Contents

Additional Problems

A1. Graphing Motion I: Bouncing Ball.

(a) Pick up your larger superball (the one in your toy kit with the bug or skull inside it), toss it straight up, and catch it when it comes back down to your hand. Watch the motion very carefully. Answer the following questions: Just after the ball leaves your hand, is the speed the largest, in the middle, the smallest, or zero? Is the velocity direction up or down?

Answer the same questions for the ball when it is halfway up to its highest point, when it is at its highest point, when it is halfway down, and when it reaches your hand again.

(b) Now, make sketches of the ball's (i) vertical position versus time (choose "up" as the positive direction and $y = 0$ as the ground) as the ball rises and then falls back to your hand, (ii) velocity versus time, and (iii) acceleration versus time. Don't worry about putting any numbers on the graphs; just make a qualitative sketch. **Make sure that your plot of velocity versus time is consistent with your answers from part (a).**

(c) Drop the ball on the ground and watch it carefully as it bounces up and down a few times. Answer the following questions: *just before* the ball hits the ground, is the speed at its peak (i.e., large), in the middle, near its slowest value, or zero, and does the velocity point upward or downward? Answer the same questions for the ball *just after* it hits the ground and starts moving upward.

(d) Now, make qualitative sketches of the ball's (i) vertical position versus time as the ball drops toward the ground before the bounce, and while the ball moves away from the ground after the bounce, (ii) velocity versus time, and (iii) acceleration versus time. **Make sure your plot of velocity is consistent with your answers from part (c).**

(e) Consider what the graphs of vertical position versus time, velocity versus time, and acceleration versus time look like for the short period of time while the ball is in contact with the ground. Sketch on your graph from part (d).

A2. Graphing Motion II: Spring and Return Ball.

(a) Take your round metal spring and your "return ball" (the little rubber ball attached to an elastic string). You want to hang the return ball from the bottom of the spring. It used to be that the return ball was the perfect size to jam into one end of the metal spring. Try jamming it in – if the ball is too small and it doesn't stay inside, then another approach is to stretch the last few turns of the spring and jam the return ball in from the side (it looks a bit like a Pac-Man when you do this).

When you have the return ball hanging from the end of the spring, hold the spring from the other end, and let the spring/ball system hang vertically and allow it to come to rest (you are welcome to use your other hand to help the end of the spring with the ball stop moving). Be careful that the string on the return ball doesn't get tangled in the spring.

(b) Call the position of the ball when it is motionless $y = 0$. Now, pull the ball end of the spring straight down approximately $6 - 10$ inches (the exact distance isn't that important) and release the ball end of the spring. Make sure that the subsequent motion of the ball and the spring is as vertical as possible.

(c) Make qualitative plots of the ball's (i) vertical position versus time for a few cycles, and (ii) vertical velocity versus time. Use your sketch of the ball's vertical velocity versus time to make a qualitative sketch of the ball's vertical acceleration vs. time. (Hint: ask the same questions as in Problem A1 if you are stuck. Specifically consider the ball's position and velocity when it is at the maximum distance from $y = 0$ and also consider the ball's position and velocity when it is passing through $y = 0$.)

A3. Falling Birdie. Consider a badminton birdie that is falling under the influence of gravity and air resistance. Assume that the vertical acceleration of this birdie is given by

$$a_y = \frac{dv}{dt} = g - bv,$$

where $g = 9.8\,\mathrm{m/s^2}$ and b is some constant that depends on the mass and shape of the birdie along with properties of the air, and v is the instantaneous velocity of the bird (positive if the velocity is down and negative if the velocity is up). Suppose that the birdie is released from rest at time $t = 0$.

(a) Discuss qualitatively how the speed of the birdie varies with time, given your knowledge of the relationship between the acceleration and the rate of change of the velocity. What is the velocity when the acceleration is zero? (This is called the terminal velocity.) What happens to the velocity after the acceleration reaches zero?

(b) Without solving the equation above, sketch $v(t)$ vs. t. This can be done as follows: at $t = 0$, v is zero and the slope is g. Sketch a straight-line segment, neglecting any change in the slope for a short time interval. At the end of the interval, the velocity is no longer zero, so the slope is no longer g. Sketch another straight-line segment with the qualitatively appropriate slope for the next short time interval. Continue until your sketch shows the birdie has reached terminal velocity. What is the slope of the line after the birdie has reached terminal velocity?

A4. Estimating Velocities: Muzzle Speed of a Blow Dart. Be careful! Students in the past have set off the sprinkler and fire alarms in their dorms! Consider doing this outside or somewhere where the ceiling is very high.

(a) To estimate the "muzzle speed" of a blow dart, fire the dart straight up into the air and time how long it takes to come back down. Roughly half of that time is the time for the object to decelerate from its initial velocity to motionless (at the top of its motion). **Warning:** students often over-estimate the time required for the trip. Remember to count "0" when you fire the dart ("0 and 1 and 2 and ...")

(b) Now, use

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

to estimate the initial velocity. **Keep this estimate handy:** you'll use it later on in this unit.

A5. Graphing Motion III: Blow darts.

(a) Fire a blow dart roughly horizontally (don't worry about the vertical motion here) at a smooth surface on which it will stick. (Windows make nice targets, but be careful with flat panel displays — students have broken them in the past with their blow darts!)

(b) Make qualitative plots of (i) the dart's horizontal position versus time up until and shortly after the dart hits its target, (ii) the dart's horizontal velocity; and (iii) the dart's horizontal acceleration. In the plot of acceleration, include the small time interval between when the dart first makes contact to when it is completely compressed and stuck. (Hint: ask the same questions as in Problem A1 if you are stuck.)

A6. Rocket Motion. A rocket lifting off from a launch pad in Florida has a vertical position given by

$$y(t) = 70 + 40t + 0.3t^3$$

during the first moments of takeoff, where y is the height of the rocket in meters and t is the time in seconds after the rocket clears the support tower. Assuming the motion of the rocket is purely vertical, determine both the speed and the acceleration of the rocket at time $t = 10\,\mathrm{s}$.

A7. Average Velocity: Round Trip of a Blow Dart.

(a) Fire a blow dart straight up into the air and wait for it to come back down to its initial position. **Question:** for the entire flight, what is the average velocity? (A very quick estimate is fine — if this question takes you more than 1 or 2 minutes to answer, then discuss this with a friend or your problem session instructor.)

(b) Now, fire the blow dart at an angle somewhere in the vicinity of 45° (the actual angle isn't critical). Watch the entire flight, and comment on the average velocity (both magnitude and direction — if not zero).

A8. Vector Addition: Walking Around.

(a) Using the method of components, add the following vectors, and determine the x-component and y-component of the total, and the magnitude and direction of the total: 10 paces at 45°, 12 paces at 90°, 17 paces at −45°, and 10 paces at −135°.

(b) **Do the experiment.** [We will do the following as a class exercise during problem session, so you don't have to do this on your own.] On a sunny day, pick a starting location in the middle of an open field or grassy area. (Lamp posts work well.) Choose the direction that your shadow casts as the 0° direction and imagine a 360° arc around that direction. Now, walk 10 paces at an angle of 45° (you'll be taking your shadow with you, so you'll be able to see that you are walking at the correct angle). Then, walk 12 paces at an angle of 90°, 17 paces at an angle of −45°, and 10 paces at −135°. Where do you end up relative to your starting location? Does this agree with your calculation? Write down how you compared your result with your calculation.

(c) Now, do the same experiment again starting from the same initial location. **But this time** walk off the vectors in a very different order, say, 10 paces at an angle of $-135°$, 17 paces at an angle of $-45°$, 12 paces at an angle of $90°$, and 10 paces at an angle of $45°$. Do you end up in the same place as you did in the previous experiment? (You should.) Comment on your results.

A9. Hello Kitty. A kitten, who has been napping in a patch of sun, sees a flying bug and sprints 3 m due North. She leaps straight up in the air, misses the bug, and comes straight back down. She then calmly strolls 5 m in a direction 40° S of W, where she spits up a hairball under the kitchen table, then walks 2 m in a direction 30° S of E where she begins to lick up some spilled milk.

(a) Draw a diagram showing all the displacements of the kitten.

(b) The kitten's sister was napping with the kitten in the patch of sun. How far and in what direction should the kitten's sister walk to go directly from the patch of sun to the spilled milk?

A10. Love Boat. Juliet is traveling on a fast gondola that is moving at a constant speed of 10 m/s down a canal. Romeo is standing still on the bank of the canal and watching the gondola go by. To attract his attention, Juliet throws her ring into the air and catches it as it falls. Relative to the gondola, the initial velocity of the ring is 15 m/s straight up. Answer the following questions according to **both** Juliet and Romeo. Neglect the effects of air resistance in this problem.

(a) What is the magnitude and direction of the initial velocity of the ring?

(b) How long is the ring in the air?

(c) What is the horizontal component of displacement of the ring while it was in the air?

(d) What is the minimum speed of the ring while it is in the air?

(e) What is the acceleration of the ring while it is in the air?

A11. Graphing Forces: The Large Superball.

(a) Continuing from Problem A1: Toss your large superball straight up in the air and catch it when it comes back down. Make a qualitative graph of vertical component of the *net force* acting on the ball versus time while in the air (you can neglect air resistance here). Compare this graph with the ones that you made for Problem A1. What is the net force acting on the ball at the top of its motion (when it stops its upward motion and starts falling back down)? How does the force at the top of its motion relate to the acceleration (from Problem A1)?

(b) Now, make a graph of vertical component of the net force acting on the ball versus time *when you include* the effects of air resistance on the ball. In doing this, you'll need to think about the direction of the resistive force, as well as how its magnitude depends on its motion.

A12. Tension: Playing with the Return Ball.

(a) Take the "return ball" (the little rubber ball attached to an elastic string) and hang it straight down. Wait for it to stop moving. Note the length of the string (you don't have to measure it). **Question:** How does the tension in the string at this moment compare to the weight mg of the ball? Is $T < mg$, does $T = mg$, or is $T > mg$? Now, add the weight of the round metal spring to the ball. (One way to do this: put the ball inside in the middle of the spring and thread the string out between the coils.) Let the system hang motionless (hold the string with the ball and spring at the bottom pulling downward). Comment on the length of the elastic string now. What is the tension in the string now (answer qualitatively — no numbers)?

(b) Qualitatively, what happens to the length of the string when the tension increases? What happens to the length of the string when the tension decreases? What do you think will happen to the string if the tension goes to zero?

(c) Now let the weight and ball hang straight down. Once the system is motionless, pull the top of the string upward, accelerating the system momentarily. What happens to the string's length? What does this mean about the tension in the string when the ball is accelerated upward? Do the same thing, but this time accelerate the ball/weight downward (quickly lower your hand holding the top of the string). What happens to the tension in the elastic string when the ball accelerates downward?

(d) Now, take the yo-yo and hang it straight down from its string. Accelerate the yo-yo upward and watch what happens. Accelerate the yo-yo downward and watch what happens. Do you observe any changes in the string in either case? What do you think is happening to the tension in the string in both cases? Note that if you accelerate the yo-yo downward hard enough, the string buckles. What does this mean about the tension in the string for a large downward acceleration?

(e) **A very common misconception is that tension for a hanging object is always equivalent to its weight mg. *This is not***

true if the object is accelerating!! If this exercise hasn't made that point clear, talk with your problem session instructor. Under what circumstances is the tension in the string equal to its weight?

A13. Components of Forces: Tension in a Horizontal String.

(a) Put a piece of string through the round metal spring. Grab the ends of the string in your hands, and pull the ends tight until you think the string is perfectly horizontal with the spring hanging from the middle. ***Is* the string perfectly horizontal?** To find out, do this right next to a good horizontal edge, such as the bottom/top edge of a bulletin board or the bottom edge of the top of a flat desk or bed frame. Describe what you see.

(b) Why is it impossible to get the string perfectly horizontal with some mass hanging from the center? The words "component" and "force" should be featured prominently in your answer to this question.

A14. Wondrous Wedge. A 2 kg block is placed on a frictionless wedge that is inclined at an angle of 60° from the horizontal as shown in Fig. 1. If you release the block, it would slide down the wedge. It turns out that if you push the wedge to the right with just the right acceleration a, the block will actually remain stationary with respect to the wedge. In other words, the block will not have any vertical acceleration, though it does have the same horizontal acceleration as the wedge.

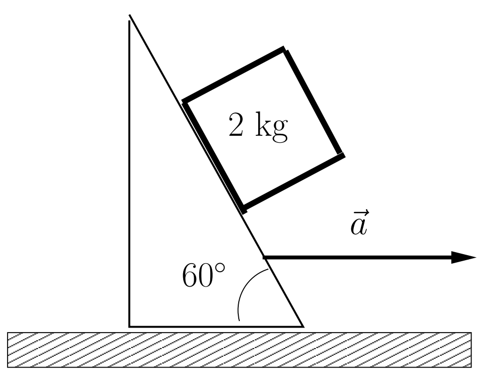

Figure 1: Figure for Problem A14.

(a) What acceleration is required for this to occur?

(b) What would happen if the wedge were given an even greater acceleration?

A15. Using Forces to Predict Motion: Dropping a Spring.

(a) Dangle your "magic spring," holding it by one end, with the other end stretched out and hanging (relatively motionless) an inch or two above your other hand. **Don't do anything yet.** Think about what will happen when you let go of the top of the spring.

(b) *Before doing the experiment*, predict the order of the following five events. Write down your prediction:. (i) The top end starts to move downward; (ii) The bottom end starts to move downward; (iii) The spring contracts halfway back to its unstretched size; (iv) The spring contracts all the way back to its unstretched size; (v) The bottom of the slinky hits your other hand.

(c) Now, once you have made your prediction, let go of the spring and see what actually happens. You might have to do this a few times to figure out what the correct order is, or have a friend or two watch with you. Write down the order of events as you actually observed them.

(d) Explain the results. In doing so, you might want to draw a force diagram for the forces acting on the bottom loop of the spring both before and after the top of the spring is released. (Treat the bottom loop as a small mass hanging from the rest of the spring.)

A16. Force and Acceleration: Unwinding Yo-Yo.

(a) Wind up your yo-yo, and hold the end of the string (or put your finger through the loop). Now, let the yo-yo fall out of your hand and unroll and drop downward (while holding the end of the string motionless). Observe what happens. Write down your observations.

(b) What can you say about the tension in the string while the yo-yo is unwinding? Is the tension (i) equal to the weight of the yo-yo; (ii) less than the weight of the yo-yo but nonzero; or (iii) zero?

(c) **Explain** how you arrived at your answer. You should be able to use the results of your observations, a force diagram and a simple use of Newton's 2^{nd} law to give a clear, definitive answer to the question. (We'll revisit this example again when we study rotations later in the semester.)

A17. Circular Motion: Around the World with Your Yo-Yo.

(a) Let the yo-yo hang at the end of the string and (holding the other end of the string) twirl it in a vertical circle so that it goes all the way around, being careful not to hit yourself or anyone around you in the head! You should realize that you have to swing the yo-yo sufficiently fast; otherwise, it won't get all the way around. Now, watch the string of the yo-yo as you swing it successively slower and slower until it falls out of the loop.

(b) Determine an expression for the theoretical minimum speed for the yo-yo at the top of its motion for it to complete the loop. Assume the string has a length l and the yo-yo has a mass m and determine the minimum speed v_{top} as a function of l, m, and any fundamental constants.

Hints: what happens to the string when the yo-yo is going just slightly too slow to be able to complete the loop? What does this imply? (The answer to this question is the key to solving this problem.) Think back to Problem A12.

A18. Friction Acting on Blow Dart.

(a) The goal of this exercise is to determine the average friction force acting on a blow dart as it slides across the floor. Find a long, smooth, carpetless floor where you can fire a blow dart and watch it slide across the floor and eventually stop. Smooth floors are the easiest surfaces to work with (the floors in Olin Science work well); for most other surfaces (e.g., carpeted floors), the dart will tend to bounce rather than slide. You also might want to crouch down low and fire at a small, glancing angle.

(b) Determine the average friction force acting on the blow dart as it slows to a stop. Make whatever measurements you deem relevant, and use the work-kinetic energy theorem — this exercise is a snap if you do it this way, and a major pain if you try it any other way. You can use your previous measurements of the initial speed of the blow dart once fired, and you might also be interested to know that the blow darts have a mass of 2.5 g and a length of 6 cm, contain roughly 1.5×10^{24} protons and neutrons, and stick nicely to the front of your glasses.

A19. Blow Dart Survivor.

(a) Let's say that you were stranded on a deserted island and a few dozen *Federal Express* boxes washed ashore. Let's say further that one of the boxes contained a *Bandito Blow Gun*. In addition to being delighted at being able to complete your PHYS 211 homework, you also realized that you could replace the suction cups with pointed tips and use this to hunt the birds that flew overhead. (You may neglect any effects of air resistance when working this problem.)

(b) Use the Work-Kinetic Energy theorem along with results of previous measurements to calculate how low a bird would have to fly for you to have any chance of hitting it.

(c) Verify your results using the principle of conservation of mechanical energy.

(d) To test your prediction, fire a dart straight up into the air, just look at its path, and see if your result seems reasonable. You might even try firing it up near a tree or building whose height you know approximately.

A20. Losing Mechanical Energy: Superballs.

(a) Take one of your superballs and release it from rest above a hard surface such as your desk or uncarpeted floor. **Questions:** Does it bounce back up to the height you released it from? Is its mechanical energy conserved?

(b) Estimate how much mechanical energy is lost in one bounce of the superball. Make whatever measurements you deem relevant, and explain your process. Some information that you may find helpful: the small and large superballs have masses 8.5 and 25 g, respectively.

A21. Spring Forward. A woman of mass m rides upward on a spring-loaded ejector pad of spring constant k. It moves upward from rest through a distance x_0 at which point the spring potential energy is zero. Right then the woman leaves the spring with speed v and flies upward reaching a maximum height h above her starting position.

(a) Make a sketch showing her starting position, launch point, and maximum height.

(b) Write down an expression for the mechanical energy for each position.

(c) Equate these expressions to determine the woman's ejection speed and maximum height above her initial position in terms of k, x_0, and m.

A22. Extreme Skiing. Lindsey Vonn has been challenged to test out a new ski event. A loop of radius R has been installed at the end of a ramp of height h as shown in Fig. 2. In this event the skier starts essentially from rest at the top of the ramp and gains enough speed going down the ramp to complete a circle on the inside of the loop. You will need to use both force and energy methods in this problem, and you may assume that the snow is frictionless.

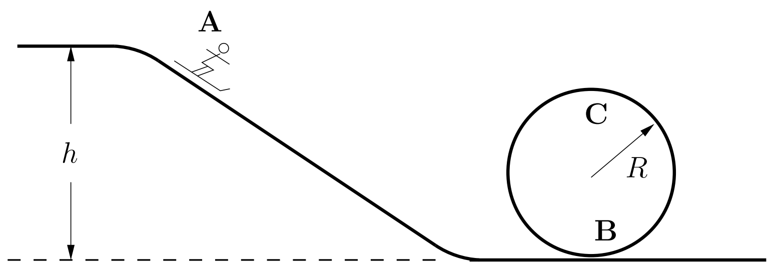

Figure 2: Figure for Problem A22.

(a) Draw a force diagram on Lindsey when she is as point **A**. Which statement about the magnitude of the normal force (**N**) of the ramp on Lindsey at point **A** is correct?

$N = mg$ $N < mg$ $N > mg$ $N = 0$ not enough info

(b) Draw a force diagram on Lindsey when she is as point **B**, just *after* she has entered the circular loop. Which statement about the magnitude of the normal force of the loop on Lindsey at **B** is correct?

$N = mg$ $N < mg$ $N > mg$ $N = 0$ not enough info

(c) Draw a force diagram on Lindsey when she is as point **C** at the top of the loop. Which statement about the magnitude of the normal force of the loop on Lindsey at **C** is correct?

$N = mg$ $N < mg$ $N > mg$ $N = 0$ not enough info

(d) Determine the minimum speed that Lindsey needs at point **C** to stay on the track and make it around the loop.

(e) Determine the minimum height h in terms of the radius R such that Lindsey can make it around the loop.

(f) What is the maximum magnitude of the force of the ground on Lindsey's skis for the minimum height, in terms of her weight?

A23. Trapped in Space. A spacecraft drifting through the center of a giant cosmic dust cloud experiences a potential energy that varies with position as shown in Fig. 3 (d is the distance from the center of the dust cloud). The spacecraft starts at the center of the dust cloud.

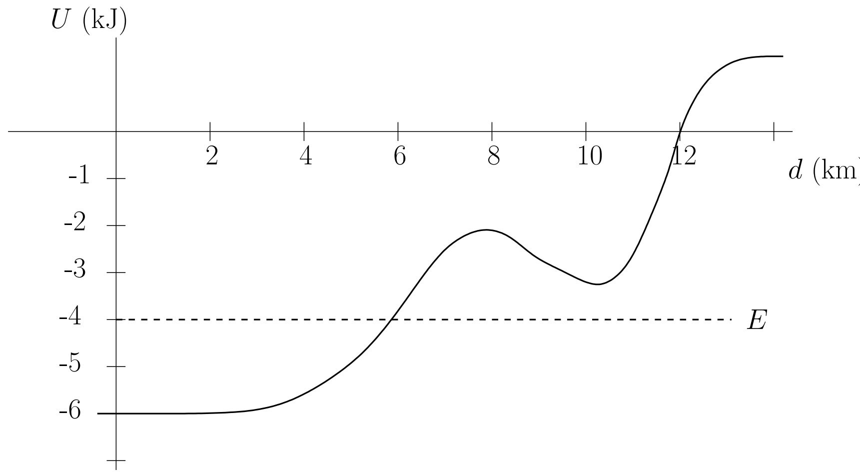

Figure 3: Figure for Problem A23.

(a) If the spacecraft has a total mechanical energy of $E = -4\,\text{kJ}$, what is the farthest distance it could drift from the cloud's center?

(b) Determine the kinetic energy of the craft when it is 2 km from the cloud center.

(c) Describe the motion of the spacecraft if it had started at the center with a mechanical energy of $-2.5\,\text{kJ}$.

A24. Hopping Popper.

(a) In your kit, there should be a small rubber hemisphere that is referred to as a "popper." If you turn it inside out and flex it for a few seconds, you can lay it on a table before it pops back into its original shape. When it pops, it will jump up off the table, giving a nice demonstration of the conversion of potential energy into kinetic energy.

(b) Using whatever means you see fit, estimate the potential energy (in J) stored in the popper just before it pops. (There's a really easy way to do this.) The diameter of the popper is 1 inch, its mass is 1.8 g, the hole in the center has a diameter of 2 mm, and there are approximately 3300 students at Bucknell.

A25. Spring Slide Stop. You push a block against a horizontal spring, compressing the spring by 15 cm. When you release the block, the spring propels it across a level tabletop. The block stops 75 cm from its release point. The spring constant is 200 N/m. Determine the magnitude of the friction force (assumed constant) between the block and the table.

A26. Superball Stack! Take your larger superball and carefully balance your smaller superball on top of it. Release both from rest, and allow them to fall perfectly vertically in a line. The large superball should hit the ground first and then collide (going up) with the small superball (still on its way down). This requires lots of patience and luck, but if you get it, the result is incredible! This can be understood by treating the various collisions as perfectly elastic collisions, which they almost are.

A27. Blow Darts and Superballs. There is nothing that makes a day more complete than firing a blow dart at a small, defenseless superball. Before doing this, though, here is some relevant information: the blow dart has a mass of 2.5 g and the small and large superballs have masses 8.5 and 25 g, respectively.

(a) Now, put your larger superball (the bug or skull ball) at the edge of a table and softly fire a blow dart directly at it (do it until the blow dart hits almost straight on). By "softly" we mean don't blow as hard as you usually do. If you do this correctly, the dart should bounce straight back, and the large ball will move forward with only a very small speed.

(b) Now, the main question: What changes do you have to make such that the blow dart will continue forward after a head-on collision with another object? Predict what you need to do, write down your prediction, and then test out your theory.

A28. Hogwarts Hijinks. Harry and Hermione are playing in the GraviFree Room at Hogwarts. Harry (mass 55 kg) is floating motionless in the center of the room. Hermione (mass 45 kg) pushes off from the wall and approaches Harry at a speed of 6.0 m/s. Neglect air resistance in this problem.

(a) As Hermione moves past Harry, he reaches out and grabs her outstretched hand, holding on tightly. Determine the speed with which Hermione and Harry move after they grab hold of each other.

(b) Harry and Hermione notice Ron giving them a funny look, so they let go of each other. Determine the speed with which Hermione moves after they let go.

(c) Determine the speed with which Harry moves after they let go.

A29. Railing in the Rain. An open railroad car of mass 2×10^4 kg is rolling without friction along a level track at 5 m/s when it starts to rain. After the car has collected 2000 kg of water, it stops raining. Assume that the rain fell perfectly vertically.

(a) What is the rail-car's speed after it stops raining?

(b) After the rain has stopped, a hole in the bottom of the rail-car is unplugged, and the rain water begins to leak out of the hole at a rate of 5 kg/s. What is the speed of the rail-car after half the rain water has leaked out?

(c) What is the speed after all the rain water has leaked out?

A30. Relative Velocities (Classical). A typical person walks with a speed of about 2 m/s relative to the ground. While you are walking between classes, watch other students who happen to be walking in the same and opposite direction as you, and answer the questions in the following parts.

(a) Choose a student walking in the same direction as you with the same approximate speed. Note how far away that person is from you. Then, after the two of you have walked for a few seconds, note again how far that person is from you. Has the distance between the two of you increased, decreased or stayed roughly the same? Assuming that you are both walking at a speed of 2 m/s relative to the ground, what does your previous answer imply about the speed of the other person as measured in your reference frame?

(b) Do the same thing for a student walking in the opposite direction as you. Answer the same questions as in part a).

(c) If you happen to see someone running to class, but going in the same direction as you, ask yourself the same questions as in part a).

A31. Measuring the Length of a Moving Object, Take 1. Measuring the length of an object that is at rest with respect to you is pretty easy: one method is to take a ruler of some kind, hold it up to the object, and note where each end of the object is with respect to the ruler.

(a) What difficulties arise if you try to measure the length of an object that is moving with respect to you using the technique described above?

(b) Other methods need to be developed to measure the length of a moving object. We'll have you try an approach that employs a group of people. [We will do the following as a class exercise during problem session, so you won't have to gather a group of your own.] Go outside, and line up in a row, parallel to a street with some automobile traffic. (Please stand a safe distance away from the street!). Stand so that there is approximately equal distance between you and your nearest neighbors

(c) Your instructor will stand across the street. When a car comes by, your instructor will yell "Now!". If the front of the car is directly in front of you, **raise your hand** and **keep it raised**. If the back of the car is directly in front of you, **raise your hand** and **keep it raised.** (By yelling "Now!" your instructor has basically synchronized your clocks, so that you are making your measurements — i.e., raising your hand — simultaneously. You'll see in Problem A33 that to make this measurement even more carefully, we would have to come up with a better method of synchronization. We'll discuss some thorny issues involving simultaneity in an upcoming lecture.)

(d) Now, measure the distance between people who have their hand raised. How is this distance related to the length of the car? Does it matter how fast the car is going in using this technique?

A32. Measuring the Length of a Moving Object, Take 2. In problem A31 you measured the length of a moving object using several people and synchronization. In this problem, you will develop a technique that you could use on your own.

(a) Assume that you know the velocity of the car (say the car is going the speed limit), and that your available tools are a ruler and a clock. Figure out a method to determine the length of the moving car. Describe what you would do and what you would measure. and how you would use the results of your measurement to determine the length.

(b) You likely measured a time interval and/or a distance. Think how the driver of the car views the situation, especially if the speed were relativistic (say if the car were going at $0.8c$ relative to you). Would the driver of the car agree with your measurement of the time interval and/or distance? Would she think your measurements are too high? too low? correct?

A33. Synchronization, Simultaneity and Spacetime Diagrams.

(a) Grab a friend or roommate (it doesn't have to be someone taking PHYS 211). Stand on opposite sides of a room or a long hallway (the larger the separation distance, the better). Throw a ball to your friend (or have that person throw the ball to you). Draw a qualitative spacetime diagram of this situation, showing world lines for you, your friend, and the ball.

(b) Your next goal is to have both of you clap your hands at precisely the same time, but you have to keep your eyes closed while doing it. Here's an approach that you might try: you could say (loudly), "On the count of three, we'll both clap our hands. One, two, THREE!" Go ahead and try this, and then comment on inaccuracies in this method (i.e., why doesn't this work?). Draw a spacetime diagram to support the argument.

(c) See if you can figure out a way that will result in you and your friend clapping at the same time. Write down the method, and draw a spacetime diagram that demonstrates that this is a good approach.

A34. Life in a Relativistic World, Part I. A typical person walks with a speed of about 3 mph relative to the ground. For this problem, imagine that the speed of light were actually 4 mph rather than 3.0×10^8 m/s.

Walk across campus, perhaps on your way to or from class or going to dinner. Choose a time when there is a lot of activity around you (cars moving around, other people walking around, etc). While you are walking, watch everything around you. Note what you see, what you feel, whatever you experience (when you are moving, waiting to cross a street, etc.), and think about how any of these things would be different if the speed of light were 4 mph. **Write a couple of paragraphs summarizing your thoughts.** And feel free to discuss this with other people in the class. (Some things to think about in particular: length contraction, time dilation, and simultaneity — all of these things would be *very* noticeable in this hypothetical scenario.)

If you really think about a lot of the things around, you should come to the conclusion that if c were really 4 mph, it would truly be a whacked-out, psychedelic, something-out-of-a-Salvador-Dali-painting experience.

A35. The Real Potential of a Superball. Pick up your largest superball and just stare at it for a little while. Does this look like it contains a lot of energy? Now, estimate its mass (or alternately look back at the Problem A27 where the mass is given) and determine the rest energy of the superball (in Joules). Now, consider that an atomic bomb releases

10^{14} to 10^{15} J of energy; consider also that a typical household uses about 10^{10} J of energy per year. Stare at your little ball again. Write a sentence or two about your thoughts. (Feel free to post your thoughts on the "Questions" page at the course web-site if you want to share them.)

A36. Life in a Relativistic World, Part II. Let's think a little more about what it would be like walking across campus if the speed of light were 4 mph rather than 3×10^8 m/s. You've already thought about time dilation, length contraction and simultaneity in problem A34. Now, think about what the relationship $E = mc^2/\sqrt{1 - v^2/c^2}$ would mean in a relativistic world.

(a) If the speed of light were 4 mph, how much *kinetic* energy would be involved in walking at a speed of 3.5 mph? (Use your own body mass in these estimates.) How much kinetic energy would you have walking at 3.8 mph? How do you think you would feel as you start trying to walk faster and faster, past 3.0 mph, past 3.5 mph, past 3.8 mph, past 3.9 mph, . . . ?

(b) What do you think might happen if you collided with another person if you were both walking with a speed of 3.8 mph but in opposite directions?

A37. Life in a Really Relativistic World. Common misconceptions about relativity abound. You'll hear people say that relativity states that "if you are on a ship traveling close to the speed of light, your mass increases to infinity, you shrink down to zero size, and you never age." Statements like this have led people to think that life would be very strange on such a spaceship. We want you to experience what it really would be like to be on such a spaceship.

So, go ahead and do this experiment. Hop on a spaceship that is traveling at a speed of at least $0.8c$ relative to some reference frame. Before you start saying that we've completely cracked up, there is a spaceship that everyone in this class has access to that meets this requirement. (Hint: the name of the ship starts with the letter E and its name rhymes with *birth*, and it is currently traveling at a speeds of greater than $0.9c$ relative to distant galaxies and quasars.)

Question: Do you feel at all strange being on such a ship? Write a sentence or two of your thoughts about this. (Feel free to post your thoughts on the "Questions" page at the course web-site if you want to share them.)

A38. Photon Absorption An elementary particle has a rest mass of $1125\,\mathrm{MeV}/c^2$, and is motionless in some reference frame. A photon with momentum $750\,\mathrm{MeV}/c$ strikes the particle and is absorbed, leaving an "excited" particle that is recoiling and nothing else. Determine the mass and recoil velocity of the excited particle after the interaction.

A39. Let There Be Light. Particle A of mass $400\,\mathrm{MeV}/c^2$ collides with the stationary particle B of mass $350\,\mathrm{MeV}/c^2$. The result of this collision is a single particle C at rest, and a 300 MeV photon. Determine the mass of particle C.

A40. What's That Skull (or Bug) Doing in My Superball? We can take advantage of the poor bugs and skulls trapped in your larger superball to comment on the rotation of the ball.

(a) Take your superball and rotate it slowly, watching the object as it rotates. You can either do this in your hand, or toss it gently with a little rotation — whichever enables you to see the object spinning easiest. Try rotating it quickly as well. What can you say about the motion of the middle portion of the object, as opposed to the motion of the part of the object farthest from the middle? How does your answer to the previous question relate to the equation $v = r\omega$ for tangential speed?

(b) Now, drop the superball straight down while spinning it very rapidly about a horizontal axis. The best way to do this is to use two hands to get it spinning as fast as you can while releasing the ball. What happens when the ball bounces? Specifically, does it bounce straight up? Why not? (You'll want to use a diagram and Newton's 2^{nd} and 3^{rd} laws to support your argument.) Also, what happens to the angular velocity of the superball after it bounces? Explain *why* this happens. (Consider the torque acting on the ball when it bounces on the floor.)

(c) (Optional) If you are good at spinning the ball, try this: toss the ball slightly away from you, but spinning with the top toward you. If you do this well, you can get the ball to bounce back and forth on the ground. Explain *why* this happens.

A41. Yo-yos (revisited) with Rotations. We're going to repeat Problem A16, but this time we're going to be quantitative and take rotation into account.

(a) Count how many turns of the string are required to wind up the yo-yo all the way. From this, calculate $\Delta\theta$ for the yo-yo to unwind completely (in radians). Now, holding the end of the string, let the yo-yo unwind all the way, and estimate the time for it to reach the bottom (within a couple of tenths of a second). Since the angular acceleration is constant during this process, you should be able to take two integrals of α to find that $\Delta\theta = \alpha t^2/2$. From this information, determine the angular acceleration of the yo-yo as it falls.

(b) Now estimate the average radius of the spool (i.e., the average distance of the point-of-contact of the string from the center of the yo-yo), and use this information to estimate the linear acceleration of the yo-yo during its fall. Then, use this information (along with a force diagram and Newton's second law) to determine the tension in the string while the yo-yo is falling. Note: the mass of the yo-yo is 52 g, its total thickness is 3.5 cm, and it fits nicely in your pocket.

(c) Is your result from part (b) for the tension consistent with the qualitative answer from Problem A16?

A42. As the Ball Turns. A solid 1.4 kg ball with diameter 15 cm rotates about its diameter at 70 revolutions per minute.

(a) Determine the kinetic energy of the solid ball.

(b) If the ball had the same mass and diameter, but all the mass was at the outer surface of the ball (in other words, the ball were hollow), would the ball have more or less kinetic energy than you calculated in part a)? Assume this hollow ball has the same angular speed.

(c) Back to the solid ball again. If you add an additional 2 J of rotational kinetic energy, determine the solid ball's new angular speed.

A43. Yo-yos, mechanical energy, and angular momentum.

(a) Unwind the yo-yo and rotate it in a vertical circle at the end of its string. Once it is going, allow the yo-yo string to wrap around your arm — the result should be that the yo-yo spirals inward until all the string is wrapped around your arm. Do this a few times and watch the yo-yo as it spirals inward. Do you think the yo-yo is speeding up, slowing down, or going at basically the same speed during this process? (Watch the yo-yo very carefully here — your eyes can easily trick you.)

(b) Now, think about this process both from a perspective of mechanical energy and angular momentum. Which of these quantities do you think are conserved during this process (or do you think that neither or both are conserved)? Justify your answers: for angular momentum, you'll need to show either that torque acting on the yo-yo is zero or non-zero, and for energy, you'll need to explain either that work is or is not being done on the yo-yo. Based on these answers, should the yo-yo be speeding up, slowing down or basically going at the same speed while spiraling inward?

(c) Now do the same thing again, but this time, instead of letting the string wind around your arm, thread the string through a PVC tube (which we'll provide in problem session) and pull the string through the tube to pull the yo-yo inward. Answer all the same questions that you did in parts a) and b).

A44. Yo-yos and torque.

(a) Take a partially-wound (i.e., partially-unwound) yo-yo and place it on a level surface such that it could roll if pushed or pulled. Now, predict which way it will roll if you pull the string straight up. (Justify your prediction with diagrams.) Try the experiment — were you correct? If not, justify what actually happened.

(b) Now, do this again, but this time let the string go over the top of the yo-yo and pull it parallel to the table. Again, first make a prediction about which direction the yo-yo will move (and justify it), then try the experiment. Again, if you were not correct in your prediction, justify what actually happened with diagrams and Newton's laws.

(c) Finally, predict which way the yo-yo will move if the string goes underneath the yo-yo, and you pull it parallel to the table. (Again, justify your prediction with a diagram. Try the experiment. Were you correct? Again, if you were not, then justify what you actually saw.

A45. When Wheels Collide. Two solid wheels of identical mass but different radii (R and $2R$) are spinning on the same axle (on very smooth bearings). The wheels are spinning in opposite directions, but with the same angular speed ω_i, as shown in Fig. 4. The two wheels are slowly brought together, and the resulting frictional interaction between the touching surfaces eventually brings the wheels to a common angular speed ω_f.

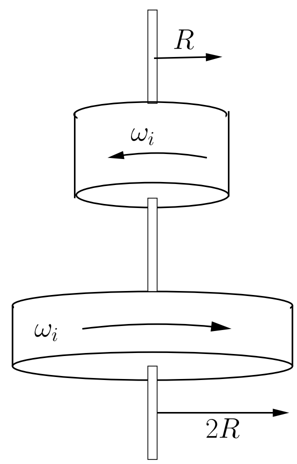

Figure 4: Figure for Problem A45.

(a) Determine ω_f in terms of ω_i.

(b) Are the wheels now rotating in the original rotation direction of the larger or the smaller wheel?

A46. How much do you suck?

(a) In problem session, you will be provided with a container that can hold water and a piece of flexible tubing. Fill the container with some water and place the container on the floor. Take your piece of flexible tubing and put one end into the water and stand up with the other end. Put the other end into your mouth and breath in, pulling the water up the tube. Don't breath in the water — that wouldn't be very fun. (That shouldn't be a problem because if you are standing up, you won't be able to get the water all the way up the tube anyway.) Estimate the maximum height above the water surface that you can hold the water by continually breathing inward. **Important:** don't use your cheeks to suck on the tube.

(b) Now, use this height to determine the "gauge pressure" of your lungs when you suck (the difference between your lungs' pressure and atmospheric pressure). To do this, determine the force required to hold up a column of water with height h in a tube with cross-sectional area A (keep things in variables — don't put the numbers in yet). Once you have the force, you should be able to get the gauge pressure (Δp) from the relation between force, pressure and area. Then, put the numbers in. What is the *absolute* pressure that your lungs achieve when you suck?

(c) If you could make a perfect vacuum with your lungs, what would be the maximum height that you could suck water up in a straw?

(d) Now, let's see how hard you can blow. Fill up the tubing about 2/3 to 3/4 of the way with water. The easiest way to do this is to bend down low and suck water again. (Do you notice how much easier it is to suck up the water when it doesn't have to climb as high?). A little before the water reaches your mouth, stop sucking and lift up the two ends of the tubing to make a "U" shape. Now, blow into one end while raising the opposite end. If you have a friend to help, that would be good — he/she can continually raise the other end to make sure that you don't blow the water out of the tubing. Estimate the maximum height (above your mouth) that you can hold the water, and use this information to estimate the gauge pressure and absolute pressure of your lungs when you are blowing.

Note: What you have just done is a technique that is used all the time in the medical industry to measure lung performance in patients. This is particularly useful for patients with lung cancer or various breathing disorders — this kind of test can quickly and easily determine how well the lungs are functioning.

(e) If you made a straw several hundred miles long, stuck one end into the ocean and stuck the other end out into the vacuum of space, would the straw suck up all of the ocean water into space? Why not? (In fact, the water wouldn't rise up at all in the tube. Try to figure out why it wouldn't.)

A47. Dunking Birds and the Ideal Gas Law.

(a) We're not actually going to use the dunking bird in its intended purpose here (don't worry — that will come). Instead, grab the Bird's bottom in the palm of your hand and wrap your hand around it. Presumably, if your hand is at normal body temperature (37° C), the fluid in the Bird will rise up toward the head, leaving a larger volume of gas than when you started.

(b) Estimate the volume of the gas in the Bird's bottom before and after you warmed it up with your hand. Actually, you really only need to approximate the ratio of the two volumes $V_{\text{after}}/V_{\text{before}}$. Now, determine the ratio of the temperature of your hand to the temperature of the air, using estimates of the room temperature and your body temperature. (What units are you using for temperature?) Using the ideal gas law, determine if the change in the volume is consistent with the change in the temperature. (Show all your work here.)

(c) What else do you think is going on inside the Bird? We're not expecting a complete answer — this is more of a set-up to help motivate the next class. But you should be able to use the ideal gas law to make some statements about what else is going on inside the Bird.

A48. Pressure and Force. It is fairly straightforward to estimate the pressure inside a blow dart's suction cup when it is sticking to something. First, we'll look at it qualitatively, then put some numbers in.

(a) Wet the suction cup on one of your darts and press it onto a flat, smooth surface so that it sticks. (It's best to have the dart wet, because this will keep air from leaking in around the suction cup.) Pull on the dart and note how much force is necessary to pull the dart off the surface. You don't have to be quantitative here; simply comment on how difficult it is to pull off.

(b) While you are pulling on the dart, what is causing the force that pulls (pushes) the dart *back onto the surface*? Of course, this is due to the pressure difference between the inside and the outside of the suction cup, but what *physically* is causing the force? (Refer to the kinetic theory of gases to answer this.)

(c) Now, let's do this semi-quantitatively. If you stuck the dart to the underside of a smooth surface, you could hang about 1 kg of mass from the dart without it coming off. Based on this, you can determine the maximum force (in N) that the dart can withstand before coming off the surface. And once you have the maximum weight that it can hold, use the definition of pressure (in terms of force and area) to estimate the pressure within the suction cup. Note: the suction cup has a diameter of about 1.8 cm. (You should estimate the pressure *difference* between the air and the inside of the cup first, then you can get the absolute pressure inside the cup.)

(d) If there were a perfect vacuum inside the suction cup, what would be the maximum weight that it could hold?

A49. Balloons and Bottles.

(a) Do the following experiment: Get a glass drink container — one of those juice/cranberry bottles will work, but a taller/deeper glass drink container is better. (You might be able to pluck something out of one of the recycling bins if needed.) You'll need to stretch a balloon across the opening of the jar, so try that out to make sure you can do it, then take the balloon off. Then, boil a small amount of water (you can use a microwave if you want, but make sure that the water is really hot and steaming). Pour a small amount of the boiling water into the jar (cover only the bottom cm or so). If the water is hot enough, there should be a noticeable amount of steam coming out of it. Then, stretch the balloon over the mouth of the jar and then watch the system as things slowly cool down.

(b) Describe what happens, and explain *why* it happens. In particular, comment on any condensation of the steam that you see on the inside of the container. Is this condensation important as far as the behavior of the balloon is concerned?

A50. Using Phase Transitions to Cool a Drink. You'll need to do this at lunch or dinner, or somewhere that you have access to ice.

(a) Get three glasses or cups. Fill one glass (let's call it glass A) with a mixture of ice and water (plenty of ice), and fill the other two glasses each halfway full with room temperature water. Let the ice/water mixture in glass A sit for a few minutes: this will ensure that both the ice and the water in the glass are at temperature 0° C.

Next, you are going to take out a fair amount of 0° C ice from glass A (a spoon is a convenient way to do this) and dump it into glass B, one of the half-filled room-temperature glasses. Then pour an equivalent amount of 0° C water from glass A into glass C, the other half-filled glass. The idea is to compare the cooling effects of 0° C ice compared to 0° C water.

Before doing the experiment **predict** whether glass B and glass C should be equally cooled, or if not, which will be cooler. Write down your prediction.

(b) Now, go ahead and do the **experiment**. Record the results in your notes. Is the result what you expected? Use heat flow arguments to **explain** why you obtained this result.

A51. The Dunking Bird, revisited. In Problem A47, you should have found that the temperature change alone wasn't enough to cause the volume change and the resulting movement of the fluid up into the Bird's head — there must have been a significant change in N, the number of gas molecules in the Bird.

(a) Explain how N is increasing in the Bird's bottom when you heat it with your hand. Explain also how N decreases in the head when the head is cooled. What about the fluid inside the Bird — why do you think the Dunking Bird is filled with methylene chloride instead of simply dyed water?

(b) Now, how is the head of the Bird cooled during its normal dunking operation? Does the water in the glass have to be cooler than the room temperature? Try the following experiment: try using water in the glass that is measured to be the same as room temperature or better yet, heat up the water to be several degrees above room temperature, and dip the Bird's head in this warm water. **Question:** does the Bird still dunk? (The result might surprise you.) So, how *does* the Bird's head cool?

(c) A Dunking Bird with a wet head is comparable in many respects to a person who is sweating on a hot day. Based on what you know about phase transitions (melting, vaporization, etc.), explain *why* it is necessary for a person to sweat on a hot day. Why doesn't a person sweat as much on cooler days?

A52. Energy Stored in a Balloon.

(a) When you blow up a balloon, you are clearly doing work on the balloon. Alternatively, you can say that the gas in the expanding balloon is doing work. And this work goes into potential energy. **Question:** where is that energy "stored"?

(b) In this problem, we're going to estimate that stored energy using $W = \int p\,dV$. We'll use the approximation that the pressure is almost constant (we'll estimate an average pressure) so that $W = p\,\Delta V$. And since the air outside the balloon is doing negative work on the balloon while it expands, and we want the *net* work done by the gas, you can use the gauge pressure to get the net work done by the air while the balloon expands.

To estimate the gauge pressure, you need to attach the balloon to the end of your flexible hose with a rubber band, blow the balloon up half-way, then make sure to pinch off that end of the balloon. Now, get some water into the tube (still holding the balloon end pinched off), and then hold up the tube (in a U-shape, with the balloon at one end and the open end of the tube at the other). Finally, release the balloon such that the gauge pressure of the balloon pushes the water in the tube.

(c) Use this technique to estimate the average gauge pressure (see problem A46 for a refresher if you have forgotten), and estimate the change in volume when the balloon is fully inflated. From this, you should be able to estimate the energy stored in the balloon.

A53. The Thermodynamics of Blow Darts. Let's think about what happens when you fire a blow dart. We'll use the results from problem A46 (in which you determined the gauge pressure that your lungs can produce) to estimate the ideal maximum speed that you could achieve when firing a blow dart.

(a) When you blow on the dart, the pressure p_{lung} from your lungs pushes the dart down the tube while atmospheric pressure p_0 pushes the dart the other way. Draw a quick sketch of the dart, and draw arrows corresponding to the forces F_{lung} and F_{atm} on the dart. Considering that the dart has a cross-sectional area A, what is the net force acting on the dart when you fire? Re-write this result in terms of the gauge pressure for your lungs.

(b) Now, use the result from A46 and the fact that the dart has a diameter of about 1.8 cm to estimate the net force acting on the dart when you blow. Now, considering that the active part of the blow gun is about 50 cm long, estimate the work done on the dart when firing. (Note: you'd get the same result by using $W = P_{\mathrm{gauge}}\,\Delta V$.) Finally, use the work to predict the exit speed for the dart when you fire.

(c) The answer that you get here might differ significantly from what you measured in Problem A4. Why do you suppose these two results could be so different? Do **not** use "human error" anywhere in your response!

A54. Path Matters. One mole of an ideal monatomic gas is heated from 300 K to 600 K.

(a) If the gas is held at constant volume, find the change in the gas's internal energy, the work done by the gas, and the heat added to the gas during this process.

(b) If the gas is held at constant pressure, find the change in the gas's internal energy, the work done by the gas, and the heat added to the gas during this process.

A55. Some Cycle. In the cycle shown below, 1.0 mole of a monatomic ideal gas is initially at a pressure of $p_A = 100\,\mathrm{kPa}$ and a temperature of $T_A = 0°\,\mathrm{C}$. The gas is heated at constant volume to $T_B = 150°\,\mathrm{C}$ and is then expanded adiabatically until its pressure is back to $p_C = 100\,\mathrm{kPa}$. Finally, the gas is compressed at constant pressure until it is back to its original state A. Find

(a) the temperature T_C after the adiabatic expansion,

(b) the heat entering or leaving the system during each process,

(c) the efficiency of this cycle, and

(d) the efficiency of a Carnot cycle operating between the temperature extremes of this cycle.

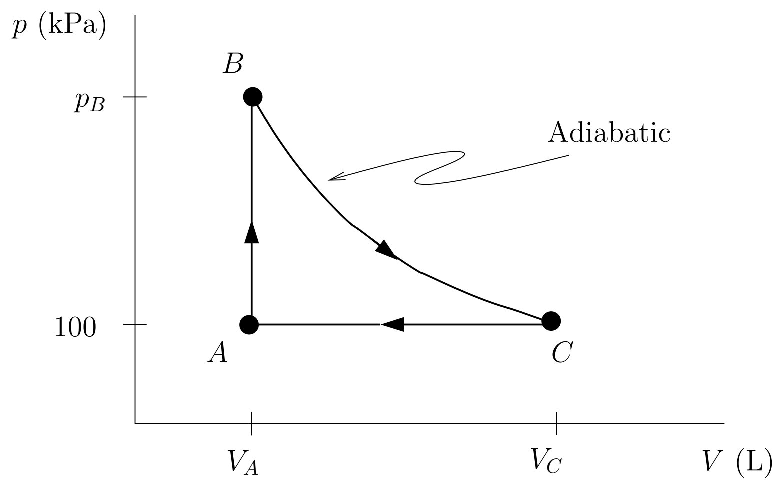

Figure 5: Figure for Problem A55.

A56. Dunking Birds as Heat Engines. First, get your Dunking Bird going. You can actually operate it one of two different ways:

(I) You can get the head wet and then just put it on a table. If the Bird is properly balanced (you may have to slide the metal piece up or down), then it should start going.

(II) You could place the Bird on top of a TV or computer monitor and let the heat from that device warm the Bird's bottom. (**Warning:** Be careful if you do it this way: people have wound up with blue or red stained monitors and desks as well as having to clean up broken glass!)

(a) If the Dunking Bird is being powered by heating of its bottom (i.e., Method II above), the gas/fluid in the bottom will undergo several steps. (The steps are actually continuous, but we'll break them up to make this easier to plot.) (i) As the bottom heats up, the pressure of the gas inside increases. (Do you remember why? It isn't just the change in temperature.) (ii) After the pressure has increased, the fluid is forced out of the bottom, and the volume of gas in the bottom increases. (iii) The Bird tips over and the air in the bottom and the air in the head are connected, causing a quick drop in the pressure of the gas in the bottom down to its initial value. (iv) The Bird stands up again, and the fluid runs down into the bottom, decreasing the volume of gas in there.

Plot the sequence described above on a P-V diagram. Do you have a cyclic process here? Show the work for the engine cycle on your P-V diagram.

(b) Repeat part (a), but for Method I (where the Bird's head is cooled by evaporation).

(c) What is the hot reservoir for this engine? What is the cold reservoir? What is the work done by the Bird (in words)? Draw an engine diagram for the Bird.

A57. Entropy and the Second Law.

(a) Scatter at least 10 coins over a large surface area. Then, carefully pick them all up and stack them into a neat pile. Has the entropy of the coins increased, decreased or stayed the same? Use arguments based on probability to answer this question. (e.g., "It is more probable that you'd . . . ").

(b) Is your answer consistent with the Second Law of Thermodynamics? (The answer must, of course, be yes, but you might have to think a bit to figure out how to reconcile this with the Second Law.) Explain your reasoning.

A58. Macrostates and Microstates. This problem gives you practice with the idea of macrostates vs. microstates, in a different context than that provided in the reading. Let's think about macrostates vs. microstates for the rolling of two six-sided dice.

When you roll two six-sided dice, each die can show a 1 through 6. The SUM of the numbers showing on the two dice is an integer from 2 through 12. That SUM is the macrostate. The microstate is the specific combination that resulted in that macrostate. So for example, if you rolled two six-sided dice, and the total of the two dice was an 8, that total could have been obtained a number of different ways: (2 and 6), or (3 and 5), etc. In this example, the MACROSTATE is the sum 8, and some of the MICROSTATES associated with that macrostate are (2 and 6) or (3 and 5).

(a) Consider all of the possible macrostates for this system of two six-sided dice. For each macrostate, write down all the possible microstates associated with that macrostate. Assume that the dice are distinguishable from each other, which means that there is a difference between (2 and 6) or (6 and 2). Which macrostates have the most microstates associated with it/them? Which macrostates have the fewest microstates associated with it/them? Use this to argue which macrostates are the most probable, and which macrostates are the least probable.

(b) Now, roll two six-sided dice 10 times, and record the macrostates that you observe. (If you don't have access to dice, you'll be able to do this part in problem session.) Does your experimental evidence support your predictions from part (a); in other words, was the most probable macrostate clearly rolled more than the least probable macrostate? You may be surprised by your results. What do you think you need to do in order for the predictions to more accurately model the results of the experiment? We'll collect data from the entire class for the number of times your most probable macrostate came up, and the number of times your least probable macrostate came up.

A59. Playing with the Period of Oscillation.

(a) Take your round metal spring, hold it by one end, and let the other end oscillate in the vertical direction. Determine the period of oscillation using the following technique: find the approximate time for 5 complete periods and divide by 5. Record the period of oscillation for the round metal spring. What is the angular frequency of the oscillator?

(b) Now, take your round metal spring and jam your return ball into one end of the spring (as you did in problem A2.) (Note: you may want to increase the mass even more). Hold the spring by one end, letting the end with the ball wedged in dangle freely so that it can oscillate in the vertical direction. **Predict** whether the period of oscillation will be *larger than*, *smaller than*, or *equal to* the period of oscillation you obtained in part (a). **Justify** your prediction. Try the **experiment** — were you correct? If not, **explain** what *actually* happened.

A60. Circular Motion Versus Oscillatory Motion.

(a) Have a partner hold one end of the round metal spring and rotate it so that the other end makes a horizontal circle. Now, you should stand back a couple of meters, and with one eye closed, watch the rotating end of the spring from the side so that the motion appears as though it is on a line. If you view it from an appropriate angle, the motion should look exactly the same as if your friend were simply oscillating the spring back and forth along a line.

(b) To develop this further, ask your friend to go ahead and swing the spring back and forth instead of in a circle. If you have one eye closed and if you are looking at it from the best angle, from your vantage point, it will look the same as if it were going in a circle. And to develop this even more, have your friend either rotate the spring in a circle or oscillate it back and forth while you try to figure out which kind of motion the spring is following. Treat it as a challenge — your friend should try to fool you into thinking it is straight line motion when it is actually going in a circle or vice-versa.

The point of this experience is to help clarify the idea that oscillatory motion can be thought of as one component of circular motion. This idea will be very important when we talk about waves and interference in PHYS 212.

A61. Resonance

(a) We're going to map out (approximately) a simple resonance curve for your round metal spring. You may use just the round metal spring, or you may use the round metal spring with the return ball jammed in one end. Just make sure you indicate what you are doing. In part (b) you will hold the spring by one end and let the other end dangle, and then oscillate your wrist at varying frequencies. Before doing this, draw a sketch of what you would expect a resonance curve (amplitude of response versus driving frequency) would look like for the spring, assuming very little damping. Based on your results from problem A59, what would you expect the frequency to be for the peak of this curve? Write that frequency down.

(b) Now, hold the spring by one end and let the other end dangle. Wait until any residual oscillations damp out, then oscillate your wrist very slightly (amplitude of only a cm or less) at a frequency significantly smaller than the one that you predicted for the peak of the curve. (Write down in your notes an approximation of what that frequency is.) Do this for a few periods of oscillation, and comment on how the motion of the bottom end of the spring relates to the motion of your hand during this procedure.

(c) Now, repeat this again for a frequency that is significantly higher than the predicted resonance frequency, and comment on the results. Finally, repeat this again for a forcing frequency close to your predicted resonant frequency. Comment on your observations.

(d) Overall, do you observe resonant behavior? Explain how your observations are consistent with ideas that we have discussed about resonance.

A62. How Attractive Are You? We often take Newton's Universal Law of Gravitation for granted, but it was far from obvious in Newton's era that every object attracted every other object in the universe. Why, for instance, don't we feel a gravitational attraction every time we come near another person or near another object?

(a) The experience part: get very close to another person or to some other object that is close to your mass. (It doesn't have to be another person — you can stand close to a wall for the experience part.) Now, try to see if you can feel any gravitational attraction. In particular, can you feel the attraction getting stronger as you get closer? Briefly comment (no more than one or two sentences) on what you feel.

(b) Based on this experience, do you think Newton's law of gravitation is obvious?

(c) Now, let's put some numbers on this experience. Estimate your mass and the mass of the other person or object, and estimate the smallest separation between the two of you (estimate the distance between the center of you and the center of the other person/object). Throw these numbers into Newton's law of gravitation to come up with a numerical estimate of the gravitational attraction that you experienced. Is this a force that is strong enough to be noticeable? (You might want to compare this force with the weight of some objects.)

A63. Orbits in a Non-Keplerian Solar System, Part I. Suppose that the gravitational force of attraction depended not on $1/r^2$, but rather was proportional to the distance between the two masses (like the force due to a stretched spring). In a planetary system that felt this different form of gravity, what would be the relationship between the period of a planet and its orbital radius? Assume circular orbits.

A64. Orbits in a Non-Keplerian Solar System, Part II This experience problem goes hand-in-hand with the previous problem, problem A63. In that problem, you work out how the period of a planet's orbit depends on radius if the force of attraction grew linearly with distance, rather than dropping off as $1/r^2$. It so happens that this is a very easy thing to test with your toy kit.

(a) Take your round metal spring, hold it by one end, and twirl it slowly so that the other end makes a circle in a horizontal plane (similar to what you did in problem A60). Determine the period of revolution using the following technique: find the time for 10 complete periods and divide by 10. Now, do it again, but this time twirl it harder so that it stretches out a lot, resulting in a circle of significantly larger radius. Again, determine the period (time for 10 revolutions divided by 10).

(b) Do your results agree with your prediction from problem A63? Specifically, when you increase the radius (by twirling faster), does the period grow linearly with radius, drop as $1/r$, remain the same, . . . ?

A65. Curved Space.

