Math 1010
Introduction to Calculus

Summer 2016 Edition, University of Lethbridge

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Contributing Textbooks

Precalculus, Version $\lfloor \pi \rfloor = 3$

Carl Stitz and Jeff Zeager

www.stitz-zeager.com

APEX *Calculus*

Gregory Hartman et al

apexcalculus.com

This version of the text assembled and edited by Sean Fitzpatrick, University of Lethbridge, May, 2016.

Contents

Preface

One of the challenges with a new course like Math 1010 is finding a suitable textbook for the course. This is made additionally difficult for a course that covers two topics – Precalculus and Calculus – that are usually offered as separate courses, with separate texts. I reviewed a number of commercially available options, but these all had two things in common: they did not quite meet our needs, and they were all very expensive (some were as much as $400).

Since writing a new textbook from scratch is a huge undertaking, requiring resources (like time) we simply did not have, I chose to explore non-commercial options. This took a bit of searching, since non-commercial texts, while inexpensive (or free), are of varying quality. Fortunately, there are some decent texts out there. Unfortunately, I couldn't find a single text that covered all of the material we need for Math 1010.

To get around this problem, I have selected two textbooks as our primary sources for the course. The first is *Precalculus*, version 3, by Carl Stitz and Jeff Zeager. The second is *APEX Calculus I*, version 3.0, by Hartman et al. Both texts have two very useful advantages. First, they're both free (as in beer): you can download either text in PDF format from the authors' web pages. Second, they're also *open source* texts (that is, free, as in speech). Both books are written using the LaTeXmarkup language, as is typical in mathematics publishing. What is not typical is that the authors of both texts make their source code freely available, allowing others (such as myself) to edit and customize the books as they see fit.

In the first iteration of this project (Fall 2015), I was only able to edit each text individually for length and content, resulting in two separate textbooks for Math 1010. This time around, I've had enough time to take the content of the Precalculus textbook and adapt its source code to be compatible with the formatting of the Calculus textbook, allowing me to produce a single textbook for all of Math 1010.

The book is very much a work in progress, and I will be editing it regularly. Feedback is always welcome.

Acknowledgements

First and foremost, I need to thank the authors of the two textbooks that provide the source material for this text. Without their hard work, and willingness to make their books (and the source code) freely available, it would not have been possible to create an affordable textbook for this course. You can find the original textbooks at their websites:

www.stitz-zeager.com, for the *Precalculus* textbook, by Stitz and Zeager, and

apexcalculus.com, for the APEX *Calculus* textbook, by Hartman et al.

I'd also like to thank Dave Morris for help with converting the graphics in the Precalculus textbook to work with the formatting code of the APEX text, Howard Cheng for providing some C++ code to convert the exercises, and the other faculty members involved with this course — Alia Hamieh, David Kaminsky, and Nicole Wilson — for their input on the content of the text.

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May, 2016

1: The Real Numbers

1.1 Some Basic Set Theory Notions

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of 'set theory' and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

Definition 1 Set

A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "pronghorns" is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set. In general, there are three ways to describe sets. They are

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

Key Idea 1 Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define a set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "pronghorns". A **roster** description of S would be $\{p, r, o, n, g, h, s\}$. Note that we listed 'r', 'o', and 'n' only once, even though they appear twice in "pronghorns." Also, the *order* of the elements doesn't matter, so $\{o, n, p, r, g, s, h\}$ is also a roster description of S. A **set-builder** description of S is:

$$\{x \mid x \text{ is a letter in the word "pronghorns".}\}$$

The way to read this is: 'The set of elements x <u>such that</u> x is a letter in the word "pronghorns."' In each of the above cases, we may use the familiar equals sign '$=$' and write $S = \{p, r, o, n, g, h, s\}$ or $S = \{x \mid x \text{ is a letter in the word "pronghorns".}\}$. Clearly r is in S and q is not in S. We express these sentiments mathematically by writing $r \in S$ and $q \notin S$.

More precisely, we have the following.

Definition 2 Notation for set inclusion

Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x$ is a consonant in the word "pronghorns"$\}$. A roster description of C is $C = \{p, r, n, g, h, s\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 3 Subset

Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{p, r, o, n, g, h, s, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' (Since the word 'different' could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.) The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 4 Intersection and Union

Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$. When it comes to their intersection, however, we run into a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 5 Empty set

The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \,|\, x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance. Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn in Figure 1.1.

In Figure 1.1 we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

All of these circles lie in a rectangle labelled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.

(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no'. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about Russell's Paradox on Wikipedia.)

1.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete definition of a real number is given below.

Definition 6 The real numbers

A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

The full extent of the empty set's role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers - mathematicians can create something from nothing! If you're interested, read about the von Neumann construction of the natural numbers or consider signing up for Math 2000.

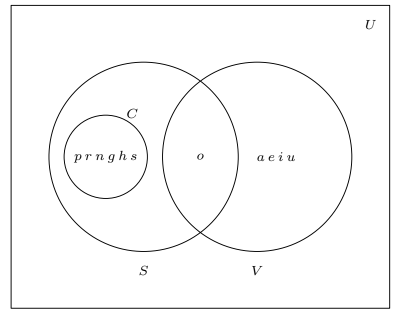
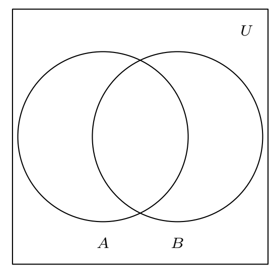
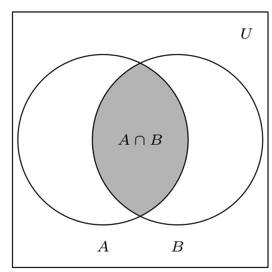


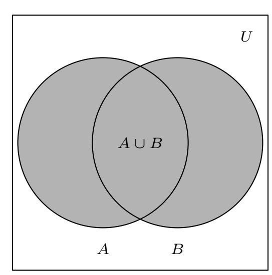
Figure 1.1: A Venn diagram for C, S, and V



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

Figure 1.2: Venn diagrams for intersection and union

Certain subsets of the real numbers are worthy of note and are listed below. In more advanced courses like Analysis, you learn that the real numbers can be *constructed* from the rational numbers, which in turn can be constructed from the integers (which themselves come from the natural numbers, which in turn can be defined as sets...).

Definition 7 Sets of Numbers

1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements. Like the number '0,' it plays a vital role in mathematics.
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\right\}$. Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers is:

   $$\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation.}\}$$

5. The **Real Numbers**: $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**: Real numbers that are not rational are called **irrational**. As a set, we have $\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$. (There is no standard symbol for this set.) Every irrational number has a decimal expansion which neither repeats nor terminates.
7. The **Complex Numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (We will not deal with complex numbers in Math 1010, although they usually make an appearance in Math 1410.)

An example of a number with a repeating decimal expansion is $a = 2.13234234234\ldots$. This is rational since $100a = 213.2342342342\ldots$, and $100000a = 213234.234234\ldots$ so $99900a = 100000a - 100a = 213021$. This gives us the rational expression $a = \dfrac{213021}{99900}$.

The classic example of an irrational number is the number π (See Section **??**), but numbers like $\sqrt{2}$ and $0.101001000100001\ldots$ are other fine representatives.

