1 Introduction to Mathematics (Continued)

1.2 Logic, Negation, Contrapositive

After diving in head first in the last section, we'll take a step back and do a more careful examination of what it is we are actually doing.

Definition 1.16. A **proposition** (or **statement**) is a sentence that is either true or false.

For example, the sentence "All liberals are hippies" is a false proposition. However, the perfectly good sentence "x=1" is *not* a proposition all by itself since we don't actually know what x is.

Exercise 1.17. Determine whether the following are propositions or not. Explain.

1. All cars are red.
2. Van Gogh was the best artist ever.
3. If my name is Joe, then my name starts with the letter J.
4. If my name starts with the letter J, then my name is Joe.
5. f is continuous.
6. All functions are continuous.
7. If f is a differentiable function, then f is continuous function.
8. The president had eggs for breakfast the morning of his tenth birthday.
9. What time is it?
10. There exists an x such that $x^2 = 4$.
11. $x^2 = 4$.
12. $\sqrt{2}$ is an irrational number.
13. For all real numbers x, $x^3 = x$.
14. There exists a real number x such that $x^3 = x$.
15. p is prime.

Given two propositions, we can form more complicated propositions using the logical connectives "and", "or", and "If..., then...".

Definition 1.18. Let A and B be propositions. The proposition "A **and** B" is true if and only if both A and B are true. The statement "A and B" is expressed symbolically as

$$A \wedge B$$

and is know as the **conjunction** of A and B.

This work is an adaptation of notes written by Stan Yoshinobu of Cal Poly and Matthew Jones of California State University, Dominguez Hills.

Definition 1.19. Let A and B be propositions. The proposition "A **or** B" is true if and only if at least one of A or B is true. The statement "A or B" is symbolically represented as

$$A \vee B$$

and is know as the **disjunction** of A and B.

Definition 1.20. Let A be a proposition. The **negation** of A, denoted $\neg A$, is true if and only if A is false.

Exercise 1.21. Describe the meaning of $\neg(A \wedge B)$ and $\neg(A \vee B)$

Definition 1.22. Let A and B represent propositions. The conditional proposition "**If** A**, then** B" is expressed symbolically as

$$A \implies B$$

and has the following truth table.

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise 1.23. Create a truth table for each of $A \wedge B$, $A \vee B$, $\neg A$ that illustrates all possible truth values. (See Definition 1.22 for an example of what a truth table looks like.)

Exercise 1.24. Let A represent "N is an even number" and B represent "N is a multiple of 4." Express the following in ordinary English sentences.

1. $A \wedge B$
2. $A \vee B$
3. $\neg A$
4. $\neg B$
5. $\neg(A \wedge B)$
6. $\neg(A \vee B)$
7. $A \implies B$

Problem 1.25. Suppose I am the coach of our co-ed dodgeball team and you all are the players. I tell you "If we win tonight, then I will buy you pizza tomorrow." After reviewing the definition of conditional proposition, determine the case(s) in which you can rightly claim to have been lied to.

Definition 1.26. Two statements are **logically equivalent** (or **equivalent** if the context is clear) if and only if they have the same truth table. That is, proposition P is true exactly when proposition Q is true, and P is false exactly when Q is false. When P and Q are logically equivalent we denote this symbolically as

$$P \Leftrightarrow Q,$$

which we read "P if and only if Q". It is common to abbreviate "if and only if" as "iff".

Theorem 1.27. If A is a proposition, then $\neg(\neg A)$ is equivalent to A.

Theorem 1.28. If A and B are propositions, then $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$. (*Note*: This theorem is referred to as DeMorgan's Law.)

Problem 1.29 (*)**.** Let A and B be propositions. Conjecture a statement similar to Theorem 1.28 for the proposition $\neg(A \vee B)$ and then prove it.

Definition 1.30. The **converse** of $A \implies B$ is $B \implies A$.

Definition 1.31. The **contrapositive** of $A \implies B$ is $\neg B \implies \neg A$.

Exercise 1.32. Let A and B represent the statements from Exercise 1.24. Express the following in ordinary English sentences.

1. The converse of $A \implies B$
2. The contrapositive of $A \implies B$

Exercise 1.33. Find the contrapositive of the following statements:

1. If n is an even natural number, then $n+1$ is an odd natural number.
2. If it rains today, then I will bring my umbrella.
3. If it does not rain today, then I will not bring my umbrella.

Exercise 1.34. Provide an example of a true conditional proposition whose converse is false.

Theorem 1.35 (*)**.** Assume A and B are statements. Then $A \Rightarrow B$ is equivalent to its contrapositive.

The upshot of Theorem 1.35 is that if you want to prove a conditional proposition, you can prove its contrapositive instead.

Problem 1.36. Let x be an integer. Prove or disprove the statement: If 6 divides x, then 3 divides x.

Problem 1.37. Let x be an integer. Prove or disprove the statement: If 6 divides x, then 4 divides x.

Theorem 1.38 (*)**.** For all integers n, x and y, if n divides x, and n divides y, then n divides $x+y$.

Theorem 1.39. For all integers n, x, and y, if n divides x, and n divides y, then n divides $x-y$.

Prove each of the next two propositions using the contrapositive of the given statement:

Theorem 1.40 (*)**.** Assume x, y are integers. If xy is odd, then both x and y are odd. (Prove using contrapositive.)

Theorem 1.41 (*)**.** Assume x, y are integers. If xy is even, then x or y is even. (Prove using contrapositive.)

Theorem 1.42. For all integers n, x, and y, if $n|x$, then $n|(xy)$.