(a) Blow up and tie off a balloon. You are going to draw a circle on the balloon with a radius of 10 cm as measured by a 2-dimensional being that lived on this surface and wasn't aware that the surface was curved in a 3-dimensional world. Mark a point on the balloon that will act as the center of the circle. Now, mark off a 10 cm portion of the string in your toy kit (or you can use your yo-yo string or dental floss). Place one of end of that 10 cm segment at the marked point on the balloon and use the other end of the 10 cm segment like a drawing compass, pulling the string so that it is tight against the surface of the balloon and swinging it around in a circle, tracing out that circle on the balloon as you go. The net result should be a reasonably clean circle with a 2-dimensional radius r_{2D} (along the surface of the balloon) of 10 cm.

(b) Now, measure the circumference of the circle. You can do this by taking the yo-yo string (or some other string) and wrapping it around the balloon until it lines up with the circle that you have just drawn. Then, straighten out the string and measure its length.

(c) Is the circumference equal to $2\pi r_{\text{2D}}$? Your result shouldn't bother you because you happen to live in a three-dimensional world, and you know therefore that the *real* center of the circle that you just drew is inside the balloon, so the *real* radius isn't 5 cm. But if you couldn't comprehend a third dimension and lived on the surface of the sphere, would you find the result surprising?

(d) Now, imagine going outward a certain well-defined distance R from the center of our sun, and drawing a circle all the way around the sun with that distance as the radius. Would you be surprised if the circumference of that circle were less than $2\pi R$? (This is, in fact, what you would find if you could do this measurement without being burned up.)

A66. What if the Pulley *Isn't* Massless? Two objects of masses m_1 and m_2, with $m_2 > m_1$, are connected by a string of negligible mass that passes over a pulley, as shown. The pulley is a uniform disk with mass m_3 and radius R and is free to rotate without friction. The string does not slip on the pulley. Find the acceleration of the mass m_2.

(Note: then tension in the string for mass 1 is **not** the same as the tension in the string for mass 2, since the pulley has a non-zero rotational inertial.)

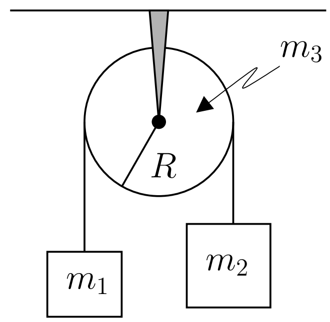

Figure 6: Figure for Problem A66.

A67. Another Massive Pulley Problem.

Two objects, each of mass m, are connected by a string of negligible mass that passes over a pulley, as shown. The surface is frictionless. The pulley is a uniform disk with radius R and mass m_p, and is free to rotate without friction. The string does not slip on the pulley. Find the acceleration of the hanging object.

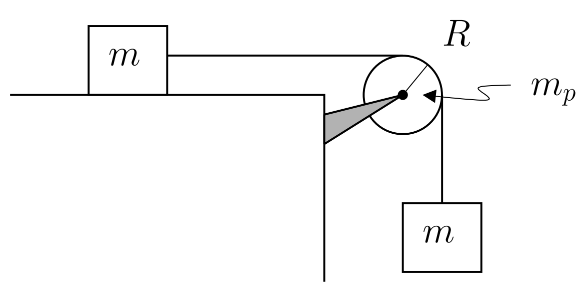

Figure 7: Figure for Problem A67.

A68. Earth's Gravity at the Moon.

(a) Calculate the magnitude g of the Earth's gravitational field at the location of the Moon.

(b) Use your result from part (a) to calculate the gravitational force of the Earth on the Moon.

(c) Use your result from part (a) to calculate the gravitational force of the *Earth* on a 70 kg astronaut standing on the surface of the Moon.

A69. Gravitational Field via Integration I. A rod lies on the x-axis with one end at $x = L_1$ and the other end at $x = L_2$. The rod is not uniform, and its mass per unit length varies as $\lambda = Cx$, where C is a constant.

(a) Determine the total mass of the rod.

(b) Find the gravitational field at the origin due to the rod.

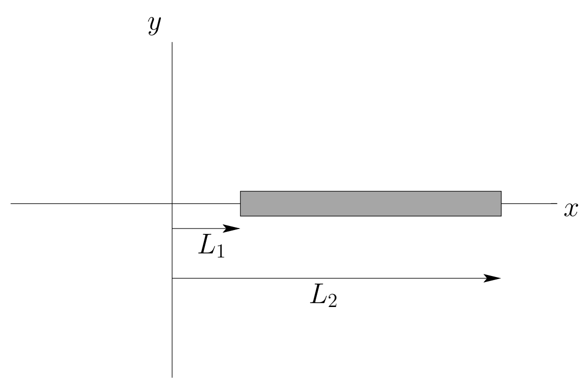

Figure 8: Figure for Problem A69.

A70. Gravitational Fields. Determine the magnitude g of the gravitational field

(a) on the surface of the Moon (due to the Moon), and

(b) at a point 2000 km above the Earth's surface (due to the Earth).

A71. Gravitational Field via Integration II A uniform rod of mass M and length L lies along the x-axis with its center at the origin. Determine the gravitational field at the point $x = d$, where $d > L/2$.

A72. Static Friction Refer to Figure 5.27 in Wolfson (3^{nd} ed.). Let's say that the guy there is pulling on the rope, but the trunk is completely motionless (and remains that way — it doesn't budge). Calculate the magnitude of the friction force acting on the trunk in terms of the mass m of the trunk, the tension T in the rope, the angle θ between the rope and the horizontal, the gravitational acceleration g, and the mass M_J of the planet Jupiter.

A73. Work, Kinetic Energy, and Dissipation You throw a 150 g baseball straight down from a sixth-story window 16 m above the ground. The initial downward speed is 7.2 m/s.

(a) Calculate the work that gravity does on the ball as it falls to the ground.

(b) Assuming that air resistance does −12 J of work on the ball, use the work-kinetic energy theorem to calculate the speed of the ball when it hits the ground.

A74. Recoil on Ice A 42 kg child stands at rest on the surface of a frozen pond (i.e., a frictionless surface). She catches a 1.1 kg ball moving horizontally at 9.5 m/s. Calculate her speed immediately after catching the ball.

A75. Angular Momentum of Point Masses The figure shows 3 objects each with mass 2.0 kg, and each moving with a speed of 4.5 m/s. But the objects are traveling in different directions, each denoted by an arrow. Determine the angular momentum about the origin for each of objects A, B and C.

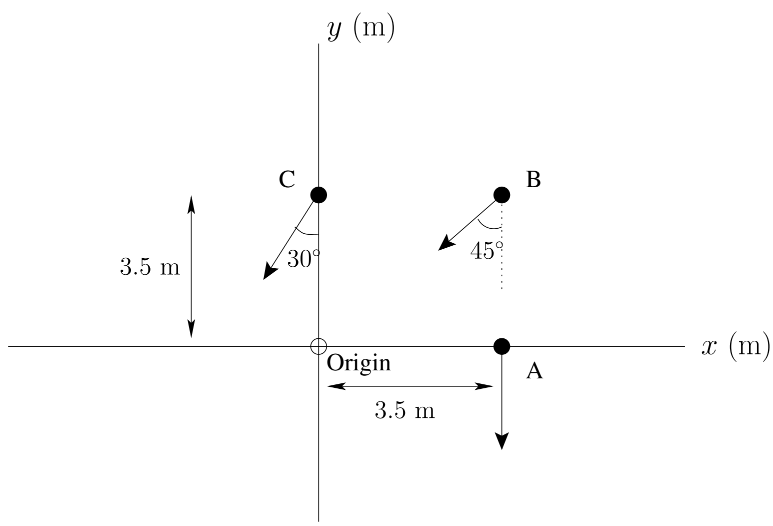

Figure 9: Figure for Problem A75.

A76. Molecular Descriptions Answer each part of this question by considering the behavior of individual molecules.

(a) Considering the motion of individual molecules in a solid, what is the difference between a colder solid and a warmer solid?

(b) What is the difference between a solid just below its melting temperature and a liquid just above this melting temperature? Again, answer this question by discussing the behavior of individual molecules in the solid/liquid.

(c) What is the difference between a cooler liquid and a hotter liquid?

(d) What is the difference between a liquid just below the boiling temperature and a gas just above this boiling temperature?

(e) What is the difference between a cooler gas and a hotter gas?

A77. Triple Star System Consider a system of three co-linear stars, each with mass M, with a distance a separating them. The two outer stars orbit in a circle about the the stationary central star. Determine the square of the orbital period, T^2.

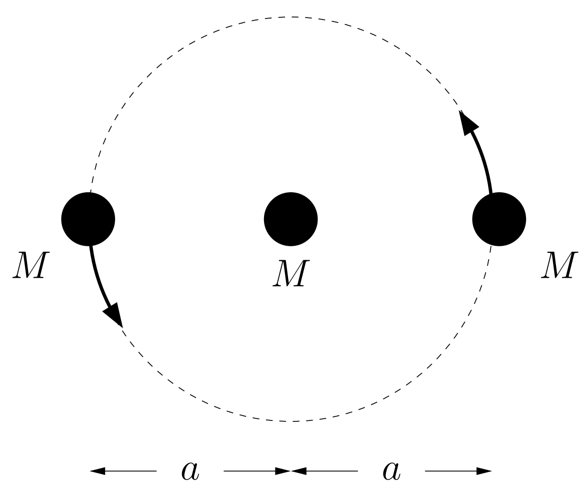

Figure 10: Figure for Problem A77.

A78. Tarzan A 17 m vine hangs vertically from a tree on one side of a 10 m-wide gorge. Tarzan wants to run toward the vine, grab ahold of it, swing over the gorge, let go of the vine, and drop vertically to the ground on the other side of the gorge. How fast must he run to make sure that he makes it across the gorge?

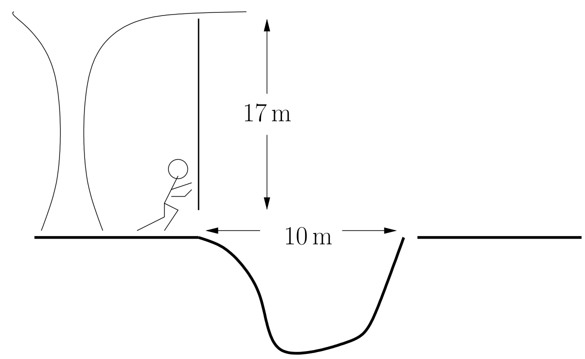

Figure 11: Figure for Problem A78.

A79. Return of Work, Kinetic Energy, and Dissipation Repeat part b) of Problem A73, but this time, instead of using the work-kinetic energy theorem, use $W_{\mathrm{nc}} = \Delta E_{\mathrm{mech}}$. Do you get the same answer for the speed of the ball?

A80. A Bleching Blarg – Checking Dimensions of Answers A blarg with mass m blechs for a time T, after which it flomps a distance d under the influence of a srof F_0 with dimensions (mass × distance)/(time)^2. For each of the following choices, determine if the expression could represent the speed (dimensions distance/time) of the blarg after all of this. **Show your work for each case** (there might be more than one correct answer).

(a) $\frac{F_0 T}{md}$ (b) $\frac{md}{F_0 T}$ (c) $\frac{F_0 T}{m}$ (d) $\frac{F_0 d}{m}$ (e) $\frac{m}{F_0 d}$ (f) $\sqrt{\frac{F_0 d}{m}}$ (g) $\sqrt{\frac{m}{F_0 d}}$.

A81. Dimensions for a Florphtl A florphtl with length L (in m) and mass m (in kg) has an initial speed v (in m/s). The florphtl is in a magnetic field B_0 (in units of T where 1 T $= 1\frac{\mathrm{kg}}{\mathrm{C \cdot s}}$) and experiences an electrical current I_0 (in C/s). Which of the following could be an expression for the acceleration (m/s^2) of the florphtl? (Don't worry about what a "T" or "C" are – you'll want these units to cancel out in the final answer anyway.)

(a) $\frac{mvLB_0}{I_0}$ (b) $\frac{mvL}{I_0 B_0}$ (c) $\frac{I_0 B_0}{mvL}$ (d) $\frac{I_0 B_0 L}{mv}$ (e) $\frac{mv}{I_0 B_0 L^2}$ (f) $\frac{I_0 L B_0}{m}$.

A82. Using ratios

(a) In a traffic jam on Interstate I-5 near Los Angeles, assume that there are 6 cars every 100 feet. How many cars would you expect to be stuck in 1.0 km of one of these traffic jams?

(b) Let's say that a 9-inch diameter pizza at Francesco's costs $12.50. How much should Francesco charge for a 12-inch diameter pizza, if the cost is determined solely by the total amount of the ingredients used to make the pizza?

A83. Ball pits The "ball pit" at Dunking Bird Amusement Park measures 11 m by 9 m with a depth of 60 cm. Assume that this ball pit contains 8000 balls. The ball pit at Fred's Amusement Park measures 13 m by 8 m with depth 40 cm, but uses balls that are half the diameter of those at Dunking Bird Park. Approximately how many balls are needed to fill the pit at Fred's Park?

A84. Period of Asteroid Orbit The asteroid *Betty* orbits the Sun with a semi-major axis of 3.8 AU. Use Kepler's Third Law (and ratios) to determine the period (in years) of Betty's orbit.

A85. Jupiter's Moons Jupiter's moon *Io* orbits Jupiter with a semi-major axis 421,700 km and an orbital period of 1.8 days. Another moon – *Ganymede* – orbits Jupiter with a semi-major axis of 1,070,000 km. Calculate the orbital period of Ganymede.

A86. Extrasolar planets The planet Zortox orbits around the star Xyl'pron with a semi-major axis of 570 klorvm and an orbital period of 2.7 flurps. Another planet – Rotnox – also orbits around Xyl'pron with an orbital period of 7.3 flurps. Determine the semi-major axis for Rotnox's orbit.

A87. Why you will do badly on tests if you don't show all work A 3.5 meter long piece of rope has a mass of 250 g. Your goal is to determine the mass of a 7.0 meter long piece of the same rope.

(a) Do this calculation in your head and then write down the answer on your paper.

(b) Do this calculation again, but write down the steps and your reasoning on the paper.

(c) Scribble out or erase every number and unit for parts (a) and (b). Now grade your work from parts (a) and (b) on a 0–10 point scale for each, basing the grade on how well someone could understand what you did and why from whatever remains visible on the page after the numbers and units have been erased or scribbled out.

A88. Science Fiction and the Laws of Physics The following is a science fiction story that is inconsistent with the known laws of physics. Read the passage, and then list 4 *different* aspects of the story that are clearly inconsistent with the laws of physics as covered in PHYS 211 this semester.

> The Starship Enterprise is on a mission 5 light years from the Earth, traveling at 7 times the speed of light while being chased by a hostile Borg ship. "At our current speed, the Borg won't catch us for another hours," says Captain Picard to Admiral Janeway (who is back on Earth) on his iPad 563 as he stares at an ice cube floating lazily in equilibrium with the liquid in his iced tea. "Well, if they catch up with you," replies Janeway, "fire a beam of anti-matter at the Borg ship. The anti-matter will annihilate part of the ship, and the kinetic energy that is produced by the resulting mass loss will blow up the rest of the ship." "Understood," replies Picard as he adds another ice cube to his tea, dropping its temperature down even more.
>
> Just at that moment, Picard is thrown from his chair as a torpedo from the Borg ship slams into the Enterprise's engines from behind. "Our engines have been destroyed " says Geordi LaForge as the ship suddenly comes to a complete halt, motionless in space as the Borg ship close in. "Borg ship," radios Picard, "this is the Starship Enterprise. We are prepared to talk with you." "Prepare to be assimilated," replies the Borg ship. "Resistance is fut – ...". "End communication," says Picard as the Enterprise fires, blowing up the Borg ship.

Chapter 1

Solving Equations of Motion Using Numerical Iteration

1.1 Introduction

The past few decades have witnessed a massive revolution in the way people live and work, due in great part to *significant* enhancements in computational power. Computers are everywhere these days in society, not just on your desktop (or on your lap) but also in your pockets (MP3 players and cell phones), in your kitchens (ranges, dishwashers and microwave ovens), and behind the scenes monitoring the money in your bank accounts, your class schedules and grades, and your music preferences at on-line music stores.

The significant enhancement in computation power has also dramatically changed all fields of science and engineering. Despite our brilliant teaching of physics in this course, there are many problems in physics and engineering that you simply will not be able to solve analytically.[1] Some problems simply don't allow a closed-form solution. But it is even more severe than that. There are a wide variety of physical systems whose equations of motion **can't** be solved, no matter how brilliant or persistent the scientist/mathematician. In fact, many real systems are "chaotic," with surprisingly complicated behavior arising from seemingly simple systems. In cases where an analytical solution is unavailable, the only option is to solve the problem *numerically*, using a computer to simulate the behavior.

Computer simulations have become among the most important techniques in science and engineering. Many of you will use numerical techniques in your career, whether you are simulating the behavior of a new passenger airline that you are designing, calculating the forces acting on an artificial joint that you are designing for a patient, or predicting the effects of a disruption in Middle East oil supply on the global economy. Nu-

[1] By "analytically" we mean using the tools from mathematics to determine a written solution in the form of an equation that can be used to describe the behavior of the system.

merical simulations also play a significant role in basic scientific research, enabling us to explore the behavior of a system that is too complicated to solve analytically and to difficult to explore experimentally. In fact, numerical simulations are so common now that they are often considered to be a third branch in scientific analysis, separate from (and complementary to) experimental and theoretical science.

The basic idea of numerical simulations is actually quite easy. In this chapter, we introduce an important technique referred to as *iteration* where we break the dynamics of the system into a series of discrete time steps. So, for example, instead of representing the motion of a ball with a continuous equation, we instead note the location of the ball, say, every tenth of a second. Given the location and velocity of the ball at a particular moment in time, we can predict its location 0.1 s later by using a very simple numerical techniques referred to as the *Euler Method*, a technique that conceptually is nothing more than a simple application of the common "distance = speed × time" approach. Despite the simplicity of the Euler method, it is a very powerful method that is used in many numerical applications. This chapter introduces the basic ideas (with some homework problems); you will use the method in lab to simulate the motion of a falling object subject to air resistance.

1.2 Solving Newton's second law analytically

Newton's second law $\vec{F}_{\rm net} = m\vec{a}$ is a *differential equation*, i.e., an equation that can be written in terms of derivatives of various quantities. Ideally, we would like to "solve" this differential equation to determine expressions (as a function of time) for the velocity and position of a particle moving under the influence of the forces. If the forces exerted on the particle are all known, then Newton's second law can be rewritten as

$$a_x = \frac{d^2x}{dt^2} = \frac{F_{{\rm net},x}}{m}, \tag{1.1}$$

where the forces are assumed to be possibly functions of position and velocity. Eq. (1.1) written in that form is known as the *equation of motion* for the system under consideration. Mathematically one would proceed by integrating Eq. (1.1) to determine the velocity as a function of time $v_x(t)$ and then integrating once again to obtain the position as a function of time $x(t)$. For example we have learned that for a particle falling from rest from a height x_0 under the force of gravity $F_{\rm net} = mg$, Eq. (1.1) becomes

$$\frac{d^2x}{dt^2} = -g, \tag{1.2}$$

and integrating we obtain the following expressions for the velocity and position:

$$v_x(t) = -gt \qquad \text{and} \qquad x(t) = x_0 - \frac{1}{2}gt^2. \tag{1.3}$$

If you don't understand how we got these expressions, then take the derivative with respect to time of $x(t)$ to get $v_x(t)$ and then $v_x(t)$ to get back to Eq. (1.2).

For the example shown above as well as a few other cases, the equation of motion is relatively straightforward to integrate to get the analytical functions for velocity and position. As discussed in the previous section, though, there are many cases where the equations of motion are not so easy to integrate and other means are necessary for determining the position and velocity of the particle as a function of time.

In the following sections we will develop a set of equations that we can use to calculate the position and velocity of a particle at specified time increments Δt, a technique called *numerical iteration*. Although this technique does not give us as a final result a neat, compact formula for the position and velocity of the particle into which any value of time can be inserted, it does allow us to map out the position and velocity of the particle for an otherwise mathematically intractable problem.

1.3 Numerical Stepping Equations

Let us incorporate the ideas mentioned above into a set of formulas that we (or better yet, a computer) could use to calculate the position and velocity of a particle moving under the influence of some forces. Call the present time t and the time a little later $t + \Delta t$. Let $x(t)$ denote the position of the particle now, then $x(t+\Delta t)$ denotes the position of the particle a short time later. Similarly $v_x(t)$ and $v_x(t+\Delta t)$ represent the present and slightly later velocities of the particle. In all of these expressions, note that $x(t + \Delta t)$ does not mean the quantity 'x' times the quantity '$t+\Delta t$' but rather means the value of the function x evaluated at the time $t + \Delta t$. This is standard functional notation used in mathematics.

Recall the definition of velocity as the rate of change of the position. Taking Δt to be very small in magnitude, we may approximate this as "velocity = displacement/time" and express the velocity at time t approximately as

$$v_x(t) \simeq \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}. \tag{1.4}$$

As you recall, $\Delta x/\Delta t$ is the definition of the average velocity, while the instantaneous velocity is actually the derivative of the position with respect to time. However, for small enough time steps, the average velocity is an excellent approximation for the instantaneous velocity.

Turning the previous expression around, we can write an expression for the position of the particle at time $t+\Delta t$ in terms of the position and velocity at time t:

$$x(t+\Delta t) = x(t) + v_x(t)\Delta t. \tag{1.5}$$

Eq. (1.5) says that the position at time $t+\Delta t$ is the position at time t plus the distance traveled $v_x(t)\Delta t$ by the particle during the short time interval Δt. Notice that this result is only approximate because the velocity v_x at time t is not necessarily equal to the average velocity during the entire time interval. However, if Δt is small enough, the approximation should be quite good.

Next we need an expression for incrementing the velocity. By analogy with the arguments leading up to Eq. (1.5), we can write

$$v_x(t+\Delta t) = v_x(t) + a_x(t)\Delta t. \tag{1.6}$$

The three equations (1.1), (1.5) and (1.6) can now be incorporated into a looping procedure in a computer program. These three equations constitute what is generally referred to as *Euler's method* of numerical approximation. Given an initial position and velocity, we calculate the initial acceleration from Eq. (1.1). Then we calculate the position and velocity a short time later from Eqs. (1.5) and (1.6). Then we repeat the process, pretending that the new values for x and v_x are the initial values. In this way we can numerically iterate the motion of the particle from instant to instant as far into the future as we care to. A spread-sheet program, such as EXCEL, can perform such calculations with very little "programming" required on your part.

A note about numerical errors is worth mentioning. Remember that although Eq. (1.1) is exact, Eqs. (1.5) and (1.6) that update x and v_x to later times are approximations that are best when Δt is small. If the calculations start going haywire, we can help the situation by choosing smaller steps. This means of course that the computer will have to run longer, but that's frequently not a serious problem.

1.4 Numerical Solution for a Mass on a Spring

Let's apply this new method to a system we will be studying more in depth later in this course. The system is a mass which moves under the influence of a force exerted on it by a spring. The spring is a device which exerts a force which is proportional to the displacement of the mass from an equilibrium position. Taking the equilibrium position to be $x = 0$, this implies that the acceleration of the mass is directly proportional to the position $x(t)$. Suppose in our particular system the acceleration is given by

$$a_x(t) = -2.00\, x(t). \tag{1.7}$$

The minus sign in this expression tells us that the force is always opposite to the displacement. We'll also assume that time is in seconds, position is in meters, velocity is in meters per second, and acceleration is in meters per second squared

To proceed, we choose time steps of size $\Delta t = 0.10$ s and start the clock at $t = 0$. We could pick any initial position and velocity; let's choose to release the mass from rest at a position 0.30 m from equilibrium, i.e. $x(0) = 0.30$ m and $v_x(0) = 0$. Let's walk through the first few steps and then show some results from a computer spreadsheet.

For our example Eqs. (1.1), (1.5) and (1.6) are written as

$$a_x(t) = -2.00\,x(t) \tag{1.8}$$
$$x(t+0.10) = x(t) + 0.10\,v_x(t) \tag{1.9}$$
$$v_x(t+0.10) = v_x(t) + 0.10\,a_x(t). \tag{1.10}$$

First calculate the initial acceleration by setting $t = 0$ in Eq. (1.8) to find

$$a_x(0) = -2.00\,x(0) = -2.00 \times 0.30 = -0.60. \tag{1.11}$$

Then update $x(t)$ and $v_x(t)$ by setting $t = 0$ in Eqs. (1.9) and (1.10):

$$x(0.10) = x(0) + 0.10\,v_x(0) = 0.30 + 0.10 \times 0 = 0.30 \tag{1.12}$$
$$v_x(0.10) = v_x(0) + 0.10\,a_x(0) = 0 + 0.10 \times (-0.60) = -0.06. \tag{1.13}$$

Since the mass was initially at rest, a short time later it is still approximately at the same location. However, since the spring is stretched at $t = 0$, a force is acting on the mass immediately, so that a short time later it has already acquired a non-zero velocity.

How would you find $x(0.20)$ and $v_x(0.20)$? Again use Eqs. (1.8) through (1.10), this time with the 'present time' $t = 0.10$. We find that

$$a_x(0.20) = -2.00\,x(0.10) = -2.00 \times 0.30 = -0.60 \tag{1.14}$$
$$\begin{aligned} x(0.20) &= x(0.10) + 0.10\,v_x(0.10) \\ &= 0.30 + 0.10 \times (-0.06) \\ &= 0.294 \end{aligned} \tag{1.15}$$
$$\begin{aligned} v_x(0.20) &= v_x(0.10) + 0.10\,a_x(0.10) \\ &= -0.06 + 0.10 \times (-0.60) \\ &== -0.12. \end{aligned} \tag{1.16}$$

We can continue this process as long as we like. You will find it convenient to organize the information for the position, velocity and acceleration for each time in the form of a table. Table 1.1 on the next page lists t, x, v_x and a_x for the motion of this mass. Note that the periodic nature of the motion is manifested in the entries of the table.

There is an unsettling aspect of the entries in Table 1.1. We started at $x = 0.30\,\mathrm{m}$, but at $t = 2.30\,\mathrm{s}$ the position of the mass is $x = -0.375\,\mathrm{m}$, and further down in the table we find that at $t = 4.5$ s, $x = 0.468\,\mathrm{m}$. What should we have expected? If we had a real mass connected to a spring and set it oscillating we would expect the amplitude of the oscillations to gradually decrease because of the presence of dissipative forces (air resistance and the imperfect elasticity of the spring). In an ideal case, with no dissipative effects, we would expect there to be no increase or decrease in the amplitude; that is, the mass should oscillate between $x = +0.30\,\mathrm{m}$ and $x = -0.30\,\mathrm{m}$. But this is not the case if we look at the data in Table 1.1. The problem is that we used too large a time increment. Why does too large a time increment lead to errors? If you recall, our stepping equations use the approximation that the average velocity is very close to the instantaneous velocity. If the time step is too large, this approximation is no longer valid and leads to errors.

We can improve our calculation of the motion by choosing a smaller time increment Δt. If we choose $\Delta t = 0.01\,\mathrm{s}$ rather than $0.10\,\mathrm{s}$, we would be calculating over a much finer time interval (10 times smaller) and while we will have to do 10 times more computations to evolve the motion out to the same time, the calculations should be more accurate. Table 1.2 lists t, x, v_x and a_x near a point of maximum displacement for this smaller time increment. The maximum displacement is now about 0.314. This is still larger than the initial displacement but not nearly as bad as before. Further reduction of the time increment would improve the result.

Table 1.1: Numerical solution for motion of mass on a spring using $\Delta t = 0.10\,\mathrm{s}$

t	$x(t)$	$v_x(t)$	$a_x(t)$
0	0.300	0	-0.600
0.100	0.300	-0.060	-0.600
0.200	0.294	-0.120	-0.588
0.300	0.282	-0.179	-0.564
0.400	0.264	-0.235	-0.528
0.500	0.241	-0.288	-0.481
0.600	0.212	-0.336	-0.424
0.700	0.178	-0.379	-0.356
0.800	0.140	-0.414	-0.281
0.900	0.099	-0.442	-0.198
1.000	0.055	-0.462	-0.109
1.100	0.008	-0.473	-0.017
1.200	-0.039	-0.475	0.078
1.300	-0.086	-0.467	0.173
1.400	-0.133	-0.450	0.266
1.500	-0.178	-0.423	0.356
1.600	-0.220	-0.387	0.440
1.700	-0.259	-0.343	0.518
1.800	-0.293	-0.292	0.587
1.900	-0.322	-0.233	0.645
2.000	-0.346	-0.168	0.692
2.100	-0.363	-0.099	0.725
2.200	-0.373	-0.027	0.745
2.300	-0.375	0.048	0.750
2.400	-0.370	0.123	0.741
2.500	-0.358	0.197	0.716
2.600	-0.338	0.268	0.677
2.700	-0.312	0.336	0.623
2.800	-0.278	0.398	0.556
2.900	-0.238	0.454	0.476
3.000	-0.193	0.502	0.386
3.100	-0.143	0.540	0.285
3.200	-0.089	0.569	0.177
3.300	-0.032	0.587	0.063
3.400	0.027	0.593	-0.054
3.500	0.086	0.587	-0.173
3.600	0.145	0.570	-0.290
3.700	0.202	0.541	-0.404
3.800	0.256	0.501	-0.512
3.900	0.306	0.450	-0.612
4.000	0.351	0.388	-0.702
4.100	0.390	0.318	-0.780
4.200	0.422	0.240	-0.844
4.300	0.446	0.156	-0.892
4.400	0.461	0.067	-0.923
4.500	0.468	-0.026	-0.936
4.600	0.465	-0.119	-0.931

Table 1.2: Data for mass on a spring near a turning point using $\Delta t = 0.01\,\mathrm{s}$

t	$x(t)$	$v_x(t)$	$a_x(t)$
4.190	0.293	0.155	-0.586
4.200	0.295	0.149	-0.589
4.210	0.296	0.143	-0.592
4.220	0.297	0.137	-0.595
4.230	0.299	0.131	-0.598
4.240	0.300	0.125	-0.600
4.250	0.301	0.119	-0.603
4.260	0.303	0.113	-0.605
4.270	0.304	0.107	-0.607
4.280	0.305	0.101	-0.610
4.290	0.306	0.095	-0.612
4.300	0.307	0.089	-0.614
4.310	0.308	0.083	-0.615
4.320	0.308	0.077	-0.617
4.330	0.309	0.071	-0.619
4.340	0.310	0.064	-0.620
4.350	0.311	0.058	-0.621
4.360	0.311	0.052	-0.622
4.370	0.312	0.046	-0.623
4.380	0.312	0.040	-0.624
4.390	0.313	0.033	-0.625
4.400	0.313	0.027	-0.626
4.410	0.313	0.021	-0.626
4.420	0.313	0.014	-0.627
4.430	0.314	0.008	-0.627
4.440	0.314	0.002	-0.627
4.450	0.314	-0.004	-0.627
4.460	0.314	-0.011	-0.627
4.470	0.313	-0.017	-0.627
4.480	0.313	-0.023	-0.627
4.490	0.313	-0.029	-0.626
4.500	0.313	-0.036	-0.626
4.510	0.312	-0.042	-0.625
4.520	0.312	-0.048	-0.624
4.530	0.312	-0.054	-0.623
4.540	0.311	-0.061	-0.622
4.550	0.310	-0.067	-0.621
4.560	0.310	-0.073	-0.619
4.570	0.309	-0.079	-0.618
4.580	0.308	-0.085	-0.616
4.590	0.307	-0.092	-0.615
4.600	0.306	-0.098	-0.613
4.610	0.305	-0.104	-0.611
4.620	0.304	-0.110	-0.609
4.630	0.303	-0.116	-0.607
4.640	0.302	-0.122	-0.604
4.650	0.301	-0.128	-0.602

Problems

1. For a certain mass-spring system the acceleration is given by $a_x(t) = -0.10\,x(t)$. Suppose the initial position and velocity are $x(0) = 10$ m and $v_x(0) = -1.0\,\mathrm{m/s}$. Calculate $x(t)$ and $v_x(t)$ at $t = 2\,\mathrm{s}$ in two different ways:

(a) Use two steps of 1 second each.

(b) Use four steps of $\frac{1}{2}$ second each. Round only your final results to three digits (keep all digits for the intermediate calculations).

(c) Why aren't the answers to a) and b) the same?

2. A drag force on an object is opposite to its velocity and is often proportional to its speed. Let's immerse the mass-spring system of problem (1) in a vat of salad oil so that the acceleration becomes

$$a_x(t) = -0.10\,x(t) - v_x(t)$$

Repeat problem 1.1 for this acceleration. Compare the results with those you originally got in problem 1.1. Are the results what you might expect when a drag force is present?

Chapter 2

Basic Postulates of Relativity

2.1 Introduction

Certain numbers immediately bring to mind thoughts or ideas. For example, "101" makes people think of spotted puppies, "747" engenders thoughts of large airplanes, "911" is the number that you call for an emergency or one of the worst dates in the history of our country, and "42" is the answer to the ultimate question of Life, the Universe and Everything. And if you mention the number "1905" to any physicist, he/she will immediately think of the year in which Albert Einstein published three papers that completely revolutionized science and fundamentally changed the way in which we view the universe. The first paper[1] introduced the idea of photons (particles of light), an idea which formed one of the cornerstones of quantum mechanics.[2] (You will learn about this next semester in PHYS 212.) The second paper[3] was the first to connect molecular diffusion — spreading of an impurity in a motionless fluid — with random Brownian motion of the individual impurity molecules, which is regarded as the first demonstration of the existence of atoms.

The third paper had a innocuous title: "On the electrodynamics of moving bodies."[4] But there is nothing even remotely innocuous about the implications of the theory, now known as Einstein's Special Theory of Relativity ("special relativity" for short), presented in that paper. Einstein's theory completely changed our conceptions of time and distance[5] and of energy and matter.[6] The theory also led to an explanation of how stars generate

[1] A. Einstein, Annalen der Physik **17**, 132 (1905).

[2] Interestingly, even though any one of these papers would be a monumental lifetime achievement for any mere mortal physicist, Einstein received the Nobel prize in physics only for his work on photons.

[3] A. Einstein, Annalen der Physik **17**, 549 (1905).

[4] A. Einstein, Annalen der Physik **17**, 891 (1905).

[5] ...and, in fact, establishes that they are profoundly related, as we shall see.

[6] ...and, in fact, establishes that they are profoundly related, as we shall see.

light — the fundamental source of energy in the universe without which life on this planet would not be possible — and led to the Earth-shattering (almost literally, unfortunately) development of nuclear weapons. The theory also holds the key to the future development of non-fossil fuel energy sources. Simply put, you cannot understand how the universe works without studying Einstein's theory of relativity.

This chapter and the following three introduce the main ideas and implications of the Special Theory of Relativity, which applies to the motion of objects in *inertial* (non-accelerating, or "free float"[7]) reference frames. At the end of the semester, we will also briefly discuss Einstein's General Theory of Relativity ("general relativity" for short), which expands the theory to account for the effects of acceleration and gravitational fields.

2.2 Preliminaries

A few definitions will be useful for the next few chapters.

An *event* is something that happens at a particular location at a particular time. It is important to be clear about this, because relativity deals with how different observers measure distances and times between events. For instance, let's say that the penguin on top of your television set explodes at 7:12 a.m. on a Saturday morning. You then run 5 km to a large tower where you capture (at 7:45 a.m.) a small platypus that inexplicably is dressed like a secret agent and who is trying to thwart your plans to take over the Tri-State Area. You could identify two events — (1) the explosion of the penguin and (2) your capture of the semi-aquatic, egg-laying mammal of action (i.e., the platypus) — and say that these events are separated by 5 km in space and 33 min in time. Relativity addresses the question of how a different observer measures the distance and time between the same two events. (Preview: not everyone will agree about the distance and time between events.)

So what do we mean, exactly, by "different observers," and what are the characteristics of these observers that will determine how their measurements will differ? We start by explaining what is meant by the term "reference frame." You can visualize a reference frame as a set of rulers (distance measuring devices) and clocks (time measuring devices) that are arrayed throughout space so that the position and time of any event can be determined directly. The distinguishing feature of a reference frame is that the set of "rulers" and "clocks" are all at rest with respect to one another. An observer IN THIS REFERENCE FRAME is at rest with respect to all the rulers and clocks. Notice that there can be many observers at different positions in this reference frame, as long as they are all at rest with respect to each other and to the measuring tools. All observers in the same reference

[7]Taylor and Wheeler, *Spacetime Physics*, 2$^{\text{nd}}$ Edition, (Freeman, 1992), p. 26.

frame will agree with each other about the distances and times between any two events, but they will not agree with observers in other reference frames moving with respect to their frame.

A particularly important kind of reference frame is an *inertial reference frame*. Observers in an inertial reference frame experience no significant acceleration, nor can they discern any gravitational effects. In an ideal inertial reference frame, the observer would be floating free (hence the name "free float" that is sometimes used to discuss an inertial reference frame), because any non-floating motion would necessarily imply either acceleration or gravitational effects. To analyze behavior in the vicinity of very strong gravitational fields, it is necessary to use general relativity.

Technically, an observer is not in a true inertial reference frame if she is standing on the surface of a planet since there is gravitation. However, there are plenty of situations where non-inertial effects are small enough as to be negligible. In fact, the gravitation from a typical planet is small enough so that the non-inertial effects are negligible, and Special Relativity works perfectly well. So, for example, we will often treat observers moving on a constant velocity train as though they are in an inertial reference frame, even though there is a small gravitational effect.

When dealing with velocities, we have to be careful. A velocity technically has meaning only if there is a reference. So, for example, if you are in a car and you are traveling 65 mph toward the West, you are really traveling 65 mph *relative to the surface of the Earth*. In fact, almost any velocity that people quote in everyday usage is defined relative to the Earth.

> **In preparation for class**, consider the following question: how fast are you *really* going if you are in the car in the previous paragraph?

Certainly, anyone who is willing to accept a non-geocentric view of the universe realizes that there is nothing inherently special about the earth as a reference frame. But scientists have long wondered if there is some preferred universal reference frame from which all velocities should be defined, some standard by which we could define *absolute velocities* for every object in the universe.

In relativity, we will use *relative velocities*, i.e., velocities will always be defined relative to some reference frame. In fact, one result of relativity is the realization that this is the best way to define velocity. There is no need to choose any special reference frame for the universe; all the results of relativity work perfectly well with velocities measured relative to any reference frame that you might choose.

The following statement applies to relative velocities: if observer A measures observer B to be moving at a (relative) velocity of $\vec{v}$ in a particular direction, then B measures A to be moving at a (relative) velocity of $-\vec{v}$; i.e., same speed but opposite direction.

2.3 Fundamental Principles of Relativity

Einstein's Special Theory of Relativity is based on a very simple premise, namely

> **The Principle of Relativity**: the laws of physics are the same for observers in different inertial reference frames.

Let's say, for example, that Michelle sets up a lab in the basement of Olin Science while Barack sets up an identical lab inside a truck that is driving on Route 15 with a constant velocity. Whatever physics equations (including fundamental constants) Michelle uses to predict and describe the behavior in her lab should work equally well for Barack in his lab.

Not only is this an intuitively reasonable statement, but the argument can be made that the whole field of physics would be useless if this statement weren't true (along with chemistry, biology and engineering as well). After all, what is the point of formulating a set of laws to describe the universe if they only apply to certain observers moving in a certain way?

The question then boils down to this: what *are* the fundamental "laws of physics" that are the same for all observers? At the beginning of the 20^{th} century, there were two main cornerstones of physics: Newton's Laws of Classical Mechanics, and Maxwell's Equations describing electrical and magnetic fields. You have already been introduced to Newton's Laws. We will be discussing electricity and magnetism in PHYS 212, but here we highlight some of the ideas relevant to our discussion of relativity.

During the 19^{th} century, there was a tremendous surge of research to describe electric and magnetic phenomena, culminating in the integration of electromagnetic theory into a set of four fundamental laws by James Clerk Maxwell in the late 1800s. Maxwell's results not only unified electricity and magnetism into a single, consistent theory, but also showed for the first time that light is an electromagnetic wave (you'll learn more about this in PHYS 212). The theory also showed how to produce a wide variety of different types of electromagnetic waves, a prescription that had been successfully tested during the period between Maxwell's theory and Einstein's work on relativity. Suffice it to say that Maxwell's equations were (and still are) considered by the scientific community to be one of the cornerstones of physical law.

But there was a problem: by the end of the 19^{th} century, some theorists attempted to generalize Maxwell's equations to apply in any reference frame and found that this could not be done within the framework of Newtonian Classical Mechanics. There arose a conflict between the two most widely-accepted cornerstones of physics: Newton's Mechanics and Maxwell's equations.

Here is where Einstein came into the picture. Whereas few people had previously had any doubts about the validity of Newtonian Mechanics, Einstein started from the assumption that Maxwell's Equations of electricity and magnetism were a fundamental law of physics that were valid in any reference frame, and then set about re-writing Newton's Laws (generalizing them, actually) to assure that Maxwell's Equations would be valid in any reference frame. (Hence the title of Einstein's third paper in 1905.)

The argument is actually fairly simple. If Maxwell's Equations are valid for observers in any inertial reference frame, then not only the form of the equations but also all the constants should be valid in any reference frame. Two of the constants in particular — the permittivity of free space ϵ_0, and the permeability of free space μ_0 combine to give a value $1/\mu_0\epsilon_0 = 9.0 \times 10^{16}\,\mathrm{m}^2/\mathrm{s}^2$, which is the square of the speed of light when it propagates through a vacuum! Based on the fundamental Principle of Relativity (above), the conclusion is staggering. If Maxwell's equations formulate a fundamental law of physics, then the Relativity Principle implies the following consequence [8] :

> **The invariance of the speed of light**: The speed of light in a vacuum c is measured to be $3.0 \times 10^8\,\mathrm{m/s}$ by any observer in any inertial reference frame.

Although verified experimentally,[9] this statement runs counter to our intuition, based on common experience. Consider the following sample problems:

Example 2.1 Classical calculation of relative velocities I.

Karen is running down the hall of Olin with a loaded blow dart gun. She is running with a constant speed of 5 m/s when she sees Brian and Jeff standing in front of their lab. While still running, she fires a blow dart in their direction. If the speed of the blow dart is 15 m/s relative to Karen, how fast is the dart moving with respect to Jeff and Brian's reference frame?

[8] Einstein stated the second postulate slightly differently: "light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body." It can be shown that the invariance of the speed of light, with respect to the motion of the *source*, and the invariance of the speed of light with respect to the motion of the *observer* are simply consequences of each other.

[9] In fact, an experiment by Michelson and Morly in 1895 already indicated the invariance of the speed of light in vacuum.

Solution: The answer is what you would think — simply add the speeds to find that the blow dart travels at a speed of 20 m/s relative to Jeff and Brian.

Example 2.2 Classical calculation of relative velocities II.

Brian now picks up his blow dart gun and aims it in Karen's direction. Karen quickly retreats, running away from Brian and Jeff with a constant speed of 5 m/s. Brian fires a dart toward Karen at a speed 15 m/s measured from his reference frame. How fast is the dart moving with respect to Karen's reference frame?

Solution: Again, the result is what you would think — simply subtract the speeds to find that the blow dart travels at a speed 10 m/s relative to Karen.

Example 2.3 Speeds of light pulses.

Lord Fa is returning to his home world of Gao. Approaching the planet at speed of 2.0×10^8 m/s (relative to the planet), he sends a beacon of light to Commander Nea stationed on Gao. This pulse of light leaves his ship with a speed 3.0×10^8 m/s relative to the ship. How fast is the pulse moving relative to Commander Nea?

Solution: Classically, you should expect that Commander Nea would view the pulse as moving with a speed of 5.0×10^8 m/s. But this is wrong. Instead, from her reference frame, the pulse is moving with a speed of 3.0×10^8 m/s! That's just the way it is with light pulses moving in a vacuum — everyone measures the same speed of 3.0×10^8 m/s, regardless of their motion.

You should find the results of the above example to be strange — there is nothing in our everyday experience that would lead us to expect such a result. But numerous experiments have measured the speed of light in a wide variety of reference frames, and the results always agree with the statement of the invariance of c.

That the speed of light (in empty space) does not depend on the speed of its source has been demonstrated so convincingly and the value of the speed measured so accurately that the value is now defined to be exactly 299,792,458 m/s. By combining this definition of c with the definition of the second (in terms of an atomic clock), we no longer need an independent definition of the meter.

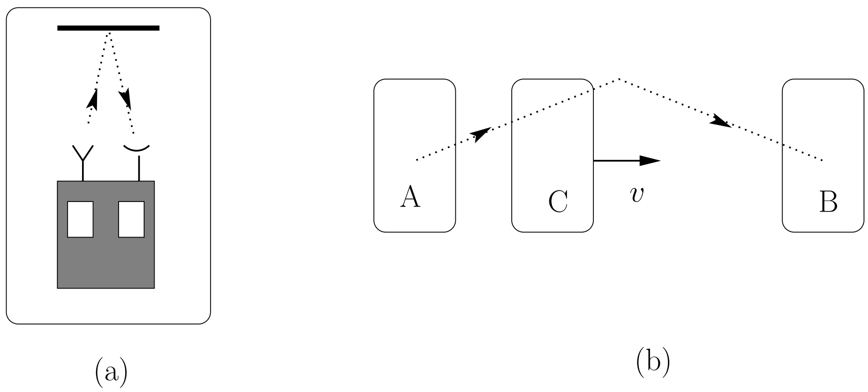

Figure 2.1: (a) A light clock used in the thought experiment described in the text. (b) Light clock C passing rest-frame clocks A and B. The dotted line shows the path of C's light pulse as observed in the rest frame of A and B.

2.4 Time dilation

The most startling consequence of the invariance of the speed of light is that it forces us to abandon the notion of absolute time. This means the time interval between two events depends on the velocity of the clocks used to measure the interval. The following *thought experiment* should help you understand this concept of the relativity of time intervals.

Imagine three identical clocks constructed as follows. Each clock contains a light source that emits a pulse of light toward a mirror some fixed distance away (see Figure 2.1a). The mirror reflects the pulse back toward the source. When the reflected pulse returns to the source and hits a triggering device, the source immediately fires a second pulse, which reflects from the mirror and triggers a third pulse, and so on. A count registers in a counter for each return pulse so the number of counts becomes a measure of elapsed time.

We place two of these light clocks, A and B, a fixed distance apart and at rest in a reference frame attached to the constant velocity Earth. We put the third clock, C, on a spaceship traveling at a constant velocity $\vec{v}$ relative to the Earth (see Fig. 2.1b), and perpendicular to the direction of travel of the light pulse in the clocks.

Suppose clock C emits a light pulse at the exact instant it passes clock A. Also suppose that the distance between A and B is such that clock C passes clock B at the precise instant clock C's reflected pulse returns to the source. We therefore have two events: Event #1 = "C passes A" and Event #2 = "C passes B." We label the time interval between these two events — measured by clock C — as Δt_C. The quantity Δt_C is called the *proper time* interval between the two events; ***proper time is defined as the time measured on a single clock that is present at both events***. In the

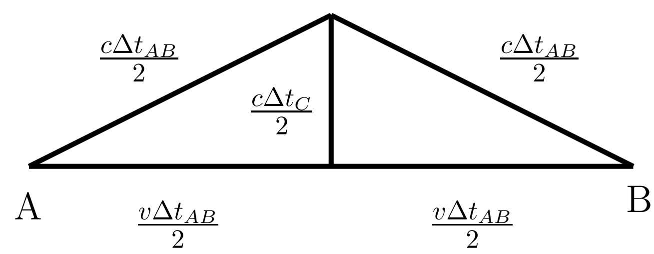

Figure 2.2: Diagram for derivation of the proper time relation. In this figure Δt_{AB} is the elapsed time determined from the clocks A and B, and Δt_C is the elapsed time on clock C.

case discussed above, clock C measures the proper time. In our particular arrangement, the proper time is exactly one tick.

We now pose the crucial question, the answer to which is the key to understanding all of special relativity.

> *What is the elapsed time Δt_{AB} as measured by clocks A and B for clock C to travel from A to B?*

"Simple," you might think. "The answer is obviously exactly one tick, the same as that measured by clock C, right?" Wrong. As we will see, the concepts of absolute time (i.e., everyone and everything measures the passage of time the say way) is a casualty of the invariance of the speed of light.

For the question posed above to have any meaning, clocks A and B must be synchronized; i.e., observers in the Earth's reference frame would say that the two clocks are reading the same time. (Note: this is not a trivial matter — we will discuss synchronization more fully in Chapter 3.) The two-clock time Δt_{AB} is then the difference between the time reading on clock A at Event #1 (clock C at clock A) and the time reading on clock B at Event #2 (clock C at clock B).

In clock C's reference frame, clock A passes C first, and then clock B passes C. The time interval between these events is Δt_C on clock C and therefore the light pulse in clock C travels a round-trip distance equal to $c\Delta t_C$. But from Fig. 2.1 this same pulse (the one inside clock C) travels a longer, zig-zag path when viewed from the frame in which clocks A and B are at rest. Because of the invariance of the speed of light, *this longer distance must translate into a longer time interval.* This means the round trip time for clock C's pulse is one tick as measured on clock C, but it is more than one tick when measured on clocks A and B. In other words, the elapsed time between the event "C passes A" and the event "C passes B" is *longer* when measured with the two clocks A and B than when measured with the single clock C.

How much longer is the time interval Δt_{AB} measured on the A and B clocks than the proper time interval Δt_C measured on clock C? We can find out by looking at the path taken by the pulse of light in clock C viewed from C's reference frame and from A and B's reference frame (see Fig. 2.2). As we've already seen, in clock C's frame the pulse travels straight up and down along the vertical line in the figure and the total round-trip distance is $c\Delta t_C$. The same pulse, traveling for time Δt_{AB} relative to A and B travels the total zigzag distance $c\Delta t_{AB}$. Clock C itself travels a distance $v\Delta t_{AB}$ relative to clocks A and B while the pulse makes one round-trip in C. Therefore, using the Pythagorean theorem on either small triangle in Fig. 2.2, we find

$$\left(\frac{c\Delta t_{AB}}{2}\right)^2 = \left(\frac{c\Delta t_C}{2}\right)^2 + \left(\frac{v\Delta t_{AB}}{2}\right)^2, \tag{2.1}$$

from which we solve for the proper time Δt_C to obtain

$$\Delta t_C = \Delta t_{AB}\sqrt{1 - v^2/c^2}. \tag{2.2}$$

This relation can be written in the general form:

$$\boxed{\Delta t_{\text{proper}} = \Delta t_{\text{two-clock}}\sqrt{1 - v^2/c^2}.} \tag{2.3}$$

This very important relation is sometimes called the "proper time relation" or the principle of "time dilation." Qualitatively, it expresses the fact that ***different observers measure the passage of time differently depending on their relative motion.***

Hidden inside Eq. (2.3) is another result from special relativity; namely, no object can travel at a speed greater than c relative to any other object or reference frame. A superluminal speed ($|v| > c$) would result in an imaginary proper time, something that has no physical meaning. You will learn later that this speed limit is imposed by energy considerations as well (it would take an infinite amount of energy to accelerate an object with mass[10] to a speed $v = c$ relative to an observer, and *more* than an infinite amount of energy to achieve a speed $v > c$). Therefore, because $|v| \le c$, the proper time interval Δt_{proper} between two events is always *smaller* than the time $\Delta t_{\text{two-clock}}$ measured in a frame that requires two synchronized clocks for measurement.

Example 2.4 Time dilation.

A father and his daughter are traveling on a train that moves with a constant speed of 1.8×10^8 m/s ($= 0.6c$) relative to the ground.

[10] Of course, a photon of light can be considered an "object" that travels at a speed c, but this is a massless object. We'll say more about this in Chapter 4.

They pass a parked VW Beetle at which point the two of them simultaneously yell, “Red Punch Buggy!” and punch each other on the shoulder. Three seconds later, the daughter yells, “Jinx!” What is the time between these two events according to a person inside the VW who is waiting for the train to pass?

Solution: The key question — who measures the proper time (i.e., the smaller time interval)? To answer this, write this down in terms of events. Event A = father/daughter punch each other; Event B = daughter jinxes her dad. In this example, the father/daughter are at both events (not the person in the car), so they measure the smaller (proper) time interval, which has already been stated to be 3 s. So, we are given Δt_{proper} and we are solving for $\Delta t_{\text{two-clock}}$, which is the time interval measured by the person in the car.

$$\begin{aligned}
\Delta t_{\text{proper}} &= \Delta t_{\text{two-clock}}\sqrt{1 - v^2/c^2} \\
\Rightarrow \Delta t_{\text{train}} &= \Delta t_{\text{VW}}\sqrt{1 - v^2/c^2} \\
\Rightarrow \Delta t_{\text{VW}} &= \frac{\Delta t_{\text{train}}}{\sqrt{1 - v^2/c^2}} \\
&= \frac{3\,\text{s}}{\sqrt{1 - (0.6c/c)^2}} \\
&= \frac{3\,\text{s}}{\sqrt{1 - 0.6^2}} \\
&= 3.75\,\text{s}.
\end{aligned} \tag{2.4}$$

In this example, the father and daughter on the train measured the proper time because they were at both events, so the time interval is smaller from their reference frame. Be careful, though: sometimes the observer standing on the Earth measures the smaller time interval — it all depends on what the events are and who happens to be present at both of them.

Note also that we expressed v as a fraction of the speed of light — it makes things a lot simpler to write v in this manner. We'll say more about this later

2.5 Length contraction

One thing that will come up repeatedly in this unit is the fact that relativity breaks down the distinction between distance and time. In fact, in relativity, distance and time are really just flip sides of the same coin. And as we will see now, you can't change our conception of time without making a similarly dramatic change in the way we view distances and length.

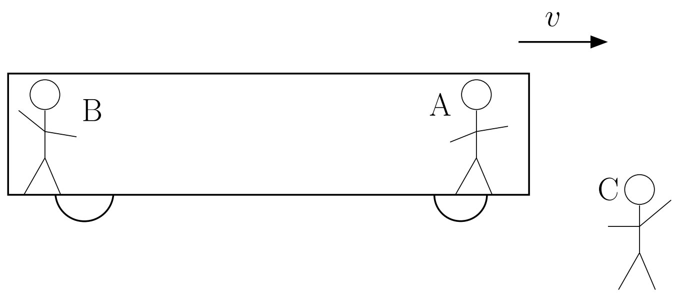

Figure 2.3: Sketch of train for length contraction thought-experiment.

Consider the following thought experiment: a train is moving on a track, with Observers A and B at the front and back end of the train. A and B have measured the length of the train with a long tape measure that they carry with them on the moving train, and find the length to be L_{train}. The train goes past Observer C who is standing next to the track with a stopwatch (see Fig. 2.3). Relative to C in the "ground reference frame," the train is moving with a speed v. From the train's reference frame, of course, it is C and the ground that are moving at a speed v in the opposite direction.

Let's say that Observer C wants to measure the length of the train. He can use his stopwatch to do this: since distance = (speed) × (time), the length of the train is simply the speed v times the time interval between when the front of the train passes and when the back of the train passes. Let's consider two events: Event A = front of train passes C; Event B = back of train passes C. According to people on the train, the time between the two events $\Delta t_{\text{train}} = L_{\text{train}}/v$, where L_{train} is the previously-measured length of the train — this is how far Observer C moves between the events according to train observers. But, as we saw in the previous section, C measures the proper time (since this observer is at both events), which is a smaller time interval:

$$\Delta t_C = \Delta t_{\text{proper}} = \Delta t_{\text{train}}\sqrt{1 - v^2/c^2}. \tag{2.5}$$

Based on this result, Observer C now says that the length of the train is

$$\begin{aligned} \text{Length} = (\text{speed}) \times (\text{time}) &= v\Delta t_C \\ &= v\Delta t_{\text{train}}\sqrt{1 - v^2/c^2} \\ &= L_{\text{train}}\sqrt{1 - v^2/c^2}. \end{aligned} \tag{2.6}$$

The length of the train as measured by an observer by the side of the track is less than the length of the same train as measured by people moving with the train.

We can write this relation (referred to as the ***Lorentz contraction*** or simply ***length contraction*** equation) in more general terms:

$$L_{\text{other}} = L_{\text{rest}}\sqrt{1 - v^2/c^2}, \tag{2.7}$$

where L_{rest} is the length of an object as measured by observers in a reference frame where that object is at rest, and L_{other} is the length as measured by observers in a different reference frame. Note that an object is always largest when viewed from its own reference frame, and shrinks when it is viewed as moving.

Some comments are in order:

- You can't have time dilation without length contraction — the two necessarily go hand-in-hand. This is a recurring theme of relativity — Einstein's theory can't be taken "a la carte"; rather, it is all or nothing. Einstein realized that if any single prediction of relativity were ever refuted, then the entire theory would have to be discarded.

- The arguments in this section apply to length components along the direction of the relative motion. Components of a length in directions perpendicular to the relative motion are not contracted.

- Length contraction is not an illusion or merely a matter of perception. In the example, the train doesn't just appear to be smaller from C's reference frame; rather, it *really is smaller* in that reference frame. Some of the homework problems and drill questions will investigate some of the curious properties of length contraction.

2.6 Experimental evidence

Most people are skeptical when they first read about the predictions of special relativity. This is to be expected, since we do not experience time dilation or length contraction effects on a daily basis. For these effects to be significant, you need relative velocities that are significant fractions of the speed of light. Looking at both Eqs. (2.3) and (2.7), the key piece is the stretch factor $\sqrt{1 - v^2/c^2}$, which is almost identically equal to 1.00 for even the fastest velocities that people ever experience. This is an important aspect of relativity; namely, that it obeys classical correspondence, i.e., the results of relativity agree with Newton's classical results for smaller velocities.

Despite the fact that relativistic effects are almost negligible in the "everyday" phenomena of our personal experience, there is copious experimental evidence that shows that Einstein's predictions are correct. In every case where an experiment has tested the theory of relativity, the experimental results have always agreed precisely with the predictions of relativity. Some examples:

- **Time dilation**. Time dilation is the most tested aspect of relativity. The most direct test was performed by taking two identical atomic clocks, flying one around the world on a plane and leaving the other on the ground, then comparing their readings after the trip. As predicted by Einstein, the clocks had ticked off different times, and by precisely the predicted amount.[11] Particle decay has also been used to test time dilation: a type of particle that typically lives for a certain period of time has been shown to live significantly longer if accelerated to high speeds (relative to the ground); again, the difference in times agrees perfectly with relativity. And the Global Positioning System (GPS) — which involves a series of satellites with precise clocks — uses relativity extensively to keep the orbiting clocks synchronized with those in the GPS units on the Earth. Without relativistic corrections, GPS wouldn't work!

- **The speed of light as a speed limit.** This result is verified daily in particle accelerators. It is fairly straightforward for scientists to accelerate subatomic particles to speeds close to the speed of light. But no matter how much energy is added, the speeds never make it to or above c. Electrons, in particular, have been accelerated to speeds $u > 0.99999999999c$, but never up to or above c.

