AMS/MAA | SPECTRUM

An Introduction to Proof via Inquiry-Based Learning

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Introduction

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2

Mathematics and Logic

2.1A Taste of Number Theory

$$\boxed{\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}}.$$

$$\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}.$$

$0\mathbb{N}\boxed{:=}=:=\boxed{\mathbb{R}}$

$\boxed{\in} n \in \mathbb{Z} n \in \boxed{a \in A} AaA \boxed{a, b \in A} a \in Ab \in Aa \in An \in$

$nn = 2kk \in \mathbb{Z} nn = 2k + 1k \in \mathbb{Z}$

$0 = 2 \cdot 00 - 1 - 1 = 2(-1) + 1 - 1 = 2(-1/2) - 1 - 1/2$

nn^2

$n, m \in \mathbb{Z} nm \boxed{n|m} k \in \mathbb{Z} m = nkn|mmnnm$

$n, m \in \mathbb{Z}$

$n|m$

$\dfrac{m}{n}$

m/n

$\frac{m}{n}$

$a, b, n, m \in \mathbb{Z}$

$a|na|mn$

nnn

$abnanbn$

$a, n \in \mathbb{Z} anan^2$

$a, n \in \mathbb{Z} ana{-}n$

$a, n, m \in \mathbb{Z} amanam + n$

$a|na|mna|mna|n$
$a|mna|nn^2n$

$a, n, m \in \mathbb{Z}$

an^2an

$a—nan$

$am + naman$

$a, b, c \in \mathbb{Z}abbcac$

$a, n, m \in \mathbb{Z}amanam - n$

$n \in \mathbb{Z}nn^2 - 1$

2.2Introduction to Logic

$x = 1x$

$x^2 = 4$

$xx^2 = 4$

$xx^2 = 4$

$\sqrt{2}$

p

AB

$AA\boxed{\neg A}A$

$ABAB\boxed{A \wedge B}AB$

$ABAB\boxed{A \vee B}AB$

$ABABA\boxed{A \implies B}ABA \implies BABABBABA$

$ABABAB\boxed{A \iff B}A \iff BAB$

AB

$BABABABABAABA$

AB

$A \wedge B$

$A \vee B$

$\neg A$

$\neg B$

$\neg(A \wedge B)$

$\neg(A \vee B)$

$A \implies B$

$ABA \wedge B$

$A \quad B \quad A \wedge B$

$ABABA \wedge BAB$

$n2^n$

$\neg A$

$A \vee B$

$\neg(A \wedge B)$

$\neg A \wedge \neg B$

$A \implies B$

$A \implies BA \implies BABA \implies B$
$PQP \iff QPQPQPQPQPQPQ$

$A\neg(\neg A)A$

$AB\neg(A \wedge B)\neg A \vee \neg B$

$AB\neg(A \vee B)$

x

$x < -1x \geq 3$

$0 \leq x < 1$

$ABA \iff B(A \implies B) \wedge (B \implies A)$

$ABC(A \vee B) \implies C(A \implies C) \wedge (B \implies C)$

$ABA \implies BB \implies A$

$ABA \implies B\neg A \implies \neg B$

$ABA \implies B\neg B \implies \neg A$

AB

$A \implies B$

$A \implies B$

$ABA \implies B$

$AA \wedge BA \vee BABA \implies B$

$ABA \implies B\neg A \vee B$

$AB\neg(A \implies B)A \wedge \neg B$

$AB\sqrt{2}$

$A \implies B$

$\neg(A \implies B)$

$.\overline{99} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots .\overline{99} \neq 1$

$A\neg A \wedge A$

$\neg\wedge\vee$

2.3 Techniques for Proving Conditional Propositions

$n \in \mathbb{Z} n + (n+1)$

$A \implies B$ $A \implies B$

A

$$...B...$$

B

$A \implies B$ $A \implies B \neg B \implies \neg A$

$\neg B$

$$...\neg A...$$

$\neg A A B$

$\neg\wedge\vee\implies\iff$

$x \in \mathbb{Z} x^2 x$

x

x^2

$x = 2kx^2 = (2k)^2 = 4k^2$

$k2k^2$

$kx = 2k$

$x \in \mathbb{Z}$

x^2

$x^2 = 2(2k^2)$

$n \in \mathbb{Z}n^2n$

$n, m \in \mathbb{Z}nmnm$

$PA \implies B\neg PQ \wedge \neg QQ\neg PPQ$

P P

$\neg P$

$$\ldots$$
$$Q\neg Q\ldots$$

P

$A \implies B\neg B\neg(A \implies B)A \wedge \neg B$

$A \implies B$ $A \implies B$

$A\neg B$

$$\ldots$$
$$Q\neg Q\ldots$$

AB

$x \in \mathbb{Z} x x$

$\mathbb{N} \mathbb{Z}$

$x, y \in \mathbb{N} x y x \leq y$

$A \Longrightarrow B \neg B B \neg Q Q Q B$
$A \Longleftrightarrow B A \Longrightarrow B B \Longrightarrow A (\Longrightarrow)(\Longleftarrow)$

$n \in \mathbb{Z} n n^2$

2.4Introduction to Quantification

$x > 0 x x \boxed{P(x)} \boxed{Q(a, b)} x a, b P Q$

- $S(x) :=$ " $x^2 - 4 = 0$"
- $L(a, b) :=$ " $a < b$"
- $F(x, y) :=$ " $x y$"

$S(x) = x^2 - 4 = 0 S(x) S(x) x^2 - 4 = 0 L(a, b) L(b, a)$
$P(x) x_0 x P(x_0)$

$S(x) L(a, b) S(0) S(-2) L(2, 1) L(-3, -2) L(2, b)$

$x \in \mathbb{R} x^2 - 4 = 0$

$x \in \mathbb{R} x^2 - 4 = 0$

$xx > 0x \in \mathbb{Z}x \in \mathbb{N}x \in \mathbb{Z}x > 0x \in \mathbb{N}x > 0$

$P(x)xP(x)$

$Q(x)xQ(x)xQ(x)$

$P(x,y)xyUx \in Uy \in UP(x,y)$

$M(x,y) :=$" xy"$xyM(x,y)$

$xyM(x,y)$

$yxM(x,y)$

$xyM(x,y)$

$xyM(x,y)$

$F(x,y) :=$" $x = y^2$"

$xyF(x,y)$

$yxF(x,y)$

$yxF(x,y)$

-
- $x = ...x \in ...x$

$fcL\lim_{x\to c} f(x) = L$

$\varepsilon > 0\delta > 0x0 < |x-c| < \delta|f(x) - L| < \varepsilon$

$\boxed{\forall}\boxed{\exists}$

$x \in \mathbb{R} x^2 - 1 = 0 (\exists x \in \mathbb{R})(x^2 - 1 = 0) x \in \mathbb{N} y \in \mathbb{N} y < x$
$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$

$x x^2 + 1$

$n n^2 = 36$

$x x^2$

$A(x) \implies B(x) A(x) B(x) n n^2 n n^2 A(x) \implies B(x) (\forall x)(A(x) \implies B(x)) x (\forall x)(A(x) \implies B(x)) (\forall x \in U') B(x) U' U A(x) n n^2$

$A(x) \implies B(x)$

$\varepsilon > 0 N \in \mathbb{N} 1/N < \varepsilon \mathbb{R}$

$(\forall x)(\forall y) \boxed{\forall x, y} x y$

$x, y \in \mathbb{R} x < y m \in \mathbb{R} x < m < y$

$(\forall n \in \mathbb{N})(n^2 \geq 5)$

$(\exists n \in \mathbb{N})(n^2 - 1 = 0)$

$(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$

$(\forall m, n \in \mathbb{Z})((2|m \wedge 2|n) \implies 2|(m+n))$

$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$

$(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

$(\forall x)(\exists y)(xy = 1)$

x

2.5 More About Quantification

$(\exists x \in U)(x^2 - 4 = 0)(\exists x \in U)(x^2 - 2 = 0)U$

$(\forall x)P(x)(\forall x)(P(x) \implies Q(x))(\forall x)(\neg Q(x) \implies \neg P(x))P(x)$
$Q(x)P(x)Q(x)$

$\neg(\forall x)P(x)(\exists x)(\neg P(x))$

$P(x)$

$\neg(\forall x)P(x)(\exists x)(\neg P(x))$

$\neg(\exists x)P(x)(\forall x)(\neg P(x))$

$(\forall x)(x > 3)$

$(\exists x)(x \wedge x)$

$x \in \mathbb{N} x^2 + x + 41$

$x \in \mathbb{Z} 1/x \notin \mathbb{Z}$

fff

$$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0),$$

$$\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0),$$

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y \neq 0)$$

$$(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)].$$

$$\neg(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)]$$

$$(\exists x)[x > 0 \wedge \neg(\exists y)(y < 0 \wedge xy > 0)],$$

$$(\exists x)[x > 0 \wedge (\forall y)(y \geq 0 \vee xy \leq 0)].$$

$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m < n)$

$y \in \mathbb{R} x \in \mathbb{R} y = x^2$

$y \in \mathbb{R} y x \in \mathbb{R} y = x^2$

$x \in \mathbb{R} y \in \mathbb{R} y = x^2$

$x \in \mathbb{R} y \in \mathbb{R} y = x^2$

$y \in \mathbb{R} x \in \mathbb{R} y = x^2$

$(\forall x, y, z \in \mathbb{Z})((xy \wedge yz) \implies xz)$

$xyxy$

$xyxy$

$xyxy$

$(\forall x)P(x)$ $(\forall x)P(x)U$

$x \in U$

......

$P(x)xxP(x)$

$(\forall x)(A(x) \implies B(x))$ $(\forall x)(A(x) \implies B(x)U$

$x \in UA(x)$

$...B(x)...$

$B(x)$

$(\forall x)P(x)$ $(\forall x)P(x)U$

$x \in U\neg P(x)$

......

$xP(x)$

$(\exists x)P(x)$ $(\exists x)P(x)U$

$\cdots$

$...xP(x)x...$

$x \in UP(x)$

$(\exists x)P(x)$ $(\exists x)P(x)U$

$x \in U \neg P(x)$

......

$x \in U P(x)$

$Q(x)(\forall x)Q(x)(\exists x)(\neg Q(x))$
$(\forall x)(P(x) \implies Q(x) x P(x) \neg Q(x)$

$n \in \mathbb{N} n^2 \geq 5$

$n \in \mathbb{N} n^2 - 1 = 0$

$x \in \mathbb{N} y \in \mathbb{N} y \leq x$

$x \in \mathbb{Z} x^3 \geq x$

$n \in \mathbb{Z} m \in \mathbb{Z} n + m = 0$

$ab 2a + 7b = 1$

$mn 2m + 4n = 7$

$a, b, c \in \mathbb{Z} abcabac$

$a, b \in \mathbb{Z} abab$

$x, y \in \mathbb{Z} xyx + y$

$x, y \in \mathbb{Z} xyx + yk \in \mathbb{Z} x + y = 2k + 1(x + y) - 2k = 1x + y$

a

$n \in \mathbb{Z} 3n^2 + n + 14$

$n, m \in \mathbb{Z} nmnm$

$nmnmnmnm$

$n, m \in \mathbb{Z} nmnk \in \mathbb{Z} n = 2k$

$$nm = (2k)m = 2(km).$$

$kmkmnmn, m \in \mathbb{Z} nmnm$

$\boxed{\exists!}$

$(\exists! x)P(x)$ $(\exists! x)P(x)U$

...

$$...xP(x)x...$$

$x \in UP(x)x_1, x_2 \in UP(x_1)P(x_2)$

$$...x_1 = x_2...$$

$xP(x)$

$c, a, r \in \mathbb{R} c \neq 0 r \neq a/cx \in \mathbb{R}(ax + 1)/(cx) = r$

3

Set Theory

3.1Sets

$Ax A$ $\boxed{x \in A}$ $\boxed{x \notin A}$ $\boxed{\emptyset}$

$AA \notin A$

$$\boxed{S = \{x \in A \mid P(x)\}},$$

$P(x) x x \in A x x x A P(x) \{x \in \mathbb{N} \mid x x \geq 8\}$

- $\boxed{\mathbb{N} := \{1, 2, 3, ...\}}$
- $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}}$
- $\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} b \neq 0\}}$
- $\boxed{\mathbb{R}}$

$\boxed{\mathbb{Z}^+}$

$A = \{x \in \mathbb{N} \mid x = 3k k \in \mathbb{N}\}$

$B = \{t \in \mathbb{R} \mid t \leq 2 t \geq 7\}$

$C = \{t \in \mathbb{Z} \mid t^2 \leq 2\}$

$D = \{s \in \mathbb{Z} \mid -3 < s \leq 5\}$

$E = \{m \in \mathbb{R} \mid m = 1 - \frac{1}{n} n \in \mathbb{N}\}$

$-\sqrt{2}$

-12

$a, b \in \mathbb{R} a < b$

$\boxed{(a, b) := \{x \in \mathbb{R} \mid a < x < b\}}$

$\boxed{[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}}$

$\boxed{[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}}$

$\boxed{(a, \infty) := \{x \in \mathbb{R} \mid a < x\}}$

$\boxed{(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}}$

$\boxed{(-\infty, \infty) := \mathbb{R}}$

$\boxed{(a, b]}\boxed{[a, \infty)}\boxed{(-\infty, b]}(a, b)(-\infty, b)(a, \infty)(-\infty, \infty)[a, b](a, b)[a, b)(a, b]$
$[a, b]ab$