It is important to note that every natural number is a whole number is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and the rational numbers are all real numbers, since they possess decimal representations (via long division!). If we take $b = 0$ in the above definition of $\mathbb{C}$, we see that every real number is a complex number. In this sense, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are 'nested' like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above. It is time for an example.

Example 1 Sets of real numbers

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.
3. Let $A = \{-117, \frac{4}{5}, 0.202\overline{002}, 0.202002000200002\ldots\}$.

   Which elements of A are natural numbers? Rational numbers? Real numbers?

SOLUTION

1. To find a roster description for these sets, we need to list their elements. Starting with $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '...' indicates the that pattern continues.

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers. A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

It shouldn't be too surprising that E is the set of all even integers, since an even integer is *defined* to be an integer multiple of 2.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper. What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way. Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number. This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number. Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

The fact that the real numbers cannot be listed is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

3. The set A contains no natural numbers. Clearly, $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number. As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown in Figure 1.3.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that

$\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line. (This intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.) Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers.

The next result tells us what types of numbers we can expect to find.

Theorem 1 Density Property of $\mathbb{Q}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, ask yourself whether there is any difference between the numbers $0.\overline{9}$ and 1.

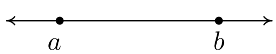


Figure 1.3: The real number line with two numbers *a* and *b*, where $a < b$.

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers *a* and *b*, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the 'Law of Trichotomy'.

The Law of Trichotomy, strictly speaking, is an *axiom* of the real numbers: a basic requirement that we assume to be true. However, in any *construction* of the real, such as the method of Dedekind cuts, it is necessary to *prove* that the Law of Trichotomy is satisfied.

Definition 8 Law of Trichotomy

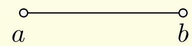
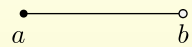
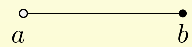
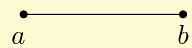
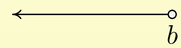
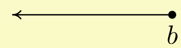
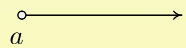
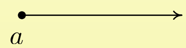
If *a* and *b* are real numbers then **exactly one** of the following statements is true:

$a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, '[' or ']', if the endpoint is included in the interval and use a filled-in or 'closed' dot to indicate membership in the interval. Otherwise, we use parentheses, '(' or ')' and an 'open' circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

Definition 9 Interval Notation

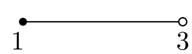
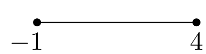
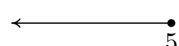
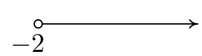
Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

The importance of understanding interval notation in Calculus cannot be overstated. If you don't find yourself getting the hang of it through repeated use, you may need to take the time to just memorize this chart.

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions.

Let's do a few examples to make sure we have the hang of the notation:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

We defined the intersection and union of arbitrary sets in Definition 4. Recall that the union of two sets consists of the totality of the elements in each of the sets, collected together. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1, 2, 3, 4, 6\}$. If $A = [-5, 3)$ and $B = (1, \infty)$, then we can find $A \cap B$ and $A \cup B$ graphically. To find $A \cap B$, we shade the overlap of the two and obtain $A \cap B = (1, 3)$. To find $A \cup B$, we shade each of A and B and describe the resulting shaded region to find $A \cup B = [-5, \infty)$.

While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

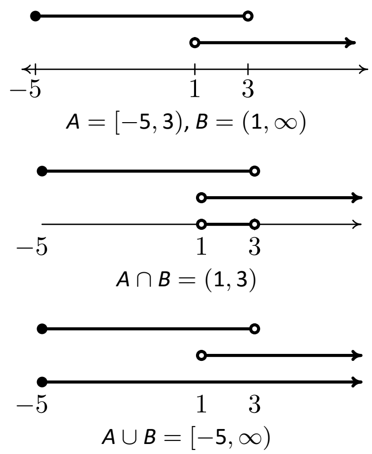


Figure 1.4: Union and intersection of intervals

Example 2 **Expressing sets as unions of intervals**

Express the following sets of numbers using interval notation.

1. $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$
2. $\{x \,|\, x \neq 3\}$
3. $\{x \,|\, x \neq \pm 3\}$
4. $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$

SOLUTION

1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. Since we are looking to describe the real numbers x in one of these *or* the other, we have $\{x \,|\, x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

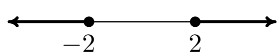


Figure 1.5: The set $(-\infty, -2] \cup [2, \infty)$

2. For the set $\{x \,|\, x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in either one of these intervals *or* the other, we have that $\{x \,|\, x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

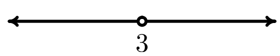


Figure 1.6: The set $(-\infty, 3) \cup (3, \infty)$

3. For the set $\{x \,|\, x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \,|\, x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

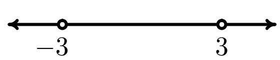


Figure 1.7: The set $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

4. Graphing the set $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$, we get one interval, $(-1, 3]$ along with a single number, or point, $\{5\}$. While we *could* express the latter as $[5, 5]$ (Can you see why?), we choose to write our answer as $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.

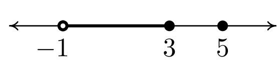
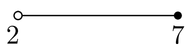
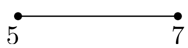
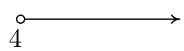


Figure 1.8: The set $(-1, 3] \cup \{5\}$

Exercises 1.1

Problems

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 – 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

2. $(-1, 5] \cap [0, 8)$

3. $(-1, 1) \cup [0, 6]$

4. $(-\infty, 4] \cap (0, \infty)$

5. $(-\infty, 0) \cap [1, 5]$

6. $(-\infty, 0) \cup [1, 5]$

7. $(-\infty, 5] \cap [5, 8)$

In Exercises 8 – 19, write the set using interval notation.

8. $\{x \mid x \neq 5\}$

9. $\{x \mid x \neq -1\}$

10. $\{x \mid x \neq -3,\ 4\}$

11. $\{x \mid x \neq 0,\ 2\}$

12. $\{x \mid x \neq 2,\ -2\}$

13. $\{x \mid x \neq 0,\ \pm 4\}$

14. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

15. $\{x \mid x < 3 \text{ or } x \geq 2\}$

16. $\{x \mid x \leq -3 \text{ or } x > 0\}$

17. $\{x \mid x \leq 5 \text{ or } x = 6\}$

18. $\{x \mid x > 2 \text{ or } x = \pm 1\}$

19. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

Definition 10 Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text.

Definition 11 Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right)$.

While most students (and some faculty) tend to skip over these properties or give them a cursory glance at best, it is important to realize that the properties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers, $1 + 2 + 3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

Definition 12 The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.
- **Factoring:** $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

Warning: A common source of errors for beginning students is the misuse (that is, lack of use) of parentheses. When in doubt, more is better than less: redundant parentheses add clutter, but do not change meaning, whereas writing $2x + 1$ when you meant to write $2(x + 1)$ is almost guaranteed to cause you to make a mistake. (Even if you're able to proceed correctly in spite of your lack of proper notation, this is the sort of thing that will get you on your grader's bad side, so it's probably best to avoid the problem in the first place.)