- **Length contraction.** No experimentalist has managed to accelerate a train to relative speeds large enough to measure length contraction effects. (Trust us: you wouldn't want to be anywhere near a train going this fast.) But there is experimental evidence for length contraction: (a) cosmic rays produced at the top of the Earth's atmosphere somehow manage to make it to the surface of the Earth before decaying, despite the fact that they are very unstable. Some of these particles have lifetimes so short that even traveling at speeds close to c, they would be expected to decay long before they reach the ground. This can be explained using length contraction: the distance from the top of the atmosphere to the Earth's surface is significantly contracted from their reference frame, so there is no problem making it to the Earth's surface before decaying.[12] (b) Another piece of experimental evidence comes from electromagnetic theory — it turns out that you can explain why an electrical current produces magnetic effects by applying relativistic length contraction to the stream of electrons. The argument is too long to present here (especially since we haven't covered electricity and magnetism yet), but suffice it to say that the results agree perfectly with an analysis based on length contraction.

[11] Note that General Relativity plays a role here because the height of a clock also affects its rate, but the experiments took account of these general relativistic effects.

[12] This result can also be explained using time dilation, of course, because time dilation and length contraction are really different aspects of the same phenomenon.

There are other experimental tests of other aspects of relativity, some of which will be discussed later in this unit (when those aspects are presented). But, in general, it is worth remembering that relativity is not a series of different theories, but rather is a single, coherent, internally consistent theory. All of the predictions are inherently related to each other. So you can't say, "Well, I'm fine with time dilation but I don't buy length contraction." You simply can't have time dilation without length contraction — they are the same thing. So even if there hadn't been any independent experimental evidence of length contraction (which there is) there would be very little doubt of its veracity since time dilation has been verified extensively.

2.7 Units and dimensionless velocities

When working with relativity, it is convenient to express lengths in terms of distance traveled by light in one unit of time. A "light year" for instance is the distance that light travels in one year. An analogy would be to say that the distance between here and New York City is "three car hours" (i.e., it takes 3 hours to get to New York in a car driving at highway speeds). In fact, you will often hear people using time directly to express a distance: "Oh, it's 3 hours to New York City from here." We will abbreviate these units as lt-s, lt-min, lt-yr, …for light-second, light-minute and light-year, respectively. Using these units for distance, we can express speeds in terms of lt-s/s, lt-min/min, lt-yr/yr, etc. Since the speed of light $c = 1\,\text{lt-s/s} = 1\,\text{lt-min/min} = 1\,\text{lt-yr/yr} = \ldots$, the speed of a particle in these units is simply the speed expressed as a fraction of the speed of light.

Example 2.5 Units Conversion from lt-s/s to m/s

A proton is traveling at a speed of 0.25 lt-s/s. Find its speed in units of m/s.

Solution: Use the fact that 1 lt-s/s is equal to about 3.00×10^8 m/s. Then convert units in the usual way:

$$0.25\,\text{lt-s/s} \times \frac{3.0 \times 10^8\,\text{m/s}}{1\,\text{lt-s/s}} = 0.75 \times 10^8\,\text{m/s}.$$

In this example, a particle has a speed $u = 0.25\,\text{lt-s/s}$. This same speed could be expressed as $u = 0.25c$. In fact, we will typically express velocities as a fraction of the speed of light c.

Problems

1. Give a one sentence definition of a meter, using the concept of a second and the defined value for c.

2. A proton is traveling at a speed of 4.0×10^7 m/s. How many lt-s/s is this?

3. A π^- meson is traveling at a speed of 0.060 lt-s/s. Convert this speed to m/s.

4. A spaceship moving at constant speed 0.80 lt-s/s travels between two planets A and B in 1000 s, as measured by synchronized clocks on the planets. Calculate the elapsed time according to a clock carried on board the spaceship.

5. How fast does a particle have to travel relative to clocks A and B, which are at rest relative to each other, in order that its elapsed time as read on a clock moving with the particle is one-tenth the elapsed time measured on clocks A and B? Express your answer both in lt-s/s and in m/s.

6. A meteorite is observed to travel a distance 1.00×10^5 lt-s relative to the Earth in a time of 6.00×10^5 s as measured by Earth observers. Calculate the elapsed time for this trip as measured by a clock carried along on the meteorite.

7. A crew of astronauts travels at a speed of $0.60c$ from Earth to the nearest star, Proxima Centauri, a distance of 4.0 lt-yr (as determined by observers on Earth).

(a) Calculate how long the trip takes according to observers at rest relative to the Earth.

(b) Calculate the time for the trip as measured by a clock on the spaceship.

(c) Based on your answer to b), calculate the distance from Earth to Proxima Centauri as determined by the astronauts using the relation "distance" = "speed" × "time", where distance, speed and time are all measured from the astronauts' reference frame.

(d) Calculate the Earth-Proxima Centauri distance from the astronauts' reference frame, but this time use length contraction. You should end up with the same result as for c).

(e) Think about the results from parts c) and d). This should convince you that length contraction and time dilation are really the same thing (i.e., you can't have one without the other).

8. Another spaceship crew wants to make the trip from Earth to Proxima Centauri in only 2.0 years as measured by clocks on board their spaceship. (Recall from the previous problem statement that the distance between Earth and Proxima Centauri is 4.0 lt-yr as determined by observers on Earth.)

(a) How long does the trip take according to Earth-frame observers?

(b) How fast must the astronauts travel relative to Earth?

Hint: First, do both parts a) and b) together. Also, you will need to express the speed in terms of the unknown travel time according to Earth-frame observers.

9. You are in a metallic red VW bug stopped at a traffic light. You see the traffic light turn green, and 2.5 μs later you hear the car behind you honk its horn. What is the time between you seeing the light change to green and your hearing of the horn honk as measured by an alien passing by in a ship at a speed $0.9c$?

10. There is a supergiant star named Betelgeuse[13] which (in the Earth's reference frame) is 80 lt-yr away.

(a) A crew of astronauts is traveling toward Betelgeuse, traveling at a speed $0.8c$ relative to the Earth-Betelgeuse reference frame. What is the separation between Earth and Betelgeuse in the astronauts' reference frame?

(b) Another crew traveling toward Betelgeuse measures the Earth-Betelgeuse distance to be 23 lt-yr. How fast is this second crew traveling relative to the Earth?

11. Betty is standing by the side of a train track when a really long train approaches traveling at a ridiculously fast speed of $0.9c$. Thinking quickly, she pulls out her stopwatch, clicks it on when the front of the train passes and clicks it off when the back of the train passes. After standing back up and smoothing down her hair, she notes that her stopwatch reads 0.0025 s. (She has really good reflexes.) Betty now makes some calculations.

(a) According to Betty, how long is the moving train? (Assume that she was warned in advance that the train was going at a speed $0.9c$.)

[13] Betelgeuse is a supergiant star located in the constellation Orion. It is very cool because it could go supernova anytime in the next million years, and that will be quite a show for us when it does.

(b) Later that day (not much later), the train reaches its destination and stops. What is the length of the train according to people standing next to the (now motionless) train?

12. During migration, two fast Arctic terns fly one behind the other over London at speed $0.8c$. A tourist sees the birds pass by while looking at the Big Ben clock tower. She notes that a time of 12.0 ms elapsed on the clock between the first bird's passing and the second bird's passing.

(a) How much time elapsed between these two events, according to the birds?

(b) How far apart are the birds according to the tourists watching the birds fly by?

13. Playing Catch on a Train I
Note: This is a non-relativistic physics problem.
You are riding in a box car of a train that is traveling along the tracks at 30 m/s. You are bored, so you start to play catch with your friend who is standing on the opposite side of the box car, 5 m away from you. You throw the ball at a speed of 10 m/s straight to your friend, who catches the ball; the given speed and direction are determined in *your* reference frame. You may ignore any vertical motion of the ball.

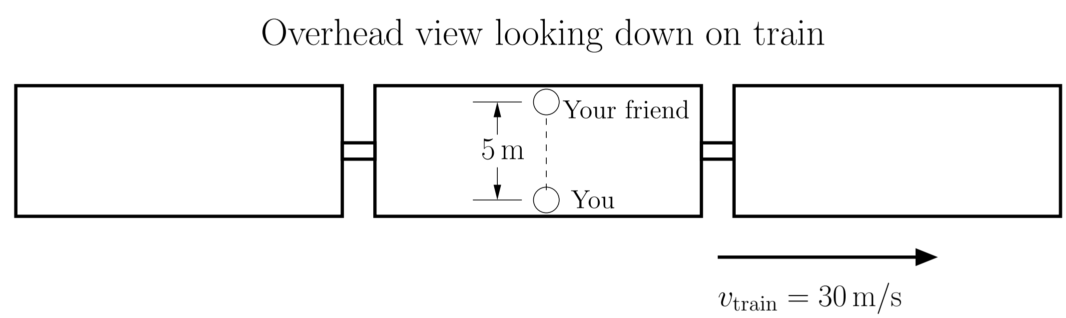

Figure 2.4: Figure for Problem 13. This is a top view looking down upon the train.

(a) How long does it take for the ball to reach your friend according to you?

(b) How long does it take for the ball to reach your friend according to an observer standing at rest on the ground? (This is a dumb question in classical physics — if the answer isn't obvious, you're thinking too hard!)

(c) How far does the ball travel according to the observer standing on the ground?

(d) What is the speed of the ball according to the observer standing on the ground?

14. Playing Catch on a Train II

Note: This is a non-relativistic physics problem.

You are riding in a box car of a train that is traveling along the tracks at an undetermined speed. Once again, you are bored, so you start to play catch with your friend who is standing on the opposite side of the box car 5 m away from you. (See figure in previous problem.) You throw the ball in a horizontal plane at a speed of 10 m/s straight to your friend, who catches the ball. According to an observer on the ground, the ball travels a distance of 13.93 m between the time you throw it and the time your friend catches it. Calculate the speed of the train along the tracks.

15. Photons on a Train

Note: This is a relativistic physics problem.

Alice is riding in a box car of a train that is traveling along the tracks at a speed $v = 0.6$ lt-ns/ns. She just happens to have a light-clock with her (just like the one illustrated in Fig. 2.1 in Chapter 2). Alice aligns the clock so that the light is aimed horizontally directly across the train car, perpendicular to the motion of the train. Alice sends a light pulse across the train, and notices that the light returns to the detector 2 ns later. Alice's friend Bob is standing at rest on the ground as Alice and her light-clock speed by. Bob measures the round-trip time for the pulse of light in Alice's light clock to be Δt_{Bob}.

(a) What is the speed of the light pulse according to Alice?

(b) Determine the distance between the emitter/detector and the mirror in the light-clock as determined by Alice.

(c) What is the speed of the light pulse according to Bob? (This is a dumb question in relativistic physics — if the answer isn't obvious, you're thinking too hard!)

(d) Determine length of the path traversed by the light pulse according to Bob **in terms of the unknown time** Δt_{Bob}.

(e) Use the Pythagorean theorem to find the distance traveled by the light pulse according to Bob. From this determine a numerical value for Δt_{Bob}.

(f) Now use Eq. (2.3) to calculate Δt_{Bob}. This answer should agree with the answer you determined in the previous part.

Chapter 3

Relativistic Spacetime

3.1 Introduction

The previous chapter introduced the basic ideas of relativity along with some of the most dramatic implications of the theory. But the predictions of time dilation and length contraction are merely special cases of a much broader theory. In this chapter, we discuss the idea of spacetime, which blends time and space together. We introduce the spacetime interval, a quantity that is one of the fundamental invariants in relativity, and we use this interval to relate distance and time measurements made in different reference frames that are moving with respect to each other. We also introduce spacetime diagrams, which provide a graphical way of illustrating relativistic phenomena, particularly the relativity of simultaneity.

3.2 Spacetime intervals

As we saw in Chapter 2, observers in different reference frames disagree about time and distance measurements. But there are a few quantities referred to as invariants upon which all observers agree regardless of their reference frames. One of these invariants was discussed in the previous chapter; namely, the invariance of the speed of light in a vacuum. It turns out that distance and time can be folded together to make another invariant, referred to as the invariant spacetime interval I, defined by

$$I^2 \equiv (c\Delta t)^2 - (\Delta x)^2 \,. \tag{3.1}$$

Note that I^2 can be positive, zero, or negative. If I^2 is positive, then the interval is called *time-like* since the first term — with Δt in it — dominates. Similarly, negative values of I^2 correspond to *space-like* intervals, and if $I^2 = 0$, the interval is called *light-like*. Qualitatively, an event is light-like if a pulse of light could travel directly between the two events. This can be seen from Eq. (3.1): if $I^2 = 0$, then

$$\begin{aligned} 0 &= (c\Delta t)^2 - (\Delta x)^2 \\ \Rightarrow \Delta x &= \pm c\Delta t, \end{aligned} \tag{3.2}$$

as would be expected for a pulse of light traveling a distance Δx in a time Δt.

Two events could be *causally-linked* (i.e., event A actually causes or contributes to event B) if the spacetime interval between them is either light-like or time-like. In fact, for time-like spacetime intervals, the interval is the proper time (multiplied by c). If two events are separated by a space-like interval, then no information can travel between the two events since it would require superluminal ($|v| > c$) information transmission, and nothing (especially information) can travel faster than light relative to any observer. So events *can't* be causally linked if the square of the spacetime interval between them is negative.

Example 3.1 Causality and intervals

In the year 2055, a father and his daughter are watching the 5$^{\text{th}}$ game of the National League Divisional Series from a Moon base at the Sea of Tranquility. The Washington Nationals lead the St. Louis Cardinals by a run with 2 outs in the bottom of the ninth inning, but the bases are loaded. The daughter sneezes and then watches in horror as 2.0 s later Drew Storen, Jr., of the Nationals throws a wild pitch that not only walks in the tying run but also allows St. Louis to score the winning run. Distraught, the daughter bursts into tears. "What's wrong?" her father asks. "It's my fault that the Nats lost! My sneeze caused Storen to throw that wild pitch!" What argument should the father use to assure his daughter that she is not personally responsible for yet another heart-breaking Nats playoff loss?

Solution: The father should first point out that the distance between the Earth and Moon is 3.84×10^8 m, or 1.3 lt-s. So, if the father/daughter received the TV signal of the strikeout 2.0 s after the sneeze, in their reference frame it must have actually occurred only 0.7 s after the sneeze. (It takes the TV signal 1.3 s to make it from the Earth to the Moon.) Now, the father should calculate the spacetime interval:

$$I^2 = (c\Delta t)^2 - (\Delta x)^2 = (1\,\text{lt-s/s} \times 0.7\,\text{s})^2 - (1.3\,\text{lt-s})^2 = -1.2\,(\text{lt-s})^2$$

So, the father should pat the daughter on the head and say, "So, you see honey — you can't have caused Storen to throw that wild pitch because the spacetime interval between your sneeze and his strikeout is a space-like interval!"[1] [2]

Don't worry about the fact that I^2 is negative for space-like intervals. The definition of I^2 in Eq. (3.1) is chosen so that for time-like intervals, the interval I itself is the proper time (multiplied by c), which is convenient for our purposes. But the interval could have equally been defined as $I^2 = (\Delta x)^2 - (c\Delta t)^2$, in which case I^2 would be negative for time-like intervals (some authors do, in fact, define I^2 this way). In fact, some physicists define two different intervals: $I^2 = (c\Delta t)^2 - (\Delta x)^2$ for time-like intervals and $I^2 = (\Delta x)^2 - (c\Delta t)^2$ for space-like intervals. We will stick with Eq. (3.1).

As stated earlier, I^2 is an invariant — observers in different reference frames will agree on the value of this interval for any two events:

$$I^2 = \left(I'\right)^2 , \tag{3.3}$$

or

$$\boxed{(c\Delta t)^2 - (\Delta x)^2 = \left(c\Delta t'\right)^2 - \left(\Delta x'\right)^2 . \qquad (3.4)}$$

This is important for several reasons. First, the invariance of the interval helps to further clarify the intimate connection between distance and time in relativity. Any disagreements between different observers about the time interval between events must be accompanied by a corresponding disagreement in the distance in order to keep the interval invariant. Second, if an interval is space-like or time-like or light-like as viewed in one reference frame, then it is the same kind of interval as viewed in any reference frame. This makes sense: if two events cannot be causally-linked in one reference frame, for instance, it would be nonsensical to think that they would be causally-linked as observed by someone in a different reference frame. Finally, the interval can be used to determine how events are viewed in one reference frame, given information in a different frame. As an example, we refer back to a problem from the previous chapter.

[1] By the way, we originally wrote this problem about the Red Sox when they hadn't won a World Series in over 80 years. The following year, they won the World Series. And **that** is a *time-like* interval, so, yes, we do take credit for the Red Sox victory.

[2] And then we switched the example to one with the Cubs because, well, it didn't work for the Red Sox anymore, and we figured, "Well, we'll be able to use this example for **decades** because the Cubs will never win the World Series!" But the Cubs messed that up this past year. (Also a time-like interval, so we take credit for the Cubs World Series championship, too.) Fans of the Nationals in PHYS 211 requested that we use the Nats now in this example.

Example 3.2 Using the interval

A spaceship crew wants to make the trip from Earth to Alpha Centauri in only 2.0 years as measured by clocks on board their spaceship.
(a) How long does the trip take according to Earth-frame observers?
(b) How fast must the astronauts travel relative to Earth?

Solution: For problem 7 in chapter 2, you used the proper time relation, but you had to express the speed in terms of the unknown travel time according to Earth-frame observers. Here, we do the same problem but using the spacetime interval.

(a) We know that the distance to Alpha Centauri is 4 lt-yr as measured by observers on the Earth, so $\Delta x = 4\,\text{lt-yr}$. From the statement of the problem, we can see that $\Delta t' = 2\,\text{yr}$. And since the astronauts are present both at the launching of the rocket and its arrival at Alpha Centauri, it follows that $\Delta x' = 0$. Using the interval, we have

$$\begin{aligned}
(c\Delta t)^2 - (\Delta x)^2 &= \left(c\Delta t'\right)^2 - \left(\Delta x'\right)^2 \\
\Rightarrow (c\Delta t)^2 &= \left(c\Delta t'\right)^2 - \left(\Delta x'\right)^2 + (\Delta x)^2 \\
\Rightarrow (\Delta t)^2 &= (2\,\text{y})^2 + \frac{(4\,\text{lt-yr})^2}{(1\,\text{lt-yr/yr})^2} - 0^2 \\
&= 20\,\text{y}^2 \\
\Rightarrow \Delta t &= 4.47\,\text{y}.
\end{aligned}$$

(b) The speed of the astronauts' ship is then simply

$$v = \frac{\Delta x}{\Delta t} = \frac{4\,\text{lt-yr/yr}}{4.47\,\text{y}} = 0.89\,\text{lt-yr/yr} = 0.89c. \tag{3.5}$$

As you might have guessed from this example, the relations from the previous chapter (time dilation and length contraction) are both special cases of the more general invariance of the spacetime interval. The proper time relation, Eq. (2.3) corresponds to a situation where one of the observers is at both events. Let's say that the observer in the primed reference frame is at both events (i.e., measures the proper time). Then $\Delta x' = 0$. The distance Δx (the distance between events as measured by the observer in the unprimed frame) is simply $\Delta x = v\Delta t$ (distance = speed × time). Eq. (3.4) then becomes:

$$\begin{aligned} & (c\Delta t)^2 - (v\Delta t)^2 = \left(c\Delta t'\right)^2 - 0^2 \\ \Rightarrow \quad & (\Delta t)^2 \left(1 - v^2/c^2\right) = \left(\Delta t'\right)^2 \\ \Rightarrow \quad & \Delta t' = \Delta t\sqrt{1 - v^2/c^2}, \end{aligned}$$

which is, in fact, the proper time relation Eq. (2.3) with $\Delta t'$ as the proper time (since the primed observer is at both events) and with Δt as the two-clock time. Similar arguments can be used to show that the length contraction relation, Eq. (2.7), is a special case of the invariance of the spacetime interval for situations where Δt or $\Delta t'$ is zero.

3.3 World Lines and Spacetime Diagrams

The motions of particles, clocks, or whatever can be represented on a spacetime diagram. A spacetime diagram consists of a pair of perpendicular axes, with the vertical axis representing time and the horizontal axis representing x position in a particular inertial reference frame. The x-axis is the direction of relative motion between this unprimed frame and another inertial frame called the primed frame.

A plot of an object's position vs. time on a spacetime diagram is called the *world line* of the object. Three world lines are shown in Fig. 3.1; a straight world line represents motion with constant velocity while a curved world line represents accelerated motion. An *event* is represented by a dot on the spacetime diagram.

When drawing a spacetime diagram, make sure you use appropriate units. (Do *not* use meters for length!) For time (which we use as the vertical axis on a spacetime diagram), we choose something appropriate to the time scale of the problem, like years, seconds, nanoseconds, etc. Then we must choose an appropriate unit of distance equal to that traveled by light in the chosen unit of time. For example, suppose we choose one second as the unit of time, then we would used one lt-s (the distance traveled by light in one second) as the unit of distance. In these units the speed of light is $c = (1\,\text{lt-s})/(1\,\text{s}) = 1\,\text{lt-s/s}$. In this method of handling the units, the world line for a pulse of light must have a slope that is numerically equal to $+1$ or -1. **And no world line can ever have a slope with magnitude less than 1; that would correspond to an object traveling faster than light.** Slopes can also be used to determine if the interval between two events is time-like, light-like or space-like. If a line were to be drawn connecting the two events, a time-like interval would correspond to a slope with magnitude greater than 1, a light-like interval would correspond to a slope with magnitude 1, and space-like interval would correspond to a

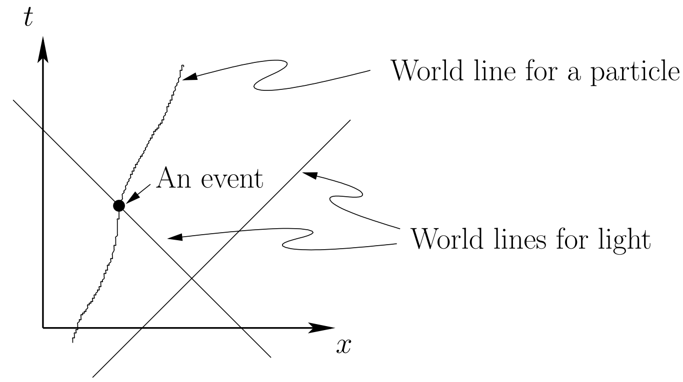

Figure 3.1: A spacetime diagram with three world lines. The two world lines for light have slopes +1 and −1.

slope with magnitude less than 1 (note that in this last case, that the "line" drawn between the two events can't be the world line of any real object, since nothing can travel faster than c).

Let's use a spacetime diagram to display the world lines of the three-clock thought experiment of Ch. 2, Section 2.4 (see Fig. 2.1 from that section and Fig. 3.2 in this section). For example, put clock A at rest at $x = 0$ and clock B at rest at $x = 0.60\,\text{lt-s}$. The world lines for the stationary clocks A and B are then vertical lines at $x = 0\,\text{lt-s}$ and $x = 0.60\,\text{lt-s}$. Let clock C travel with speed $0.60c$ in the positive x-direction. Because $c = 1\,\text{lt-s/s}$, clock C passes through $x = 0$ at time $t = 0\,\text{s}$, and it passes through $x = 0.60\,\text{lt-s}$ at time $t = 1.0\,\text{s}$. It has traveled a distance of $0.60\,\text{lt-s}$ in a time $1.0\,\text{s}$.

Notice in Fig. 3.2 that we have labeled the world line of clock C as the t' axis. This is a general result: the world line of a particular observer (say, someone traveling in a space ship) is the t' axis for that observer. This can be understood by considering a person on a spaceship holding a ball. The world line for the ball is the same as the world line of the ship and person since they are all moving together. From the perspective of the astronaut, the ball remains right in front of him and isn't moving anywhere, so it makes sense that that astronaut will say that the location of the ball remains at $x' = 0$. And just as it is true that the points where $x = 0$ in the unprimed frame define the t-axis, so it is that the points where $x' = 0$ in the primed frame define the t'-axis.

Some comments are in order:

1. A world line is nothing more than a plot of time versus position. If you ever find yourself stumped about how to plot a world-line, ask yourself: "Where is the (whatever) at time $t = 0$ (i.e., what is its initial x-coordinate)? Where is it at time $t = 1$? At time $t = 2$? ..." Then simply plot those points and connect them.

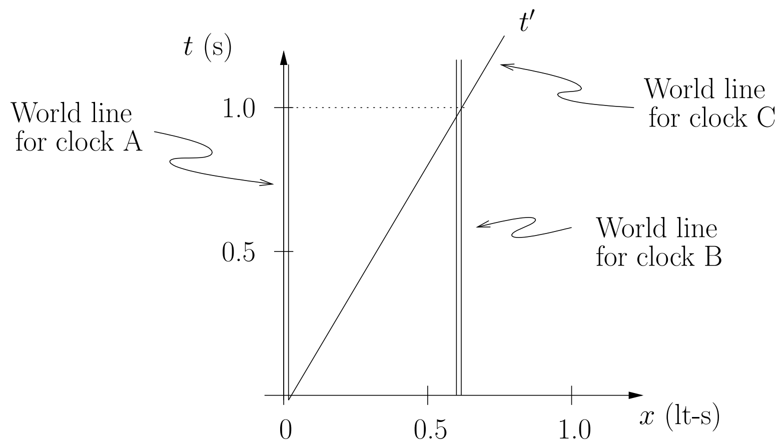

Figure 3.2: World lines for the three clocks in the thought experiment of Section 3.3

2. The slope of a world line is simply $1/v$. This comes from the standard relation: distance = speed × time, or equivalently, $\Delta x = v\Delta t$. So $\Delta t = \frac{1}{v}\Delta x$. Practically, this means that if you have a ship moving at a speed of, say, $0.5c$, then the slope will be $1/v$ or $2.0\,\mathrm{s/lt\text{-}s}$. When plotting a world line, this means that you go up 2 and over 1 (or over 0.5 and up 1).

3. Don't **ever** forget — nothing can travel faster than light, so there should **never** be a world line on a spacetime diagram with a slope whose magnitude is less than 1.

4. Remember: events are plotted as dots.

5. Label everything clearly.

Example 3.3 Spacetime diagram corresponding to Example 1

Draw the spacetime diagram for the baseball scenario (Cubs losing the World Series) discussed in Example 1, using the reference frame of the Earth/Moon. Show the world lines for Anthony Rizzo, Jr., the girl and her father, and the TV signal. Also, show and label the following events: A — girl sneezes, B — Rizzo strikes out, and C — girl and father see Rizzo striking out.

Solution: The world lines for Rizzo and the girl/father are simply straight vertical lines since they aren't moving in the Earth-Moon

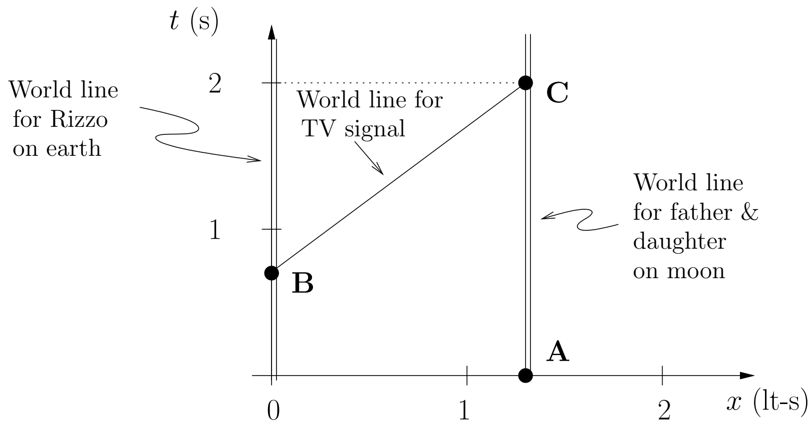

Figure 3.3: Spacetime diagram for situation discussed in Examples 1 and 3

reference frame. If this isn't clear, then answer these questions: If we put the Earth at $x = 0$ at time $t = 0$, where is the Earth at time $t = 1\,\text{s}$? Answer: still at $x = 0$. At $t = 2\,\text{s}$? Answer: still at $x = 0$. The Earth's world line is nothing more that a set of points where x is always zero. As for the girl/father on the Moon, we already said in Example 1 that they are about 1.3 lt-s away from the Earth.

We know from the problem that the girl/father see the strikeout 2 s after she sneezes. So, if she sneezes at $t = 0$ (it is arbitrary as to what we choose as the $t = 0$ time), then the TV signal arrives at $t = 2\,\text{s}$. It must have been sent from the Earth at an earlier time, and since it travels at the speed of light, then the world line for the TV signal is a 45° line. The only thing left is to plot the three dots for the events.

Note that if you imagine a line between A and B, that line would have a slope with magnitude less than 1 (i.e., too shallow), indicating that nothing can travel between these two events, consistent with the result in Example 1 that the interval is space-like and the corresponding events can't be causally linked.

3.4 Ordering of events — the relativity of simultaneity

Every event has a set of space and time coordinates. In Example 3 above, we would say that the event A (girl sneezes) occurs at time $t = 0$ and location $x = 1.3\,\text{lt-s}$. Similarly, we can determine the location and times of events B and C, all as measured by observers in the Earth-Moon reference frame.

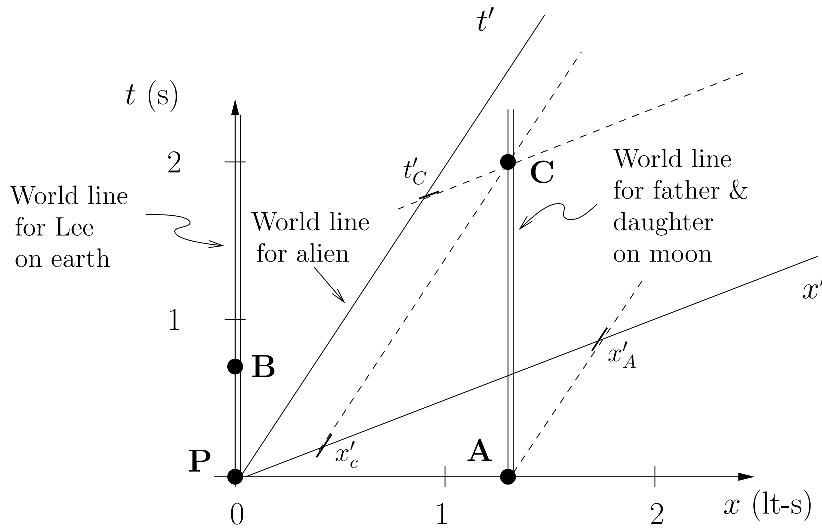

Figure 3.4: Extension of spacetime diagram in Example 3.

Let's add one more event to the scenario: let's say that at time $t = 0$, the pitcher Masahiro Tanaka, Jr., pitches the ball toward Castro. In Fig. 3.4 we have added this event and labeled it P. In the Earth-Moon frame, we can say quite definitively that A and P are simultaneous and come first, then B, then C. Also, A and C happen at the same location, and P and B happen at the same location.

Special relativity helps us answer the following question: how does an observer moving in a different reference frame view these same events? We won't worry here about the actual numerical values of x' and t' (the position and time as measured by a different observer), but we can say quite a lot about the ordering of events in space and time by looking at the spacetime diagrams.

We have added another world line to Fig. 3.4, namely, the world line for a hypothetical alien whizzing past the Earth just as the pitch is thrown. This alien is monitoring the game to try to understand human culture. We assume the alien is traveling at a speed $0.5c$; hence, the world line has a slope of 2.

We have already commented that the world line of an observer in a primed frame is simply the t' axis for that frame, so we have labeled the alien's world line t'. But where should we put the x'-axis and what scale should we put on it? It turns out that to satisfy the invariance of the speed of light, we must draw the x'-axis at the same angle relative to the x-axis as the angle of the t'-axis relative to the t-axis. This means the slope of the x'-axis is equal to the speed v of the primed frame relative to the unprimed frame.

Recall that the t'-axis represents points where $x' = 0$. It turns out that x' is constant along any line parallel to the t'-axis. In other words, lines parallel to the t'-axis are equal-location lines for the primed frame of reference, just as the t-axis and all lines parallel to it are each lines of equal location for the unprimed frame of reference. The same ideas work for events on lines parallel to the x or x' axes; events on a line parallel to the x-axis are simultaneous in the unprimed frame, and events on a line parallel to the x'-axis are simultaneous in the primed reference frame.

We can use these ideas to "read off" coordinates for events in both reference frames. As an example, let's look at event C in Fig. 3.4. We have already commented that in the unprimed frame, its x location is 1.3 lt-s and its time is 2 s. The coordinates of this event in the alien's reference frame are determined by drawing lines parallel to the x' and t' axes (shown as dotted lines in Fig. 3.4). The intersections of these construction lines with the opposing primed axis gives the x_C and t_C coordinates. The rules for determining coordinates can be summarized as follows:

1. To find x_C, draw a straight line through C parallel to the t-axis and read off where it crosses the x-axis.

2. To find t_C draw a straight line through C parallel to the x-axis and read off where it crosses the t-axis.

3. To find x'_C, draw a straight line through C parallel to the t'-axis and read off where it crosses the x'-axis.

4. To find t'_C, draw a straight line through C parallel to the x'-axis and read off where it crosses the t'-axis.

Using this type of construction, we can see that although events A and C occur at the same place in the unprimed (Earth-Moon) reference frame, event C happens to the left of the event A in the primed (alien) reference frame. This is easy to understand: the alien is far from the Moon when event A happens, so A is far "to the right," whereas the alien is close to the Moon when event C happens, so from the alien's perspective, C isn't so far to the right, i.e., smaller x' coordinate.

But what about the ordering of events in time? We have commented that the invariance of the spacetime interval says that if two observers disagree about distances, then they will have to disagree about time intervals as well.

In preparation for class: Look at the t' coordinates for events P and A. In the Earth-Moon reference frame, these events are simultaneous. What about in the alien reference frame?

We have said that any two events on a line parallel to the x'-axis are simultaneous in the primed frame of reference. Similarly two events that

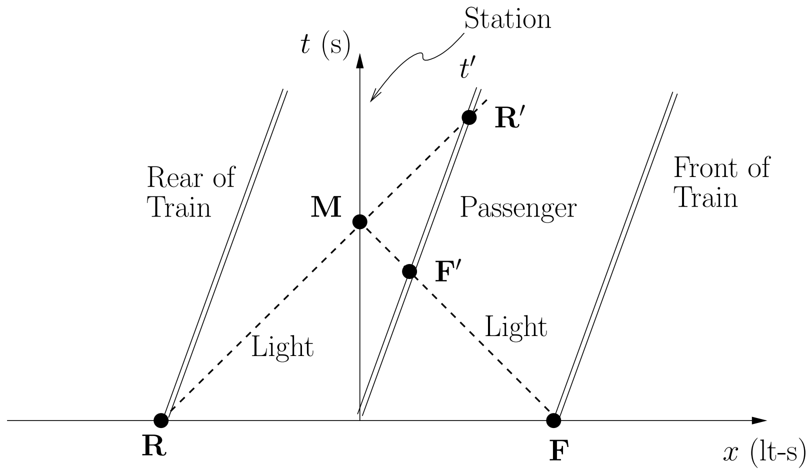

Figure 3.5: Spacetime diagram for train of Example 4. (Light world lines shown as dashed lines.)

lie on a line parallel to the x-axis are simultaneous in the unprimed frame. However two different events cannot lie both on a line parallel to the x-axis and parallel to the x'-axis. Thus two events that are simultaneous in one frame cannot be simultaneous in the other frame. We explore this idea in the following example.

Example 3.4 Simultaneity is Relative

Einstein showed, with the following thought experiment, that two events which occur at the same time but at different places in one frame, occur at different times in another frame.

Imagine a train moving past a station. By chance, lightning happens to strike the front and back of the train at the same time according to observers on the station platform. Light pulses from these strikes travel toward the middle of the train, where a passenger observes their times of arrival. Do the light pulses arrive simultaneously or does one arrive before the other, and if so, which one?

Solution: Use a spacetime diagram, Fig. 3.5, with the station at rest in the unprimed frame and the train at rest in the primed frame. The x- and x'-axes both lie along the track. (Note: this does **not** mean that the x- and x'-axes are the same thing on a spacetime plot. The x'-axis is not shown in Fig. 3.5, but remember that it is the mirror image of the t'-axis about a 45° line; i.e., the angle between the t-

and t'-axes is the same as the angle between the x- and x'-axes.) The world line for the middle of the station is shown as the t-axis.

Because all parts of the train are at rest in the primed frame, we draw the world lines for the front and the rear ends of the train parallel to the t'-axis. Also, in Fig. 3.5, we have chosen the world line for the passenger riding in the exact middle of the train to be the t'-axis. In the primed frame the front and rear world lines are then equidistant from the passenger, by definition.

The lightning strikes occur at points R and F on the world lines of the rear and front of the train. Because each strike represents an event and because these two events occur simultaneously in the station frame, R and F must be drawn on the same horizontal line. We arbitrarily choose this line to be at $t = 0$.

The light pulses produced by the lightning strikes travel with speed $c = 1$ lt-whatever per whatever from the event F back toward the passenger and from R forward toward the passenger. The pulse from F is represented by a world line of slope -1 and the pulse from R is represented by a world line of slope $+1$. Figure 3.5 shows that the pulse from F arrives at the passenger's world line (at F$'$) earlier (i.e., at a smaller value of t') than does the pulse from R, which arrives at R$'$.

The passenger must conclude that the front strike occurred before the rear strike because she is sitting in the middle of the train, equidistant from R and F, and she knows the light pulses must have taken the same time (in her frame) to reach her. By the same argument, an observer on the station platform who was at the exact middle of the train at $t = 0$ when the strikes occurred, sees the pulses at the same time. This is shown on the spacetime diagram by the fact that the world lines of the pulses cross the world line of the middle of the station at $x = 0$ (event M) at the same time.

Problems

1. If two events are separated by a time-like interval in one frame of reference, are they separated by a time-like interval in all frames of reference? Explain.

2. A simple way of synchronizing two clocks at rest relative to one another is to stand exactly halfway between them and emit light pulses toward each of them at the same instant of time. Each clock is then set to 0 when the synchronizing pulse reaches it.

(a) How does this scheme ensure that the clocks are started simultaneously?

(b) On a spacetime diagram show the world lines of two clocks at rest in the unprimed frame of reference at $x = 0$ and at $x = L$, along with the world lines of two synchronizing light pulses that start from the midpoint and reach each of the two clocks at $t = 0$.

3. Events A, B, and C are shown on the spacetime diagram in Fig. 3.6.

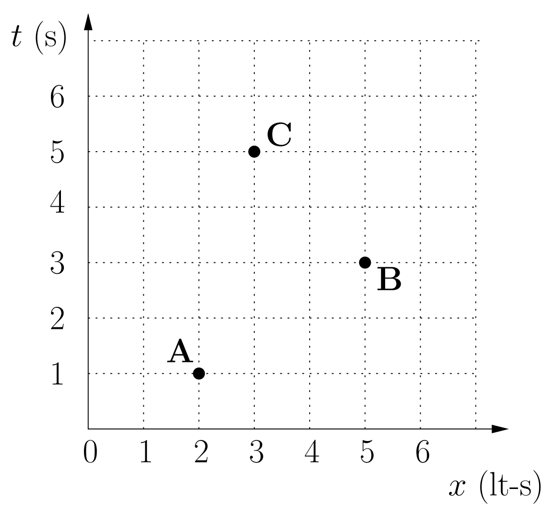

Figure 3.6: Figure for Problem 3.

(a) Calculate the value of the squared interval for each pair of events, i.e., find I^2_{AB}, I^2_{AC}, and I^2_{BC}.

(b) Label each interval as time-like, space-like, or light-like.

(c) In the frame shown, event A occurs before B, which occurs before C. Which pairs of events could have their time-order reversed (switching before and after) by choosing an appropriate reference frame?

(d) In the frame shown, event B occurs to the right of C, which occurs to the right of A. Which pairs of events could have their space-order reversed (switching left and right) by choosing an appropriate reference frame?

(e) Which events could be a "cause" for which other events?

4. Fig. 3.7 shows a spacetime diagram with seven straight lines through the origin labeled with capital letters A through G. Various events are marked as points with small letters a through e. The x-t axes belong to the Earth's reference frame.

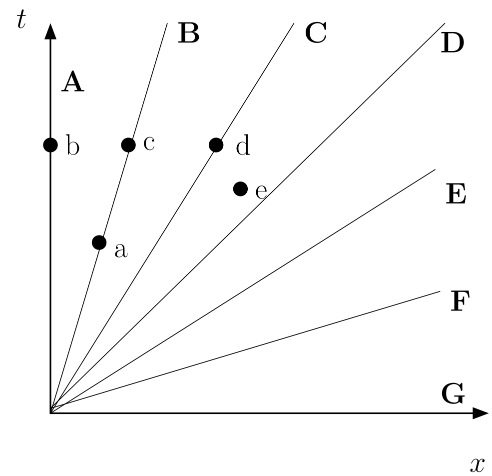

Figure 3.7: Figure for Problem 4.

(a) Which line is a world line of an object at rest relative to the Earth?

(b) Which line is a world line of a spaceship traveling at speed $+0.3c$ relative to the Earth?

(c) Which line is a world line of a light pulse emitted by the spaceship as it passes the Earth?

(d) Which events happen simultaneously in the Earth frame?

(e) Which events happen simultaneously in the spaceship frame?

(f) Which pairs of events are clearly separated by space-like intervals? Which are clearly separated by time-like intervals?

5. For Example 4 in the text, show from the spacetime diagram in Fig. 3.5 that lightning hit the front of the train at a negative value of t', but that lightning hit the rear of the train at a positive value of t'. Use the rule for finding the t' coordinate of an event to solve this problem. [Hint: you might want to extend some of the axes in the negative direction.]

6. Farmer Brown, at rest in his frame, carries a ladder through a barn. According to Farmer Brown, the ladder measures 20 lt-ns. According to observers at rest with respect to the barn, Farmer Brown and his ladder are moving at a speed $0.80c$ (alternately, Farmer Brown sees the barn moving at speed $0.80c$). In the barn's frame, the front door of the barn is at $x = 0$ lt-ns and the back door is at $x = 16$ lt-ns.

(a) Calculate the length of the ladder as measured by observers in the barn's reference frame. According to these observers, will the ladder fit within the barn?

(b) Calculate the length of the barn as measured by Farmer Brown. According to Farmer Brown, will the ladder fit within the barn?

(c) Draw a careful spacetime plot for this situation, with appropriate tick marks on the axes (labeled with numbers). Starting with the barn frame, draw world lines for the entrance and exit of the barn (i.e., the front and back doors). Draw also world lines for the front and back of the ladder (which is moving as viewed in the barn frame). The distance between the front and back of the ladder in your plot should agree with your answer to part (a), and the slopes of these lines should be consistent with the known velocities.

(d) Label the following events on your diagram: A = front of ladder enters the barn; B = front of ladder leaves the barn; C = back of ladder enters barn; D = back of ladder leaves barn. Determine the order of these events in time as viewed from the barn's reference frame. Is this result consistent with your answer to (a), i.e., whether or not the ladder fits in the barn, according to barn-frame observers? (Consider whether the back of the ladder enters the barn before or after the front of the ladder leaves the barn.)

(e) Determine the order of events A, B, C and D as viewed from Farmer Brown's reference frame. Is this result consistent with your answers to (b)?

(f) Based on your answers for this problem, can you see how relativistic time-ordering (i.e, the fact that different observers do not necessarily agree on the ordering of events in time) is necessarily linked with length contraction?

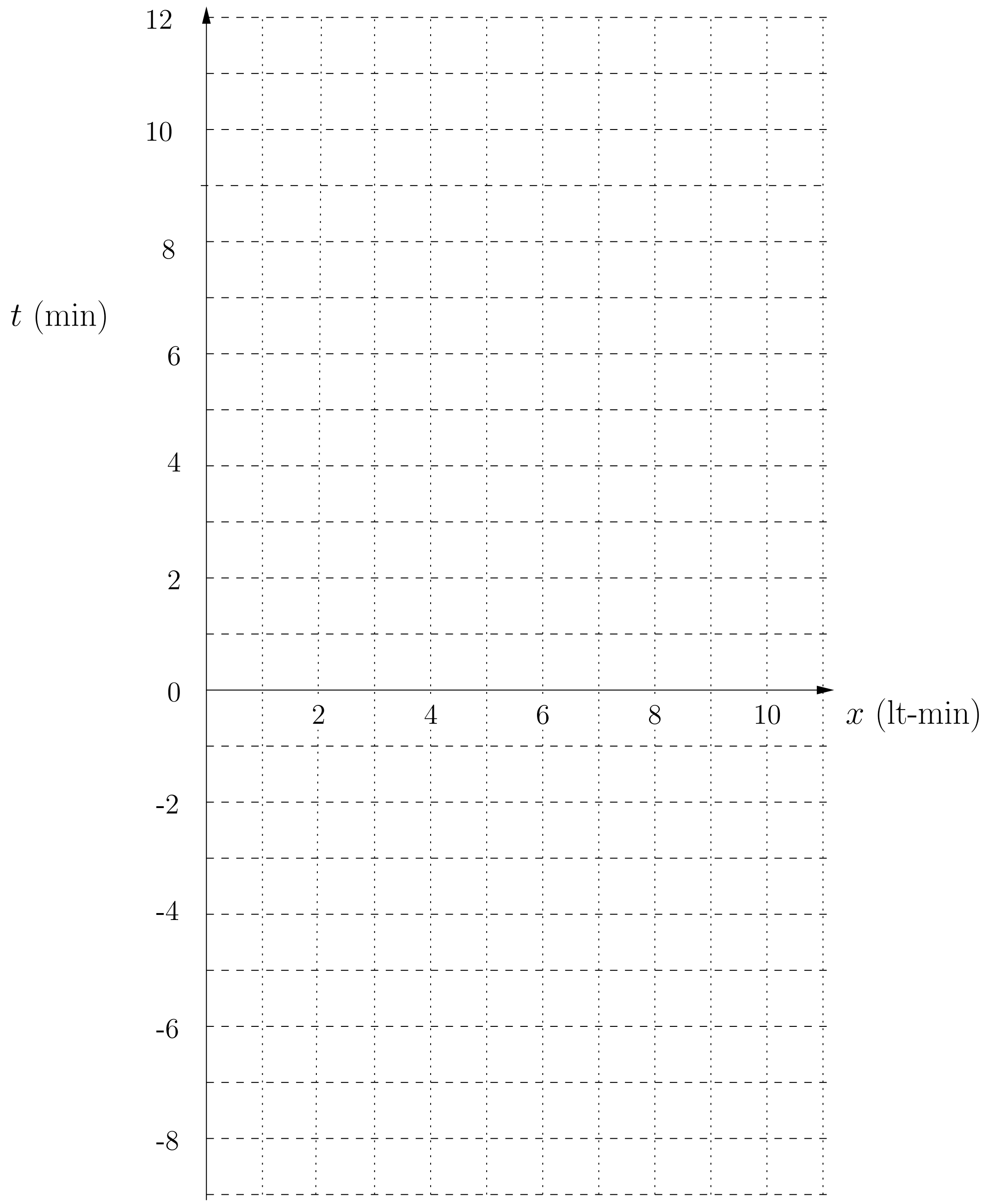

Figure 3.8: Figure for Problem 7.

7. The Earth is 8 lt-min from the Sun. An astronomer on Earth, looking through a telescope, notices the sudden appearance of a giant solar flare on the Sun's surface. At precisely that instant (when the astronomer detects the flare), a Klingon space ship whizzes over his head at speed $0.8c$, heading straight for the Sun.

(a) On the facing page, construct a spacetime diagram for this situation. Label the following three events: **A**: Klingon ship hits Sun, **B**: flare occurs on Sun, and **C**: Klingon ship passes Earth. (Note that event B is the occurrence of the flare *on* the Sun, not the detection of the flare by an astronomer on Earth.)

(b) Order the events A, B, C, from earliest to latest, according to Earth-based observers.

(c) Calculate the time intervals Δt between each pair of events (AB, AC, and BC), according to Earth observers.

(d) Calculate the interval $\Delta t'_{BA}$ between events B and A, but now according to Klingon ship observers.

(e) Classify each of the intervals as space-like, time-like or light-like.

8. Two spacecraft, *Aaaak* and *Blech*, are carrying aliens from the planet Zortox to Earth. Both spacecraft are traveling at a speed of $0.6c$ relative to the Earth, with *Aaaak* in front. In the Earth's frame, *Aaaak* and *Blech* are a distance 8.0 lt-s apart. At the moment that *Aaaak* passes Earth, the Zortoxians aboard *Aaaak* dump its garbage. At a time 10.0 s later (as measured by earthlings) *Blech* dumps its garbage.

(a) Draw a spacetime diagram in Earth's frame that includes worldlines for Earth, *Aaaak*, and *Blech*. Indicate and label the events corresponding to the garbage dumps.

(b) How far has *Blech* traveled during the 10.0 s between the two garbage dumps measured by earthlings?

(c) What is the distance between the two garbage dump events measured by earthlings?

(d) What is the distance between the two garbage dump events measured by the Zortoxians on board their ship?

(e) Calculate the time interval between garbage dump events measured by the Zortoxians on board their ship.

9. A train of rest length 40 lt-ns moves along the tracks at $0.8c$ and is struck by two lightning bolts. One bolt hits the front of the train and the other hits the back. According to observers on the tracks the bolts are simultaneous.

(a) How far apart did the lightning bolts strike according to observers on the tracks?

(b) According to riders on the train, how much time passed between the striking of the lightning bolts?

(c) According to riders on the train, which lightning bolt struck first?

10. Joe holds and lights a sparkler, and one minute later, it goes out. Cheri, riding in a rocket past these events, notes that, as measured in her frame, the sparkler burned for 100 seconds.

(a) How far apart in Cheri's frame did these two events (lighting and going out) occur?

(b) As measured by Cheri, how far did the lit sparkler travel, and how fast was it moving?

(c) As measured by Joe, how fast was Cheri traveling during the one minute of sparkler light, and how far did she travel?

11. The spacetime diagram in the figure shows the world lines of the Earth, a star, and a rocket, as well as several labeled events.

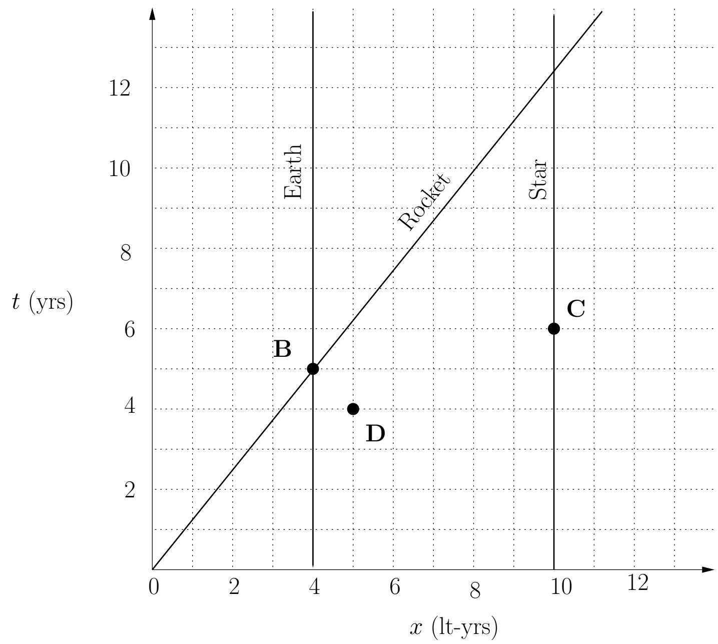

Figure 3.9: Figure for Problem 11.

(a) On the diagram, label as "A" the event "Rocket arrives at Star."

(b) Determine the speed of the Rocket, as measured by Earth observers.

(c) Determine the time between passing the Earth and passing the Star, as measured by Rocket observers.

(d) Determine the distance between the Earth and the Star, as measured by Rocket observers.

(e) Draw the world line of a lost satellite passing the Earth at the same time as the Rocket, but going away from the Star at a speed that is $\frac{1}{2}$ of the Rocket speed (as determined by Earth observers.) Label this line "Satellite."

(f) Order the events A, B, C, D, from earliest to latest, as observed in the Earth-Star reference frame.

(g) Order the events A, B, C, D, from earliest to latest, as observed in the Rocket reference frame.

(h) In some reference frame, the events C and D are simultaneous. In that frame, what is the distance between events C and D?

(i) Explain why no one could ever measure the proper time between events C and D.

Chapter 4

Relativistic Momentum and Energy

4.1 Introduction

So far in our discussions of relativity, we have taken a very simple principle — the *Principle of Relativity*, which states that the laws of physics are the same for observers in any inertial reference frame — and have used this principle to change completely our notions of how time and space work. But we are not yet done looking at the implications of this principle. It will be necessary to generalize the classical relations for energy and momentum to account for the strange behavior that we have already seen at relativistic velocities. And the new relativistic equations for energy and momentum carry significant implications that change our notions of energy and matter. This discussion leads to what is probably the most famous equation in all of physics — namely $E = mc^2$ — as well as the basic principle behind nuclear power generation. Of course, this is also the principle behind nuclear weapons, so it can be argued that this result fundamentally changed society. But this is also the principle behind energy generation in stars (including our own Sun); there would be no life on this planet without this principle.

But before we discuss relativistic energy and momentum, we will take a closer look at the concept of velocity. If observers in different reference frames don't agree on the results of measurement of lengths and time intervals, they won't agree on the results of their determinations of velocities.

4.2 Relativistic Velocity Transformations

Let's say that two spaceships leave Earth. The *USS Zaphod* leaves the Earth going in one direction with a speed $0.8c$ relative to Earth. The *USS Beeblebrox* leaves Earth going in the opposite direction with a speed $0.8c$ relative to Earth. What is the speed of the *Zaphod* from the reference frame

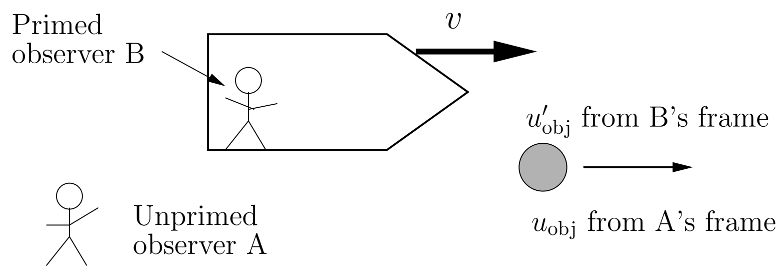

Figure 4.1: An object moving in the x-direction relative to both unprimed and primed frames. The speed of the object is measured to be u_{obj} from the reference frame of observer A and u'_{obj} from the reference frame of observer B.

of the *Beeblebrox*? Based on classical assumptions, you might expect the answer to be 1.6c. But this conflicts with Einstein's theory of relativity which states that no object can travel faster than the speed of light relative to any other observer. It is clear that it is necessary to replace classical laws for addition and subtraction of velocities with a more general, relativistic transformation.

A relativistic approach to velocity addition and subtraction was already hinted at earlier in Chapter 2. The principle of invariance of the speed of light requires that all observers (in any reference frame) measure the same speed for a pulse of light — we can't simply add and subtract velocities. On the other hand, common experience shows us that for non-relativistic speeds, simple addition and subtraction work fine. So, we need a velocity transformation relation that reduces to the classical result for small speeds, but which prevents anything from traveling faster than the speed of light. It turns out that this can be accomplished by taking the classical result and dividing by a relativistic correction that becomes significant (i.e., not just 1) for speeds close to c.

Figure 4.1 shows the scenario that we are discussing. Two reference frames are defined: an unprimed frame denoted by observer A and a primed frame denoted by observer B on a spaceship moving with a speed v relative to observer A. They are both measuring the speed of the same object. Observer A says the object is moving with a speed u_{obj} while observer B says the ball is moving with a speed u'_{obj}.

The question is: How are u_{obj}, u'_{obj} and v related? The answer is given by a *velocity transformation* equation,

$$\boxed{u_{\text{obj}} = \frac{u'_{\text{obj}} + v}{1 + u'_{\text{obj}} v/c^2}. \qquad (4.1)}$$

Equation (4.1) is used to relate an object's velocity in one frame to that as viewed in another frame.

We won't derive Eq. (4.1) rigorously here. Rather, note that if the object is a pulse of light, then $u'_{\text{obj}} = c$, and Eq. (4.1) reduces to

$$u_{\text{obj}} = \frac{c+v}{1+cv/c^2} = \frac{c+v}{1+v/c} = c\left(\frac{c+v}{c+v}\right) = c. \tag{4.2}$$

Both observers measure the object to have a speed c, so, the invariance of the speed of light is preserved in this transformation.

Note that the numerator in Eq. (4.1) is the result that you would get classically, whereas the denominator is a relativistic correction. Also, note that if either the object or the primed observer are traveling at speeds that aren't a significant fraction of the speed of light, then the denominator of Eq. (4.1) is very nearly 1, so we recover the classical result for "everyday" speeds.

We'll show how to work with this relation in the next example.

Example 4.1 Baseball velocity addition

Bucknell Bison baseball pitcher Christy Mathewson throws a blazing fastball while riding on a really fast train. From his reference frame (i.e., the train's frame) the ball moves toward the front of the train with a speed $u_{\text{ball}} = 0.7c$. The train itself is moving relative to the ground with speed $v = 0.8c$. How fast is the ball moving relative to someone on the ground?

Solution: Classically, the speed as viewed from the ground would be $u_{\text{ball}} + v$ or $1.5c$. (This is the numerator of Eq. (4.1).) But, of course, this isn't possible in a relativistic universe where nothing goes faster than c. Using Eq. (4.1) we find

$$u_{\text{ball}} = \frac{u'_{\text{ball}} + v}{1 + u'_{\text{ball}} v/c^2} = \frac{0.7c + 0.8c}{1 + 0.7 \times 0.8} = 0.96c. \tag{4.3}$$

Note that relativistic correction keeps the speed less than c.

Exercise: What if the ball were a beam of light? How fast would it be moving from the train's reference frame? How fast from the ground's reference frame? Show that Eq. (4.1) gives the correct result for this case.

If the problem gives you the speed as measured by the unprimed observer, you can use the following inverse transformations to get the speed as measured by the primed observer:

$$\boxed{u'_{\text{obj}} = \frac{u_{\text{obj}} - v}{1 - u_{\text{obj}} v/c^2}.} \tag{4.4}$$

You don't really need to write down these relations or try to figure out which speed is v, which speed is u_{obj}, and which speed is u'_{obj}. There is a very simple way of handling all of these problems. No matter which velocity you are looking for, the answer is always:

$$\frac{\text{classical result}}{\text{relativistic correction}},$$

where the relativistic correction is simply "1+(product of other two speeds divided by c^2)" or "1−(product of other two speeds divided by c^2).". You will be given two velocities, and you'll be looking for the third one. Just figure out the answer classically, then divide by the correction. The only question then is whether to use the "+" or "−" in the correction. The rule: if you *added* magnitudes of velocities in the numerator, then you use the "+" in the denominator, and if you *subtracted* magnitudes in the numerator, then you use the "−" in the denominator. This will take care of any velocity addition or subtraction that you need.