$(a, b), [a, b], (a, b][a, b)a < b$

$AB AB \boxed{A \subseteq B} AB$

$A = \{1, 2, 3\}$

$A = \emptyset$

A

$A \subseteq A$

$\emptyset \subseteq A$

$$A \subseteq B x x \in A x \in B A \subseteq B A x x \in A x \in B A A B$$

$ABA \subseteq B$

$ABCA \subseteq BB \subseteq CA \subseteq C$

AB $\boxed{A = B}$

$$A = BA \subseteq BB \subseteq AA \subseteq BB \subseteq AA = B$$

$ABA = BA \subseteq BB \subseteq A$

$$A = BA \subseteq BB \subseteq A(\subseteq)(\supseteq)$$

$A \subseteq BAA \neq B$ $\boxed{A \subset B}$ $\boxed{A \subsetneq B}$

$$\subset \subseteq$$

ABU

AB $\boxed{A \cup B := \{x \in U \mid x \in Ax \in B\}}$

AB $\boxed{A \cap B := \{x \in U \mid x \in Ax \in B\}}$

AB $\boxed{A \quad B := \{x \in U \mid x \in Ax \notin B\}}$

AU $\boxed{A^c := U \quad A = \{x \in U \mid x \notin A\}}$

$ABA \cap B = \emptyset AB$

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} A = \{1, 2, 3, 4, 5\} B = \{1, 3, 5\} C = \{2, 4, 6, 8\}$

$A \cap C$	$A \quad B$
$B \cap C$	$B \quad A$
$A \cup B$	$C \quad B$

B^c

$(A \cup B)^c$

A^c

$A^c \cap B^c$

$U = \mathbb{R} A = [-3, -1) B = (-2.5, 2) C = (-2, 0]$

A^c

$(A \cup B)^c$

$A \cap C$

$A \quad B$

$A \cap B$

$A \cup B$

$A \quad (B \cup C)$

$(A \cap B)^c$

$B \quad A$

$U = \{x, y, z, \{y\}, \{x, z\}\} S = \{x, y, z\} T = \{x, \{y\}\}$

$S \cap T$

$(S \cup T)^c$

$T \quad S$

$ABA \subseteq BB^c \subseteq A^c$

$ABA \quad B = A \cap B^c$

AB

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

ABC

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \not\subseteq B$

$A \cap B = \emptyset$

$(A \cup B)^c \neq \emptyset$

$(A \cap B)^c = \emptyset$

$(\forall x)(x \in A \wedge x \in B)$

$(\forall x)(x \in A \implies x \notin B)$

$(\exists x)(x \notin A \wedge x \notin B)$

$(\exists x)(x \in A \vee x \in B)$

$(\exists x)(x \in A \wedge x \notin B)$

3.2 Russell's Paradox

$\mathcal{U}\mathcal{U} := \{A \mid A\}$

$\mathcal{U}\mathcal{U}$

XY

YXY

3.3 Power Sets

$SSSS$ $\boxed{\mathcal{P}(S)}$

$SSSS$

$S = \{a, b\}\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, S\}A \subseteq SA \in \mathcal{P}(S)$

$A = \{\circ, \triangle, \square\}$

$B = \{a, \{a\}\}$

$C = \emptyset$

$D = \{\emptyset\}$

$nn = 0$

$A = \{x, y\} x A \{x\} A \mathcal{P}(A) B \{a\} B B \mathcal{P}(B) \subseteq \in$
$S \subseteq T \mathcal{P}(S) \subseteq \mathcal{P}(T) \mathcal{P}(S) \subseteq \mathcal{P}(T) S \subseteq T$

$STS \subseteq T \mathcal{P}(S) \subseteq \mathcal{P}(T)$

ST

$\mathcal{P}(S \cap T) \subseteq \mathcal{P}(S) \cap \mathcal{P}(T)$

$\mathcal{P}(S) \cap \mathcal{P}(T) \subseteq \mathcal{P}(S \cap T)$

$\mathcal{P}(S \cup T) \subseteq \mathcal{P}(S) \cup \mathcal{P}(T)$

$\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$

3.4Indexing Sets

$$(0, 1), (0, 1/2), (0, 1/4), \dots, (0, 1/2^{n-1}), \dots$$

$$I_1 = (0, 1), I_2 = (0, 1/2), \dots, I_n = (0, 1/2^{n-1}), \dots$$

$\mathbb{N} I_n$

$$\{a\}, \{a, b\}, \{a, b, c\}, \dots, \{a, b, c, \dots, z\}$$

$$A_1 = \{a\}, A_2 = \{a, b\}, A_3 = \{a, b, c\}, \dots, A_{26} = \{a, b, c, \dots, z\}$$
$\{1, 2, \dots, 26\}$

$\mathbb{R}$

- $\Delta \Delta \{S_\alpha\}_{\alpha \in \Delta} S$
- $\mathbb{N} \{U_n\}_{n \in \mathbb{N}} \{U_n\}_{n=1}^{\infty}$
- $\{A_1, \dots, A_{26}\} \{A_n\}_{n=1}^{26}$

$\{A_\alpha\}_{\alpha \in \Delta}$

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha := \{x \mid x \in A_\alpha \alpha \in \Delta\}}.$$

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha := \{x \mid x \in A_\alpha \alpha \in \Delta\}}.$$

$\Delta = \mathbb{N}$

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n n \in \mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

$\Delta = \{1, 2, 3, 4\}$

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

$\bigcup \cup \bigcap \cap$

$\{I_n\}_{n\in\mathbb{N}}$

$\bigcup_{n\in\mathbb{N}} I_n$ $\bigcap_{n\in\mathbb{N}} I_n$

$\{A_n\}_{n=1}^{26}$

$\bigcup_{n=1}^{26} A_n$ $\bigcap_{n=1}^{26} A_n$

$S_n = \{x \in \mathbb{R} \mid n - 1 < x < n\} n \in \mathbb{N}$

$\bigcup_{n=1}^{\infty} S_n$ $\bigcap_{n=1}^{\infty} S_n$

$T_n = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n}\} n \in \mathbb{N}$

$\bigcup_{n=1}^{\infty} T_n$ $\bigcap_{n=1}^{\infty} T_n$

$r \in \mathbb{Q} N_r r$

$$\bigcup_{r\in\mathbb{Q}} N_r \qquad \bigcap_{r\in\mathbb{Q}} N_r$$

$\{A_\alpha\}_{\alpha\in\Delta} A_\alpha \cap A_\beta = \emptyset \alpha \neq \beta$

$\{A_\alpha\}_{\alpha\in\Delta} \bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$

$\mathbb{R}\mathbb{R}$

$\mathbb{R}\mathbb{R}$

$\mathbb{R}\mathbb{R}$

$\{A_\alpha\}_{\alpha\in\Delta} B$

$$B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$$

$$B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$$

$\{A_\alpha\}_{\alpha\in\Delta}$

$$\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$$

$$\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$$

$\{A_\alpha\}_{\alpha\in\Delta} \{a_\alpha\}_{\alpha\in\Delta} a_\alpha \in A_\alpha \alpha \in \Delta$

$A_\alpha a_\alpha A_\alpha$

3.5Cartesian Products of Sets

$n \in \mathbb{N} n n$ $\boxed{(a_1, a_2, \dots, a_n)}$ $a_i i (a_1, a_2, \dots, a_n) n (a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n)$
$a_i = b_i 1 \leq i \leq n 2 (a, b) 3 (a, b, c)$

$n [] \langle\rangle \{\}$
n

$A B A B A \times B A B A B A B$

$$\boxed{A \times B := \{(a, b) \mid a \in A, b \in B\}}.$$

$n A_1, \dots, A_n$

$$\boxed{\prod_{i=1}^{n} A_i := A_1 \times \cdots \times A_n := \{(a_1, \dots, a_n) \mid a_j \in A_j 1 \leq j \leq n\}},$$

$A_i i$

$$\underbrace{A \times \cdots \times A}_{n}$$

A^n

$A = \{a, b, c\} B = \{☺, ☹\}$

$$A \times B = \{(a, ☺), (a, ☹), (b, ☺), (b, ☹), (c, ☺), (c, ☹)\}.$$

$\mathbb{R}^2 \mathbb{R}^3$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

$A B$

$B \times A$

$B \times B$

$A B A \times B$

$A B A \times B$

$A = \{1, 2, 3\} B = \{1, 2\} C = \{1, 3\} A \times B \times C$

$X = [0, 1] Y = \{1\}$

$X \times Y$

$Y \times X$

$X \times X$

$Y \times Y$

$AA \times \emptyset$

$ABA \times BB \times A$

$\mathbb{N} \times \mathbb{R}\mathbb{R}^2$

$ABCDA \subseteq CB \subseteq DA \times B \subseteq C \times D$

$A \times B \subseteq C \times DA \subseteq CB \subseteq D$

$C \times DA \times BA \subseteq CB \subseteq D$

$ABCA \times BA \times B \times C$

$A = [2,5]B = [3,7]C = [1,3]D = [2,4]$

$(A \cap B) \times (C \cap D)$

$(A \times C) \cap (B \times D)$

$(A \cup B) \times (C \cup D)$

$(A \times C) \cup (B \times D)$

$A \times (B \cap C)$

$(A \times B) \cap (A \times C)$

$A \times (B \cup C)$

$(A \times B) \cup (A \times C)$

$ABCD$

$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

$(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \quad C) = (A \times B) \quad (A \times C)$$

$AB(A \times B)^C A^C B^C$

4

Induction

$(\forall n \in \mathbb{N}) P(n) (\forall n \in \mathbb{Z})(n \geq a \implies P(n)) P(n) a \in \mathbb{Z}$

4.1Introduction to Induction

$n \in \mathbb{N} 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

$n \in \mathbb{N} n^2 + n + 41$

$n = 1 1 = \frac{1(1+1)}{2} n = 2 1 + 2 = 3 = \frac{2(2+1)}{2} n = 3 1 + 2 + 3 = 6 = \frac{3(3+1)}{2}$

$n = 1 n^2 + n + 41 = 43 n = 2 n^2 + n + 41 = 47 n = 3 n^2 + n + 41 = 53$

$$n = 41 n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1)$$

$S \subseteq \mathbb{N}$

$1 \in S$

$k \in S k + 1 \in S$

$S = \mathbb{N}$

$Sk(k+1)\mathbb{N}$
$S = \{k \in \mathbb{N} \mid P(k)\} S\mathbb{N}$

$P(1), P(2), P(3), \ldots$

$P(1)$

$P(k)P(k+1)$

$P(n)n \in \mathbb{N}$

$n \in \mathbb{N}P(n)P(n)n$

$(\forall n \in \mathbb{N})P(n)$

$P(1)n = 1$

$k \in \mathbb{N}P(k)P(k+1)k \in \mathbb{N}P(k)P(k+1)P(k+1)$

$P(n)n \in \mathbb{N}$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$n \in \mathbb{N}\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

$n \in \mathbb{N}4^n - 1$

$n \in \mathbb{N}n^3 - n$

$p_1, p_2, \ldots, p_n n \frac{n^2-n}{2}$

$2^n 2^n n \in \mathbb{N}3n \in \mathbb{N}n = 2$

4.2More on Induction

$(\forall n \in \mathbb{N})P(n)$

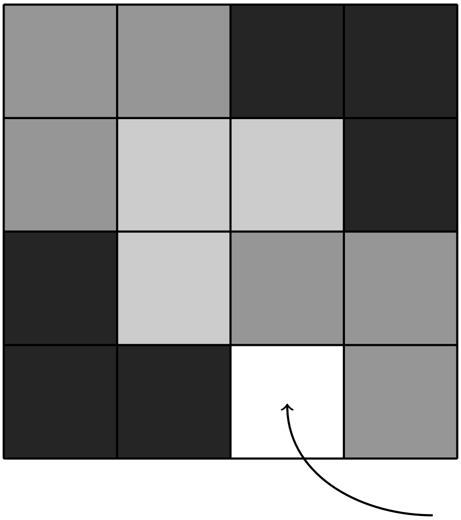

Figure 4.1$n = 2$

$$(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$$

$a \in \mathbb{Z} a = 1 S = \{k \in \mathbb{N} \mid P(a+k-1)\}$

$P(a), P(a+1), P(a+2), ... a$

$P(a)$

$P(k) P(k+1)$

$P(n) n \geq a$

$n \geq a P(n)$

$(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$ $a = 1$

$P(a) n = a$

$k \in \mathbb{Z} P(k) P(k+1) k \geq a P(k) P(k+1) P(k+1)$

$P(n) n \geq a$

$A n \mathcal{P}(A) 2^n$

$n \geq 0n < 2^n$

$n \geq 049^n - 5$

$n \geq 046 \cdot 7^n - 2 \cdot 3^n$

$n \geq 22^n > n + 1$

$n \geq 01 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

$r \neq 1n \geq 0$

$$1 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

$n \geq 32 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \dfrac{(n-2)(n^2 + 2n + 3)}{3}$

