Definitions

Improper integral of type I: infinite intervals. Below we define definite integrals over infinite intervals. These are called **improper integrals of type I**, or **integrals over infinite intervals**.

Half-infinite intervals

Definite integrals over intervals of the form $[a,\infty)$ or $(-\infty,a]$ are defined via the limit expressions below. When the relevant limit exists, we say the improper integral **converges** (or **exists**); otherwise we say the improper integral **diverges**.

- Let f be continuous on the interval $I=[a,\infty)$. We define the integral of f over I, denoted $\int_a^\infty f(x)\,dx$, as the following limit, assuming it exists:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx.$$

- Let f be continuous on the interval $I=(-\infty,a]$. We define the integral of f over I, denoted $\int_{-\infty}^a f(x)\,dx$, as the following limit, assuming it exists:

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty}\int_R^a f(x)\,dx.$$

Real line

Let f be continuous on the interval $I=(-\infty,\infty)$, and let a be an element of I. We say the integral of f over I **converges** (or **exists**) if *both* of the half-infinite integrals $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ converge, and define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

in this case. If *either* (or both) of the half-infinite integrals diverge, we say that the integral of f over $(-\infty,\infty)$ **diverges**.

Improper integrals of type II: discontinuities. Assume f is continuous on the interval $I=[a,b]$, except possibly at one point.

- Assume f is not continuous at $x=a$. We define the integral of f over $[a,b]$ as

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx,$$

assuming this limit exists.

- Assume f is not continuous at $x=b$. We define the integral of f over $[a,b]$ as

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx,$$

assuming this limit exists.

- Assume f is not continuous at $c\in(a,b)$. We define the integral of f over $[a,b]$ as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad (*)$$

assuming both improper integrals on the right side of $(*)$ exist.

Area interpretation of improper integrals. Let f be defined on an interval I for which the corresponding integral is improper, and let $\mathcal{R}$ be the (potentially unbounded) region between the graph of f and the x-axis over the interval I.

- We define the **area** (or **total area**) of $\mathcal{R}$ to be the integral of $|f|$ over I, assuming this integral converges.
- We define the **signed area** of $\mathcal{R}$ to be the integral of f over I, assuming this interval converges.

Theory

Direct comparison test. Let f and g be *nonnegative* functions on an interval I, and suppose $f(x) \leq g(x)$ for all x in I. If the integral of g over I converges, then the integral of f over I converges. Using logical notation:

$$\text{integral of } g \text{ over } I \text{ converges} \quad \Longrightarrow \quad \text{integral of } f \text{ over } I \text{ converges.}$$

Equivalently,

$$\text{integral of } f \text{ over } I \text{ diverges} \quad \Longrightarrow \quad \text{integral of } g \text{ over } I \text{ diverges.}$$

Limit comparison test. Let f and g be continuous and *positive* on the interval I.

- If $I = [a, \infty)$ and $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$, then
$$\int_a^\infty f(x)\,dx \text{ converges} \quad \Longleftrightarrow \quad \int_a^\infty g(x)\,dx \text{ converges.}$$
- If $I = (-\infty, a]$ and $\displaystyle\lim_{x\to-\infty} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$, then
$$\int_{-\infty}^a f(x)\,dx \text{ converges} \quad \Longleftrightarrow \quad \int_{-\infty}^a g(x)\,dx \text{ converges.}$$
- If $I = (a, b]$ and $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$, then
$$\int_a^b f(x)\,dx \text{ converges} \quad \Longleftrightarrow \quad \int_a^b g(x)\,dx \text{ converges.}$$
- If $I = [a, b)$ and $\displaystyle\lim_{x\to b^-} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$, then
$$\int_a^b f(x)\,dx \text{ converges} \quad \Longleftrightarrow \quad \int_a^b g(x)\,dx \text{ converges.}$$

Examples

1. Evaluate $\int_{-2}^{\infty} e^{-x}\, dx$.
2. Evaluate $\int_{0}^{\infty} xe^{-x}\, dx$.
3. Evaluate $\int_{1}^{\infty} x^r\, dx$ for $r \neq 0$.
4. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$.
5. Decide whether $\int_{2}^{\infty} \frac{1}{x^5+\sqrt{x+3}}\, dx$ converges.
6. Decide whether $\int_{1}^{\infty} \frac{2+\sin x}{x}\, dx$ converges.
7. Let $f(x) = ax^2 + bx + c$ be any fixed irreducible quadratic polynomial with $a > 0$. Decide whether $\int_{-\infty}^{\infty} \frac{1}{f(x)}\, dx$ exists.
8. Evaluate $\int_{0}^{2} \frac{1}{x-1}\, dx$.
9. Evaluate $\int_{0}^{1} \ln x\, dx$.
10. Evaluate $\int_{1}^{4} \frac{x}{\sqrt[3]{x^2-4}}\, dx$.
11. Decide whether $\int_{0}^{\infty} \frac{1}{\sqrt{x}+3x^5}\, dx$ converges.