4.3 New definitions for energy and momentum

You have learned previously that in interactions among low velocity particles in which the only forces are the interparticle forces (i.e. no *external* forces), the total momentum $\sum_i m_i \vec{u}_i$ and the total mass $\sum_i m_i$ are conserved. (As in Chapter 3, we use the symbol u to refer to the velocity of some particle as viewed from a reference frame, reserving v for the velocity of the reference frame itself.) For example, when particle 1 collides with particle 2 and particles 3, 4, and 5 emerge from the point of collision, we have two conservation laws:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 + \vec{p}_5 \tag{4.5}$$

and

$$m_1 + m_2 = m_3 + m_4 + m_5 \qquad \textbf{Caution: Only valid classically!} \tag{4.6}$$

After Einstein discovered the velocity transformation law, Eq. (4.1) and (4.4), he recognized that the classical definition of momentum ($\vec{p} = m\vec{u}$) was incompatible with Eq. (4.5) and the Relativity Principle. That is, for

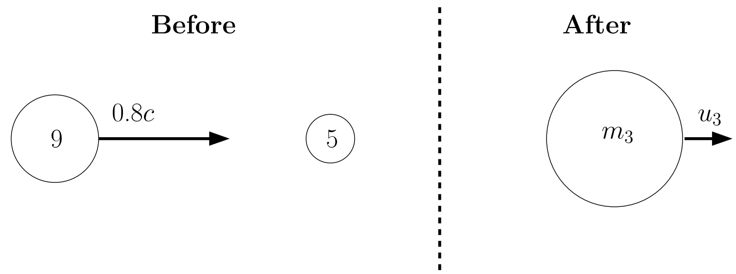

Figure 4.2: Collision discussed in Example 2.

a given collision, classical momentum could be conserved in one frame but not another. An example will illustrate this:

Example 4.2 Say goodbye to the classical expression for momentum.

(For this example, we use the natural units of MeV, MeV/c, MeV/c^2 and c. More detail on these units appears section 4.8.) Figure 4.2 shows a particle of mass $9\,\mathrm{MeV}/c^2$ and speed $0.8c$ striking a stationary particle of mass $5\,\mathrm{MeV}/c^2$, producing a single particle. (a) Calculate the mass and speed of the single particle after the collision. (b) Show that classical momentum is *not* conserved in the frame in which the final particle is at rest.

Solution: The classical laws Eqs. (4.5) and (4.6) would yield

$$\begin{aligned}\text{Eq. (4.5)} \quad &\Rightarrow \quad 9\,\mathrm{MeV}/c^2 \times 0.8c + 5\,\mathrm{MeV}/c^2 \times 0 = m_3 u_3 \\ &\Rightarrow \quad u_3 = \frac{7.2\,\mathrm{MeV}/c}{14\,\mathrm{MeV}/c^2} = 0.514c\end{aligned}$$

and

$$\begin{aligned}\text{Eq. (4.6)} \quad &\Rightarrow \quad 9\,\mathrm{MeV}/c^2 + 5\,\mathrm{MeV}/c^2 = m_3 \\ &\Rightarrow \quad m_3 = 14\,\mathrm{MeV}/c^2.\end{aligned}$$

Transform now to a frame in which the final particle is at rest. This clearly means that we should view the collision from a spaceship traveling with particle 3 at $0.514c$ to the right, relative to the original

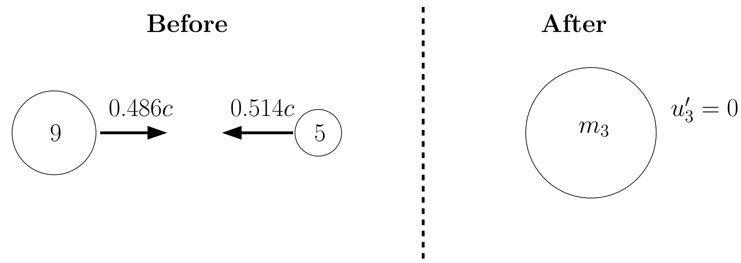

Figure 4.3: Collision discussed in Example 2 as viewed from rest frame of product particle.

observer. Equation (4.4) gives

$$\begin{aligned}
u'_3 &= \frac{u_3 - v}{1 - u_3 v/c^2} = \frac{0.514c - 0.514c}{1 - 0.514^2} = 0 \\
u'_5 &= \frac{u_5 - v}{1 - u_5 v/c^2} = \frac{0 - 0.514c}{1 - 0 \times 0.514} = -0.514c \\
u'_9 &= \frac{u_9 - v}{1 - u_9 v/c^2} = \frac{0.8c - 0.514c}{1 - 0.8 \times 0.514} = 0.486c
\end{aligned}$$

In the new primed frame, the collision appears as in Fig. 4.3. Checking the classical momentum conservation law in the new frame gives

$$9\,\mathrm{MeV}/c^2 \times 0.486c + 5\,\mathrm{MeV}/c^2 \times (-0.514c) = m_3 \times 0. \qquad (4.7)$$

But the left side of this equation here works out to be $1.80\,\mathrm{MeV}/c$ which is NOT equal to the right side (which is 0). So, classical momentum is not conserved in this new frame. Therefore, either (a) conservation of momentum isn't a valid law of physics; (b) the Relativity Principle (invariance of the laws of physics) is violated; or (c) we need a new definition for momentum.

You probably won't be surprised to hear that Einstein wasn't about to give up on the Relativity Principle because of this argument. After all, he had already redefined time and space to make the Principle work. And although the classical expression for momentum does not lead to invariance for high velocity collisions, there are attributes of particles involving their masses and velocities that do produce invariant conservation laws. These quantities are called relativistic momentum and relativistic energy, or more simply, momentum and energy. They are defined by

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \qquad \textbf{Definition of relativistic momentum}, \tag{4.8}$$

and

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \textbf{Definition of relativistic energy}. \tag{4.9}$$

Einstein was motivated to define momentum and energy in this way because conservation of momentum and energy defined in this new way are invariant, as we will show with an example below. Of course, motivation is all very nice, but the most compelling reason that the momentum and energy of a particle must be defined this way instead of in the classical way is because experiments with high-speed particles it is these new relativistic quantities that are conserved, and not those given by the classical definitions.

Let's explore this invariance by redoing Example 2 using Einstein's new definitions and the relativistic conservation laws:

$$\vec{p}_{\text{before}} = \vec{p}_{\text{after}} \tag{4.10}$$

$$E_{\text{before}} = E_{\text{after}} \tag{4.11}$$

Example 4.3

Figure 4.4 shows a particle of mass $9\,\text{MeV}/c^2$ and speed $0.8c$ striking a stationary particle of mass $5\,\text{MeV}/c^2$, producing a single particle of mass $16\,\text{MeV}/c^2$. You might not be too happy here with the final particle having a mass of $16\,\text{MeV}/c^2$, but hold on a little longer — we'll explain this shortly. (A little preview — this might be a good time to take a pen and scribble Eq. (4.6) out of existence.) In the next chapter, we'll learn more rigorously how to determine the correct attributes of the final particle. Here we just want to check the conservation laws. (a) Check the conservation of relativistic momentum and relativistic energy in the rest frame of the $5\,\text{MeV}/c^2$ particle. (b) Check the conservation of relativistic momentum and relativistic energy in the rest frame of the $16\,\text{MeV}/c^2$ particle.

Solution: (a) We will use the relativistic definitions and conservation laws given in Eqs. (4.8)–(4.11). Conservation of momentum gives

$$\frac{9\,\text{MeV}/c^2 \times 0.8c}{\sqrt{1-0.8^2}} + 0 = \frac{16\,\text{MeV}/c^2 \times 0.6c}{\sqrt{1-0.6^2}}, \tag{4.12}$$

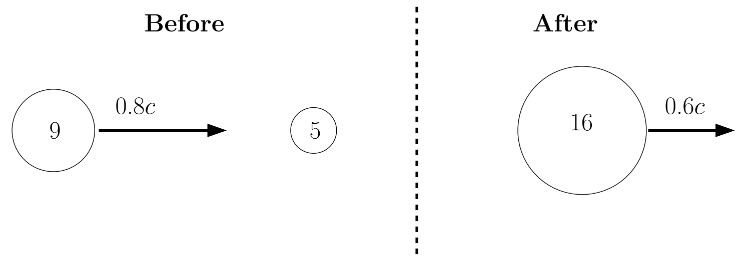

Figure 4.4: Collision discussed in Example 3.

and conservation of energy gives

$$\frac{9\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-0.8^2}} + \frac{5\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-0^2}} = \frac{16\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-0.6^2}}. \tag{4.13}$$

The momentum equation gives $12\,\mathrm{MeV}/c = 12\,\mathrm{MeV}/c$, while the energy equation gives $15\,\mathrm{MeV} + 5\,\mathrm{MeV} = 20\,\mathrm{MeV}$. So the conservation laws are satisfied in this frame.

(b) Now, transform to a frame in which the final particle is at rest, by viewing from a spaceship moving at $0.6c$ to the right. The velocity transformations give

$$\begin{aligned} u'_{16} &= \frac{u_{16}-v}{1-u_{16}v/c^2} = \frac{0.6c-0.6c}{1-0.6^2} = 0 && (4.14)\\ u'_5 &= \frac{u_5-v}{1-u_5v/c^2} = \frac{0-0.6c}{1-0\times 0.6} = -0.6c = -\frac{3}{5}c && (4.15)\\ u'_9 &= \frac{u_9-v}{1-u_9v/c^2} = \frac{0.8c-0.6c}{1-0.8\times 0.6} = \frac{5}{13}c \simeq 0.385c && (4.16) \end{aligned}$$

In this new frame, the collision appears as in Fig. 4.5. When we check

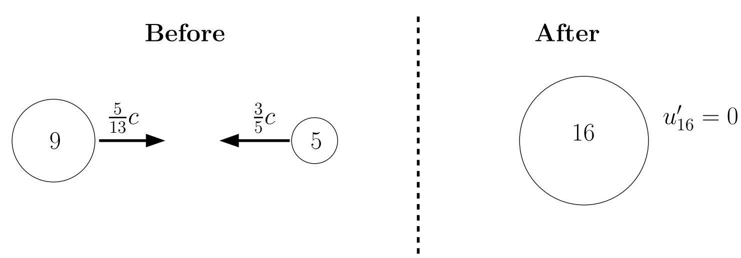

Figure 4.5: Collision discussed in Example 3 as viewed from rest frame of product particle.

the relativistic conservation laws in this new frame, we find:

$$\frac{9\,\mathrm{MeV}/c^2 \times 5c/13}{\sqrt{1-(5/13)^2}} - \frac{5\,\mathrm{MeV}/c^2 \times 3c/5}{\sqrt{1-(3/5)^2}} = \frac{16\,\mathrm{MeV}/c^2 \times 0}{\sqrt{1-0^2}} \tag{4.17}$$

$$\frac{9\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-(5/13)^2}} + \frac{5\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-(3/5)^2}} = \frac{16\,\mathrm{MeV}/c^2 \times c^2}{\sqrt{1-0^2}} \tag{4.18}$$

The momentum equation gives

$$\frac{15}{4}\,\mathrm{MeV}/c - \frac{15}{4}\,\mathrm{MeV}/c = 0, \tag{4.19}$$

which checks out, while the energy equation gives

$$\frac{39}{4}\,\mathrm{MeV} + \frac{25}{4}\,\mathrm{MeV} = 16\,\mathrm{MeV} \tag{4.20}$$

which also checks out. This means the conservation laws are true in both the original and the new frame, and the Relativity Principle is upheld with Einstein's new definitions.

This may be a nice argument on paper, but does it work in practice? Are relativistic momentum and energy, rather than classical momentum and mass, really conserved in particle interactions? The answer is an emphatic **YES**! In countless interactions observed in high-energy particle accelerators, relativistic momentum and energy are always found to be conserved.

4.4 Another Invariant

We now have relativistic expressions for energy and momentum. It turns out that these can be combined to form an invariant, just like we combined distance and time to get the invariant spacetime interval. Recall from chapter 4, we defined the square of the interval as

$$I^2 = (c\Delta t)^2 - (\Delta x)^2\,. \tag{4.21}$$

We can combine energy and momentum of an object or particle in a similar manner to get an invariant:

$$m^2 = \left(\frac{E}{c^2}\right)^2 - \left(\frac{p}{c}\right)^2. \tag{4.22}$$

Given any object or particle with energy E and momentum p as measured by an observer in a reference frame, this observer can easily calculate the value of m for that object. If a different observer is in another reference frame (labeled with "primes") and determines E' and p' for the same particle, she will find that if she calculates

$$(m')^2 = \left(\frac{E'}{c^2}\right)^2 - \left(\frac{p'}{c}\right)^2, \tag{4.23}$$

then she will get exactly the same value for m' that the first observer found for m. In other words

$$m = m', \tag{4.24}$$

or

$$E^2 - (pc)^2 = (E')^2 - (p'c)^2. \tag{4.25}$$

In the same manner that we used the invariant spacetime interval to relate Δx and Δt as measured in one reference frame to $\Delta x'$ and $\Delta t'$ in another reference frame, we can use the invariance of m to relate E and p in one frame to E' and p' in a different frame.

What is this invariant m? This is simply the mass of the object. In words, the invariance expressed in Eqs. (4.22–4.25) states that all observers agree about the mass of an object.[1]

We can rewrite Eq. (4.22) in a slightly more convenient form

$$E^2 - (pc)^2 = (mc^2)^2. \tag{4.26}$$

As we'll see in the next chapter, this is actually the most useful of all the energy and momentum relations. It applies to *every* particle in *every* situation. (We'll see that Eqs. (4.8) and (4.9) aren't very useful for 'particles' of light.) In the homework for tonight, you'll show that this relation comes very easily from the relativistic definitions for energy and momentum (4.8) and (4.9).

4.5 Rest Energy and Kinetic Energy

Let's look more closely at what we called the relativistic energy of a particle in Eq. (4.9). If the particle is at rest, so that $u = 0$, the energy reduces to $E = mc^2$, perhaps the most famous formula in all of physics. So we discover that a particle has energy even when it's not moving! This energy is called the *rest energy*, E_0. That is

$$E_0 = mc^2. \tag{4.27}$$

[1]You may hear people saying that in relativity, "a person's mass increases as (s)he approaches the speed of light." (In fact, it used to be common for physicists to say this.) This is an unfortunate claim. What they are doing is saying, "Well, since $p = mu/\sqrt{1-u^2/c^2}$, we're going to artificially call $m/\sqrt{1-u^2/c^2}$ the relativistic mass so that we can hold on to the $p = mu$ definition of momentum." There is no reason to do this — there is nothing in relativity that requires us to redefine mass and, in fact, mass is an invariant.

This is a remarkable result! Consider an apple with a mass of $100\,\mathrm{g} = 0.1\,\mathrm{kg}$. Einstein says that this apple has an energy at rest of $E_0 = 0.1\,\mathrm{kg} \times (3 \times 10^8)^2 = 9 \times 10^{15}\,\mathrm{J}$! This is a huge amount of energy. Let's compare this to a classical energy.

You know that the classical kinetic energy of a particle is $K_{\text{class.}} = \frac{1}{2}mv^2$. For the 0.1 kg apple falling at 10 m/s we get $K_{\text{class.}} = 5\,\mathrm{J}$. This is the energy that the apple has because it is in motion. In relativistic physics, kinetic energy is not expressed as $K = \frac{1}{2}mv^2$, but it is still defined as the energy that a particle has *because it is in motion*. The relativistic kinetic energy is the difference between a particle's energy when it is moving and its rest energy,

$$K = E - mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2. \tag{4.28}$$

This doesn't resemble the classical expression for kinetic energy $K_{\text{class.}} = \frac{1}{2}mv^2$ (or, since we use v for the velocity of the frame and u as the velocity of the particle, $K = \frac{1}{2}mu^2$). Since we know that the classical expression for kinetic energy is valid in the low-speed regime, the relativistic kinetic energy given be Eq. (4.28) must somehow reduce to the classical expression when the speed of the particle is small compared to the speed of light. We show the connection between the relativistic and classical forms of kinetic energy in the next example.

The relativistic expressions for energy and kinetic energy have amazing consequences. In collisions of high speed particles, or in radioactive decays, it is the total energy of a system of particles that is conserved, not the mass. In these processes, rest energy (mc^2) can be converted into kinetic energy, and vice versa. In a decay, the loss of a small amount of mass corresponds to the loss of a huge amount of rest energy, which will be manifested in a huge increase in kinetic energy.

Example 4.4 Relating relativistic and classical expressions for kinetic energy

Use the binomial approximation in Eq. (4.28) to find an approximate expression for K when u is much smaller than c, i.e., when $u/c \ll 1$. (The binomial expansion states that $(1-\epsilon)^{-1/2} \simeq 1 + \frac{1}{2}\epsilon + \cdots$ if ϵ is small.)

Solution: We write $1/\sqrt{1 - u^2/c^2}$ as $(1-u^2/c^2)^{-1/2}$ so that Eq. (4.28) becomes

$$\begin{aligned} K &= mc^2\left(1-u^2/c^2\right)^{-1/2} - mc^2 \\ &= mc^2\left(1+\frac{1}{2}\frac{u^2}{c^2}+\cdots\right) - mc^2 \\ &\simeq \frac{1}{2}mu^2, \end{aligned} \tag{4.29}$$

where in the last line we have assumed the classical limit in which $u/c \ll 1$. Thus we see that the classical expression for kinetic energy is only a low-velocity approximation to the correct expression, given by Eq. (4.28).

4.6 Photons: Particles with Zero Mass

How do we deal with the energy and momentum of light? As you will see next semester in PHYS 212, the same year that Einstein published his first paper on Special Relativity, he also proposed that light must be considered to be composed of particles which are now called photons. (This was the first of the three 1905 papers that we discussed at the beginning of Chapter 2.) Since light always travels at a speed c in a vacuum, then photons in a vacuum must travel at that speed regardless of the reference frame of the observer. But if we look back at Eqs. (4.8) and (4.9), we find that the denominators of both equations are zero for a particle moving at the speed of light, and clearly a photon cannot have infinite momentum or infinite energy.

The only way to resolve this dilemma is to postulate that photons are particles with zero mass ($m = 0$). Equations (4.8) and (4.9) are still not very useful in this case, since a fraction which has zero in both the numerator and denominator is undefined. However, with zero mass these equations no longer imply infinite energy and momentum for particles moving at light speed.

For a massless particle (such as photons), Eq. (4.26) can be rewritten for $m = 0$ as

$$E = |p|c \qquad \textbf{for massless particles only.} \tag{4.30}$$

Remember, Eq. (4.30) is valid only for massless particles.

4.7 More experimental evidence

Now that we have introduced the relativistic relations for energy and momentum, we can discuss some additional pieces of evidence that Einstein's theory of relativity is, in fact, correct. The following examples can be added to those presented in Chapter 2. Remember that if even one of these experiments had disagreed with Einstein's theory, then the entire theory would have to be thrown out since everything is internally consistent.

- **Particle accelerators**. As we already discussed in Chapter 2, subatomic particles are frequently accelerated in high energy experiments to speeds very close to c, but no one has ever managed to accelerate a particle with mass to a speed greater than c. There's more here, though: as the particle's speed (relative to the laboratory) gets closer and closer to c, the amount of energy that has to be added to increase the speed further gets larger and larger, diverging as the speed approach c. For instance, the amount of energy that needs to be added to accelerate a particle from $0.98c$ to $0.99c$ has been found experimentally to be much larger than the energy to accelerate the same particle from $0.97c$ to $0.98c$, and in fact, much larger than that predicted classically. As is the case with all other tests of relativity, the amount of energy to be added agrees perfectly with Einstein's predictions. Homework problem 7 investigates this further.

- **Collisions of high-energy particles**. When subatomic particles are slammed into each other with high energies, new particles are actually created that weren't there before the collision. These collisions are converting kinetic energy (KE) into matter, and this is done all the time in particle accelerators. (This is, in fact, the main tool that physicists use to study massive subatomic particles.) This is an experimental result that simply cannot be explained classically. Once again, though, the results agree perfectly with Einstein's theory. We will be discussing this in more detail in Chapter 5, and you will be doing (or have already done) a lab on this (the Relativistic Energy and Momentum lab).

- **Matter-to-KE conversions**. One of the most convincing and most dramatic tests of Einstein's theory of relativity occurred on July 16, 1945, in New Mexico when the first atomic bomb was exploded, converting matter into a horrifying amount of kinetic energy (don't forget that factor of c^2 in the famous $E = mc^2$ equation). Since then, there have been quite a few additional such demonstrations of Einstein's theory. (And again, the quantitative aspects of these demonstrations agree perfectly with the theory.)

  It isn't necessary to explode a bomb to convert matter into energy. Nuclear energy has found peaceful applications in the area of power generation. (There is a nuclear power plant in Berwick, PA, in fact, which you can see easily if you drive on Rt. 80 toward New Jersey, shortly after passing Bloomsburg). We will discuss nuclear power generation more in the next chapter (including fusion power — still being developed — which doesn't produce any long-lasting radioactive waste).

4.8 A note on units

When working with energy and momentum for small, subatomic particles (the ones that are most typically traveling at relativistic speeds), it is convenient to define a unit of energy called the "electron volt" (eV for short). One electron volt is the kinetic energy gained by an electron when accelerated through a 1 volt potential difference. (You'll learn more about this in PHYS 212.) Quantitatively, $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$. An analogous energy unit might be a "superball-meter" — the amount of kinetic energy gained by a superball when dropped 1 m.

For high energy particles, the energies can get into the thousands, millions or billions of electron volts, so we also define $1\,\mathrm{keV} = 10^3\,\mathrm{eV}$, $1\,\mathrm{MeV} = 10^6\,\mathrm{eV}$, $1\,\mathrm{GeV} = 10^9\,\mathrm{eV}$.

Units for mass and momentum are also defined in terms of energy in relativity. For mass, we use eV/c^2 — "electron volts per c^2" — or keV/c^2, MeV/c^2, GeV/c^2. For momentum, we use eV/c (or keV/c, MeV/c, GeV/c). For example, an electron has a mass of $511\,\mathrm{keV}/c^2$; conceptually, this means that an electron has a rest energy of 511 keV, or that its mass — if converted completely into kinetic energy — would produce 511 keV of kinetic energy.

Warning: when using these units, don't throw any numbers in for the c — it is part of the unit. So, the mass of an electron should be written as "$511\,\mathrm{keV}/c^2$" (or $0.511\,\mathrm{MeV}/c^2$), **not** as $\cancel{511\,\mathrm{keV}/(3.0 \times 10^8\,\mathrm{m/s})^2}$ or $\cancel{511\,\mathrm{keV}/(1\,\mathrm{lt\text{-}s/s})^2}$.

4.9 Summary

The various equations introduced in this chapter are summarized as in Table 4.1.

Table 4.1: Relativistic formulas for energy and momentum.

Definition of momentum:	$\vec{p} = \dfrac{m\vec{u}}{\sqrt{1-u^2/c^2}}$	**Definition of energy:**	$E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}}$
Energy in terms of momentum and mass:	$E^2 = p^2c^2 + m^2c^4$	**Velocity in terms of energy and momentum:** (See Problem 4.6.)	$\vec{u} = \dfrac{\vec{p}c^2}{E}$
Definition of kinetic energy:	$K = E - mc^2$	**Energy in terms of momentum for zero-mass particle:**	$E = \lvert\vec{p}\rvert\, c$

Problems

1. Duck Dodgers hops in his spaceship and leaves the Earth at a speed $0.6c$ in an attempt to reach the newly discovered Planet X before aliens from Mars.

(a) Mission control on Earth sends an encoded message (a flashing beacon) to Duck Dodgers warning him about the progress of the Martian ship. The light pulses travel at a speed c relative to observers on the Earth. How fast are the pulses traveling relative to Duck Dodgers?

(b) Duck Dodgers doesn't understand the message that he received, so he sends a radio message back toward the Earth asking for clarification. The radio signal is traveling at a speed c relative to the Duck. How fast is the signal traveling relative to observers on the Earth?

(c) The radio message is intercepted by the Martian who is behind Duck Dodgers but traveling in the same direction at a speed $0.8c$ relative to the Earth. How fast is the radio message going relative to the Martian?

(d) The radio message is also intercepted by one of the Martian's monsters who is traveling back toward the Earth to attack. The monster is traveling at a speed $0.9c$ relative to the Martian. How fast is the radio signal relative to the monster?

2. A particle travels at speed $0.50c$ relative to Captain Kirk. Mr. Spock is traveling at a speed $0.70c$ relative to Captain Kirk, in the same direction as the particle. Calculate the velocity of the particle relative to Mr. Spock.

3. A proton's velocity is measured at $0.6c$ relative to an observer on earth, and $0.8c$ relative to an observer passing by in a rocket. Determine the speed of the rocket relative to earth. (There are two possible correct answers that correspond to two different physical situations.)

4. After traveling on vacation to Betelgeuse to witness a supernova, Fred and Ethel are returning home, traveling at a speed $0.75c$ relative to and toward the Earth. Ethel is particularly anxious to get home and see her new great-great-great-great-great-great-great-great grandson, so she hops on the emergency shuttlecraft, which leaves Fred's ship traveling at a speed of $0.75c$, relative to Fred. How fast is Ethel's shuttle traveling relative to the Earth?

5. A particle of mass $3m$, moving at speed $0.60c$ in the positive x-direction, collides with and sticks to a particle of mass $2m$ originally at rest. Assume a head-on collision.

(a) Calculate the initial total momentum before impact, using the classical definition, $p = mu$, for momentum.

(b) Assuming conservation of mass as well as classical momentum, find the velocity of the composite particle of mass $5m$ after the collision.

(c) Now transform to a primed frame in which the particle of mass $3m$ is at rest. Use the relativistic velocity transformation to compute the velocities u'_{3m}, u'_{2m}, and u'_{5m} in the primed frame.

(d) Still in the primed frame, check whether momentum mu' is conserved by computing the total momentum before the collision and the total momentum after the collision.

6. Show that the velocity of a particle expressed in terms of relativistic energy and momentum is $u = pc^2/E$.

7. An electron is accelerated from a velocity $u_1 = 0.98c$ to a velocity $u_2 = 0.99c$. Calculate the change in the electron's kinetic energy in units of MeV. ($m_{\text{electron}} = 0.511\,\text{MeV}/c^2$.)

8. Electron A has a total energy of 1.0 MeV. Electron B has a kinetic energy of 0.25 MeV. Electron C has a kinetic energy of 0.75 MeV. Electron D has a momentum of $1.0\,\text{MeV}/c$. For each of the electrons A through D, determine its energy, momentum, kinetic energy, and speed.

9. A certain particle has a total energy of 1.20 MeV and a momentum of $0.95\,\text{MeV}/c$. Calculate the particle's mass, kinetic energy, and velocity.

10. Compute the momentum and velocity of a proton that has a total energy equal to 7 times its rest energy. ($m_{\text{proton}} = 938\,\text{MeV}/c^2$.)

11. Combine Eqs. (4.8) and (4.9) to derive Eq. (4.26).

12. Show, from Eq. (4.26) and the result of problem (6) that any massless particle moves at the speed of light and that if a particle moves at the speed of light it must have zero mass.

13. A proton (mass $938\,\text{MeV}/c^2$) is traveling at velocity $0.60c$ in the $+x$ direction relative to a spaceship which itself is traveling at velocity $0.80c$ in the $+x$ direction relative to Earth. Calculate the velocity and then the energy and momentum of the proton as measured in the Earth frame.

14. A particle's energy and momentum in one frame are 41 MeV and 40 MeV/c respectively. Find the particle's energy and momentum as measured in a different frame in which the particle's speed is $u' = 0.8c$.

15. Given a particle with $E_A = 21\,\text{MeV}$ and $p_A = 15\,\text{MeV}/c$ as measured in reference frame **A**, and $E_B = 20\,\text{MeV}$ as measured in frame **B**, determine the mass m_B and momentum p_B of the particle as measured in frame **B**.

16. The Fermi National Accelerator Laboratory (Fermilab) is located outside Chicago, Illinois, and is one of the world's largest particle accelerator facilities. At Fermilab, protons (mass $938\,\text{MeV}/c^2$) are given huge amounts of energy and achieve velocities that are nearly the speed of light.

(a) Prior to a recent upgrade, protons at Fermilab could reach speeds that were only 163 m/s slower than the speed of light. How much energy is required to get a proton from rest up to the speed $u = c - 163\,\text{m/s}$?

(b) After the upgrade, the protons were able to reach the speed $u = c - 132\,\text{m/s}$ (a whopping increase of 31 m/s). How much additional energy is required to get this 31 m/s increase?

(c) The Large Hadron Collider currently being developed at CERN (a particle accelerator facility in Europe) is designed to get protons up to an energy of 7.0 TeV. ($1\,\text{TeV} = 10^{12}\,\text{eV}$). Determine u/c, the ratio of the speed of the proton to the speed of light (note that u/c is NOT equal to 1!)

17. A certain J-boson has mass of $150\,\text{MeV}/c^2$, speed of $0.8c$, and total energy of 250 MeV. Determine the J-boson's momentum and kinetic energy.

18. A proton (mass $938\,\text{MeV}/c^2$) traveling down a beam pipe at Fermilab is determined to have kinetic energy of 1.2 GeV. Determine this proton's momentum and speed.

19. An evil genius fires a rocket into a star, destroying the star. Ten minutes later, as measured in the reference frame of the star, debris thrown out from the explosion demolishes a populated planet a distance 7 lt-min from the star, as measured in the star/planet frame. Simultaneous with the explosion (according to the star/planet reference frame), the Starship Enterprise is 5 lt-min from the star, on the opposite side from the planet. The Enterprise is heading toward the star/planet system with a speed of $0.6c$ relative to the planet and star.

(a) Draw a spacetime diagram for this situation. Take the star/planet reference frame as the unprimed frame, and draw the world lines of the star and the planet, showing that they are 7 lt-min apart. Indicate the destruction of the star as event A on your spacetime diagram, and indicate the destruction of the planet as event B on your spacetime diagram. Also, draw the world line for the debris sent from the star to the planet.

(b) Draw the world line of the Starship Enterprise on your spacetime diagram, showing both the velocity of the spaceship and the correct location of the Enterprise when the star explodes.

(c) Calculate the speed of the matter thrown out from the explosion as measured by Starfleet officers on board the Enterprise.

(d) Calculate the time interval between the explosion of the star and the destruction of the planet, as measured by Starfleet officers on board the Enterprise. (Hint: use your result from (c) to determine an expression for the distance between events — according to the Enterprise — in terms of the unknown time between events.)

20. Are momentum and energy conserved?

A mad scientist at rest in a lab on the earth is colliding particles together to make more massive particles. She creates head-on collisions of particles with a mass of $4\,\mathrm{GeV}/c^2$ and speed $0.6c$ to create particles at rest with a mass of $10\,\mathrm{GeV}/c^2$.

In the rest frame of the scientist, the $4\,\mathrm{GeV}/c^2$ particles have equal and oppositely directed momenta, and the $10\,\mathrm{GeV}/c^2$ particle is at rest, so momentum is conserved.

An observer in a rocket flying over the lab at a speed $v = 0.6c$ views this experiment.

(a) Test whether **relativistic** momentum is conserved according to the observer in the rocket. Your test should include detailed numerical calculations demonstrating the conservation (or non-conservation) of **relativistic** momentum.

(b) Test whether **relativistic** energy is conserved according to the scientist in her lab. Your test should include detailed numerical calculations demonstrating the conservation (or non-conservation) of **relativistic** energy.

(c) Test whether **relativistic** energy is conserved according to the observer in the rocket. Your test should include detailed numerical calculations demonstrating the conservation (or non-conservation) of **relativistic** energy.

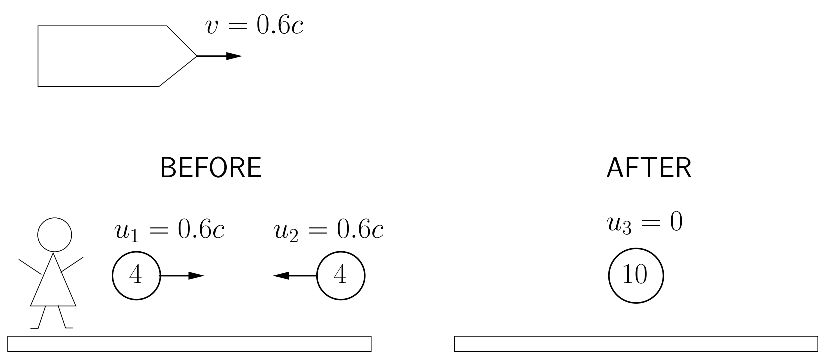

Figure 4.6: Figure for Problem 20.

Chapter 5

Applications of the Relativistic Conservation Laws

5.1 Introduction

You should now understand why Einstein's postulates require new definitions of momentum and energy. The classical momentum is not conserved, nor in general is the total mass of the particles in an interaction. In place of these, relativistic momentum and relativistic energy are conserved, and they are conserved in any inertial frame.

In this chapter, we apply these new, relativistic conservation laws to analyze collisions and decays of subatomic particles. The key result in these applications is the ability for matter to be converted into kinetic energy and vice-versa. In relativistic collisions, the amount of matter that you start with is not the same as the amount of matter that you finish with! We also discuss the principles behind nuclear fission and nuclear fusion.

5.2 Changes of Rest Energy

Much of the light you see comes from changes in rest energy of atoms. Examples are sunlight, light from a candle flame, a lightning flash, light emitted by a fluorescent lamp, light from the phosphor coating on the screen of a television set or a video monitor, and laser light. In all these examples, the basic mechanism is that an atom in an "excited" state releases its energy in the form of a photon, with the atom going into its ground (lowest possible) state, or into an excited state of lower energy. We can represent the emission process by the simple reaction equation

$$\mathbf{A}^* \to \mathbf{A} + \gamma. \tag{5.1}$$

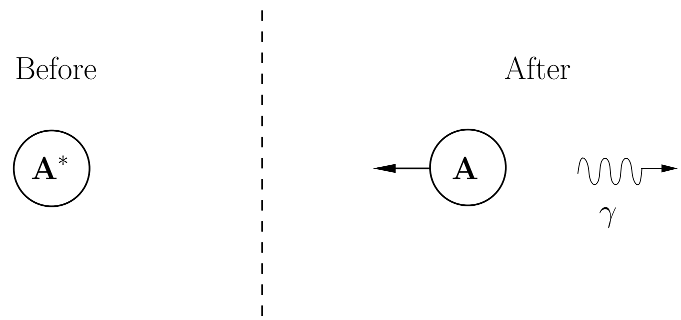

Figure 5.1: An excited atom emits a photon and recoils.

Here $\mathbf{A}^*$ represents the excited atom, $\mathbf{A}$ the atom in its ground or lowest state, and γ (Greek *gamma*) the photon.

In Fig. 5.1, the excited atom is shown at rest, so all of its energy is rest energy and it has no momentum. But the photon has energy, and from the relation $E = pc$, it also has momentum. And because momentum must be conserved, the atom recoils. We can write the conservation of energy equation for the reaction in Eq. (5.1) as follows

$$\text{Rest Energy of } \mathbf{A}^* = \text{Energy of } \mathbf{A} + \text{Energy of photon.} \tag{5.2}$$

Because both the kinetic energy of $\mathbf{A}$ and the photon energy are positive numbers, the rest energy (i.e., the mass) of the excited-state atom must be greater than that of the ground-state atom. Therefore, in the emission process rest energy, i.e., mass, is converted to kinetic energy.

When light is absorbed by an atom, exactly the opposite effect occurs. The atom begins in its ground state, absorbs the photon energy and goes into an excited state. Again, by conservation of energy, the excited atom must have more rest energy than the ground-state atom.

Another everyday example of changing rest energy occurs in chemical reactions. For example, the reaction for the oxidation of a carbon atom

$$\mathrm{C} + \mathrm{O}_2 \to \mathrm{CO}_2 \tag{5.3}$$

is known to release energy in the form of one or more photons. Therefore the sum of the masses of C and O_2 must be greater than the mass of the carbon dioxide molecule. The change in rest energy in the case of chemical reactions is typically on the order of 1 eV (or 1.6×10^{-19} J). Much larger energies, on the order of 1 MeV, are involved in nuclear reactions. An example of a nuclear reaction is the decay of a neutron into a proton, an electron, and an electron antineutrino:

$$n \to p + e^- + \bar{\nu}_e. \tag{5.4}$$

Here the excess mass of the neutron over the mass of the proton plus electron (the electron antineutrino has very small mass) is converted to the kinetic energy of the three reaction products.

Another important example of changes in rest mass is the production of new particles in a high energy particle accelerators. In these accelerators high-speed particles are shot at target particles and some of the kinetic energy of the incoming particles is converted to rest energy. In this way hundreds of new particles, most with lifetimes between 10^{-10} and 10^{-23} s, have been produced. You'll learn more about these new particles next semester in PHYS 212.

5.3 General Strategy for Applying the Relativistic Conservation Laws

In a typical problem you are given information about the particles before an interaction and asked to compute certain properties of the outgoing particles after the interaction. You do this by writing down equations that express the fact that the sum of the incoming momenta is equal to the sum of the outgoing momenta and the sum of the incoming energies is equal to the sum of the outgoing energies. What quantities should be used in writing these equations? Here is some time-saving advice.

> Always write the conservation of momentum and conservation of energy equations in terms of momentum and energy or mass variables, never in terms of velocity or kinetic energy.

This rule keeps the algebra as simple as possible — it gets around having to solve simultaneous equations with the $\sqrt{1 - v^2/c^2}$ terms that can make the algebra messy. For example, if you are given the velocity of one or more particles in the problem statement, first calculate the momentum and energy of each particle from the given velocities.

A second piece of advice:

> When working with "eV" units (e.g., MeV for energy, MeV/c for momentum, MeV/c^2 for mass), don't ever put any numbers in for the speed of light c. Just leave it as "c." The units will then automatically take care of themselves.

For example, if you have an motionless electron, its energy can be obtained from $E^2 = p^2c^2 + m^2c^4$. For a motionless electron $p = 0$, so $E = mc^2 = 0.511\,\text{MeV}/c^2 \times c^2 = 0.511\,\text{MeV}$. (See section 4.8.)

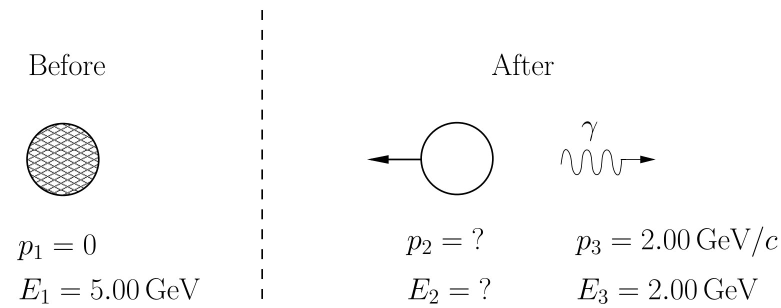

Figure 5.2: Emission of a photon by a nucleus as discussed in Example 1.

Example 5.1 Emission of a photon by a nucleus.

An excited atomic nucleus, of mass $5.00\,\mathrm{GeV}/c^2$ and at rest, as in Fig. 5.2, decays to its ground state by emitting a photon of energy $2.00\,\mathrm{GeV}$. Calculate the recoil velocity and mass of the ground-state nucleus.

Solution: First draw a picture, and label each particle with its value of energy and momentum. Before the decay the excited nucleus has zero momentum because it is at rest. And from $E^2 = p^2c^2 + m^2c^4$, with $p = 0$, we know its energy is the same as its rest energy, namely $5.00\,\mathrm{GeV}$.

After the decay the ground-state nucleus recoils with unknown energy and momentum, E_2 and p_2. Also, the emitted photon has an energy of $2.00\,\mathrm{GeV}$, as specified in the problem. And because the photon's mass is zero its momentum has the same numerical value as its energy. Notice that in the diagram there are two unknowns, the energy and momentum of the recoiling ground-state nucleus. We plan to solve for these two unknowns with two equations, the energy and momentum conservation equations.

Looking at the diagram, we write down the energy conservation equation in terms of the symbols and numerical quantities shown in the diagram:

$$5.00\,\mathrm{GeV} = E_2 + 2.00\,\mathrm{GeV}.$$

Similarly, we write the momentum conservation equation in terms of symbols and numerical quantities shown in the diagram:

$$0 = p_2 + 2.00\,\mathrm{GeV}/c.$$

From these conservation-law equations we easily solve for the energy and momentum of the recoiling nucleus to obtain $E_2 = 3.00\,\mathrm{GeV}$ and

$p_2 = -2.00\,\mathrm{GeV}/c$. Now that we've obtained expressions for the energy and momentum of the recoiling ground-state nucleus, we can find its velocity using a formula from Problem 4.6 (and from Table 4.1):

$$u_2 = \frac{p_2c^2}{E_2} = \frac{(-2.00\,\mathrm{GeV}/c) \times c^2}{3.00\,\mathrm{GeV}} = -\frac{2}{3},$$

and its mass from

$$\begin{aligned} m_2c^2 &= \sqrt{E_2^2 - p_2c^2} \\ &= \sqrt{(3.00\,\mathrm{GeV})^2 - (2.00\,\mathrm{GeV}/c)^2 \times c^2} \\ &= \sqrt{5}\,\mathrm{GeV}, \end{aligned}$$

so the mass is $m_2 = \sqrt{5}\,\mathrm{GeV}/c^2 \simeq 2.24\,\mathrm{GeV}/c^2$.

Notice that even though we were asked to find the velocity and mass of the recoiling nucleus, we didn't use these variables in our analysis until the very end, after we solved for its energy and momentum.

5.4 Nuclear masses, fusion and fission

A particularly important application of the material in this chapter is nuclear power generation. There are two main approaches: fusion and fission. Nuclear fusion involves the merging (fusing) of two light nuclei (usually hydrogen) to form a more massive nucleus (usually helium), whereas fission[1] involves the splitting of a very massive nucleus (e.g., uranium) into two or more lighter nuclei. For the process to release kinetic energy, conservation of relativistic energy requires that the end product(s) have a smaller total mass than the initial nucleus or nuclei.

Figure 5.3 shows a plot of the masses of the elements, divided by the total number of protons and neutrons (nucleons) in the nucleus of each atom. This plot is very illuminating when considering fusion and fission processes. The fusion of two ^{2}H nuclei to form a single ^{4}He nucleus results in a lower overall mass, since the number of nucleons does not change; consequently,

[1]The story of how fission was discovered is quite interesting. It starts with Lise Meitner and Otto Hahn, who conducted "transuranium" experiments where they bombarded massive nucleii with the goal of making **more** massive nucleii (more massive than uranium). But the experiments produced puzzling results. Meitner – with her nephew Otto Frisch – later provided an explanation. Instead of making more massive nucleii, they realized that the nucleii were breaking up with a resulting loss of mass, and Meitner used Einstein's theory of relativity to explain the increase in energy observed in the process. Meitner – who was inexplicably overlooked for the Nobel Prize for the fission discovery – also discovered a radiation process which was named the *Auger effect* after a scientist who also discovered this process, a couple of years **after** Meitner had discovered it.

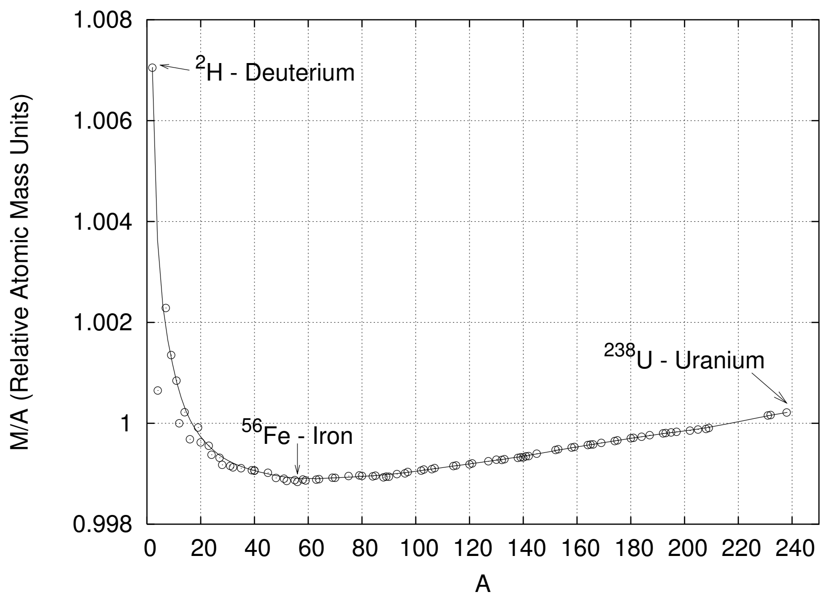

Figure 5.3: Plot of mass per nucleon (proton and neutrons) for the elements versus the number of nucleons A. (Data from NIST: http://physics.nist.gov/PhysRefData/Compositions/)

this process releases kinetic energy. On the other hand, elements with large atomic number A have a larger mass/nucleon than those with intermediate values of A; consequently, kinetic energy can also be released by splitting up one of these heavier atoms (fission).

Of the two processes — fission and fusion — fission is a much easier process to achieve in the laboratory or in industrial processes. Many large nuclei are naturally unstable, e.g., ^{235}U can spontaneously decay via the fission process $^{235}\mathrm{U} \rightarrow {}^{134}\mathrm{Xe} + {}^{100}\mathrm{Sr} + {}^{1}n$. Practically, then, the issue boils down to setting things up such that the process can be accelerated when desired, and can be inhibited when unwanted. From that perspective, the concept of a chain reaction is relevant. The idea of combining multiple nuclear fissions into chain reactions —which was pioneered by Lise Meitner, Otto Hahn, Fritz Strassmann, and Enrico Fermi in the 1930s — is straightforward: if the neutrons that are released in a fission process bombard another nearby (unstable) nucleus, they can trigger the fission of that nucleus as well. Practically, all that is needed is a large enough density of the unstable nucleus (e.g., ^{235}U) and a chain reaction will start. This idea was pursued by the Manhattan Project in the 1940s to develop an atomic bomb, the detonation of which was achieved by explosively compressing a uranium sample to increase its density above the critical value for a chain reaction. Alternatively, the strength of the fission chain reaction can be controlled by absorbing some of the neutrons produced in the fission reaction. Graphite rods (which absorb neutrons) are commonly used to "moderate" the reaction in this way to allow the reaction to proceed in a controlled manner in power generators.

Nuclear fission power has a few serious drawbacks: (a) the fuel (uranium, plutonium, etc.) is expensive and limited in supply. If society were to switch entirely to uranium-fission-based power generation, it is estimated that the supply of uranium would last for only 50-100 years. (b) The by-products of the fission reaction are nuclei which themselves are unstable and radioactive; consequently, the material poses a health hazard unless properly stored. (Note: some countries use techniques to extract additional energy from this nuclear "waste.")

Another drawback of nuclear fission reactors — which is diminishing with improved technology — is the concern that they could "melt down" and release massive amounts of radiation (this actually happened to the Chernobyl 4 reactor in the Soviet Union in 1986). This threat has been lessened recently by the development of much better systems, including a "melt-proof" system with an expandable core; if the temperature of the core exceeds a defined value, the core expands, dropping the density of the fissile material down below its critical value and stopping the chain reaction. (This works even if all cooling is stopped.) But even this system isn't perfect, as there is always the concern that a terrorist attack or gross human error could result in the release of disastrous amounts of radioactive waste into the environment.

In contrast to fission reactors, nuclear fusion reactors use water as their fuel (actually the ^{2}H isotope of hydrogen, also called deuterium, which is found in small amounts in water) and produce helium as a by-product, so waste disposal is less of a problem.[2] The nuclear energy production is also much more efficient for this process than for fission, as can be inferred from the steepness of the curve in Fig. 5.3. It is estimated that there is enough ^{2}H (deuterium) in ocean water to power the world's needs for many thousands of years (if not millions). In fact, nuclear fusion is the power source in stars, including our own Sun. It can be argued that almost all of the Earth's energy sources can be traced back to nuclear fusion.

Nuclear fusion is not without its problems, though. Specifically, it is very difficult to achieve in a controlled manner. Making a fusion bomb unfortunately isn't that difficult (relatively speaking), as a fission explosion can be (and has been) used to compress hydrogen together and cause explosive fusion. But to achieve a controlled fusion reaction is a very difficult procedure that will require a significant amount of ingenuity over the next few decades. If physicists and engineers manage to overcome the technical

[2] Some radioactive tritium is released in the process as well, but it is short-lived with a half-life of only 12 minutes; consequently there is no long-term waste problem with the tritium. The only long-term waste would be the activated material in the reactor containment vessel itself.

hurdles, earth-based fusion reactors could prove to be an important source of abundant and relatively clean power. And in the mean time, we can continue to make use of the giant fusion reactor in space that beams energy down to earth.

Problems

1. A nucleus with mass $\sqrt{5}\,\mathrm{GeV}/c^2 \simeq 2.24\,\mathrm{GeV}/c^2$ in its ground state and initially at rest absorbs a photon of energy E_1. After absorbing the photon, the nucleus is raised to an excited state, with mass $5.00\,\mathrm{GeV}/c^2$, and recoils with unknown momentum p_3.

(a) Draw a picture of this interaction.

(b) Write down the two conservation laws in terms of E_1, p_3, and given numerical values.

(c) Solve for E_1 and p_3.

2. Calculate the speed of the recoiling excited-state nucleus in problem 1. Compare with the case of photon emission, done as example 1 in the text. Are the recoil velocities the same for emission and absorption?

3. A particle of mass $m_1 = 9\,\mathrm{GeV}/c^2$ and energy $E_1 = 15\,\mathrm{GeV}$ approaches a stationary particle of mass $m_2 = 5\,\mathrm{GeV}/c^2$. The particles collide and form a single particle of mass m_3. Determine m_3 by using the conservation laws.

4. An incident proton, mass $m = 938.27\,\mathrm{MeV}/c^2$, strikes a target proton at rest with just enough energy to create an electron-positron pair. (The two protons are still present after the collision.) A positron is the antiparticle of an electron; both the electron and positron have masses $0.511\,\mathrm{MeV}/c^2$. Calculate the minimum energy needed by the incident proton in the frame where the target proton is initially at rest. (Hint: After the collision, both protons and the electron-positron pair all move together with the same velocity.)

5. A particle of mass $3.0\,\mathrm{MeV}/c^2$ and momentum $1.0\,\mathrm{MeV}/c$ hits and sticks to a particle of mass $2.0\,\mathrm{MeV}/c^2$, initially at rest.

(a) Find the mass of the composite particle and its velocity.

(b) How much kinetic energy is converted to mass?

6. A deuteron (mass $1875.61\,\mathrm{MeV}/c^2$) absorbs a photon and splits into a proton (mass $938.27\,\mathrm{MeV}/c^2$) and a neutron (mass $939.57\,\mathrm{MeV}/c^2$). What is the minimum energy of the photon required to do this?

7. The easiest and most immediately promising nuclear reaction to be used for fusion power is the fusion of a deuterium (^{2}H) nucleus, with mass $1875.61\,\mathrm{MeV}/c^2$, and a tritium (^{3}H) nucleus, with mass $2808.92\,\mathrm{MeV}/c^2$. The fusion reaction produces a ^{4}He nucleus of mass $3727.38\,\mathrm{MeV}/c^2$, and a free neutron of mass $939.57\,\mathrm{MeV}/c^2$:

$$^2_1\mathrm{H} + ^3_1\mathrm{H} \to ^4_2\mathrm{He} + n.$$

(a) Is rest energy converted to kinetic energy or vice-versa? Support your answer.

(b) Calculate the amount of energy that is converted.

8. In a fission process, a slow neutron causes a uranium nucleus (mass $= 218,943.42\,\mathrm{MeV}/c^2$) to split into a barium nucleus (mass $= 131,261.73\,\mathrm{MeV}/c^2$) and a krypton nucleus (mass $= 85,629.32\,\mathrm{MeV}/c^2$), plus two excess neutrons (actually 3 including the original neutron, but that is present before the process as well), each of mass $939.57\,\mathrm{MeV}/c^2$. Calculate the energy converted from mass to kinetic energy in this process.

9. A photon of momentum $2.0\,\mathrm{MeV}/c$ traveling along the positive x-axis strikes a particle of mass $4.0\,\mathrm{MeV}/c^2$, which is initially at rest. The result of the collision is simply two photons: photon γ_1 travels backward, along the negative x-axis and photon γ_2 travels forward, along the positive x-axis.

(a) Draw before and after pictures of the interaction.

(b) Find the energies of γ_1 and γ_2 after the collision.

10. Based on the plot in Fig. 5.3, answer the following questions:

(a) Why is a fusion reaction a more efficient power source ("pound for pound") than a fission reaction?

(b) A supermassive star goes supernova after it has run out of hydrogen to fuse, at which point it starts fusing helium into heavier elements, then fusing those into heavier elements, etc., until it gets to iron (Fe). Up until this point, the fusion reactions produce kinetic energy and heat, maintaining the star. But after the star has fused its materials into iron, it stops producing kinetic energy, collapses very suddenly and goes "Blammo!!" (This is a supernova.) What is so special about iron, and why can't the star produce additional kinetic energy after this point?

11. A flower absorbs a (higher energy) photon of ultraviolet light and emits a (lower energy) red photon. Describe what happens to the mass of the flower first when it absorbs the ultraviolet photon and then later when it emits the red photon. Would you expect any mass changes during this process to be noticeable? Explain why or why not.

12. It's not just nuclear reactions that involving converting mass-energy to kinetic energy. Chemical reactions, such as combustion, also do this, although the effect on the masses is hardly noticeable. For instance, when a car burns one gallon of gasoline, 132 MJ of energy is released. Consider the total mass of the reactants (i.e., all the molecules of the gasoline and oxygen before the reaction) versus the total mass of all the molecules in the chemical products after the reaction. How much mass (in kg) is lost in this reaction (i.e., total mass of reactants minus total mass of products)?

Chapter 6

Thermal Energy and Solids

6.1 Introduction

In our everyday lives we experience many examples where mechanical energy is not conserved: brakes slow down a car, a bouncing superball returns to a lower height than it started from, and blow darts slide to a stop along the corridors of Bucknell residence halls. Because friction takes mechanical energy away from an object, historically it was not at all obvious that energy should be conserved. But some physicists in the 19th century noticed that when friction acts to slow an object and take away some mechanical energy, the object invariably becomes hotter. This suggested that temperature is connected to some kind of internal energy of the object — let's call it *thermal energy* — and that friction has acted to convert some of the object's mechanical energy to this thermal energy. Careful experiments by Joule and others confirmed the hypothesis that total energy is conserved even when mechanical energy is gained or lost, and now energy conservation is one of the most fundamental principles in physics.

But what is thermal energy? As we shall see, it is nothing more than the kinetic and potential energy of the individual molecules that make up the objects in our everyday world. In this unit we will begin by distinguishing mechanical energy from thermal energy.

6.2 Thermal Kinetic Energy

First, let's consider molecular kinetic energy. Consider a set of N molecules, each with the same mass m, with velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$. We will show that the kinetic energy associated with these moving molecules can be separated into mechanical kinetic energy and thermal kinetic energy. The first step is to calculate the motion of what is called the *center of mass.* At some particular instant in time, the positions of each particle are given by the vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. The location of the center of mass, denoted by the vector $\vec{r}_{cm}$, is the average of these positions

$$\vec{r}_{cm} = \frac{1}{N}(\vec{r}_1 + \vec{r}_2 + \cdots + \vec{r}_N). \tag{6.1}$$

Taking a time derivative of this equation gives

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{N}(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_N), \tag{6.2}$$

so the center of mass velocity is simply the average velocity (in this simplified case of equal masses).

For a rigid object, like a solid, the velocity of the center of mass is simply the velocity of the object. If the center of mass velocity of some object is zero, then that object, viewed macroscopically, is at rest. A ball sitting on a table has a stationary center of mass, and therefore no mechanical kinetic energy. However, the individual molecules of the ball are certainly not at rest and do have kinetic energy. The motion appears random, with molecules moving in every direction; some molecules moving faster and some slower. It is this molecular kinetic energy which we identify as thermal energy.[1]

> The thermal kinetic energy of an object is simply the molecular kinetic energy when the center of mass is at rest.

What about the case where the center of mass is moving? For example, if the ball is not sitting on the table but rather flying through the air. It is still possible to identify the thermal kinetic energy, because there is always some co-moving reference frame in which the ball is at rest. The molecular motion as viewed in that frame will again be the thermal kinetic energy.

But nevertheless we may ask if it is possible to identify the thermal kinetic energy in a frame where the ball is moving. And indeed, it is possible. The velocity $\vec{v}_i$ of the ith molecule can be written as a sum of the center of mass velocity $\vec{v}_{cm}$ and the velocity of the particle relative to the center of mass $\vec{v}_{i,rel}$. That is, $\vec{v}_i = \vec{v}_{cm} + \vec{v}_{i,rel}$. Then the total kinetic energy is the sum over all particles:

$$\begin{aligned} K_{\text{total}} &= \tfrac{1}{2}\sum_i m(\vec{v}_{cm} + \vec{v}_{i,rel}) \cdot (\vec{v}_{cm} + \vec{v}_{i,rel}) \\ &= \tfrac{1}{2}\sum_i m\, v_{cm}^2 + \sum_i m\, \vec{v}_{cm} \cdot \vec{v}_{i,rel} + \tfrac{1}{2}\sum_i m\, v_{i,rel}^2. \end{aligned} \tag{6.3}$$

Note that $\vec{v}_{cm}$ is the same for all particles, so it can be brought outside the sum over particles. Then the first term becomes

$$\tfrac{1}{2}\sum_i m\, v_{cm}^2 = \tfrac{1}{2}v_{cm}^2 \sum_i m = \tfrac{1}{2}M v_{cm}^2, \tag{6.4}$$

[1] For simplicity, we are excluding the possibility of rotations.

where M is the total mass of all the particles. This is exactly the mechanical kinetic energy we have already encountered. The second term can be written as

$$m\vec{v}_{cm} \cdot \left(\sum \vec{v}_{i,rel} \right). \tag{6.5}$$

Since $\vec{v}_{i,rel}$ is the particle velocity relative to the center of mass frame, the sum $\sum \vec{v}_{i,rel} = 0$, and this term vanishes. The last term in Eq. (6.3) is just the total kinetic energy measured in a frame moving with the center of mass, which is the thermal kinetic energy. Putting this all together,

$$K = K_{\text{mech}} + K_{\text{therm}}, \tag{6.6}$$

thus the kinetic energy divides cleanly into mechanical and thermal kinetic energy.