$n \geq 1\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

$n \geq 1\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

$n \geq 03^{2n} - 18$

$n \geq 22^n < (n+1)!$

$n \geq 22 \cdot 9^n - 10 \cdot 3^n 4$

n

$R(n)nR(n)$

$U(n)nU(n)$

$B(n)nB(n)$

4.3Complete Induction

$P(1), P(2), P(3), \ldots$

$P(1)$

$k \in \mathbb{N} P(j) j \in \mathbb{N} j \leq k P(k+1)$

$P(n) n \in \mathbb{N}$

$P(k) P(j) jk$
$P(k+1) P(k) P(k-1) P(k) P(k+1) P(1), P(2), \ldots, P(k)$

$(\forall n \in \mathbb{N}) P(n)$

$P(1) P(k) k$

$k \in \mathbb{N} k \in \mathbb{N} P(j) j \in \mathbb{N} j \leq k P(k+1) k \in \mathbb{N} P(j) j \leq k P(k+1)$
$P(k+1)$

$P(n) n \geq a$

$a_1 = 1 a_2 = 3 a_n = 3a_{n-1} - 2a_{n-2} n \geq 3 a_n = 2^n - 1 n \in \mathbb{N}$

$a_1 = 3, a_2 = 5, a_3 = 9 a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} n \geq 4 a_n = 2^n + 1$
$n \in \mathbb{N}$

$f_1 = 1 f_2 = 1 f_n = f_{n-1} + f_{n-2} n \geq 3 \left(\frac{3}{2}\right)^{n-2} \leq f_n \leq 2^n n \in \mathbb{N}$

$P(1)$

1245

$n \geq 44n$

$2nn12$

$nn011101011011$

$011101 \to 11101$

$011011 \to 010011$

$n \in \mathbb{N} n$

$n f_{n+2}$

4.4 The Well-Ordering Principle

$A \subseteq \mathbb{R} m \in A m A a \in A a \leq m m A a \in A m \leq a$

$A \subseteq \mathbb{R} A A$

$A \boxed{\max(A)} A \boxed{\min(A)}$

$\{5, 11, 17, 42, 103\}$

$\mathbb{N}$

$\mathbb{Z}$

$(0, 1]$

$(0, 1] \cap \mathbb{Q}$

$(0, \infty)$

$\{42\}$

$\{\frac{1}{n} \mid n \in \mathbb{N}\}$

$\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

$\emptyset$

$S \mathbb{N} P(n) := n S$

$A\ell \in \mathbb{Z}\ell \leq aa \in AA$

$Au \in \mathbb{Z}a \leq ua \in AA$

ℓAuA

5

The Real Numbers

5.1Axioms of the Real Numbers

$+ \cdot \mathbb{R}$

$a, b, c \in \mathbb{R}(a+b)+c=a+(b+c)$

$a, b \in \mathbb{R}a+b=b+a$

$0 \in \mathbb{R}a \in \mathbb{R}0+a=a$

$a \in \mathbb{R}-a \in \mathbb{R}a+(-a)=0$

$a, b, c \in \mathbb{R}(ab)c=a(bc)$

$a, b \in \mathbb{R}ab=ba$

$1 \in \mathbb{R}1 \neq 0a \in \mathbb{R}1a=a$

$a \in \mathbb{R} \quad \{0\} a^{-1} \in \mathbb{R} a a^{-1} = 1$

$a, b, c \in \mathbb{R} a(b + c) = ab + ac$

$\mathbb{R}\mathbb{R} \quad \{0\} 0 1 \mathbb{R}\mathbb{R} 0 0' \mathbb{R} 0 = 0'$

$\mathbb{R}$

$\mathbb{R}$

$a \in \mathbb{R} - a a^{-1} a \neq 0$

$\mathbb{R}$

$\mathbb{N}\mathbb{R}$

$1 \in \mathbb{N}$

$n \in \mathbb{N} n + 1 \in \mathbb{N}$

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\}$
$\mathbb{Z}$

$a, b \in \mathbb{R} n \in \mathbb{Z}$

$$\boxed{a - b := a + (-b)}$$

$$\boxed{\frac{a}{b} := ab^{-1}} b \neq 0$$

$$\boxed{a^n := \begin{cases} \overbrace{aa \cdots a}^{n}, n \in \mathbb{N} \\ 1, \qquad n = 0 a \neq 0 \\ \dfrac{1}{a^{-n}}, \quad -n \in \mathbb{N} a \neq 0 \end{cases}}$$

$\mathbb{Q}\mathbb{R} \quad \mathbb{Q}$

$a, b, c \in \mathbb{R}$

$a = ba + c = b + c$

$0a = 0$

$-a = (-1)a$

$(-1)^2 = 1$

$-(-a) = a$

$a \neq 0(a^{-1})^{-1} = a$

$a \neq 0ab = acb = c$

$ab = 0a = 0b = 0$

$a, b \in \mathbb{R}(a + b)(a - b) = a^2 - b^2$

$a, b, c \in \mathbb{R}\boxed{<}\mathbb{R}$

$a \neq ba < bb < a$

$a < bb < ca < c$

$a < ba + c < b + c$

$a < b0 < cac < bc$

$<>\leq\geq$

$a, b \in \mathbb{R}$

$\boxed{a > b}b < a$

$\boxed{a \leq b}a < ba = b$

$\boxed{a \geq b}b \leq a$

$<>\leq\geq$

$a, b \in \mathbb{R}a, b > 0a + b > 0a, b < 0a + b < 0$

$a, b, c, d \in \mathbb{R}a < bc < da + c < b + d$

$a \in \mathbb{R}a > 0-a < 0$

$abcda < bc < dac < bd$

$a, b \in \mathbb{R}$

$ab > 0a, b > 0a, b < 0$

$ab < 0a < 0 < bb < 0 < a$

$aba < ba^2 < b^2$

$a \in \mathbb{R}a^2 \geq 0$

$0 < 1$

$-1 < 0n \in \mathbb{Z}n < n + 1nn + 1$

$a \in \mathbb{R}a > 0a^{-1} > 0a < 0a^{-1} < 0$

$a, b \in \mathbb{R}a < b-b < -a$

$a, b, c \in \mathbb{R}a < bc < 0bc < ac$

$a \in \mathbb{R}|a|$

$$\boxed{|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}}$$

$a \in \mathbb{R}|a| \geq 0a = 0$

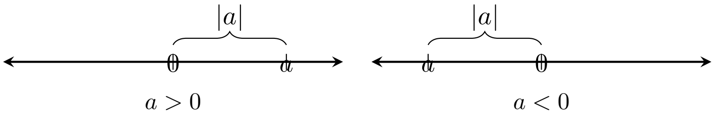

Figure 5.1$|a|$

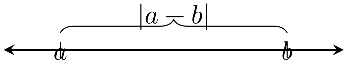

Figure 5.2$|a - b|$

$|a|a$

$a, b \in \mathbb{R}|a - b| = |b - a|$

$ab|a - b||b - a|ab$

$a, b \in \mathbb{R}|ab| = |a||b|$

$\pm a \leq ba \leq b{-}a \leq b$

$a, b \in \mathbb{R}\pm a \leq b|a| \leq b$

$a \in \mathbb{R}|a|^2 = a^2$

$a \in \mathbb{R}\pm a \leq |a|$

$a, r \in \mathbb{R}r|a| \leq r{-}r \leq a \leq r$

$rr(-r, r)r$

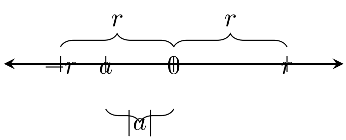

Figure 5.3$|a| \leq r$

$a, b, r \in \mathbb{R}r|a - b| \leq rb - r \leq a \leq b + r$

$|a - b|ab|a - b| \leq rabrarb$

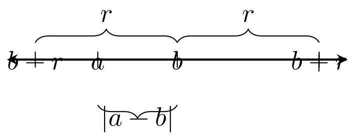

Figure 5.4$|a-b| \leq r$

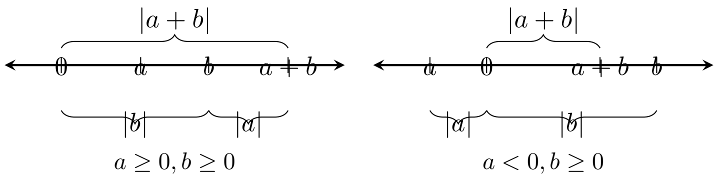

Figure 5.5

$a, b \in \mathbb{R}|a+b| \leq |a|+|b|$

$xyzz \leq x+y\mathbf{a}\mathbf{b}\mathbb{R}^n\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\|+\|\mathbf{b}\|\|\mathbf{a}\|\mathbf{a}$

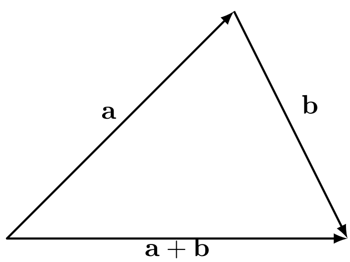

Figure 5.6

$a, b \in \mathbb{R}|a-b| \geq ||a|-|b||$

$A \subseteq \mathbb{R}bAa \in Aa \leq bA$

$\{5, 11, 17, 42, 103\}$

$\mathbb{N}$

$\mathbb{Z}$

$(0, 1]$

$(0, 1] \cap \mathbb{Q}$

$(0, \infty)$

$\{42\}$

$\{\frac{1}{n} \mid n \in \mathbb{N}\}$

$\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

$\emptyset$

$A \subseteq \mathbb{R} A$

$A \subseteq \mathbb{R}$

$A \subseteq \mathbb{R} p A p A p \leq b b A p A p A p \geq b b A$

$A \subseteq \mathbb{R} A A$

$A \boxed{\sup(A)} A \boxed{\inf(A)}$

$A \subseteq \mathbb{R} A A \sup(A) \in A \max(A) = \sup(A)$

$A A$

$A \subseteq \mathbb{R} A b A b A \varepsilon > 0 a \in A b - \varepsilon < a$

$A\mathbb{R}\sup(A)$

$A\mathbb{R}\inf(A)$

$x \in \mathbb{R}n \in \mathbb{N}x < n$

$x \in \mathbb{R}k, n \in \mathbb{Z}k < x < n$

$xN \in \mathbb{N}0 < \frac{1}{N} < x$

$x \in \mathbb{R}L = \{k \in \mathbb{Z} \mid k \leq x\}L$

$x \in \mathbb{R}n \in \mathbb{Z}n \leq x < n + 1$

$a < bb - aN \in \mathbb{N}\frac{1}{N} < b - aNan \in \mathbb{N}n \leq Na < n + 1\frac{n+1}{N}$

$(a, b)pp \in (a, b)$

$\pi\sqrt{2}\sqrt{2}\sqrt{2} \approx 1.41421356237 \in (1, 2)$

$(a, b)pp \in (a, b)$

5.2Standard Topology of the Real Line

$Ux \in U(a, b)x(a, b) \subseteq U$

$(1,2)$	$\{\frac{1}{n} \mid n \in \mathbb{N}\}$
$(1,\infty)$	$\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$
$(1,2) \cup (\pi,5)$	$\mathbb{R}$
$[1,2]$	$\mathbb{Q}$
$(-\infty,\sqrt{2}]$	$\mathbb{Z}$
$\{4,17,42\}$	$\emptyset$

