MA 151: Applied Calculus

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Contents

Chapter 0

Brief Review

0.1 Linear Equations

Example 1. Most people's favorite version of a linear equation is this:

$$y = mx + b \qquad \text{“slope-intercept form”}$$

where

$$\begin{aligned} m &= \text{slope} \quad &&\text{(i.e. the ratio of how much the line rises, divided} \\ & &&\text{by how much the line goes horizontally),} \\ b &= y\text{-intercept} \quad &&\text{(i.e. where the line hits the } y\text{-axis).} \end{aligned}$$

Graph the following lines on the graphs paper below.

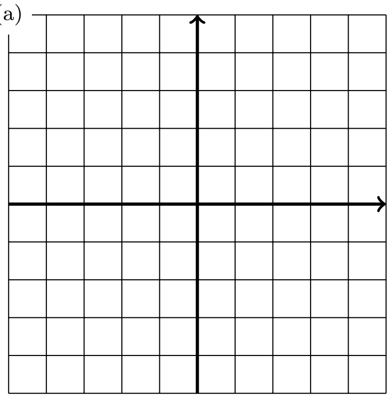
(a) $y = 3x + 2$

(b) $y = -\frac{1}{2}x - 5$

(c) $y = -\frac{2}{3}x + 5$

(d) $y = 5x - 7$

(a)



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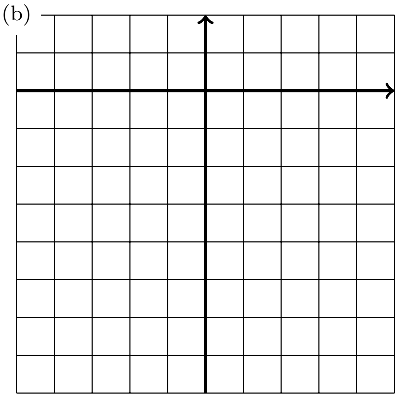

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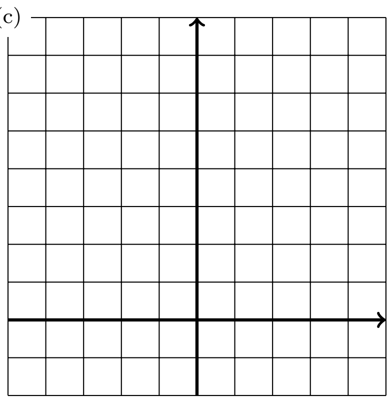

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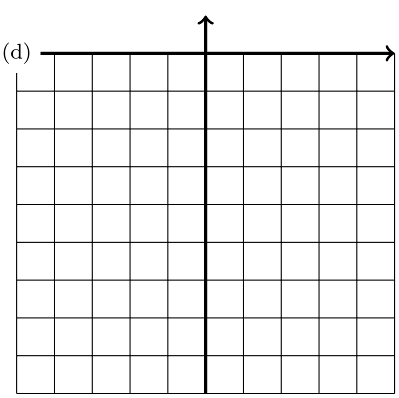


Example 2. The following equations all define a line, but are not in the usual slope-intercept form, i.e. of the form $y = mx + b$.

Turn the following equations into slope-intercept, and then graph them below

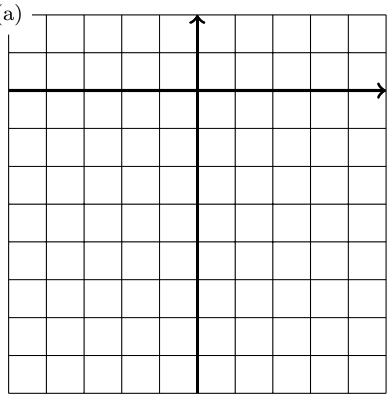
(a) $2y + x = -10$

(b) $3y + 2x = 15$

(c) $y = 5(x + 2) - 17$

(d) $y - 10 = 3x - 8$

(a)



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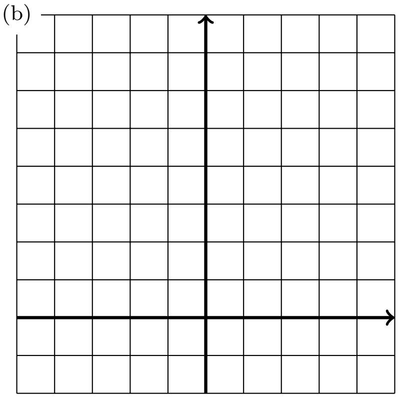

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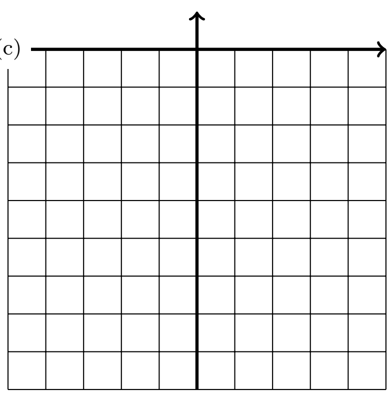

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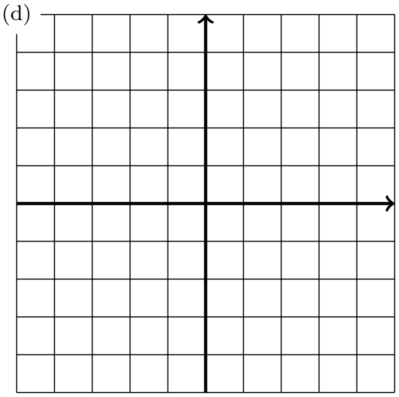


Example 3. In some problems the quickest way to write a linear equation is like this

$$y = m(x - x_0) + y_0 \qquad \text{"point-slope form*"}$$

where

$$m = \text{a given slope,}$$
$$(x_0, y_0) = \text{a given point.}$$

(a) Find the point-slope form equation of the line through the point $(-2, 3)$ with slope 5.

(b) Turn the equation from (a) into slope-intercept form.

(c) Find the point-slope form equation of the line through the point $(-2, 3)$ with slope $-1/2$.

(d) Turn the equation from (c) into slope-intercept form.

*Sometimes people write point-slope as $y - y_0 = m(x - x_0)$. That's ok, there's more than one way to write it. But the version I've given is more useful because it's written as an explicit function, and in any case it's the version I want you to use.

Example 4. This example is meant to show that sometimes it makes sense to think about a problem using the point-slope form of a line.

Suppose that today my son is 52 inches tall and growing at 1.5 inches per year.

(a) Roughly speaking, how tall will he be tomorrow?

(b) How tall will he be in one year?

(c) How tall will he be in two years?

(d) Write a formula for y (=height) as a function of t (=the calendar year), using point-slope form.

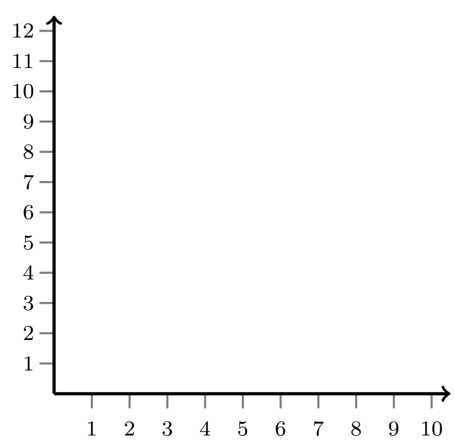
Example 5. This example is meant to show that it's actually quite easy to graph a line in point-slope form.

(a) On the graph paper below, graph the point $(5, 7)$ with a circle about like this •

(b) Add to the graph a second large point, • that is 2 places to the right right and 3 places up; mark the distances of 2 and 3 with dashed lines.

(c) Draw a line through the two points you have labeled.