6.3 Thermal Potential Energy

Thermal energy is not just kinetic, but also involves potential energy. Molecules exert forces on each other, pushing and pulling.[2] These forces are conservative, so there is a potential energy associated with each pair of molecules. To understand this potential energy, we must first consider the force between an isolated pair of molecules. There are three distinct regimes, depending on how far apart the two molecules are.

- When the molecules are closer than a molecular diameter, they exert a strong repulsive force on each other.
- When the molecules are within a few molecular diameters, they exert attractive forces on each other.
- When the molecules are more than a few molecular diameters away from each other, the force becomes negligibly small.

This behavior is captured by what is called the *pair potential*, shown in Fig. 6.1, which is the potential energy $U_{\text{pair}}(r)$ due to a pair of molecules separated by a distance r. The diagram illustrates the three regions. Notice that at a separation $r = d$, where d is the molecular diameter, there is an equilibrium point dividing the regions of attractive and repulsive forces.

As we shall see in the next two chapters, this pair potential explains the existence of solid, liquid, and gas phases, and many details of the phases and of the transitions between the phases. It is one of the remarkable triumphs of the atomic theory of matter that so much behavior can be explained by such a simple model of the forces between atoms!

[2]The origin of these forces is the electric force interactions between the charges of the atoms combined with quantum mechanics, which governs the location of the charges. Both of these topics will be covered in PHYS 212.

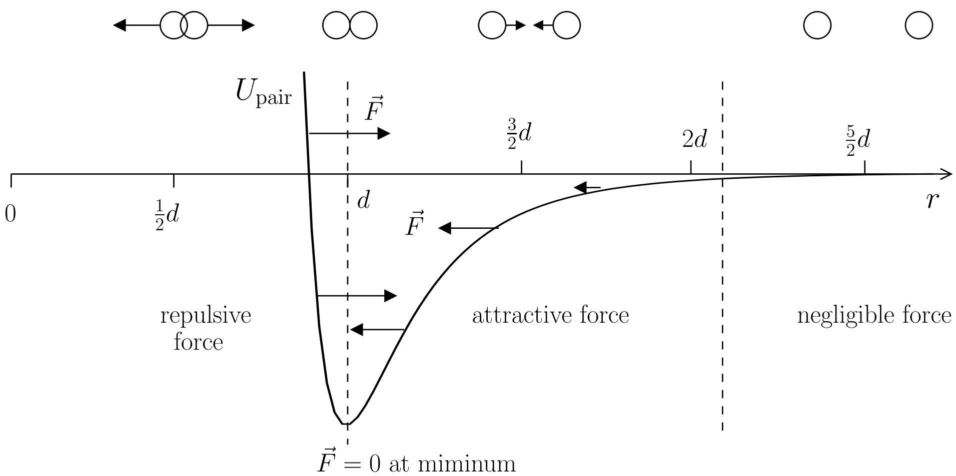

Figure 6.1: The pair potential energy U_{pair} as a function of r, the separation between the pair of molecules. The dashed lines separate the regions of repulsive force, attractive force, and negligible force. At separation $r = d$ the pair is in equilibrium, which defines the molecular diameter.

In principle, the *system* potential energy for a system of N molecules contains the pair potential energy for every single pairing of the molecules. This is a very large number of pairs! Fortunately, only those molecules which are immediate neighbors are close enough to have an appreciable force and potential energy, so we only need to consider the potential energy due to neighboring molecules.

What about other sources of potential energy besides the intermolecular forces? For example, gravitational potential energy. When a ball is thrown upwards, the gravitational potential energy of each molecule increases. But the height of each molecule is increased by the same amount that the height of the center of mass is increased. Therefore this change in potential energy has the form

$$\Delta U_{\text{grav}} = M_{\text{object}} g \Delta y_{cm}, \tag{6.7}$$

where M_{object} is again the total mass of the object. This is the familiar mechanical potential energy. Therefore, gravitational potential energy is always part of the mechanical energy, whereas the molecular interaction energy makes up the thermal potential energy.

In summary, here is the big picture for thermal energy:

- For both potential energy and kinetic energy, it is the 'organized' motion that makes up the *mechanical energy*, such as all molecules increasing their height together or all molecules having a net alignment of their velocities.

- The remaining disorganized motion, such as the wiggling of the molecules and their individual pushes and pulls on each other, corresponds to the *thermal energy.*
- Friction is an agent that takes organized motion and disorganizes it, taking away mechanical energy and increasing thermal energy.
- Going the other way — taking away thermal energy and increasing mechanical energy — is more difficult, since molecules are not likely to spontaneously start moving together. Nevertheless, we can capture some amount of thermal energy and convert it to mechanical energy with a device called a heat engine, which is the topic of Supplementary Reading Chapter 10.

6.4 The Solid State

Molecules interacting via the pair potential can be solids, liquids, or gases. The remainder of this chapter is concerned with the thermal energy of the solid state, while liquid and gas states will be presented in Chapter 7.

Most inorganic solids are crystalline, which means the molecules are arranged in a symmetric way, such as a cubic lattice. (Organic solids instead are constructed from long carbon chains.) In this lattice, each pair of neighboring molecules is separated by roughly the equilibrium distance, that is, the minimum of the pair potential well, and only makes small excursions from this location. As illustrated in Fig. 6.2, the pair potential in this region is identical to a parabolic potential. We have previously encountered a parabolic potential energy curve as the potential energy for a mass on a spring. Evidently, as long as the molecules in a solid are not deviating significantly from their equilibrium position, we may regard their interactions with their nearest neighbors as equivalent to being attached by a spring.

> CHECKPOINT: What is the main difference between the pair potential and the spring potential? What would I need to do to a pair of molecules to see this difference? (Push together? Pull apart? How far?)

This leads to what we call the *ideal solid*: the molecules are balls of mass m, they are connected by springs of spring constant k_{sp}, and the springs have an equilibrium length of d. This model is illustrated in Fig. 6.3. We may think of the spring constant as determining the bond strength and the equilibrium length d as the bond length. These three parameters (m, d, and k_{sp}) define the model, and we shall see that for many solids determining these parameters describes much of the behavior of the solid.

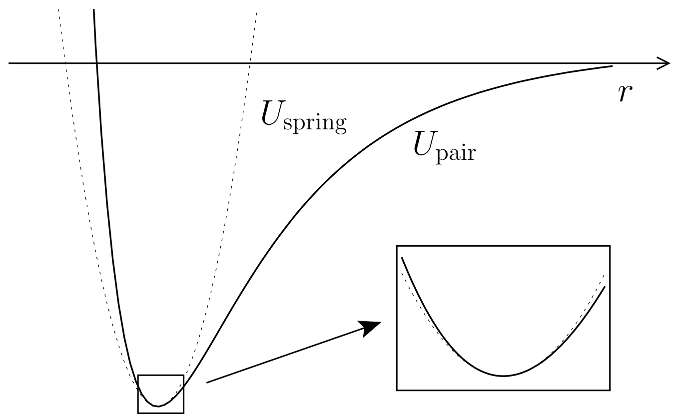

Figure 6.2: The solid curve is the pair potential U_{pair} as a function of separation r. The dashed line is the spring potential U_{spring} with the spring constant k_{sp} chosen to match U_{pair} near the minimum. The inset shows the match.

One may imagine packing the molecules together in different ways. The arrangement illustrated in Fig. 6.3, is called a *simple cubic* lattice. In fact, most solids are packed differently, for example, in the way a grocer would stack oranges (which is called a *face-centered cubic* lattice). Fortunately, this distinction has little impact on the quantities we will study, so we will stick with the simple cubic lattice.

Given the properties of some solid, how are the ideal solid parameters determined? Let's derive these for a specific case, namely copper. Molecular properties, such as mass, are usually not specified for a single molecule, but

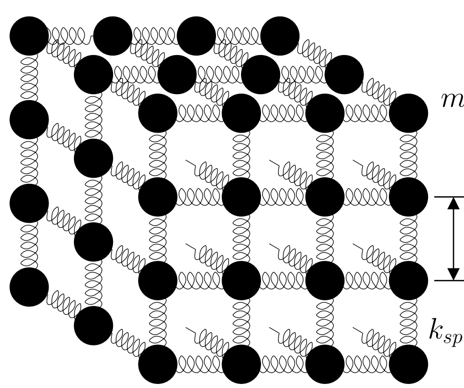

Figure 6.3: The ball-spring model of a solid.

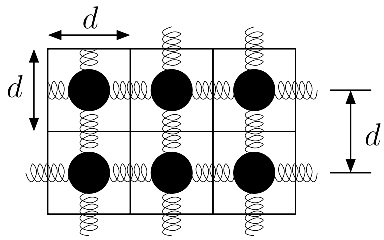

Figure 6.4: For the ideal solid with a separation d between the balls, the volume per ball is given by a cube of side d (shown here in two-dimensions).

rather for a *mole*. One mole equals Avogadro's number

$$N_A = 6.02 \times 10^{23} \tag{6.8}$$

of molecules. Avogadro's number is chosen so that roughly one mole of protons has a mass of one gram. The precise definition is that one mole of carbon atoms has a mass of 12 g. The mass of one mole of a material would logically be called the molar mass, but instead it is usually called the *molecular weight*.[3]

One mole of copper has mass 64 g, so we may conclude that a single molecule of copper (the ball in our model) has mass

$$m_{\mathrm{Cu}} = \frac{64\,\mathrm{g}}{6.02 \times 10^{23}} = 1.06 \times 10^{-22}\,\mathrm{g}. \tag{6.9}$$

This is the first of our three parameters.

Next, we can get the equilibrium spacing d between the molecules by knowing the density of copper, which is $8.94\,\mathrm{g/cm^3}$. In Fig. 6.4 shows that each molecule of copper occupies its own cubical region with volume d^3. We can relate the density ρ of the solid to the mass per volume of a unit cell. That is,

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{m}{d^3} \qquad \Rightarrow \qquad d = \left(\frac{m}{\rho}\right)^{1/3}. \tag{6.10}$$

For copper this gives

$$d_{\mathrm{Cu}} = \left(\frac{m_{\mathrm{Cu}}}{\rho_{\mathrm{Cu}}}\right)^{1/3} = \left(\frac{1.06 \times 10^{-22}\,\mathrm{g}}{8.94\,\mathrm{g/cm^3}}\right)^{1/3} = 2.28 \times 10^{-8}\,\mathrm{cm}, \tag{6.11}$$

or equivalently, $2.28 \times 10^{-10}\,\mathrm{m}$. And so we have the second parameter.

The final step is to determine the spring constant. Fortunately, it is not necessary to try pulling on a single molecule and measuring the force that it pulls back with. Rather, a macroscopic chunk of material can be fixed at

[3]It's not our fault.

one end and pulled on the other, and by measuring how much the object stretches, the bond spring constant can be measured.

Imagine a piece of copper wire with cross-sectional area A and length L. The applied force required to stretch the wire by an amount ΔL is given by the following relation:

$$F_{\text{app}} = \frac{YA}{L}\Delta L. \tag{6.12}$$

This relation indicates that the amount of stretch ΔL is proportional to the amount of force applied; doubling the force will double the amount of stretch from the equilibrium length. A larger cross-sectional area A makes the wire harder to stretch, which accounts for the factor of A in the numerator. The longer the wire, the easier it is to stretch, accounting for the factor of L in the denominator. The final parameter Y is called Young's modulus, and is a property of the material but not dependent on the geometry of the wire. For example, Young's modulus for copper is $Y_{\text{Cu}} \approx 130 \times 10^9\,\text{N/m}^2$.

Example 6.1 Stretching an Extension Cord

Consider 16 gauge copper wire, commonly used in power cables, which has a cross-sectional area of $1.3 \times 10^{-6}\,\text{m}^2$. How much force is required to stretch a 2-meter length of wire a distance of 1 centimeter?

Solution: According to Eq. (6.12), the force is given by

$$\begin{aligned} F_{\text{app}} = \frac{Y_{\text{Cu}}A}{L}\Delta L &= \frac{(130 \times 10^9\,\text{N/m}^2)(1.3 \times 10^{-6}\,\text{m}^2)}{2\,\text{m}}(0.01\,\text{m}) \\ &= 845\,\text{N}. \end{aligned} \tag{6.13}$$

Now we need to calculate Young's modulus for the ideal solid. Consider a rectangular solid of $N_x \times N_y \times N_z$ molecules. The object is stretched in the z-direction by an applied force F, with a resulting stretch ΔL. The stretch is shared equally among each of the N_z springs aligned in the z-direction, so each spring is stretched an amount $\Delta L/N_z$. The plane of molecules where the force is applied consists of N_xN_y molecules, each connected to a spring pulling with force $k_{\text{sp}}\Delta L/N_z$. This spring force is balancing the applied force, so we can conclude that

$$F_{\text{app}} = N_xN_y\left(\frac{k_{\text{sp}}\Delta L}{N_z}\right). \tag{6.14}$$

If we multiply top and bottom by d^2 we can identify the cross-sectional area $A = (N_x d)(N_y d)$, and the length $L = N_z d$:

$$F_{\text{app}} = \frac{d^2 N_x N_y k_{\text{sp}}}{d^2 N_z} \Delta L = \frac{k_{\text{sp}}}{d} \frac{A}{L} \Delta L \tag{6.15}$$

from which we conclude that Young's modulus for the ideal solid is

$$Y = \frac{k_{\text{sp}}}{d}. \tag{6.16}$$

This can be used to find the spring constant, $k_{\text{sp}} = Yd$. For example, for copper

$$k_{\text{sp}} = (130 \times 10^9\,\text{N/m}^2)(2.28 \times 10^{-10}\,\text{m}) = 29.6\,\text{N/m}. \tag{6.17}$$

In this way, we can find all three ideal solid parameters from knowing the molecular weight, the density, and Young's modulus.

6.5 Speed of Sound

How well does this ball-spring picture of a solid work? One way to test the model is to study the speed of sound in a solid. Sound is a compression wave, much like a compression pulse sent down a stretched slinky. If an ideal solid is suddenly struck at one end, how fast does the compression wave travel toward the other end? We can almost guess the answer. The wave "hops" from one molecule to its neighbor and each hop moves the wave a distance d. Since the molecules are harmonic oscillators, the time it takes for a hop must be related to the period of oscillation T, so we could guess $v_{\text{sound}} \approx d/T$. It is not difficult to do the full calculation for the ideal solid[4] and find that the answer differs from this guess by a factor of 2π:

$$v_{\text{sound}} = \frac{2\pi d}{T} = d\omega = d\sqrt{\frac{k_{\text{sp}}}{m}} \tag{6.18}$$

Thus, the speed of sound in the ideal solid depends on all three parameters. The values we obtained for copper (be careful to use SI units here!) give

$$v_{\text{sound}} = 2.28 \times 10^{-10}\,\text{m}\sqrt{\frac{29.6\,\text{N/m}}{1.06 \times 10^{-25}\,\text{kg}}} = 3810\,\text{m/s} \tag{6.19}$$

which is exactly the measured value for the speed of sound in copper. Evidently the ideal solid model works quite well. You will make more comparisons in the homework.

[4] This is done in PHYS 221/222.

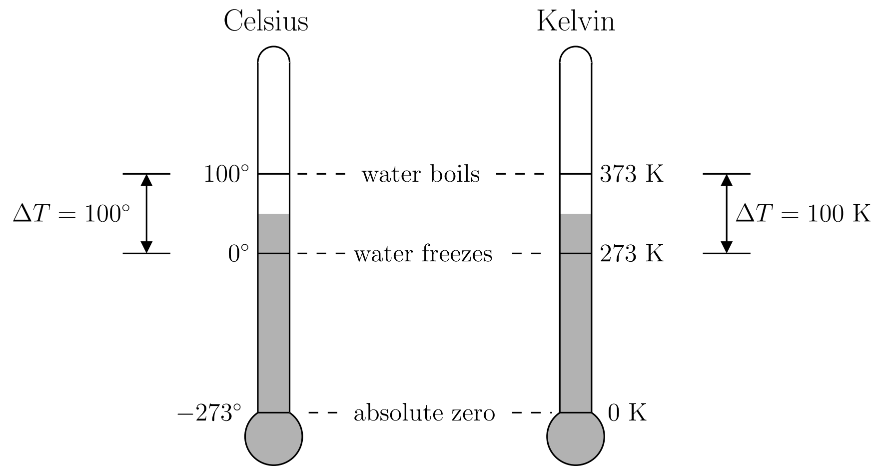

Figure 6.5: The size of the degree is the same for Celsius and Kelvin temperature scales. They differ by a shift: $T_K = T_C + 273$

6.6 Temperature

We began the chapter mentioning that when mechanical energy gets converted to thermal energy, the temperature increases. But what is temperature? Everyone has an intuitive feel for it: we know that a high temperature corresponds to something that is "hot" and a low temperature corresponds to something that is "cold." We also know from experience that "heat" — which we will define shortly — flows from hot (high temperature) to cold (low temperature) objects.

It is common in many introductory text books to define temperature as a measure of the average thermal kinetic energy K_{therm}/N of a material.[5] This definition of temperature is not valid for all situations (we will provide a more rigorous definition of temperature in Chapter 9); however, there are many situations that can benefit from simple view of temperature as a measure of thermal kinetic energy.

We must talk about temperature units. The Celsius temperature scale is defined so that water freezes at a temperature of $0°$ C and water boils at $100°$ C. Considering temperature as a measure of kinetic energy, we can define *absolute zero* as the temperature where all molecular motion stops. This occurs at $-273.15°$ C in the Celsius scale.

A important variation on the Celsius temperature scale is called the Kelvin scale, illustrated in Fig. 6.5. The "size" of the degree is the same for Kelvin and Celsius, that is, a change in temperature of 1 K is the same as

[5] Historically, this approach was used successfully in the 18$^{\text{th}}$ and 19$^{\text{th}}$ centuries to develop the first successful quantitative theories of thermodynamics, referred to as the *kinetic theory* of thermodynamics.

a change in temperature of 1° C. The boiling temperature of water is still 100 K higher than the freezing temperature. The difference in the scales is the location of zero: in the Kelvin scale, absolute zero corresponds to 0 K, water freezes at 273 K and boils at 373 K. The Kelvin temperature scale is the one best suited for most thermodynamics problems.

6.7 Molar Specific Heat

Now we are prepared to address the question, "If a certain amount of thermal energy is added to a system, how much will the temperature increase?" This is a question with far-reaching applications. For instance, how much energy needs to be added to a swimming pool to heat it up to a comfortable temperature? How much cooling water or antifreeze is needed to keep a car from overheating? How much will the temperature of a bucket of water increase if a hot ingot of lead is tossed into it?

We mentioned in the previous section that temperature is often associated with the thermal energy of a system, i.e., increases in temperature are associated with increases in the thermal energy. But *how much* does the temperature increase with a certain amount of energy is added to a material? For that, we define the *molar specific heat* C:

$$C = \frac{\Delta E_{\text{therm}}}{n\Delta T}. \tag{6.20}$$

In words, the specific heat is defined as the energy required to raise the temperature of one mole of a material by a temperature 1 K. So, a material with a large specific heat requires a lot of heat to increase its temperature significantly, whereas a material with a small specific heat requires less energy to raise its temperature by the same amount.

Turning Eq. (6.20) around, the energy required to raise the temperature of a material is given by

$$\Delta E_{\text{therm}} = nC\Delta T. \tag{6.21}$$

Measurements have been made of the molar specific heat for a wide variety of materials. A few of these values are listed in Table 6.1 for some common metals.

Note that the relation between the thermal energy and the temperature depends on the amount of material via the number of moles n. Having more stuff will require more thermal energy to get the same temperature change. The molar specific heat C, however, is defined such that it does *not* depend on the amount of material.

Example 6.2 Temperature increases

How much thermal energy must be added to 5.0 kg of lead to increase its temperature from 25° C to 40° C?

Solution: First, it is convenient to determine how many moles of lead we have here. From Table 6.1, we see that the molar mass of lead is 207 g/mol, so the number of moles is given by

$$n = 5.0\,\mathrm{kg} \cdot \frac{1000\,\mathrm{g}}{\mathrm{kg}} \cdot \frac{1\,\mathrm{mol}}{207\,\mathrm{g}} = 24.2\,\mathrm{mol}$$

From Table 6.1, we see that the molar specific heat of lead is $26.6\,\frac{\mathrm{J}}{\mathrm{mol}} \cdot \mathrm{K}$. Using Eq. (6.21), we find

$$\Delta E_{\mathrm{therm}} = nC\Delta T = 24.2\,\mathrm{mol} \cdot 26.6\,\frac{\mathrm{J}}{\mathrm{mol}{\cdot}\mathrm{K}} \cdot 15\,\mathrm{K} = 9640\,\mathrm{J}.$$

Note that the conversion between Celsius and Kelvin is trivial; a temperature difference of 15° C is the same as a temperature difference of 15 K.

Looking at Table 6.1, it is apparent that the molar specific heat for metals is fairly consistent — around 25 J/mol·K — for several common metals. A pattern like this indicates that there might be a simple, common explanation. That explanation is provided by the ideal solid model and what is known as the equipartition theorem.

6.8 The Equipartition Theorem

We now discuss a remarkable relationship between temperature and thermal energy, referred to as the *equipartition theorem*. The basic idea is that in thermodynamic systems, thermal energy is equally ("equi") divided ("partition") between certain types of molecular energy, both kinetic and potential. At first glance, you might think that this would mean that half of the energy is kinetic and half is potential (and sometimes this is true), but it is not quite that simple. For one, only those energy terms which are quadratic in a dynamical variable (such as $\frac{1}{2}mv_x^2$) get the equal shares. For any terms more complicated than that, like the pair potential, we cannot so easily say how the energy is divided. Also, the number of these egalitarian quadratic energy terms, often called *degrees of freedom*, depends on details such as whether there is rotational as well as translational kinetic energy.[6]

Maybe it will help to see the theorem:

> EQUIPARTITION THEOREM:
> Any term in the energy of a molecule that is quadratic, such as $\frac{1}{2}mv_x^2$ or $\frac{1}{2}k_{\mathrm{sp}}x^2$ or $\frac{1}{2}I\omega^2$, averages to $\frac{1}{2}k_BT$.

This amazing result says that when some 10^{23} particles push and pull and collide with each other, all the messy forces involved will result in every quadratic energy term averaging to the same value. It doesn't matter if the molecule is heavier or lighter, or what the spring constant is. It also doesn't matter whether we are talking about potential energy or translational kinetic energy or rotational kinetic energy. As long as the energy is quadratic in the dynamical variable, the thermal energy will depend only on T and Boltzmann's constant,

$$k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}, \tag{6.22}$$

which is another constant of nature. Notice how the units work out: k_BT is an energy.

In the next sections, we'll use the equipartition theorem to analyze thermal energy of an ideal solid.

6.9 Ideal Solid Specific Heat

To determine the molar specific heat of an ideal solid, let us make the approximation that the neighbors of a particular molecule remain fixed. This turns out to be a reasonable approximation for most solids. Then the energy describing that particular molecule is

$$E_{\mathrm{molecule}} = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2 + \tfrac{1}{2}k_{\mathrm{sp}}x^2 + \tfrac{1}{2}k_{\mathrm{sp}}y^2 + \tfrac{1}{2}k_{\mathrm{sp}}z^2. \tag{6.23}$$

There are six terms contributing to the energy, all of which are quadratic. Each term is fluctuating up and down as the molecule interacts with its neighbors. The equipartition theorem tell us, then, that the average energy of this ball over time will be

$$\langle E_{\mathrm{molecule}}\rangle = 6\left(\tfrac{1}{2}k_BT\right) = 3k_BT. \tag{6.24}$$

Now consider an N molecule ideal solid. The number of moles is given by $n = N/N_A$, and the thermal energy will be

$$E_{\mathrm{therm}} = N(3k_BT) = \frac{N}{N_A}(3k_BN_A)T = n3RT, \quad \text{(ideal solid)} \tag{6.25}$$

[6] For a solid the molecules essentially do not rotate, but rotational kinetic energy can have a significant effect on the thermal energy of a liquid or gas.

Material	M (g/mol)	ρ (g/cm^3)	Y (GN/m^2)	C (J/mol·K)	v_s (m/s)
Aluminum	27.0	2.70	70	24.2	5000
Iron	55.8	7.87	211	25.1	5120
Copper	63.5	8.96	130	24.4	3810
Gold	197	19.3	78	25.4	2030
Lead	207	11.3	16	26.6	1190
ideal solid	mN_A	m/d^3	$k_{\rm sp}/d$	3R = 24.9	$d\sqrt{k_{\rm sp}/m}$

Table 6.1: Material properties for a few selected substances.

where $R = N_A k_B = 8.31\,\mathrm{J/mol{\cdot}K}$ is called the gas constant, although it has nothing in particular to do with gases.[7]

Comparing Eqs. (6.25) and (6.21), the molar specific heat of the ideal solid is

$$C = 3R = 24.9\,\mathrm{J/mol{\cdot}K} \qquad \text{(ideal solid)} \tag{6.26}$$

regardless of the material. This relation is known as the Dulong-Petit law. The molar specific heats of most solids agree with the Dulong-Petit result to within a few percent accuracy. For example, the molar specific heat of copper is $C_{\rm Cu} = 24.4\,\mathrm{J/mol{\cdot}K}$, and comparable values for a variety of other solids are given in Table 6.1. This provides more evidence that the ball-spring model of the ideal solid is reasonable.

Finally, note that the specific heat is only defined in terms of $\Delta E_{\rm therm}$ and ΔT; it relates *changes* in the temperature to changes in thermal energy. However, if we assume that the specific heat is independent of temperature, which is a reasonable approximation down to some low temperature, then we can also estimate

$$E_{\rm therm} \approx nCT = n3RT. \qquad \text{(ideal solid)} \tag{6.27}$$

6.10 Heat and the First Law of Thermodynamics

There are many ways to add thermal energy to an object or to remove it from the object. We have already discussed how friction can increase the thermal energy of a blow dart as it slides across the floor. Another way to change the thermal energy is to bring the object into *thermal contact* with something hotter or colder. For a pair of solid objects, thermal contact occurs when they are physically in contact. Then the molecules at the boundary exert forces on each other and energy is transferred from the object with the higher temperature to the object with the lower temperature. As we have already discussed, temperature directs the flow of thermal energy, determining which objects will spontaneously give off energy and which objects will receive it.

[7] This isn't our fault either.

The energy transferred spontaneously by molecular motion is given the name *heat.* Similar to work, heat is an energy transfer and not an energy. Think of thermal energy as a bank balance and heat and work as deposits and withdrawals. The distinction between heat and work is the mechanism for the energy transfer.

> Heat is the thermal energy transferred spontaneously due to a temperature difference.

All other forms of energy transfer into a system are lumped together as thermodynamic "work." For example, the term work can refer to energy transfer due to external forces that act on system (but do not change the motion of the center of mass of the system)[8], but it also encompasses energy transfers due to other things, such as the warming of a piece of frozen broccoli in a microwave oven. To distinguish the two forms of energy transfer, it is common to use the symbol Q for heat, and W for work.

Now we can state the first law of thermodynamics, which is simply a statement of energy conservation: the change in thermal energy is equal to how much work is done on the system plus how much heat flows into the system:

$$\Delta E_{\text{therm}} = Q + W \qquad \text{(1st Law of Thermodynamics)} \tag{6.28}$$

Note that Q, like W, can be positive or negative, depending on whether heat is flowing in or out. The convention used here is to define the heat Q as being positive if heat is flowing *into* the material (and negative if heat is flowing out), and to define the work W as the work done *on* the system. Some people find it convenient to write the first law with these conventions stated explicitly:

$$\Delta E_{\text{therm}} = Q_{\text{in}} + W_{\text{on}} \qquad \text{(1st Law of Thermodynamics)} \tag{6.29}$$

From our perspective today, with energy conservation a fundamental principle, the 1st law may seem to be pretty obvious. But historically it was a very significant discovery, showing that indeed heat was just an energy transfer, rather than some new substance.[9] And the importance of the first law cannot be overstated – this seemingly simple result forms the foundation of much of what we will be doing during the next couple of weeks.

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[8] In Unit 1 in this course we discussed the work done on a single particle, and the resulting change in the kinetic energy of the particle. In thermodynamics we are interested in composite systems, such as gases liquids, and solids. In composite systems, work can result in changes in thermal energy as well as kinetic energy. In the thermodynamic systems we study, there will be no changes in the bulk kinetic energy K_{mech}.

[9] Early theories of thermodynamics proposed – incorrectly – that heat was some sort of fluid (called *caloric*) that flows between hot and cold materials. We now know, of course, that heat is simply the "flow" of energy.

It is worth appreciating what is not heat. Rubbing your hands together when they are cold certainly does increase their thermal energy, but not due to heat. There is not a higher temperature object making energy flow into your hands spontaneously, so there is no heat flow. Rather, you are doing work with your muscles, and the friction force between your hands converts the mechanical energy of your moving hands into thermal energy.

Example 6.3 First law

You hold a 35 mol iron anvil in place on a moving conveyor belt so that the belt slides under the stationary anvil. The belt does 12, 000 J of work on the anvil, and and it gets warmer. During this process, the anvil loses 7, 000 J to the cooler surrounding air and to the belt. If the anvil had an initial temperature of 22.0° C, what is its temperature at the end of this process?

Solution: First, we can use the first law to determine the change in the anvil's thermal energy. Conceptually, 12, 000 J is added in the form of work and 7, 000 J is removed in the form of a heat flow. In terms of Eq. (6.29), $W_{\text{on}} = 12000\,\text{J}$ and $Q_{\text{in}} = -7000\,\text{J}$ (negative since heat is flowing *out* of the anvil). So,

$$\Delta E_{\text{therm}} = Q_{\text{in}} + W_{\text{on}} = -7000\,\text{J} + 12000\,\text{J} = 5000\,\text{J}.$$

We can now use Eq. (6.21) and the molar specific heat of iron (see Table 6.1 to find the temperature change of the iron anvil:

$$\Delta E_{\text{therm}} = nC\Delta T.$$

Solving for the rise in temperature gives

$$\begin{aligned} \Delta T &= \frac{\Delta E_{\text{therm}}}{nC} \\ &= \frac{5000\,\text{J}}{35\,\text{mol} \times 25.1\frac{\text{J}}{\text{mol}\cdot\text{K}}} \\ &= 5.7\,\text{K} \end{aligned}$$

The final temperature is therefore 22.0° C + 5.7° C = 27.7° C.

Note that we could get a very good approximation of the result here by using the ideal solid approximation for the molar specific heat.

When a pair of objects is in thermal contact but is otherwise thermally isolated, we can say that ΔE_{therm} is equal and opposite for the two objects, since the thermal energy lost by the hotter object is gained by the colder object. This brings the objects closer together in temperature, until finally they have the same temperature and no more heat flows. This situation is called *thermal equilibrium*.

Example 6.4 Hot Meets Cold

One mole of an ideal solid at temperature 70° C is brought into thermal contact with two moles of an ideal solid at temperature 10° C. How much heat will flow out of the hotter object before thermal equilibrium is reached?

Solution: We will need to determine the final equilibrium temperature, T_f. This is done by balancing the heat flows in and out:

$$\Delta E_{\text{therm},1} = -\Delta E_{\text{therm},2} \;\Rightarrow\; n_1 C_1 \underbrace{(T_f - T_{1,i})}_{\Delta T_1} = -n_2 C_2 \underbrace{(T_f - T_{2,i})}_{\Delta T_2} \tag{6.30}$$

where $T_{1,i}$ and $T_{2,i}$ are the initial temperatures of objects 1 and 2. Putting in values:

$$(1\,\text{mol})(3R)(T_f - 70^\circ\,\text{C}) = -(2\,\text{mol})(3R)(T_f - 10^\circ\,\text{C}) \tag{6.31}$$

Note that we have used Celsius temperature. This is because ΔT is the same whether measured in Kelvin or Celsius (see Fig. 6.5). Now we solve:

$$T_f - 70 = -2(T_f - 10) \quad \Rightarrow \quad 3T_f = 70 + 20 \tag{6.32}$$

so

$$T_f = \frac{90}{3} = 30^\circ\,\text{C}. \tag{6.33}$$

To complete the calculation, we go back to the change in thermal energy, Eq. (6.30),

$$\Delta E_{\text{therm},1} = (1\,\text{mol})(24.9\,\text{J/mol}\cdot\text{K})(30^\circ\,\text{C} - 70^\circ\,\text{C}) = -996\,\text{J}. \tag{6.34}$$

So 996 J of heat flowed out of object 1 and into object 2.

Problems

1. To understand better how the ideal solid thermal energy is derived, consider the following scenario. A mad scientist creates a new material, flattium, in which the molecules can only move in the x-y plane, while their z coordinates remain fixed. Consider how Eq. (6.23) would be changed, and then use the equipartition theorem to derive an expression for the thermal energy of flattium.

2. Here is some practice with the ideal solid model.

(a) Using the data in Table 6.1, determine the ideal solid parameters, m, d, and k_{sp}, for iron.

(b) Use these values to estimate the speed of sound in iron. Compare your answer with the measured value.

3. Determine the thermal energy of one mole of a solid at a temperature of 100° C. You can use the ideal solid approximation for the molar specific heat.

4. Calculate the thermal energy required to raise the temperature of iron by 25 K for the amounts given below.

(a) One mole of iron.

(b) One gram of iron.

(c) One cubic centimeter of iron.

5. A two-mole ideal solid at temperature 40° C is brought into thermal contact with a one-mole ideal solid at temperature 10° C. Energy flows from the hotter solid to the colder solid until they reach the same final temperature.

(a) Calculate the final temperature.

(b) Calculate the amount of thermal energy transferred in this process.

6. Using your results from Problem 2, calculate the typical period of oscillation for an iron molecule at 50° C.

7. A 20 kg brick of lead is dropped from a height of 5.0 m above the sidewalk. It falls to the ground where it comes to rest. Assume that 60% of the mechanical energy of the brick is converted to thermal energy of the brick (the remaining energy went into thermal energy of the sidewalk and a big crack). Determine the temperature increase of the brick. *Hint:* you will need to calculate how many moles of lead the brick contains.

8. For silver, the ideal solid parameters are $m = 1.79 \times 10^{-25}\,\mathrm{kg}$, $d = 2.58 \times 10^{-10}\,\mathrm{m}$, and $k_{\mathrm{sp}} = 21.4\,\mathrm{N/m}$. Based on this information, calculate the density and Young's modulus for silver.

9. In the following list of processes, the thermal energy of an object is increasing (and so the temperature is increasing as well). For which processes is this increase due to heat flow?

(a) a drill bit which has been used to bore a hole
(b) an ice cube placed in a glass of water
(c) a cup of coffee warming in a microwave
(d) the filament in a light bulb in a lamp that is plugged in and turned on
(e) cookies placed into an oven to bake

10. Consider three bricks, all with mass 10 kg and at room temperature. The first brick is made of aluminum, the second brick copper, and the third brick lead. Which will have the largest thermal energy? Rank from from highest to lowest.

11. (a) Using the data in Table 6.1, determine the ideal solid parameters, m, d, and k_{sp}, for aluminum.
(b) Use these values to estimate the speed of sound in aluminum. Compare your answer with the measured value.

12. For an ideal solid at temperature T, determine the ratio of thermal kinetic energy to thermal potential energy. Use the equipartition theorem to justify your answer.

13. Specific heats are often given by the amount of thermal energy required to raise the temperature of *one kilogram* of material by a degree, rather than *one mole* of material. The per-kilogram specific heat c satisfies $\Delta E_{\mathrm{therm}} = m_{\mathrm{obj}} c \Delta T$, where m_{obj} is the mass of some object. Calculate the per-kilogram specific heat of iron.

14. Ideal solid **A** containing one-mole at some initial temperature T_A is brought into contact with ideal solid **B** containing three moles at temperature 20° C. The system equilibrates at a temperature of 75° C.

(a) Calculate the initial temperature of the solid **A**.
(b) Calculate the amount of thermal energy transferred.

15. Thermal energies are large! Calculate (roughly) the thermal energy of an 8 kg brick of lead at room temperature, say 22° C. Compare this to the gravitational potential energy of lifting this brick a height of 2 m.

16. In Section 6.4, an equation was derived to determine the spring constant k_{sp} for the ball-spring model from the value of the Young's modulus for a material: $k_{\mathrm{sp}} = Yd$.

(a) Show that this equation gives the proper units for the spring constant, given the units for Y and d.

(b) Write a sentence explaining why it makes sense that a material with a large Young's modulus is associated with a large spring constant k_{sp} for interactions between adjacent atoms.

17. In Section 6.5, an equation was derived to determine the sound speed:

$$v_{\mathrm{sound}} = d\sqrt{\frac{k_{\mathrm{sp}}}{m}}.$$

(a) Show that this equation gives the proper units for the speed of sound, given the units for d, k_{sp} and m.

(b) Write a sentence explaining why it makes sense that a material with a large k_{sp} is associated with a large speed of sound.

(c) Write a sentence explaining why it makes sense that the speed of sound is smaller for a material whose atoms have a larger molar mass.

18. You do 275 J of work on a system, and its thermal energy increases by 530 J. Calculate the heat that flows into or out of the system, and specify which direction the heat flows (i.e., in or out).

19. You are polishing a 5.0 g gold wedding ring. After doing this for a minute, you find that the ring is hot, having warmed up 20° C. Assuming that the ring loses 210 J to the air while you are polishing it, calculate the work that you did on the ring while polishing it.

Chapter 7

Liquids, Gases, and Phase Transitions

In this chapter we study the liquid and gas states of matter, as well as the phase transitions that occur when going from solid to liquid or from liquid gas. As before with the solid state, our tools for understanding these states will be the molecular pair potential and the equipartition theorem.

7.1 Phases of Matter

Since the dawn of human existence, people have noticed that matter could be solid, liquid, or gas. What cavewoman Thag and her contemporaries did not realize is that these quite different phases are made up of the same stuff: molecules that are pushing and pulling on each other, sometimes in a solid phase, sometimes a liquid and sometimes a gas. She was not the only one who didn't understand this. Plato didn't know about molecules. Nor did Dante, or even Newton. Only after the seminal work of Boltzmann and Einstein around the turn of the 20^{th} century did our process of scientific discovery lead us to understand the molecular form of matter.

This discovery is one of the crowning achievements of our species, and is also very practical, having provided the basis for most of our modern technology. The great 20th century physicist Richard Feynman[1] once remarked

> "If, in some cataclysm, all scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis . . . that all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one

[1] You're going to be hearing more about him in PHYS 212.

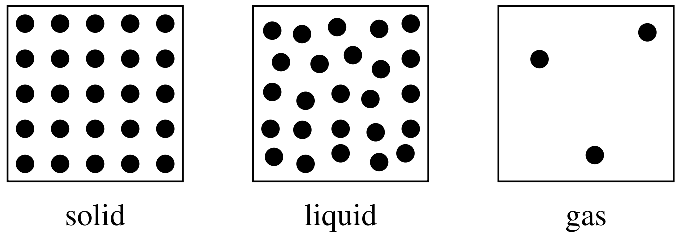

Figure 7.1: The molecular picture of the phases of matter.

> another. In that one sentence you will see an enormous amount of information about the world, if just a little imagination and thinking are applied."

Let us now follow Feynman's suggestion and apply a little imagination and thinking.

We have been discussing the solid phase of matter, where the particles are arranged in a regular lattice pattern, and the forces between the particles can be well-modeled as springs attached between neighbors. But we know from everyday experience that a solid can be melted when heated enough. Consider a lattice of vibrating molecules. Once the molecular excursions become large enough, the molecules start slipping past one another. As a result, the regular arrangement of the molecules in the lattice breaks down and the molecules are now disordered. They are still very closely packed and the density is comparable to the solid state, but the object has no rigidity. This is a liquid.

For most substances, there exists a boiling point separating a liquid phase from a gas phase. The picture you should have for the gas state is molecules moving about freely, far from their neighbors, and moving in a straight line until they collide with another molecule or with the walls of the container. Like a liquid, the gas phase is disordered. But the density of gases is much lower than liquids. Another difference between liquids and gases that we can understand immediately from the molecular viewpoint is their compressibility. Because the gas is dilute, we can compress a gas if we push the walls of the container inward. The molecules end up a bit closer together and bounce around a little faster, but otherwise they don't object. Liquids are essentially incompressible: you can't squeeze water to fit into a smaller volume. The liquid molecules are already packed together, albeit in a messy way, and any further squeezing is resisted by the repulsive forces of the pair potential.

Our basic understanding of the thermal energy of matter, developed in sections 6.2 and 6.3 for solids applies for liquids and gases as well: in particular, thermal kinetic energy is associated with motion of the molecules. And the potential energy associated with the forces between the molecules — i.e., the pair potential — provides the thermal potential energy. However, there are significant differences in the thermal potential energy for the different phases:

- in the solid phase are the molecules very near their equilibrium separation, allowing us to approximate their forces with springs
- in the liquid phase, however, the potential energy is complicated, since the molecules are pushed closer together and pulled farther apart as the molecules squeeze by each other, making the spring approximation invalid, and
- in the gas phase, the molecules are so far apart that, except for very brief collisions, there is no potential energy.

In contrast to these differences, the thermal *kinetic energy* is identical in form in all three phases. This allows us to develop the notion of thermal speed.

7.2 Thermal Speed

For all phases of matter, the translational kinetic energy of a single molecule is

$$K_{\text{trans}} = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2. \tag{7.1}$$

We can take advantage of this to determine how fast the molecules are moving at some given temperature T. Via the equipartition theorem, we can say that the average translational kinetic energy is

$$\langle K_{\text{trans}}\rangle = \left\langle \tfrac{1}{2}mv_x^2\right\rangle + \left\langle \tfrac{1}{2}mv_y^2\right\rangle + \left\langle \tfrac{1}{2}mv_z^2\right\rangle = \tfrac{3}{2}k_BT, \quad \text{(single molecule)} \tag{7.2}$$

since there are three quadratic terms in the energy. Recalling that $v^2 = v_x^2 + v_y^2 + v_z^2$, we can write

$$\frac{3}{2}k_BT = \left\langle \tfrac{1}{2}mv^2\right\rangle = \tfrac{1}{2}m\langle v^2\rangle \quad \Rightarrow \quad \langle v^2\rangle = 3k_BT/m. \tag{7.3}$$

Now we define the thermal speed, which indicates the typical speed of the molecules, as[2]

$$v_{\text{therm}} = \sqrt{\langle v^2\rangle} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3RT}{M}} \tag{7.4}$$

where m is the molecular mass and M is the molar mass.[3] The second equality above comes from multiplying numerator and denominator by N_A and

[2]Why didn't we simply define v_{therm} as the average velocity $\langle\vec{v}\rangle$? Because the average velocity is zero, which tells us nothing about the typical magnitude of the velocity.

[3]A.k.a. what is often horribly called the "molecular weight" (horrible because its a mass, not a weight, and it's for a mole of molecules, not just one). From here on, we'll abandon that nonsensical term and call it molar mass.

then using $M = N_A m$ and $R = N_A k_B$. So we have shown, via equipartition, a direct connection between the temperature and the translational kinetic energy.

Example 7.1 Speed of Nitrogen in the Atmosphere

What is the typical speed of a nitrogen molecule in the atmosphere at room temperature of 22° C?

Solution: First we need to convert the temperature to Kelvin: $T = 273 + 22 = 295\,\mathrm{K}$. The molar mass of elemental nitrogen is $14\,\mathrm{g/mol}$. However, nitrogen in the air is in molecular form, N_2, which has two nitrogen atoms per molecule, and a molar mass of $28\,\mathrm{g/mol}$. We may use Eq. (7.4), but to be consistent with SI units, we should convert the molar mass to kilograms:

$$\begin{aligned} v_{\mathrm{therm}} &= \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.31\,\mathrm{J/mol{\cdot}K})(295\,\mathrm{K})}{0.028\,\mathrm{kg/mol}}} \\ &= 512\,\mathrm{m/s} \end{aligned} \tag{7.5}$$

which is the same as about 1150 m.p.h. Room temperature molecules are fast!

A variation of this approach (i.e., using the equipartition theorem) can be used to determine how much a typical molecule is displaced from its equilibrium position but in the *solid phase only!* This is discussed (along with an example) in Sec. 7.5.

7.3 The Liquid State

In the liquid state, the pair potential "springs" are continually pushing and pulling and then getting stretched to the distance where they weaken and let go. In this way, molecules freely change their neighbors and slide past one another, which is why a liquid can flow. In this complicated picture, it is not possible to make a simple calculation for the thermal potential energy. The molar specific heat depends in a complicated way on the details of the pair potential, and so it varies considerably from material to material. While physicists and chemists have developed advanced theories for describing the liquid state, these are beyond the scope of this course.

Liquid	molecule	C (J/mol·K)
water	H_2O	75.3
methanol	CH_3OH	79.5
ethanol	C_2H_5OH	112.4
acetone	$(CH_3)_2CO$	125.5
benzene	C_6H_6	134.8

Table 7.1: Molar specific heats of selected liquids. Data is taken at room temperature.

However, if we are given a measured specific heat, such as that of water, we can still relate changes in the thermal energy to temperature changes via

$$\Delta E_{\text{therm}} = nC_{\text{liq}}\Delta T \tag{7.6}$$

Measured values of the specific heat are given for a few different liquids in Table 7.1.

Note that C_{liq} is different than the specific heat of the solid state for the same material. It is typically larger. For example, for copper molar specific heat in the liquid phase is $C_{\text{liq}} = 36.3\,\text{J/mol·K}$, compared to $C_{\text{s}} = 24.4\,\text{J/mol·K}$ in the solid phase.

7.4 The Gas State

In the gas phase, most of the time the molecules are so far apart that they exert no force on each other. The exception is the brief molecular collision, where the pair potential plays a role in determining the forces that they exert on each other during the collision. However, after the collision the pair of molecules head off in their new directions with new speeds, moving rather quickly out of range of each other and feeling no force. The details of the molecular collisions are complicated, but we can avoid having to worry about them by making use of the equipartition theorem.

The total thermal energy for a gas is, to an excellent approximation, purely kinetic, as the molecules are too far apart to have an appreciable potential energy. However, the kinetic energy may consist of both translational and rotational kinetic energies. For a *monatomic* gas molecule, such as argon, there is no contribution to rotational kinetic energy and so

$$E_{\text{molecule}} = K_{\text{trans}} = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2. \quad \text{(monatomic ideal gas)} \tag{7.7}$$

There are three quadratic contributions to the energy. The equipartition theorem says, therefore, that the average thermal energy per molecule is $3(\frac{1}{2}k_BT)$. Therefore, the total thermal energy is

$$E_{\text{therm}} = N\langle E_{\text{molecule}}\rangle = \tfrac{3}{2}Nk_BT = \tfrac{3}{2}nRT. \quad \text{(monatomic ideal gas)} \tag{7.8}$$

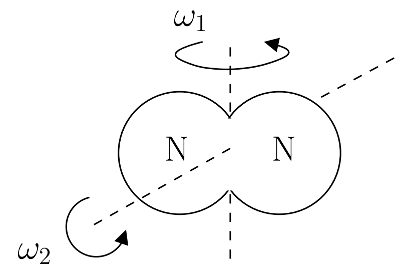

Figure 7.2: Molecular nitrogen, N_2, has two distinct axes of rotation, both of which contribute to the molecular kinetic energy.

However, many gas molecules are diatomic, such as N_2, which makes up about 78% of our atmosphere, and O_2, which makes up most of the rest. These dumbbell-shaped molecules have significant rotational kinetic energy as well as translational kinetic energy, as shown in Fig. 7.2. Their molecular energy is given by

$$E_{\text{molecule}} = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2 + \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 \quad \text{(diatomic ideal gas)} \tag{7.9}$$

where ω_1 and ω_2 are the angular frequencies of the molecular rotation about the two axes indicated in Fig. 7.2. The details of the molecular mass or rotational inertia again do not matter, and the equipartition theorem now gives

$$E_{\text{therm}} = N\langle E_{\text{molecule}}\rangle = \tfrac{5}{2}Nk_BT = \tfrac{5}{2}nRT. \quad \text{(diatomic ideal gas)} \tag{7.10}$$

These results can be combined into one expression,

$$E_{\text{therm}} = \frac{f}{2}Nk_BT = \frac{f}{2}nRT. \qquad \text{(ideal gas)} \tag{7.11}$$

where f is the number of *degrees of freedom*, or the number of quadratic terms appearing in the energy of a molecule. So $f = 3$ for a monatomic ideal gas, and $f = 5$ for a diatomic ideal gas.

Given the usual definition of molar specific heat, $\Delta E_{\text{therm}} = nC\Delta T$, we may identify

$$C_{\text{monatomic}} = \tfrac{3}{2}R = 12.5\,\text{J/mol·K}, \qquad C_{\text{diatomic}} = \tfrac{5}{2}R = 20.8\,\text{J/mol·K}. \tag{7.12}$$

As Table 7.2 shows, these values are highly accurate.

Here is an application: Argon gas is the most abundant monatomic gas in the atmosphere, and is technologically useful since monatomic gases conduct heat slower than diatomic gases, due to the smaller heat capacity. Modern double-glazed windows have argon gas between the two sheets of glass, to minimize the rate of heat transfer through the window.

Gas	Type	C (J/mol·K)
Neon (Ne)	monatomic	12.5
Argon (Ar)	monatomic	12.5
Hydrogen (H_2)	diatomic	20.5
Oxygen (O_2)	diatomic	21.1
Nitrogen (N_2)	diatomic	20.8

Table 7.2: Molar specific heats (at constant volume) of selected gases.

In solids and liquids, sound waves move by molecules pushing on their neighbors. In a gas, the molecules are not in contact with each other to push and pull, and so sound propagation is fundamentally different: the molecules must travel from collision to collision for the sound wave to move. Thus, the speed of the sound wave is essentially the thermal speed of the molecules themselves, which makes for much slower speed of sound. While we will skip the derivation, one can show that the speed of sound in an ideal gas is given by

$$v_{\text{sound}} = \sqrt{\frac{\gamma}{3}} v_{\text{therm}} = \sqrt{\frac{\gamma RT}{M}}. \tag{7.13}$$

We have introduced the parameter γ, which is often called the "adiabatic exponent" and is defined as

$$\gamma = \frac{f+2}{f}, \tag{7.14}$$

where f is again the number of degrees of freedom. This gives us

$$\gamma = \frac{5}{3} \quad \text{(monatomic)}, \qquad \gamma = \frac{7}{5} = 1.4 \quad \text{(diatomic)}. \tag{7.15}$$

We will see γ again in the next lecture, since it also plays a role in fundamental gas processes.

Notice that the speed of sound in a gas is dependent on the temperature, since the thermal speed depends on temperature, while for liquids and solids the speed of sound is essentially independent of temperature.

Example 7.2 The Speed of Sound in Air

Estimate the speed of sound in air at room temperature of 22° C $=$ 295 K. Use the average molar mass based on an approximate composition of 78% N_2 and 22% O_2.

Solution: For the molar mass, we use the molar masses of nitrogen and oxygen to find

$$M = 0.78(28\,\mathrm{g/mol}) + 0.22(32\,\mathrm{g/mol}) = 28.9\,\mathrm{g/mol}. \tag{7.16}$$

Both of these gases are diatomic, so we get the sound speed

$$\begin{aligned} v_{\text{sound}} &= \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4(8.31\,\mathrm{J/mol{\cdot}K})(295\,\mathrm{K})}{0.0289\,\mathrm{kg/mol}}} \\ &= 345\,\mathrm{m/s}. \end{aligned} \tag{7.17}$$

That's a pretty accurate value. To get a more precise value we would need to know the amount of water vapor in the air and a few other details.

7.5 Phase Transitions

Heat up an ice cube and it melts. Heat up a chunk of copper and it melts also, albeit at a much higher temperature. What is melting, and what determines the temperature at which a substance melts? Our ball-spring model, taken literally, cannot exhibit melting, because no matter how energetically the molecules vibrate, they are still connected to the same neighbors. But we should recall that the spring was only an approximation to the molecular pair potential, valid when the thermal energy was low enough that the molecules were mostly near the minimum of their potential well. Once a pair of molecules is stretched far enough apart, their interaction differs from a spring in that the attractive force between them weakens and ultimately becomes negligible (see Fig. 6.1). So we need a mental picture of a "spring" that weakens and "gives up" under too much stretching.

We can use the ball-spring model to make a rough estimate of when melting should occur. The equipartition theorem tells us how far the molecules move in their vibrations. Let x be the displacement of a molecule from its equilibrium position in the x-direction. The average value of x is zero, because the molecule is displaced equal amounts of time in the $+x$ and $-x$ directions. But the average value of x^2 is not zero and is given by (using the equipartition theorem)

$$\left\langle \tfrac{1}{2}k_{sp}x^2 \right\rangle = \tfrac{1}{2}k_{sp}\langle x^2\rangle = \tfrac{1}{2}k_BT. \quad \Rightarrow \quad \langle x^2\rangle = \frac{k_BT}{k_{sp}}. \tag{7.18}$$

This tells us the typical size of the excursions. We can define a thermal displacement magnitude

$$x_{\text{therm}} = \sqrt{\langle x^2\rangle} = \sqrt{\frac{k_BT}{k_{sp}}}, \tag{7.19}$$

which exhibits the reasonable behavior that the higher the temperature, the farther the molecule moves (on average) from equilibrium.

Example 7.3 Wiggling copper atoms.

Copper is a solid at room temperature, $T = 295\,\mathrm{K}$. As the copper atom oscillates about its equilibrium position, what is the typical magnitude of its displacement in the x-direction?

Solution: From Eq. (7.19), we get

$$x_{\text{therm}} = \sqrt{\frac{(1.38 \times 10^{-23}\,\mathrm{J/K})(295\,\mathrm{K})}{29.6\,\mathrm{N/m}}} = 1.17 \times 10^{-11}\,\mathrm{m}. \qquad (7.20)$$

where we used the copper spring constant from chapter 6. Note that we used SI units, so x_{therm} comes out in meters.

Is this answer reasonable? Recall that the equilibrium distance between copper atoms is $d = 2.28 \times 10^{-10}\,\mathrm{m}$, which is about 20 times larger. So at room temperature, copper molecules are vibrating somewhere around 5% of the distance of their separation. That sounds plausible.

Melting occurs when the molecular excursions become an appreciable fraction of the bond length d, an idea is known as the *Lindemann criterion.* Lindemann found empirically[4] that a reasonable estimate for the melting temperature can be obtained by setting $x_{\text{therm}} \approx d/10$. Note that if $x_{\text{therm}} \approx d/10$, it does not mean that each atom in the solid is vibrating precisely $d/10$ from the equilibrium; some are going further, and some are going less. The Lindemann criterion implies

$$x_{\text{therm}} = \sqrt{\frac{k_B T_m}{k_{sp}}} \approx d/10 \quad \Rightarrow \quad T_m \approx \frac{d^2 k_{sp}}{100 k_B} \qquad (7.21)$$

This is not an highly accurate estimate, but it does capture some general features. For example, lead has a relatively low melting temperature, which is evidently due to its weak spring constant. An estimate for copper, based on the ball-spring parameters, is

$$T_m \approx \frac{(2.28 \times 10^{-10}\,\mathrm{m})^2 (29.6\,\mathrm{N/m})}{100(1.38 \times 10^{-23}\,\mathrm{J/K})} = 1120\,\mathrm{K} \qquad (7.22)$$

Material	T_m (K)	L_f (kJ/mol)	T_v (K)	L_v (kJ/mol)
Oxygen	54.4	0.444	90.2	6.82
Nitrogen	63.2	0.72	77.4	5.56
Ethanol	159	5.02	352	38.6
Water	273	6.01	373	40.6
Lead	600	4.77	2022	180
Copper	1358	13.3	2835	300
Iron	1811	13.8	3134	340

Table 7.3: Melting and vaporization temperatures for a few materials, along with the latent heats of fusion and vaporization.

which is comparable to the measured value of 1358 K. Iron has a stronger spring constant that copper, and correspondingly a higher melting temperature.

In the solid state, whenever thermal energy is added to an object, the temperature increases. However, when the temperature of a solid reaches the melting temperature, additional thermal energy no longer causes temperature increase but rather phase change. Adding a little thermal energy to a solid at the melting temperature causes a few of the molecules to break from the lattice structure and become liquid. Adding more thermal energy causes even more molecules to become liquid. While this is happening, the temperature of the material is not changing. Rather, a solid with temperature T_m is being converted to a liquid at temperature T_m, as shown in Fig. 7.3. The amount of thermal energy required to convert one mole of a solid to a liquid is called the *latent heat of fusion*, and denoted by the symbol L_f. Given the latent heat of fusion for a material, it is straightforward to determine how much energy is needed to melt a certain amount of that material:

$$|\Delta E_{\text{therm}}| = nL_f, \qquad \text{(melt/solidify)} \tag{7.23}$$

where n is the number of moles of the material. This same relation can be used to determine how much energy is released when a certain amount of a liquid is frozen into solid form. Energy must be added to melt something, and energy is released when something freezes. Latent heats for a few materials are given in Table 7.3.

[4]i.e., simply by looking at the experimental data

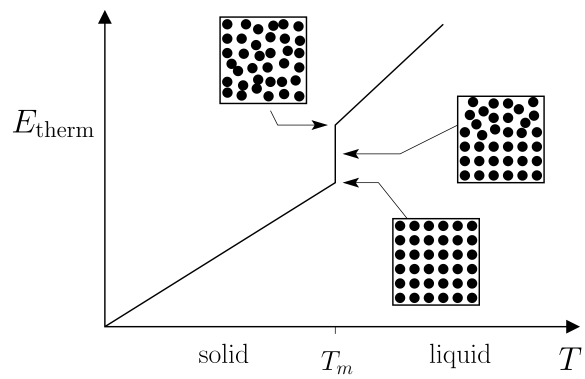

Figure 7.3: Shown is E_{therm} versus T for one mole of a typical material. The slopes in the solid phase and liquid phase are the molar specific heats, which are not typically equal to each other. The vertical jump at T_m represents the latent heat of fusion, i.e., the amount of thermal energy required to change phase.

Example 7.4 Melting lead

Calculate the amount of thermal energy that has to be added to 3.0 moles of lead at room temperature to melt $\frac{2}{3}$ of the lead.

Solution: This is a two-part process. First the temperature of the solid lead solid must be raised to its melting temperature. *Then* the lead can be melted.