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2+x-6=0$ is by factoring the left hand side of this equation to get $(x-2)(x+3)=0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x-2=0$ or $x+3=0$ so $x=2$ or $x=-3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter **??**.

The expression $\frac{0}{0}$ is technically an 'indeterminate form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter **??**.

Theorem 2 Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab=0$ if and only if $a=0$ or $b=0$ (or both)

  Note: This not only says that $0 \cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0 \cdot \left(\dfrac{1}{a}\right) = 0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.

We now continue with a review of arithmetic with fractions.

It's always worth remembering that division is the same as multiplication by the reciprocal. You'd be surprised how often this comes in handy.

Note: A common denominator is **not** required to **multiply** or **divide** fractions!

Note: A common denominator **is** required to **add or subtract** fractions!

Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.

We reduce fractions by 'cancelling' common factors - this is really just reading the previous property 'from right to left'.**Caution:** We may only cancel common **factors** from both numerator and denominator.

Key Idea 2 Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \dfrac{a}{1}$ and $\dfrac{a}{a} = 1$.
- **Fraction Equality:** $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$. In particular: $\dfrac{a}{b} \cdot c = \dfrac{a}{b} \cdot \dfrac{c}{1} = \dfrac{ac}{b}$
- **Division of Fractions:** $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$.

  In particular: $1 \Big/ \dfrac{a}{b} = \dfrac{b}{a}$ and $\dfrac{a}{b} \Big/ c = \dfrac{a}{b} \Big/ \dfrac{c}{1} = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{a}{bc}$
- **Addition and Subtraction of Fractions:** $\dfrac{a}{b} \pm \dfrac{c}{b} = \dfrac{a \pm c}{b}$.
- **Equivalent Fractions:** $\dfrac{a}{b} = \dfrac{ad}{bd}$, since $\dfrac{a}{b} = \dfrac{a}{b} \cdot 1 = \dfrac{a}{b} \cdot \dfrac{d}{d} = \dfrac{ad}{bd}$
- **'Reducing' Fractions:** $\dfrac{a\cancel{d}}{b\cancel{d}} = \dfrac{a}{b}$, since $\dfrac{ad}{bd} = \dfrac{a}{b} \cdot \dfrac{d}{d} = \dfrac{a}{b} \cdot 1 = \dfrac{a}{b}$.

  In particular, $\dfrac{ab}{b} = a$ since $\dfrac{ab}{b} = \dfrac{ab}{1 \cdot b} = \dfrac{a\cancel{b}}{1 \cdot \cancel{b}} = \dfrac{a}{1} = a$ and $\dfrac{b-a}{a-b} = \dfrac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.

Next up is a review of the arithmetic of 'negatives'. On page 10 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5 - 3 = 5 + (-3)$.

It might be junior high (elementary?) school material, but arithmetic with fractions is one of the most common sources of errors among university students. If you're not comfortable working with fractions, we strongly recommend seeing your instructor (or a tutor) to go over this material until you're completely confident that you understand it. Experience (and even formal educational studies) suggest that your success handling fractions corresponds pretty well with your overall success in passing your Mathematics courses.

Key Idea 3 Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:** $-a - b = -(a + b)$ and $b - a = -(a - b)$.

In this text we do not distinguish typographically between the dashes in the expressions '$5 - 3$' and '-3' even though they are mathematically quite different. In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a+b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a)+(-b) = -a-b$. Negatives do not 'distribute' across multiplication: $-(2\cdot 3) \neq (-2)\cdot(-3)$. Instead, $-(2\cdot 3) = (-2)\cdot(3) = (2)\cdot(-3) = -6$. The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$. It's about time we did a few examples to see how these properties work in practice.

Example 3 Arithmetic with fractions

Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4} + \dfrac{6}{7}$
2. $\dfrac{5}{12} - \left(\dfrac{47}{30} - \dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{12}{5} - \dfrac{7}{24}}{1 + \left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
4. $\dfrac{(2(2)+1)(-3-(-3)) - 5(4-7)}{4 - 2(3)}$
5. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right) - \left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no

factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned}
\frac{1}{4}+\frac{6}{7} &= \frac{1}{4}\cdot\frac{7}{7}+\frac{6}{7}\cdot\frac{4}{4} && \text{Equivalent Fractions}\\
&= \frac{7}{28}+\frac{24}{28} && \text{Multiplication of Fractions}\\
&= \frac{31}{28} && \text{Addition of Fractions}
\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30}-\frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step. The lowest common denominator for all three fractions is 60.

We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

$$\begin{aligned}
\frac{5}{12}-\left(\frac{47}{30}-\frac{7}{3}\right) &= \frac{5}{12}-\frac{47}{30}+\frac{7}{3} && \text{Distribute the Negative}\\
&= \frac{5}{12}\cdot\frac{5}{5}-\frac{47}{30}\cdot\frac{2}{2}+\frac{7}{3}\cdot\frac{20}{20} && \text{Equivalent Fractions}\\
&= \frac{25}{60}-\frac{94}{60}+\frac{140}{60} && \text{Multiplication of Fractions}\\
&= \frac{71}{60} && \text{Addition and Subtraction of Fractions}
\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction. The longest division line (also called a 'vinculum') acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions):

$$\frac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. There are two ways to proceed. One is to simplify the numerator and denominator separately, and then use the fact that division is the same thing as multiplication by the reciprocal, as follows:

$$\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} = \frac{\left(\frac{12}{5}\cdot\frac{24}{24}-\frac{7}{24}\cdot\frac{5}{5}\right)}{\left(1\cdot\frac{120}{120}+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} \quad \text{Equivalent Fractions}$$

$$= \frac{288/120-35/120}{120/120+84/120} \quad \text{Multiplication of fractions}$$

$$= \frac{253/120}{204/120} \quad \text{Addition and subtraction of fractions}$$

$$= \frac{253}{\cancel{120}}\cdot\frac{\cancel{120}}{204} \quad \text{Division of fractions and cancellation}$$

$$= \frac{253}{204}$$

Since $253 = 11\cdot 23$ and $204 = 2\cdot 2\cdot 3\cdot 17$ have no common factors our result is in lowest terms which means we are done.

While there is nothing wrong with the above approach, we can also use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24\cdot 5 = 120$.

$$\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} \quad \text{Equivalent Fractions}$$

$$= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)}$$

Distributive Property

$$= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} \quad \text{Multiply fractions}$$

$$= \frac{\frac{12\cdot 24\cdot\cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot\cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot\cancel{5}\cdot\cancel{24}}{\cancel{5}\cdot\cancel{24}}} \quad \text{Factor and cancel}$$

$$= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)}$$

$$= \frac{288-35}{120+84} = \frac{253}{204},$$

which is the same as we obtained above.

4. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\ &= \frac{(5)(0)+15}{-2} \\ &= \frac{15}{-2} \\ &= -\frac{15}{2} \quad \text{Properties of Negatives}\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

5. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\ &= \frac{15}{65}-\frac{-48}{65} \\ &= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\ &= \frac{15+48}{65} && \text{Add numerators} \\ &= \frac{63}{65}\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done.

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favourite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication.