(d) Describe what the graph you made has to do with the line $y = \frac{3}{2}(x - 5) + 7$.



0.2 Fractions

Example 1. (a) Add the fractions, and simplify if possible: $\frac{5}{14} + \frac{7}{14}$.

(b) Add the fractions, and simplify if possible: $\frac{17}{x} + \frac{3}{x}$.

(c) Get a common denominator and combine the fractions: $\frac{3}{10} + \frac{8}{15}$.

(d) Get a common denominator and combine the fractions: $\frac{3}{7} + \frac{2}{11}$.

(e) Get a common denominator and combine the fractions: $\frac{3}{7} + \frac{x}{11}$.

(f) Get a common denominator and combine the fractions: $\frac{3}{x} + \frac{x}{11}$.

(g) Multiply the fractions, and simplify if possible: $\frac{-5}{3} \cdot \frac{7}{10}$

(h) Multiply the fractions, and simplify if possible: $\frac{x}{2} \cdot \frac{x}{7}$.

(i) Multiply the fractions, and simplify if possible: $\frac{3x}{2} \cdot \frac{-13}{5x}$.

(j) Simplify until you get a single fraction, with no compound fractions: $x\left(\frac{1+\frac{1}{x}}{x+\frac{1}{x}}\right)$

0.3 Exponents

Example 1. Recall:

$$a^{-b} \text{ means } \frac{1}{a^b} \qquad (a^n)^m = a^{nm} \qquad \frac{a^n}{a^m} = a^{n-m}$$
$$a^{1/b} \text{ means } \sqrt[b]{a} \qquad a^n a^m = a^{n+m} \qquad (ab)^n = a^n b^n$$

Using the above properties, simplify the following.

(a) $(-2)^5$

(b) $\dfrac{x^{17}}{x^{22}}$

(c) $4^{-3/2}$

(d) $\sqrt{36x^4}$

Example 2. (a) Simplify the following

$$\frac{-2x^{-4}y^6}{3x^3y^{-3}}$$

so that your final is written using only exponents, no fractions, and each base, 2, 3, x and y, appears only once.

(b) Challenge Problem: Simplify the following

$$\left(\frac{(-2x^{-4}y^6)^{-8}}{(3x^3y^{-3})^{-2}}\right)^{-2}$$

so that your final answer has no fractions, and each base, 2, 3, x and y, appears only once.

0.4 Square roots

Example 1. Recall that $(5 \cdot 7)^2 = 5^2 \cdot 7^2$. Since this is true, a similar result holds for square roots: $\sqrt{5 \cdot 7} = \sqrt{5} \cdot \sqrt{7}$.

(a) Simplify the following: $\sqrt{4 \cdot 3}$

(b) Simplify the following: $\sqrt{49x}$ (assume that $x > 0$)

(c) Simplify the following: $\sqrt{7x^2}$ (assume that $x > 0$)

0.5 Grouping and expanding terms

Example 1. Simplify the following:

$$(3y^3 + 9y^2 - 11y + 8) - (-4y^2 + 10y - 6)$$

Example 2. Simplify the following:

$$(3x - 1)(x + 2) - (2x + 5)^2$$

0.6 Quadratics

Example 1. A quadratic function has the following form

$$y = ax^2 + bx + c$$

Match the following quadratics with their graphs: see if you can do this without using your calculator.

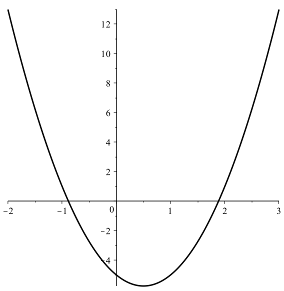
(a) $y = x^2$

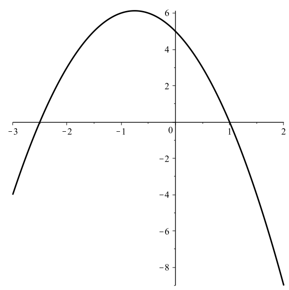
(b) $y = (x + 2)^2$

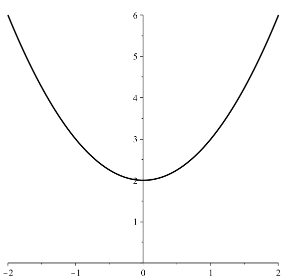
(c) $y = x^2 + 2$

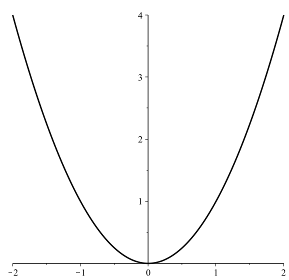
(d) $y = -2x^2 - 3x + 5$

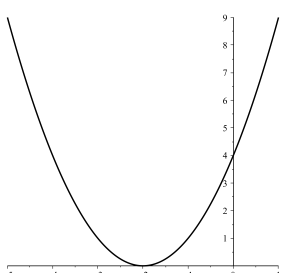
(e) $y = 3x^2 - 3x - 5$











Example 2. Factoring a quadratic means to write it as a product. Usually you shouldn't bother to factor a quadratic unless the x^2-coefficient equals 1. In that case, you're trying to write it like this:

$$y = x^2 + bx + c = (x + d)(x + e)$$

There are various tricks in finding d and e, but honestly, in this case, I usually just guess and check as follows: (1) guess two values, d and e, that multiply together to give you c, and then (2) check to see if they add up to b. Note: when I write "$+b$" and "$+c$" and say "add" I'm also including negative numbers in there.

(a) Factor and solve $x^2 + 2x + 1 = 0$.

(b) Factor and solve $x^2 + 3x + 2 = 0$.

(c) Factor and solve $x^2 - 3x + 2 = 0$.

(d) Factor and solve $x^2 - x - 2 = 0$.

(e) Factor and solve $x^2 - x - 12 = 0$.

(f) Factor and solve $x^2 - 8x + 12 = 0$.

Example 3. A lot of times it's not worth factoring a quadratic, or it may be impossible. In these cases, just use the quadratic formula

$$ax^2 + bx + c = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(a) Apply the quadratic formula to $x^2 - x - 12 = 0$.

(b) Apply the quadratic formula to $2x^2 + 3x - 2 = 0$.

Chapter 1

Functions and Change

1.1 What is a Function?

Example 1. (a) $f(x) = x^2$, then $f(5) = ?$

(b) $g(t) = \sqrt{t^2 + 1}$, $\qquad g(0) = ?$, $\qquad g(-3) = ?$

(c) $f(t + 1) = ?$

(d) $g(x + h) = ?$

Example 2. The historic Senator Theater is the nearest movie theater to Loyola. Their ticket prices for adults (non-students) seeing a 3D movie are \$13.50 for movies after 6 PM, \$11 for a matinee (noon – 6 PM), and \$9 for an early bird show (before 11 AM). Write a piecewise function $P(t)$ for the price of the ticket where t is the time of the showing.

Example 3. Since July 1, 2016, the Maryland minimum wage is \$8.75 per hour (FYI: it will increase to \$9.25 on July 1, 2017). Suppose someone is able to make time and a half per hour of overtime (over 40 hours). If x is the number of hours worked for this week and $f(x)$ is the income function for (gross) income earned that week, answer the following:

(a) Make up an integer A such that $A < 40$, and then calculate $f(A) =?$

(b) Make up an integer B such that $B > 40$, and then calculate $f(B) =?$

(c) Make up a non-integer C that $C < 40$, and then calculate $f(C) =?$ (For instance, $f(10.5) =?$)

(d) Make up a non-integer D that $D > 40$, and then calculate $f(D) =?$

(e) $f(x) =?$ in general

Example 4. What are the domains of these functions?

