Theory

Polynomial facts.

1. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ with $a_n \neq 0$. We call n the **degree** of f, denoted $\deg f$.

2. A polynomial of degree n has at most n distinct roots.

3. *Equating coefficients.* Given polynomials $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} \cdots + b_1 x + b_0$ we have

$$f(x) = g(x) \iff n = m \text{ and } a_i = b_i \text{ for all } i.$$

4. A nonzero polynomial is **irreducible** if it cannot be factored into two polynomials of smaller degree. If $f(x)$ is an irreducible polynomial with real coefficients, then $\deg f = 1$ or $\deg f = 2$.

Partial fraction decomposition. Let $f(x)/g(x)$ be a rational function (i.e, $f(x)$ and $g(x)$ are both polynomials), and suppose that $\boxed{\deg f < \deg g}$.

- If $g(x)$ factors into non-repeated linear factors as

$$g(x) = D(x - a_1)(x - a_2)\cdots(x - a_r),$$

then there is a unique choice of constants $A_1, A_2, \ldots, A_r$ such that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_r}{x - a_r}.$$

- If $g(x)$ factors into non-repeated irreducible linear and quadratic factors as

$$g(x) = D(x - a_1)(x - a_2)\cdots(x - a_r)(x^2 + b_1 x + c_1)(x^2 + b_2 x + c_2)\cdots(x^2 + b_s x + c_s),$$

there there is a unique choice of constants $A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_s, C_1, C_2, \ldots, C_s$ such that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_r}{x - a_r} + \frac{B_1 x + C_1}{x^2 + b_1 x + c_1} + \frac{B_2 x + C_2}{x^2 + b_2 x + c_2} + \cdots + \frac{B_s x + C_s}{x^2 + b_s x + c_s}.$$

Comment.

1. If $\deg f \geq \deg g$, then we can perform long polynomial division to write

$$f(x) = h(x) + \frac{r(x)}{g(x)},$$

where $h(x)$ is a polynomial and $\deg r(x) < \deg g(x)$, and then apply partial fraction decomposition to $r(x)/g(x)$.

2. There is a more general statement of partial fraction decomposition covering the case where $g(x)$ has repeated irreducible linear and quadratic factors, but we will not use it. See the text if you are interested.

Procedures

Partial fraction decomposition. Let $f(x)/g(x)$ be a quotient of polynomials, and suppose $\deg f < \deg g$. To compute the partial fraction decomposition of $f(x)/g(x)$ proceed as follows.

1. Factor $g(x)$ into powers of distinct irreducible polynomials.

   Factoring trick. If $g(x)$ has integer coefficients and a leading coefficient equal to 1, then any integer roots of $g(x)$ will be factors of the constant term.

2. Set up the partial fraction decomposition equation with as yet unknown constants (A_i, B_i, etc.). Clear the denominators of both sides of the equation, resulting in an identity between two polynomials. The polynomial on the right will be expressed in terms of the unknowns (A_i, B_i, etc.).

3. To solve for the undetermined constants (A_i, B_i, etc.) set up and solve a linear system of equations using one of the following techniques.

   (a) *Equate coefficients.* Express the polynomial on right in "standard form" by collecting like terms. For the left and right polynomials to be equal, their corresponding coefficients must all be equal. This yields a system of equations in the unknowns (A_i, B_i, etc.) that you must now solve.

   (b) *Evaluate equality at choices of x.* Evaluate the polynomial equation at various explicit choices of x. Each evaluation at a specific $x = c$ yields a new linear equation in the unknowns (A_i, B_i, etc.). Do this enough times so that your system of equations determines the unknowns uniquely. As far as possible, make judicious choices for x to make your algebra easier.

Examples

1. Compute $\displaystyle\int \frac{x+2}{x^2-1}\,dx$
2. Compute $\displaystyle\int \frac{1}{x^4+3x^2+2}$
3. Compute $\displaystyle\int \frac{x^2+1}{x^3+2x^2-x-2}\,dx$
4. Compute $\displaystyle\int \frac{x^2}{x^2+2x-1}\,dx$