In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Definition 13 Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

Note: The expression 0^0 is an indeterminate form. See the comment regarding '$\frac{0}{0}$' on page 12.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you **cannot** do this sort of thing with addition. That is, in general, $(a+b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Definition 14 Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols.)
2. Evaluate **e**xponents.
3. Evaluate **d**ivision and **m**ultiplication as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

For example, $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -(a^n)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Order of operations follows the "PEDMAS" rule some of you may have encountered.

Example 4 Operations with exponents

Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 3, the division here acts as a grouping symbol, so we save the division to the end.

$$\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} = \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4}$$
$$= \frac{16-16}{4} = \frac{0}{4} = 0$$

2. As before, we simplify what's in the parentheses first, then work our way

through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
12(-5)(-5+3)^{-4} &+ 6(-5)^2(-4)(-5+3)^{-5} \\
&= 12(-5)(-2)^{-4} + 6(-5)^2(-4)(-2)^{-5} \\
&= 12(-5)\left(\frac{1}{(-2)^4}\right) + 6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 12(-5)\left(\frac{1}{16}\right) + 6(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-60)\left(\frac{1}{16}\right) + (-600)\left(\frac{1}{-32}\right) \\
&= \frac{-60}{16} + \left(\frac{-600}{-32}\right) \\
&= \frac{-15\cdot \cancel{4}}{4\cdot \cancel{4}} + \frac{-75\cdot \cancel{8}}{-4\cdot \cancel{8}} \\
&= \frac{-15}{4} + \frac{-75}{-4} \\
&= \frac{-15}{4} + \frac{75}{4} \\
&= \frac{-15+75}{4} \\
&= \frac{60}{4} \\
&= 15
\end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\frac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} &&= \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} = \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} &&= 3^2\cdot 4^{-2} = 3^2\cdot\left(\frac{1}{4^2}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) &&= \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we

did in Example 3. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} &= \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^{2}} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)} \\
&= \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} = \frac{\left(\frac{24}{5}\right)\cdot 25}{\left(1-\frac{144}{25}\right)\cdot 25} \\
&= \frac{\left(\frac{24\cdot 5\cdot \cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\frac{144\cdot \cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Since 120 and 119 have no common factors, we are done.

It's important that you understand the difference between the statements $y = \sqrt{x}$ and $y^2 = x$. As we'll discuss in Chapter **??**, the equation $y = \sqrt{x}$ defines y as a **function** of x, which means that for each value of $x \geq 0$ there is only one value of y such that $y = \sqrt{x}$. For example, $y = \sqrt{4}$ is equivalent to $y = 2$. On the other hand, there are **two** solutions to $y^2 = x$; namely, $y = \sqrt{x}$ and $y = -\sqrt{x}$. For example, the equation $y^2 = 4$ is equivalent to the two equations $y = 2$ and $y = -2$ (or, more concisely, $y = \pm 2$). Since these two equations are closely related, it's easy to mix them up. The main thing to remember is that $\sqrt{x}$ always denotes the *positive* square root of x.

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 15 The principal n^{th} root

Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 15 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically, (see Chapter **??**) and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

Theorem 3 Properties of Radicals

Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 3 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with. We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to **never consider 'distributing' roots or exponents**. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Things get more complicated once complex numbers are involved. Fortunately (disappointingly?), that's not a can of worms we'll be opening in this course.

Definition 16 Rational exponents

Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number. (If n is even we need $a \geq 0$.)
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

It would make life really nice if the rational exponents defined in Definition 16 had all of the same properties that integer exponents have as listed on page 17 - but they don't. Why not? Let's look at an example to see what goes wrong.

Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 16, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.

Example 5 **Combining operations**

Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{\sqrt{3}}{3}\right)^2}$

3. $(\sqrt[3]{-2} - \sqrt[3]{-54})^2$

4. $2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3}$

Solution

1. We begin in the numerator and note that the radical here acts a grouping symbol, so our first order of business is to simplify the radicand. (The line extending horizontally from the square root symbol '$\sqrt{}$ is, you guessed it, another vinculum.)

$$\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} = \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)}$$
$$= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)}$$
$$= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)}$$
$$= \frac{-(-4)-\sqrt{16+24}}{2(2)}$$
$$= \frac{-(-4)-\sqrt{40}}{2(2)}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\frac{-(-4)-\sqrt{40}}{2(2)} = \frac{-(-4)-2\sqrt{10}}{2(2)} = \frac{4-2\sqrt{10}}{2(2)}$$
$$= \frac{2\cdot 2-2\sqrt{10}}{2(2)} = \frac{2(2-\sqrt{10})}{2(2)}$$
$$= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} = \frac{2-\sqrt{10}}{2}$$

Since the numerator and denominator have no more common factors, we are done. (Do you see why we aren't 'cancelling' the remaining 2's?)

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)}$$
$$= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)}$$
$$= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} = \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}}$$
$$= \frac{2\sqrt{3}}{3-1} = \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube, (of an integer, that is!) we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, -1 *is* a perfect cube, and we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2} + 3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 = (2\sqrt[3]{2})^2 \\ &= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} = 4\sqrt[3]{4}\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned}2\left(\frac{9}{4} - 3\right)^{1/3} &+ 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} \\ &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\ &= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\ &= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\ &= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\ &= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\ &= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

Exercises 1.2

Problems

In Exercises 1 – 33, perform the indicated operations and simplify.

1. $5 - 2 + 3$

2. $5 - (2 + 3)$

3. $\dfrac{2}{3} - \dfrac{4}{7}$

4. $\dfrac{3}{8} + \dfrac{5}{12}$

5. $\dfrac{5 - 3}{-2 - 4}$

6. $\dfrac{2(-3)}{3 - (-3)}$

7. $\dfrac{2(3) - (4 - 1)}{2^2 + 1}$

8. $\dfrac{4 - 5.8}{2 - 2.1}$

9. $\dfrac{1 - 2(-3)}{5(-3) + 7}$

10. $\dfrac{5(3) - 7}{2(3)^2 - 3(3) - 9}$

11. $\dfrac{2((-1)^2 - 1)}{((-1)^2 + 1)^2}$

12. $\dfrac{(-2)^2 - (-2) - 6}{(-2)^2 - 4}$

13. $\dfrac{3 - \frac{4}{9}}{-2 - (-3)}$

14. $\dfrac{\frac{2}{3} - \frac{4}{5}}{4 - \frac{7}{10}}$

15. $\dfrac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$

16. $\dfrac{1 - \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1 + \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$

