the situation where $T_0 = 15°\text{C}$, the object's initial temperature is $100°\text{C}$, and after 5 minutes the object's temperature is $80°\text{C}$.

2. Solve the following differential equations using separation of variables. If an initial condition is given, provide the corresponding particular solution. Otherwise, give the general solution.
   (a) $f'(x) = xf(x) + x$
   (b) $\dfrac{dy}{dx} = \dfrac{x^3}{y^2}$
   (c) $\cot x\, f'(x) + f(x) = 2,\ f(0) = 0.$