(a) $f(x) = 3x - 5$

(b) $g(x) = \sqrt{x+5}$

(c) $h(x) = \dfrac{1}{x}$

(d) $F(t) = \dfrac{\sqrt{5-t}}{t+2}$

1.2 Linear Functions

Example 1. For the two points $(1, 2)$ and $(-5, 4)$ what is the slope of the line connecting them?

Example 2. Find an equation of the line that passes through $(1, 2)$ and $(-5, 4)$.

Example 3. A cab company has an initial charge of \$4.00 plus \$2.20 per mile. Find a formula for the cab fare, C, in dollars, as a function of the number of miles, m.

Example 4. ACME company has seen a decline in sales of their product. In 2010 they sold 28.4 million, while in 2016 they sold 22.7 million.

(a) Find a formula for annual sales S, in millions of items, as a linear function of the years t, since 2010.

(b) Predict the sales in 2019.

1.3 Rates of change

Example 1 (Problem 12)**.** Table 1.14 shows world bicycle production (from `http://www.earth-policy.org/indicators/C48/`, accessed April 19, 2005.)

Table 1.14 *World bicycle production, in millions*

Year	1950	1960	1970	1980	1990	2000
Bicycles	11	20	36	62	92	101

(a) Find the change in bicycle production between 1950 and 2000. Give units.

(b) Find the average rate of change in bicycle production between 1950 and 2000. Give units and interpret your answer in terms of bicycle production.

Example 2. Find the average rate of change of $f(x) = 4x^2 - 2$ between $x = -1$ and $x = 3$.

Example 3 (Problem 46*)**.** Consider two situations: (1) A company has an increase in sales from \$100,000 to \$500,000; (2) A company has an increase in sales from \$20,000,000 to \$20,500,000.

(a) Which absolute change is bigger?

(b) Which relative change is bigger? Justify your answer.

1.4 Applications of Functions to Economics

Example 1 (Problem 20)**.** A company producing jigsaw puzzles has fixed costs of \$6000 and variable costs of \$2 per puzzle. The company sells the puzzles for \$5 each.

(a) Find formulas for the cost function, the revenue function, and the profit function.

(b) Sketch a graph of $R(q)$ and $C(q)$ on the same axes. What is the break-even point, q_0, for the company?

(c) What is the marginal cost?

Example 2 (Problem 32)**.** The demand curve for a product is given by $q = 120,000 - 500p$ and the supply curve is given by $q = 1000p$ for $0 \leq q \leq 120,000$, where price is in dollars.

(a) At a price of \$100, what quantity are consumers willing to buy and what quantity are producers willing to supply? Will the market push prices up or down?

(b) Find the equilibrium price and quantity. Does your answer to part (a) support the observation that market forces tend to push prices closer to the equilibrium price?

1.5 Exponential Functions

Example 1 (Problem 6)**.** The gross domestic product, G, of Switzerland was 310 billion dollars in 2007. Give a formula for G (in billions of dollars) t years after 2007 if G increases by

(a) 3% per year

(b) 8 billion dollars per year

Example 2 (Loyola's Tuition part I)**.** For school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. For school year 2016–2017, the annual tuition at Loyola was \$45,030. Over this time the tuition grew exponentially with an annual percentage rate of growth of 2.47%.

Assuming that the tuition continues to grow at the same rate, what will it be for the 2019–2020 school year?

Example 3 (Loyola's Tuition part II)**.** For school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. For school year 2016–2017, the annual tuition at Loyola was \$45,030.

Find r, the relative growth rate, so that this growth is modeled by an exponential equation.

1.6 Natural Logarithm

Example 1 (Problem 2)**.** Solve

$$10 = 2^t$$

using natural log.

Example 2 (Loyola tuition part III)**.** In the school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. Since then it has had an annual growth rate of $r = 2.47\%$. Assuming this growth rate continues, when will the tuition reach \$52,000?

Example 3. A city's population starts at 600,000 in 2010 and has a continuous growth rate of 5%. What is the population size in 2017?

1.7 Exponential Growth and Decay

Example 1 (Problem 14*)**.** A population, currently 200, is growing at 5% per year.

(a) Write a formula for the population, P, as a function of time, t, years in the future.

(b) Graph P against t.

(c) Estimate the population 10 years from now.

(d*) Find the doubling time of the population algebraically.

(e*) Model the same population using a continuous growth rate, compare the graph of this model with the graph from part (b).

Example 2. (a) Find the future value in 8 years of a \$7,000 payment today, if the interest rate is 3.5% compounded continuously.

(b) Find the present value of a \$7,000 payment that will be made 8 years from now if the interest rate is 3.5% compounded continuously.

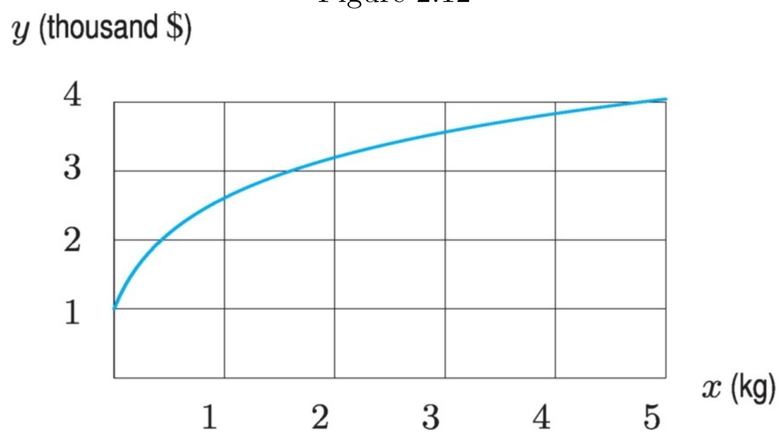
Chapter 2

The Derivative

2.1 Tangent and Velocity Problems

Example 1 (Problem 5)**.** Figure 2.12 shows the cost, $y = f(x)$, of manufacturing x kilograms of a chemical.

Figure 2.12



(a) Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$? Explain your answer graphically.

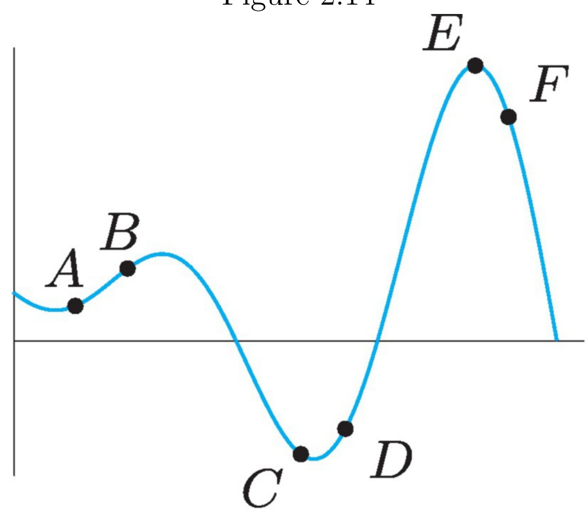
(b) Is the instantaneous rate of change of the cost of producing x kilograms greater at $x = 1$ or at $x = 4$? Explain your answer graphically.

(c) What are the units of these rates of change?

Example 2 (Problem 12)**.** Match the points labeled on the curve in Figure 2.14 with the given slopes.

Slope	Point
-3	
-1	
0	
$1/2$	
1	
2	

Figure 2.14



Example 3. Suppose we drop a penny from the roof of a very tall building. Then the distance fallen is given by

$$s(t) = 4.9t^2,$$

where s is measured in meters, and t is the number of seconds since the penny has been dropped.

(a) Find the average velocity from $t = 0$ to $t = 7$.

(b) Estimate the instantaneous velocity at $t = 7$.