17. $\left(\dfrac{2}{3}\right)^{-5}$

18. $3^{-1} - 4^{-2}$

19. $\dfrac{1 + 2^{-3}}{3 - 4^{-1}}$

20. $\dfrac{3 \cdot 5^{100}}{12 \cdot 5^{98}}$

21. $\sqrt{3^2 + 4^2}$

22. $\sqrt{12} - \sqrt{75}$

23. $(-8)^{2/3} - 9^{-3/2}$

24. $\left(-\frac{32}{9}\right)^{-3/5}$

25. $\sqrt{(3 - 4)^2 + (5 - 2)^2}$

26. $\sqrt{(2 - (-1))^2 + \left(\frac{1}{2} - 3\right)^2}$

27. $\sqrt{(\sqrt{5} - 2\sqrt{5})^2 + (\sqrt{18} - \sqrt{8})^2}$

28. $\dfrac{-12 + \sqrt{18}}{21}$

29. $\dfrac{-2 - \sqrt{(2)^2 - 4(3)(-1)}}{2(3)}$

30. $\dfrac{-(-4) + \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$

31. $2(-5)(-5 + 1)^{-1} + (-5)^2(-1)(-5 + 1)^{-2}$

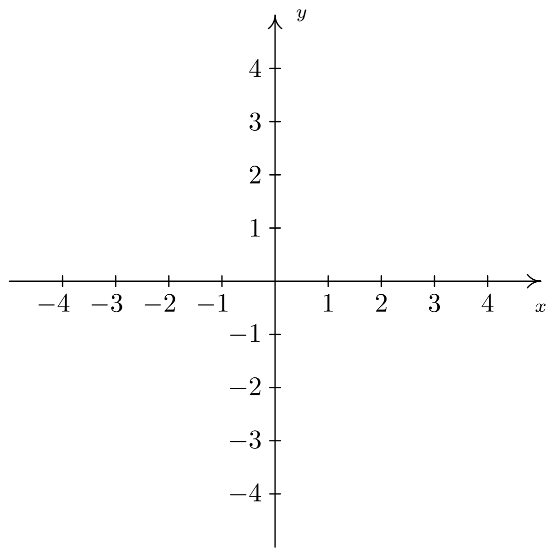
32. $3\sqrt{2(4) + 1} + 3(4)\left(\frac{1}{2}\right)(2(4) + 1)^{-1/2}(2)$

33. $2(-7)\sqrt[3]{1 - (-7)} + (-7)^2\left(\frac{1}{3}\right)(1 - (-7))^{-2/3}(-1)$

1.3 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as drawn below.

The Cartesian Plane is named in honour of René Descartes.

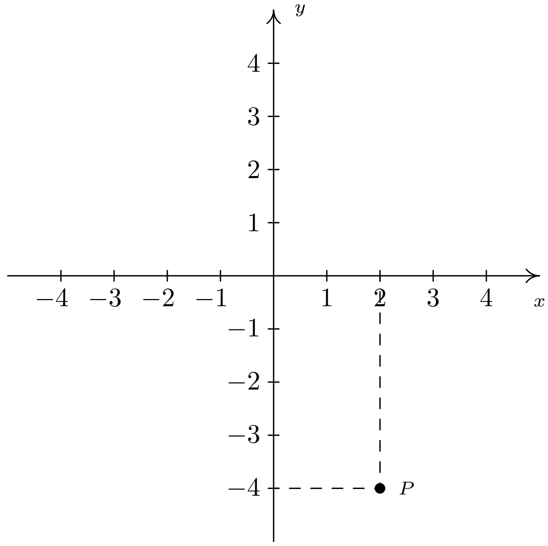


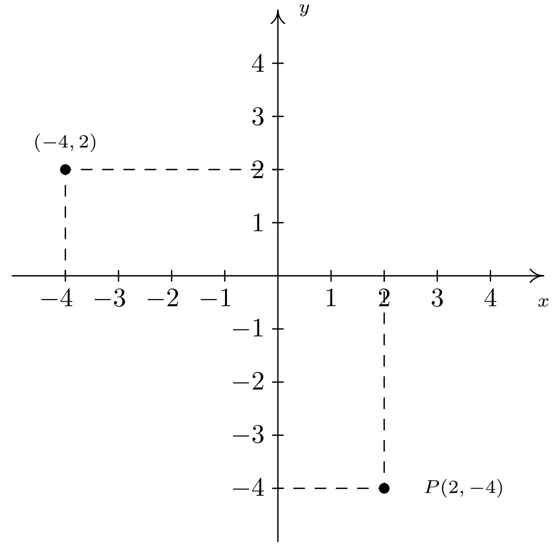
Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y.

The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. This process is sometimes called 'projecting' the point P to the x- (respectively y-) axis. We then describe the point P using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**. Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point P. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach P from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important − if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the t-axis. The first number in the ordered pair would then be the t-coordinate.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

Cartesian coordinates are sometimes referred to as *rectangular coordinates*, to distinguish them from other coordinate systems such as *polar coordinates*.

Key Idea 4 Important Facts about the Cartesian Coordinate Plane

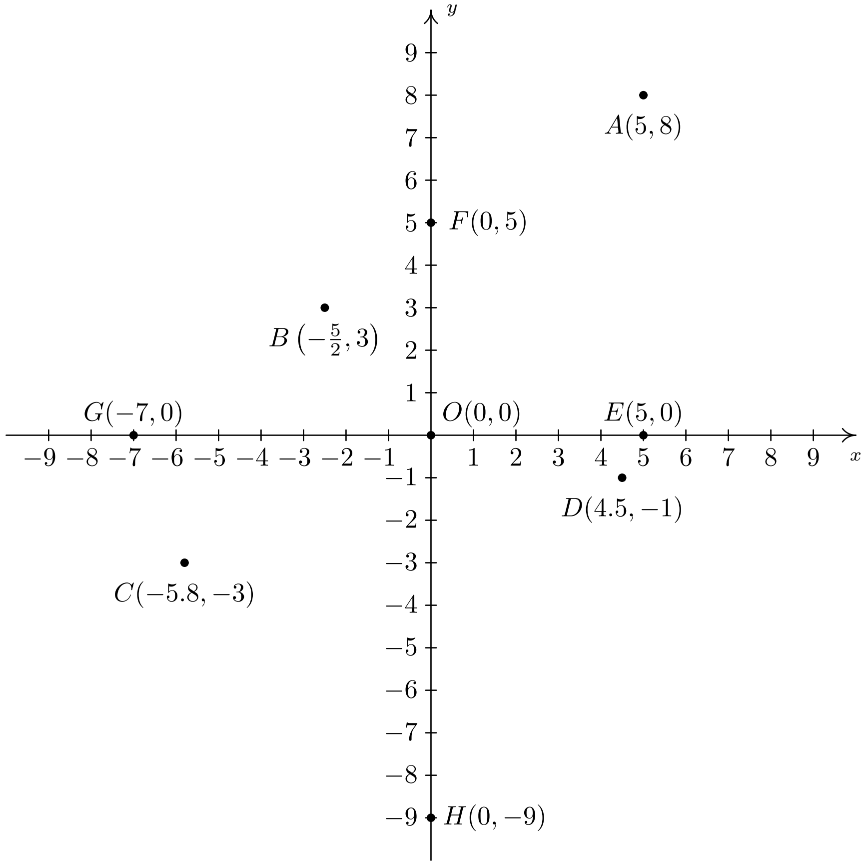
- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

Example 6 Plotting points in the Cartesian Plane

Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.

The letter O is almost always reserved for the origin.

Solution To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.9.

For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative *x*-axis (if $y = 0$) or on the positive or negative *y*-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive *y*-axis whereas $(-117, 0)$ lies on the negative *x*-axis. Such points do not belong to any of the four quadrants.