$(a,b)(-\infty,b)(a,\infty)(-\infty,\infty)$

UV

$U \cup V$

$U \cap V$

$\{U_\alpha\}_{\alpha\in\Delta}\bigcup_{\alpha\in\Delta} U_\alpha$

$\{U_i\}_{i=1}^{n} n \in \mathbb{N}\bigcap_{i=1}^{n} U_i$

$\{U_\alpha\}_{\alpha\in\Delta}\bigcap_{\alpha\in\Delta} U_\alpha$

$\{U_\alpha\}_{\alpha\in\Delta}\bigcap_{\alpha\in\Delta} U_\alpha$

$A \subseteq \mathbb{R} p \in \mathbb{R} A(a,b) pq \in (a,b) \cap A q \neq p$

$pApApAAppA$

$I = (1, 2)$

$12I$

$p \in IpI$

$p < 1p > 2pI$

$p(a, b)(a, b][a, b)[a, b]p \in [a, b]$

$p = 0A = \{\frac{1}{n} \mid n \in \mathbb{N}\}A$

A

$p \in \mathbb{R}p\mathbb{Q}$

$A \subseteq \mathbb{R}A$

$\mathbb{R}\emptyset(-\infty, \infty)$

$[a, b]$

$[a, b](-\infty, b][a, \infty)(-\infty, \infty)$

$\mathbb{R}$

$U \subseteq \mathbb{R}UU^C$

AB

$A \cup B$

$A \cap B$

$\{A_\alpha\}_{\alpha\in\Delta}\bigcap_{\alpha\in\Delta} A_\alpha$

$\{A_i\}_{i=1}^n n \in \mathbb{N}\bigcup_{i=1}^n A_i$

$\{A_\alpha\}_{\alpha\in\Delta}\bigcup_{\alpha\in\Delta} A_\alpha$

$$V = \bigcup_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$$

$$W = \bigcap_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$$

$$X = \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

$$Y = \bigcap_{n=1}^{\infty}(-n, n)$$

$Z = (0,1) \cap \mathbb{Q}$

$\mathbb{R}$

$K \subseteq \mathbb{R}K$

$[0,1) \cup [2,3]$	$\mathbb{R}\quad\mathbb{Q}$
$[0,1) \cup (1,2]$	$\mathbb{Z}$
$[0,1) \cup [1,2]$	$\{\frac{1}{n} \mid n \in \mathbb{N}\}$
$\mathbb{R}$	$[0,1] \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$
$\mathbb{Q}$	$\{17, 42\}$

$\{17\}$ $\emptyset$

$K \mathbb{R} \sup(K), \inf(K) \in K$

$A \subseteq \mathbb{R} U_1 U_2 A \cap U_1 A \cap U_2 A \subseteq U_1 \cup U_2 A = (A \cap U_1) \cup (A \cap U_2)$

$a \in \mathbb{R}\{a\}$

$[a, b]$

$\mathbb{R}$

6

Three Famous Theorems

$\sqrt{2}$

6.1The Fundamental Theorem of Arithmetic

$2^2 \cdot 32^2 \cdot 32 \cdot 3 \cdot 23 \cdot 2^2 12 = 2 \cdot 612 = 3 \cdot 4$

$n \in \mathbb{Z}$

$a \in \mathbb{Z} anan$

$n \in \mathbb{N} nnn$

$n > 1nn$

$n \in \mathbb{N} 4^n - 1n$

$n \in \mathbb{N} n^2 - n + 11$

$SS \neq \emptyset Snnnabn = abnabn$

nn

$$n = p_1 p_2 \cdots p_k,$$

$p_1, p_2, \dots, p_k$

$n, d \in \mathbb{Z} d > 0 q, r \in \mathbb{Z} n = dq + r 0 \leq r < d$

$n, d \in \mathbb{Z} d > 0 n > 0 qr$

$d = 1 q = n r = 0 n = 1 \cdot n + 0 = dq + r d > 1$

$$S := \{n - dk \mid k \in \mathbb{Z} n - dk \geq 0\}.$$

$S \neq \emptyset S r$

$n \geq 0 k = 0 n - dk = n - d \cdot 0 = n \geq 0 n \in S$

$n < 0 k = n n - dk = n - dn = n(1-d) n < 0 d > 1 n(1-d) > 0$
$n - dn \in S$

$S \neq \emptyset S r = n - dq q \in \mathbb{Z} n = dq + r r \geq 0 r \geq d r' \in \mathbb{Z} r = d + r'$
$0 \leq r' < r$

$$n = dq + r = dq + d + r' = d(q+1) + r'.$$

$r' = n - d(q+1) 0 \leq r' < r S r r S r < d$

$q r q_1, q_2, r_1, r_2 \in \mathbb{Z} n = dq_1 + r_1 n = dq_2 + r_2 0 \leq r_1, r_2 < d$
$r_2 \geq r_1 0 \leq r_2 - r_1 < d dq_1 + r_1 = dq_2 + r_2 r_2 - r_1 = d(q_1 - q_2) d$
$r_2 - r_1 r_2 - r_1 > 0 r_2 - r_1 \geq d 0 \leq r_2 - r_1 < d r_2 - r_1 = 0 r_1 = r_2$
$q_1 = q_2 q r$

$n d q r d 0 \leq r < n 0 \leq r < |n|$

$n d n d r d q n d n d r - q d q$

$n = 27 d = 5 q, r \in \mathbb{Z} 0 \leq r < n n = dq + r$

$q r n$

$n = -26 d = 3 q, r \in \mathbb{Z} 0 \leq r < n n = dq + r$

$m, n \in \mathbb{Z} m n m n \boxed{\gcd(m, n)} m n \gcd(m, n) = 1 m n$

$\gcd(54, 72)$

$S := \{ps + at > 0 \mid s, t \in \mathbb{Z}\} p \in S s = 1 t = 0 S S d s_1, t_1 \in \mathbb{Z} d = ps_1 + at_1 d = 1 m \in S s_2, t_2 \in \mathbb{Z} m = ps_2 + at_2 d d \leq m q, r \in \mathbb{N} \cup \{0\}$ $m = qd + r 0 \leq r < d r m d ps_1 + at_1 ps_2 + at_2 r p, a, s_1, s_2, t_1 t_2 r p a p a d$ $r m d S d p \in S p p a d$

$p, a \in \mathbb{Z} p p a s, t \in \mathbb{Z} ps + at = 1$

$s t s, t \in \mathbb{Z} 2s + 7t = 1$

$p a p a p a p a b p b$

$p p a b a, b \in \mathbb{N} p a p b$

p

$a, b, d d a b d a d b$

$n n p_1 p_2 \cdots p_k q_1 q_2 \cdots q_l n k = l p_i q_j$

6.2 The Irrationality of $\sqrt{2}$

$\sqrt{2}$

$r \in \mathbb{R}$

$r r = \frac{m}{n} m, n \in \mathbb{Z} n \neq 0$

r

$a a \sqrt{2} \sqrt{2}$
$\sqrt{2} m, n \in \mathbb{Z} n \neq 0 \sqrt{2} = \frac{m}{n}$

$\sqrt{2}$

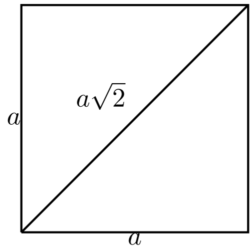

Figure 6.1

$p\sqrt{p}$

$pq\sqrt{pq}$

π

thelearnerlab.com

6.3 The Infinitude of Primes

11

$m1m \geq 1m1k \in \mathbb{N}1 = mkk \geq 1mk \geq m1 = mk1 \geq m1 \leq m \leq 1$
$m = 1$

$pn \in \mathbb{Z}pnpn + 1$

$p_1, p_2, \dots, p_k$

$np_1, \ldots, p_n abn = 5\{2,7\}\{3,5,11\}a = 14b = 165a + ba - b$

7

Relations and Partitions

7.1Relations

$ABA \times B(a,b)a \in Ab \in BA \times B = \{(a,b) \mid a \in A, b \in B\}$

$ABRABA \times BRAB(a,b) \in Rab\boxed{aRb}(a,b) \in RRAARA$

$\mathbb{N} \times \mathbb{R}\mathbb{R}\mathbb{N} \times \mathbb{R}\mathbb{R} \times \mathbb{R}$

$RABaRbbRa$

$A = \{a,b,c,d,e\}B = \{1,2,3,4\}$

$$R = \{(a,1),(a,2),(a,4),(c,2),(d,2),(e,2),(e,4)\}$$

$AB(c,2) \in RcR2a$

$A = \{a,b,c,d,e\}A$

$$R = \{(a,a),(a,b),(a,c),(b,b),(b,a),(b,c),(c,d),(c,e),(d,d),(d,a),(d,c),(e,a)\}.$$

$AATAxTyxy$

$$T = \{(x,y) \in A \times A \mid xy\}.$$

$= \leq < (3,\pi) \leq < 3 \leq \pi 3 < \pi(3,\pi) = 3 \neq \pi \leq < = (-\sqrt{2},4) \leq (4,-\sqrt{2})$

$S\{-1,1\}\mathbb{Z}1Sxx-1Sxx1-1$

$A\emptyset \subseteq A \times AAA$

$RABABa \in Ab \in B(a,b)RaRb$

AB

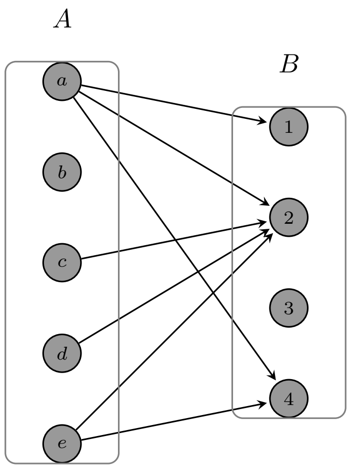

Figure 7.1$A = \{a,b,c,d,e\}B = \{1,2,3,4\}$

$A = \{1,2,3,4,5,6\}B = \{1,2,3,4\}DAB(a,b) \in Da - bD$

$RAAAA$

AAA

$A = \{1,2,3,4,5,6\}|Ax|yxy|$

$A = \{a,b,c,d\}RA$

$$R = \{(a,a),(a,b),(a,c),(b,b),(b,a),(b,c),(c,c),(c,a),(c,b),(d,d)\}.$$

R

A

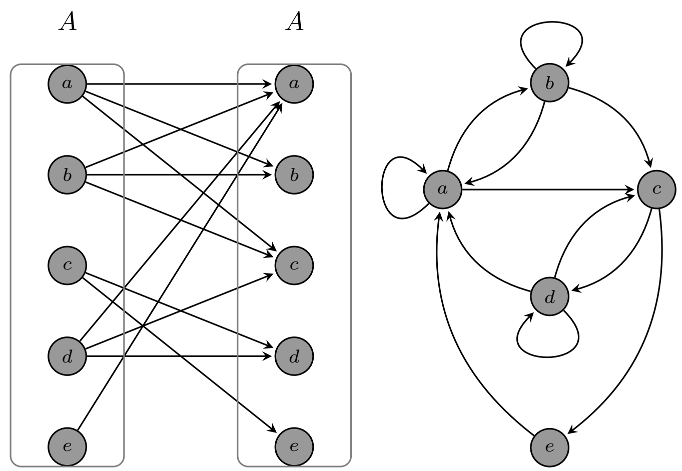

Figure 7.2$A = \{a, b, c, d, e\}$

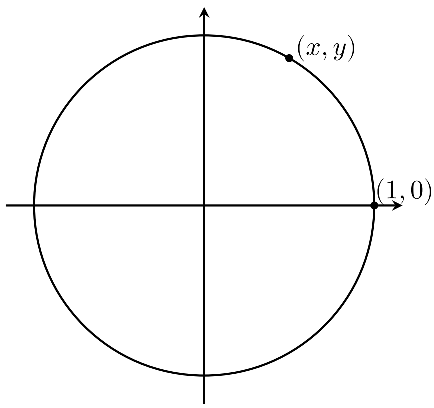

Figure 7.3$x^2 + y^2 = 1$

$RABaRb(a, b)aRbAB$

$x^2 + y^2 = 1(x, y)x^2 + y^2 = 1\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}\mathbb{R}^2$

$\mathbb{R}^2$

$\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

$\{(x, y) \in \mathbb{Z}^2 \mid y = x^2\}$

$\{(x, y) \in \mathbb{R}^2 \mid y^2 = x\}$

$\{(x, y) \in \mathbb{N} \times \mathbb{R} \mid y^2 = x\}$

$\leq \mathbb{R}$

$R A a \in A a R$

$$\boxed{\operatorname{rel}(a, R) := \{b \in A \mid aRb\}}.$$

R

$$\boxed{\operatorname{Rel}(R) := \{\operatorname{rel}(a) \mid a \in A\}}.$$

$R \boxed{\operatorname{rel}(a)} \operatorname{rel}(a, R) \operatorname{rel}(a) a a \operatorname{Rel}(R) \operatorname{Rel}(R) A \mathcal{P}(A)$

R

$\operatorname{rel}(a) = \{a, b, c\}, \operatorname{rel}(b) = \{a, b, c\}, \operatorname{rel}(c) = \{d, e\}, \operatorname{rel}(d) = \{a, c, d\}, \operatorname{rel}(e) = \{a\},$

$\operatorname{Rel}(R) = \{\{a, b, c\}, \{d, e\}, \{a, c, d\}, \{a\}\}$

$\operatorname{Rel}(R) \operatorname{rel}(x) x \in A$

$P F P x F y x y \operatorname{rel}() \operatorname{Rel}(F)$

$\equiv_5 \mathbb{Z} a \equiv_5 b a - b \operatorname{rel}(1) \operatorname{rel}(2) \operatorname{rel}(6) \operatorname{Rel}(\equiv_5) \operatorname{Rel}(\equiv_5)$

$\leq \mathbb{R} x \in \mathbb{R} \operatorname{rel}(x)$

$R A = \{1, 2, 3, 4, 5\} \operatorname{rel}(1) = \{1, 3, 4\} \operatorname{rel}(2) = \{4\} \operatorname{rel}(3) = \{3, 4, 5\}$
$\operatorname{rel}(4) = \{1, 2\} \operatorname{rel}(5) = \emptyset R$