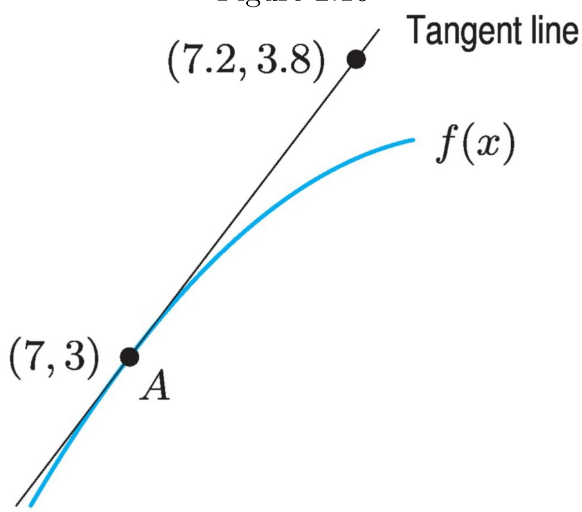
Example 4. Using a calculator or an equivalent app, estimate $f'(1)$ for $f(x) = 3x^2$.

Example 5 (Problem 18)**.** Use Figure 2.16 to fill in the blanks in the following statements about the function f at point A.

(a) $f(\_\_\_\_\_\_) = \_\_\_\_\_\_$

(b) $f'(\_\_\_\_\_\_) = \_\_\_\_\_\_$

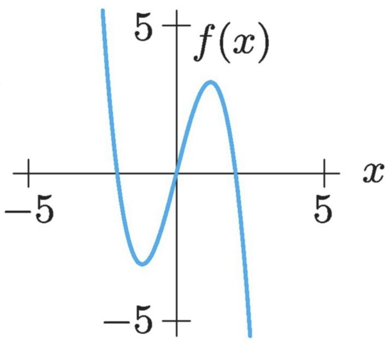
Figure 2.16



2.2 The Derivative as a Function

Example 1 (Problems 18–21)**.** Match the functions in Problems 18–21 with one of the derivatives in Figure 2.25.

#18



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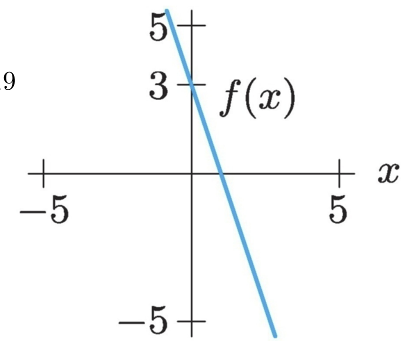

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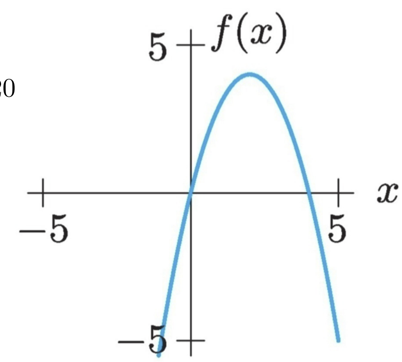

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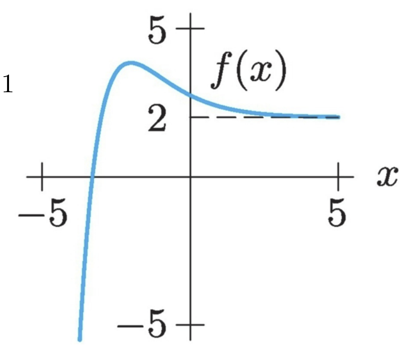
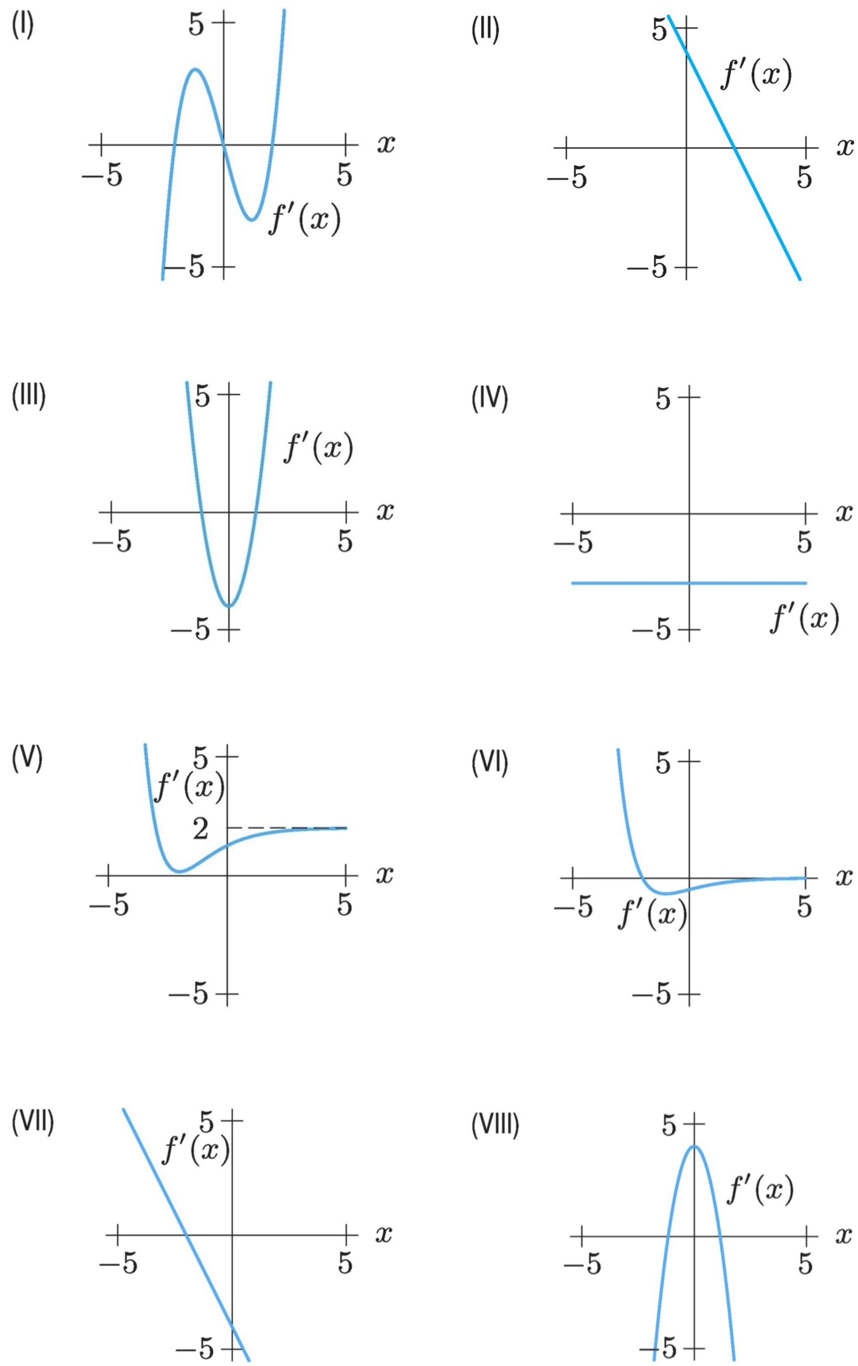


Figure 2.25


(I)
5
−5
5
−5
x
$f'(x)$
(II)
5
−5
5
−5
x
$f'(x)$
(III)
5
−5
5
−5
x
$f'(x)$
(IV)
5
−5
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−5
x
$f'(x)$
(V)
5
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−5
5
−5
x
$f'(x)$
(VI)
5
−5
5
−5
x
$f'(x)$
(VII)
5
−5
5
−5
x
$f'(x)$
(VIII)
5
−5
5
−5
x
$f'(x)$

2.3 Variations on the derivative

Example 1 (Problems 2 and 4)**.** Write the Leibniz notation for the derivative of the given function and include units.