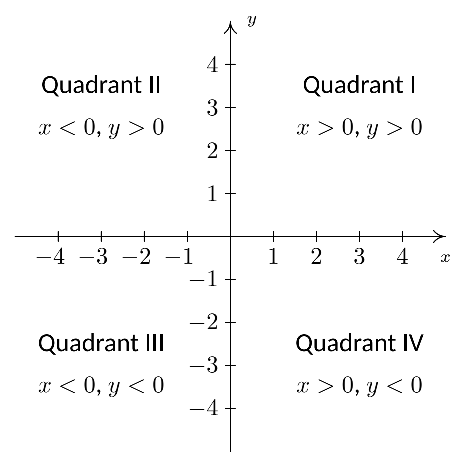


Figure 1.9: The four quadrants of the Cartesian plane

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 17 Symmetry in the Cartesian Plane

Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the *x*-axis** if $a = c$ and $b = -d$
- **symmetric about the *y*-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

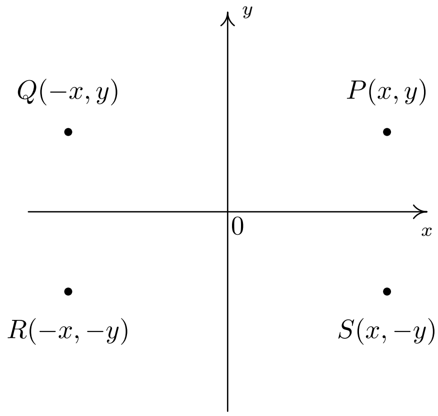


Figure 1.10: The three types of symmetry in the plane

In Figure 1.10, *P* and *S* are symmetric about the *x*-axis, as are *Q* and *R*; *P* and *Q* are symmetric about the *y*-axis, as are *R* and *S*; and *P* and *R* are symmetric about the origin, as are *Q* and *S*.

Example 7 Finding points exhibiting symmetry

Let *P* be the point $(-2,3)$. Find the points which are symmetric to *P* about the:

1. *x*-axis
2. *y*-axis
3. origin

Check your answer by plotting the points.

Solution The figure after Definition 17 gives us a good way to think about finding symmetric points in terms of taking the opposites of the *x*- and/or *y*-coordinates of $P(-2,3)$.

1. To find the point symmetric about the *x*-axis, we replace the *y*-coordinate with its opposite to get $(-2,-3)$.
2. To find the point symmetric about the *y*-axis, we replace the *x*-coordinate with its opposite to get $(2,3)$.
3. To find the point symmetric about the origin, we replace the *x*- and *y*-coordinates with their opposites to get $(2,-3)$.

   The points are plotted in Figure 1.11.

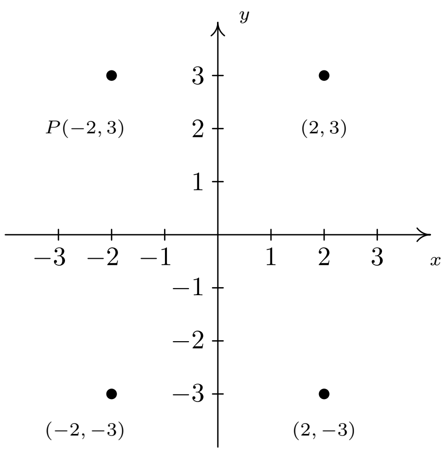


Figure 1.11: The point $P(-2,3)$ and its three reflections

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2,3)$ and pretend that the *x*-axis is a mirror, then the reflection of $(-2,3)$ across the *x*-axis would lie at $(-2,-3)$. If we pretend that the *y*-axis is a mirror, the reflection of $(-2,3)$ across that axis would be $(2,3)$. If we reflect across the *x*-axis and then the *y*-axis, we would go from $(-2,3)$ to $(-2,-3)$ then to $(2,-3)$, and so we would end up at the point symmetric to $(-2,3)$ about the origin. We summarize and generalize this process below.

Key Idea 5 Reflections in the Cartesian Plane

To reflect a point (x,y) about the:

- *x*-axis, replace *y* with $-y$.
- *y*-axis, replace *x* with $-x$.
- origin, replace *x* with $-x$ and *y* with $-y$.

1.3.1 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0,y_0)$ and $Q(x_1,y_1)$, in the plane. By the **distance** *d* between *P* and *Q*, we mean the length of the line segment joining *P* with *Q*. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.12.

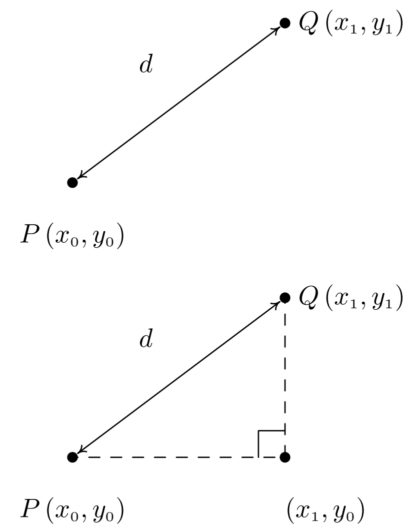


Figure 1.12: Distance between P and Q

With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Key Idea 6 The Distance Formula

The distance d between the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 6 for these cases.

Example 8 Distance between two points

Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

Solution

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$.

Example 9 Finding points at a given distance

Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$.

We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 6, yields

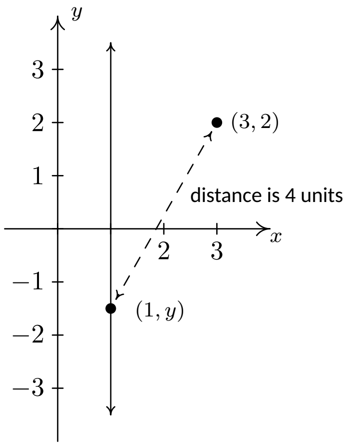


Figure 1.13: Diagram for Example 9

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}
$$

We obtain two answers: $(1, 2 + 2\sqrt{3})$ and $(1, 2 - 2\sqrt{3})$. The reader is encouraged to think about why there are two answers.

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.

If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

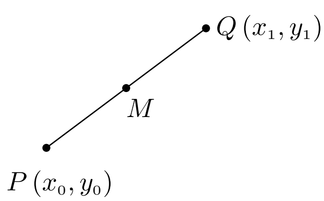


Figure 1.14: The midpoint of a line segment

Key Idea 7 The Midpoint Formula

The midpoint M of the line segment connecting $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

If we let d denote the distance between P and Q, we leave it as Exercise 17 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Key Idea 7 gives the coordinates of the midpoint.

Example 10 Finding the midpoint of a line segment

Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution

$$
\begin{aligned}
M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\
&= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\
&= \left(-\frac{1}{2}, 0\right)
\end{aligned}
$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$.

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise **??** in Section **??**.