$R A$

$R a \in A a R a$

$R a, b \in A a R b b R a$

$R a, b, c \in A a R b b R c a R c$

$=\mathbb{R}$

$\leq\mathbb{R}<\mathbb{R}$

$S\subseteq\mathcal{P}(S)$

RA

R

R

R

$A = \{a, b, c, d, e\}$

RA

SA

TA

RA

$RARa \in A$

$$...RaRa...$$

RA

$RARa, b \in AaRb$

$$...aRb$$
$$RbRa...$$

RA

$RARa, b, c \in AaRbbRc$

$$...aRbbRc$$
$$RaRc...$$

RA

T

F

$\equiv_5$

$PHxHyxy$

$PTxTyxy$

$\mathbb{N}$

$L||l_1||l_2 l_1 l_2$

$C[0,1][0,1]f \sim g$

$$\int_0^1 |f(x)|dx = \int_0^1 |g(x)|dx.$$

$R\mathbb{N}nRmn + m$

$D\mathbb{R}(x, y) \in Dx = 2y$

$F\mathbb{Z} \times (\mathbb{Z} \quad \{0\})(a, b)F(c, d)ad = bc$

$\sim\mathbb{R}^2(x_1, y_1) \sim (x_2, y_2)x_1^2 + y_1^2 = x_2^2 + y_2^2$

$S\mathbb{R}xSy\lfloor x\rfloor = \lfloor y\rfloor\lfloor x\rfloor x\lfloor \pi\rfloor = 3\lfloor -1.5\rfloor = -2\lfloor 4\rfloor = 4$

$C\mathbb{R}xCy|x - y| < 1$

7.2Equivalence Relations

$\sim A \sim A \sim$

$\sim a \sim babab$

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1),(1,6),(2,2),(2,3),(2,4),(3,3),(3,2),(3,4),(4,4),(4,2),(4,3),(5,5),(6,6$

R

R

RA

$\mathrm{Rel}(R)\mathrm{rel}(x)x \in A$

$A = \{a, b, c, d, e\}$

$\sim Aabcb$

$\mathrm{Rel}(\sim)\mathrm{rel}(x)x \in A$

$A \sim A$

$\mathcal{T} \sim \mathcal{T} T_1 \sim T_2 T_1 T_2 \sim \mathcal{T}$

$\sim Aa, b \in A\mathrm{rel}(a) = \mathrm{rel}(b)a \sim b$

$\sim A$

$$\bigcup_{a \in A} \mathrm{rel}(a) = A$$

$a, b \in A\mathrm{rel}(a) = \mathrm{rel}(b)\mathrm{rel}(a) \cap \mathrm{rel}(b) = \emptyset$

$\sim Aa \in A\mathrm{rel}(a)a$

$\sim A\mathrm{rel}(a)\boxed{[a]}\,\overline{a}a[a][1] = [6]\mathrm{Rel}(\sim)\boxed{A/\sim}A\sim A\sim A/\sim A\sim$

$P{\sim}Pa \sim bab{\sim}P[]P/{\sim}[] \in P/{\sim}$

$\equiv_5\mathbb{Z}$

${\sim}AA/{\sim}{\sim}$

$RSAR \cap SA$

$RSAR \cup SA$

7.3Partitions

${\sim}A{\sim}A[a]a \in A$

$\Omega AA\Omega$

$X \in \Omega$

$X, Y \in \Omega X \cap Y = \emptyset X \neq Y$

$$\bigcup_{X \in \Omega} X = A$$

$\Omega AX \in \Omega$

${\sim}PPP/{\sim}P$

$A = \{a, b, c, d, e, f\}\Omega = \{\{a\}, \{b, c, d\}, \{e, f\}\}\Omega AA\Omega A$

A

A

A

$\mathbb{Z}$

$\mathbb{Z}$

$\mathbb{Z}$

$\mathbb{Z}$

$\sim A A/\sim A$

A

$\sim = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(5,4),(5,5),(6,6),(5,6),(6,5),(4,6$

$A = \{1,2,3,4,5,6\} \mathrm{Rel}(\sim)$

$A \Omega A \boxed{R_\Omega} A a R_\Omega b X \in \Omega a b A \Omega$

$A = \{a,b,c,d,e,f\} \Omega = \{\{a,c\},\{b,c\},\{d,f\}\} R_\Omega$

$A \Omega R_\Omega$

$A \mathrm{Rel}(\sim) R$

$A \Omega \mathcal{P}(A) R_\Omega$

$A = \{1,2,3,4,5,6\} \Omega = \{\{1,3,4\},\{2,4\},\{3,4\},\{6\}\}$

ΩA

R_Ω

R_Ω

$\mathrm{Rel}(R_\Omega) A R_\Omega \Omega \mathrm{Rel}(R_\Omega)$

ΩA

$$\bigcup_{X \in \Omega} X = A,$$

R_Ω

A

$\Omega A R_\Omega$

$\Omega A \Omega R_\Omega$

A

$A A$

$\Omega A R_\Omega$

$R A R A R$

$A = \{\circ, \triangle, \ , \square, \blacksquare, \ , ☺, ☹\} \Omega A R_\Omega$

7.4Modular Arithmetic

$n \in \mathbb{N} n \mathbb{Z} n$

$$\boxed{n\mathbb{Z} := \{m \in \mathbb{Z} \mid m = nk k \in \mathbb{Z}\}}.$$

$5\mathbb{Z} = \{\ldots, -10, -5, 0, 5, 10, \ldots\} 2\mathbb{Z}$

$3\mathbb{Z} 5\mathbb{Z} 15\mathbb{Z} 20\mathbb{Z}$

$3\mathbb{Z} \cap 5\mathbb{Z} = n\mathbb{Z} n \in \mathbb{N} n 15\mathbb{Z} \cap 20\mathbb{Z}$

$3\mathbb{Z} 5\mathbb{Z} 15\mathbb{Z}$

$5\mathbb{Z} 15\mathbb{Z} 20\mathbb{Z}$

$n \in \mathbb{N} a, b \in n\mathbb{Z} -a a + b a b n\mathbb{Z}$

$n \in \mathbb{N} \boxed{\equiv_n} \mathbb{Z} a \equiv_n b a - b \in n\mathbb{Z} a \equiv_n b a b n$

$a - b \in n\mathbb{Z} n a - b a \equiv_n b n a - b$

$\equiv_5$

$$\begin{aligned} \mathrm{rel}(0) &= \{\dots, -10, -5, 0, 5, 10, \dots\} \\ \mathrm{rel}(1) &= \{\dots, -9, -4, 1, 6, 11, \dots\} \\ \mathrm{rel}(2) &= \{\dots, -8, -3, 2, 7, 12, \dots\} \\ \mathrm{rel}(3) &= \{\dots, -7, -2, 3, 8, 13, \dots\} \\ \mathrm{rel}(4) &= \{\dots, -6, -1, 4, 9, 14, \dots\}. \end{aligned}$$

$\mathbb{Z} \equiv_5$

$\equiv_n$

$n \in \mathbb{N} \equiv_n \mathbb{Z}$

$\equiv_n$

$n \in \mathbb{N} \boxed{[a]_n} a \equiv_n [a]_n a n \equiv_n \boxed{\mathbb{Z}/n\mathbb{Z}} \mathbb{Z} n\mathbb{Z}$

$[2]_7$

$$\begin{aligned} m \in [2]_7 &\iff m \equiv_7 2 \\ &\iff m - 2 \in 7\mathbb{Z} \\ &\iff m - 2 = 7k k \in \mathbb{Z} \\ &\iff m = 7k + 2 k \in \mathbb{Z}. \end{aligned}$$

$7 7\mathbb{Z} = \{\dots, -14, -7, 0, 7, 14, \dots\} [2]_7 2 7\mathbb{Z} [2]_7 = \{\dots, -12, -5, 2, 9, 16, \dots\}$

7070

$[4]_7$

$[-3]_7$

$[7]_7$

$[0]_3[1]_3[2]_3[4]_3[-2]_3\mathbb{Z}/3\mathbb{Z}$

$n \in \mathbb{N}a, b \in \mathbb{Z}[a]_n = [b]_n na - b$

$n \in \mathbb{N}a \in \mathbb{Z}[a]_n = [0]_n na$

$n \in \mathbb{N}a, b \in \mathbb{Z}[a]_n = [b]_n abn$

$a_1b_1 - a_2b_2 = a_1b_1 - a_2b_1 + a_2b_1 - a_2b_2$

$n \in \mathbb{N}a_1, a_2, b_1, b_2 \in \mathbb{Z}[a_1]_n = [a_2]_n[b_1]_n = [b_2]_n$

$[a_1 + b_1]_n = [a_2 + b_2]_n$

$[a_1 \cdot b_1]_n = [a_2 \cdot b_2]_n$

$\mathbb{Z}/n\mathbb{Z}$

$n \in \mathbb{N}\mathbb{Z}/n\mathbb{Z}$

$$[a]_n + [b]_n := [a+b]_n[a]_n \cdot [b]_n := [a \cdot b]_n.$$

$[2]_7 + [6]_7 = [2+6]_7 = [8]_7[8]_7 = [1]_7[2]_7 + [6]_7 = [1]_7[2]_7 \cdot [6]_7 = [2 \cdot 6]_7 = [12]_7 = [5]_7$

$\mathbb{Z}/n\mathbb{Z}[a]_n \cdot [b]_n = [0]_n[a]_n \neq [0]_n[b]_n \neq [0]_n$

$n \in \mathbb{N}\mathbb{Z}/n\mathbb{Z}[a]_n, [b]_n, [c]_n \in \mathbb{Z}/n\mathbb{Z}$

$[a]_n + [b]_n = [b]_n + [a]_n$

$([a]_n + [b]_n) + [c]_n = [a]_n + ([b]_n + [c]_n)$

$n \in \mathbb{N}\mathbb{Z}/n\mathbb{Z}[a]_n, [b]_n, [c]_n \in \mathbb{Z}/n\mathbb{Z}$

$[a]_n \cdot [b]_n = [b]_n \cdot [a]_n$

$([a]_n \cdot [b]_n) \cdot [c]_n = [a]_n \cdot ([b]_n \cdot [c]_n)$

k

$n \in \mathbb{N} k \in \mathbb{N} [a_1]_n, [a_2]_n, \dots, [a_k]_n \in \mathbb{Z}/n\mathbb{Z}$

$[a_1]_n + [a_2]_n + \cdots + [a_k]_n = [a_1 + a_2 + \cdots + a_k]_n$

$[a_1]_n [a_2]_n \cdots [a_k]_n = [a_1 a_2 \cdots a_k]_n$

$n \in \mathbb{N} a \in \mathbb{Z} k \in \mathbb{N} ([a]_n)^k = [a^k]_n$

$[8^{179}]_7$

$$\begin{aligned} [8^{179}]_7 &= ([8]_7)^{179} && () \\ &= ([1]_7)^{179} && () \\ &= [1^{179}]_7 && () \\ &= [1]_7. \end{aligned}$$

$[6]_7 = [-1]_7 [2^3]_7 = [1]_7$

$a 0 \leq a \leq 6 [a]_7$

$[6^{179}]_7$

$[2^{300}]_7$

$[2^{301} + 5]_7$

$ab [a]_6 \cdot [b]_6 = [0]_6 [a]_6 \neq [0]_6 [b]_6 \neq [0]_6$

$n \in \mathbb{N} n [a]_n, [b]_n \in \mathbb{Z}/n\mathbb{Z} [a]_n \cdot [b]_n = [0]_n [a]_n \neq [0]_n [b]_n \neq [0]_n$

$2x = 1 \mathbb{Z} [2]_7 [x]_7 = [1]_7 x \mathbb{Z} [14]_7 [x]_7 = [1]_7$

$m \in \mathbb{N}$

$$m = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0,$$

$a_k, a_{k-1}, \dots, a_1, a_0 \in \{0, 1, \dots, 9\} a_k, a_{k-1}, \dots, a_1, a_0 m$

$$[m]_3 = [a_k + a_{k-1} + \cdots + a_1 + a_0]_3.$$

33

$n \geq 0 3^{2n} - 18n = 0$

$n 4n + 3$

8

Functions

8.1Introduction to Functions

$f(x) = x^2 - 1 XYXY$

$XYfXYXYx \in Xy \in Y(x,y) \in fXf$ $\boxed{\mathrm{Dom}(f)}$ Yf $\boxed{\mathrm{Codom}(f)}$

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid x(x,y) \in f\}}$$

fXf

$\boxed{f : X \to Y}$ $fXYf : X \to Yf : X \to Yf(a,b) \in ff$ $\boxed{f(a) = b}$ f $abfabafbfbfaf$

$f : X \to YYf$

$fg\mathrm{Dom}(f) = \mathrm{Dom}(g)\mathrm{Codom}(f) = \mathrm{Codom}(g)f(x) = g(x)x \in X$

