Course Notes for Calculus I

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Contents

Chapter 1

Introduction

1.1 Organization of the Course

This calculus courses has a slightly unconventional method of organization. We are using a teaching model called the 'Flipped Classroom'. In this model, the majority of the information will be delivered outside of class time through two avenues: these notes and a series of short videos. You will be required to watch one of the short videos before most lecture periods. These videos will deliver the main content of the course. We will begin each lecture period assuming you have watched the required video.

The videos are used to introduce the main topics of the course. They are the explanation. Along with each video and lecture period, there will be a short section of notes. The notes are provided for reference, so that after you've watched the video, you can use the notes to remind yourself of the content and refer to useful ideas, concepts and formulae. The notes are not primarily written to explain the material; instead, they are written to provide a record of the ideas in the video for your reference. The notes are light on examples. The lecture time will be mostly devoted to necessary practice and examples.

The course is organized into 30 lectures; the videos and the activities are numbered to match these lectures. There is a detailed schedule on the course website showing when the various lectures happen over the term. Please use this schedule to ensure you watch the appropriate videos before class and bring the appropriate sections of the notes.

1.2 Other Resources

In addition to the notes associated to each lecture, there are a number of other resources which are distributed via the course website. These resources have been developed to assist you, so please make use of them.

- A set of pre-calculus notes. This is a document which I have prepared to answer the question: what should I know from high-school mathematics in order to succeed at calculus? If you are unsure about your background, please read these notes and use them to identify potential difficulties with high-school level mathematics.
- A complete formula sheet. All of the basic formulas and rules for calculation are included on this sheet. There are sections on algebra, trigonometry, derivatives and integrals.
- A library of functions. This reference gives archetypical examples of all of the elementary functions that we use in calculus. There are also small example graphs, so that you can develop a visual intuition for common functions.
- A notation reference. Though many of the idea in the course have been covered in high-school mathematics, often our notations differ. This reference covers a number of notations used in the course which may be unfamiliar.
- A course outcomes sheet. This was originally developed as a study aid for students. It summarizes the main definitions and concepts of the course, as well as the types of questions your will encounter on assignments and exams. In particular, it gives a guide to the material on the exam. If you want to know whether a definition, topic or type of problem might show up on the exam, consult this sheet. There will be nothing on the exam that isn't referenced on this sheet.

1.3 Perspectives and Philosophy

In addition to the strictly mathematical material of the course, I will try to share some ideas about the philosophy of mathematics, its aesthetics, and how our own worldview and assumptions influence mathematical thought. There will be short introductions to each lecture using aquotation and a particular question about the nature, existence and role of mathematics. I hope that you engage with these questions and find them intriguing. The perspectives and philosophy aspect of the course will not be part of the exam.

Chapter 2

Functions & Models

2.1 Analytic Geometry

2.1.1 The Cartesian Plane

Analytic geometry was a huge breakthrough for mathematics. Prior to the 17th century, algebra and geometry were haphazardly connected branches of mathematics. Various attempts had been made to give algebraic descriptions to geometric objects (including some primitive versions of cartesian coordinates long before Descartes), but none of these ideas and systems had managed to give a systematic and thorough connection. In the 17th century, Descartes proposed the coordinate system which now bears his name: the cartesian coordinates. Assigning numerical values to points in the plane (2 dimensional) or space (3 dimensional) allowed geometric problems to be interpreted algebraically, and vice-versa. Moreover, this connection was complete and systematic: in theory, any geometry in the plane or space could be described by Descartes' coordinates. This breakthrough was fundamental to the use of mathematics in the sciences, starting in the early modern age and continuing into the present. Cartesian coordinates must be numbered among a very short list of the most important mathematical advances in history.

The basic idea of cartesian coordinates should be familiar to you from high school level mathematics. In the plane, cartesian coordinates are formed by choosing a center point, drawing two perpendicular directions called axes (usually, but not necessarily, labelled x and y), and describing each point in the plane by two real numbers, written (a, b), representing how far along each axis the point falls.

We must point out that a and b are always real numbers, at least by default. Though we frequently deal with integer or rational coordinates, the system allows for any real numbers to represent coordinates. Since there are two dimensions in the plane, we refer to the plane with cartesian coordinates as $\mathbb{R}^2$.

Similarly, for space with cartesian coordinates (a, b, c), we write $\mathbb{R}^3$. The axes in $\mathbb{R}^3$ are typically called x, y and z.

Though we won't deal with higher dimensions in this course, the cartesian coordinate system generalizes easily to any number of dimensions. We can no longer visualize or draw the geometry in higher dimensions, but we can still work with the algebraic representation according to the same principles. For example, in $\mathbb{R}^5$, there are five axes, all perpendicular to each other, and points are represented by five real numbers (a, b, c, d, e), showing the distance along each of the axes. We write $\mathbb{R}^n$ for a cartesian system in general n dimensions. This is one of the most powerful aspects of cartesian coordinates – we can now do geometry in many dimensions and transcend the limitations of our three-dimensional vision and perception.

2.1.2 Loci

Cartesian coordinate are useful for giving algebraic definitions for various geometric shapes and objects. For an equation in x and y, the coresponding shape consists of all points (a, b) in the plane which satisfy the equation when x is replaced by a and y is replaced by b. Such an collection of points is called *locus* of the equation. The plural of locus is *loci*.

2.1.3 Lines

The simplest these loci are lines. Lines are given by equations $ax + by + c = 0$ for constants a, b, c. If $b \neq 0$, then this can be converted into the more familiar slope-intercept form $y = mx + n$ by solving for y. Along with planes and other flat objects, lines are called the *linear loci*. Linear loci can are loci whose equations in x and y (or in more variables in higher dimensional $\mathbb{R}^n$) involve only degree one polynomials. The subject of linear algebra is the study of such linear loci and their many properties.

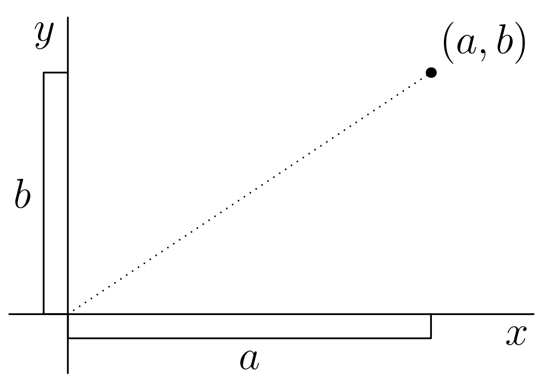


Figure 2.1: Cartesian Coordiantes in $\mathbb{R}^2$

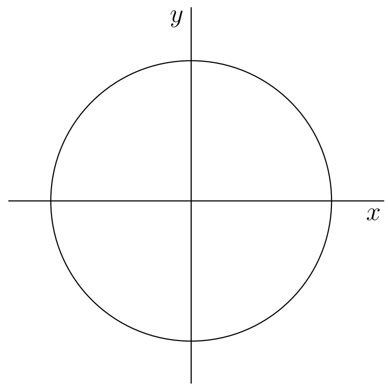


Figure 2.2: The Circle: A Locus

You will be expected to work with the equations of lines. In particular, you will need to be able to derive the equation of the line, both in standard form and slope-intercept form, given two points or given a point and a slope. This will be particularly important when we describe tangent lines to graphs.

You will also be expected to find the intersections of lines. In Figure 2.3, we have the lines $y = x + 1$ and $y = 3 - 3x$. Finding the intersetion point is equivalent to finding two coordinates x and y which satisfy both equations. Algebraically, that is solving the system composed of both equation.

$$\begin{aligned}
y &= x + 1 \\
y &= 3x - 3 \\
x + 1 &= 3x - 3 \\
-2x &= -4 \\
x &= 2 \\
y &= x + 1 = 2 + 1 = 3 && \implies \text{The intersection point is } (2, 3)
\end{aligned}$$

2.1.4 Conics

The conics are the second important class of loci. Unlike the linear loci, these are objects where we allow degree two polynomials in the coordinate variables. Conics are a very old topic in mathematics; their names and definitions come from ancient Greece. They are called conics (short for conic sections) since they can be formed by taking slices of a hollow cone at various angles. We will give three equivalent definitions for each conic: one instrinsically geometric definition, one as a the slice of a cone, and one by algebraic equations. For the algebraic equations, we will assume that the conic is centred at the origin.

The Circle

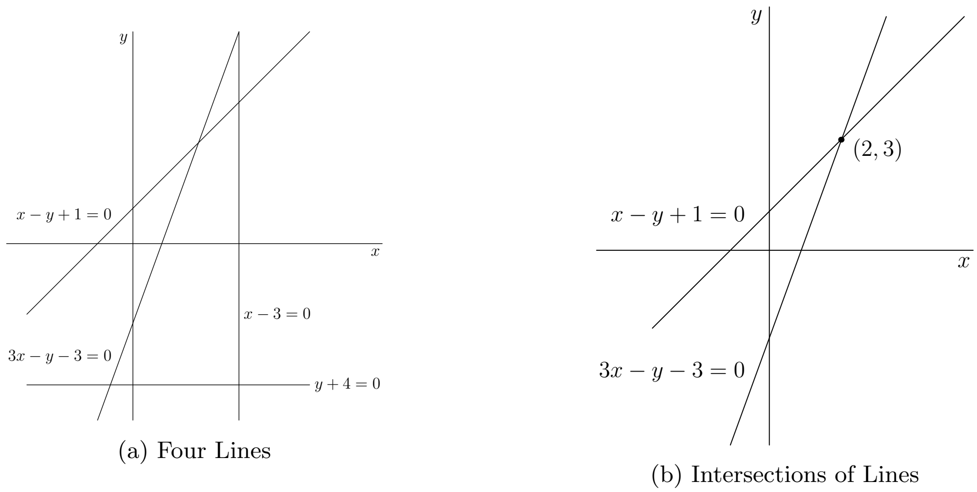


Figure 2.3: Lines as Loci

Intrinsic Definition Given a positive real number r, a circle is the set of all points in the plane which are exactly r units away from the origin. r is called the radius of the cicle.

Conic Slice Definition A circle is a perfectly horizontal slice of a cone.

Algebraic Definition A circle is described by the equation $x^2 + y^2 = r^2$.

The Ellipse

Intrinsic Definition Given two points p and q and a positive real number r, the ellipse is the set of all points in the plane where the sum of the distances to p and q is exactly r. The points are called the foci (singular focus).

Conic Slice Definition An ellipse is a slice of a cone at an angle greater than zero but less than the angle of the cone.

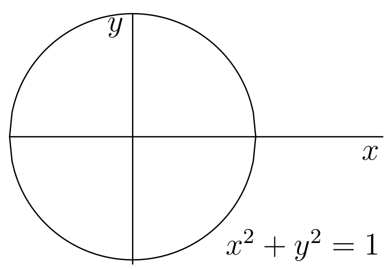
Algebraic Definition An ellipse is described by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Assuming that $a > b$, the foci are at $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$. The length a is called the semi-major axis and the length b is called the semi-minor axis.

