Definitions

Inverse trigonometric functions. The following are examples of what are called **inverse trigonometric functions**.

- On the restricted domain $[-\pi/2, \pi/2]$ the function $f(x) = \sin x$ is one-to-one, with range $[-1, 1]$. The inverse function of f restricted to this domain is called the **arcsine function**, denoted $f^{-1}(x) = \arcsin x$.
- On the restricted domain $[0, \pi]$ the function $g(x) = \cos x$ is one-to-one, with range $[-1, 1]$. The inverse function of g restricted to this domain is called the **arccosine function**, denoted $g^{-1}(x) = \arccos x$.
- On the restricted domain $(-\pi/2, \pi/2)$ the function $h(x) = \tan x$ is one-to-one, with range $(-\infty, \infty)$. The inverse function of h restricted to this domain is called the **arctangent function**, denoted $h^{-1}(x) = \arctan x$.

Comment. Occasionally an alternative notation is used to denote inverse trig functions: namely,

$$\arcsin x = \sin^{-1} x \qquad \arccos x = \cos^{-1} x \qquad \arctan x = \tan^{-1} x.$$

We will avoid this alternative notation as it misleadingly suggests these inverse trigonometric functions are *reciprocals* of the corresponding trigonometric functions. They are not!

Theory

Properties of inverse trigonometric functions.

- The function arcsin is an increasing function with domain $[-1, 1]$ and range $[0, \pi]$. It satisfies the following properties:

$$\begin{aligned} &\arcsin(x) = \theta \iff \sin\theta = x \text{ and } -\pi/2 \le \theta \le \pi/2 \\ &\arcsin(\sin\theta) = \theta \text{ for all } -\pi/2 \le \theta \le \pi/2 \\ &\sin(\arcsin x) = x \text{ for all } -1 \le x \le 1. \end{aligned}$$

- The function arccos is a decreasing function with domain $[-1, 1]$ and range $[0, \pi]$. It satisfies the following properties:

$$\begin{aligned} &\arccos(x) = \theta \iff \cos\theta = x \text{ and } 0 \le \theta \le \pi \\ &\arccos(\cos\theta) = \theta \text{ for all } 0 \le \theta \le \pi \\ &\cos(\arccos x) = x \text{ for all } -1 \le x \le 1. \end{aligned}$$

- The function arctan is an increasing function with domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$. It satisfies the following properties:

$$\begin{aligned} &\arctan(x) = \theta \iff \tan\theta = x \text{ and } -\pi/2 < \theta < \pi/2 \\ &\arctan(\tan\theta) = \theta \text{ for all } -\pi/2 < \theta < \pi/2 \\ &\tan(\arctan x) = x \text{ for all } x \\ &\lim_{x\to\infty} \arctan x = \pi/2, \quad \lim_{x\to-\infty} \arctan x = -\pi/2 \end{aligned}$$

Derivative formulas for inverse trigonometric functions. The following derivative/antiderivative formulas hold:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \iff \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \text{(for all } x \text{ in } (-1,1))$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \iff \int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C \qquad \text{(for all } x \text{ in } (-1,1))$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \iff \int \frac{1}{1+x^2}\,dx = \arctan x + C \qquad \text{(for all } x).$$

Examples

1. Compute the following values of trigonometric functions by hand.
   (a) $\arcsin(-1)$
   (b) $\arccos(-\sqrt{2}/2)$
   (c) $\arctan(-1/\sqrt{3})$
   (d) $\arcsin\left(\sin\left(\frac{10\pi}{11}\right)\right)$
   Hint. The answer is not $10\pi/11$.
2. Find all solutions to the following trigonometric equations lying within the interval $[0, 2\pi]$. You may express your answer in terms of values of inverse trigonometric functions.
   (a) $3\sin 2\theta + 4 = 6$
   (b) $\tan(\theta + \pi) = -10$
3. Find the equation of the tangent line to $f(x) = \arccos x$ at $x = 1/2$.
4. Compute $\lim_{x\to 1^-} \frac{\arccos(x^2)}{\sqrt{1-x}}$
5. Compute $\int \frac{x+1}{\sqrt{1-(x+2)^2}}\,dx$.