In the first part of the process the temperature of all of the lead increases to 600 K. The thermal energy change corresponding to this is given by

$$\begin{aligned}\Delta E^{(1)}_{\text{therm}} &= n_1 C \Delta T \\ &= 3.0\,\text{mol} \times 26.6\,\text{J/mol·K} \times (600\,\text{K} - 295\,\text{K}) \\ &= 24.3\,\text{kJ} \end{aligned} \tag{7.24}$$

In the second part of the process the phase changes, so the thermal energy change necessary to melt two moles of the lead is given by

$$\begin{aligned}\Delta E^{(2)}_{\text{therm}} &= n_2 L_f \\ &= 2.0\,\text{mol} \times 4.77 \times 10^3\,\text{J/mol} \\ &= 9.5\,\text{kJ} \end{aligned} \tag{7.25}$$

Combining these two thermal energy changes gives

$$\begin{aligned}(\Delta E_{\text{therm}})_{\text{total}} &= \Delta E^{(1)}_{\text{therm}} + \Delta E^{(2)}_{\text{therm}} \\ &= 24.3\,\text{kJ} + 9.5\,\text{kJ} \\ &= 33.8\,\text{kJ} \end{aligned} \tag{7.26}$$

In this calculation, we used the measured value for the molar specific heat for lead (see Table 6.1). We could have used the Dulong-Petit (ball-spring) approximation ($C = 3R$) which would have given us a result very close to the value that we calculated here.

Note also that if we wanted to start with 2 moles of liquid lead and 1 mole of solid lead both at 600 K, and cool it down to a solid at room temperature, the same calculation would tell us how much thermal energy we would need to *remove*. It would also be 33.8 kJ.

For most substances, there exists a boiling point separating a liquid phase from a gas phase. At the molecular level, the liquid state has nearly solid-like density, with molecules packed close together and fairly near the minimum of the potential well. The transition to the gas phase requires pulling these molecules apart and setting them free, where they have no neighbors. Let E_{bind} be the depth of the potential well, that is, E_{bind} is the amount of energy needed to bring a pair of molecules from their equilibrium separation to far apart from each other (see Fig. 7.4(a)). Vaporization (or boiling) occurs roughly when $k_B T \approx E_{\text{bind}}$, so we can estimate the boiling temperature as

$$T_v \approx \frac{E_{\text{bind}}}{k_B}. \tag{7.27}$$

Example 7.5 Molecular Binding Energy

Estimate the molecular pair binding energy E_{bind} for copper, using the information in Table 7.3.

Solution: Copper vaporizes at $T_v = 2835\,\text{K}$, so we may estimate

$$E^{\text{Cu}}_{\text{bind}} \approx k_B T_v = (1.38 \times 10^{-23}\,\text{J/K})(2835\,\text{K}) = 3.91 \times 10^{-20}\,\text{J}. \tag{7.28}$$

A convenient energy unit for describing molecular bonds is the electron volt (eV), defined as

$$1\,\text{eV} = 1.60 \times 10^{-19}\,\text{J}. \tag{7.29}$$

In terms of electron volts, then,

$$E^{\text{Cu}}_{\text{bind}} = \frac{3.91 \times 10^{-20}\,\text{J}}{1.60 \times 10^{-19}\,\text{J/eV}} = 0.245\,\text{eV} \tag{7.30}$$

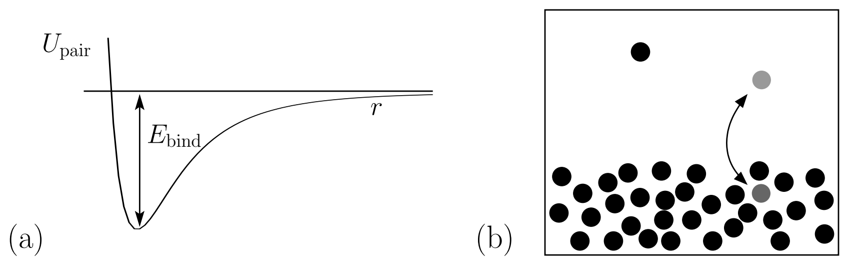

Figure 7.4: (a) The pair potential depth E_{bind}. (b) Removing a particle from the liquid state requires an energy of about 10–12E_{bind}.

Like with melting, vaporization requires an input of thermal energy. As thermal energy is added, the temperature remains fixed at the vaporization temperature, while an increasing amount of liquid gets converted to gas. The amount of energy required to convert a mole of a substance from liquid to gas is called the *latent heat of vaporization*. This is used much the same way as the latent heat of fusion:

$$|\Delta E_{\text{therm}}| = nL_v. \qquad \text{(vaporize/condense)} \tag{7.31}$$

As before, ΔE_{therm} is positive if we are adding thermal energy to vaporize, and it is negative if we are removing thermal energy to condense.

One final comment about latent heat and phase transitions: the amount of heat needed to melt or boil most common materials is quite large. Just looking at Tables 6.1 and 7.3, you can see that it is necessary to add a “k” to the units for latent heats L_f and L_v (versus the units for molar specific heat C) because we are usually talking about thousands of Joules of energy to cause a phase transition for each mole of the substance. This is a very important result with *lots* of practical applications. For example, this is the reason why ice is so good at cooling your drink; it isn’t the low *temperature* of the ice that is important, rather it is the large amount of energy that the ice absorbs when it melts that does such a good job of cooling your drink. Phase transitions are used *all the time* in cooling and heating applications. A standard air conditioner or refrigerator typically uses some substance (e.g., freon) whose condensation and vaporization play a key role in the cooling process. And your body uses phase transitions to keep cool on hot summer days. Sweat (water) on your skin vaporizes, and most of the energy needed for this phase transition comes from your body. This is how you can manage not to overheat even if the surrounding air temperature is greater than your body temperature. So, we would all be dead were it not for phase transitions.

7.6 Pressure

Liquids and gases push outwardly on their surroundings. To describe this push we introduce the concept of *pressure*. Consider a liquid or a gas enclosed in some container, and focus on one wall of the container with area A, such as shown in Fig. 7.5. The fluid pushes on the wall in the perpendicular direction with a force of magnitude F. Pressure is then the force per area

$$p = \frac{F}{A}, \tag{7.32}$$

and has units $\mathrm{N/m^2}$. This combination of units is given the name pascal (Pa), that is, $1\,\mathrm{Pa} = 1\,\mathrm{N/m^2}$. Atmospheric pressure is given by

$$p_{\mathrm{atm}} = 1.01 \times 10^5\,\mathrm{Pa} = 101\,\mathrm{kPa}. \tag{7.33}$$

Note that pressure applies to more than just liquids and gases. Any time that there is a force exerted on a surface, you can define a pressure simply by dividing that force by the surface area. Conceptually, the pressure indicates how "spread out" the force is, i.e., if the same force is exerted over a larger surface area, then there is less force per unit area (smaller pressure) and each part of the surface experiences a smaller force. That is why, for example, it is useful to wear snowshoes with a large surface area when walking over fresh snow — the downward force you exert on the ground is spread over a larger area, resulting in a smaller pressure on the snow, so the snow doesn't collapse.

Pressure does not have a direction. If we consider some point in the fluid, pressure is an outward push in all directions. This outward push is balanced by an identical outward push from a neighboring region of the gas. Only at the boundaries of the gas is there an imbalance — here the enclosed gas is only pushing from inside — and that is where we can measure the pressure. And note: **a gas can ONLY push on a surface; it NEVER pulls!!!**

We can use our picture of the gas state to derive the pressure of a gas. In the *ideal gas* approximation we ignore collisions between the molecules entirely, and treat each molecule as bouncing back and forth between the walls of the container. The particles bouncing off the walls exert a force on the wall, and this is precisely the origin of pressure.

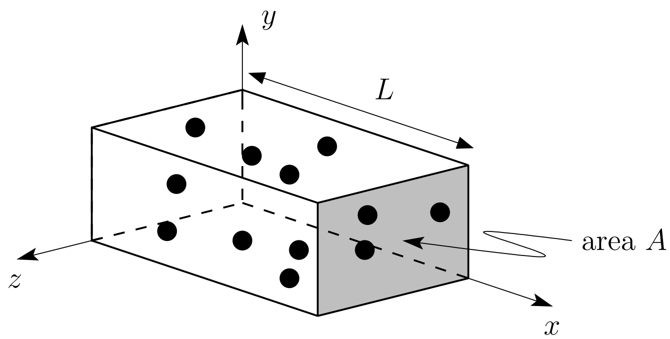

Figure 7.5: A gas enclosed in a container of length L, with the shaded wall having area A.

Example 7.6 Pressure and Balloons

Explain qualitatively how the gas inside an inflated balloon prevents the balloon from collapsing.

Solution: When a balloon is inflated, the rubber is stretched in all directions, resulting in an increased tension. At every point along the surface of the balloon fabric, the tension pulls in all directions along the surface. Because of the curvature of the balloon, these tension forces add up to give a net inward tension force.

The inward components of the tension do not cause the balloon to collapse because there is an outward force due to air molecules trapped within the balloon bouncing off the inner surface of the balloon. Each time a molecule bounces off a piece of the balloon, it gives that piece a small outward kick. Of course, there are also molecules outside the balloon, and each time one of these bounces off the balloon, it gives a small inward kick. If the outward and inward forces were balanced, which is what happens with an open balloon, then the net force would be the tension force, and the balloon would rapidly shrink.

But what if the collisions from the inside molecules are more frequent and/or harder collisions? Consider that there are around 10^{23} gas molecules in a typical balloon, and they are moving quite fast at typical room temperatures (see Example 1). There are a **lot** of collisions occurring each second between gas molecules and the inside of the balloon. The result of all of these collisions is an outward force exerted on the balloon fabric by the gas molecules. This outward gas force opposes the inward components of the tension *and* the inward force

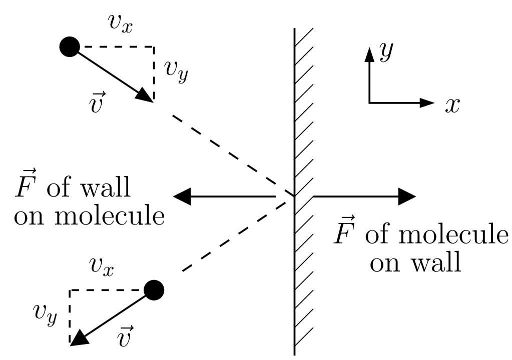

Figure 7.6: The x-component of velocity changes sign, while the y-component of velocity is unchanged.

due to the collisions of all the gas molecules on the outside, and as a result the balloon does not collapse.

7.7 The Ideal Gas Law

Now that we know how it is that a gas can exert a pressure, we can use these ideas — along with the previous discussion of thermal velocities of gas molecules — to calculate how the pressure relates to the temperature of the gas, the number of gas molecules (or moles) and the geometry of the container holding the gas. From this we will derive one of the most useful relations in thermodynamics: the *ideal gas law*.

A collision with a wall is shown in Fig. 7.6. Note that the x-component of velocity changed sign, while the y-component of the velocity was unchanged. If we focus on v_x, we can see that the molecules in Fig. 7.5 will bounce off the shaded wall and change the sign of v_x, then a time L/v_x later they will bounce off the back wall and head back toward the shaded wall. They will likely bounce off the side walls en route, but this has no effect on v_x, so we can ignore it. Thus the time between collisions on the shaded wall is $\Delta t = 2L/v_x$ (the factor of two coming from the trip away and then back).

The force on the wall due to the particle is zero in between collisions, and then very abruptly some non-zero value during the collision. Viewed as a function of time, the force would be a series of spikes, since each collision is short in duration, and then there is no force while the particle travels to the other side of the container and back again. What we feel as steady pressure is the time average of many collisions, so we would like to obtain an average of this force over time.

We can calculate this time averaged force first by noting that the force of the molecule on the wall is, by Newton's third law, equal and opposite the force of the wall on the molecule. The force of the wall on the molecule causes a change in the x-component of momentum, p_x (see Fig. 7.6). Note that the symbol p here, and in the next two equations, refers to momentum, not pressure. We apologize for the fact that the standard symbols for these quantities are the same (but there are only so many letters to use). The momentum change is given by

$$\Delta p_x = -mv_x - mv_x = -2mv_x. \tag{7.34}$$

This momentum change happens once every time interval of $\Delta t = 2L/v_x$ (the travel time between collisions), so we can conclude the average force of the wall on the molecule is

$$F_{\text{avg},x} = ma_{\text{avg}} = \frac{m\Delta v_x}{\Delta t} = \frac{\Delta p_x}{\Delta t} = -\frac{2mv_x}{2L/v_x} = -\frac{mv_x^2}{L}. \tag{7.35}$$

The average force of the *molecule on the wall* is then equal and opposite. If we consider all N molecules, then we can conclude

$$F_{\text{on wall},x} = \sum_i \frac{mv_{i,x}^2}{L} \tag{7.36}$$

where $v_{i,x}$ is the x-component of the velocity $\vec{v}_i$ of the ith particle. We can appeal to the equipartition theorem, which tells us

$$\left\langle \sum_i \tfrac{1}{2} m v_x^2 \right\rangle = N\left(\tfrac{1}{2} k_B T\right), \tag{7.37}$$

and so the average force on the wall is given by

$$F_{\text{on wall},x} = \frac{Nk_BT}{L}. \tag{7.38}$$

Finally, we use the definition of pressure to conclude

$$p = \frac{F_{\text{on wall},x}}{A} = \frac{Nk_BT}{AL} = \frac{Nk_BT}{V} \tag{7.39}$$

where we have used $V = AL$ (see Fig. 7.5). This last relation holds regardless of the shape of the container, and we have thus derived the *ideal gas law*

$$pV = Nk_BT \qquad \text{or} \qquad pV = nRT. \tag{7.40}$$

This is another universal law where the details of the molecules are irrelevant: the molecular mass canceled out, and any interaction forces between the molecules negligible as long as the gas is dilute enough.

This is an extremely accurate law for most gases at room temperature and higher temperatures. Note also that it does not depend on the mass of the gas molecule, or whether it is diatomic or monatomic.

In Eq. (7.40), both sides of the equations have units of energy. If we use SI units, pressure should be measured in pascal, and volume should be measured in cubic meters. However, we may take advantage of the relation

$$1\,\mathrm{J} = (1\,\mathrm{Pa})(1\,\mathrm{m}^3) = (10^{-3}\,\mathrm{kPa})(10^3\,\mathrm{L}) = (1\,\mathrm{kPa})(1\,\mathrm{L}), \tag{7.41}$$

where L is liters. This tells us we can use kilopascals and liters and the product pV will turn out to be joules.

Example 7.7 Volume of a Mole of Gas

Calculate the volume in liters occupied by a mole of ideal gas at a temperature of 22° C and atmospheric pressure.

Solution: Starting from Eq. (7.40), we solve for V:

$$V = \frac{nRT}{p} = \frac{(1\,\mathrm{mol})(8.31\,\mathrm{J/mol{\cdot}K})(295\,\mathrm{K})}{101\,\mathrm{kPa}} = 24.2\,\mathrm{L}. \tag{7.42}$$

Note that we had to convert temperature from Celsius to Kelvin in this calculation. **This is important:** you **always** have to use Kelvin for the temperature when using the ideal gas law.

Example 7.8 Using Ratios in the Ideal Gas Law

An ideal gas at a temperature 50° C is in a car piston. The piston compresses the gas to 1/3 of its original volume. The pressure increases by a factor of 5 during this process. Calculate the new temperature of the gas in the piston.

Solution: We don't know the volume, pressure or number of moles of gas at any point in this problem, so we will have to solve this problem using ratios.

First, we need to convert temperature to Kelvin: $T = (50 + 273)\,\mathrm{K} = 323\,\mathrm{K}$. Next, write down the ideal gas law: $pV = nRT$. We are interested ultimately in the final temperature, so re-write this as:

$$T = \frac{pV}{nR} \tag{7.43}$$

This holds both initially and after the compression, so $T_1 = p_1V_1/(n_1R)$ and $T_2 = p_2V_2/(n_2R)$.

$$\frac{T_2}{T_1} = \frac{\left(\dfrac{p_2V_2}{n_2R}\right)}{\left(\dfrac{p_1V_1}{n_1R}\right)} = \frac{p_2}{p_1}\frac{V_2}{V_1}, \tag{7.44}$$

since $n_2 = n_1$ and R is a constant. So, the final temperature is

$$T_2 = T_1 \cdot \frac{p_2}{p_1} \cdot \frac{V_2}{V_1} = (323\,\mathrm{K}) \cdot 5 \cdot \frac{1}{3} = 538\,\mathrm{K}, \tag{7.45}$$

or $(538 - 273)^\circ$ C $= 265^\circ$ C.

Note that when using ratios to determine a new value for the pressure or volume, it doesn't matter what units we use for those quantities because the units will cancel between numerator and denominator. Though it is still always necessary to use Kelvin for temperature.

Problems

1. Melting iron

(a) Use your results from Problem 6.2 to estimate the typical thermal displacement for atoms in a chunk of solid iron at room temperature. (Assume a room temperature of 22° C.) Compare your result to the typical lattice separation for the iron atoms. Based on this result and the Lindemann criterion, explain why it is reasonable that iron is a solid at room temperature.

(b) Now, assuming that iron melts when the typical displacement is one-tenth the lattice separation (i.e., $x_{\text{therm}} \approx d/10$, which is the Lindemann criterion), estimate the melting temperature of iron. Compare your result to the experimental value.

(c) Write a sentence explaining in your own words why the melting temperature should be related to the typical thermal displacement x_{therm}. Don't worry so much about the factor 1/10 in the Lindemann melting criterion, but your explanation **should** state why melting occurs when x_{therm} gets sufficiently large.

2. Calculate the thermal speed at temperature 22° C of

(a) molecular oxygen (O_2)

(b) methane (CH_4)

(c) carbon dioxide (CO_2)

3. A 100 g piece of ice at 0° C is placed into a container holding 200 g of water, initially at temperature 25° C. Heat flows from the water to the ice, cooling the water and melting the ice.

(a) Calculate how many moles of ice and how many moles of water are initially present.

(b) Determine how much heat flows out of the water in cooling to 0° C.

(c) Determine how many moles of ice are melted by this added heat.

4. Compare two containers of the same ideal gas; each container has the same volume and the same number of molecules. The temperature of the gas in the first container is twice the temperature in the second container, $T_1 = 2T_2$. Find the following ratios

(a) $v_{\text{therm},1}/v_{\text{therm},2}$.

(b) $K_{\text{molec},1}/K_{\text{molec},2}$.

(c) p_1/p_2.

5. How many moles are in 3 liters of ideal gas at pressure 200 kPa and temperature 100° C?

6. Using the vaporization temperature of water to estimate the pair binding energy for water molecules.

7. Calculate the speed of sound in a gas of pure molecular hydrogen at a temperature of 22° C.

8. One mole of water at 20° C has 20 kJ of thermal energy added. Calculate the number of moles which remain in the liquid state.

9. Rank the following according to the speed of the molecules at room temperature, from fastest to slowest:

(a) copper (solid)
(b) water (liquid)
(c) krypton (gas)
(d) molecular nitrogen (gas)

10. A fixed amount of ideal gas is at temperature 25° C, volume 4.0 L and pressure 100 kPa. The temperature of the gas is increased to 80° C while the volume is decreased to 3.2 L. Determine the new pressure.

11. For silver, the ball-spring parameters are $m = 1.79 \times 10^{-25}$ kg, $d = 2.58 \times 10^{-10}$ m, and $k_{sp} = 21.4$ N/m. Based on this information, estimate the melting temperature and latent heat of fusion for silver.

12. Let's play microwave. This is just fun, and you've got to do it. Load up the molecular dynamics applet, select the solid preset, and click "Start". Now we're going to melt the solid without adding heat. This is exactly what a microwave oven does: the microwaves do work, pushing and pulling molecules around, and this gets converted to thermal energy. So let's do the same thing. You can "pull" a molecule by clicking on it and dragging it. Reach in and pull on a molecule, and then wait and watch how the system responds. Now do it again. Keep doing it until you've fully melted the solid. (Notes: it might help to reduce the "Animation speed" so that you can see what is going on. You also might have to pull your mouse a large distance quickly before letting go when "pulling" a molecule.)

(a) Describe what you observe in the simulation and what you had to do to melt the solid by pulling on individual atoms. **Question:** Why would pulling on just one or two individual atoms melt the entire solid?

(b) Now think of something else cool to do with the applet. Write a few sentences describing what you did and what you found.

13. Aquaman buys a balloon filled with a fixed amount of helium from a street vendor in New York City on a hot $37°$ C day. He measures the pressure inside the balloon to be 1.1 atm (1 atm = 101 kPa). When he arrives at the underwater city of Atlantis, he discovers that the balloon is now 0.40 times the original volume. His thermometer indicates that the ocean has a temperature of $2°$ C. Determine the pressure inside the balloon.

14. Let's consider the ideal gas law $pV = nRT$ qualitatively from a perspective of molecules of the gas hitting the shaded side of a container, as shown in Fig. 7.7.

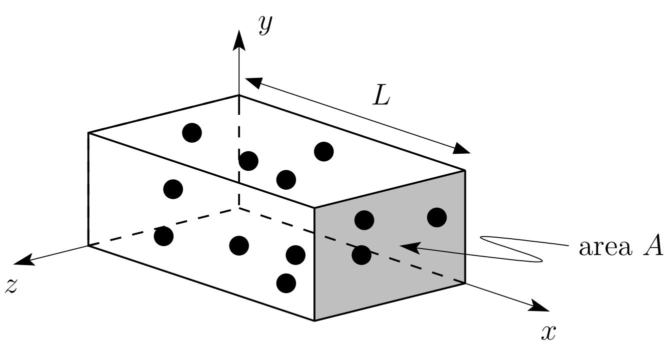

Figure 7.7: For problem 14. Gas in a container of length L and cross section A.

(a) Considering collisions of gas molecules with the wall, explain qualitatively why the pressure of a gas increases if the temperature of the gas increases, with everything else constant. There are two different reasons why increasing the temperature increases the pressure.

(b) Explain qualitatively why the pressure of a gas increases if the number of moles of gas molecules in the gas increases, with everything else constant.

(c) Explain qualitatively why the pressure increases if the volume of the container holding the gas decreases, with everything else constant. There are actually two different reasons why decreasing the volume increases the pressure: one is a result of decreasing L and the other is a result of decreasing A.

15. In this problem you will make some simplifying assumptions and estimate the pressure of a gas of nitrogen molecules from a microscopic picture of the gas. Assume that the gas is in a 10 cm × 10 cm × 10 cm box at room temperature, $T = 22^\circ$ C. The assumptions are:

- All the molecules travel at the speed v_{therm} derived in Example 1. This is not actually true — there is a spread in molecular speeds around the average — but v_{therm} is a typical speed.
- One third of the molecules in the gas travel in the $\pm x$-direction, one third travel in the $\pm y$-direction, and one third travel in the $\pm z$-direction. This is obviously not true, but this assumption will simplify the calculations.

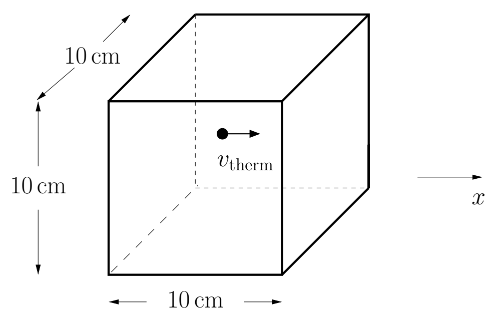

Figure 7.8: Figure for problem 15.

(a) Calculate the numerical value of the *change* in the momentum of a single nitrogen molecule traveling in the x-direction after it collides elastically with the right wall of the container. (If you need help determining the speed of the nitrogen molecules, see Example 1.)

(b) Calculate the number of times this single nitrogen molecule collides with the right wall of the container in 1 second.

(c) Calculate the total change in momentum of the molecule in 1 second due to collisions with the right wall of the container.

(d) Calculate the average force on the right wall of the container due to collisions with the single molecule.

(e) At room temperature and atmospheric pressure, the number density, (i.e., the number of molecules per unit volume) of nitrogen molecules is 2.49×10^{19} molecules/cm^3. Use the second of our simplifying assumptions and calculate the average force on the right wall of the container due to all of the molecules in the gas.

(f) Calculate the pressure that the gas exerts on the right wall of the container. Compare your answer to atmospheric pressure ($1\,\text{atm} = 1.01 \times 10^5\,\text{Pa}$).

Chapter 8

Gas Processes

There are a lot of very useful and important things that can be done with gases. Burning gases in an internal combustion engine can power a car. Boiling water (and its vapor) can turn a turbine in an electrical power generation plant. Expanding lungs enable your body to draw in air and extract oxygen to enable you to live. Vaporization and expansion/compression of methylene chloride and water allows a dunking bird to bob up and down indefinitely over your glass of water in your dorm room. Numerous gas processes in the atmosphere drive the weather system on the Earth. Expanding and contracting gaseous bladders enable fish to swim without sinking or floating to the surface. Adiabatic heating of a collapsing cloud of hydrogen in space provides the spark that triggers nuclear fusion and enables the birth of stars (including our Sun). Hot gases shooting out of an engine can power an airplane or a rocket. And many chemical reactions produce gases that do work on systems (e.g., inflating airbags, firing ammunition, . . .).

These are just a small sample of the ways in which thermodynamic processes play a critical role in practical applications and in the basic functioning of the universe as we know it. In all of these cases, it is critical to understand several things: (1) how much work is required to perform the process or, alternately, how much work is produced by the process; (2) how much energy in the form of heat must be added to the gas (or released from the gas) during the process; and (3) how the thermal energy of the gas either increases or decreases during the process. From this perspective, by far the most important mathematical relation required for analyzing gas processes is one that we have already seen: the First Law of Thermodynamics (Eq. 6.29)

$$\Delta E_{\text{therm}} = Q_{\text{in}} + W_{\text{on}}.$$

In this chapter, we will discuss in detail how we can calculate the work, heat and thermal energy changes that occur in thermodynamic processes.

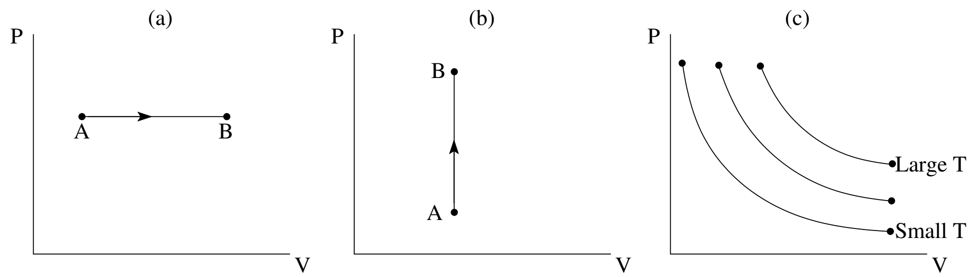

Figure 8.1: Processes on a p-V diagram. (a) Constant pressure expansion. (b) Constant volume process. (c) Three constant temperature (isothermal) processes.

8.1 p-V diagrams

An important tool used in analysis of thermodynamics processes is a plot of pressure versus the volume of a gas, referred to in short as a *p-V diagram.* These diagrams are very useful first because they are directly related to the work done on or by a gas and second because they are convenient for showing the time evolution of a thermodynamic process, particularly *cyclic* processes that repeat over and over again, as is the case with many engineering systems.

Given a fixed amount of a gas, a point on a p-V diagram represents the state of the system. A changing volume and/or pressure results in a curve on a p-V diagram. Figure 8.1 shows a few different processes. Figure 8.1(a) shows a constant pressure process, with an arrow that indicates an increasing volume (expansion). If the arrow were turned around, the same graph would show a constant pressure compression. In the literature, you may see a constant pressure process referred to as an *isobaric* process ("iso" meaning "same" and "baric" meaning "pressure"). Figure 8.1(b) shows a constant volume process, with an arrow that indicates an increasing volume.

Figure 8.1(c) shows a few constant temperature (*isothermal*) processes. Assuming the number of moles of the gas doesn't change during a process, the ideal gas law can be used to determine the relationship between the pressure p and volume V for an isothermal process: $pV = nRT$ implies $p = nRT/V$. The result is a swoopy curve (yes, "swoopy" is a valid scientific expression) since the pressure depends inversely on the volume if n and T are constant. The resulting curve on a p-V diagram is referred to as an "isotherm" and the same graph can show several different isotherms, depending on the magnitude of the temperature. Figure 8.1(c) shows that the isotherms corresponding to higher temperatures are higher and to the right on a p-V diagram. It can also be seen that either a constant volume increase in pressure or an isobaric expansion result in a larger temperature (indicated by moving to a hotter isotherm). These results, of course, are

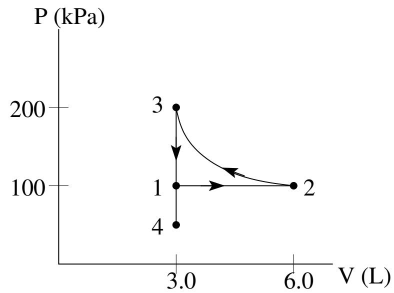

Figure 8.2: Solution for Example 1. Constant pressure expansion ($1 \to 2$) to twice the initial volume, followed by isothermal compression ($2 \to 3$) back to initial volume, followed by cooling at constant volume ($3 \to 4$) to temperature half of the initial value.

both consistent with the ideal gas law.

Example 8.1 Sketching p-V diagrams.

A gas in a cylinder starts at a temperature $T_1 = 300$ K, volume $V_1 =$ 3.0 L and pressure $p_1 = 100$ kPa. It is expanded at constant pressure to twice its initial volume. It is then compressed isothermally back to the original volume, after which it is cooled down to a temperature which is half of the initial temperature. Sketch a p-V diagram for the all three processes.

Solution: See Fig. 8.2. The constant pressure expansion ($1 \to 2$) also results in a temperature increase. Since $pV = nRT$, a doubling of the volume (to 6.0 kPa) at constant pressure results in a doubling of the temperature (to 600 K) as well. So, after the isothermal compression ($2 \to 3$), the gas is at twice its initial temperature and therefore twice the initial pressure; i.e., $p_3 = 200$ kPa. (You could also easily determine the pressure p_3 by noting that with n and T both constant, $p_2V_2 = p_3V_3$.) The final process ($3 \to 4$) brings the gas to a temperature half of its initial value, and since the final volume is the same as the initial volume, the final pressure must by half of the initial pressure; i.e., $p_4 = 50$ kPa.

8.2 Calculating work done in gas processes

Okay, we now have the tools to enable us to calculate the work done on or by a gas during a process. The key is the volume. It turns out that an

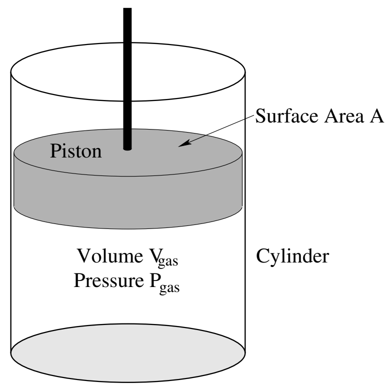

Figure 8.3: Piston with surface area A in a cylinder containing a gas with volume V_{gas} and p_{gas}.

expanding gas does work on its environment, whereas it is necessary for an external agent to do work **on** a gas for it to be compressed. This can be understood qualitatively by considering a balloon. If you want to compress the gas on a balloon, you have to squeeze the balloon, doing work on the gas. But if you stick a pin in the balloon, it will pop and do work on the environment, as evidenced both by the loud "POP!" sound and the pieces of balloon fabric that go flying all over the place after the explosion.

Quantitatively, the work done in an expansion or compression process can be analyzed as a p-V process. For simplicity, let's consider a cylinder with a fitted piston with mass m that seals in the gas but which can slide up and down without friction (Fig. 8.2).

The work done on the piston by the gas inside the cylinder[1] can be calculated from the standard relation for the work done by a spatially-varying force: $W_{\text{by}} = \int_{x_i}^{x_f} F_{\text{gas}}\, dx$. Since the pressure p is the force divided by the surface area, $F_{\text{gas}} = p_{\text{gas}} A$ where A is the cross-sectional area of the piston. Consequently, $W_{\text{by}} = \int_{x_i}^{x_f} pA\, dx = \int_{V_i}^{V_f} p\, dV$ where we are dropping the "gas" subscripts. This result deserves its own line and an equation number:

$$W_{\text{by}} = \int_{V_i}^{V_f} p\, dV \tag{8.1}$$

So, the work done by the gas is simply the integral of the pressure, integrated over the volume. And the work done *on* a gas is just the opposite:

$$W_{\text{on}} = -\int_{V_i}^{V_f} p\, dV \tag{8.2}$$

[1]Note of course that the gas in the cylinder isn't the only thing doing work on the piston in this example: gravity also does work, as does the gas *outside* the cylinder, along with any friction or air resistance.

Note also that we can also calculate the work by determining the area under a curve of p versus V, i.e., the area in a p-V diagram. Here is another example of how useful p-V diagrams can be, since they display quite directly the work done on or by a gas process.

NOTE: the sign of the work is critically important. There is a big difference between a positive and a negative value, e.g., the work done by a process W_{by}. For a car engine, a positive value of W_{by} means that the car is able to drive, whereas a negative value for W_{by} means that you have to push the car to get it to go.

Special cases:

- **Constant volume processes**. If the volume remains constant during a process, the work done on and by the gas is zero.
- **Isobaric (constant pressure) processes**. In this case, the work done by the gas is

$$W_{\text{by}} = \int_{V_i}^{V_f} p\,dV = p\int_{V_i}^{V_f} dV = p\Delta V. \tag{8.3}$$

  So, the work done **on** the gas $W_{\text{on}} = -W_{\text{by}} = -p\Delta V$. This can also been seen easily from a p-V diagram for an isobaric process (Fig. 8.1a): the area under the curve is simply the area of the rectangle $p\Delta V$.
- **Isothermal (constant temperature) processes**. In this case, the work done by the gas

$$\begin{aligned} W_{\text{by}} &= \int_{V_i}^{V_f} p\,dV = \int_{V_i}^{V_f} \frac{nRT}{V}\,dV = nRT\int_{V_i}^{V_f} \frac{dV}{V} = nRT[\ln V]_{V_i}^{V_f} \\ &= nRT(\ln V_f - \ln V_i) = nRT\ln\left(\frac{V_f}{V_i}\right). \end{aligned} \tag{8.4}$$

  And, of course, the work done **on** the gas $W_{\text{on}} = -W_{\text{by}} = -nRT\ln(V_f/V_i)$.

  Useful trick: note that since the ideal gas law states that $pV = nRT$, the work done by a gas during an isothermal process can also be written as $W_{\text{by}} = pV\ln(V_f/V_i)$. This is really useful if you know the pressure and volume at any moment during the isothermal process but don't know the temperature.
- **Straight-line processes on a p-V diagram**. In these cases, the work can be determined by determining the area under the curve.
- If you can't figure it out from any of the four approaches, there is one other way in which you will be able to calculate the work done on or by a gas during a processes: use the First Law of Thermodynamics. If you have a process where you can determine the heat flowing in Q_{in} and the change in the thermal energy ΔE_{therm}, then you will be able to find $W_{\text{on}} = \Delta E_{\text{therm}} - Q_{\text{in}}$.

Remember that the sign of the work depends on whether the volume is increasing or decreasing. If V increase, W_{by} is positive and W_{on} is negative, and if V decreases, then W_{by} is negative and W_{on} is positive.

A note about units: in standard units, if pressure is in N/m^2 (i.e., Pa) and volume is in m^3, the resulting calculated work wil be in J. But in many realistic problems, the pressure is given in kPa and the volume is given in L. Since $1\,\text{L} = 10^{-3}\,\text{m}^3$ and 1 kPa = 10^3 Pa, a calculation of work using L and kPa also gives an answer in J. So, it isn't necessary to convert to Pa and m^3 when doing these calculations.

Example 8.2 Calculating work for gas processes

Calculate the work done on the gas for each of the three processes in Example 1 (see Fig. 8.2).

Solution: Let's deal with each of the three processes in turn:

- **Process** $1 \rightarrow 2$**:** This is a constant pressure expansion. First, we need to recognize that since the volume is increasing, the work done **by** the gas W_{by} is positive, so W_{on} is negative. We can figure this out easily enough by recognizing that W_{by} is simply the area under the curve

$$W_{\text{by}} = \int_{V_1}^{V_2} p\,dV = (100\,\text{kPa})(6.0\,\text{L} - 3.0\,\text{L}) = 300\,\text{J}. \tag{8.5}$$

  Alternately, this could also be determined by $W_{\text{by}} = \int_{V_1}^{V_2} p\,dV = p\Delta V$ (since this is a constant pressure process), giving

$$W_{\text{by}} = (100\,\text{kPa})(6.0\,\text{L} - 3.0\,\text{L}) = 300\,\text{J}. \tag{8.6}$$

  So, $W_{\text{on}} = -W_{\text{by}} = -300$ J.

- **Process** $2 \rightarrow 3$**:** This is an isothermal compression, so $W_{\text{on}} = -W_{\text{by}} = -nRT\ln(V_3/V_2)$. We don't know how many moles of gas there are, but we could figure it out using the Ideal Gas Law. **But we aren't going to do that**, because we can replace nRT with either p_2V_2 or p_3V_3. So,

$$W_{\text{on}} = -p_2V_2\ln\left(\frac{V_3}{V_2}\right) = -(100\,\text{kPa})(6.0L)\ln\left(\frac{3.0L}{6.0L}\right) = 416\,\text{J}. \tag{8.7}$$

  Note that this is a positive number, which makes sense since it is a compression, and we know that W_{on} is positive for a compression. Note also that the work for the process $2 \rightarrow 3$ has a larger magnitude than that for process $1 \to 2$, which also makes sense because there is a larger area under the curve $2 \to 3$ than under the curve $1 \to 2$.

- **Process** $3 \to 4$**:** This is a constant volume process, so $W_{\text{on}} = 0$. This is also consistent with work as area under the curve, as there is no area under the process $3 \to 4$.

8.3 Adiabatic processes

In many real gas processes, there is no heat flow either into or out of the system; i.e., $Q_{\text{in}} = 0$. This occurs if (a) the system is completely isolated from its environment so there is no way for heat to be exchanged with its surroundings (e.g., if it is surrounded by insulation); or (b) if the processes occurs so rapidly that there simply isn't enough time for there to be an appreciable flow of heat into or out of the system.

A process with no heat flow (i.e., $Q_{\text{in}} = 0$) is referred to as an *adiabatic* process. The calculations of Q_{in}, W_{on} and ΔE_{therm} are easier for adiabatic processes, predominately because $Q_{\text{in}} = 0$ (by definition), so if you can determine one of either W_{on} or ΔE_{therm}, you can get the other from the First Law of Thermodynamics. Calculating W_{on} is non-trivial for an adiabatic process (it isn't one of the four special cases discussed in the previous section), so for an adiabatic process, we will typically find ΔE_{therm} first. We'll discuss how to do that in the next section.

First, however, we can quickly get an idea of how an adiabatic process looks on a p-V diagram. Since $Q_{\text{in}} = 0$ for an adiabatic process, the First Law becomes: $\Delta E_{\text{therm}} = W_{\text{on}}$; i.e., any work done on the system goes into increasing the thermal energy. And recall that the thermal energy of a gas increases with the temperature of the gas. For an adiabatic compression, $W_{\text{on}} > 0$ which means that $\Delta E_{\text{therm}} > 0$. This means that for any adiabatic compression, the gas heats up (i.e., the temperature increases). This makes sense: we are doing work on the gas to compress it, and that work goes into heating up the gas and increasing the temperature.

Since an adiabatic compression results in an increase in temperature, the curve on a p-V diagram for an adiabatic compression must go to larger and larger temperature, which means that and adiabatic process is associated with a curve (an "adiabat") that is steeper than those for isothermal processes (i.e., "isotherms"). This can be seen in Fig. 8.4, which shows curves for both adiabatic and isothermal processes.

To make a quantitative statement about the functional form of the adiabatic curves, we need to consider the type of gas. As discussed in section 7.4, the thermal energy of a gas depends on the type of molecules in that

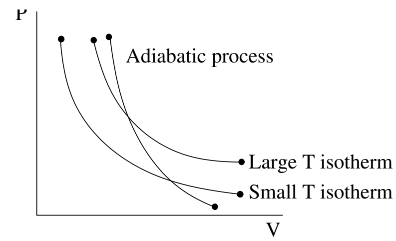

Figure 8.4: p-V diagram showing both isotherms and adiabats.

gas. (See Eqs. 7.8, 7.10 and 7.11.) In particular, the behavior of the gas depends on the number of degrees of freedom f and the "adiabatic exponent" $\gamma = \frac{f+2}{f}$. (Recall that $\gamma = 5/3$ for a monatomic gas with 3 degrees of freedom, and $\gamma = 7/5$ for a diatomic gas with 5 degrees of freedom.) Without going through the full derivation,[2] an adiabatic process has pressure and volume that are related via:

$$pV^\gamma = (\text{constant}). \tag{8.8}$$

From a practical perspective, this means that given three of the four quantities p_i, V_i, p_f and V_f, for an adiabatic process the fourth can be determined from the relation $p_i V_i^\gamma = p_f V_f^\gamma$.

Similarly, a relation can be found between the temperature T and volume V during an adiabatic process[3]: $TV^{\gamma-1} = (\text{constant})$.

Example 8.3 Adiabatic expansion

A diatomic gas with an initial pressure, temperature and volume of 150 kPa, 300 K and 3.5 L is compressed to a volume of 0.5 L; the compression is so fast that there isn't enough time for heat to flow into or out of the gas. Calculate the pressure and temperature of the gas after the compression.

Solution: Since there isn't enough time for heat to flow into or out of the gas, this is an adiabatic process. And since the gas is composed of diatomic molecules at a temperature that isn't too high, there are 5 degrees of freedom for the gas and $\gamma = \frac{f+2}{f} = 7/5$. Since this is an adiabatic process, pV^γ =constant, so so $p_i V_i^\gamma = p_f V_f^\gamma$, which implies

$$p_f = p_i \left(\frac{V_i}{V_f}\right)^\gamma = 150\,\text{kPa}\left(\frac{3.5\,\text{L}}{0.5\,\text{L}}\right)^{7/5} = 2290\,\text{kPa}. \tag{8.9}$$

[2]The full derivation involves solving a differential equation, but differential equations are not required for PHYS 211.

[3]Simply combine Eq. 8.8 with the Ideal Gas Law.

As for the temperature: since this is an adiabatic process, $TV^{\gamma-1} =$ constant, so $T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$ which implies

$$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} = (300\,\mathrm{K}) \left(\frac{3.5\,\mathrm{L}}{0.5\,\mathrm{L}}\right)^{7/5-1} = 653\,\mathrm{K}. \qquad (8.10)$$

Note that the gas heats up, which is always the case for an adiabatic compression.

8.4 Calculating heat and thermal energy changes

Okay, we now have everything that we need to determine the heat Q_{in}, the work W_{on} or W_{by} and the change in thermal energy ΔE_{therm} for an arbitrary process. This will prove to be quite important when we talk about heat engines in a couple of chapters.

We have already discussed the different approaches that you can use to determine the work done on or by a gas (see the bulleted list at the end of Section 8.2). We can make similar bulleted lists for ΔE_{therm} and W_{on} or W_{by}.

To determine the change in the thermal energy for a process:

- The easiest case is if the process is isothermal or if it ends up at the same temperature that it started (e.g., if it is a *cyclic* process), then $\Delta E_{\text{therm}} = 0$.

- If you know how many degrees of freedom the gas molecules have ($f =$ 3 or 5 for monatomic and diatomic gases, respectively, for example), then $\Delta E_{\text{therm}} = \frac{f}{2} nR\Delta T$. This is convenient if you know the starting and ending temperature.

  Useful trick redux: the same $pV = nRT$ trick can be used here. If you don't know the initial and final temperatures but you **do** know the initial and final pressure and volume, then $\Delta E_{\text{therm}} = \frac{f}{2}(p_2V_2 - p_1V_1)$.

- If neither of the above work, but you can figure out Q_{on} and W_{on}, then you can get ΔE_{therm} from the first law: $\Delta E_{\text{therm}} = Q_{\text{on}} + W_{\text{on}}$.

We can make a similar bulleted list for how to determine the heat flow Q_{in}:

- The easiest case is if the problem states that the process is adiabatic[4]: in this case, $Q_{\text{in}} = 0$.

[4] The problem might be subtle about this: it might say "... for a well-insulated system ..." or "... for a very rapid process ...," either of which imply no heat exchange between the system and the environment.

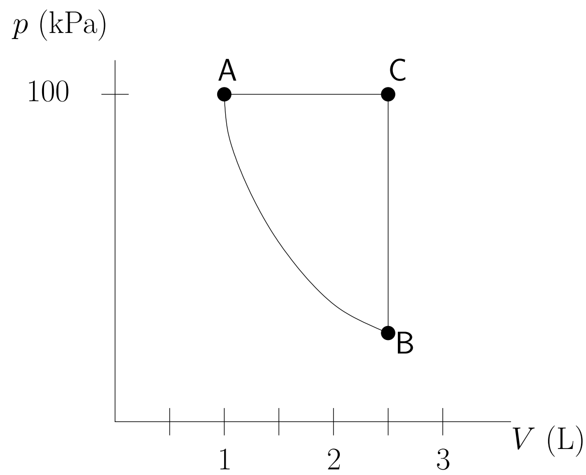

Figure 8.5: For Example 4.

- If the process is not adiabatic, then you will be using the first law of thermodynamics to find Q_{in}: $Q_{\text{in}} = \Delta E_{\text{therm}} - W_{\text{on}}$.

For all of these calculations, **make sure that you check the signs** of everything. There is a really big difference between positive and negative values for each of ΔE_{therm}, Q_{in} and W_{on} for a gas process.[5]

Example 8.4 A complete cycle

Consider a fixed amount of an ideal gas of diatomic molecules undergoing the processes shown in Fig. 8.5. The gas starts at point **A** in the diagram and expands along the path **A**$\longrightarrow$**B** with no heat flowing into or out of the gas.

(a) Calculate the pressure of the gas at point **B** after the expansion.
(b) Determine the net work done **on** the gas and the heat flow Q_{in} in one complete cycle (**A**$\longrightarrow$**B**$\longrightarrow$**C** $\longrightarrow$ **A**).

Solution: (a) Since the process $A \to B$ is adiabatic, pV^γ = constant and $p_A V_A^\gamma = p_B V_B^\gamma$ so

$$p_B = p_A \left(\frac{V_A}{V_B}\right)^{7/5} = 100\,\text{kPa} \times \left(\frac{1.0\,\text{L}}{2.5\,\text{L}}\right)^{7/5} = 27.7\,\text{kPa}. \qquad (8.11)$$

[5]If you ever forget this, set your car in neutral and push it from here to New York — that's what you would have to do for a car whose engine is described by a positive value of W_{on}.

(b) We will calculate the work done on the gas for each of the three processes and add up the results to get the net work. To figure out the total heat flow Q_{in} we could do something similar — i.e., calculate Q_{in} for each process and then add them up. But there is an easier way: since we know that $\Delta E_{\text{therm}} = 0$ for the complete cycle, once we have determined the total W_{on} for the complete cycle, we can use the First Law of Thermodynamics to get Q_{in} for the complete cycle.

Let's start with **process** $A \to B$. This is an adiabatic process, so $Q_{\text{in}} = 0$. Calculating work is tricky for adiabatic processes, so we'll find ΔE_{therm} first and then use the First Law to get W_{on}. This is a diatomic gas, so $f = 5$ and $\Delta E_{\text{therm}} = \frac{f}{2} nR\Delta T$. We could calculate T_B, but we won't bother, because ΔE_{therm} can be found with the pressures and volumes given:

$$\begin{aligned}\Delta E_{\text{therm}} &= \frac{f}{2}(p_B V_B - p_A V_A) = \frac{5}{2}\Big(27.7\,\text{kPa} \times 2.5\,\text{L} - 100\,\text{kPa} \times 1.0\,\text{L}\Big) \\ &= -76.7\,\text{J}.\end{aligned} \tag{8.12}$$

We can now find the work done on the gas via the First Law:

$$W_{\text{on}} = \Delta E_{\text{therm}} - Q_{\text{in}} = -76.7\,\text{J} - 0 = -76.7\,\text{J}. \tag{8.13}$$

Process $B \to C$**:** this is a constant volume process, so $W_{\text{on}} = 0$.

Process $C \to A$**:** this is a constant pressure (isobaric) process, so

$$W_{\text{on}} = -p\Delta V = -p(V_A - V_C) = -100\,\text{kPa}(1.0\,\text{L} - 2.5\,\text{L}) = 150\,\text{J}. \tag{8.14}$$

So, the net work W_{on} done on the gas for the complete cycle $A \to B \to C \to A$ is

$$\begin{aligned}W_{\text{on}}^{\text{net}} &= W_{\text{on}}^{A\to B} + W_{\text{on}}^{B\to C} + W_{\text{on}}^{C\to A} \\ &= -76.7\,\text{J} + 0\,\text{J} + 150\,\text{J} = 73.3\,\text{J}.\end{aligned} \tag{8.15}$$

As for the total heat flow Q_{in} for the entire cycle, we can use the First Law:

$$Q_{\text{in}} = \Delta E_{\text{therm}} - W_{\text{on}} = 0 - 73.3\,\text{J} = -73.3\,\text{J}. \tag{8.16}$$

So, what this means is that for the entire cycle, 73.3 J of heat flows **out** of the gas.

Problems

1. Three identical gas-cylinder systems are compressed from the same initial state to final states that have the same volume, one isothermally, one adiabatically, and one isobarically. Which system has the most work done on it? The least?

2. The relation between thermal energy and temperature depends on the type of gas molecule: for a monatomic gas, $\Delta E_{\mathrm{therm}} = \frac{3}{2}nR\Delta T$ and for a diatomic gas, $\Delta E_{\mathrm{therm}} = \frac{5}{2}nR\Delta T$. In your own words — and discussing explicitly the microscopic picture of gases — explain why it is that you need to put more thermal energy into a diatomic gas than for a monatomic gas to raise the temperature by 1 K.

3. Consider a piston in a car engine that is compressing an air-gasoline mixture before ignition, all in the gas state.

(a) Under what circumstances would the compression be adiabatic? (There are a couple of ways to get an adiabatic compression, one of which is more relevant to a car engine.)

(b) Use the First Law of Thermodynamics to argue that the gas heats up if the compression is adiabatic. Where does that additional thermal energy come from during this process?

4. A gas with n moles of molecules starts with a temperature T_1, pressure p_1 and volume V_1. Draw a p-V diagram for the following multi-step processes (and label each process with its letter a, b, c or d):

(a) The gas expands at a constant pressure to twice its initial volume.

(b) The gas is then compressed isothermally back to its initial volume.

(c) The gas then expands adiabatically back to its initial pressure.

5. A gas with n moles of molecules starts with a temperature T_1, pressure p_1 and volume V_1. Draw a p-V diagram for the following multi-step processes:

(a) The gas is heated up at constant volume, doubling its initial temperature.

(b) The gas expands adiabatically until the pressure returns to $1.5p_1$.

(c) A vent is opened, allowing gas to escape until the pressure returns to the original pressure p_1.

(d) The gas is compressed isothermally until the volume returns to the original volume V_1.

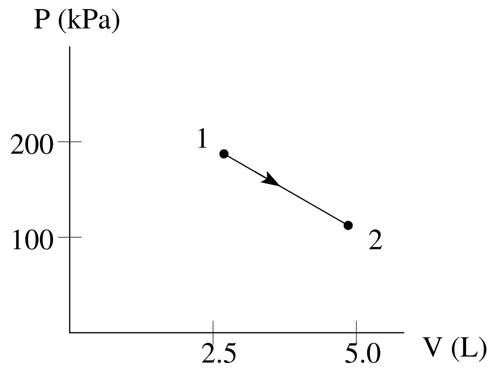

Figure 8.6: For Prob. 6.

6. Calculate the work done on a gas as it undergoes the process shown in the p-V diagram (Fig. 8.6).

7. A piston initially contains 1.2 moles of a monatomic gas at a pressure of 210 kPa and a volume of 0.25 L. The gas expands at constant pressure to a volume of 0.47 L. Calculate the work done on the gas during this process.

8. A piston initially contains 0.75 moles of a diatomic gas at a pressure of 105 kPa, a temperature 350 K and a volume of 0.33 L. The gas is ignited, raising the temperature rapidly to 550 K without a significant change in the volume. Calculate the work done on the gas during this process.

9. A monatomic ideal gas undergoes an isothermal compression from an initial volume of 7.0 L and pressure 100 kPa to a final volume and pressure of 2.0 L and 350 kPa, respectively. The temperature along the isotherm is 120° C.

(a) Determine the number of moles of this gas.
(b) Determine the work done *on the gas* during this process.
(c) Determine the heat flowing into the gas during this process.

10. Consider a fixed amount of an ideal monatomic gas undergoing the *adiabatic* process illustrated in Fig. 8.7.

(a) Calculate the number of moles in the gas.
(b) Calculate the work done on the gas between points **A** and **B**.

11. (a) Use the Equipartition Theorem to determine a relation between the thermal energy change ΔE_{therm} and the temperature change ΔT for a monatomic *flattium* gas where the molecules can move only in two dimensions.

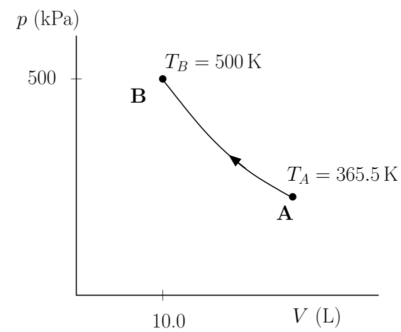

Figure 8.7: For problem 10.

(b) Do the same thing, but this time for a diatomic flattium gas.

12. The "5/2" in the ΔE_{therm} relation for a diatomic gas (as opposed to "3/2" for a monatomic gas) comes from the fact that a diatomic molecule can also rotate in two different directions, adding two terms to the Equipartition Theorem. But if the gas is *really* hot, it can also excite *vibrations* for diatomic molecules, which provides two more terms in the equation for the energy of a diatomic molecule: one for the kinetic energy of the vibration and one for the potential energy of the vibrating molecule. What might you expect for the relation between ΔE_{therm} and ΔT for a diatomic gas where the molecules are also vibrating?

13. A piston contains 0.65 moles of diatomic Nitrogen gas (N_2), with an initial pressure and volume of 110 kPa and 0.15 L, respectively. The piston compresses the gas adiabatically to a volume 0.062 L. Calculate the pressure of the gas after this compression.

14. A piston contains 0.45 moles of monatomic Helium gas, with an initial volume and temperature of 0.12 L and 285 K, respectively. The piston expands the gas adiabatically to a volume of 0.31 L. Calculate the temperature of the gas after this expansion.

15. A fixed amount of diatomic ideal gas goes through the cycle as shown in Fig. 8.8.

$A \to B$: adiabatic compression

$B \to C$: constant pressure expansion

$C \to A$: cooling at constant volume

Fill in the chart below for the change in thermal energy, heat flow in, and work done *on the gas* for each of the processes and for the complete cycle. **Show all work.**

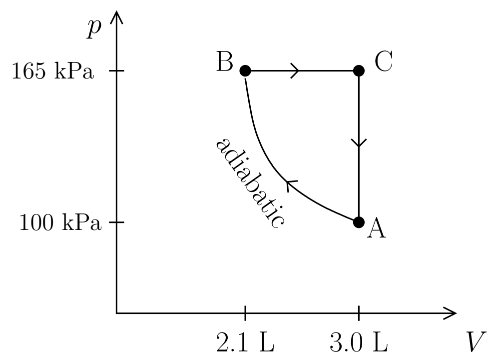

Figure 8.8: For problem 15.

	ΔE_{therm} (J)	Q_{in} (J)	W_{on} (J)
$A \to B$	116 J		
$B \to C$			
$C \to A$			
cycle			

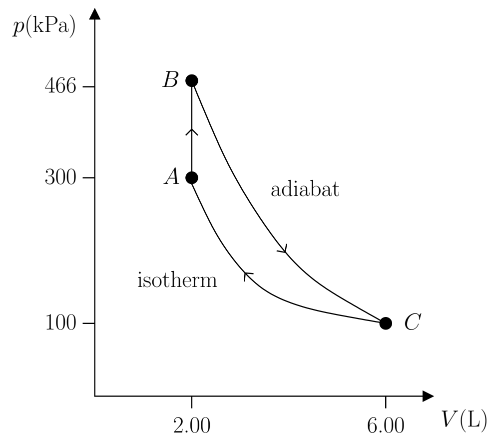

Figure 8.9: For problem 16.

16. An amount 0.20 moles of diatomic ideal gas undergoes the processes shown in the p-V diagram (Fig. 8.9). The table below has boxes for the change in thermal energy of the gas, the heat added to the gas, and the work **done on** the gas, all in joules. Energy values have already been entered in two of the boxes. Fill in the remaining entries with values accurate to three significant digits. Be sure to show your work for each entry.

	ΔE_{therm}	Q_{in}	W_{on}
$A \to B$	830 J		
$B \to C$			-830 J
$C \to A$			

17. The pV diagram (Fig. 8.10) shows a cyclic process for 2 moles of an ideal monatomic gas.

(a) Determine the temperature of the gas at state **A**.

(b) Determine the heat flow into the gas during the process **A→B**.

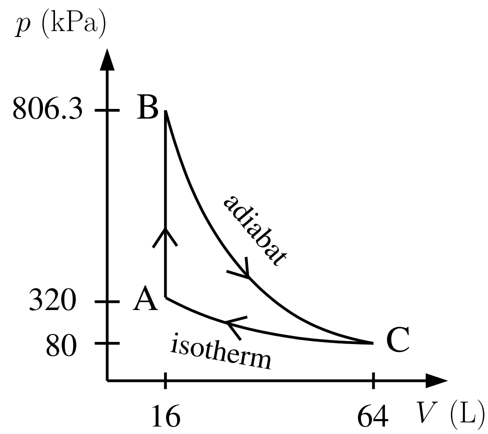

Figure 8.10: For problem 17.

(c) Determine the heat flow into the gas during the process **B**→**C**.

(d) Is the work done on the system for the complete cycle positive, negative or zero? (No calculation required.)

Chapter 9

Second Law of Thermodynamics and Entropy

In this chapter we will discuss one of the most significant developments in the history of science — the development of a *statistical* theory of thermodynamics. Here is the question: if a chunk of ice, or a glass of water, or an air-filled balloon is composed of 10^{22} or 10^{23} molecules, isn't it necessary to describe the dynamics of each individual molecule? To determine the force on each molecule and solve Newton's second law to figure out its motion? The answer is no, thankfully. Instead, we can treat each of these molecules as though they are behaving randomly, and recover all the results of thermodynamics from a probabilistic treatment.

The importance of this statistical approach cannot be overstated. The idea that we can treat thermodynamic systems probabilistically led to a revolution in scientific thought that ranks up there with Newton's development of classical physics, Pasteur's development of germ theory of disease, Einstein's theory of relativity, and the development of quantum mechanics (which you'll see in PHYS 212).