The Parabola

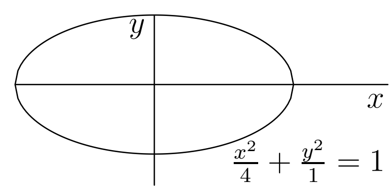
Intrinsic Definition Given a positive real number r, a point p and a line L, the parabola is the set of all points in the plane which are equidistant to the point p and the line L. The point p is called the focus and the line L is called the directrix

Conic Slice Definition A parabola is a slice of the cone at exactly the angle of the cone.

Figure 2.4: Slicing a Cone



(a) The Circle



(b) The Ellipse

Algebraic Definition A parabola is described by the equation $y = ax^2$. The focus is $(0, \frac{a}{4})$ and the directrix is $y = \frac{-a}{4}$.

The Hyperbola

Intrinsic Definition Given two points p and q and a positive real number r, the hyperbola is the set of all points in the plane where the difference of the distances to p and q is exactly r. The points are called the foci (singular focus).

Conic Slice Definition A hyperbola is slice of a cone at an angle steeper than the angle of the cone.

Algebraic Definition A hyperbola is described by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

One of the major motivating problems for conics and analytic geometry is the problem of celestial motion – how planets, moons, stars, comets and other celestial objects move and orbit around each other. The Greeks assumed, erroneously, that orbits ought to be perfect circles. Johannes Kepler, in the 16th century, correctly observed that orbit take non-circular shapes. He put forward a very convincing theory that orbits have shapes which are conics with the larger object fixed at one of the foci of the conic. This leads to ellipses for objects without escape velocity and hyperbolas for those with escape velocity. Though not perfect (particularly in complicated multi-body problems or when relativistic corrections are included), Kepler's model is remarkably accurate. Conics are still used as the basic models of orbital paths.

2.1.5 Other Loci

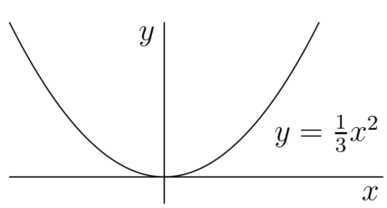
Lines and conics are just two of many classes and families of loci in $\mathbb{R}^2$. Here are some others.

- A curve in $\mathbb{R}^2$: While curves include lines and conics, the term 'curve' is a general term for curving paths. Many equations give rise to complicated curves, which can be very unpredicable. Future calculus courses will study the calculus of cuves.
- Instead of lines, we can restrict to portions of lines. For example, the piece of the line $y = x$ that has only positive x coordinates is the ray pointing out from the origin at an angle of $\pi/4$ radians. By taking finite pieces, we get line segments.
- With line segments, we can form straight-edged objects such as polygons. They are also loci not defined by one particular equation, but by the equations of several lines and restriction of those lines.

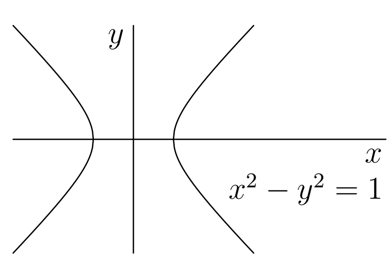
These are far from the only examples. Curves and loci can be very complicated: they can double back and self intersect, they can be jagged and disconnected, and they can even be very strange fractal-like space filling curves. Consider the locus of the equation $x^2+y^2 = 0$. Though this is a reasonable equation in the coordinate variables x and y, this locus only has one point: $(0, 0)$. Since x^2 and y^2 are always positive, no other values satisfy. Worse, consider $x^2 + y^2 = -1$. This locus has no points at all.

Graphs of functions are also loci. If $f(x)$ is a real-valued function, then the locus $y = f(x)$ is the graph of that function. Note that the graph itself is not the function, but just a geometric picture or representation of what the function does.

Since each value x leads to at most one function value $f(x)$, the graph of a function has the important property that for any fixed value x, there is only one point of the locus with that x coordinate. This is usually refered to a a vertical line test – a vertical line can cross the graph of a function at most once. Many of the loci we've seen above do not satisfy this; therefore, they cannot be graphs of functions. In addition, graphs never double back and never self-intersect.



(a) The Parabola



(b) The Hyperbola

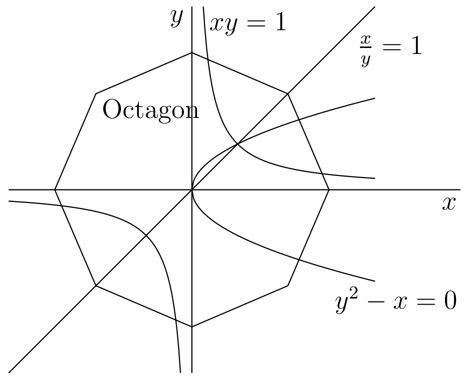


Figure 2.7: Several Other Loci

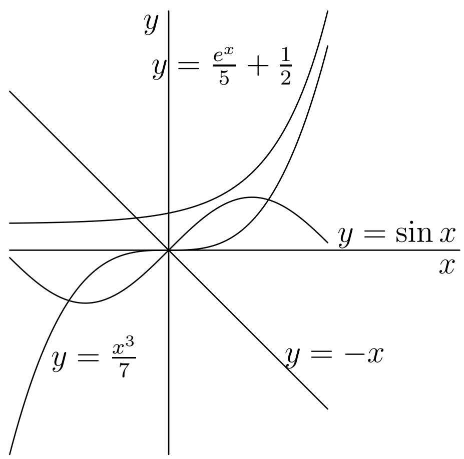


Figure 2.8: Four Graphs

2.1.6 Intersection

Intersection is a key idea in analytic geometry. Given two loci in $\mathbb{R}^n$, we often want to know if they intersect. In diagram 2.9, the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ and the line $y = -x$ have two intersection points: $p_1 = (\frac{-2}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $p_2 = (\frac{2}{\sqrt{5}}, \frac{-2}{\sqrt{5}})$

As with intersection of lines, finding the intersection of loci is the same as asking for common solutions of the equations. Solving systems of equations is one of the most basic tasks in mathematics. and intersection is the geometric interpretation.

2.1.7 Shifts

For conics, I noted that all of the conic equations were centered at the origin. By understanding shifts of loci, we learn how to write conics centered at any point. Shifts are a process of changing the equation of a locus to move the locus around the plane.

Let a be a positive real number and consider the equation of a locus.

- If we replace all instance of x by $x-a$, we move the locus a units in the positive x direction.
- If we replace all instance of x by $x+a$, we move the locus a units in the negative x direction.
- If we replace all instance of y by $y-a$, we move the locus a units in the positive y direction.
- If we replace all instance of y by $y+a$, we move the locus a units in the negative y direction.

Consider the cirlce $x^2+y^2=1$, which is a circle of radius one centered at the origin. Figure 2.10 shows the following shifts.

- The locus $(x-1)^2+(y-2)^2=1$ is a circle of radius 1 centered at $(1,2)$.
- The locus $(x+3)^2+(y-3)^2=1$ is a circle of radius 1 centered at $(-3,3)$.
- The locus $(x+4)^2+(y+6)^2=1$ is a circle of radius 1 centered at $(-4,-6)$.
- The locus $(x-5)^2+(y+2)^2=1$ is a circle of radius 1 centered at $(5,-2)$.

Shifts do not only apply to conics: we can shift any locus in this way.

2.2 Functions

In order to define functions, we need to define sets. A *set* is any collection of things; the things in a set are called its *elements* While most of our sets will be sets of numbers, apriori a set can have any kind of elements. Particularly useful sets for this course are the standard number sets ($\mathbb{N}$, the natural

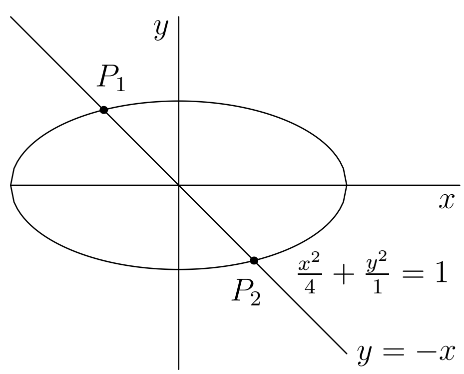


Figure 2.9: Intersection of a Line and an Ellipse

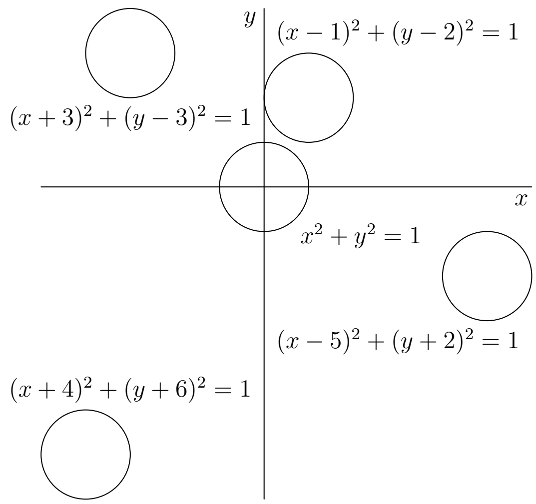


Figure 2.10: Shifts of a Circle

numbers; $\mathbb{Z}$, the integers; $\mathbb{Q}$, the rational number, $\mathbb{R}$, the real numbers), open inverals (a, b) and closed interval $[a, b]$.

Let A and B be sets. A *function* $f : A \to B$ is a rule that assigns an element of B to each element of A. The set A is called the domain. The range of the function is the subset of B consisting of all outputs of the function.

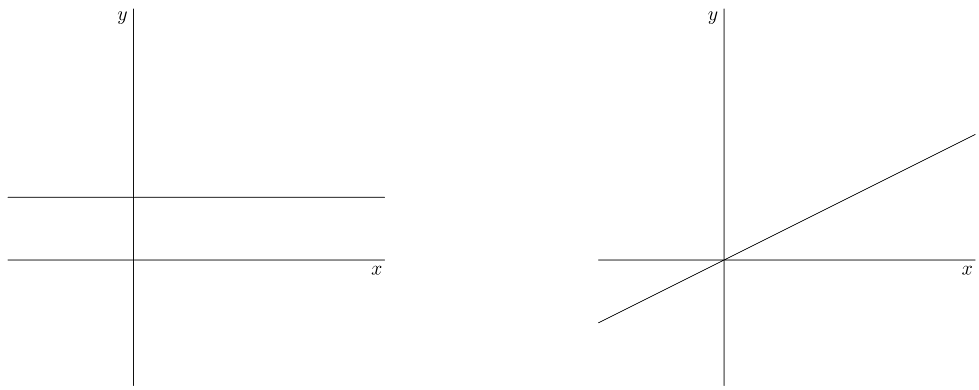
There are three major interpretations of functions. First, we can think of a function as a machine which acts on things. The function $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is a rule which assignes to each number its square. However, it's often more natural to think of f as the thing that squares numbers. Functions are agents which perform actions.