#2. The cost, C, of a steak, in dollars, is a function of the weight, W, of the steak, in pounds.

#4. An employee's pay, P, in dollars, for a week is a function of the number of hours worked, H.

Example 2 (Problem 6)**.** An economist is interested in how the price of a certain item affects its sales. At a price of \$$p$, a quantity, q, of the item is sold. If $q = f(p)$, explain the meaning of each of the following statements:

(a) $f(150) = 2000$

(b) $f'(150) = -25$

Example 3. The cost, C (in dollars), to produce ℓ liters of a chemical can be expressed as $C = f(\ell)$. Using units, explain the meaning of the following statements in terms of the chemical:

(a) $f(350) = 1750$

(b) $f'(350) = 9$

Example 4. For the function $f(x) = 2\ln(x)$ first

(a) Use a table of numbers to approximate $f'(1)$, and to write the equation of the tangent line at the point $(1, 0)$.

(b) Using linear approximation and your answer to part (a), to approximate $f(1.01)$, $f(1.001)$.

Example 5 (Problem 46)**.** The area of Brazil's rain forest, $R = f(t)$, in million acres, is a function of the number of years, t, since 2000.

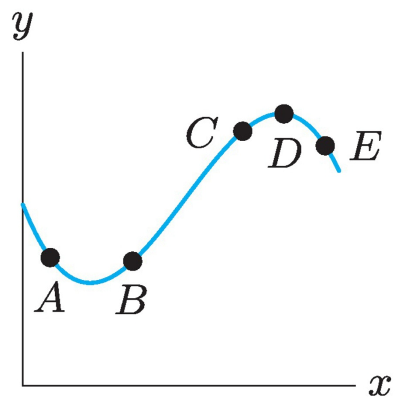
(a) Interpret $f(9) = 740$ and $f'(9) = -2.7$ in terms of Brazil's rain forests.

(b) Find and interpret the relative rate of change of $f(t)$ when $t = 9$.

Example 6 (Problem 50(b*))**.** The world population in billions is predicted to be approximately $P = 7.1e^{0.011t}$ where t is in years since 2013. Estimate the relative rate of change of population in 2018 using this model and $\Delta t = 0.1$.

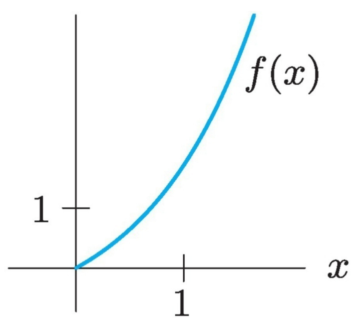
2.4 The second derivative

Example 1 (Problem 2). At which of the labeled points, if any, are both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ positive?

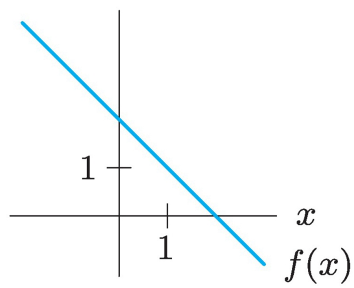


Example 2 (Problems 4, 6, 8). Give the signs of the first and second derivatives for the following functions. Each derivative is either positive everywhere, zero everywhere, or negative everywhere.

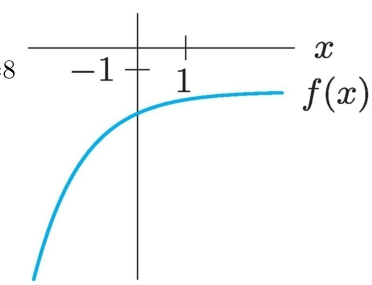
#4



#6



#8



Example 3. The temperature outside on a given day is given by $f(t)^\circ C$, where t is in hours since midnight. From 6 AM until noon, the first derivative was negative and the second was positive. Which of the following is correct? You may choose more than one.

This poll should be done through Poll Everywhere and then discussed online.

(a) The temperature was below freezing but getting warmer.

(b) The temperature was below freezing and getting colder.

(c) We do not no whether the temperature was above or below freezing.

(d) The temperature was higher at noon than at 6 AM.

(e) The temperature was lower at noon than at 6 AM.

(f) The temperature was rising but at a slower rate as the morning progressed.

(g) The temperature was rising but at a faster rate as the morning progressed.

(h) The temperature was falling and at a faster rate as the morning progressed.

(i) The temperature was falling but at a slower rate as the morning progressed.

Example 4. Let $P(t)$ represent the price of a share of stock of a corporation at time t. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

(a) "The price of the stock is falling faster and faster."

(b) "The price of the stock is getting close to its peak, at which it will remain for a little while."

(c) "The price of the stock is skyrocketing."

2.5 Marginal Cost and Revenue

Example 1. It costs \$2500 to produce 1350 items and it costs \$2545 to produce 1360 items. What is the approximate marginal cost when producing 1350 items?

Example 2 (Problem 4*)**.** Figure 2.55 shows a total cost function, $C(q)$:

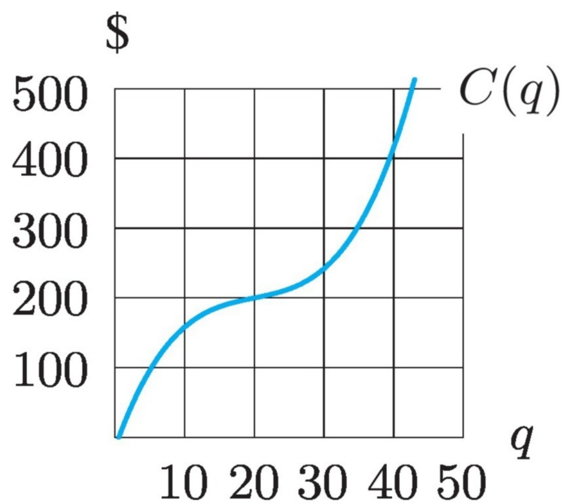


Figure 2.55

(a) Estimate the marginal cost when the production level is 20 and interpret it.

(b) Is the marginal cost greater at $q = 5$ or at $q = 30$? Explain.

(c) Is the marginal cost greater at $q = 20$ or at $q = 40$? Explain.

Example 3 (Problem 8)**.** Figure 2.57 shows part of the graph of cost and revenue for a car manufacturer. Which is greater, marginal cost or marginal revenue, at

(a) q_1?

(b) q_2?

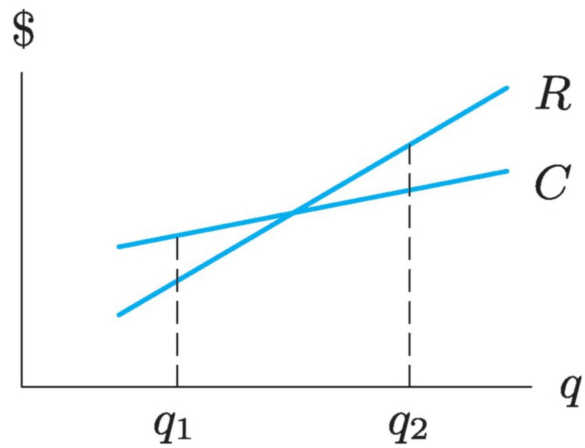


Figure 2.57

Example 4. To produce 2000 items, the total cost is \$4000 and the marginal cost is \$15 per item. Estimate the costs of producing:

(a) 2001 items

(b) 1999 items

(c) 2050 items

Example 5 (Problem 12)**.** Cost and revenue functions for a charter bus company are shown in Figure 2.58. Should the company add a 50th bus? How about a 90th? Explain your answers using marginal revenue and marginal cost.

