Definitions

First-order linear equation. A **first-order linear differential equation** in the unknown $f(x)$ is a differential equation that can be written in the form

$$f'(x) + p(x)f(x) = q(x) \tag{$*$}$$

Equation $(*)$ is called the **standard form** of the equation.

Integrating factor. Consider a first-order linear equation in the unknown $f(x)$ with standard form

$$f'(x) + p(x)f(x) = q(x).$$

An **integrating factor** for this equation is any function of the form

$$v(x) = e^{P(x)},$$

where $P(x)$ is an antiderivative of $p(x)$. Using indefinite integral notation, we have

$$v(x) = e^{\int p(x)\,dx}.$$

Procedures

Solving first-order linear equations. Suppose p, q are continuous on the interval I. To solve the differential equation with standard form

$$f'(x) + p(x)f(x) = q(x),\ x \in I, \tag{$*$}$$

proceed as follows:

1. Compute an antiderivative $P(x)$ of $p(x)$.
2. Set $v(x) = e^{P(x)}$: i.e., $v(x) = e^{\int p(x)\,dx}$.
3. The function $f(x)$ is a solution of $(*)$ if and only if it is a solution of
$$(v(x)f(x))' = v(x)q(x).$$
4. Find an antiderivative $G(x)$ of $v(x)q(x)$. Then the general solution of $(*)$ is
$$f(x) = \frac{G(x)}{v(x)} + \frac{C}{v(x)},$$
where C is any constant. Using indefinite integral notation:
$$f(x) = \frac{1}{v(x)} \int v(x)q(x)\,dx.$$

Examples

1. Use the integrating factor method to find the general solution to $y' = ky$, where k is any fixed constant.

2. Consider the differential equation

$$(x-2)f' = e^{-x} - 3f, \;\; x \in (-\infty, 2).$$

   (a) Find the general solution to the differential equation.

   (b) Find the solution satisfying $f(1) = -1$.

3. Find the general solution to the differential equation

$$(x^2+1)f'(x) - x = x^3 - xf(x), \;\; x \in (-\infty, \infty).$$