Second, we think of functions as relationships and dependencies. When we have a function $f : A \to B$, we can thing of the elements of B depending on the elements of A. If we say that population growth is a function of food supply, we mean that there is a function which goes from numbers representing food suply to numbers representing growth rates. That function encodes the dependance of growth on food supply.

Lastly, we often visualize a function by its graph. See the library of functions reference sheet for graphs of many common functions. Visualizing the action of a function by its graph is a very convenient way to remember the function and its properties.

2.2.1 Functions on $\mathbb{R}$

For the purposes of calculus, we will deal with functions defined on $\mathbb{R}$ and its subsets. For most functions, we will not explicitly stipulate a domain; the domain of the function will implicitly be the



(a) A Constant Function (b) A Linear Function

largest subset of real numbers where the function applies. Likewise the range will be the subset of all possible outputs of the function.

Determining the domain of a function on $\mathbb{R}$ means avoiding illegal mathematical actions. There are three common restrictions.

- We cannot divide by zero.
- We cannot take even roots of negative numbers.
- We cannot take logarithms of negative numbers or zero.

There are special domain restrictions for certain functions, such as inverse trig functions, but the main three cover the vast majority of functions we will be dealing with. Determining the domain of a function f means excluding real numbers which would lead to one of the three problems.

In addition, should we wish to, we can put additional domain restrictions on a function. Functions of time cannot extend infinitely back in time, so we usually stipulate a starting time; the domain of the function will be after that starting time. Restricting domains is also useful to make a function invertible.

2.3 Types of Functions

We want to give a catalog of the common types of functions we encounter in calculus.

2.3.1 Constant Functions

The simpliest kind of function is a constant function. Its output is the same regardless of input. The function $f(x) = 5$ is constant at 5: it will give the value 5 no matter what value of x we specify. Constant functions have no domain restrictions.

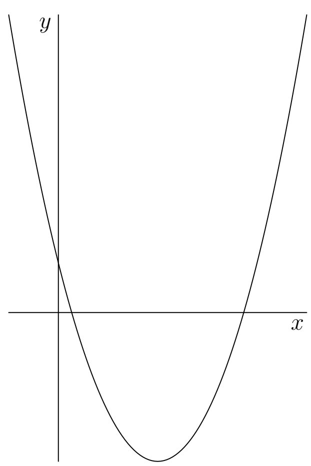
2.3.2 Linear Functions

Linear functions have the form $f(x) = ax + b$ for real numbers a and b. Their graphs which are straight lines, hence the name 'linear'. Linear functions includes constant functions, since we allow $a = 0$. All the tools from analytic geometry for understanding lines are useful for understanding linear functions. Linear functions have no domain restrictions.

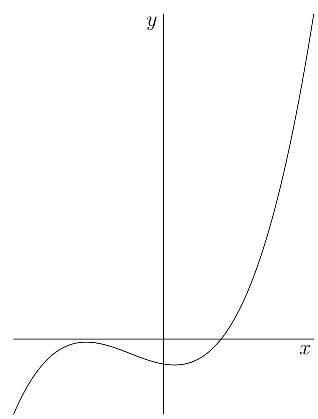
A particularly important linear function is the function $f(x) = x$, which is called the *Identity Function*. It is the unique function which takes any input and gives that input back without any action.

2.3.3 Quadratic Functions

Quadratic functions have the form $f(x) = ax^2 + bx + c$. Their graphs are parabolas. We have a large array of tools to understand parabolas, including the vertex-form (to find the highest/lowest point of the function) and the quadratic equation (to find the roots). Quadratic functions have no domain restrictions.



(a) A Quadratic Function



(b) A Polynomial Function

2.3.4 Polynomial Functions

Polynomial functions have the following form, where the a_i are anyreal numbers.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

The number n of the largest power in the polynomial is called the *degree* of the polynomial. The previous cases are all polynomials: constant functions have degree zero, linear functions have degree one, and quadratics have degree two. We have names for the next few degress as well: cubics have degree three, quartics have degree four and quintics have degree five. Polynomial functions have a familiar standard shape involving a graph that curves up a down some number of times. The maximum number of times the graph of polynomial can change directions is one less than the degree. Polynomials of degree n can have at most n roots, though they may have fewer. Polynomial functions have no domain restrictions.

2.3.5 Rational Functions

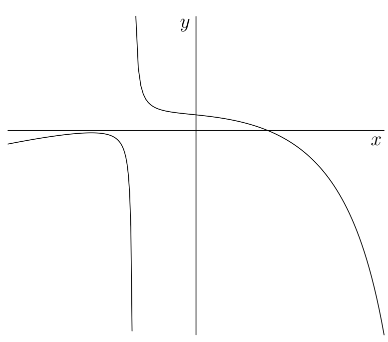
Rational numbers are fractions involving integers, In the same way, rational functions are fractions involving polynomials. Rational functions have the follwoing form, where $p(x)$ and $q(x)$ are polynomials.

$$f(x) = \frac{p(x)}{q(x)}$$

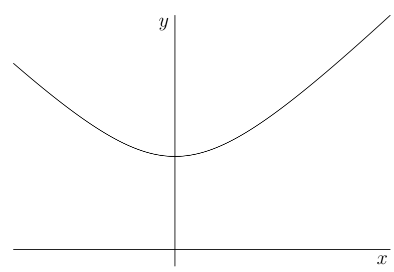
Rational functions may have domain restrictions. Into order to avoid dividing by zero, we must avoid x where $q(x) = 0$. Rational functions may have *vertical asymptotes* near their underfined points. A vertical asymptote is a vertical line which the graph of the function approaches near an undefined point.

2.3.6 Algebraic Functions

We are now moving into much broader categories of functions, where it is impossible to give a cohesive sense of the behaviour of the functions. Terminology is still useful for grouping these functions. Algebraic functions are functions which involve the four basic operations of addition, subtraction, multiplication and division as well as any real exponent. This includes polynomials and rational functions, but also roots, since roots can be written as fractional exponents. ($\sqrt{x} = x^{\frac{1}{2}}$). Algebraic functions can be very complicated conglomerations of these operations.



(a) A Rational Function



(b) An Algebraic Function

2.3.7 Trigonometric Functions

The first type of non-algebraic functions are trigonometric functions. Sine and cosine have the familiar sinusoidal wave shape: infinitenly many oscilation which repeat perfectly with some period. Sine and cosine waves can be analyzed by their amplitude and period. The other trigonometric functions (tangent, cotangent, secant and cosecant) are also periodic, but they have undefined points and vertical asymptotes at regular intervals.

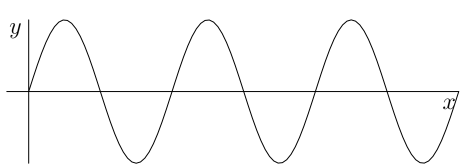
Though the trigonometric functions are usually associated to triangles, it is more natural to define them using the circle. If θ is the angle from the positive x axis, then $\cos\theta$ is exactly the x coordinate of the circle and $\sin\theta$ is exactly the y coordinate of the circle. All of the important properties of trigonometric functions can be derived from circle geometry, including the many trigonometric identities. Please see the formlua reference sheet for definitions and identities involving trigonometric functions.

Each trigonometric function has a inverse function. There are two standard notations; the inverse of $\sin(x)$ is written either as $\sin^{-1}(x)$ or $\arcsin(x)$. Even though the former notation is familiar from calculators, we will use the latter in these notes to avoid confusions with $\frac{1}{\sin(x)}$.

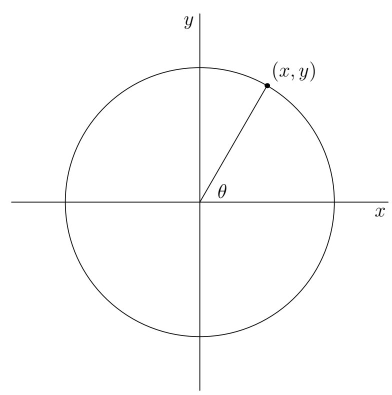
2.3.8 Exponential and Logarithmic Functions

If a is a positive real number, then functions of the form $f(x) = a^x$ are called exponential functions. They differ from roots and polynomials in that the variable is in the exponent. This distinction is very important but easy to confuse: $f(x) = x^a$ is an algebraic function but $f(x) = a^x$ is an exponential function, which is not algebraic. The laws of exponents (see the formula sheets) help us work with exponential functions.

The exponential bases you've most likely seen to date have been 2 and 10. These are useful bases, but in calculus (for reasons that will become clear later) we prefer a different base. The irrational number



(a) A Trigonometric Function



(b) Trigonometry and the Circle

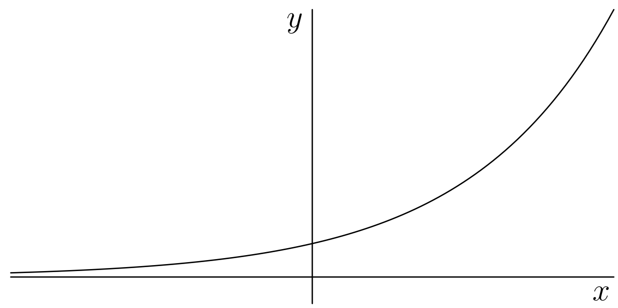


Figure 2.15: An Exponential Function

e is called Euler's number. It has an approximate value $e = 2.71828\ldots$. It is by far the most common exponential base; you will be seeing the exponential function $f(x) = e^x$ very frequently. It is reasonable to claim that $f(x) = e^x$ is the most important function in calculus.

The inverse of the exponential function is the logarithm. If the exponential has base a, its logarithm is written $\log_a x$. However, the inverse of the exponential $f(x) = e^x$ is instead written $\ln x$ and called the natural logarithm. We will almost exclusively deal with the natural logarithm in calculus. Logarithms (in any base) can only act on positive numbers.

2.3.9 Hyperbolic Functions

Though we won't define or investigate these function until Calculus II, it is useful to mention that there is a third important family of non-algebraic functions called the hyperbolics functions. There are a strange family, in some ways similar to trigonometric functions and in some ways similar to exponentials. They have inverse functions as well, called the inverse hyperbolics.

2.3.10 Elementary Functions

Everything we've defined so far comes together to form the large family refered to as the elementary functions. We can think of elementary functions as those function for which we have a familiar and succinct notation. Everything we've defined so far, as well as any combination of these functions, is an elementary function. As we will see later, calculus gives us methods for defining many non-elementary functions. Depending on how far you proceed through the calculus stream, you will see a few or many of these non-elementary functions.