$X = \{a,b,c,d\}Y = \{1,2,3,4\}fXY$

$$f = \{(a,2),(b,4),(c,4),(d,1)\}.$$

$XfXYf : X \to Y\mathrm{Rng}(f) = \{1,2,4\}f(a) = 2c \mapsto 4fbdf$

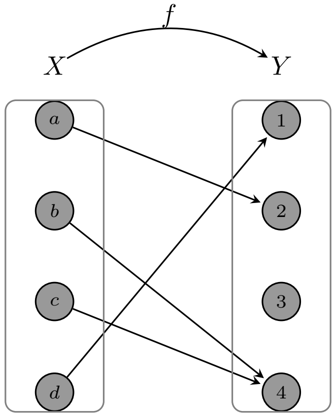

Figure 8.1$X = \{a, b, c, d, \}Y = \{1, 2, 3, 4\}$

$X = \{\circ, \square, \triangle, \smiley\}Y = \{a, b, c, d, e\}XYYXXX$

$f = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, d)\}$

$g = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, c)\}$

$h = \{(\circ, a), (\square, b), (\triangle, c), (\circ, d)\}$

$k = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, d), (\square, e)\}$

$l = \{(\circ, e), (\square, e), (\triangle, e), (\smiley, e)\}$

$m = \{(\circ, a), (\triangle, b), (\smiley, c)\}$

$i = \{(\circ, \circ), (\square, \square), (\triangle, \triangle), (\smiley, \smiley)\}$

$\text{happy}YX(y, \smiley) \in \text{happy}y \in Y$

$\text{nugget} = \{(\circ, \circ), (\square, \square), (\triangle, \triangle), (\smiley, \square)\}$

$\sin\cos\log\ln f \sin\sin(x)\ln(a)$

XY

$f(x) = x^2 - 1 x f x^2 - 1$

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2 - 1 g : \mathbb{N} \to \mathbb{R} g(x) = x^2 - 1$

$\mathbb{R}\mathbb{R}$

$f(x) = x^2 - 1 g(x) = \sqrt{x} h(x) = \frac{1}{x} f : \mathbb{R} \to \mathbb{R} g : [0, \infty) \to \mathbb{R}$
$h : \mathbb{R} \quad \{0\} \to \mathbb{R}$

$f \mathrm{Rng}(f) = \mathrm{Codom}(f)$

$g \mathrm{Rng}(g) \mathrm{Codom}(g)$

$f : X \to Y XY nmn < m \mathrm{Rng}(f) = \mathrm{Codom}(f)$

$XY X \subseteq Y \iota : X \to Y \iota(x) = x XY$

ι

$X = \{a, b, c\} Y = \{a, b, c, d\} XY$

$X i_X : X \to X i_X(x) = x X$

$X = \{\circ, \square, \triangle, \smiley\}$

$\mathbb{R}\mathbb{R}^2$

A

$RARRAA$

$f : A \to A f A$

$f : X \to Y f(x) = c c \in Y$

$\text{happy}(y) = ☺$

$$\text{nugget}(x) = \begin{cases} x, x, \\ \square,. \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 - 1, x \geq 0, \\ 17, \quad -2 \leq x < 0, \\ -x, \quad x < -2 \end{cases}$$

$f : \mathbb{R} \quad \{0\} \to \mathbb{R} f(x) = \frac{|x|}{x} f$

$f : \{1, 2, 3\} \to \{1, 2, 3\} f(a) = a - 1$

$g : \mathbb{N} \to \mathbb{Q} g(n) = \frac{n}{n-1}$

$A_1 = \{1, 2, 3\} A_2 = \{3, 4, 5\} h : A_1 \cup A_2 \to \{1, 2\}$

$$h(x) = \begin{cases} 1, x \in A_1 \\ 2, x \in A_2. \end{cases}$$

$s : \mathbb{Q} \to \mathbb{Z} s(a/b) = a + b$

$a, b, c \in \mathbb{R} abc(ab)ca(bc)$

$$[a_1]_n + [a_2]_n + \cdots + [a_k]_n$$

$$[a_1]_n [a_2]_n \cdots [a_k]_n$$

$\mathbb{Z}/n\mathbb{Z}$

$f : X \to Y abX f(a) = f(b)$

$f : \mathbb{Z}/5\mathbb{Z} \to \mathbb{N}$

$$f([a]_5) = \begin{cases} 0, a \\ 1, a. \end{cases}$$

$g : \mathbb{Z}/6\mathbb{Z} \to \mathbb{N}$

$$g([a]_6) = \begin{cases} 0, a \\ 1, a. \end{cases}$$

$m : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} m([x]_8) = [6x]_{10}$

$h : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} h([x]_{10}) = [6x]_{10}$

$k : \mathbb{Z}/43\mathbb{Z} \to \mathbb{Z}/43\mathbb{Z} k([x]_{43}) = [11x - 5]_{43}$

$\ell : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z} \ell([x]_{15}) = [5x - 11]_{15}$

$k, n \in \mathbb{N} m \in \mathbb{Z} f_m : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} f_m([x]_n) = [mx]_k$

8.2Injective and Surjective Functions

f

$f : X \to Y$

$f y \in \mathrm{Rng}(f) x \in X y = f(x)$

$f y \in Y x \in X y = f(x)$

$f f$

$x \in X y \in Y f(x) = y$

$y \in \mathrm{Rng}(f) x \in X y = f(x)$

XY

$f : X \to Y$

$f : X \to Y$

$f : X \to Y$

$f : X \to Y$

$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$X \subseteq \mathbb{R} f : X \to \mathbb{R}$

$f : X \to \mathbb{R} f$

$X \subseteq \mathbb{R} f : X \to \mathbb{R}$

$f : X \to \mathbb{R} f$

$X \subseteq \mathbb{R} f : X \to \mathbb{R}$

$f : X \to \mathbb{R} f$

$f f(x) x f(x_1) = f(x_2) x_1 x_2 x_1 = x_2 f(x_1) = f(x_2) x_1 x_2 x_1 x_2 x_1, x_2 \in X x_1 x_2$

$f : X \to Y f x_1, x_2 \in X f(x_1) = f(x_2)$

$$...f x_1 = x_2...$$

f

f

$f : X \to Y f y \in Y$

$$...f x \in X f(x) = y...$$

f

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2$

$g : \mathbb{R} \to [0, \infty) g(x) = x^2$

$h : \mathbb{R} \to \mathbb{R} h(x) = x^3$

$k : \mathbb{R} \to \mathbb{R} k(x) = x^3 - x$

$c : \mathbb{R} \times \mathbb{R} \to \mathbb{R} c(x, y) = x^2 + y^2$

$f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} f(n) = (n, n)$

$g : \mathbb{Z} \to \mathbb{Z}$

$$g(n) = \begin{cases} \frac{n}{2}, & n \\ \frac{n+1}{2}, & n \end{cases}$$

$\ell : \mathbb{Z} \to \mathbb{N}$

$$\ell(n) = \begin{cases} 2n + 1, & n \geq 0 \\ -2n, & n < 0 \end{cases}$$

h

k

ℓ

$X Y m n m \leq n X Y$

$\iota : X \to Y X \subseteq Y$

$i_X : X \to X$

$A B S A \times B \pi_1 : S \to A \pi_2 : S \to B \pi_1(a, b) = a \pi_2(a, b) = b \pi_1 \pi_2 S A$
B

π_1

$S\pi_1\pi_1\pi_2$

$\sim A f : A \to A/\sim f(x) = [x]$

$\sim$

RA

$Rf : A \to \mathrm{Rel}(R) f(a) = \mathrm{rel}(a)$

Rf

$f : X \to Y \sim X a \sim b f(a) = f(b) \sim X$

f

c

$f f \overline{f}$

$f : X \to Y \sim X \overline{f} : X/\sim \to \mathrm{Rng}(f) \overline{f}([a]) = f(a)$

$f \overline{f f}$

$X = \{a, b, c, d, e, f\} Y = \{1, 2, 3, 4, 5\} \varphi : X \to Y$

$$\varphi = \{(a, 1), (b, 1), (c, 2), (d, 4), (e, 4), (f, 4)\}.$$

$\varphi \mathrm{Rng}(\varphi) = \{1, 2, 4\} \overline{\varphi \varphi} \varphi(a) = \varphi(b) \varphi(d) = \varphi(e) = \varphi(f) [a] = [b]$ $[d] = [e] = [f] [a] [d] [b] [c] [d] X = \{a, b, c, d, e, f\} Y = \{1, 2, 3, 4, 5\}$ $X/\sim = \{[a], [c], [d]\} \mathrm{Rng}(\varphi) = \{1, 2, 4\} [a] a a b [b] a$

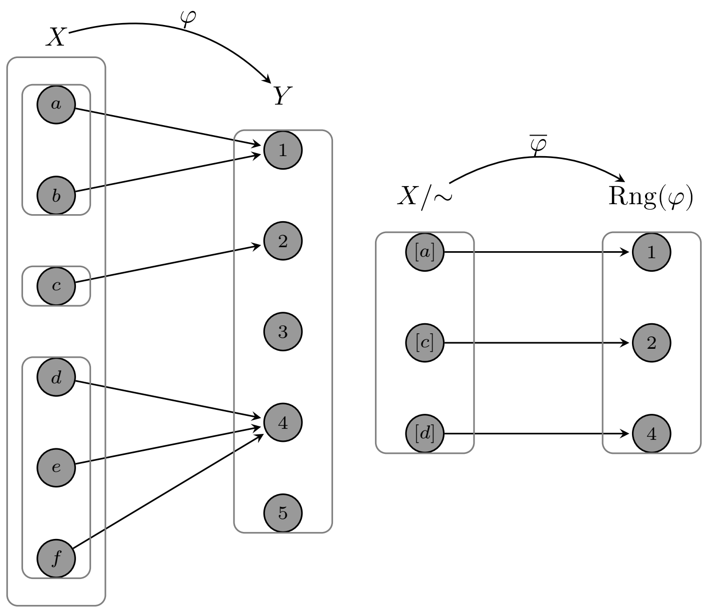

Figure 8.2

$\overline{f}f$

$\overline{c}c$

$Y = \{0, 1, 2, 3\} f : \mathbb{Z} \to Y f(n) n f(11) = 3 11 = 4 \cdot 2 + 3 f(n) = n \pmod 4 \{0, 1, 2, 3\} f f$

$\mathbb{Z}/\sim$

$\overline{f}$

f

h

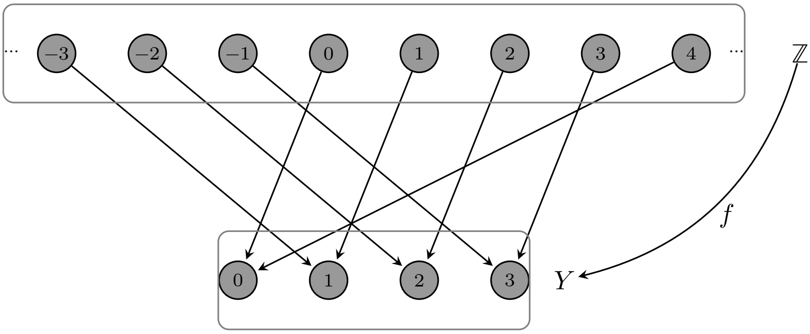

Figure 8.3

h

$\overline{h}$

8.3Compositions and Inverse Functions

$f : X \to Y g : Y \to Z g \circ f : X \to Z$ $\boxed{(g \circ f)(x) = g(f(x))}$ $g \circ f f g$

$f \circ g g \circ f$

$X = \{1, 2, 3, 4\} f : X \to X g : X \to X$

$$f = \{(1, 1), (2, 3), (3, 3), (4, 4)\}$$

$$g = \{(1, 1), (2, 2), (3, 1), (4, 1)\}.$$

$g \circ f$

$f \circ g$

$f \circ g g \circ f$

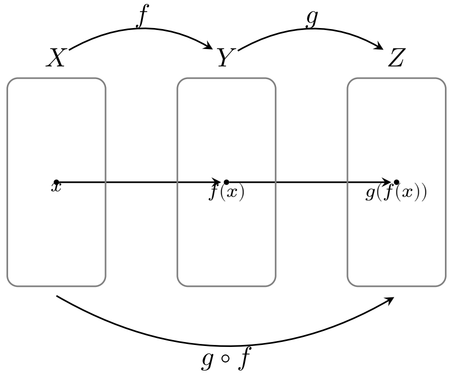

Figure 8.4

$\iota : X \to Y X Y f : Y \to Z f \circ \iota : X \to Z$

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

$x \in X f \circ \iota f f \circ \iota f X$ $\boxed{f|_X}$

$f : \mathbb{R} \to \mathbb{R} g : \mathbb{R} \to \mathbb{R} f(x) = x^2 g(x) = 3x - 5 f \circ g g \circ f$

$f : \mathbb{R} \to \mathbb{R} g : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 5x + 7, x < 0 \\ 2x + 1, x \geq 0 \end{cases}$$