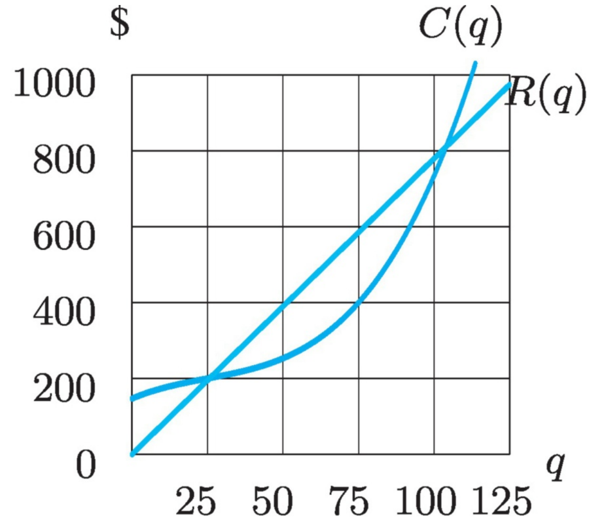


Figure 2.58

Chapter 3

Rules for Derivatives

3.1 Shortcuts for powers of x, constants, sums, and differences

Example 1. Let $f(x) = 3x^2 - 5x + 8$. Find $f'(x)$.

Example 2. (a) For $f(x) = 6\sqrt{x}$ find $f'(x)$.

(b) For $C(q) = q^{13} - \dfrac{5}{q^3} + 7$, find the marginal cost.

Example 3. (a) For $y = 2.5q^2 - 0.75q + 9.23$, find y''.

(b) For $C(q) = q(q^2 + q^{-2})$, find $C''(q)$.

Example 4. Let $f(x) = 3x^2 - 4x + 1$.

(a) Find the equation of the tangent line to f at $(1, 0)$

(b) Find when f has a horizontal tangent line.

3.2 Derivatives of exponentials and logarithms

Example 1. Let $f(x) = 3x^3 + 2e^x$

(a) Find $f'(x)$.

(b) Find the equation of the tangent line at $x = 0$.

(c) Compare the graph of $f(x)$ and the graph of the tangent line.

Example 2. The human population of the entire world can be modeled by $P = 6.8(1.011)^t$ where P is in billions, and t is the year with $t = 0$ corresponding to 2010 (source Wikipedia).

Find the estimated rate of growth in 2020, and interpret your answer, with units.

Example 3. Find the marginal revenue function if $R(q) = 4q^2 + 7\ln(q)$.

3.3 The Chain Rule

Example 1. Suppose we are given $A(t) = 1000e^{0.149t}$. Find $A'(1)$.

Example 2. Find the derivatives of

(a) $R = (q^3 - 5q + 7)^5$

(b) $h(x) = \dfrac{17}{\sqrt{3 + 5x^2}}$

Example 3. Find the derivatives of

(a) $y = 5e^{6x} + e^{-x}$

(b) $y = e^{3x^2 - 7x + 11}$

Example 4. Find the derivative of $f(x) = \ln(x^2 + 5)$.

3.4 Product and Quotient Rules

Example 1. Use the Product Rule to differentiate $y = x^2 e^{5x-3}$.

Example 2. Find the derivatives of the following:

(a) $y = x^3(2x - 7)^4$

(b) $y = 3t^4 \ln(t)$

Example 3. Use the Quotient Rule to differentiate $y = \dfrac{4t+5}{2-3t^2}$.

Example 4. Let $f(x) = \dfrac{e^x}{2x + e^x}$.

(a) Find $f'(x)$.

(b) Find the equation of the tangent line at $x = 0$.

Chapter 4

Using the Derivative

4.1 Local Max and Mins

4.1.1 Definition. Suppose c is in the domain of f:

- f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for x near c.
- f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for x near c.

4.1.2 Definition. The point $(c, f(c))$ is a **critical point** of a function f if either

- $f'(c) = 0$ or
- $f'(c)$ is undefined.

The x-value, c, is a **critical number** of f.
The y-value, $f(c)$, is a **critical value** of f.

Example 1 (Problem 5*). (a) Sketch the graph of a function with two local maxima and one local minimum.

(b) Sketch the graph of a function that has two critical points. One should be a local maximum and one should be neither a local maximum nor local minimum.

4.1.3 Test (First Derivative Test). Suppose c is a critical point of a continuous function f. When moving from left to right:

- If $f'(x)$ changes from positive to negative at c, then f has a *local maximum* at c.
- If $f'(x)$ changes from negative to positive at c, then f has a *local minimum* at c.
- If $f'(x)$ does not change sign from at c, then f does not have a local extremum at c.

4.1.4 Test (Second Derivative Test)**.** Suppose c is a critical number for f and $f'(c) = 0$.

- If $f''(c) < 0$, then f has a *local minimum* at c.
- If $f''(c) > 0$, then f has a *local maximum* at c.
- If $f''(c) = 0$, then the Second Derivative Test tells us nothing.

Example 2. Find all local extrema of the function below, using the Second Derivative Test:

$$f(x) = \frac{2}{3}x^3 - 4x^2 - 42x.$$

Example 3. Find and classify all the critical points of the function

$$f(x) = 2x^5(2x-1)^4 + 7.$$

4.2 Inflection points

4.2.1 Definition. An **inflection point** for a function $f(x)$ is a point on the graph of $f(x)$ where the concavity changes.

An inflection point is where $f''(x)$ changes from positive to negative, or from negative to positive.

Example 1. Find all critical points and inflection points of the function

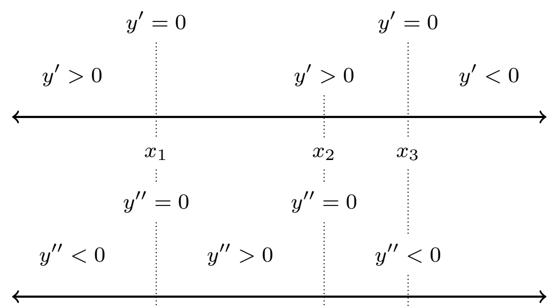
$$f(x) = x^3 - 12x + 8.$$

Identify each critical point as a local max, local min, or neither.

Example 2. Find all inflection points of each of the functions below

$$f(x) = x^9 \quad \text{and} \quad g(x) = x^6$$

Example 3 (Problem #26)**.** Sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. (Assume that the function is defined and continuous for all real x.)



Example 4. Graph a function with the given properties.

(a) Has local minimum and global minimum at $x = 3$ but no local or global maximum.

(b) Has local minimum at $x = 3$, local maximum at $x = 8$, but no global maximum or minimum.

(c) Has no local or global maxima or minima.

(d) Has local and global minimum at $x = 3$, local and global maximum at $x = 8$.

4.3 Global max and min

4.3.1 Test (Global Max/Min Test)**.** To find the global max and global min of $f(x)$ on an interval:

(a) Find the critical numbers

(b) Compare the values for $f(x)$ at the critical numbers and at the ends of the interval.

"Values for $f(x)$" means you plug the critical numbers and ends of the interval into $f(x)$ and calculate the result.

Example 1. For the function

$$f(x) = x^5 - 2x^4, \quad -1 \leq x \leq 2$$

identify any global maxima and minima of f in the given interval.

Example 2. The energy expended by a bird per day, E, depends on the time spent foraging for food per day, F hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense. Find the foraging time that minimizes energy expenditure if

$$E = 0.25F + \frac{1.7}{F^2}$$

4.4 Optimizing Cost and Revenue

Class Discussion. Suppose you're looking at a demand curve. What does the point on the curve where $p = 0$ mean? What does the point where $q = 0$ mean?