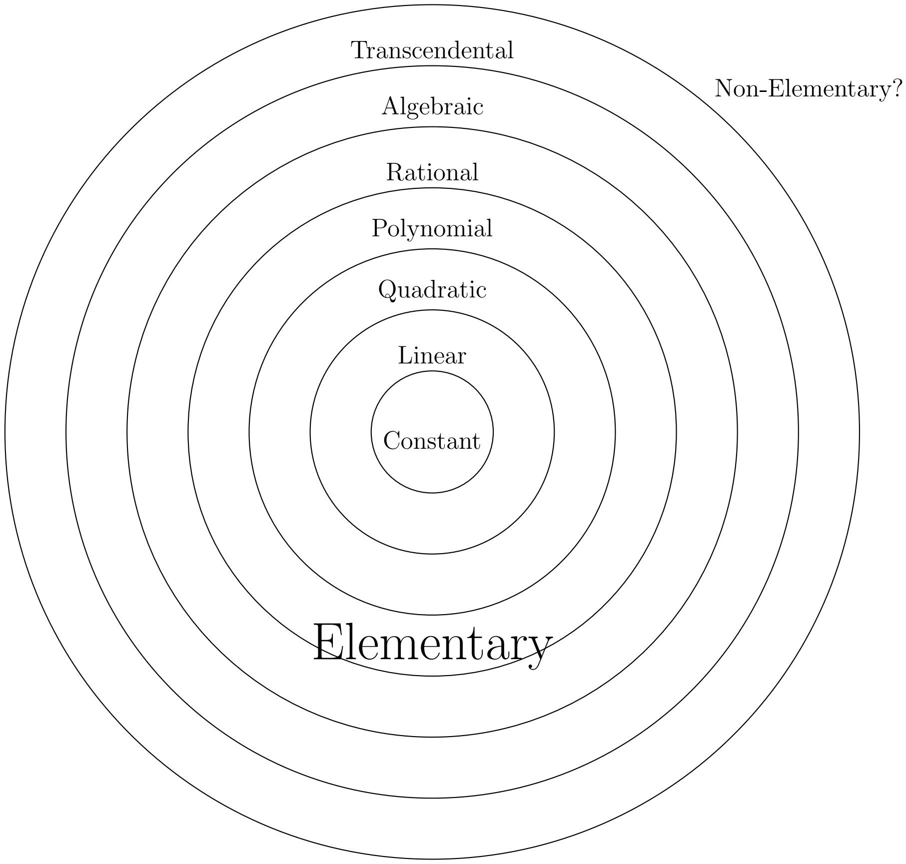


Figure 2.16: The Universe of Functions

2.4 Properties of Functions

We use a number of properties of functions to describe their behaviour.

- The *domain* of a function is all possible real inputs to the function. The domain can be restricted to a smaller subset if desired.
- The *range* of a function is all possible real outputs of the function.
- A function is *even* if $f(-x) = f(x)$. Visually, the graph of the function has a mirror-image symmetry over the y axis.
- A function is *odd* if $f(-x) = -f(x)$. Visually, the graph of the function is the preserved under a half-turn rotation about the origin.
- A function is *periodic* if there is a real number a such that $f(x+a) = f(a)$ for all x. The smallest positive real number a that satisfies is called the period.

- A function is called *increasing* if $a < b$ implies $f(a) < f(b)$. Visually, the graph of the function is growing upwards as the input increases.
- A function is called *decreasing* if $a < b$ implies $f(a) > f(b)$. Visually, the graph of the function is declining downwards as the input increases.
- A function is called *monotonic* if it is either always increasing or always decreasing.
- A function is *bounded above* if there is a number A such that $f(x) < A$ for all x. Visually, the graph of the function never gets above the height $y = A$.
- A function is *bounded below* if there is a number B such that $f(x) > B$ for all x. Visually, the graph of the function never gets below the height $y = B$.
- A function is *bounded* if it is both bounded above and below. That is, there exists numbers A and B such that $B < f(x) < A$ for all x. Visually, the function remains within the range $y \in [A, B]$ for any input.
- A *y-intercept* of a function is a place where it crosses the y axis. Since a function satisfies a vertical line test, there is only one possible y intercept found at $f(0)$ (if 0 is in the domain).
- An *x-interceopt* or *root* of a function is a place where it crosses the x axis or, equivalently, has the output $f(x) = 0$.

2.5 Operations on Functions

2.5.1 Pointwise Operations

The most familiar way to put functions together is through the conventional operations of arithmetic: addition, subtraction, multiplication, division and exponentiation. If f and g are functions, then $f + g$, $f - g$, $f \cdot g$, $\frac{f}{g}$ and f^g are all functions. They may have additional domain restrictions, such as avoiding the roots of g in $\frac{f}{g}$. We call these combinations *pointwise operations* on function.

2.5.2 Composition

A more novel way to combine function is *composition*. In pointwise operations, both the functions act seperately on the input and then we combine the result. In composition, we let one function act first and then let the second act on the output of the first. If f and g are functions, we write $f \circ g$ or $f(g(x))$ for their composition. In this notation, somewhat counterintuitively, the function on the right acts first. $f \circ g$ means we let the function g act first and then let the function f act second. Additional domain restrictions may result from composition, since the output of g must be acceptable input for the function f. In $f \circ g$, we call g the *inside function* and f the *outside function*.

Example 2.5.1. If $f(x) = e^x$ and $g(x) = x^2 + 1$, then $f \circ g(x) = e^{x^2+1}$ and $g \circ f = (e^x)^2 + 1$.

Example 2.5.2. Often is it important to recognize a composed functions and identify the pieces of the composition. The function $h(x) = \sqrt{x + 7}$ is a composition with inside function $g(x) = x + 7$ and outside function $f(x) = \sqrt{x}$. For complicated functions, there may be several ways to decompose the function as a composition; finding various decompositions and knowing which composition to use is an important skill.

Composition can be iterated. If f, g and h are functions, $f \circ g \circ h(x)$ is the composition of all three, with h acting first, g second and f third.

2.5.3 Inversion

The last important operation on functions is inversion. Inversion attempts to undo what a function has accomplished, to go backwards and return the output to the original input. If $f(x)$ is a function, we write $f^{-1}(x)$ for its inverse. This notation *does not* mean the reciprocal function $\frac{1}{f(x)}$.

In order for a function to be invertible, each output needs to go back to a unique input. This restrictions implies that no two inputs can be sent to the same output. This property is called *injectivity*, but for our purposes, it is enough that a function is *monotonic.* A function which always increases or decreases can only have one input sent to any particular output; therefore, it is invertible. When we want to invert a function, we must make sure that it is monotonic.

When we compose a function and its inverse, we get the original input back. That is $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$ on the approriate domains. This is very useful for manipulating equations, since we can remove any function by compositing with its inverse.

Example 2.5.3. We use the natural logarithm to remove the exponential with base e.

$$\begin{aligned} e^x &= y \\ \ln e^x &= \ln y \\ x &= \ln y \end{aligned}$$

Example 2.5.4. Likewise, we use inverse trig to remove trig function.

$$\begin{aligned} \sin x &= y \\ \arcsin(\sin x) &= \arcsin y \\ x &= \arcsin y \end{aligned}$$

For a monotonic function, the domain of the inverse will be the range of the original function. If we can calculate the range of the original function, we get the domain of the inverse for free. In the first example above, the range of e^x is $(0, \infty)$, so that becomes the domain of $\ln x$. In the second second above, $\sin x$ is not monotonic on its whole domain, so we have to make an adjustment before we can understand the range.

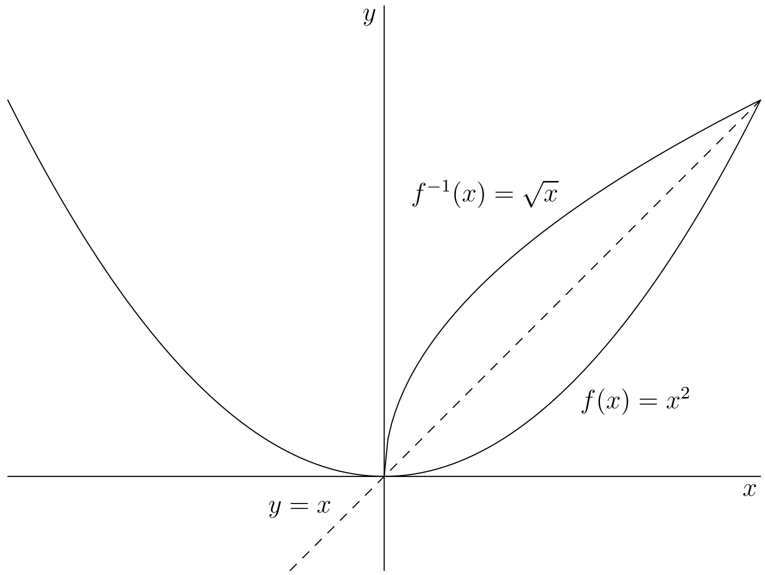


Figure 2.17: Restricting the Domain of $f(x) = x^2$

2.5.4 Restriction of Domain

Many functions are not monotonic, but we would like to invert them anyway. We solve this problem by restricting the domain. The classic example is the quadratic $f(x) = x^2$. This function has a domain of $\mathbb{R}$, but it is not invertible, since both x and $-x$ are sent to x^2. Going backwards, we do not know whether to sent x^2 to x or $-x$. Therefore, we restrict the domain. The function $f(x) = x^2$ on the restricted domain $[0, \infty)$ is increasing, therefore it is invertible. Its inverse is $f^{-1}(x) = \sqrt{x}$, returning the positive square root.

The example of $\sin x$ we used above is desperately not monotonic: it isolates up and down constantly. We have to make a severe restriction to invert it, by choosing a very small piece where it is monotonic. For $\sin x$, the conventional choice is the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, where $\sin x$ is an increasing function. On this interval, the range of $\sin x$ is $[-1, 1]$, so the domain of $\arcsin x$ is also $[-1, 1]$.

2.6 Models

In this course, we use the term 'model' in the broadest possible mathematical sense: a model is any application of a mathematical concept to an external situation. Since calculus is the study of functions, our models will be functions chosen to describe an observed connection between two quantities. The branch of mathematics that deals with models is called applied mathematics.

Models are the mathematical versions of the scientific method. If we observe a relationship between two quantities, choosing a function to describe that relationship is a scientific hypothesis. We can then work with the mathematics to understand the function. The behaviour of the function can be then tested against data, either the original data or new data. We can re-evaluate to see if the model fits the data to an acceptable precision.

Our analysis of functions helps us understand models. We can ask questions about functions inspired by their applied context. We'll start with three important questions.

- Do we restrict the domain to make it reasonable? A mathematical model where the independant variable is human population would reasonably impose restriction that the population must be positive and bounded above by some maximum population. These restrictions of the model exist in addition to whatever natural mathematical restrictions we already have on domain.
- We can ask abour starting values. If our model depends on time, we can set a starting time (often but not necessarily $t = 0$) and specify a value for the function at the starting time. Even though there is a mathematical description of what happens before the starting time, we can ignore it since it doesn't apply to the situation we are modelling.
- We can ask about the constants in our model. Functions often involve constants apart from the variables, such at the coefficients of a polynomial. In a model, the constants should have reasonable meaning and interpretation.