Example 11 **An abstract midpoint problem**

If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

Solution To prove the claim, we use Equation 7 to find the midpoint

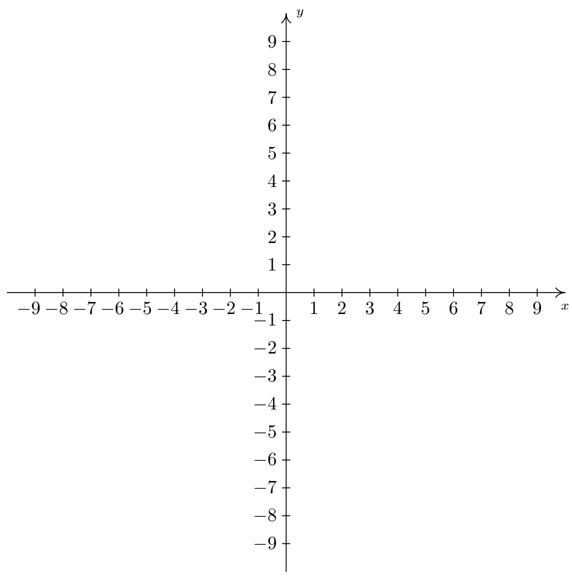
$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required.

Exercises 1.3

Problems

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 – 10, find the distance d between the points and the midpoint M of the line segment which connects them.

3. $(1,2)$, $(-3,5)$
4. $(3,-10)$, $(-1,2)$
5. $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$
6. $\left(-\frac{2}{3},\frac{3}{2}\right)$, $\left(\frac{7}{3},2\right)$
7. $\left(\frac{24}{5},\frac{6}{5}\right)$, $\left(-\frac{11}{5},-\frac{19}{5}\right)$.
8. $(\sqrt{2},\sqrt{3})$, $(-\sqrt{8},-\sqrt{12})$
9. $(2\sqrt{45},\sqrt{12})$, $(\sqrt{20},\sqrt{27})$.
10. $(0,0)$, (x,y)
11. Find all of the points of the form $(x,-1)$ which are 4 units from the point $(3,2)$.
12. Find all of the points on the y-axis which are 5 units from the point $(-5,3)$.
13. Find all of the points on the x-axis which are 2 units from the point $(-1,1)$.
14. Find all of the points of the form $(x,-x)$ which are 1 unit from the origin.
15. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
16. Verify the Distance Formula 6 for the cases when:
    (a) The points are arranged vertically. (Hint: Use $P(a,y_0)$ and $Q(a,y_1)$.)
    (b) The points are arranged horizontally. (Hint: Use $P(x_0,b)$ and $Q(x_1,b)$.)
    (c) The points are actually the same point. (You shouldn't need a hint for this one.)
17. Verify the Midpoint Formula by showing the distance between $P(x_1,y_1)$ and M and the distance between M and $Q(x_2,y_2)$ are both half of the distance between P and Q.
18. Show that the points A, B and C below are the vertices of a right triangle.
    (a) $A(-3,2)$, $B(-6,4)$, and $C(1,8)$
    (b) $A(-3,1)$, $B(4,0)$ and $C(0,-3)$
19. Find a point $D(x,y)$ such that the points $A(-3,1)$, $B(4,0)$, $C(0,-3)$ and D are the corners of a square. Justify your answer.
20. Discuss with your classmates how many numbers are in the interval $(0,1)$.
21. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

1.4 Complex Numbers

We conclude our first chapter with a review the set of **Complex Numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number x with $x^2 = -1$, since for any real number $x^2 \geq 0$. However, we could formally extract square roots and write $x = \pm\sqrt{-1}$. We build the complex numbers by relabelling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**. The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2-7i)+(3+4i) = 5-3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition 18 The imaginary unit

The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Historically, the lack of solutions to the equation $x^2 = -1$ had nothing to do with the development of the complex numbers. Until the 19th century, equations such as $x^2 = -1$ would have been considered in the context of the analytic geometry of Descartes. The lack of solutions simply indicated that the graph $y = x^2$ did not intersect the line $y = -1$. The more remarkable case was that of *cubic* equations, of the form $x^3 = ax + b$. In this case a **real** solution is *guaranteed*, but there are cases where one needs **complex** numbers to find it! For details, see the excellent book *Visual Complex Analysis*, by Tristan Needham.

Property 1 in Definition 18 establishes that i does act as a square root of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 18 in place, we are now in position to define the **complex numbers**.

Note the use of the indefinite article 'a' in Definition 18. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

Some Technical Mathematics textbooks label the imaginary unit 'j', usually to avoid confusion with the use of the letter i to denote electric current. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

Definition 19 Complex number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

To use the language of Section 1.1.1, $\mathbb{R} \subseteq \mathbb{C}$.

Complex numbers include things you'd normally expect, like $3 + 2i$ and $\frac{2}{5} - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers, so 0 and $\pi - \sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 18. The next example should help recall how these animals behave.

Example 12 Arithmetic with complex numbers

Perform the indicated operations.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

Solution

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We distribute and combine like terms:

$$\begin{aligned}(1-2i)-(3+4i) &= 1-2i-3-4i && \text{Distribute}\\ &= -2-6i && \text{Gather like terms}\end{aligned}$$

   Technically, we'd have to rewrite our answer $-2-6i$ as $(-2)+(-6)i$ to be (in the strictest sense) 'in the form $a+bi$'. That being said, even pedants have their limits, and we'll consider $-2-6i$ good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{aligned}(1-2i)(3+4i) &= (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i) && \text{F.O.I.L.}\\ &= 3+4i-6i-8i^2 \\ &= 3-2i-8(-1) && i^2=-1\\ &= 3-2i+8 \\ &= 11-2i\end{aligned}$$

3. How in the world are we supposed to simplify $\dfrac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing two terms, one of which is a square root: we and multiply both numerator and denominator by $3+4i$, the (complex) conjugate of $3-4i$. Doing so produces

$$\begin{aligned}\frac{1-2i}{3-4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)} && \text{Equivalent Fractions}\\ &= \frac{3+4i-6i-8i^2}{9-16i^2} && \text{F.O.I.L.}\\ &= \frac{3-2i-8(-1)}{9-16(-1)} && i^2=-1\\ &= \frac{11-2i}{25}\\ &= \frac{11}{25}-\frac{2}{25}i\end{aligned}$$

4. We use property 2 of Definition 18 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12} = \left(i\sqrt{3}\right)\left(i\sqrt{12}\right) = i^2\sqrt{3\cdot 12} = -\sqrt{36} = -6$.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}(x-[1+2i])(x-[1-2i]) &= x^2 - x[1-2i] - x[1+2i] + [1-2i][1+2i] \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &= x^2 - 2x + 1 - 4(-1) \\ &= x^2 - 2x + 5\end{aligned}$$

This type of factoring will be revisited in Section **??**.