Example 1. Let $C(q)$ be the total cost of producing a quantity q of a certain product. See Figure below (Figure 4.52 in the text).

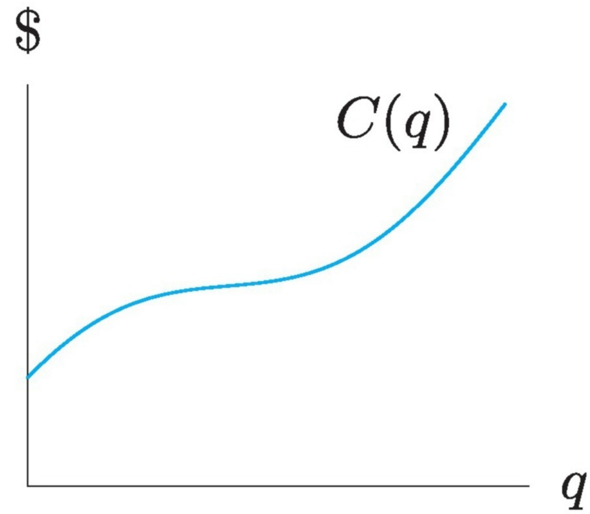
(a) What is the meaning of $C(0)$?

(b) Describe in words how the marginal cost changes as the quantity produced increases.

(c) Explain the concavity of the graph (in terms of economics).

(d) Explain the economic significance (in terms of marginal cost) of the point at which the concavity changes.

(e) Do you expect the graph of $C(q)$ to look like this for all types of products?



Example 2. A demand function is $p = 400 - 2q$, where q is the quantity of the good sold for price $\$p$.

(a) Find an expression for the total revenue R, in terms of q.

(b) Find the marginal revenue, MR, in terms of q. Calculate the marginal revenue when $q = 10$.

(c) Compare with the change in total revenue when production changes from $q = 10$ to $q = 11$ using the revenue function to the approximation in change in revenue using MR.

Example 3. The demand equation for a product is $p = 295 - 0.2q$. Write the revenue as a function of q and find the quantity that maximizes revenue. What price corresponds to this quantity? What is the total revenue at this price?

Example 4. (a) Production of an item has fixed costs of \$9,500 and variable costs of \$175 per item. Express the cost, C, of producing q items.

(b) The relationship between price, p, and quantity, q, demanded is linear. Market research shows that 10,500 items are sold when the price is \$280 and 13,000 items are sold when the price is \$250. Express p as a function of price q.

(c) Find the profit function $P(q)$.

(d) How many items should the company produce to maximize profit? (Give your answer to the nearest integer.) What is the profit at that production level? What is the price charged at that production level?

Example 5. Suppose you are making something, say T-shirts, and you want to model how much revenue you'll bring in. You know the demand curve of your T-shirts, i.e. how many T-shirts you'll sell at a given price. You can write the demand curve in two ways, $q = Q(p)$, i.e. quantity sold depends on the price you set, or $p = P(q)$, i.e. the price you set should depend on how many you want to sell.

Which do you think makes more sense: use $q = Q(p)$ and write R as a function p, so $R(p) = p \times Q(p)$, or use $p = P(q)$ and write R as a function of q, so $R(q) = P(q) \times q$?

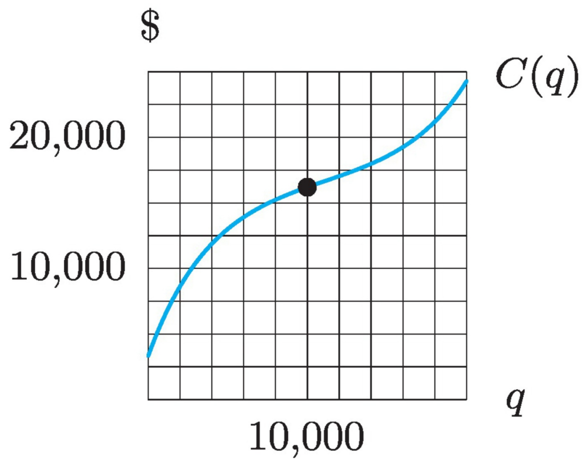
4.5 Average Cost

Example 1 (Problem 2)**.** Figure 4.63 shows cost with $q = 10,000$ marked.

(a) Find the average cost when the production level is 10,000 units and interpret it.

(b) Represent your answer to part (a) graphically.

(c) At approximately what production level is average cost minimized?



Example 2. The cost function is $C(q) = 1000 + 20q$. Find the marginal cost to produce the 200th unit and the average cost of producing 200 units.

Example 3 (Problem 9)**.** The average cost per item to produce q items is given by

$$a(q) = 0.01q^2 - 0.6q + 13, \text{ for } q > 0.$$

(a) What is the total cost, $C(q)$, of producing q goods?

(b) What is the minimum marginal cost? What is the practical interpretation of this result?

(c) At what production level is the average cost a minimum? What is the lowest average cost?

(d) Compute the marginal cost at $q = 30$. How does this relate to your answer to part (c)? Explain this relationship both analytically and in words.

4.6 Elasticity of Demand

4.6.1 Definition. 4.6 Formulas for Elasticity Let q be the quantity of some product demanded (bought) when the price is p (so q is a function of p).

- **elasticity** E defined as
$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right|$$
- E approximated by
$$E \approx \left| \frac{\Delta q/q}{\Delta p/p} \right|$$
- E interpreted as: percentage change in demand, compared to percentage change in price.
- predicting percentage change in demand:
$$\frac{\Delta q}{q} \approx -E \frac{\Delta p}{p}$$
- $E > 1$ means **elastic demand**
  $E < 1$ means **inelastic demand**

Example 1. The elasticity of the demand for eggs is 0.43 and the elasticity of fresh tomatoes is 2.22. What is the effect on the quantity demanded of both eggs and tomatoes of

(a) a 10% increase in price?

(b) a 15% decrease in price?

Example 2. In Fall 2013, the undergraduate enrollment at Loyola University Maryland was 3875 and the tuition was \$41850 per year (information taken from the 2013–2014 Loyola Catalogue). The elasticity of demand for a 4 year college is 0.10 (according to `http://centerforcollegeaffordability.org/archives/1336`).

(a) Will a 5% increase in tuition cause total revenue to go up or go down?

(b) Can you find a way to predict this answer without repeating all the calculations?

4.6.2 Test. 4.6 Critical Points in Elasticity In general, the elasticity determines whether R is an increasing function of p or not:

If $E < 1$ then increasing p will increase R
If $E > 1$ then increasing p will decrease R
If $E = 1$ then R is at a critical point.

Example 3. The demand function of Loyola T-shirts is $q = 1500 - 125p$.

(a) Find R when $p = \$5$.

(b) Find E when $p = \$5$.

(c) When $p = \$5$, find out if R is increasing or decreasing (i.e. will increasing p make R increase or decrease?). Do the problem in two different ways: by using the Elasticity, and by finding R as a function of p and using the derivative.

Chapter 5

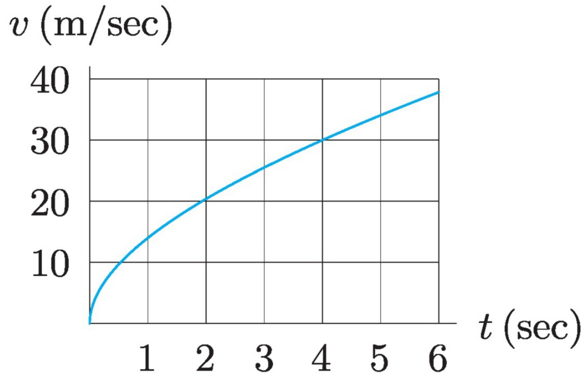
Accumulated Change: the Definite Integral

5.1 Distance and Accumulated Change

Example 1. The odometer on our car is broken, but we really need an estimate of how far we're driving. The speedometer readings are shown below; use them to estimate the distance traveled over the first 30 minutes. Find a lower estimate and an upper estimate.