The analysis of models thus consists of everything we used to analyze functions in Sections 2.2 and 2.5 as well as these new questions. Often we will want a qualitative analysis of the model. Understanding the general shape and behaviour of the functions involved is invaluable to this qualitative analysis. A major goal for this course is to develop tools and competencies to allow students to analyze functions and models both quantitatively and qualitatively. The qualitative analysis can often be expressed in a narrative: what story is the function telling about the relationship? If the independant variable is time, as it often is, what is the story of the dependant variable over a time span? What happens to it?

2.6.1 Regression

One of the greatest challenges of applied mathematics is choosing appropriate models. If the real world provides us with a set of data points, how do we choose a function? Often we can intuit what kind of function (linear, polynomial, exponential, etc) we think might fit the data. However, it is very difficult to be more specific just through intuition. Once we've chosen a type of function for our model, *regression* is a set of techniques that help to find the best specific function for the data. In this course, we will only be looking at regressions graphically. The hard work of calculating functions in a regression is left for future courses.

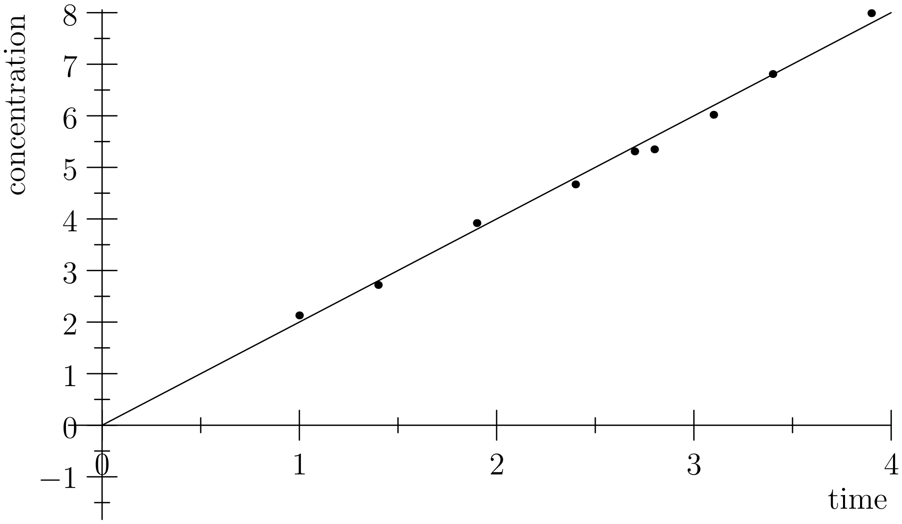


Figure 2.18: A Linear Regression

2.7 Population Growth

A very natural motivating problem is population growth. Here are some reasonable mathematical questions we might ask about population.

- How do we encode population growth as a function?
- What factors affect the growth of population?
- What is the long-term behaviour of a population?
- What is the growth rate of a population?

To encode population growth as a function, we chose a function $p(t)$ representing population (p) in terms of the independant variable time (t).

Consider this data set for a population (where t is counted in years). This growth looks exponential. We could apply exponential regression to find an exponential function, as in Figure 2.21. However, there is another way of finding a function to fit this model, using the growth rates to find a function to fit the data. Algebraic techniques have a hard time understanding growth rates, or rates of change in general. In calculus, we will develop a tool called the *derivative* of a function which will measure its rate of change. For now, we are going to avoid the technical definition but work with the idea. We need some notation. For a function $p(t)$, there are two common ways of writing the derivative.

$$p'(t) \qquad \frac{dp}{dt}$$

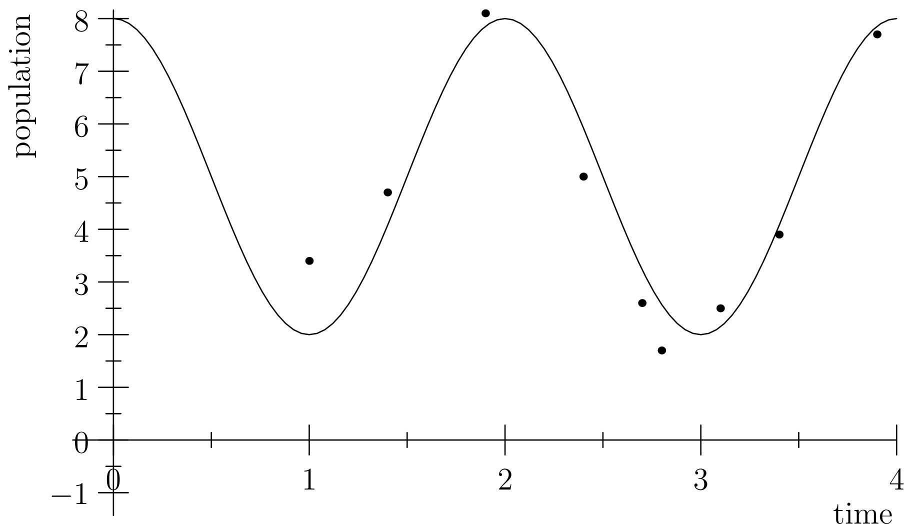


Figure 2.19: A Sinusoidal Regression

The first notation is called Newton's notation, and the second is called Leibniz's notation. We will use both in this course but we will rely on Leibniz's notation most of the time.

The derivative is the rate of change of the function, so for this model, $\frac{dp}{dt}$ is the growth rate of the population. The average annual increase in the data above is 20% growth. That means that the growth rate (the amount of population added in a year) is $\frac{20}{100}$ (or $\frac{1}{5}$) times the current population. We can express this as an equation.

$$\frac{dp}{dt} = \frac{1}{5}p(t)$$

This is a mathematical translation of the understanding of percentage growth. An equation of this form, involving a function and its derivative, is called a *differential equation* (DE, in short). Very often in applied mathematics, it is easier to observe the rate of change of a function than the function itself. This leads to models expressed as differential equations. Many of the most important mathematical models in history are expressed as differential equations.

The regression provided the function $p(t) = 2^{\frac{t}{4}}$. We will see later in the course that the solution to this differential equation is a very similar function.

$$p(t) = e^{\frac{1}{5}t}$$

Solving differential equations is one of the major goals of calculus. The solution to this equation leads to the profound connection between percentage growth rate and exponential growth: if we observe a percentage growth rate, then the quantities we are observe should be described by an exponential function. This is how we know that many populations grow exponentially, as do functions involving

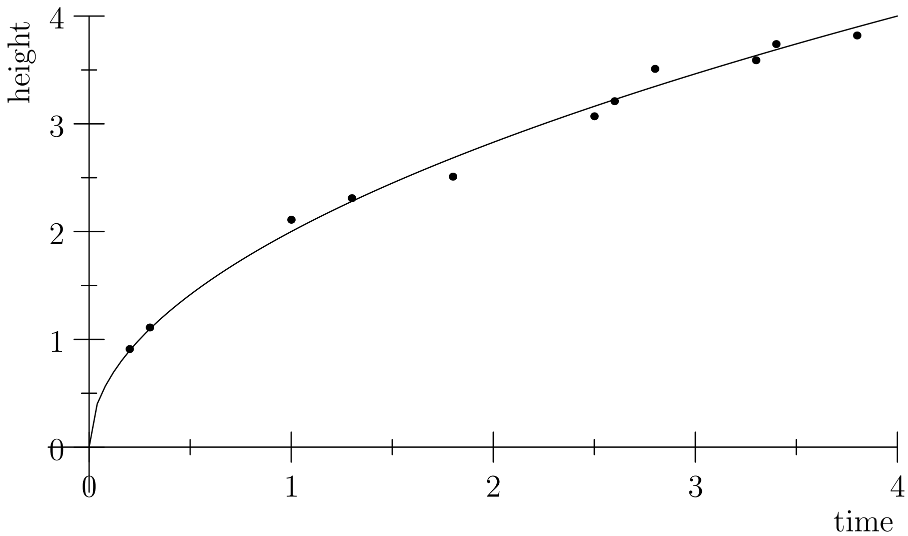


Figure 2.20: A Square-Root Regression

resource use, radioactive decay, heat dissipation, debt repayment and investment interest. The importance of the exponential function is strongly related to its use as the solution to problems of percentage growth.

DEs gives us a different approach to fiting a function to data. Instead of just using intuition to choose a class of functions and regression to get the specific function, we can look at the growth rates of the function in the data. If we can make a consistent observation about the growth rates, such as the observation that they grow by a consistent percentage, we can write that observation as a differential equation. We can then (hopefully) solve the differential equation to find the appropriate function.

Before we worry about solving DEs, we simply want to understand what they say. Our goals is essentially translation: a differential equation is a mathematical translation of a statement about the growth rate of a quantity. We want to be able to pass both ways between the equation and the associated statement. We have seen how percentage growth with percentage $c\%$ translates into an equation.

$$\frac{dp}{dt} = \frac{c}{100}p$$

We can translate other statements as well. Percentage decay says that we loose a percentage $c\%$ of the quantity per unit time.

$$\frac{dp}{dt} = -\frac{c}{100}p$$

Not all differential equations are percentage growth. Perhaps the rate of change is proportional to the square of the current value.

$$\frac{dp}{dt} = c(p(t))^2$$

Year	Population	Growth Rate
0	1032	Not Applicable
1	1214	17.6 %
2	1372	13.0 %
3	1629	18.7 %
4	2143	31.6 %
5	2520	17.6 %
6	2940	16.7 %
7	3292	12.0 %
8	3813	15.8 %
9	4757	24.8 %
10	5632	18.4 %
11	6842	21.5 %
12	8010	17.1 %

2.8 Autonomous Differential Equations

2.8.1 Phase Line Analysis

At this point in the course, we lack the tools to solve differential equations. Instead, we want to do some qualitative analysis of the differential equation. We will restrict this analysis to a particular type. A differential equation where the left-side is just $\frac{dp}{dt}$ and the right-side is some algebraic expression in p is called an *autonomous differential equation*. Many natural models are described by autonomous equations, including population growth. There is a lovely piece of analysis for autonomous equations called the *phase line analysis*.

Phase line analysis looks at the right side of the equation and asks for values of the function p which set the right side to zero. What does this mean? When the right side of our differential equation is zero, so the left side is zero as well. The left side is the growth rate, so that means the growth rate is (momentarily) zero. These are values of the population p where there is no growth. We call these values *steady states* of the population. If the population is exactly at its steady state, it will not change; steady states are constant popluations which do no grow or decline. For population models, we can make the reasonable assumptions that $p \geq 0$ and $p = 0$ is always a steady state.