In the previous example, we used the idea of a 'conjugate' to divide two complex numbers. (You may recall using conjugates to rationalize expressions involving square roots.) More generally, the **complex conjugate** of a complex number $a+bi$ is the number $a-bi$. The notation commonly used for complex conjugation is a 'bar': $\overline{a+bi} = a-bi$. For example, $\overline{3+2i} = 3-2i$ and $\overline{3-2i} = 3+2i$. To find $\overline{6}$, we note that $\overline{6} = \overline{6+0i} = 6-0i = 6$, so $\overline{6} = 6$. Similarly, $\overline{4i} = -4i$, since $\overline{4i} = \overline{0+4i} = 0-4i = -4i$. Note that $\overline{3+\sqrt{5}} = 3+\sqrt{5}$, not $3-\sqrt{5}$, since $\overline{3+\sqrt{5}} = \overline{3+\sqrt{5}+0i} = 3+\sqrt{5}-0i = 3+\sqrt{5}$. Here, the conjugation specified by the 'bar' notation involves reversing the sign before $i = \sqrt{-1}$, not before $\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem 4 Properties of the Complex Conjugate

Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{z^n} = \left(\overline{z}\right)^n$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Essentially, Theorem 4 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out $z = a+bi$ and $w = c+di$ for real numbers a, b, c and d. Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise. To prove the second property, we compare $\overline{z+w}$ with $\overline{z}+\overline{w}$. We have $\overline{z}+\overline{w} = \overline{a+bi} + \overline{c+di} = a-bi+c-di$. To find $\overline{z+w}$, we first compute

$$z+w = (a+bi)+(c+di) = (a+c)+(b+d)i$$

so

$$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = a-bi+c-di = \overline{z}+\overline{w}$$

As such, we have established $\overline{z+w} = \overline{z}+\overline{w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction. The last property is a characterization of real numbers. If z is real, then $z = a + 0i$, so $\bar{z} = a - 0i = a = z$. On the other hand, if $z = \bar{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

Proof by Mathematical Induction is usually taught in Math 2000.

We now consider the problem of solving quadratic equations. Consider $x^2 - 2x + 5 = 0$. The discriminant $b^2 - 4ac = -16$ is negative, so we know by Theorem **??** there are no *real* solutions, since the Quadratic Formula would involve the term $\sqrt{-16}$. Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

We're assuming some prior familiarity on the part of the reader where quadratic equations are concerned. If you feel that it would be unfair to tackle quadratic equations with complex solutions before the case of real solutions has been properly addressed, you may want to briefly skip ahead to Section **??**.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Example 13 **Finding complex solutions**

Find the complex solutions to the following equations.

1. $\dfrac{2x}{x+1} = x + 3$
2. $2t^4 = 9t^2 + 5$
3. $z^3 + 1 = 0$

Solution

Remember, all real numbers are complex numbers, so 'complex solutions' means both real and non-real answers.

1. Clearing fractions yields a quadratic equation so we collect all terms on one side and apply the Quadratic Formula.

$$\begin{aligned}
\frac{2x}{x+1} &= x + 3 && \\
2x &= (x+3)(x+1) && \text{Clear denominators}\\
2x &= x^2 + x + 3x + 3 && \text{F.O.I.L.}\\
2x &= x^2 + 4x + 3 && \text{Gather like terms}\\
0 &= x^2 + 2x + 3 && \text{Subtract } 2x
\end{aligned}$$

From here, we apply the Quadratic Formula

$$\begin{aligned}
x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} && \text{Quadratic Formula}\\
&= \frac{-2 \pm \sqrt{-8}}{2} && \text{Simplify}\\
&= \frac{-2 \pm i\sqrt{8}}{2} && \text{Definition of } i\\
&= \frac{-2 \pm i2\sqrt{2}}{2} && \text{Product Rule for Radicals}\\
&= \frac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} && \text{Factor and reduce}\\
&= -1 \pm i\sqrt{2}
\end{aligned}$$

We get two answers: $x = -1 + i\sqrt{2}$ and its conjugate $x = -1 - i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2), we have a Quadratic

in Disguise. We proceed accordingly.

$$\begin{aligned} 2t^4 &= 9t^2 + 5 & \\ 2t^4 - 9t^2 - 5 &= 0 & \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2+1)(t^2-5) &= 0 & \text{Factor} \\ 2t^2 + 1 = 0 \quad &\text{or} \quad t^2 = 5 & \text{Zero Product Property} \end{aligned}$$

From $2t^2 + 1 = 0$ we get $2t^2 = -1$, or $t^2 = -\frac{1}{2}$. We extract square roots as follows:

$$t = \pm\sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2 = 5$, we get $t = \pm\sqrt{5}$. In total, we have four complex solutions - two real: $t = \pm\sqrt{5}$ and two non-real: $t = \pm\frac{i\sqrt{2}}{2}$.

3. To find the *real* solutions to $z^3 + 1 = 0$, we can subtract the 1 from both sides and extract cube roots: $z^3 = -1$, so $z = \sqrt[3]{-1} = -1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{aligned} z^3 + 1 &= 0 & \\ (z+1)(z^2 - z + 1) &= 0 & \text{Factor (Sum of Two Cubes)} \\ z + 1 = 0 \quad &\text{or} \quad z^2 - z + 1 = 0 & \end{aligned}$$

From $z + 1 = 0$, we get our real solution $z = -1$. From $z^2 - z + 1 = 0$, we apply the Quadratic Formula to get:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to $z^3 + 1 = 0$ - one real: $z = -1$ and two non-real: $z = \frac{1 \pm i\sqrt{3}}{2}$. As always, the reader is encouraged to test their algebraic mettle and check these solutions.

It is no coincidence that the non-real solutions to the equations in Example 13 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem **??** which we state on the next page.

Theorem 5 Discriminant Theorem

Given a Quadratic Equation $AX^2 + BX + C = 0$, where A, B and C are real numbers, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one (repeated) real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B \pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section **??** and we will revisit Theorem 5 then.

Exercises 1.4

Problems

In Exercises 1 – 10, use the given complex numbers z and w to find and simplify the following:

- $z+w$
- zw
- z^2
- $\frac{1}{z}$
- $\frac{z}{w}$
- $\frac{w}{z}$
- $\overline{z}$
- $z\overline{z}$
- $(\overline{z})^2$

1. $z=2+3i$, $w=4i$
2. $z=1+i$, $w=-i$
3. $z=i$, $w=-1+2i$
4. $z=4i$, $w=2-2i$
5. $z=3-5i$, $w=2+7i$
6. $z=-5+i$, $w=4+2i$
7. $z=\sqrt{2}-i\sqrt{2}$, $w=\sqrt{2}+i\sqrt{2}$
8. $z=1-i\sqrt{3}$, $w=-1-i\sqrt{3}$
9. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
10. $z=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $w=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In Exercises 11 – 18, simplify the quantity.

11. $\sqrt{-49}$
12. $\sqrt{-9}$
13. $\sqrt{-25}\sqrt{-4}$
14. $\sqrt{(-25)(-4)}$
15. $\sqrt{-9}\sqrt{-16}$
16. $\sqrt{(-9)(-16)}$
17. $\sqrt{-(-9)}$
18. $-\sqrt{(-9)}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)\cdot i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 19 – 26, use this information to simplify the given power of i.

19. i^5
20. i^6
21. i^7
22. i^8
23. i^{15}
24. i^{26}
25. i^{117}
26. i^{304}

In Exercises 27 – 35, find all complex solutions.

27. $3x^2+6=4x$
28. $15t^2+2t+5=3t(t^2+1)$
29. $3y^2+4=y^4$
30. $\frac{2}{1-w}=w$
31. $\frac{y}{3}-\frac{3}{y}=y$
32. $\frac{x^3}{2x-1}=\frac{x}{3}$
33. $x=\frac{2}{\sqrt{5}-x}$
34. $\frac{5y^4+1}{y^2-1}=3y^2$
35. $z^4=16$
36. Multiply and simplify: $\left(x-[3-i\sqrt{23}]\right)\left(x-[3+i\sqrt{23}]\right)$