Definitions

Indeterminate forms. Consider a limit expression of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where a is either a finite number or $\pm\infty$.

The expression is an **indeterminate form of type** $0/0$ if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

The expression is an **indeterminate form of type** ∞/∞ if

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Comment. A limit expression having an indeterminate form does *not* mean that the limit does not exist. You should interpret this conclusion as saying simply that our current analysis is not detailed enough to determine whether the limit exists and/or what that limit is.

In this spirit, we will be careful *not* to write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty},$$

as this suggests we are asserting something more definitive about the limit.

Further indeterminate forms. Assume a is either a finite number or $\pm\infty$.

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty$, then $\lim\limits_{x \to a} f(x) - g(x)$ is an **indeterminate form of type** $\infty - \infty$.

If $\lim\limits_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, then $\lim\limits_{x \to a} f(x)g(x)$ is an **indeterminate form of type** $0 \cdot \infty$.

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} f(x)^{g(x)}$ is an **indeterminate form of type** 0^0.

If $\lim\limits_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} f(x)^{g(x)}$ is an **indeterminate form of type** ∞^0.

If $\lim\limits_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, then $\lim\limits_{x \to a} f(x)^{g(x)}$ is an **indeterminate form of type** 1^∞.

Theory

L'Hôpital's rule. Let f and g be differentiable on an open interval I containing a, where a is either a finite number or $\pm\infty$, and suppose $g'(x) \neq 0$ for all $x \neq a$ in the interval.

If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $0/0$ or ∞/∞, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists or is equal to $\pm\infty$.
The same result holds if we replace the limit with a one-sided limit.

Comment. Students tend to fall madly in love with l'Hôpital's rule after seeing it for the first time. Some comments to temper your passion:

1. Make sure the necessary conditions hold: (a) f, g differentiable on an interval about a, $g(x) \neq 0$ on for $x \neq a$, and the limit expression is indeterminate of type $0/0$ or ∞/∞.

2. As magic as the rule appears, there are many examples where either the application of this rule does not help, and/or it is easier to use a different technique. Consider the following limits, for example:
$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \lim_{x\to\infty} \frac{x^4 - x^2 + 5x + 7}{2x^4 + x^3 + x^2 + x + 1}$$

Procedures

Examples

1. Decide whether the following limit expressions have determinate or indeterminate forms. If determinate, compute the limit.

   (a) $\displaystyle\lim_{x\to 0+} \frac{\sin x}{\ln x}$

   (b) $\displaystyle\lim_{x\to(\pi/2)^-} \frac{\tan x}{\cos x}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{e^x}{2^x + 3^x}$

2. Compute the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1000}}$

   (b) $\displaystyle\lim_{x\to 0} \frac{2^x - 3^{-x}}{4^x - 5^{-x}}$

   (c) $\displaystyle\lim_{x\to 1} \frac{\cos(\pi x/2)}{\log_2(x)}$

   (d) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x \sin x}$

3. Compute the following limits.

   (a) $\displaystyle\lim_{x\to 0^+} \frac{1}{\sin x} - \frac{1}{x}$

   (b) $\displaystyle\lim_{x\to\infty} 2x - \sqrt{4x^2 - 13x}$

   (c) $\displaystyle\lim_{x\to -\infty} x^2 2^x$

   (d) $\displaystyle\lim_{x\to 0^+} (1 + x)^{1/x}$

   (e) $\displaystyle\lim_{x\to\infty} (1 + x^2)^{2/x}$