Once we have the steady states, we ask what happens between each steady state. Assuming that the DE is reasonable, then between the steady states, the right side will be either positive or negative. When it is positive, we have a positive growth rate and the population increases. When it is negative, we have a negative growth rate and the population decreases. This direction of growth, negative or positive, is called the *trajectory* of the popluation between the steady states.

Amazingly, this gives us an impressively complete understanding of the population.

- If the popluation is at a steady state, it doesn't change.

- If the popluation is not at a steady state, we look at the trajectory.
- If the trajectory is positive, the popluation grows to the closest larger steady state or infinity.
- If the trajectory is neagative, the population declines to the closest smaller steady state or zero.

We summarize this information is a phase line diagram. We take a ray representing $p \geq 0$ and put dots on the ray for the steady states. In between, we put arrows to show the trajectories. Its best to see the phase line diagrams through examples.

Example 2.8.1.

$$\frac{dp}{dx} = p^2 - p$$

The right side is zero when $p = 0$ or $p = 1$, so those are the steady states. When $p \in (0, 1)$ the derivative is negative, so the trajectory is decreasing. When $p \in (1, \infty)$, the derivative is positive, so the trajectory is increasing. This phase line encodes this information.

Example 2.8.2. This is an example of the logistic equation.

$$\frac{dp}{dt} = 4p - p^2$$

The right side is zero when $p = 0$ or $p = 4$, so those are the steady states. When $p \in (0, 4)$ the derivative is positive, so the trajectory is increasing. When $p \in (4, \infty)$, the derivative is negative, so the trajectory is decreasing.

The logistic equation leads to logistic growth. We can see from the Figure 2.23 that the trajectories all point towards the steady state $p = 4$. In logistic growth, the population wants to approach to some non-zero steady state. From below, this is growth up to some maximum and from above it is decay down to a minimum. (After exponential growth, logistic growth is the most commonly used model for populations.) Figure 2.24 shows both exponential and logistic growth (where the steady state for the logistic model is at $p = 6$.)

Example 2.8.3.

$$\frac{dp}{dt} = p^3 - 7p^2 + 10p$$

The right side factors as $p(p - 2)(p - 5)$, so it is zero then $p = 0$, $p = 2$ or $p = 5$. Those are the steady states. When $p \in (0, 2)$ the derivative is positive, so the trajectory is increasing. When $f \in (2, 5)$, the derivative is negative, so the trajectory is decreasing. When $p \in (5, \infty)$, the derivative is positive, so the trajectory is increasing.

2.9 Two Motivating Problems

2.9.1 The Velocity Problem

Moving from biological examples to physics, the velocity problem is one of the basic motivating problems of calculus. Assume we have an object moving in one dimension. We describe its position as a function $p(t)$ where p is position in terms of time t. We want to know its velocity.

If $p(t) = at + b$ is a linear function, then algebra can answer this question. With this linear function, for each unit of time we gain a units of distance. The value a is the velocity. Geometrically, the velocity is measured by the slope of the straight line graph of $p(t)$. The slope is measured by the ratio of the change in p, Δp, to the change in t, Δt.

If slope is the way to measure velocity, then we need a notion of slope for non-linear functions as well. The notion comes from the idea of tangent lines. A tangent line to a graph is a line which touches the graph at one point without crossing it (as opposed to a secant line, which crosses the graph twice).

The slope of a graph is defined to be the slope of its tangent, should such a line exist. The velocity problem is reduced to the problem of finding the tangent line (or, more particularly, its slope). How do we find a tangent line? Again, algebra has trouble with this. The best that algebra can do is find secant lines which approximate a tangent line. We adjust the approximation by letting the two points of the secant line come closer and closer together, as in the Figure 2.28. In this way, we build an approximation process which can get better an better. However, algebra an never finish the process – it can just supply improved approximations.

2.9.2 The Distance Travelled Problem

Let's consider the opposite problem. Say we have a function $v(t)$ which tells us the velocity of an object at any point of time. Can we determine the distance the object has covered over a period of time?

Again, algebra can only answer this question for very simple situations. Let's assume that velocity $v(t) = c$ is constant. In this case, if we have constant velocity of c unit distance for unit time and we have travelled for t_0 units of time, the distance is just the product ct_0. Graphically, the situation is summarized in Figure 2.29. We can see that the distance travelled under constant velocity is the area under the velocity graph. To solve the distance travelled problem, we have to find areas under curves. Like tangent lines, this is not a problem that algebra can easily tackle. However, it can approximate areas. Algebra is good at areas of rectangles, so we can use rectanges to approximate areas.

The sum of the areas of all the rectangles is reasonable approximation to the area under the curve. If we want a better approximation, we can divide into smaller rectangles. In this way, we set up an approximation process to understand areas under curves. Algebra can never completely answer the question, but it can get better and better approximations by using more and more rectangles in its approximation.

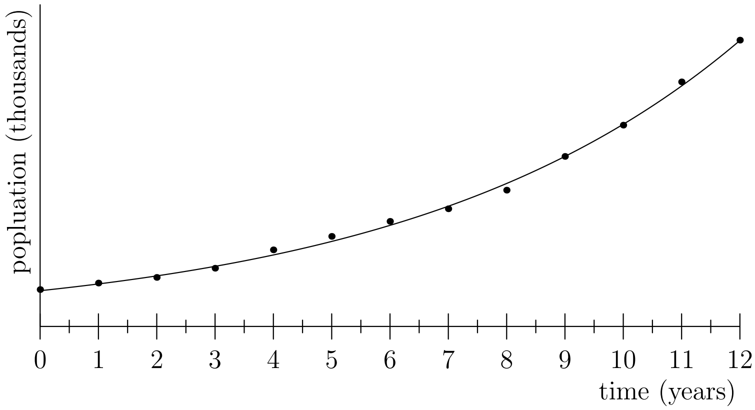


Figure 2.21: An Exponential Regression

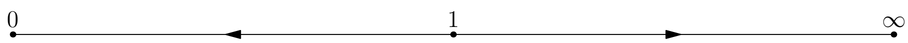


Figure 2.22: The Phase Line Diagram for $\frac{dp}{dx} = p^2 - p$

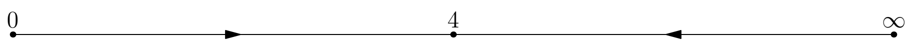


Figure 2.23: The Phase Line Diagram for $\frac{dp}{dt} - 4p - p^2$

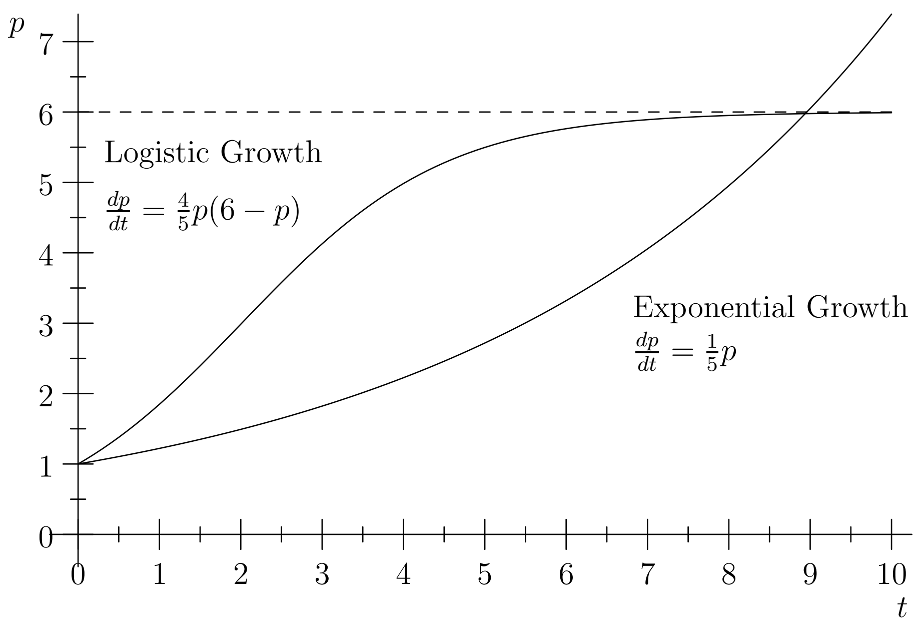


Figure 2.24: Exponential and Logistic Growth

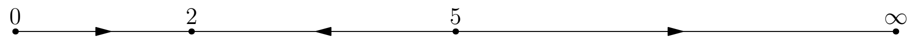


Figure 2.25: The Phase Line Diagram for $\frac{dp}{dt} = p^3 - 7p^2 + 10p$

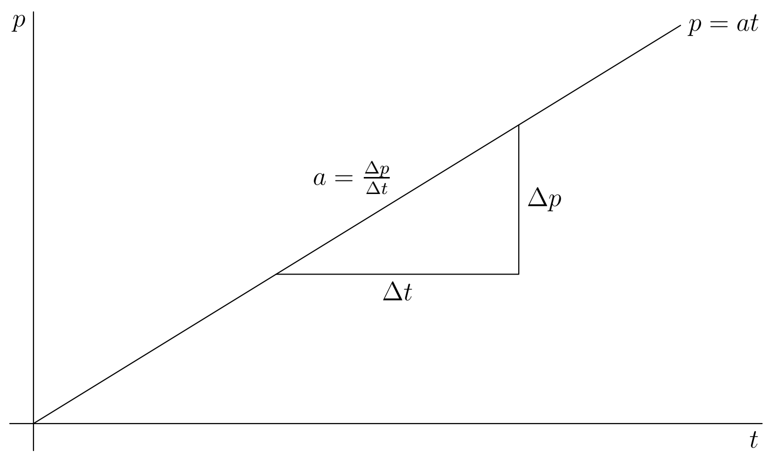
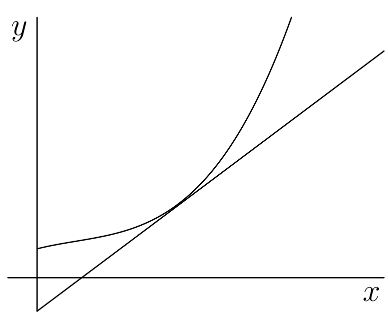
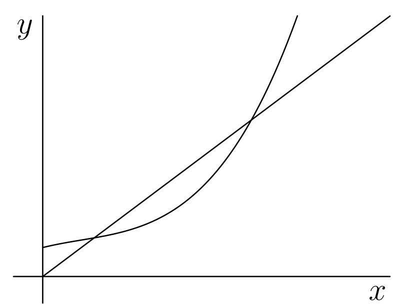