$g(x) = 7x - 11 g \circ f$

$f : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/23\mathbb{Z} g : \mathbb{Z}/23\mathbb{Z} \to \mathbb{Z}/32\mathbb{Z} f([x]_{15}) = [3x + 5]_{23}$ $g([x]_{23}) = [2x + 1]_{32} g \circ f$

$f : X \to Y f \circ i_X = f = i_Y \circ f i_X i_Y X Y$

$f : X \to Y g : Y \to Z h : Z \to W (h \circ g) \circ f = h \circ (g \circ f)$

$X Y Z f : X \to Y g : Y \to Z$

$f g \circ f$

$gg \circ f$

$fg \circ f$

$gg \circ f$

$f : X \to Y g : Y \to Z g \circ f$

$f : X \to Y g : Y \to Z g \circ f$

$f : X \to Y g : Y \to Z g \circ f$

$f : X \to Y g : Y \to Z$

$g \circ f f$

$g \circ f g$

$g \circ f f$

$g \circ f g$

$f : X \to Y f g : Y \to X g \circ f = i_X i_X X$

gf

$f : X \to Y f g : Y \to X f \circ g = i_Y i_Y Y$

gf

$X = \{a, b\} Y = \{1, 2\}$

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2 f$

$f : \mathbb{R} \to [0, \infty) f(x) = x^2 g : [0, \infty) \to \mathbb{R} g(x) = \sqrt{x}$

f

$g f f \circ g(x)$

$f : X \to Y g : Y \to X g \circ f = i_X f \circ g = i_Y f$

$f g g f$

$f : X \to Y f^{-1} Y X f$

$$\boxed{f^{-1} = \{(f(x), x) \in Y \times X \mid x \in X\}}.$$

$f^{-1} f^{-1} f f$

$f f^{-1} f^{-1}$

$f : X \to Y f^{-1}$

$X \subseteq \mathbb{R} f : X \to \mathbb{R} f f^{-1}$

$f : X \to Y f^{-1} : Y \to X f$

$f : \mathbb{R} \to \mathbb{R}$

$f^{-1} f$

$f : X \to Y$

$f^{-1} \circ f = i_X$

$f \circ f^{-1} = i_Y$

$f : X \to Y f^{-1} : Y \to X$

$f : X \to Y g : Y \to X g \circ f = i_X f \circ g = i_Y f^{-1} g = f^{-1}$

$f^{-1} f^{-1} f$

$X \subseteq \mathbb{R} f : X \to \mathbb{R} f^{-1}(x) [f(x)]^{-1}$

$X, Y \subseteq \mathbb{R} f : X \to Y f(x) = e^x g : Y \to X g(x) = \ln(x) X Y f g$

$f : X \to Y (f^{-1})^{-1} = f$

$f^{-1} (f^{-1})^{-1} R X Y (R^{-1})^{-1} = R$

$f : X \to Y g : Y \to Z (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

8.4Images and Preimages of Functions

$f : X \to Y$

$S \subseteq X S f$

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

$T \subseteq Y T f$

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

$S S f X Y f X Y S \subseteq X f(S) \subseteq Y f(X) = \mathrm{Rng}(f)$
$f^{-1} f f f^{-1} : Y \to X f^{-1}(y) y \in Y f^{-1} T \subseteq Y f^{-1}(T) T f^{-1}(\{y\})$ $y \in Y y \notin \mathrm{Rng}(f) f^{-1}(\{y\}) = \emptyset y \in Y f^{-1}(\{y\}) f^{-1}(y) f^{-1} f^{-1}(Y) = X$

$f : \mathbb{Z} \to \mathbb{Z} f(x) = x^2$

$f(\{0, 1, 2\})$

$f^{-1}(\{0, 1, 4\})$

$f : \mathbb{R} \to \mathbb{R} f(x) = 3x^2 - 4$

$f(\{-1, 1\})$

$f([-2, 4])$

$f((-2, 4))$

$f^{-1}([-10, 1])$

$f^{-1}((-3, 3))$

$f(\emptyset)$

$f(\mathbb{R})$

$f^{-1}(\{-1\})$

$f^{-1}(\emptyset)$

$f^{-1}(\mathbb{R})$

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2$

$A B \mathbb{R} A \cap B = \emptyset f^{-1}(A) = f^{-1}(B)$

$AB\mathbb{R}A \cap B = \emptyset f(A) = f(B)$

$f : X \to Y ABX f(A) f(B) Y$

$fgST f(f^{-1}(T)) \neq T g^{-1}(g(S)) \neq S$

$f : X \to Y A, B \subseteq X C, D \subseteq Y$

$A \subseteq B f(A) \subseteq f(B)$

$C \subseteq D f^{-1}(C) \subseteq f^{-1}(D)$

$f(A \cup B) \subseteq f(A) \cup f(B)$

$f(A \cup B) \supseteq f(A) \cup f(B)$

$f(A \cap B) \subseteq f(A) \cap f(B)$

$f(A \cap B) \supseteq f(A) \cap f(B)$

$f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$

$f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$

$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$

$f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$

$A \subseteq f^{-1}(f(A))$

$A \supseteq f^{-1}(f(A))$

$f(f^{-1}(C)) \subseteq C$

$f(f^{-1}(C)) \supseteq C$

$f : X \to Y \{A_\alpha\}_{\alpha \in \Delta} X$

$$f\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) = \bigcup_{\alpha \in \Delta} f\left(A_\alpha\right)$$

$$f\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \subseteq \bigcap_{\alpha \in \Delta} f\left(A_\alpha\right)$$

$f : X \to Y \{C_\alpha\}_{\alpha\in\Delta} Y$

$$f^{-1}\left(\bigcup_{\alpha\in\Delta} C_\alpha\right) = \bigcup_{\alpha\in\Delta} f^{-1}\left(C_\alpha\right)$$

$$f^{-1}\left(\bigcap_{\alpha\in\Delta} C_\alpha\right) = \bigcap_{\alpha\in\Delta} f^{-1}\left(C_\alpha\right)$$

$[a] f^{-1}(\{f(a)\})$

$f : \mathbb{R} \to \mathbb{R} f(x+y) = f(x) + f(y) x, y \in \mathbb{R}$

$f(0) = 0$

$f(-x) = -f(x) x \in \mathbb{R}$

$f f^{-1}(\{0\}) = \{0\}$

$f(x) = mx m \in \mathbb{R} f(x+y) = f(x) + f(y) f(x) = mx$

8.5Continuous Real Functions

$f : A \to \mathbb{R} A \mathbb{R}$

$\mathbb{R} |a-b| < r a b r$

$f a \in \mathrm{Dom}(f) f a \varepsilon > 0 \delta > 0 x \in \mathrm{Dom}(f) |x-a| < \delta |f(x)-f(a)| < \varepsilon$
$f B \subseteq \mathrm{Dom}(f) f B f f$

$f a \in \mathrm{Dom}(f) f(x) f(a) x \in \mathrm{Dom}(f) a \varepsilon f(a) \delta a \varepsilon f(a) a x \in \mathrm{Dom}(f)$
$|x-a| < \delta 2\delta 2\varepsilon (a, f(a)) a \varepsilon > 0 \delta > 0$

$\delta\varepsilon$

$f : \mathbb{R} \to \mathbb{R} f(x) = 3x+2 f a \in \mathbb{R} \varepsilon > 0 \delta = \varepsilon/3 \delta x \in \mathbb{R} |x-a| < \delta$

$$|f(x)-f(a)| = |(3x+2)-(3a+2)| = |3x-3a| = 3\cdot|x-a| < 3\cdot\delta = 3\cdot\varepsilon/3 = \varepsilon.$$

$f a a f$

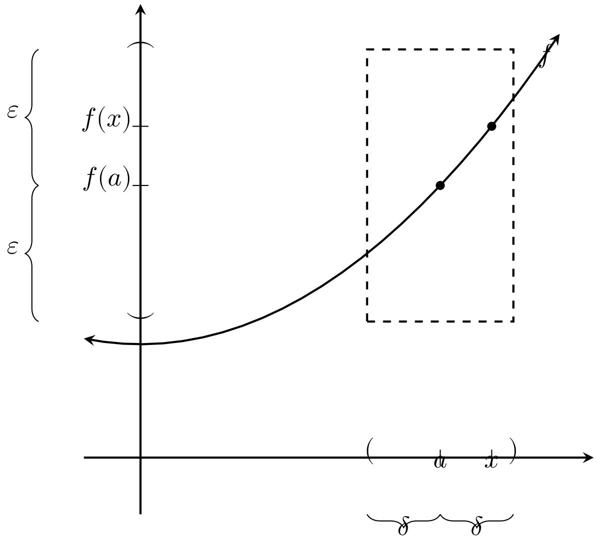

Figure 8.5fa

$f : \mathbb{R} \to \mathbb{R} f(x) = x$

$g : \mathbb{R} \to \mathbb{R} g(x) = x + 42$

$h : \mathbb{R} \to \mathbb{R} h(x) = 5x$

$m = 0 m = 0$

$f : \mathbb{R} \to \mathbb{R} f(x) = mx + b m, b \in \mathbb{R} f$

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2$

f

f

$f : \mathbb{R} \to \mathbb{R} f(x) = \sqrt{x} f$

$f f a \in \mathrm{Dom}(f)$

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1, x = 0 \\ x, . \end{cases}$$

f

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, . \end{cases}$$

f

$f : \mathbb{N} \to \mathbb{R} f(x) = 1 f$

$f f f^{-1}(U) U f$

$f f f^{-1}(I) I f$

$f : \mathbb{R} \to \mathbb{R} f(x) = x^2 f$

$f : \mathbb{R} \quad \{0\} \to \mathbb{R} f(x) = \frac{1}{x} f$

$f I f^{-1}(I)$

$f U \mathrm{Dom}(f) f(U)$

$f C f^{-1}(C)$

$f [a, b] \mathrm{Dom}(f) f([a, b])$

$f C \mathrm{Dom}(f) f(C)$

$f B \mathrm{Dom}(f) f(B)$

$f B f^{-1}(B)$

$f K f^{-1}(B)$

$fC\mathrm{Dom}(f)f(C)$

$fCf^{-1}(C)$

$fKff(K)$
$f[a,b]ff(a)f(b)[a,b]$

$ff[a,b]f(a) < 0 < f(b)f(a) > 0 > f(b)r \in [a,b]f(r) = 0$

$ff[a,b]f(a) < c < f(b)f(a) > c > f(b)c \in \mathbb{R}r \in [a,b]f(r) = c$

9

Cardinality

9.1Introduction to Cardinality

$\mathbb{N}2\mathbb{N} := \{2n \mid n \in \mathbb{N}\}2\mathbb{N}\mathbb{N}\mathbb{N}2\mathbb{N}f : \mathbb{N} \to 2\mathbb{N}f(n) = 2nf\mathbb{N}2\mathbb{N}$

$ABABAB\boxed{\operatorname{card}(A) = \operatorname{card}(B)}$

$\operatorname{card}(A)\operatorname{card}(A) = \operatorname{card}(B)\operatorname{card}(A) \leq \operatorname{card}(B)\operatorname{card}(A) < \operatorname{card}(B)$
AB

$fABf^{-1}BA\operatorname{card}(A) = \operatorname{card}(B)$

$A = \{1, 2, 3, 4, 5\}B = \{, , , , \}f : A \to B$

$$f = \{(1,), (2,), (3,), (4,), (5,)\}$$

$AB\operatorname{card}(A) = \operatorname{card}(B)AB5! = 120ABfABBA\operatorname{card}(A) = \operatorname{card}(B)$

$f : \mathbb{Z} \to 6\mathbb{Z}f(n) = 6nf\operatorname{card}(\mathbb{Z}) = \operatorname{card}(6\mathbb{Z})f^{-1} : 6\mathbb{Z} \to \mathbb{Z}f^{-1}(n) = \frac{1}{6}n\mathbb{Z}6\mathbb{Z}$

$\mathbb{R}^+f : \mathbb{R} \to \mathbb{R}^+f(x) = e^x\operatorname{card}(\mathbb{R}) = \operatorname{card}(\mathbb{R}^+)f^{-1} : \mathbb{R}^+ \to \mathbb{R}$
$f^{-1}(x) = \ln(x)$

$m,n \in \mathbb{N} \cup \{0\}(0,0)(m,n)(0,0)(m,n)(0,0)(m,n)m+n(0,0)(4,3)$
$\mathcal{L}_{m,n}(0,0)(m,n)\mathcal{L}_{4,3}kk01011000101001\mathcal{S}_kk\mathcal{S}_3 = \{000,100,010,001,110,101,011,11$
$\mathcal{L}_{m,n}\mathcal{S}_{m+n}(0,0)(m,n)0101100\mathcal{S}_{m+n}\mathcal{L}_{m,n}\mathcal{L}_{m,n}\mathcal{S}_{m+n}\mathrm{card}(\mathcal{L}_{m,n}) = \mathrm{card}(\mathcal{S}_{m+n})$