We will introduce statistical mechanics by revisiting the basic phenomenon of heat flow, the spontaneous thermal energy transfer from hotter objects to colder objects. The direction of the heat flow is determined by what is known as the *second law of thermodynamics.* We can derive the second law of thermodynamics from probability arguments; essentially, thermal energy flow is dictated by moving from an improbable to a probable situation. Entropy is introduced as a measure of probability. And along the way to understanding the second law, we will provide a general definition of temperature.

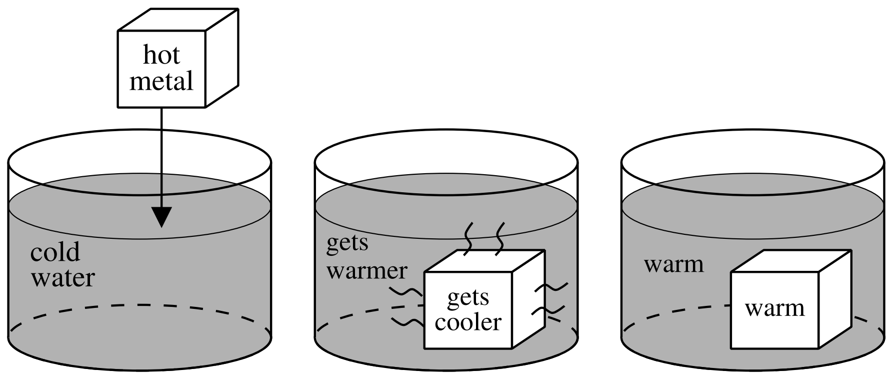

Figure 9.1: A hot piece of metal is placed into cold water. Thermal energy is transferred from the hot metal to the cold water until they are in thermal equilibrium.

9.1 Heat Flow Revisited

Consider the following process, illustrated in Fig. 9.1: a hot piece of metal is placed into a container holding cold water. As time passes, thermal energy flows from the metal to the water, making the metal colder and the water warmer. Eventually, the two are at the same temperature and no more thermal energy is transferred. This is *heat*: the spontaneous thermal energy transfer due to the temperature difference, as we identified in section 6.10.

In a heat flow scenario, such as this one, the first law of thermodynamics states that energy is conserved, and so we must have

$$\Delta E_{\text{therm,water}} = -\Delta E_{\text{therm,metal}}. \tag{9.1}$$

However, energy conservation would be equally well satisfied if the heat flowed the other way. Imagine putting the hot metal into cold water and finding that the metal becomes increasingly hotter while the water becomes increasingly cooler, beginning to freeze. Absurd! This is never observed to happen. And yet it would be perfectly consistent with the first law of thermodynamics.

What this process illustrates is that there must be an additional law of nature involved that determines the direction of heat flow. In a fit of creativity, physicists decided to call this the *second* law of thermodynamics. There are many equivalent ways to state the second law. We will begin with the *Clausius statement* of the second law, since it is the most intuitive.

> 2nd Law of Thermodynamics (Clausius):
> Heat cannot flow spontaneously from a material at lower temperature to a material at higher temperature.

Let's examine this. First, note that the law is, at this point, empirical, which means it is a statement about the observed behavior of nature. The second law rules out the absurd scenario whereby heat flowed from the cold water to the hot metal. But, note that the second law makes no statement about whether the heat will actually flow from the metal to the water. According to the second law, this heat flow is allowed, but not required. That is exactly what we want from a general law, since after all the metal and the water may or may not be thermally coupled.

Temperature plays a crucial role in the second law, since the question of whether heat is allowed to flow from A to B or instead from B to A is answered by the temperatures T_A and T_B. Temperature plays the role of nature's traffic cop, enforcing thermodynamic "one-way streets."[1] The primary topic of this chapter is the explanation of why temperature plays this role.

Another interesting aspect of the second law is the phenomenon of *irreversibility.* Many processes in nature are reversible. A movie of the flight of a ball thrown straight up into the air, turning around and coming back down, looks the same whether played forward or backward. This is because Newton's law are *reversible* as long as friction is negligible. But once heat flows from the hot metal to the cold water, it will never spontaneously flow back again. A movie of the process (with some thermometers used to make the temperature visible) would look different played backward versus forward. Physicists believe the second law is the origin of any irreversibility observed in nature, which is to say, the second law of thermodynamics plays a crucial role in determining the direction of time flow.

Interestingly, the second law is unique among laws of physics. Most laws are simply inferred from the behavior of nature. We don't know why energy conservation happens; we just know it does. The second law is different because we can essentially derive it. We know *why* it happens. It is ultimately a statement about probability: thermal energy flows spontaneously from hotter objects to colder objects because that brings the system to a state with a more likely arrangement of energy.

The rest of this chapter is concerned with expanding our probabilistic understanding of the second law and temperature.

9.2 Microstates, Macrostates, and Multiplicity

To explain how probabilities work in thermodynamics — and ultimately to explain entropy and how it relates to the second law of thermodynamics — it is necessary to discuss some fundamental concepts of probability. We start with definitions of *microstates* and *macrostates*:

[1] However, nature needs no traffic court since its one-way streets, like its speed limit, are self-enforcing.

> A *macrostate* is a specification of the macroscopic state of the system. For example, the pressure, temperature, and number of moles of an ideal gas would specify a macrostate.
>
> A *microstate* is the detailed specification of the microscopic state of the system. In the ideal gas example, the microstate would be precise values for the position and velocity of every single molecule.

A macrostate can have many microstates associated with it. In the ideal gas example, there are many possible arrangements of the molecules that are consistent with having, say, one mole of gas with atmospheric pressure and room temperature. This brings us to multiplicity:

> The *multiplicity* Ω of a macrostate is the number of microstates associated with that macrostate.

Let's explore these ideas with a specific example: a pair of six-sided dice, one red and one green.[2] There are 36 possible outcomes of rolling these dice, listed in Table 9.1, and the sum of the two dice can be any number between two and twelve. Not every sum is equally probable, however. If you roll the dice many times, you will notice you get a sum of seven much more often than, say, a sum of twelve.

The 36 possible outcomes are the microstates. The red dice showing '5' and the green die showing '3' would be a particular microstate (labeled 5-3 in Table 9.1). The sum of the dice, eight in this case, represents a macrostate. Notice that there are many ways to roll a sum of eight; or stated another way, there are multiple microstates associated with the macrostate '8.' The number of ways to roll an '8' is the multiplicity Ω. Looking at Table 9.1, we see there are five different ways to roll an '8', so the multiplicity $\Omega = 5$.

The multiplicity of a macrostate is useful to know because it tells us the probability of obtaining that particular macrostate. Each of the 36 microstates for a pair of dice is equally likely. The reason that a sum of seven is a more likely outcome than a sum of twelve is not because 4-3 is more likely than 6-6 (it's not!), rather, there are more ways to roll a '7.'

Now let's come back to physics. The macrostate of a collection of molecules could be defined in terms of the number of particles and the amount of energy E_{therm} they have. A microstate would correspond to a particular arrangement of the energy among the molecules. Since there are many possible ways to arrange the energy among the molecules, there are many microstates associated with this macrostate. The number of possible

[2]Having dice of the same color wouldn't change anything. We just use different colors to help label the dice.

sum	rolls (red die–green die)						Ω	probability
2	1-1						1	1/36
3	1-2	2-1					2	2/36 = 1/18
4	1-3	2-2	3-1				3	3/36 = 1/12
5	1-4	2-3	3-2	4-1			4	4/36 = 1/9
6	1-5	2-4	3-3	4-2	5-1		5	5/36
7	1-6	2-5	3-4	4-3	5-2	6-1	6	6/36 = 1/6
8	2-6	3-5	4-4	5-3	6-2		5	5/36
9	3-6	4-5	5-4	6-3			4	4/36 = 1/9
10	4-6	5-5	6-4				3	3/36 = 1/12
11	5-6	6-5					2	2/36 = 1/18
12	6-6						1	1/36

Table 9.1: The 36 possible results from rolling a pair of dice (one red, one green).

ways to arrange the given amount of energy would then be the multiplicity Ω.

To go from multiplicity to probability we need one more piece of information. In the case of the dice, each of the 36 possible outcomes was equally likely, assuming that the dice were fair, returning each of the six values with equal probability. Does this apply as well for our system of N particles sharing a total energy E_{therm}? In general, we cannot prove this, but to make progress we will assume that it is true.

THE FUNDAMENTAL ASSUMPTION OF STATISTICAL MECHANICS:
All of a system's accessible microstates are equally likely.

"Accessible microstates" here means simply those which are allowed by energy conservation. The motivation for this assumption is that whatever the specific dynamics are, however the molecules are colliding and sloshing energy back and forth among each other, they eventually visit every possible state allowed by energy conservation. So a sequence of snapshots of the system would look like randomly selected examples of possible microstates. In the end, nature has confirmed that starting with the fundamental assumption leads to predictions that match experiments extremely well. Now we shall see what the fundamental assumption buys us.

9.3 Einstein Solid

We now develop the ideas of the previous section in the context of a specific model. The simplest model to work with, it turns out, is not the ball-spring model or the ideal gas, but rather a variation of the ball-spring solid called the Einstein solid. Experiments on very cold metals showed that their specific heats could fall well below the value $3R$, suggesting something not contained in the ball-spring model was occurring at low temperatures. Einstein showed that a quantum mechanical version of the ball-spring model could explain this result.[3] To begin, notice that a three-dimensional oscillator, such as the molecule in the ball-spring model, can be written as a sum of three independent, one-dimensional oscillators:

$$\begin{aligned} E_{\text{ball}} &= \left(\tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2\right) + \left(\tfrac{1}{2}k_{sp}x^2 + \tfrac{1}{2}k_{sp}y^2 + \tfrac{1}{2}k_{sp}z^2\right) \\ &= \left(\tfrac{1}{2}mv_x^2 + \tfrac{1}{2}k_{sp}x^2\right) + \left(\tfrac{1}{2}mv_y^2 + \tfrac{1}{2}k_{sp}y^2\right) + \left(\tfrac{1}{2}mv_z^2 + \tfrac{1}{2}k_{sp}z^2\right) \end{aligned} \tag{9.2}$$

In the second grouping, each term in parentheses is an oscillator moving in one particular direction and independent of the motion in the other perpendicular directions. Thus a set of N molecules in the ball-spring model is equivalent to $3N$ one-dimensional oscillators. In what follows we will be working primarily with the one-dimensional oscillators so we let N represent the number of oscillators instead of the number of molecules. The number of molecules is then $N/3$.

Einstein proposed to treat the one-dimensional oscillators quantum mechanically, which should be appropriate when the temperature is low enough. We will not discuss quantum mechanics here — that is a topic for PHYS 212 — but we will summarize the main results of interest to us. The energy levels of the quantum harmonic oscillator are not continuous but rather discrete (or *quantized*). This is illustrated in Fig. 9.2. At very low energies we cannot vary the oscillator energy up or down by arbitrarily small amounts, but rather can only add energy in discrete chunks. Furthermore, for the quantum harmonic oscillator, these energy levels are equally spaced. Therefore we can write the energy level of an oscillator as

$$E_{\text{osc}} = E_0 + n\epsilon \qquad \text{where } n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{9.3}$$

Here E_0 is the lowest energy level possible, and we may increase the energy by adding an integer number of "energy units" of size ϵ.

Now consider a system of two oscillators, with a total energy of three "energy units." These oscillators bounce energy back and forth and so one of the oscillators may have at a given instant anywhere from zero to all three of the energy units. Let n_1 be the number of energy units that the first oscillator has, and n_2 the number of energy units for the second oscillator. Specifying n_1 and n_2 determines a particular microstate. The total energy

[3]The complete description of very cold metals requires an additional modification, worked out by a Dutch physicist named Peter Debye. We will not consider the Debye theory here.

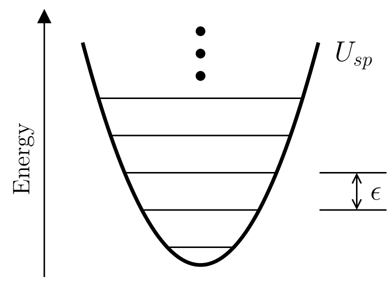

Figure 9.2: The quantum harmonic oscillator has discrete energy levels, shown as horizontal lines. The energy difference between successive levels is ϵ.

of three units implies $n_1 + n_2 = 3$, so the possible microstates, written as (n_1, n_2), are

$$(3,0), \quad (2,1), \quad (1,2), \quad (0,3).$$

Evidently, the multiplicity of the macrostate with two oscillators and a total of three energy units is $\Omega = 4$. That is, there are four different microstates with this total energy.

Example 9.1 Three oscillators, two energy units

Write down all the microstates for a system of three oscillators and a total of two energy units, and determine the multiplicity.

Solution: For microstates written as (n_1, n_2, n_3), we need to have $n_1 + n_2 + n_3 = 2$, so the possible microstates are

$$(2,0,0), \quad (0,2,0), \quad (0,0,2), \quad (1,1,0), \quad (1,0,1), \quad (0,1,1),$$

and the multiplicity $\Omega = 6$.

It is feasible to determine the multiplicity directly by counting the microstates when the number of oscillators and energy units is small. But this becomes unwieldy very quickly as the number of oscillators and energy units is increased. Fortunately, we can derive the general result for N oscillators and q total energy units, which is

$$\Omega = \frac{(q+N-1)!}{q!\,(N-1)!}. \tag{9.4}$$

The factorial function is defined as $n! = n(n-1)(n-2)\cdots 2\cdot 1$. For example, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$. A special case is the factorial of the number zero:

by definition, $0! = 1$. The meaning of $n!$ is that it is the number of distinct ways to order n objects. The number of ways to order zero objects is taken to be 1.

Example 9.2 Checking the multiplicity formula.

Verify the Einstein solid multiplicity formula, Eq. (9.4), for the cases of two oscillators with three energy units and three oscillators with two energy units.

Solution: For two oscillators and three energy units ($N = 2$ and $q = 3$) the multiplicity formula gives

$$\Omega = \frac{(3+2-1)!}{3!\,(2-1)!} = \frac{4!}{3!\,1!} = \frac{24}{6 \cdot 1} = 4, \tag{9.5}$$

which matches our result above. For the second case, $N = 3$ and $q = 2$, giving

$$\Omega = \frac{(2+3-1)!}{2!\,(3-1)!} = \frac{4!}{2!\,2!} = \frac{24}{2^2} = 6, \tag{9.6}$$

verifying the second case.

Factorials become very large very quickly. For example, $100! \approx 10^{157}$, which is an amazingly large number. An Einstein solid with 100 oscillators and 200 energy units has a multiplicity $\Omega = 2.8 \times 10^{82}$. Now you can appreciate having Eq. (9.4) to work with instead of counting all possible microstates. And imagine how large the result would be for Avogadro's number of oscillators!

9.4 Coupled Einstein Solids

Our original goal was to understand heat flow. That is, why thermal energy spontaneously goes from hotter objects to colder objects. To that end, we will now consider two Einstein solids, solid A with a number N_A oscillators and q_A energy units, and solid B with N_B oscillators and q_B energy units. If solids A and B are brought into thermal contact, then they will be able to pass energy units back and forth while maintaining a fixed total $q_{\text{tot}} = q_A + q_B$. But which way will the energy go, on average? And when will it come to thermal equilibrium? Let us try to address these questions.

q_A	q_B	Ω_A	Ω_B	Ω_{AB}
0	6	1	28	28
1	5	3	21	63
2	4	6	15	90
3	3	10	10	100
4	2	15	6	90
5	1	21	3	63
6	0	28	1	28

Table 9.2: Possible macrostates for system A and B sharing six units of energy, with $N_A = 3$ and $N_B = 3$.

Once the two Einstein solids are thermally coupled and exchanging energy, A and B should be regarded as *subsystems* of the combined system. For a particular division of energy among the two subsystems, we have a multiplicity Ω_A that depends on N_A and q_A, and a multiplicity Ω_B that depends on N_B and q_B.

How do we calculate the combined multiplicity of the system? If you have three pairs of pants and five shirts, then you have $3 \cdot 5 = 15$ possible combinations you can make, at least in polite company. Similarly, subsystem A may be in any of the number Ω_A microstates and subsystem B in any of Ω_B microstates, so the number of paired microstates we can make is the product $\Omega_{AB} = \Omega_A \Omega_B$. This is the combined multiplicity of the system.

Let's consider a specific case. Let $N_A = 3$ and $N_B = 3$, and $q_{\text{tot}} = q_A + q_B = 6$. The two systems may divide up the six energy units a variety of ways, as shown in Table 9.2. For each choice, the multiplicities Ω_A and Ω_B and the combined multiplicity Ω_{AB} are given. Note that the most probable arrangement of energy, the one with the largest multiplicity, is the one with three energy units in each subsystem. If subsystem A started with zero energy units and subsystem B with six units, then simple random energy exchanges would move the coupled systems toward the more probable state with $q_A = q_B = 3$. This is a clue about the origin of the second law.

From Table 9.2 we see that the most probable situation is only slightly more probable than the other possibilities. This changes dramatically as the system size is increased. In Fig. 9.3 we plot the combined multiplicity as a function of q_A for various numbers of oscillators and energy units. As the figure shows, when the numbers become larger, say in the thousands, the multiplicity function becomes sharply peaked. Some particular division of energy between the two subsystems is vastly, hugely, awesomely, mind-bogglingly more probable[4] than all others. This we identify as the equilibrium division of energy. Now imagine what occurs when you approach Avogadro's number

[4] I.e., it isn't just a little more probable, it is a **lot** more probable.

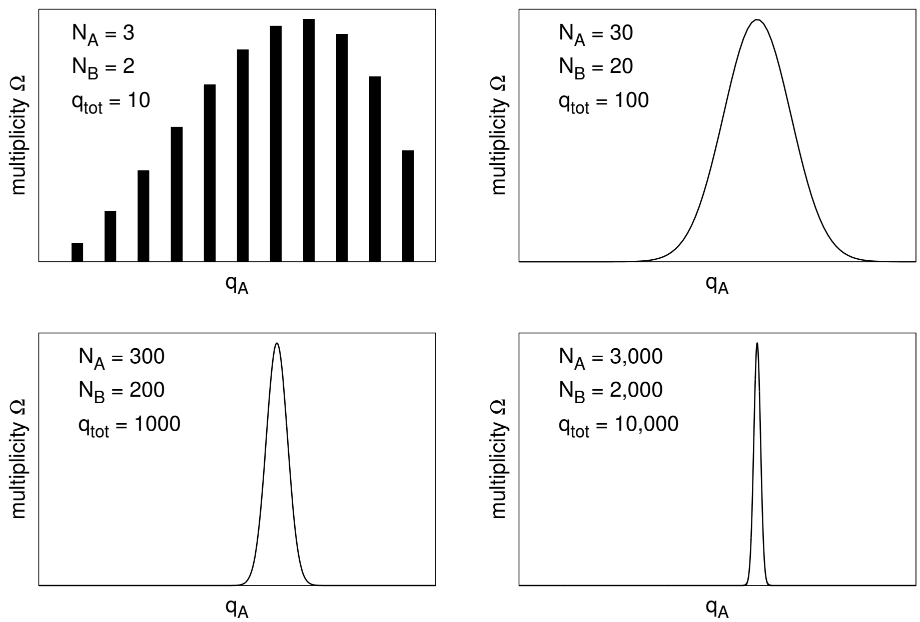

Figure 9.3: Plots of the multiplicity as a function of q_A for a variety of system sizes. Note that q_B is determined by $q_A + q_B = q_{\text{tot}}$.

of energy units. The multiplicity function becomes completely sharp. There is some particular division of the energy between subsystems A and B that is ridiculously, overwhelmingly, staggeringly[5] more probable than any other.

Now we have the probabilistic origin of the second law. Subsystems A and B, before they are thermally coupled, can be prepared with any thermal energy we would like. We made the metal object hot and the water cold before plunging the metal into the water. But once the subsystems are thermally coupled, they will move from whatever division of energy they started with toward the maximally probable arrangement of energy for the coupled system. They are irresistibly led to it by essentially random exchanges of energy between the subsystems. The energy transferred along the way is what we had previously identified as heat.

To summarize:

> The second law of thermodynamics is a result of a system prepared in an improbable initial state then moving to a vastly more probable final state.

[5] "...vastly, hugely, awesomely, mind-bogglingly, ..." and that doesn't even begin to cover it!

This is an incredibly important result!!!! With this statement, we don't have to worry at all about the detailed, Newtonian mechanics of the (many, many) individual molecules or atoms in a solid, liquid or gas. We treat all the motion as though it is random and then simply figure out the probabilities.

9.5 Entropy

Entropy is part of the title of the chapter; perhaps it is time we introduced it. The fact is, we have already been discussing the entropy, because entropy is simply the multiplicity cast into a more convenient form, by means of a logarithm. We define entropy as

$$S = k_B \ln \Omega. \tag{9.7}$$

The factor of Boltzmann's constant plays little role here, apart from giving entropy units (which are J/K).[6] The logarithm is a monotonic function, which means that the larger Ω gets, the larger S gets. So being the most probable state is the same as being the highest entropy state. This is a really important statement, so important that we will elevate it to box-dom:

> Entropy is a measure of probability: the more probable a state, the higher its entropy.

Entropy is often incorrectly described as a measure of the disorder of a system. This is simply not true; entropy is measure of probability and probability only. It **is** true that higher entropy states are often more disordered than lower-entropy states, but this is not always true; there are many examples of systems that become **more** ordered as their entropy increases.

We can now write the second law of thermodynamics rather concisely as a statement of probability, given in the boxed statement at the end of the previous section:

$$\Delta S_{\text{total}} \geq 0. \qquad \text{(Entropic version of 2nd law)} \tag{9.8}$$

Starting from some initial state that is not the maximum entropy state, the combination of all our subsystems will exchange thermal energy and move spontaneously toward the maximum entropy state. And for large systems, it moves irreversibly: there is a negligibly small probability of moving away from the maximum entropy state (think about the sharply peaked multiplicity).

[6] By the way, Boltzmann was the one who realized that the second law had a probabilistic origin, and Eq. (9.7) is engraved on his tombstone. Check it out if you're ever in Vienna.

Note that the entropic form of the second law refers to the **total** entropy of a system, i.e., the **total** entropy cannot decrease. But the entropy of part of a system **can** decrease. So, for instance, it is very possible to have a chemical reaction where the stuff inside your beaker ends up with a lower entropy, as long as there is a corresponding increase in entropy somewhere else (most likely in the air around the beaker whose entropy increases when heated up by heat flowing from the beaker).

We could have expressed all this with the multiplicity, so why take a logarithm and call it entropy? There are three reasons. First, since multiplicities become very, very large for even modest sized systems, we find more workable expressions if we use the logarithm. For example, in the previous case of 100 oscillators with 200 energy units, we get an entropy of

$$S/k_B = \ln \Omega = \ln\left(\frac{299!}{200!\,99!}\right) = 190, \tag{9.9}$$

which is much nicer to manipulate and plot than 10^{82}.

The second reason is that the combined entropy of two systems is simply the sum,

$$S_{AB} = S_A + S_B, \tag{9.10}$$

which you will show in Problem 14. When we are trying to identify the maximum entropy state, we can combine the contributions S_A and S_B from subsystems A and B by simply adding them together (like we would for energies). That will turn out to be handy now as we finally come to the definition of temperature.

The third reason is historical: it so happens that entropy was defined by Clausius a few years before Boltzmann developed a probabilistic theory for thermodynamics. Clausius defined the quantity that he called *entropy*[7] in terms of energy flow in a thermodynamical system (to be discussed in the next chapter). He even stated the entropic form of the second law of thermodynamics, though no one at the time understood that this is really a statement of probability. So, taking the logarithm of multiplicity was needed to keep the entropic statement of the second law consistent with that proposed by Clausius.

9.6 The Definition of Temperature

As we discussed in Chapter 6, temperature is often defined in terms of the thermal kinetic energy. Certainly thermal kinetic energy and temperature are related, via the equipartition theorem, so it is a useful and convenient

[7] Clausius chose the word *entropy* partially after the Greek word *trope* which means *transformation* and partially because he wanted a word that sounded similar to *energy* since he defined entropy in terms of an energy flow.

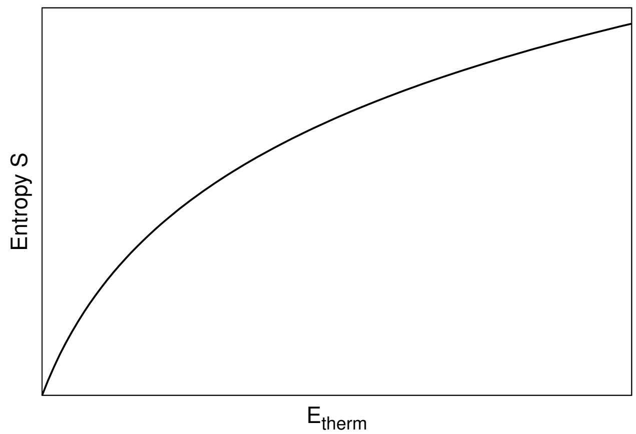

Figure 9.4: Entropy as a function of thermal energy.

picture to have. But defining temperature this way leaves its most fundamental role — namely, that it is the traffic cop dictating which way thermal energy will spontaneously flow — completely unexplained. In this section we will introduce a definition of temperature that naturally explains its presence in the Clausius statement of the second law. Conveniently, this *second law temperature* turns out to be the same temperature we know and love from the ideal gas law, the equipartition theorem, and the ball-spring solid.

Let's think of the entropy of a system as a function of its thermal energy. Adding more thermal energy to a system gives more ways to distribute the energy, and so increases the multiplicity. This means an increase in entropy, so S should be an increasing function of E_{therm}. A typical dependence of entropy on E_{therm} is shown in Fig. 9.4. Note that the entropy is increasing with E_{therm}, but also note that the rate of increase slows down with increasing energy. That is, the slope is steadily decreasing as E_{therm} increases. This can be understood as a type of diminishing returns: systems with very low E_{therm} can gain a lot of multiplicity by adding energy. Once the thermal energy is high, additional thermal energy has less impact on the entropy.

Now let's couple two subsystems, A and B. The combined energy is fixed, $E_{\text{total}} = E_A + E_B$. Consequently, as system A gains energy, system B loses energy, and vice-versa. In Fig. 9.5 we plot both S_A and S_B, but notice that the S_B curve is flipped over left to right. This is because $E_B = 0$ occurs at the right side of the plot, where E_A is at its maximum, and E_B increases as you move to the left. The reason for plotting it this way is that we can, for a particular choice of E_A, read off both $S_A(E_A)$ and $S_B(E_B)$. Also shown on the plot is the combined entropy $S_{\text{total}} = S_A + S_B$.

Now imagine starting with a relatively small value of E_A, where the

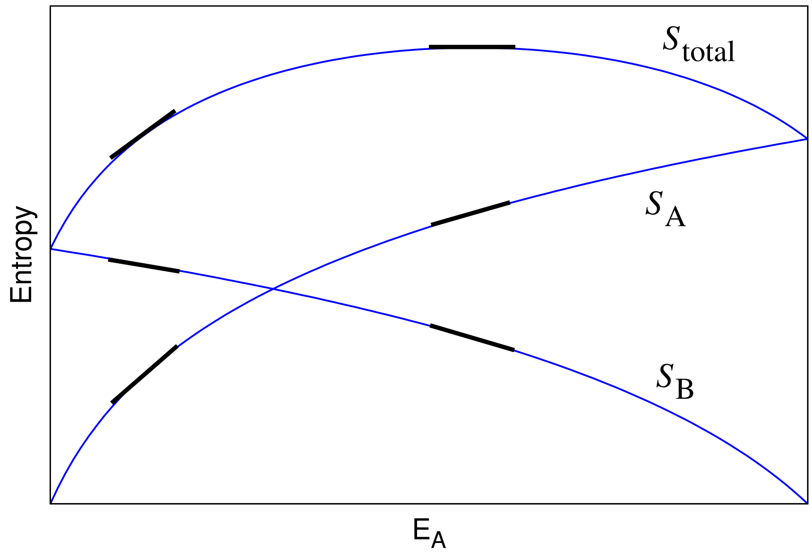

Figure 9.5: Entropies of subsystems A and B, as well as the combined system entropy S_{total}, all plotted versus E_A.

heavy lines are drawn on the left. What would be the net effect on the entropy if we were to take some energy from system B and give it system A? The plot shows that S_B would decrease and S_A would increase. The plot also shows that, since the S_A curve in this region is steeper than the S_B curve, system A would gain more entropy than system B would lose. In other words, S_{total} would increase. Therefore, the "force" of probability pushing towards a (vastly) more probable state dictates that energy flows from system B to system A.

What the previous analysis should make clear is that the question of which way the energy will flow is determined by the magnitude of the *slope* on an entropy versus energy graph. Whichever system, A or B, has the steeper slope will be the one to receive the energy.

Let's carry this analysis further. After some energy has flowed from A to B, we find that E_A has increased to where the second set of heavy lines are drawn. Here, the slopes of the S_A and S_B curves are equal in magnitude and opposite in sign. Any entropy change of system A is canceled by the entropy change of system B, so there is no longer entropy gained by increasing E_A (or decreasing it). Thermal energy will no longer be transferred because we are at the maximum combined entropy, which can be seen from the plot of S_{total}, and we have reached thermal equilibrium. Any additional transfer of energy (in either direction) will result in a decrease in total entropy.

All this discussion leads to the notion that the slope dS/dE_{therm} is directing the thermal energy traffic. Whichever subsystem has the smaller slope will give up energy to the subsystem which has the larger slope. Hence, we define temperature as

$$\boxed{\frac{1}{T} \equiv \frac{dS}{dE_{\text{therm}}}, \qquad (9.11)}$$

and our probability analysis becomes equivalent to the Clausius statement.

This definition, then, explains the role of temperature in the second law, but does it match our previous notions of temperature? And what does it mean intuitively? First, yes, it does match the ideal gas temperature, etc. This can be shown by deriving the equipartition theorem from this definition of temperature; all our previous uses for temperature (such as the ideal gas) had their origin in the equipartition theorem.

As for an intuitive meaning, think of it this way: inverse temperature (that is, $1/T$) is a measure of how much use a system has for energy. When a system can find many ways to divide up the energy, then adding some energy will increase S a lot. That is a low temperature system. A high temperature system is one where diminishing returns has set in, and additional energy does not result in a substantial entropy increase.

Finally, note that for large systems we can add some amount of energy without significantly changing the temperature (for example, adding 10 joules of thermal energy to a cup of water). In this case, we can approximate Eq. (9.11) as

$$\frac{1}{T} \approx \frac{\Delta S}{\Delta E_{\text{therm}}} \qquad \text{or} \qquad T \approx \frac{\Delta E_{\text{therm}}}{\Delta S}. \qquad (9.12)$$

This is often a handy way to *estimate* temperature from entropy change or vice-versa.

Example 9.3 The Temperature of my Coffee

Adding 50 J of thermal energy to my coffee cup caused its entropy to increase by an amount of 0.17 J/K. Estimate the temperature of my coffee.

Solution: According to Eq. (9.12) we have

$$T \approx \frac{\Delta E_{\text{therm}}}{\Delta S} = \frac{50\,\text{J}}{0.17\,\text{J/K}} = 294\,\text{K}. \qquad (9.13)$$

That's room temperature. Yuck!

Problems

1. Consider an Einstein solid with three oscillators and four units of energy.

(a) Calculate the multiplicity for this macrostate.

(b) Write out the triplet for each possible microstate. For example, the microstate where the first oscillator has all the units of energy can be written as $(4, 0, 0)$. Confirm that you find the correct number of microstates.

2. Calculate the multiplicity of an Einstein solid with 24 oscillators and 15 energy units.

3. Suppose you roll a fair six-sided die three times in a row.

(a) Determine the probability of getting exactly the sequence 1–3–2?

(b) Now determine the probability of getting any other particular sequence (hint: no calculation necessary).

(c) What is the probability of rolling a sum of 6?

4. For two Einstein solids with $N_A = 3$ and $N_B = 3$ and six energy units, how many times more probable is the macrostate with equally shared energy than the macrostate where system A has all the energy? Use Table 9.2.

5. Is it really true that the entropy of an isolated system consisting of two Einstein solids never decreases? Consider a pair of very small solids. Explain why this statement is more accurate for large systems than for small systems.

6. A large object's entropy is observed to increase by 0.15 J/K when we add 45 J of thermal energy. Assume that this causes a negligible increase in the temperature of the object. Determine the approximate temperature of the object.

7. The idea of "diminishing returns" says that while the entropy does increase with increasing thermal energy, the slope is decreasing (see Fig. 9.4). The Einstein solid multiplicity, like most materials, shows this behavior. Here is how to see it:

(a) For an Einstein solid with 10 oscillators and 5 energy units, calculate how much the entropy increases, i.e. ΔS, if you add one more energy unit (you may leave your answer in terms of k_B).

(b) Now consider an Einstein solid with 10 oscillators and 15 energy units, and calculate how much the entropy increases if you add one more energy unit.

(c) Do your answers to (a) and (b) confirm the diminishing returns? Explain why.

8. For two Einstein solids A and B, the entropy as a function of thermal energy is given by

$$S_A = k_B\,400\ln(E_A/300) \qquad S_B = k_B\,100\ln(E_B/800)$$

where E_A and E_B are the thermal energies of systems A and B. If the two solids are brought to thermal equilibrium, what relation, if any, can be made between the final energies $E_{A,f}$ and $E_{B,f}$?

9. Consider a very strange system whose multiplicity is $\Omega_A = 1$ regardless of how much energy it has. Imagine starting this system with some amount of energy and bringing it into thermal contact with system B, an Einstein solid.

(a) In which direction will the energy flow, or will no energy flow?

(b) What can you say about the energies of the final state? For example, will they be equal? If they are unequal, which is larger? Is there anything more you can conclude?

10. A substance has entropy $S = c\sqrt{E_{\text{therm}}}$, where c is some constant. Use the definition of temperature to find E_{therm} as a function of T.

11. Consider two Einstein solids with $N_A = 3$ and $N_B = 3$ and eight energy units.

(a) Make a table like Table 9.2. Note that many of the multiplicities you will need are already in Table 9.2, so there is no need to re-calculate everything.

(b) How many times more probable is the macrostate with equally shared energy than the macrostate where system A has all the energy?

12. An Einstein solid has four oscillators and three units of energy.

(a) Calculate the multiplicity of the solid.

(b) Identify all the possible microstates using the parenthesis notation of Example 1.

13. System A and system B are both large. For system A, adding 250 J of thermal energy causes an entropy increase of 0.80 J/K. For system B, adding 250 J of thermal energy causes an entropy increase of 0.60 J/K.

(a) Without mentioning temperature, use probability arguments to determine which way thermal energy will flow when systems A and B are thermally coupled.

(b) Estimate the temperature of each object and check that your result is consistent with part (a).

14. Show that $S_{AB} = S_A + S_B$ follows from the definition of entropy.

15. Entropy applies to more than just heat flow. We can use entropy and the second law of thermodynamics to discuss movement of air in a room.

(a) Consider a room with only 100 gas molecules. Theoretically, the gas molecules can move anywhere in the room. Calculate the probability that all 100 of the molecules will be found on one particular side of the room.

(b) Now, consider a real room with a realistic amount of gas in it – let's say that there are 10^{26} gas molecules in the room. Calculate the probability that all of these gas molecules will be found in one particular side of the room. (Note: the probability is **so** small that your calculator or computer might simply give "0" for the answer.)

(c) Is it reasonable to say that you will "never" find all the air in one side of the room?

(d) Now, write a couple of sentences explaining why it is (from a probability perspective) that when a perfume bottle is opened, the scent of the perfume will spread throughout the room.

(e) After the perfume smell has spread throughout the room, would you expect all of the perfume molecules to go back into the bottle? Discuss this using the entropic form of the second law of thermodynamics.

16. The graphs in the figure below give plots of entropy S vs. E_{therm} for two different solids, A and B. Solid A starts with indicated energy E_A and entropy S_A, and Solid B starts with E_B and S_B. When Solid A has energy E_A, the slope of the entropy vs. energy curve is $dS_A/dE = 0.2\,\text{K}^{-1}$, and when Solid B has energy E_B, the slope of the entropy vs. energy curve is $dS_B/dE = 0.4\,\text{K}^{-1}$.

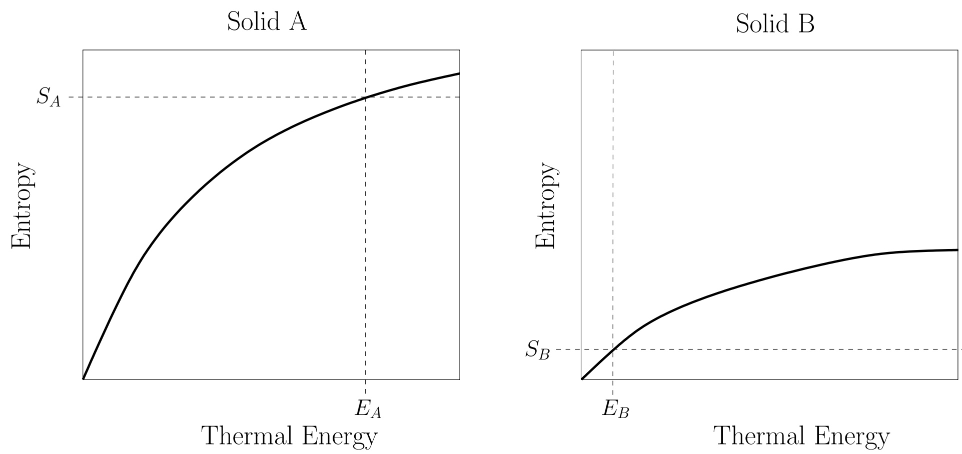

Figure 9.6: Figure for Problem 16

The two solids are brought into thermal contact with each other so that energy can flow between them.

(a) Which way will the energy flow: from A to B, from B to A, or will no energy flow? Give qualitative reasoning to support your answer.

(b) Now let's get quantitative. Calculate the approximate entropy changes ΔS_A and ΔS_B, and ΔS_{total} if 3 J of energy flow between the two solids in the direction that you chose in part (a).

(c) By what factor has the multiplicity for the total system increased from this energy transfer? In other words, calculate the ratio of multiplicities $\Omega_{\text{after}}/\Omega_{\text{before}}$.

Note: The answer you get will be a ridiculously, mind-boggling, impossible-to-put-into-words-just-how-huge-it-really-is number that you will not be able to calculate — you'll have to express it as $e^{\text{something really big}}$. To give you and idea of just how large this number is, if you were to write it as a digit followed by a bunch of zeros, and if each digit were 5 mm wide, the number would fill up several *light years.*

(d) Explain in your own words why heat flows in this system when the two solids are brought into contact. Don't use the words "entropy" or "second law" but rather explain it based on probabilities.

17. The graphs in the figure below give plots of entropy S vs. E_{therm} for two different solids, A and B. Solid A starts with indicated energy E_A and entropy S_A, and Solid B starts with E_B and S_B. When Solid A has energy E_A, the slope of the entropy vs. energy curve is $dS_A/dE = 0.5\,\mathrm{K}^{-1}$, and when Solid B has energy E_B, the slope of the entropy vs. energy curve is $dS_B/dE = 0.1\,\mathrm{K}^{-1}$.

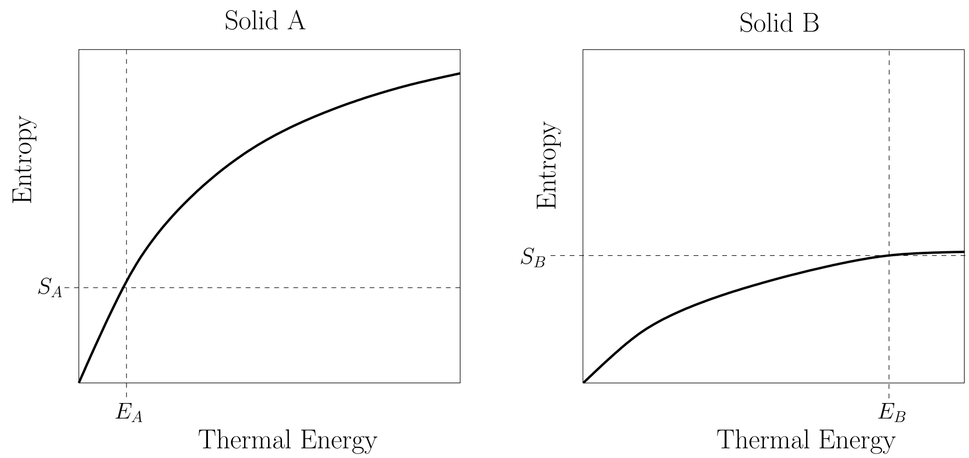

Figure 9.7: Figure for Problem 17

The two solids are brought into thermal contact with each other so that energy can flow between them.

(a) Which way will the energy flow: from A to B, from B to A, or will no energy flow? Give qualitative reasoning to support your answer.

(b) Now let's get quantitative. Calculate the approximate entropy changes ΔS_A and ΔS_B, and ΔS_{total} if 2 J of energy flow between the two solids in the direction that you chose in part (a).

(c) By what factor has the multiplicity for the total system increased from this energy transfer? In other words, calculate the ratio of multiplicities $\Omega_{\text{after}}/\Omega_{\text{before}}$.

Note: The answer you get will be a ridiculously, mind-boggling, impossible-to-put-into-words-just-how-huge-it-really-is number that you will not be able to calculate — you'll have to express it as $e^{\text{something really big}}$. To give you and idea of just how large this number is, if you were to write it as a digit followed by a bunch of zeros, and if each digit were 5 mm wide, the number would fill up several *light years*.

(d) Explain in your own words why heat flows in this system when the two solids are brought into contact. Don't use the words "entropy" or "second law" but rather explain it based on probabilities.

18. System A and System B are brought into thermal contact when the energy in A is $E_A = 1000\,\mathrm{J}$ and the energy in B is $E_B = 1100\,\mathrm{J}$. Using the table below, listing energies and corresponding entropies of the two systems, determine whether heat will flow from A to B, or from B to A. Show all your work.

E_A (J)	E_B (J)	S_A (J/K)	S_B (J/K)
950	1150	6.76	10.34
975	1125	6.84	10.21
1000	**1100**	**6.93**	**10.08**
1025	1075	7.02	9.95
1050	1050	7.10	9.82

Chapter 10

Heat Engines

10.1 Introduction

Mechanical energy is essential for our every day life: cars move along roads and highways, electrons flow through semiconductor devices in our iPods, and blood flows through our arteries. Mechanical energy makes matter do things, and converting other forms of energy to mechanical energy is an essential technological challenge. Batteries and our bodies convert chemical bond energy into mechanical energy. And nuclear reactors convert mass into mechanical energy.

But we have seen that there is a considerable amount of energy contained in the disorganized thermal motion of the molecules and the disorganized pushes and pulls on their molecular neighbors. Harnessing some of this thermal energy and converting it to organized mechanical energy provides yet another source of mechanical energy. But just how do we go about doing this?

It is tempting to imagine some kind of molecular referee who could convince the all the molecules in a material to align their motion. If the molecules in your textbook could do this, your book would zip away from you at many hundreds of miles per hour, so it would be a very useful trick. However, no such microscopic referee exists. In fact, this trick would violate the second law of thermodynamics, moving from a more probable to a less probable arrangement of velocities.[1]

Nevertheless, it is still possible to convert some (but not all) thermal energy to mechanical energy. That is, we can design devices to do this while still satisfying the second law. These devices are called *heat engines*, and they played an essential role in the industrial revolution and continue to play a vital role in modern society.

[1]This microscopic referee was first pondered by Maxwell, and is commonly referred to as Maxwell's Demon. He showed that the referee could make heat flow from a colder object to a hotter one — in contradiction to the second law, which of course is impossible.

In this chapter we will study the basic physics behind heat engines. We will discuss how the basic principle of a heat engine can be understood using the arguments of statistics and entropy discussed in Chapter 9. We will also describe the basic gas cycles that many such engines employ. As a fundamental starting point, any heat engine must satisfy the second law of thermodynamics, $\Delta S_{\text{total}} \geq 0$, so we begin with developing a convenient and powerful relationship between entropy change and heat.

10.2 Entropy Change and Heat

As discussed in the previous chapter, entropy is a measure of probability; specifically, it is Boltzmann's constant times the logarithm of the multiplicity. While we can work with the multiplicity and take logarithms for simple enough models, we often want to know the entropy (actually, the entropy *change* ΔS) for more complicated situations without having to sort out exactly what is going on with the multiplicity.

In many situations it is possible to do this. We begin with our result from the previous chapter:

$$\frac{1}{T} = \frac{dS}{dE_{\text{therm}}}, \tag{10.1}$$

which we can rewrite as a relation between a small (infinitesimal) entropy change dS and a small thermal energy change dE_{therm},

$$dS = \frac{dE_{\text{therm}}}{T} \qquad (W = 0). \tag{10.2}$$

In our development of the definition of temperature, we only considered energy transfers between subsystems A and B that happened spontaneously, due to the increased probability associated with the new energy distribution. In other words, we only allowed for thermal energy changes due to *heat* and not due to *work*. Hence the $W = 0$ label in Eq. (10.2).

Recall that the first law of thermodynamics, for small amounts of heat and work, says

$$dE_{\text{therm}} = dQ + dW. \tag{10.3}$$

If no work is being done, dE_{therm} is the same thing as dQ: the thermal energy has changed by however much heat flow has occurred. Thus, we could equally well write Eq. (10.2) with a dQ in the numerator.

So the question is, what is the appropriate generalization of Eq. (10.2) to cases where there is both heat flow and work done? Should the numerator still be dE_{therm}, or should it be dQ, or something else entirely?

The answer is: it depends. If the work is being done slowly enough that the system remains in thermal equilibrium (which means that the basic hypothesis of all microstates being equally likely is at all times still true), then we have a clear answer: the numerator should be dQ, and so

$$dS = \frac{dQ}{T} \qquad \text{(whether or not } W = 0\text{).} \tag{10.4}$$

A good rule of thumb for "slow enough" is that whatever moving object is doing the work should move slower than the speed of sound. In many cases this isn't much of a limitation. For our purposes, we will assume that we remain in equilibrium for all the processes we consider.[2]

But you may be wondering why doing work (slowly enough) doesn't affect the entropy, that is, ΔS only depends on the heat flow. The full explanation is beyond the scope of this course, but here is the flavor of it. We could do work on a solid by squeezing it, and this would certainly increase the thermal energy. The Einstein solid of the previous chapter would respond to the squeezing by having an increased energy spacing ϵ, but not by having more "energy units." So the multiplicity wouldn't change, even though the thermal energy has gone up. And of course if the multiplicity doesn't change, the entropy doesn't change.

In practical terms, Eq. (10.4) is a very handy tool for calculating entropy changes. Of course, we usually have more than a small amount of heat flow, so we will need to use calculus to add up the net entropy change:

$$\Delta S = S_B - S_A = \int_A^B dS = \int_A^B \frac{dQ}{T}. \qquad \text{(equilibrium processes)} \tag{10.5}$$

Often we are considering constant temperature situations, and then this result simplifies even further:

$$\Delta S = \frac{1}{T}\int dQ = \frac{Q}{T} \qquad \text{(constant temperature)} \tag{10.6}$$

The sign of Q is important here! When Q is positive, ΔS is positive, and when Q is negative, ΔS is negative. Or to put it another way: heat flow in increases the entropy, and heat flow out decreases the entropy. **Do not forget this!** The second law is commonly misunderstood to say that all entropies must always increase. This is simply not true. The second law only tell us the *total* entropy must increase.

There are three common situations where the temperature is constant, even though heat is flowing in or out of the system.

- *for an isothermal process* — isothermal expansion or contraction of a gas is, by definition, at constant temperature (iso = "equal", thermal = "temperature").

[2] When this isn't the case, and the system goes out of equilibrium, we do not have a general expression for the change in entropy. This is an active area of research today!

- *during a phase change* — the latent heat at a phase transition (e.g., melting/solidifying or vaporizing/condensing) keeps the temperature constant.
- *for a thermal reservoir* — if a system is very large, modest amounts of heat flow will not affect the temperature. For example, dumping a cup of coffee into the ocean will not change the ocean's temperature measurably.

Example 10.1 Entropy Change of Melting Ice

Consider an 18 g ice cube at 0° C. Heat flows in until is has changed phase to 0° C water. The water molecules are now free to wander, which increases their number of possible microstates. How much has the entropy increased?

Solution: Since the molar mass of H_2O is 18 g, our ice cube contains one mole. Thus the heat required to melt it is (via Table 7.3)

$$Q = nL_f = 1\,\mathrm{mol} \cdot 6.01\,\mathrm{kJ/mol} = 6010\,\mathrm{J}. \tag{10.7}$$

Now we can find the entropy change

$$\Delta S = S_{\mathrm{water}} - S_{\mathrm{ice}} = \frac{Q}{T} = \frac{6010\,\mathrm{J}}{273\,\mathrm{K}} = 22.0\,\mathrm{J/K}. \tag{10.8}$$

Note that we have to use Kelvin for this to work.

In some cases, however, T is changing while the heat flows. In this case we must evaluate some kind of integral to find ΔS. We will restrict ourselves to the cases where no work is being done and there is no phase change (i.e., nothing melting, solidifying, condensing or vaporizing). That is, heat is flowing in or out of a solid or liquid, whose volumes are essentially constant. In this case,

$$dQ = dE_{\mathrm{therm}} = nC\,dT \tag{10.9}$$

that is, we can relate the small heat flow to a small temperature change. Putting this into our integral expression gives

$$\Delta S = \int_A^B \frac{dQ}{T} = \int_{T_A}^{T_B} \frac{nC\,dT}{T} \tag{10.10}$$

Since the specific heat usually is nearly constant with respect to temperature, we may bring it outside of the integral:

$$\Delta S = nC \int_{T_A}^{T_B} \frac{dT}{T} = nC(\ln T_B - \ln T_A) = nC \ln(T_B/T_A). \tag{10.11}$$

Note that when the temperature increases, ΔS is positive, while when the temperature decreases, ΔS is negative, since the natural log of a number less than one is negative.

Example 10.2 Entropy Change from Heating Water

Let's pick up with that 18 g of 0° C water, and now add heat until it has become 100° C water (but not yet started to boil). How much does the entropy increase?

Solution: We have one mole of water, and we get the molar specific heat of water from Table 7.1. We need to use Kelvin for our temperature units, so

$$\begin{aligned}\Delta S &= nC \ln(T_f/T_i) = 1\,\text{mol} \cdot 75.3\,\text{J/mol·K} \cdot \ln\left(\frac{373\,\text{K}}{273\,\text{K}}\right) \\ &= 23.5\,\text{J/K}\end{aligned} \tag{10.12}$$

Notice in this case we didn't need to calculate the amount of heat flow involved.

One final special case in which entropy changes are easy to calculate is for a *cyclic* process, i.e., a process where the system (or part of the system) ends up in the same state that it started in. In this case,

$$\Delta S_{\text{cyclic}} = 0 \qquad \text{(cyclic processes)} \tag{10.13}$$

10.3 Second Law and Heat Flow

Our new relatively simple relation between heat flow and entropy change can be directly brought back to the second law. Recall the Clausius statement of the second law, that heat can only spontaneously flow from higher temperature to lower temperature. Suppose we have a pair of bricks A and B with temperatures $T_A = 400\,\text{K}$ and $T_B = 300\,\text{K}$. We bring the bricks into thermal contact and let 20 J of heat flow from the higher temperature brick to the lower temperature brick. This is a small enough amount of energy that the temperature of the bricks is essentially unchanged (they are acting as reservoirs), so we can use the constant temperature approximation (Eq. 10.6) for entropy changes.

Then the total entropy change is

$$\begin{aligned}\Delta S_{\text{total}} &= \Delta S_A + \Delta S_B = \frac{Q_A}{T_A} + \frac{Q_B}{T_B} \\ &= \frac{-20\,\text{J}}{400\,\text{K}} + \frac{+20\,\text{J}}{300\,\text{K}} = -0.050\,\text{J/K} + 0.067\,\text{J/K} \\ &= 0.017\,\text{J/K}. \end{aligned} \tag{10.14}$$

Notice the signs of Q_A and Q_B: since heat was flowing out of brick A, Q_A is negative. We find that even though the entropy of brick A went down, the total entropy of bricks A and B went up, as required by the second law.

If we had tried, as a thought experiment, to send the 20 J in the other direction, this would have changed all the signs, and we would be confronted with a $\Delta S_{\text{total}} < 0$, violating the second law.

More generally, for some amount Q flowing from reservoir A to reservoir B, we have

$$\Delta S_{\text{total}} = \frac{-Q}{T_A} + \frac{Q}{T_B} = Q\left(\frac{1}{T_B} - \frac{1}{T_A}\right) \tag{10.15}$$

Since the second law requires $\Delta S_{\text{total}} \geq 0$, we see that T_A must be greater than T_B for this to happen. And so we have recovered the Clausius statement of the second law, i.e., that heat can flow spontaneously only from higher to lower temperature.

10.4 Heat Engines

Let's return to the question at the beginning of the chapter: how can we harness some of the thermal energy of some object and convert it to mechanical energy? We know how to spontaneously decrease the thermal energy of an object: put it into contact with something at a lower temperature. So we can extract thermal energy as *heat*, but that doesn't yet give us *mechanical energy*, which is what we are after. We need somehow to transform heat to work: start with the spontaneous energy flow due to a temperature difference, but convert it to something that will turn a crank or generate electricity or lift a weight. Once we have energy available in the form of work, we can use it to manipulate mechanical energy however we like.

Can we simply convert all the heat to work? The first law of thermodynamics, a.k.a. energy conservation, would have no problem with this. This hypothetical engine is illustrated in Fig. 10.1. In both scenarios in

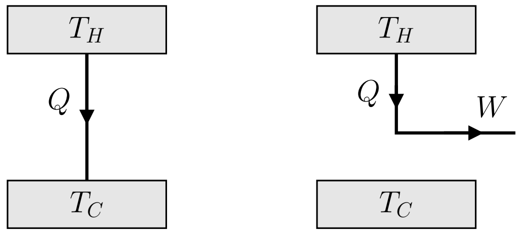

Figure 10.1: On the left, heat flows from a hot reservoir at temperature T_H to a cold reservoir at T_C. On the right, how the diagram would be altered if we could convert the heat Q into work W.

this figure, the hot reservoir is giving off heat Q, and so has negative entropy change. Since we are dealing with a reservoir, we use the constant temperature approximation (Eq. 10.6) to find

$$\Delta S_H = -\frac{|Q|}{T_H}. \tag{10.16}$$

(We use absolute value bars so that there no confusion about the sign of Q). For the figure on the left, this negative entropy change is allowed, because it is offset by the positive entropy change of the cold reservoir, $\Delta S_C = |Q|/T_C$. But for the figure on the right, there is no compensating positive entropy change. And so if we could convert all heat to work, then

$$\Delta S_{\text{total}} = \Delta S_H = -\frac{|Q|}{T_H} < 0 \tag{10.17}$$

which violates the second law! So we cannot do this.

But, you may have noticed that we could still satisfy the second law in the previous example even if we didn't dump all of the heat Q into the cold reservoir. Suppose we only dumped enough heat to make the positive ΔS_C large enough to compensate for the negative ΔS_H. That would leave a little energy that we could conceivably convert to work, while satisfying the second law (and the first, for that matter).

A schematic diagram of this process — called an *engine diagram* — is shown in Fig. 10.2. An amount of heat Q_H is pulled from the hot reservoir, and an amount Q_C is dumped into the cold reservoir. In between, some device which we'll call the working substance intercepts this heat and produces work. For now, don't worry about how this might actually be accomplished; we'll discuss that in detail in the next section. Instead, focus on the big picture: such a device will be consistent with the first law as long as we conserve energy. Looking at the arrows for energy flow, this tells us

$$|Q_H| = |Q_C| + |W|. \qquad \text{(first law)} \tag{10.18}$$

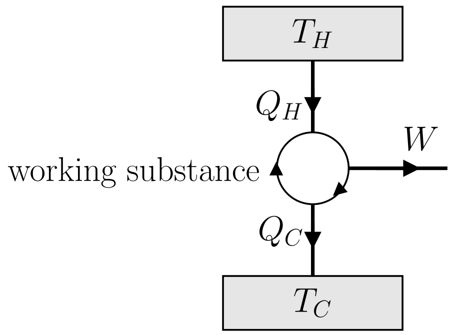

Figure 10.2: Engine diagram.

Again, we have used absolute value bars everywhere to avoid ambiguity about whether the symbol Q_H represent the heat flow out of the hot reservoir (in which case it is a negative value), or the heat flow in to the work substance (in which case it is positive).

And continuing with the big picture: such a device will be consistent with the second law as long as the total entropy doesn't decrease. So where do entropy changes happen? Certainly in the reservoirs. The hot reservoir has an entropy decrease (since heat leaves the reservoir) and the cold reservoir has an entropy increase (since heat is added to the reservoir). But what about the working substance? This gets at an essential point: in order to be a heat engine, the working substance is *not* a source of energy. It's not a battery stuck in between the reservoirs, or anything else consuming chemical energy. Rather, it must be returned back to the same state it started from, so the process can be repeated indefinitely. But if this is the case, if the working substance undergoes some cyclic process, then we can use Eq. (10.13), which tells us $\Delta S_{\text{w.s.}} = 0$, since the final and initial states are the same.

And now we can do the complete entropy accounting:

$$\Delta S_{\text{total}} = \Delta S_H + \Delta S_C + \cancelto{0}{\Delta S_{\text{w.s.}}} = -\frac{|Q_H|}{T_H} + \frac{|Q_C|}{T_C} \geq 0 \quad \text{(second law)} \tag{10.19}$$

One way to think of Eq. (10.19) is that it gives a lower bound on how much heat we have to dump to the cold reservoir:

$$|Q_C| \geq \frac{T_C}{T_H}|Q_H|. \tag{10.20}$$

That lower bound isn't zero, so we must dump some heat. *This is an important result*: any heat engine *must* dump some of its heat into a cold reservoir. It is impossible to turn all heat into usable mechanical energy.

But the good news is that the lower bound on the dumped heat $|Q_C|$ is smaller than $|Q_H|$, so we do get to "skim off" some of the energy and generate some work. Notice that the bound on how much heat we have to dump becomes smaller for very large T_H or small T_C. Evidently, the more extreme the difference in temperature between the reservoir, the more work we will be able to extract.

Let's quantify that. Let's introduce a dimensionless quantity called the efficiency, which is simply the fraction of heat pulled out of the hot reservoir that we are able to convert to work:

$$\epsilon \equiv \frac{|W|}{|Q_H|}. \tag{10.21}$$

From the first law we know $|W| = |Q_H| - |Q_C|$, so we can substitute this in and write the efficiency equivalently as

$$\epsilon = \frac{|Q_H| - |Q_C|}{|Q_H|} = 1 - \frac{|Q_C|}{|Q_H|}. \tag{10.22}$$

Obviously, the efficiency can't ever be greater than 1; that would correspond to an engine that produces more mechanical energy than the amount of heat that flows into it, and that would violate the first law of thermodynamics. But since $|Q_C|$ can never be zero in a real engine — we must always dump some heat — the efficiency can never reach 1. That would violate the second law of thermodynamics. We'll say more about this in a bit, after we have done a simple example using efficiency.