Figure 2.26: Slope of a Linear Function



(a) A Tangent Line



(b) A Secant Line

Figure 2.27: Tangent and Secant Lines

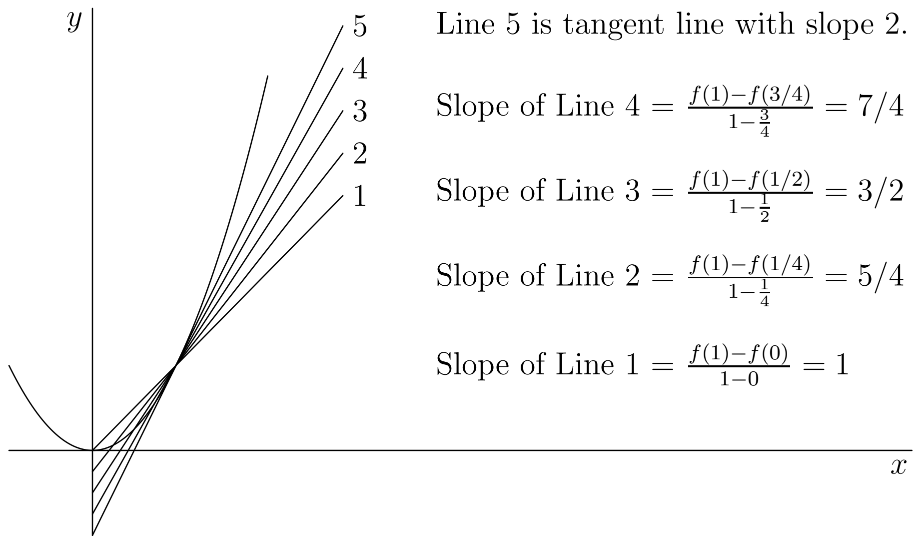
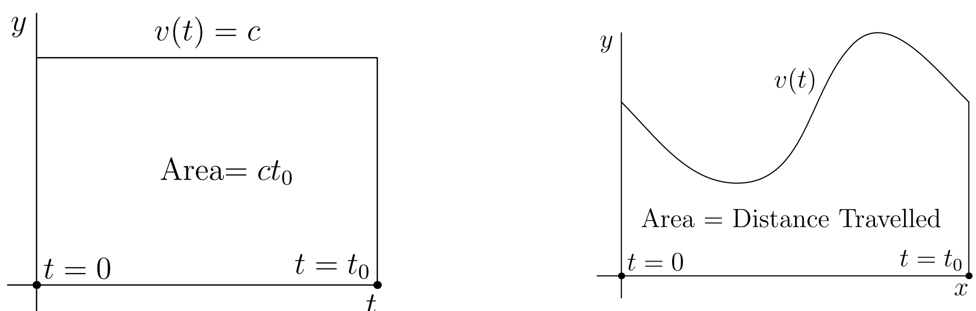


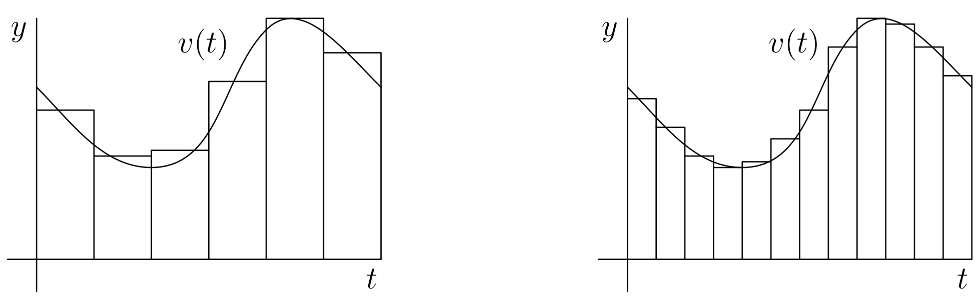
Figure 2.28: Secant Approximations to the Tangent Line



(a) Constant Velocity

(b) Non-Constant Velocity

Figure 2.29: Distance Travelled Problem



(a) Approximation of Area by Rectangles

(b) Improved Approximation with More Rectangles

Figure 2.30: Approximating Areas under Curves

Chapter 3

Limits

3.1 Limits at Finite Values

In the prevous section, the two motivating problems of velocity and distance-travelled led us to approximation processes. Inside the field of algebra, we couldn't get exact answers to these problems, only approximate answers. However, algebra found a process for refining the approximation.

Calculus starts by asking if we can somehow calculate the end of an approximation processs. Is there an exact answer at the end of all the approximation? Calclulus uses a new tool, the limit, to find such an exact answer. A limit is a way of understanding an infinite process and asking where the process eventually leads. The limit is the key new tool that transcends algebra and creates calculus.

3.1.1 Definitions

Our definition of limits will be restricted to limits of functions. In a limit, we are comparing the input and output of a function during an approximation process. The process starts with moving the input towards a specific value and observing what happens to the output.

Let $f(x)$ be a function and a a point either in the domain of the function or on the boundary of its domain. Then the statement

$$\lim_{x \to a} f(x) = L$$

means that as x (the input) gets closer and closer to a, f(x) (the output) gets closer and closer to L. If such an L exists, we say the limit of $f(x)$ exists at $x = a$; we also say that the limit converges. Otherwise we say the limit diverges.

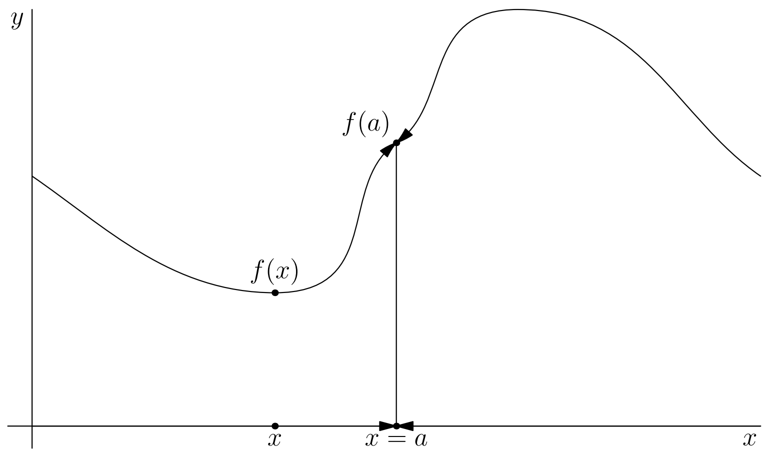


Figure 3.1: A Convergent Limit

There are several ways in which the limit can diverge. The statement

$$\lim_{x \to a} f(x) = \infty$$

means that as x gets closer and to a, the function value $f(x)$ gets larger and larger without bound. The statement

$$\lim_{x \to a} f(x) = -\infty$$

means that as x gets closer and to a, the function value $f(x)$ becomes a larger and larger negative value without bound. The statement

$$\lim_{x \to a} f(x) \qquad \text{DNE}$$

means that the limit does not exist; it doesn't approach any number at all.

3.1.2 Vertical Asymptotes

For limits which approach $\pm\infty$, we see that the graph of the function approaches a verticle line. These lines are called *vertical asymptotes* for the functions. Vertical asymptotes are shown as the dotted lines in the Figure 3.2.

3.1.3 One-Sided Limits

In the above definitions, we assume that $x \to a$ means x approaches a from both sides, considering x slightly larger and x slightly smaller than a. Sometimes it is convenient to only use one side. These

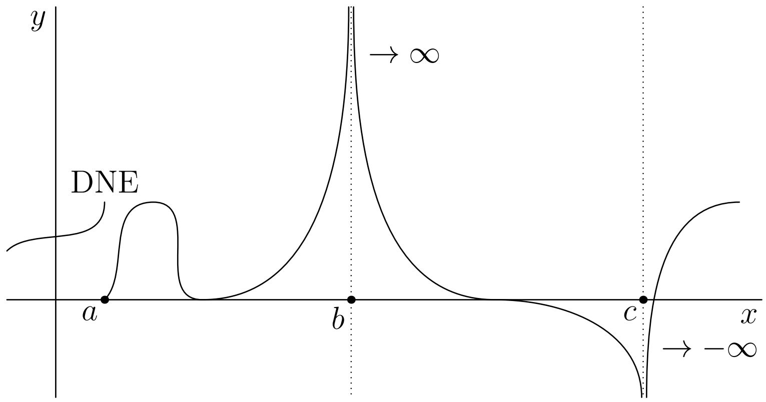


Figure 3.2: Three Divergent Limits

are called one sided limits. If we want to approach from the left (from x slightly smaller than a), we adjust the limit notation slightly by writing a^-..

$$\lim_{x \to a^-} f(x)$$

If we want to approach from the right (from x slightly larger than a), we write a^+.

$$\lim_{x \to a^+} f(x)$$

3.1.4 Calculating Limits

There is a three step procedure to calculating limits.

- First, try to evaluate the function at the limit point. For reasonable funciton (specifically, continuous funcitons, which will be defined later), if the function can be evaluated directly, the function value will be your limit value.
- If the first method fails, try to work out logically what the limit should do.
- If both methods fail, the limit is called an *indeterminante form.* In this case, we need to use various algebraic tricks (factoring, expanding, multiplying by conjugates, trig identities) and rules for manipulating limits to change the limit into a more approachable form.

3.1.5 Limits Rules

The third step referred to the rules for manipulating limits, which we will now state. Let f and g be functions with limits defined at $x = a$ and let c be a constant. (In the quotient limit, assume $g(x) \neq 0$, and in the exponential limit, assume $f(x) > 0$.)

$$\begin{aligned}
\lim_{x\to a} f(x) + g(x) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
\lim_{x\to a} f(x) - g(x) &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\
\lim_{x\to a} cf(x) &= c \lim_{x\to a} f(x) \\
\lim_{x\to a} f(x)g(x) &= \lim_{x\to a} f(x) \lim_{x\to a} g(x) \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \\
\lim_{x\to a} (f(x))^c &= (\lim_{x\to a} f(x))^c
\end{aligned}$$

In addition to the limit rules, there are two limits that are frequently useful. The second limit can be taken as a definition of the exponential base e.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x} &= 1 \\
\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} &= e
\end{aligned}$$

3.2 Limits at Infinity

3.2.1 Definitions

In addition to asking what happens to a function as the input approaches paticular finite values, we can also ask what happens as the input approaches infinity, that is, at the input gets larger and larger.

The statement

$$\lim_{x\to\infty} f(x) = L$$

means that as x gets larger and larger without bound, $f(x)$ gets closer and closer to L. The statement

$$\lim_{x\to\infty} f(x) = \infty$$

means that as x gets larger and larger without bound, $f(x)$ also gets larger and larger without bound. The statement

$$\lim_{x\to\infty} f(x) = -\infty$$

means that as x gets larger and larger without bound, $f(x)$ becomes a larger and larger negative number without bound.

These statements are defined the same way for $x \to -\infty$ when x becomes a larger and larger negative number without bound.