Time (min)	0	10	20	30
Velocity (mi/h)	17	32	35	37

Example 2. The figure below shows the velocity, v, of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.



Example 3. The velocity of a vehicle on a track is given by $v(t) = 9t$ m/s. Find the exact distance traveled by this vehicle from $t = 2$ to $t = 10$ seconds.

5.2 The Definite Integral

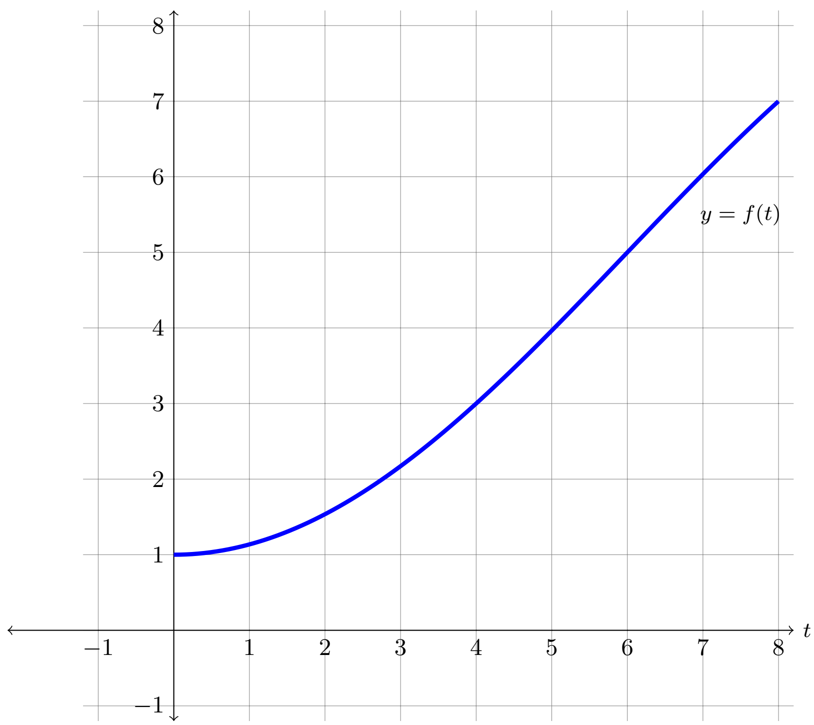
Example 1. Using the graph of $f(t)$ below, draw rectangles representing each of the following Riemann sums for the function $f(t)$ on the interval $0 \leq t \leq 8$ (or $t \in [0, 8]$). Calculate the value of each sum.

(a) Left-hand sum with $\Delta t = 4$ ($n =$?)

(b) Right-hand sum with $\Delta t = 4$ ($n =$?)

(c) Left-hand sum with $\Delta t = 2$ ($n =$?)

(d) Right-hand sum with $\Delta t = 2$ ($n =$?)



5.3 The Definite Integral as Area

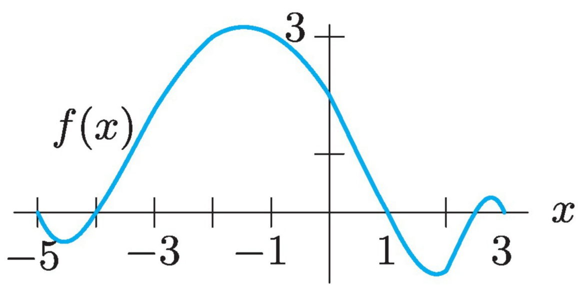
Example 1 (Problem 7)**.** Using the figure below (Figure 5.36 in the text), decide whether each of the following definite integrals is positive or negative.

(a) $\displaystyle\int_{-5}^{-4} f(x)\,dx$

(b) $\displaystyle\int_{-4}^{1} f(x)\,dx$

(c) $\displaystyle\int_{1}^{3} f(x)\,dx$

(d) $\displaystyle\int_{-5}^{3} f(x)\,dx$



Example 2. The following graph shows the function f. Evaluate the integrals.

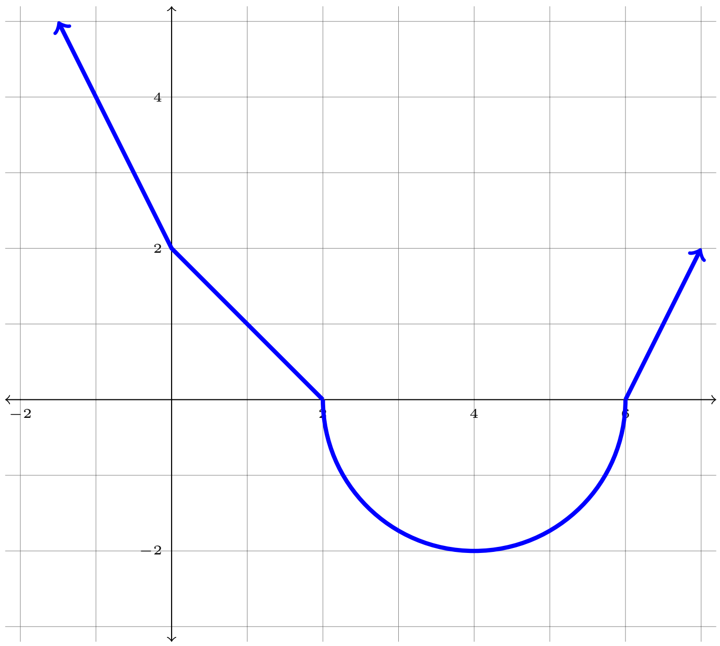
(a) $\displaystyle\int_{-1}^{0} f(x)\,dx$

(b) $\displaystyle\int_{0}^{2} f(x)\,dx$

(c) $\displaystyle\int_{2}^{4} f(x)\,dx$

(d) $\displaystyle\int_{0}^{4} f(x)\,dx$

(e) $\displaystyle\int_{0}^{6} f(x)\,dx$



5.4 Interpretations of the Definite Integral

Example 1. Explain in words what each integral represents and give the units

(a) $v(t)$ is velocity in mph and t is time in hours,

$$I = \int_2^5 v(t)\ dt.$$

(b) $a(t)$ is acceleration in m/s^2 and t is in seconds,

$$I = \int_3^4 a(t)\ dt.$$

(c) $f(t)$ is the rate at which water is flowing out of a water main break in liters per seconds, and t is in seconds,

$$I = \int_0^3 f(t)\ dt.$$

5.5 Total Change and the Fundamental Theorem of Calculus

Example 1. The marginal cost $C'(q)$ of making T-shirts is shown below. Suppose the fixed cost is \$100.

q	0	20	40	60	80	100
$C'(q)$	10	4.67	3.95	3.58	3.33	3.15

(a) Estimate the total cost of making 60 T-shirts.

(b) What is the total variable cost of making 60 T-shirts?

(c) Estimate the difference in cost between making 60 T-shirts and 100.

Example 2. A cup of coffee is put into a 70 °F room when $t = 0$. The temperature (in °F) of the coffee t minutes after being in the room is given by

$$H(t) = 110e^{-0.1672t} + 70.$$

(a) Find $H'(t)$ and explain in words what this represents.

(b) What is $H(0)$ and what does it represent?

(c) What does $\int_2^4 H'(t)\,dt$ represent, and what is that value?

(d) How much does the temperature change in the first 5 minutes in the room?