Example 10.3 A simple engine problem

An engine is described by the engine diagram in Fig. 10.3. Determine the work output by this engine and the efficiency of the engine.

Solution: We can straightforwardly find the work done by this engine using the first law of thermodynamics, i.e., energy conservation. The

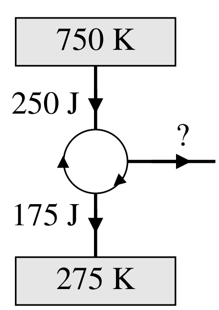

Figure 10.3: Diagram for Example 3

energy going into the working substance is equal to the energy going out of the working substance, so

$$|Q_H| = |Q_C| + |W| \tag{10.23}$$

and

$$|W| = |Q_H| - |Q_C| = 250\,\mathrm{J} - 175\,\mathrm{J} = 75\,\mathrm{J}. \tag{10.24}$$

Then from the definition of efficiency, Eq. (10.21), we have

$$\epsilon = \frac{|W|}{|Q_H|} = \frac{75\,\mathrm{J}}{250\,\mathrm{J}} = \boxed{0.30}.$$

Okay, so we have a definition for efficiency, which can never be greater than one (that would violate the first law of thermodynamics). But it can never **be equal to** 1 either.[3] The maximum value of ϵ corresponds to $|Q_C|$ being at its minimum (since it is being subtracted). Evidently,

$$\epsilon_{\mathrm{max}} = 1 - \frac{|Q_{C,\mathrm{min}}|}{|Q_H|} = 1 - \frac{(T_C/T_H)|Q_H|}{|Q_H|} = 1 - \frac{T_C}{T_H}. \tag{10.25}$$

Warning: do not confuse this result for the maximum efficiency ϵ_{max} with the similar looking expression Eq. (10.22) for efficiency ϵ in general. Also, this is only valid if heat is drawn from and dumped into isothermal reservoirs.

It is worth keeping in mind that Eq. (10.25) for the maximum efficiency is really a special case of an engine operating between two thermal reservoirs. A more general approach is just to use the first and second laws of thermodynamics. The first law just says balance the energy in and the energy out, and the second law says

$$\Delta S_{\mathrm{total}} = 0 \qquad \text{(maximum efficiency)} \tag{10.26}$$

Ultimately, that's the one equation you need to remember to solve problems involving a maximally efficient engine.

Let's consider some examples:

[3] This is important: if you ever calculate an efficiency equal to or greater than one, then it is wrong.

Example 10.4 Automobile Efficiency

The internal combustion engine of an automobile is a heat engine. Yes, there is gas being consumed, but that's being burned to provide the high temperature of the hot reservoir. From there on, the engine functions as a heat engine.

The hot reservoir is about 820° C and the cold reservoir is almost air temperature, but typically more like 70° C. Burning a gallon of gasoline provides about 120 MJ of heat. What is the upper limit on how much work can be extracted from a gallon of gas, and how much heat must be dumped?

Solution: The moment you see the words "upper limit" (or similar language), then you can pull out the second law of thermodynamics: $\Delta S_{\text{total}} = 0$. For this problem, $\Delta S_{\text{total}} = \Delta S_H + \Delta S_C$, since the reservoirs are the only parts of this system whose entropy changes.

To calculate ΔS for the reservoirs, we need to convert the temperatures to Kelvin, so $T_H = 820 + 273 = 1093\,\text{K}$ and $T_C = 70 + 273 = 343\,\text{K}$. The entropy change of the hot reservoir is

$$\Delta S_H = \frac{-|Q_H|}{T_H} = -\frac{120 \times 10^6\,\text{J}}{1093\,\text{K}} = -1.10 \times 10^5\,\text{J/K} \tag{10.27}$$

Therefore, since $\Delta S_{\text{total}} = 0$, it follows that ΔS_C must be positive by at least this amount.

$$\Delta S_{C,\text{min}} = \frac{|Q_{C,\text{min}}|}{T_C} = 1.10 \times 10^5\,\text{J/K} \tag{10.28}$$

so

$$|Q_{C,\text{min}}| = 343\,\text{K} \cdot 1.10 \times 10^5\,\text{J/K} = \boxed{37.7\,\text{MJ.}} \tag{10.29}$$

We get the maximum work by subtracting this from $|Q_H|$:

$$|W_{\text{max}}| = 120 - 37.7 = \boxed{82\,\text{MJ.}} \tag{10.30}$$

In reality, the work output is only about 1/3 of this amount, due to (i) design choices to get the work out faster, i.e. to get more power, (ii) friction, and (iii) imperfect combustion of the fuel.

We could have solved the previous example by using the equation for ϵ_{max} for engines with thermal reservoirs (although there was no need to do so). The following is an example where the ϵ_{max} approach would not work; i.e., you **must** start from the second law of thermodynamics:

Example 10.5 A Two-Brick Heat Engine

A hot brick initially at temperature $T_H = 400\,\mathrm{K}$ is used as the source for a heat engine. An equal sized cold brick made of the same material, initially at temperature $T_C = 300\,\mathrm{K}$, is used to dump the waste heat. As the heat flows, the two bricks come to thermal equilibrium. Assuming the heat engine was maximally efficient, what is the final temperature?

Solution: You might think the final temperature should just be 350 K, but that would be true only if we didn't intercept any of the heat and convert it to work.

There is a lot we aren't given. We don't know the amounts or the composition of the bricks, but we know they are identical. The hotter brick is going to cool from 400 K down to some T_f, so its entropy change will be $\Delta S_H = nC \ln(T_f/400)$, which is negative since $T_f/400 < 1$. The colder brick will warm from 300 K to T_f, so its entropy change will be $\Delta S_C = nC \ln(T_f/300)$. For maximum efficiency, then,

$$\Delta S_{\text{total}} = \Delta S_H + \Delta S_C = nC \ln\left(\frac{T_f}{400}\right) + nC \ln\left(\frac{T_f}{300}\right) = 0. \quad (10.31)$$

The nC factors divide out, and we get

$$\ln T_f - \ln 400 + \ln T_f - \ln 300 = 0 \quad (10.32)$$

and so

$$\begin{aligned} 2 \ln T_f = \ln(400 \cdot 300) \qquad &\Rightarrow \qquad T_f^2 = 400 \cdot 300 \\ &\Rightarrow \qquad \boxed{T_f = 346.4\,\mathrm{K}.} \end{aligned} \quad (10.33)$$

We see that the bricks end up slightly cooler than 350 K, which makes sense since we skimmed off some of the energy as it was flowing from hot to cold.

10.5 Gas Cycles for Heat Engines

We have made the case that skimming off some heat flow and generating work is possible, that is, consistent with energy conservation and the second law. But how do we actually make a heat engine? We need to find some working substance that can take in heat, do work, and dump heat. We do

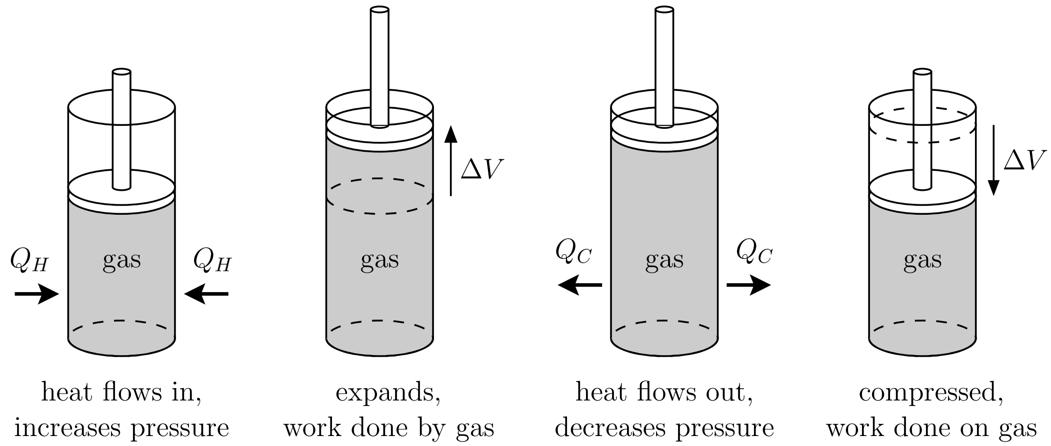

Figure 10.4: Caption for gas-piston figure

not have to look very far: an ideal gas will do the job nicely. In fact, gases are the most commonly used substance in heat engines today.

Consider a gas enclosed in a cylinder with a movable piston, as shown in Fig. 10.4. Recall that the work done by a gas is given by

$$W_{\mathrm{by}} = \int_A^B p\,dV, \tag{10.34}$$

so the gas can do work if we let the piston expand. We can put heat into the gas by bringing it into contact with something at a higher temperature, and we can dump heat out of the gas by bringing it into contact with something at a lower temperature. To be useful, we will want to complete a full cycle, to bring the gas back to its starting point. This means contracting the piston at some point in the cycle, which will cost work (i.e., work is done **on** the piston, or work done **by** the piston is negative). But part of the cycle will involve an expansion, and that produces positive work by the piston. Overall, we can get work out of the process (i.e., the net work by the piston is positive) if the expansion happens with higher pressure (so more force) than the contraction.

A typical cycle is illustrated in Fig. 10.4. Notice that the expansion happens at higher pressure than the compression, leading to a net work being done by the gas. These gas cylinders are often paired up, as in an automobile, so that the expansion of one cylinder causes the compression of the other cylinder, with power left to spare.

To quantify the cyclic gas process, it is useful to plot it on a p-V diagram. One such cyclic process is illustrated in Fig. 10.5, which is a sequence of constant pressure and constant volume processes that leads to a rectangular cycle on the p-V diagram. Let's analyze this case in some detail, starting

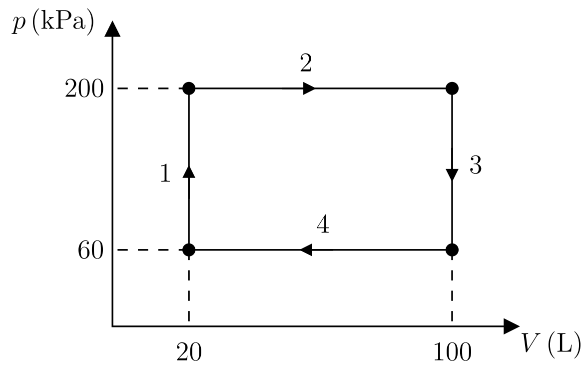

Figure 10.5: Cyclic process made up of constant pressure and constant volume processes.

with a calculation of the net work for the whole cycle.

The constant volume steps do no work, while for constant pressure, $W_{\text{by}} = p\Delta V$. Adding this up around the whole cycle gives

$$\begin{aligned} W_{\text{cycle}} &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + 200\,\text{kPa}(+80\,\text{L}) + 0 + 60\,\text{kPa}(-80\,\text{L}) = 140 \cdot 80\,\text{kPa}{\cdot}\text{L} \\ &= 11.2\,\text{kJ} \end{aligned} \tag{10.35}$$

Notice that the work done in the cycle is just the area of the enclosed rectangle, which is true for any cycle.

Now let's figure out what $|Q_H|$ and $|Q_C|$ are. For this we will need to know whether the gas is monatomic or diatomic. Let's assume the gas is diatomic, so $f = 5$ and $\gamma = 1.4$. For the constant volume processes (steps 1 and 3 in Fig. 10.5) there is no work done by the gas, so the first law says

$$Q = \Delta E_{\text{therm}} + 0 = \tfrac{5}{2}\Delta(pV) = \tfrac{5}{2}V\Delta p, \tag{10.36}$$

where we've used the fact that V is a constant in the last step. Now we can calculate the heat flow into the gas for steps 1 and 3:

$$\begin{aligned} Q_1 &= \tfrac{5}{2} \cdot 20\,\text{L} \cdot (200\,\text{kPa} - 60\,\text{kPa}) = 7.0\,\text{kJ} \\ Q_3 &= \tfrac{5}{2} \cdot 100\,\text{L} \cdot (60\,\text{kPa} - 200\,\text{kPa}) = -35.0\,\text{kJ}. \end{aligned} \tag{10.37}$$

For a constant pressure process (steps 2 and 4 in Fig. 10.5) the work is $p\,\Delta V$ and so the first law says

$$Q = \Delta E_{\text{therm}} + W_{\text{by}} = \tfrac{5}{2} \cdot \Delta(pV) + p\Delta V = \tfrac{7}{2} \cdot p\Delta V. \tag{10.38}$$

where we've used the fact that p is a constant in the last step. Now we can compute

$$\begin{aligned} Q_2 &= \tfrac{7}{2} \cdot 200\,\text{kPa} \cdot (100\,\text{L} - 20\,\text{L}) = 56.0\,\text{kJ} \\ Q_4 &= \tfrac{7}{2} \cdot 60\,\text{kPa} \cdot (20\,\text{L} - 100\,\text{L}) = -16.8\,\text{kJ} \end{aligned} \tag{10.39}$$

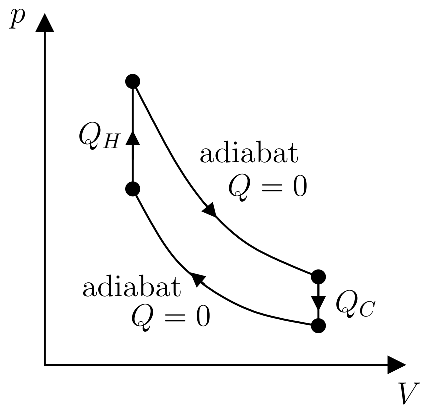

Figure 10.6: Otto cycle.

We have calculated the heat flow into the gas for each step. Now we can identify Q_H as coming from all the steps with positive Q, where heat really does flow in. In our example, this would be steps 1 and 2. In contrast, Q_C comes from all the steps with negative Q, where heat is actually flowing out of the gas, which is steps 3 and 4 for our example. So we can calculate

$$\begin{aligned} |Q_H| &= Q_1 + Q_2 = 7 + 56 = 63.0\,\text{kJ} \\ |Q_C| &= |Q_3| + |Q_4| = 35 + 16.8 = 51.8\,\text{kJ}. \end{aligned} \tag{10.40}$$

We can check our calculation, since we have already computed the work in Eq. (10.35):

$$|W| = |Q_H| - |Q_C| = 63.0 - 51.8 = 11.2\,\text{kJ} \quad \checkmark \tag{10.41}$$

And finally we can compute the efficiency,

$$\epsilon = \frac{|W|}{|Q_H|} = \frac{11.2}{63.0} = 0.178. \tag{10.42}$$

This rectangular gas cycle engine is not actually practical. Automobiles use instead something called the Otto cycle, which is a sequence of constant volume and adiabatic processes, as illustrated in Fig. 10.6.

The heat for the constant volume steps is again given by

$$Q = \Delta E_{\text{therm}} = \frac{f}{2} V \Delta p \qquad \text{(constant volume)} \tag{10.43}$$

which is positive when the pressure increases and negative when it decreases. And now for the nice part: by definition there is no heat flow during either of the adiabatic steps, so these constant volume processes give Q_H and Q_C

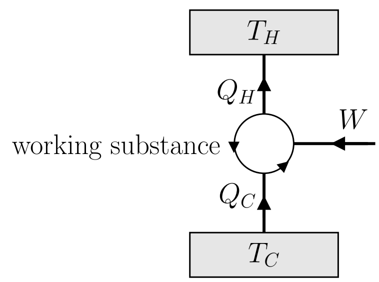

Figure 10.7: A refrigerator diagram, made by reversing all the arrows on a heat engine diagram. Note that the working substance cycle is now counterclockwise instead of clockwise.

as labeled in Fig.10.6. In the homework problems you will work out the details of the Otto cycle.

The adiabats are exactly what makes this a more practical engine. The adiabatic expansion and compression steps happen very quickly, which is why no heat flows because it simply doesn't have enough time to flow. But this means a cycle is completed relatively quickly, and so we are getting the work of the cycle out in a short amount of time. Recalling that work per time is power, we see that having a fast cycle can lead to more power output, which is often desired.

10.6 Refrigerators

An interesting thing happens if we take a heat engine and run it backwards: we make a refrigerator. By refrigerator, we mean a device which requires work as input and is able to make heat flow from a colder object to a hotter object. This includes, of course, the refrigerator that keeps your milk from getting spoiled, but also includes air conditioners and even heat pumps (which is basically an air conditioner hooked up backwards to cool the outdoors and warm your house).

Making heat flow from cold to hot may sound like a violation of Clausius's statement of the second law, but it is not as long as something else is going on in the process. Heat will not *spontaneously* flow from cold to hot; rather, we must cleverly engineer the refrigerator to make it happen and, most importantly, we must plug it in, to give the necessary work as input.

Let's start with an engine diagram in reverse, as shown in Fig. 10.7, and focus on the first and second laws. Energy conservation now requires

$$|Q_C| + |W| = |Q_H| \qquad \text{(first law)} \tag{10.44}$$

which is exactly the same equation as before, since we reversed *all* the arrows. Now the signs of the entropy change in the reservoir are reversed, so the second law says

$$\Delta S_{\text{total}} = \Delta S_H + \Delta S_C = \frac{|Q_H|}{T_H} - \frac{|Q_C|}{T_C} \geq 0. \quad \text{(second law)} \tag{10.45}$$

This now provides an *upper* bound on $|Q_C|$, namely

$$|Q_C| \leq \frac{T_C}{T_H}|Q_H|. \tag{10.46}$$

What does this mean? It means we can run a heat engine backwards and satisfy the first and second laws and actually make some heat go from cold to hot. But there is a limitation on how much heat we can pull out of the cold reservoir, and this limitation depends on the reservoir temperatures and on how much work we provide. If we eliminate $|Q_H|$ from the upper bound, Eq. (10.46), we can show

$$|Q_C| \leq \frac{T_C}{T_H - T_C}|W|, \tag{10.47}$$

which you will derive in the homework. This expression says that we need work to pull heat out of the cold reservoir, and also that the effectiveness of pulling out heat will depend on the reservoir temperatures. We can quantify this "effectiveness" with what is called the coefficient of performance, CP, defined as the ratio of what want ($|Q_C|$) to what we must pay ($|W|$):

$$CP = \frac{|Q_C|}{|W|} \leq \frac{T_C}{T_H - T_C}. \tag{10.48}$$

As before, the maximum coefficient of performance is obtained from the borderline case of the second law $\Delta S_{\text{total}} = 0$. Notice that reservoirs that are close in temperature yield a larger CP. This makes sense, since the closer T_H and T_C are, the less "uphill" we are making the heat flow.

In the refrigerator diagram the working substance now goes through a counterclockwise gas cycle. This allows for the heat to flow into the gas while it is at a lower temperature (lower than T_C) and heat to flow out of the gas while it is at a higher temperature (higher than T_H). But this counterclockwise cycle also means compression happens at higher pressure than the expansion. This will require some outside source of work to drive the gas through this cycle, which is exactly what we found from general first law and second law considerations: work input is necessary to make the refrigerator function. In other words, your refrigerator won't function properly if you forget to plug it in.

Problems

1. A 36 g ice cube at 0° C is dropped into 90 g of water at 22° C. Heat flows from the water to the ice, bringing both to equilibrium at 0° C.

(a) Determine the number of moles of ice melted in cooling the water to 0° C.

(b) Calculate the entropy change of the melted ice.

(c) Calculate the entropy change of the 90 g of water in this process. Is your answer consistent with the second law?

2. Consider a pair of 5.0 mol ideal solids. Solid A initially has a temperature of 500 K, while solid B has a temperature of 200 K. The solids are brought into thermal contact, and heat flows until the system reaches equilibrium. Determine the entropy change of each solid, and the total entropy change.

3. By what factor is the multiplicity increased in melting 18 g of 0° C ice into 0° C water?

4. A heat engine draws 600 J of heat per cycle from a hot reservoir at 800 K, and dumps 300 J of heat per cycle into a cold reservoir at 200 K. Determine the efficiency of this engine.

5. Consider a 10 mol brick of ideal solid initially at temperature 500 K. This brick is used as the heat source for a heat engine, which is dumping heat to a room-temperature reservoir at 295 K.

(a) Determine the entropy change of the brick as it cools to room temperature.

(b) Determine the minimum amount of heat that must be dumped by this heat engine.

(c) Calculate the efficiency of this best-case engine.

6. Consider the two-brick heat engine of Example 5. Calculate the maximum amount of work that could be obtained from this engine. For this problem, assume that the bricks are each an ideal solid with 2.0 mol.

7. Susquehanna Valley Limousine has modified their automobile engine to provide heat for an oven to bake fresh chocolate chip cookies while you cruise Lewisburg.[4] For each gallon of gas, 12 MJ of heat is dumped from the engine into the oven at a temperature of 190° C, and the remainder of the heat coming from the engine is dumped to the environment at 70° C, as described in Example 4. Calculate the maximum work possible from a gallon of gas for this engine. As in Example 4, assume that the hot reservoir has a temperature of 820° C and the heat coming from the burning gas totals 120 MJ.

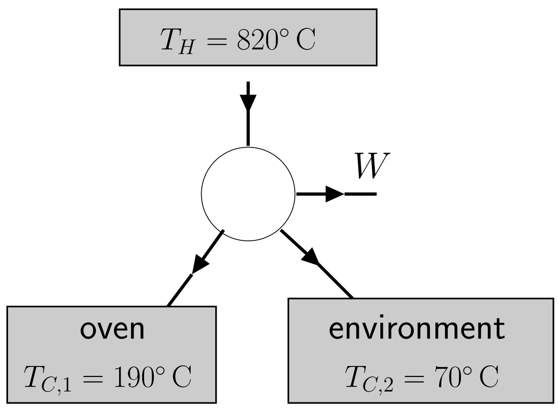

Figure 10.8: Engine diagram for Problem 7

[4]Not really.

8. In the cycle shown below, 1.0 mol of a monatomic ideal gas is initially at a pressure of $p_A = 100\,\text{kPa}$ and a temperature of $T_A = 0°\,\text{C}$. The gas is heated at constant volume to $T_B = 150°\,\text{C}$ and is then expanded adiabatically until its pressure is back to $p_C = 100\,\text{kPa}$. Finally, the gas is compressed at constant pressure until it is back to its original state A. Find

(a) the pressure, volume and temperature for each of the three labeled points (A, B and C),

(b) the heat entering or leaving the system during each process, and

(c) the efficiency of this cycle.

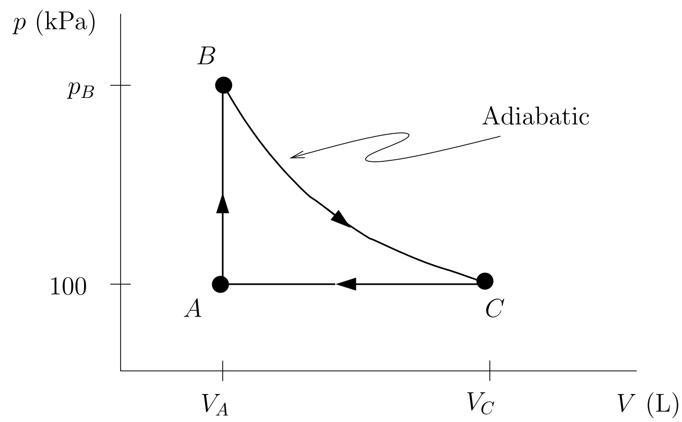

Figure 10.9: Cycle for Problem 8

9. For the rectangular gas cycle in Fig. 10.5, assume that there is 0.50 mol of diatomic gas.

(a) If this cycle is to operate between two reservoirs, calculate the minimum possible value for T_H and the maximum value for T_C.

(b) Compare the efficiency of this gas cycle to the maximum efficiency possible for a cycle operating between these two reservoirs.

10. Explain why, for any gas cycle that is clockwise on a p-V diagram, the area enclosed by the loop gives the net work by the gas in the cycle.

11. Your problem session instructor will provide this problem.

12. An Otto cycle for a fixed number of moles of a diatomic ideal gas is shown in Fig. 10.10.

(a) Calculate the pressure at points C and D.

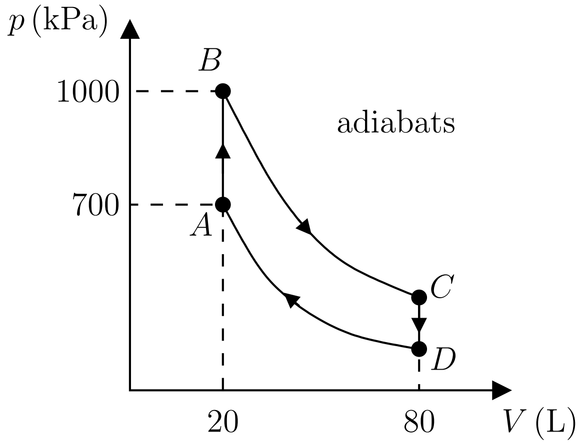

Figure 10.10: Otto cycle for Problem 12.

(b) Determine $|Q_C|$, $|Q_H|$, and $|W|$ for one cycle. *Hint:* save $|W|$ for last.

(c) Calculate efficiency of this cycle.

13. Derive Eq. (10.47), the upper bound for $|Q_C|$ in a refrigerator, by starting from the first and second laws.

14. Can you cool off your house by opening the door to the refrigerator? Explain why or why not.

15. A refrigerator is designed to keep its interior at a constant 5° C while dumping heat into a room temperature environment at 22° C. The interior of the refrigerator may be regarded as a cold reservoir, while the air around it is a hot reservoir. Now suppose a pitcher containing 1800 g of room-temperature water is put into the refrigerator. We can break down what happens next into two steps: (1) the water will cool down to the temperature of the cold reservoir (fridge interior) as its heat flows into cold reservoir, and (2) the refrigerator will turn on and do work (run a compressor, circulate refrigerant, etc.) to extract this heat from the cold interior and dump it to the air outside.

We'll calculate what happens in each of these steps.

Step #1:

(a) Calculate the change in entropy of the water as it cools from 22° C to 5° C.

(b) Calculate the change in entropy of the cold reservoir (i.e. the inside of the refrigerator) due to this same heat transfer.

(c) Is the second law of thermodynamics satisfied in this step? Explain briefly why or why not, using your results above.

Step #2

(a) Now we must get rid of the extra heat in the cold reservoir by transferring it to the hot reservoir. Calculate the change in the entropy of the cold reservoir (refrigerator) as the excess heat from step #1 is removed from the refrigerator's interior.

(b) Determine the minimum amount of work required to remove this excess heat from the cold reservoir.

Chapter 11

Gravity and Geometry

11.1 Introduction

Previously we have described the force of gravity on an object by a field model. In this Newtonian model, masses cause gravitational fields and these fields act on other masses to cause forces.

In this chapter we examine another model of gravity which is Einstein's explanation in terms of curved spacetime. In this model masses cause spacetime to be curved and other masses, moving in 'straight' lines in this curved spacetime, seem to accelerate relative to the source mass. As described succinctly by John A. Wheeler, the leading spokesman for Einstein's theory of gravity:

> Mass tells spacetime how to curve; spacetime tells mass how to move.

We begin by looking again at the concept of the invariant spacetime interval. We present the modifications of the interval that Einstein introduced to account for the effects of gravity. This leads us to examine how the gravitational effects of a spherical mass (like planets and stars) show up in clock rates and the measurement of distances.

We also develop the idea of curved space and describe how one might determine whether the three-dimensional space we live in is curved. We apply this idea to the curvature of spacetime to examine how a particle moves in curved spacetime and present an unusual principle called the principle of maximum proper time which governs how an object moves in the presence of a gravitational field.

11.2 The Interval Revisited

In developing his conception of gravity as a manifestation of curved spacetime, Einstein sought to extend his notion of invariance from special rel-

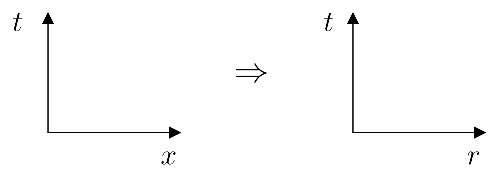

Figure 11.1: Cartesian to polar coordinates

ativity. Recall that for special relativity all *inertial* reference frames were treated as equally valid for describing physics. In 1915, Einstein created the theory of general relativity, removing the inertial frame restriction: *all arbitrarily* moving reference frames were to be equally valid. This enabled him to include gravity into spacetime physics.

Recall that in Sec. 3.2 we introduced the spacetime interval I^2 of special relativity, defined by

$$I^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{11.1}$$

The interval was shown to be invariant: observers in any inertial reference frame would calculate the same value for the interval separating any given pair of events.

Let's work on generalizing this expression to include the effects of gravity. The first thing to do is convert from Cartesian to polar coordinates, shown in Fig. 11.1, as these will be more appropriate to describe spacetime in the presence of spherically symmetric masses, like Earth, stars, or black holes. Here "r" is a radial coordinate: it decreases or increases as you move toward or away from the given spherical mass. The interval expressed in these coordinates, not surprisingly, takes the form

$$I^2 = (c\Delta t)^2 - \Delta r^2 \tag{11.2}$$

when motion along only the radial direction is considered.

Next, let's see how Einstein includes the effects of gravity in the invariant spacetime interval. How is it that "mass tells spacetime how to curve"? The full answer is very hard, far beyond the scope of this course: solve a set of nasty coupled nonlinear differential equations! But for the case of intervals in the spacetime outside a spherical mass, the result is surprisingly simple. Eq. (11.2) becomes

$$I^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\Delta t)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \Delta r^2. \tag{11.3}$$

Here "M" is the mass of the central body, and G is Newton's universal gravitation constant.

The appearance of the coefficients $(1 - \frac{2GM}{c^2 r})$ and $(1 - \frac{2GM}{c^2 r})^{-1}$ in the general relativistic expression for the spacetime interval tells you that we're now dealing with curved spacetime. The presence of a nearby mass curves spacetime!

Finally, recall from Sec. 3.2 that when I^2 is positive the interval I is timelike. In fact, the quantity I/c represents the proper time interval $\Delta\tau$ between the two given events. Dividing Eq. (11.3) by c^2 and replacing I/c with $\Delta\tau$, we get

$$\Delta\tau^2 = \left(1 - \frac{2GM}{r}\right)\Delta t^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{r}\right)^{-1}\Delta r^2. \tag{11.4}$$

On the other hand, when I^2 is negative, we are dealing with a spacelike interval. It is convenient to introduce Δs, the proper distance, by

$$\Delta s^2 = -I^2 > 0. \tag{11.5}$$

In this case, Eq. (11.3) becomes

$$\Delta s^2 = \left(1 - \frac{2GM}{r}\right)^{-1}\Delta r^2 - \left(1 - \frac{2GM}{r}\right)(c\Delta t)^2. \tag{11.6}$$

Let's explore how these gravity-related factors lead to the warping of time and space.

11.3 Gravity's Effect on Clock Rates

An example will show how clock rates are affected by the curvature of spacetime. Consider two clocks, clock A parked near a large mass, and clock B parked very far away. Note that both clocks are at rest: there is NO relative motion. Yet general relativity predicts that the clocks still run at different rates.

We start from Eq. (11.4)

$$\Delta\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)\Delta t^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\Delta r^2 \tag{11.7}$$

and insert the conditions for clock B and then for clock A. For clock B, it is at rest, so $\Delta r = 0$. Also it is very far away (say $r \to \infty$), so that $\frac{2GM}{c^2 r} \to 0$. Then for B, Eq. (11.7) becomes

$$\Delta\tau_B^2 = (1-0)\Delta t^2 - \frac{1}{c^2}(1-0)^{-1}(0) = \Delta t^2 \quad \text{or} \quad \Delta\tau_B = \Delta\tau. \tag{11.8}$$

This shows that Δt is the proper time as recorded on far-away clocks.

Now consider clock A, say at rest at radial location r_A. Similar to the above calculation we find

$$\Delta\tau_A = \left(1 - \frac{2GM}{c^2 r_A}\right)^{1/2}\Delta t = \left(1 - \frac{2GM}{c^2 r_A}\right)^{1/2}\Delta\tau_B. \tag{11.9}$$

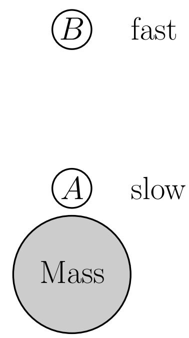

Figure 11.2: Low clocks are slow clocks!

Since $(1-\frac{2GM}{c^2r})^{1/2} < 1$, we see that $\Delta\tau_A < \Delta\tau_B$. This shows that clocks near a mass record *less* elapsed proper time than those far away. As illustrated in Fig. 11.2, a nice mnemonic is “Low clocks are slow clocks”.

Example 11.1 Clocks on the Earth’s Surface.

How much slower does a clock at sea level run than one far away from Earth?

Solution:

Use Eq. (11.9), with $M = 5.97 \times 10^{24}\,\text{kg}$ and $r_A = R_E = 6.37 \times 10^6\,\text{m}$

$$\begin{aligned}\Delta\tau_A &= \left(1 - \frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(3.00 \times 10^8)^2 \times 6.37 \times 10^6}\right)^{1/2} \Delta\tau_B \\ &= (1 - 1.39 \times 10^{-9})^{1/2}\Delta\tau_B \\ &\approx (1 - 6.95 \times 10^{-10})\Delta\tau_B. \end{aligned} \tag{11.10}$$

A useful way to express this result is the rate at which clock A gets behind:

$$\Delta\tau_B - \Delta\tau_A = \Delta\tau_B - (1 - 6.95 \times 10^{-10})\Delta\tau_B = 6.95 \times 10^{-10}\Delta\tau_B. \tag{11.11}$$

Thus clock A falls behind clock B an amount 6.95×10^{-10} seconds every second. Thus, for clock A to get behind clock B by one second, it takes

$$\frac{1}{6.95 \times 10^{-10}}\text{seconds} = 1.44 \times 10^9\,\text{s} \approx 46\,\text{yr}. \tag{11.12}$$

We don’t notice these time effects much near Earth, but these small difference are essential to be accounted for in the design and operations of GPS devices. For a more dramatic example, see Problem 2.

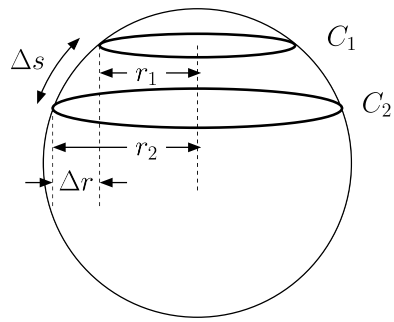

Figure 11.3: Two concentric circles drawn on a sphere.

11.4 Curved Spacetime

We now need to give a careful definition of the radial coordinate r. We begin with an example in ordinary three-dimensional space. Imagine drawing two large circles on the earth with the north pole as the common center (see Fig. 11.3). Let Δs be the distance between one circle and the other, measured along the earth's surface as we walk out from the north pole. If we were sphere-dwellers who lived entirely in 2-dimensions, that is, on the surface of the earth, and knew nothing about a 3^{rd} dimension (up or down), we could (having studied Euclidean geometry) *define* the radial coordinates, r_1 and r_2, in terms of the circumferences of the two circles, by

$$C_1 = 2\pi r_1 \qquad \text{and} \qquad C_2 = 2\pi r_2. \tag{11.13}$$

The actual radii of the circles, r_1 and r_2, are shown in Fig. 11.3; they clearly are not measured along the surface of the earth. But these radii are not available for direct measurement by the sphere-dwellers; they must *calculate* the radii using Eq. (11.13).

If we lived on a flat space, such as a flat piece of paper, the radial coordinates r_1 and r_2 would be related by

$$\Delta r = r_2 - r_1 = \Delta s \qquad\qquad \text{(flat space expectation)}, \tag{11.14}$$

where Δs is the shortest distance between the circles measured along the paper. But when the circles are actually drawn on the curved surface of the earth, r_2 is larger than r_1 by the amount Δr, which is less than Δs.

We humans, who can comprehend the spherical nature of the earth's surface, can understand this deviation from flat-space geometry by studying Fig. 11.3, where the actual difference in radius, Δr, between the two circles on the earth is clearly smaller than Δs. Thus, on a sphere, the flat-space expectation of Eq. (11.14) is incorrect. We can therefore use Eq. (11.14) in an experimental test, performed entirely on the surface of the earth, to determine whether that surface is curved or flat.

Figure 11.4: An outward displacement Δs in the gravitational field of a star.

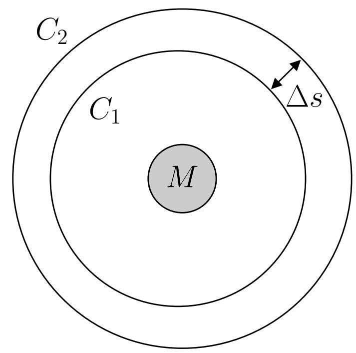

Figure 11.5: Two concentric circles drawn around a star of mass M.

The test requires that we measure the circumferences of the two circles and the distance Δs between them. Then we must divide each circumference by 2π to get the radial coordinate, substitute these calculated values for r_1 and r_2, along with the measured Δs, into Eq. (11.14) and check for equality. If we get equality then the surface is flat. If Δs is larger than $r_2 - r_1$, then the surface is curved like a sphere or a bowl. (If Δs is smaller, the surface is curved like a saddle.)

Now let's consider the region around a star as shown in Fig. 11.4. We ask whether the space around the star is curved. How can we tell? We begin by drawing two circles centered on the star, C_1 and C_2 in Fig. 11.5. We then measure their circumferences with meter sticks (which have been properly calibrated with clocks and light pulses) and calculate r_1 and r_2. We also measure the distance Δs, by laying meter sticks radially along a path from C_1 to C_2. Then to test whether space is curved or flat, we substitute our measurements into Eq. (11.14). If Eq. (11.14) is satisfied, then space is flat; otherwise it is curved.

When we do this experiment (say with radar ranging around the sun) we find that actual measurements of phenomena occurring near the sun indicate that Δs is, in fact, slightly larger than Δr. Thus Eq. (11.14) is *not* satisfied, and the space around the sun is curved like a bowl. The source of the curvature is the mass M of the sun. It's like placing a large mass on a

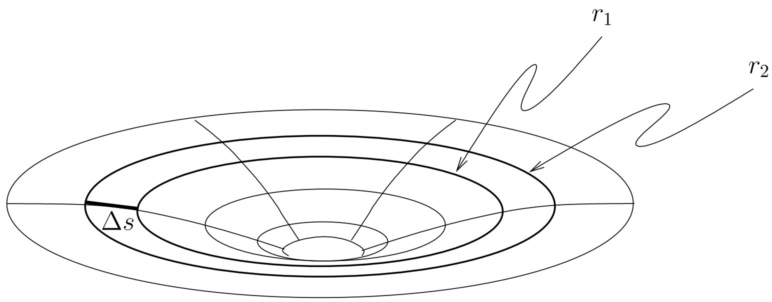

Figure 11.6: Curved space near a black hole. The distance Δs measured radially along the surface between circles of radial coordinate r_1 and r_2 is larger than their difference Δr.

rubber sheet; the added mass distorts and curves the sheet so that nearby smaller masses are attracted to it. Near a black hole, the curvature effect is quite pronounced. Figure 11.6 shows a popular representation of the curved space near a black hole.

Now imagine the motion of a small object in this curved space. It will try to follow the shortest path. If it goes on a path that takes it deep into the hole, where Δs is much bigger than Δr, the total distance will be larger than if the object skirts around the hole, trading a lot of "down-and-back-up" distance for just a little extra "sideways" distance. This is why light bends around large mass stars or galaxies (see Fig. 11.7).

It's time to get quantitative. Can Einstein's conception of curved space-time near the sun tell us that Δs is bigger than Δr, and by how much? Start with Eq. (11.6)

$$\Delta s^2 = \left(1 - \frac{2GM}{r}\right)^{-1} \Delta r^2 - \left(1 - \frac{2GM}{r}\right)(c\Delta t)^2. \tag{11.15}$$

We would like to consider a snapshot of the space around the sun; that is locate the two points at the *same time*. Thus set $\Delta t = 0$ and solve for Δs.

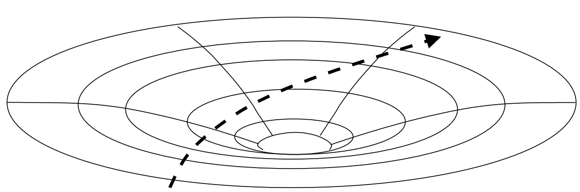

Figure 11.7: Light being bent by curved space near a black hole.

We find

$$\Delta s = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \Delta r. \tag{11.16}$$

Since $(1 - \frac{2GM}{c^2 r})^{-1/2} > 1$ for any $M > 0$ and $r < \infty$ we have that $\Delta s > \Delta r$ near any mass — the space near stars and planets is curved like a bowl!

Example 11.2

How curved is space near the sun?

Solution: Use Eq. (11.16) with $M = M_{\text{sun}} = 1.99 \times 10^{30}\,\text{kg}$ and $r = R_{\text{sun}} = 6.96 \times 10^8\,\text{m}$.

$$\begin{aligned}\Delta s &= \left(1 - \frac{2 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{(3.00 \times 10^8)^2 \times 6.96 \times 10^8}\right)^{-1/2} \Delta r \\ &= (1 - 4.24 \times 10^{-6})^{-1/2} \Delta r \\ &= 1.0000021\,\Delta r. \end{aligned} \tag{11.17}$$

That's not looking very curved, but it is big enough to direct the motion of all the planets in their orbits about the sun!

11.5 Law of Motion for a Freely Falling Body

Imagine taking a trip, in gravity-free spacetime, from the point $x = 0$, $t = 0$ to the point $x = 0$, $t = 1$. Not a very adventurous trip, to be sure, but some would opt for the lazy way; simply remain at rest at $x = 0$ and you'll get there! But there are other ways of making the trip (see Fig. 11.8). For example you could take off at some high speed, say $v = 0.6c$ for half a second (in the rest frame) and travel about 0.3 light-sec and then suddenly reverse direction and travel for another half second back to $x = 0$ at a velocity of $-0.6c$. This hyperactive approach gets you to your destination quicker, in the sense that the time elapsed on your watch (proper time) will be only

$$\Delta t_{\text{proper}} = \Delta t\sqrt{1 - v^2/c^2} = 1.0 \times \sqrt{1 - 0.6^2} = 0.8\,\text{s}. \tag{11.18}$$

This effect is precisely what happens in the twin paradox.

Of all the ways of making the trip, the one taken at zero velocity will give the longest proper time. This is because any non-zero velocity during any part of the trip reduces the proper time required for the trip. What is the 'natural' route for the trip? By 'natural' we mean the route taken by a particle with no forces acting on it. Clearly a particle makes the trip the lazy way, it simply 'crawls' at constant velocity $v = 0$ and it gets there. The 'natural' way is also the way that gives the largest possible proper time for the trip (1 second in this case).

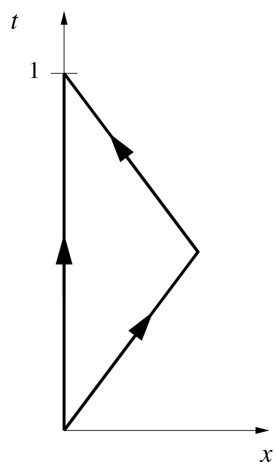

Figure 11.8: Two ways of taking a spacetime trip.

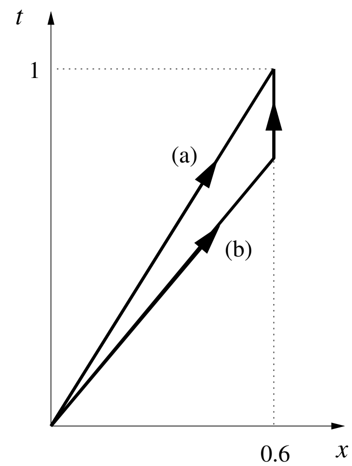

Figure 11.9: Two ways of taking a different spacetime trip.

Let's see if this principle, which we call the *principle of maximum proper time*, can be generalized. For example, suppose we want to go from the point $x = 0$, $t = 0$ to the point $x = 0.6$, $t = 1$. If our principle is correct then we should expect that the way to take the trip in the greatest proper time is to go the 'natural' way, i.e., at a constant velocity $v = 0.6c$, shown as route (a) in Fig. 11.9. We already know that the proper time for this constant velocity trip is $(1 - 0.6^2)^{1/2} = 0.8\,\mathrm{s}$. Can we find a way that gives a longer proper time?

Figure 11.9 shows an alternate route to get to $x = 0.6$, $t = 1$, labeled as path (b). In homework problem 4, you will calculate the proper time for this route. What you should find is that the total proper time is less for the two-step trip, route (b), than the 0.8 second 'natural' trip at a single velocity, route (a). In fact there is no other route that gives a longer proper time than the natural route. Our principle shows an encouraging ability to "explain" Newton's first law, the law of inertia.

This same principle works in curved spacetime. The path between two events actually taken by a freely falling body is that path that MAXIMIZES the elapsed proper time. This rule leads to the "straightest" possible path through the curved spacetime, and answers the question: How is it that "spacetime tells mass how to move"?

Here is a qualitative example. Throw a ball straight up and catch it one second later at the same original height. What is the "naturally chosen"

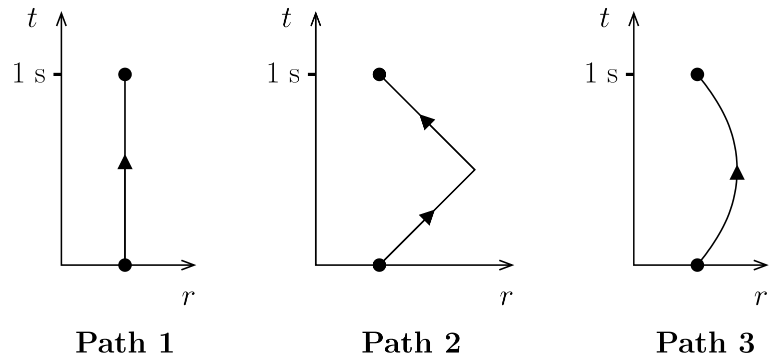

Figure 11.10: Trajectories

path through spacetime that maximizes proper time? We have just seen that, because of time dilation, paths that speed up, slow down, or reverse direction tend to *decrease* proper time. On the other hand, we've learned that clocks at higher altitudes measure *more* elapsed proper time than those at lower altitudes. So now consider these three paths, shown in Fig. 11.10:

Path 1 The ball stays just above your hand the whole time. No speeding to decrease proper time — BUT also not taking advantage of increasing proper time by going high where clocks run faster. Not the best.

Path 2 Zoom high at almost light-speed, zoom back down. This gets the ball high where clocks run fast, but time dilation at near light speed means almost no elapsed proper time! No good.

Path 3 Spend most of the trip at a higher place where proper time elapses rapidly. But don't go very fast or change speed quickly, so the time dilation is not too severe.

Result: Best path for maximizing proper time is parabolic motion — fast up, slow and stop smoothly at the top, faster on the way down. This is the motion we actually observe!

Problems

1. Refer to Fig. 11.3. Suppose C_1 is the 50° north latitude circle on the earth, and C_2 is the 40° north latitude circle on Earth. The measured circumferences are $C_1 = 25850\,\text{km}$ and $C_2 = 30800\,\text{km}$.

(a) Determine r_1, r_2 and Δr.

(b) Determine Δs as a direct measurement on the earth's surface of the distance between the 40° and 50° latitude circles. Compare your result to Δr.

(c) Explain how you could use your results to convince a member of the Flat Earth Society that the earth's surface is actually curved.

2. Consider clock C on the surface of a neutron star, and clock D far away. How do their rates compare? How much time elapses on clock D before clock C is one second behind? (The mass and radius of the neutron star are $2 \times 10^{30}\,\text{kg}$ and $10\,\text{km}$, respectively.)

3. For the same neutron star as in Problem 2, calculate the height above the surface for which the circumference of a circle concentric with the star is $2\pi \times 10.001\,\text{km}$.

4. Following path (b) in Fig. 11.9 means you go at $v = 0.8c$ for the first leg, and $v = 0$ for the next. Calculate the t-coordinate at the junction point; then determine the proper time for each leg and add them to get the total proper time. Compare to the $0.8\,\text{s}$ of the direct route.

Answers to Selected Problems

Additional Problems
A3 (a) $v_{\text{terminal}} = g/b$. **A6** 130 m/s; 18 m/s^2. **A7** (a) $v_{\text{avg}} = 0$. **A21** $v = \sqrt{kx_0^2/m - 2gx_0}$; $h = kx_0^2/2mg$. **A22** (d) $\sqrt{gR}$; (e) $2.5R$; (f) $6mg$. **A23** (a) 6 km; (b) 2 kJ; c) drifts as far as ∼7 km. **A28** (a) 2.7 m/s; (b) 2.7 m/s; c) 2.7 m/s. **A38** $m = 1718\,\text{MeV}/c^2$; $u = 0.4c$. **A39** $550\,\text{MeV}/c^2$. **A42** (a) 0.085 J; (b) more kinetic energy (0.141 J); (c) 347 rev/min.
A55 (a) 82° C or 355 K; (b) $A \to B$: 1873 J, $B \to C$: 0 J, $C \to A$: −1701 J; (c) 0.092; d) 0.355. **A66** $a = g(m_2 - m_1)/(m_1 + m_2 + m_3/2)$, downward for m_2. **A68** (a) 0.0027 N/kg; (b) 2.0×10^{20} N; (c) 0.19 N.
A69 (a) $m = C(L_2^2 - L_1^2)/2$; (b) $GC\ln(L_2/L_1)$. **A72** $T\cos\theta$. **A73** (a) +23.52 J; (b) 14.3 m/s. **A75** $\vec{L}_A = -31.5\,\hat{k}\,\text{kg·m}^2/\text{s}$, $\vec{L}_B = 0$, $\vec{L}_C = +15.75\,\hat{k}\,\text{kg· m}^2/\text{s}$.
A80 (c) and (f) are possible; the others have incorrect dimensions. **A82** (a) 197 cars; (b) \$22.22. **A84** 7.4 y. **A85** 7.3 d.

Chapter 1
1.1 (a) $x(2) = 7.0$ m, $v(2) = -2.9$ m/s; (b) $x(2) = 6.56$ m, $v(2) = -2.80$ m/s.

Chapter 2
2.2 0.133 lt-s/s. **2.3** 1.8×10^7 m/s. **2.5** 0.995 lt-s/s $= 2.98 \times 10^8$ m/s. **2.6** 5.92×10^5 s. **2.7** (a) $6\frac{2}{3}$ yr; (b) $5\frac{1}{3}$ yr; (c) 3.2 lt-yr; (d) 3.2 lt-yr.
2.8 (a) 4.47 yr; (b) $0.894c$. **2.10** (a) 48 lt-yr; (b) $0.958c$. **2.12** (a) 20 ms; (b) 9.6 lt-ms. **2.13** (a) 0.50 s; (b) 0.50 s (Duh!); (c) 15.8 m; (d) 31.6 m/s; **2.14** 26.0 m/s.

Chapter 3
3.1 Yes; **3.4** (a) A; (b) B; (c) D, (d) b,c,d; (e) a,e; (f) space-like: bc, bd, cd, ae, be, ce; time-like: ab, ac, de. **3.7** (a) calendar page; (b) B, then C, then A; (c) $\Delta t_{\text{BA,Earth}} = 18$ min, $\Delta t_{\text{AC,Earth}} = 10$ min, $\Delta t_{\text{BC,Earth}} = 8$ min; (d) 30 min; (e) BC light-like, CA time-like, AB time-like. **3.6** (a) 12 lt-ns, yes; (b) 9.6 lt-ns, no; (c) discuss with prob session group and instructor; (d) A, C, B, D; it should be consistent with (a); (e) A, B, C, D; it should be consistent with (b); (f) Discuss with problem session group and instructor.

3.8 (b) 6 lt-s; (c) 2 lt-s; (d) 10 lt-s; (e) 14 s. **3.10** (a) 80 lt-s; (b) 80 lt-s, 0.8 lt-s/s = 0.8c; (c) 0.8 lt-s/s = 0.8c, 48 lt-s. **3.11** (b) 0.8c; (c) 4.5 yr; (d) 3.6 lt-yr; (f) D, B, C, A; (g) C, D, B, A; (h) 4.58 lt-yr.

Chapter 4
4.2 0.308c. **4.3** 0.385c or 0.946. **4.7** 1.05 MeV. **4.10** $p = 6499\,\mathrm{MeV}/c$, $u = 0.9897c$. **4.14** $E' = 15\,\mathrm{MeV}$, $p' = 12\,\mathrm{MeV}/c$. **4.16** (a) 899 GeV; (b) 100 GeV; (c) $u/c \simeq 1 - 8.98 \times 10^{-9} \simeq 0.999999991$. **4.17** $p = 200\,\mathrm{MeV}/c$; $K = 100\,\mathrm{MeV}$. **4.18** $p = 1.921\,\mathrm{GeV}/c$; $u = 0.899c$.

Chapter 5
5.1 (c) $E_1 = 10/\sqrt{5} \simeq 4.47\,\mathrm{GeV}$, $p_3 = 10/\sqrt{5} \simeq 4.47\,\mathrm{GeV}/c$. **5.2** $u = 2c/3$, same magnitude as Example 1. **5.5** (a) $m = 5.06\,\mathrm{MeV}/c^2$, $u = 0.19c$; (b) 0.06 MeV. **5.7** (a) rest energy to kinetic energy; (b) 17.6 MeV.

Chapter 6
6.1 $2nRT$. **6.2** (a) $m = 9.27 \times 10^{-26}\,\mathrm{kg}$, $d = 2.28 \times 10^{-10}\,\mathrm{m}$, $k_{\mathrm{sp}} = 48.0\,\mathrm{N/m}$; (b) 5180 m/s. **6.3** 9.30 kJ. **6.4** (a) 628 J; (b) 11.2 J; (c) 88.5 J. **6.5** (a) 30° C; (b) 499 J. **6.7** 0.229° C. **6.9** b, e. **6.15** 303,000 J versus 157 J (the word "wow!" would be appropriate here). **6.19** 223 J.

Chapter 7
7.1 (a) 9.2×10^{-12} m; (b) 1810 K. **7.2** (a) 479 m/s; (b) 678 m/s; (c) 409 m/s. **7.3** (a) $n_{\mathrm{ice}} = 5.6\,\mathrm{mol}$, $n_{\mathrm{liq}} = 11.1\,\mathrm{mol}$; (b) 20.9 kJ, (c) 3.5 mol. **7.4** (Note: assume $m_1 = m_2$.) (a) $\sqrt{2}$; (b) 2, (c) 2. **7.5** 0.194 mol. **7.13** 2.44 atm. **7.15** (a) 4.76×10^{-23} kg·m/s; (b) 2,560 collisions/s; (c) 1.22×10^{-19} kg·m/s; (d) 1.22×10^{-19} N; (e) 1.01×10^3 N; (f) 1.01×10^5 Pa.

Chapter 9
9.1 (a) 15. **9.2** 1.5×10^{10}. **9.3** (a) 1/216; (b) 1/216; (c) 10/216. **9.4** $25/7 \simeq 3.6$. **9.6** 300 K. **9.7** (a) $0.916k_B$; (b) $0.446k_B$. **9.8** $E_{A,f} = 4E_{B,f}$ **9.9** (a) From A to B; (b) All energy will flow to system B. **9.16** (a) From A to B; (b) $\Delta S_A = -0.6\,\mathrm{J/K}$, $\Delta S_B = +1.20\,\mathrm{J/K}$, $\Delta S_{\mathrm{total}} = +0.6\,\mathrm{J/K}$;
(c) $\Omega_{\mathrm{after}}/\Omega_{\mathrm{before}} = e^{4.35 \times 10^{22}}$.

Chapter 10
10.1 (a) 1.38 mol melted; (b) +30.3 J/K; (c) −29.2 J/K; yes (it had **better** be consistent with the second law!) **10.3** this is e raised to the power 1.59×10^{24}. (Don't bother trying to calculate that with your calculator.) **10.4** 1/2. **10.5** (a) −131.5 J/K; (b) 38.8 kJ; (c) 0.241. **10.6** 358 J (Note: this assumes you keep enough digits in the final temperature. If you round the final temperature to 346 K, you'll get 399 J). **10.8** (a) A: 100 kPa, 22.7 L, 273 K; B: 155 kPa, 22.7 L, 423 K; C: 100 kPa, 29.5 L, 355 K; (b) A to B: 1870 J; B to C: 0; C to A: −1700 J; (c) 0.092. **10.9** (a) $T_H = 4813\,\mathrm{K}$, $T_C = 289\,\mathrm{K}$; (b) $\epsilon = 0.18$, as compared with maximum of 0.94.

Chapter 11

11.1 (a) $r_1 = 4114\,\mathrm{km}$, $r_2 = 4902\,\mathrm{km}$, $\Delta r = 788\,\mathrm{km}$; (b) $1112\,\mathrm{km}$; (c) $\Delta s \neq \Delta r$, which means it is curved. **11.2 (a)** $\Delta\tau_C = 0.839\Delta\tau_D$**;** 6.20 **s.** **11.3** 1.19 **m**

Tables of Thermodynamic Properties

Selected Properties of Solids

Material	M (g/mol)	ρ (g/cm^3)	Y (GN/m^2)	C (J/mol·K)	v_s (m/s)
Aluminum	27.0	2.70	70	24.2	5000
Iron	55.8	7.87	211	25.1	5120
Copper	63.5	8.96	130	24.4	3810
Gold	197	19.3	78	25.4	2030
Lead	207	11.3	16	26.6	1190
ideal solid	mN_A	m/d^3	k_{sp}/d	3R = 24.9	$d\sqrt{k_{\mathrm{sp}}/m}$

Liquid Specific Heats

Liquid	molecule	C (J/mol·K)
water	H_2O	75.3
methanol	CH_3OH	79.5
ethanol	C_2H_5OH	112.4
acetone	$(CH_3)_2CO$	125.5
benzene	C_6H_6	134.8

Gas Specific Heats

Gas	Type	C (J/mol·K)
Neon (Ne)	monatomic	12.5
Argon (Ar)	monatomic	12.5
Hydrogen (H_2)	diatomic	20.5
Oxygen (O_2)	diatomic	21.1
Nitrogen (N_2)	diatomic	20.8

Latent Heats

Material	T_m (K)	L_f (kJ/mol)	T_v (K)	L_v (kJ/mol)
Oxygen	54.4	0.444	90.2	6.82
Nitrogen	63.2	0.72	77.4	5.56
Ethanol	159	5.02	352	38.6
Water	273	6.01	373	40.6
Lead	600	4.77	2022	180
Copper	1358	13.3	2835	300
Iron	1811	13.8	3134	340