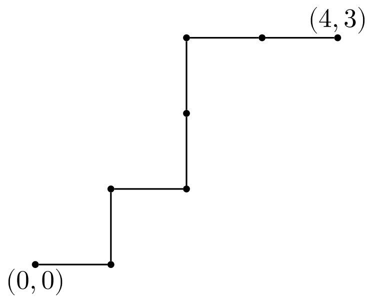

Figure 9.1$(0,0)(4,3)$

$f:(a,b)\to(c,d)$

$\mathrm{card}(\{a,b,c\}) = \mathrm{card}(\{x,y,z\})$

$\mathrm{card}(\mathbb{N}) = \mathrm{card}(\{2n+1 \mid n \in \mathbb{N}\})$

$\mathrm{card}(\mathbb{N}) = \mathrm{card}(\mathbb{Z})$

$\mathrm{card}((a,b)) = \mathrm{card}((c,d))(a,b)(c,d)$

$\mathrm{card}(\mathbb{N}) = \mathrm{card}(\{\frac{1}{2^n} \mid n \in \mathbb{N}\})$

$AAA \times \{x\}$

$\mathcal{D}_n(0,0)(n,n)y=x()()()()(())())(()()(()(\mathcal{B}_n n\mathcal{B}_3 = \{()()(),()(()),(()()),(())(),((()$
$\mathcal{D}_3$

$\mathrm{card}(\mathcal{D}_n) = \mathrm{card}(\mathcal{B}_n)$

$\varphi:\mathcal{F}\to\mathcal{P}(\mathbb{N})\varphi(f)\mathbb{N}f$

$\mathcal{F}\mathbb{N}\{0,1\}$

$\mathcal{F}$

$\mathcal{F}\mathcal{P}(\mathbb{N})$

ABC

$\mathrm{card}(A) = \mathrm{card}(A)$

$\mathrm{card}(A) = \mathrm{card}(B) \mathrm{card}(B) = \mathrm{card}(A)$

$\mathrm{card}(A) = \mathrm{card}(B) \mathrm{card}(B) = \mathrm{card}(C) \mathrm{card}(A) = \mathrm{card}(C)$

$X \mathcal{P}(X)$

$ABCD \mathrm{card}(A) = \mathrm{card}(C) \mathrm{card}(B) = \mathrm{card}(D)$

$ABCD \mathrm{card}(A \cup B) = \mathrm{card}(C \cup D)$

$\mathrm{card}(A \times B) = \mathrm{card}(C \times D)$

$ABABAB$ $\boxed{\mathrm{card}(A) \leq \mathrm{card}(B)}$

ABC

$A \subseteq B \mathrm{card}(A) \leq \mathrm{card}(B)$

$\mathrm{card}(A) \leq \mathrm{card}(B) \mathrm{card}(B) \leq \mathrm{card}(C) \mathrm{card}(A) \leq \mathrm{card}(C)$

$C \subseteq A \mathrm{card}(B) = \mathrm{card}(C) \mathrm{card}(B) \leq \mathrm{card}(A)$

$AB \mathrm{card}(A) \leq \mathrm{card}(B) \mathrm{card}(A) \neq \mathrm{card}(B)$

$AB \mathrm{card}(A) = \mathrm{card}(B) AB$

AB $\boxed{\mathrm{card}(A) < \mathrm{card}(B)}$ $\mathrm{card}(A) \leq \mathrm{card}(B) \mathrm{card}(A) \neq \mathrm{card}(B)$

$\mathrm{card}(A) = \mathrm{card}(B) \mathrm{card}(A) \leq \mathrm{card}(B) AB \mathrm{card}(A) < \mathrm{card}(B)$ $f : A \to B \mathrm{card}(A) < \mathrm{card}(B) AB \mathrm{card}(A) \neq \mathrm{card}(B) \mathrm{card}(A) < \mathrm{card}(B)$

9.2Finite Sets

$n \in \mathbb{N}$ $\boxed{[n] := \{1, 2, \ldots, n\}}$

$[5] = \{1, 2, 3, 4, 5\}$

$AA = \emptyset\text{card}(A) = \text{card}([n])n \in \mathbb{N}A = \emptyset A\text{card}(A) = \text{card}([n])An$

$A\text{card}(A) = \text{card}(B)B$

$An \in \mathbb{N} \cup \{0\}x \notin AA \cup \{x\}n + 1$

$n \in \mathbb{N}[n]$

$An \in \mathbb{N}x \in AA \quad \{x\}n - 1$

$A\text{card}(B) < \text{card}(A)BA$

$A_1, A_2, \ldots, A_k \bigcup_{i=1}^{k} A_i$

$nkn > kn$

$n, k \in \mathbb{N}f : [n] \to [k]n > kf$

9.3Infinite Sets

AA

$n \in \mathbb{N}\mathrm{card}([n]) = \mathrm{card}(\mathbb{N}) f : [n] \to \mathbb{N} m := \max(f(1), f(2), \ldots, f(n)) + 1$

$\mathbb{N}$

$f : A \to B g : B \to [n] n \in \mathbb{N}$

$A\mathrm{card}(A) = \mathrm{card}(B) B$

$\mathbb{Z}$

$R = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$

$\mathbb{N} \times \{a\}$

$AA[n] n \in \mathbb{N}$

$1, 2, 3, 4, \ldots$

$g_1, g_2, g_3, \ldots$

$fB = A \quad \{f(1), f(2), \ldots\} AB \cup \{f(2), f(3), \ldots\} g : A \to CCA$
$a \in A \quad Cf : \mathbb{N} \to Af(n) = g^n(a) g^n gn$

A

$f : \mathbb{N} \to A$

$AABA\mathrm{card}(B) = \mathrm{card}(A)$

$A \operatorname{card}(\mathbb{N}) \leq \operatorname{card}(A)$

$\mathbb{Z}$

$\mathbb{N} \times \mathbb{N}$

$\mathbb{Q}$

$\mathbb{R}$

$\mathbb{N}$

$(0, 1)$

$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$

9.4Countable Sets

$A = \emptyset A \operatorname{card}(A) = \operatorname{card}([n]) n \in \mathbb{N} A n$

$A \operatorname{card}(A) = \operatorname{card}(\mathbb{N}) A \aleph_{\mathbf{0}}$

$A 1, 2, ... \aleph_0 A$

AA

$\{a, b, c\}$

$\{\frac{1}{2^n} \mid n \in \mathbb{N}\}$

$\mathbb{N}$

$\mathbb{Z}$

$\mathbb{N} \times \{a\}$

$ABAf : A \to BB$

AA

$f : \mathbb{N} \to AA$

$0, 1, -1, 2, -2, ...1, 2, 3, 4, 5, ...mnm/n\mathbb{N}$

$\mathbb{Q}$

$ABA \cup B$

$\{A_n\}_{n=1}^{\infty} B_1 := A_1 n > 1$

$$B_n := A_n \quad \bigcup_{i=1}^{n-1} A_i.$$

$\{B_n\}_{n=1}^{\infty}$

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

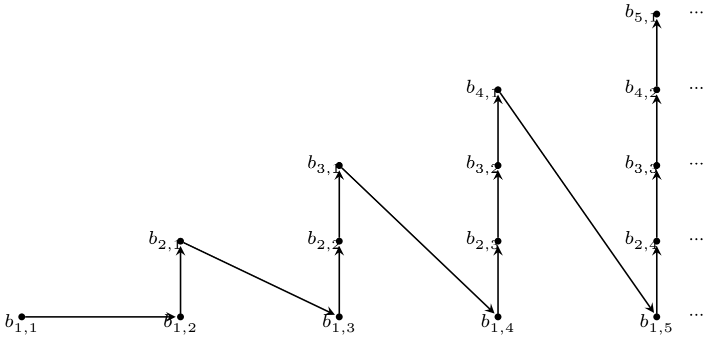

Figure 9.2

$\{B_n\} B_n B_n n B_n = \{b_{n,1}, b_{n,2}, b_{n,3}, ...\} \mathbb{N} \bigcup_{n=1}^{\infty} B_n \{B_n\} f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N} f(b_{n,m}) = 2^n 3^m$

$\Delta \mathbb{N} [k] k \in \mathbb{N} \{A_n\}_{n\in\Delta} A_n \bigcup_{n\in\Delta} A_n$

$ABA \times B$

0110010

9.5Uncountable Sets

$AAA\mathbb{N}A$

$(0,1)(0,1)(0,1)$

$(0,1)0.20.1\overline{99}0.a_1 a_2 a_3 ...(0,1) ki > ka_i = 90.20.1\overline{99}$

$(0,1) f : \mathbb{N} \to (0,1) n \in \mathbb{N} f(0,1) f(n) = 0.a_{1n} a_{2n} a_{3n} ... a_{1n} n a_{2n} f(n)$
$kb = 0.b_1 b_2 b_3 ...$

$$b_i = \begin{cases} 2, a_{ii} \neq 2 \\ 3, a_{ii} = 2. \end{cases}$$

b

$n \in \mathbb{N} f(n) \neq b$

f

$(0,1)$

$(0,1)$

$(0,1)(0,1)$

$ABA \subseteq BAB$

$ABABA \quad B$

$f : A \to BAB$

$(0,1)(0,1) \subseteq \mathbb{R}\mathbb{R}(0,1)\mathbb{R} f : (0,1) \to \mathbb{R} f(x) = \tan(\pi x - \frac{\pi}{2})$

$\mathrm{card}((0,1)) = \mathrm{card}(\mathbb{R})$

$a, b \in \mathbb{R} a < b (a,b)[a,b](a,b][a,b)$

$\mathbb{C}$

$ABA A \cup B$

$ABA A \cap B$

$ABA A \times B$

$ABA A \quad B$

SS

$S \mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(S)$

$\mathbb{N}\mathcal{P}(\mathbb{N})$

$\mathbb{R}\mathcal{P}(\mathbb{N})\mathrm{card}(\mathcal{P}(\mathbb{N})) = \mathrm{card}(\mathbb{R})A\mathcal{P}(A)AA\mathcal{P}(A)\mathrm{card}(A) \leq \mathrm{card}(\mathcal{P}(A))$ $\mathrm{card}(A) < \mathrm{card}(\mathcal{P}(A))f : A \to \mathcal{P}(A))B = \{x \in A \mid x \notin f(x)\}$

$A\mathrm{card}(A) < \mathrm{card}(\mathcal{P}(A))$

Appendix A

Elements of Style for Proofs

$a^3 = b^{-1}x < 55 \big| 107 \in \mathbb{Z}$

$x^2 + 3x^2 + 3 < 7 = \leq \in$

$= A = BABf(x) = x^2 = 2xf(x) = x^2 f'(x) = 2x$

$= \Longrightarrow x^2 = b \Longrightarrow a + b = a \Longrightarrow b = 0 \Longrightarrow$

$\Longrightarrow \forall, \exists, \vee, \wedge \Longleftrightarrow$

$= A \in BA \subseteq Ba_{ij} \in Aa_{ij}A$

$$A = B \leq C = D,$$

$A = BB \leq CC = DA \leq Dd(12, 5)$

$$d = \sqrt{12^2 + 5^2} = 13.$$

$d = 13$

$\sqrt{12^2 + 5^2} = 13 = dd = 1313\sqrt{12^2 + 5^2} = d$

$\tan^2(x) = \sec^2(x) - 1$

$$\begin{aligned} \tan^2(x) &= \sec^2(x) - 1 \\ \left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\ \frac{\sin^2(x)}{\cos^2(x)} &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\ \sin^2(x) &= 1 - \cos^2(x) \\ \sin^2(x) + \cos^2(x) &= 1 \\ 1 &= 1 \end{aligned}$$

$1 = 1\tan^2(x) = \sec^2(x) - 11 = 1$

$$\begin{aligned} \sec^2(x) - 1 &= \frac{1}{\cos^2(x)} - 1 \\ &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\ &= \frac{\sin^2(x)}{\cos^2(x)} \\ &= \left(\frac{\sin(x)}{\cos(x)}\right)^2 \\ &= (\tan(x))^2 \\ &= \tan^2(x). \end{aligned}$$

$kkn \in \mathbb{N}kaA \subseteq \mathbb{R}$ ...

$x \in Sx \in Sx \in Sy = x^2 x \in S$

$x \in Sx \in Sx \in Sx \in Sx \in Sx \in Sx \in S$

$x \in Sxx \in Sx \in S$

xx

$x = \ldots \cdots = xa^2bb = a^2ba^2ba^2$

$a^2 = ba^2 \neq b$

ff

Appendix B

Fancy Mathematical Terms

Appendix C

Paradoxes

-
-

$nnnn$

$37^{50} S \mathbb{N} t \in \mathbb{N} \quad StStt \in S$

Appendix D

Definitions in Mathematics

$f : A \to \mathbb{R} c \in A \varepsilon > 0 \delta > 0 |x - c| < \delta x \in A |f(x) - f(c)| < \varepsilon f A f A$

$< \leq > \mathbb{R} \mathbb{Z}$

$|x - c| < \delta - \delta < x - c < \delta x c \delta$