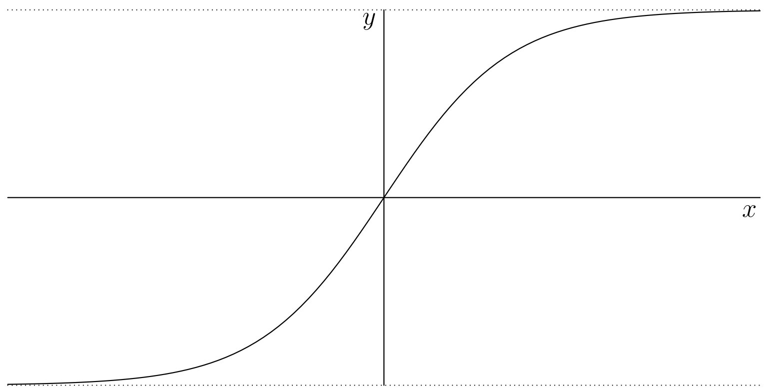


Figure 3.3: Horizontal Asymptotes

3.2.2 Horizontal Asymptotes

If we have a function $f(x)$ where the limit

$$\lim_{x \to \pm\infty} f(x) = L$$

exists either at positive or negative infinity, that means that the graph of the function approaches the line $y = L$. Such lines are called *horizontal asymptotes.*

3.3 Asymptotic Analysis

3.3.1 A Novel Technique

The calculation of infinite limits is similar to the same steps as finite limits except that the first step, evaluation, is impossible. We start at the second step and look for a simple logical explanation. If such an explanation is not forthcoming, the limit is an indeterminate form and we use algegra and the limit rules. The limit rules apply to infinite limits as they did to finite limits.

In addition to the algebraic methods already discussed, for infinite limits there is a powerful technique called asymptotic analysis. In practice, this is the most commonly used approach to infinite limits.

Asymptotic analysis interprets limit at infinity as a measurement of the growth of functions. The functions $f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = e^x$ all get very large as x gets very large; they all grow. Asymptotic analysis asks which of these functions grows faster.

The limit of a ratio of function $\frac{f(x)}{g(x)}$ is asking essentially the same question. By looking at the ratio of two functions as $x \to \infty$, we ar implicitly asking which grows faster. If g grows faster, then the denominator should outpace the numerator, and the limit should tend to 0. If f grows faster, then the numerator should outpace the denominator and the limit should tend to ∞. If f and g has roughly the same growth, then the limit should settle to some finite value larger than 0. This leads to the notion of asymptotic order.

3.3.2 Asymptotic Order

In asymptotic analysis, we start with a quotient limit.

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$$

- If this limit is 0, then we say that g has greater asymptotic order than f. Alternatively, we say that g grows faster than or g dominates f as $x \to \infty$.
- If this limit is ∞, then we say that f has greater asymptotic order than g. Alternatively, we say that f grows faster than or f dominates g.
- If this limit is finite but non-zero, we say that f and g have the same asymptotic order. Alternatively, we say that f and g grow at the same asymptotic rate and neither dominates.

With this definition, we can evaluate many limits by just knowing which functions have greater or lesser asymptotic order.

3.3.3 An Asymptotic Ranking of Functions

Here are several rules for asymptotic order of functions. Many of these are obvious, but some require more work to establish. The proofs of these statements are not included in this course.

- A constant function $f(x) = c$ has a lower asymptotic order than any increasing function.
- Any multiple of a function $cf(x)$ has the same asymptotic order as the original function $f(x)$.
- The logarithm $f(x) = \ln x$ grows slower than any function $f(x) = x^r$ for $r > 0$.
- The function $f(x) = x^r$ grows slower than $g(x) = x^s$ as long as $0 < r < s$. In particular, polynomials of lower degree grow slower than polynomials of higher degree.
- The exponential function $f(x) = e^x$ grows faster than $g(x) = x^r$ for any r.
- The function $f(x) = x^x$ grows faster than $g(x) = e^x$.

These are the most common types of functions we will consider. We can summarize this in a list of asymptotic orders: in the following list '$f < g$' means that f grows slower than g.

$$c < \ln x < \ldots < x^{\frac{1}{3}} < x^{\frac{1}{2}} < x < x^2 < x^3 < \ldots < e^x < x^x < \ldots$$

This is the basic list and is something you will either memorize or frequently reference for many limits in this course. There are other functions at the top of this list which grow faster than x^x, but they are not frequently used.

3.4 Determining Asymptotic Order

In quotient limits, f and g may be more complicated than the simple functions in our asymptotic ranking. We have a two useful rules to help us work with f and g which are combinations of pieces of various asymptotic order.

- If $f = f_1 + f_2 + f_3 + \ldots + f_n$ then the asymptotic order of f is the maximum of the asymptotic order of the f_i. This means that in a sum or difference, we only need to consider the fastest growing pieces. We can simply ignore all the rest.
- If $f_1 < g_1$ and either $f_2 < g_2$ or f_2 and g_2 have the same asymptotic order, then $f_1 f_2 < g_1 g_2$. Basically, this says that relationships of asymptotic order are preserved in products.

3.5 Ratios with Equivalent Asymptotic Order

In a quotient limit where f and g have the same asymptotic order, we look at the *leading coefficients*. The leading coefficients are the coefficients which sit in front of the term with the highest asymptotic order.

Example 3.5.1.

$$\lim_{x\to\infty} \frac{8x^4 + 3x^2 + 4}{14x^4 - 9x^3 - 50x^2 - 4x - 1}$$

The leading coefficient in the numerator if 8, and in the denominator is 14. Only these terms matter, which radically simplifies the limit

$$\lim_{x\to\infty} \frac{8x^4 + 3x^2 + 4}{14x^4 - 9x^3 - 50x^2 - 4x - 1} = \frac{8}{14} = \frac{4}{7}$$

Note that none of the information associated to the pieces of lower asymptotic order matters at all. No matter how large the constants may be, only the asymptotic order tells us which pieces are important.

3.6 Limits, Asymptotics and Models

We can use our tools of limits and asymptotic analysis to analyze models. Limits at finite values tell us how models behave near their undefined points; in particular, whether they diverge to infinity or remain bounded. This leads to an understanding of the limitations of a model and how it behaves at its own extreme situations.

Limits at infinity and asymptotic analysis tells us about the growth and long term behaviour of a model. We can compare population growth models asymptotically to get an idea of which is fundamentally a faster growth model. The analysis of algorithms in computing science is also done almost entirely with asymptotic analysis; the asymptotic order of an algorithm is a very good measure of how fast it can operate.

The notion of *stability* is frequently the focus of our study of limits and models. The word 'stability' has various technical definitions in pieces of applied mathematics, but it always relates to the limits of the model and the behaviour near those limits.

If we consider popluation models, we have four categories of long term (asymptotic) behaviour.

- The function can grow without bound, as in the exponential growth function $p(t) = p_0 e^{at}$.
- The function can decay to zero, as in the exponential decay function $p(t) = p_0 e^{-at}$.
- The function can approach a steady state, as in the logistic growth function
$$p(t) = \frac{p_0 K e^{at}}{K + p_0(e^{at} - 1)}$$
- The function can osscilate without ever reaching a steady state, possibly with chaotic behaviour. A non-chaotic osscilating function is this periodic version of logistic growth:
$$p(t) = \left(\frac{p_0 K e^{at}}{K + p_0(e^{at} - 1)} \right) \left(1 + \frac{1}{5} \sin(bt) \right)$$

Asymptotic analysis can also give information about the extreme values in a model. Consider the ideal gas law (where P is pressure, V is volume, n is the amount of gas, T is temperature and R is a constant. (We assume that P and V are the variables, and n, R and T are constant.)

$$PV = nRT$$

We can ask what happens to pressure at low volumes, expressed as a limit as $V \to 0$.

$$\lim_{V \to 0} P = \lim_{V \to 0} \frac{nRT}{V} = \infty$$

This tells us that low volumes result in high pressures. We could equivalently ask what happens at very high volumes.

$$\lim_{V \to \infty} P = \lim_{V \to \infty} \frac{nRT}{V} = 0$$

Unsurprisingly, very large volume result in very low pressures.

The behaviour at extreme values depends on the particulars of the models. A more complicated gas law is the Vander Waal's gaw law (where a and b are new positive constants).

$$\left(P + \frac{n^2 a}{V^2} \right) (V - nb) = nRT$$

What happens for these gasses at high pressures? It's difficult to solve directly for volume, but we can analyze the equation in its current form. If the term involving pressure grows very large, the term only involving volume must grow to zero, so that the product remains the same. That happens when V is very close to the value nb.

$$\lim_{P \to \infty} V = nb$$

Therefore, in this model, extreme pressures happen near the fixed, but non-zero, volume $V = nb$.

3.7 Continuity

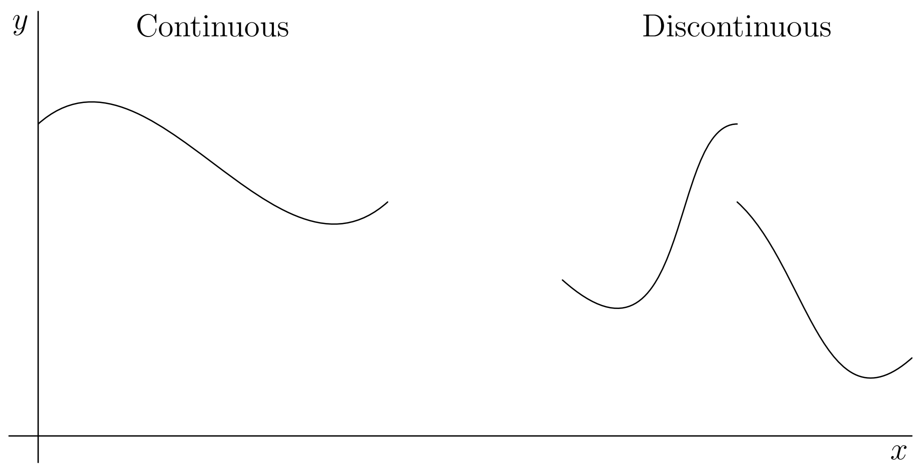


Figure 3.4: Continuity

A function is continuous at a point a in its domain if

$$f(a) = \lim_{x \to a} f(x)$$

We have already encounted these kinds of limit: in the calculation of limits, the first step was to try to evaluate the function at the limit point. We were implicitly relying on the continuity of our standard functions. Happily, we were justified in doing so: all the standard elementary functions (polynomials, rational function, algebraic functions, trig, exponentials) are continuous on their domains. The phrase 'on their domains' is important; where there is a break in the domain, the function cannot be continuous. Continuity only happens inside the domain of a function. Visually, the graph of a continuous function is connected; it can be drawn without lifting your pen (pencil, chalk, etc).

3.7.1 Intermediate Value Theorem

Though it has a formal name, the Intermediate Value Theorem is a very sensible and obvious result of continuity. Formally stated, the theorem says that if f is a continuous function on an interval $[a, b]$ and α is a real number with $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = \alpha$.

Rephrased, the theorem says that a continuous function cannot skip any values. If $f(a) = 0$ and $f(b) = 1$, then the function must output all values between 0 and 1 as the input goes from a to b. Continuity means the graph is connected; it can't jump over any intermediate values.

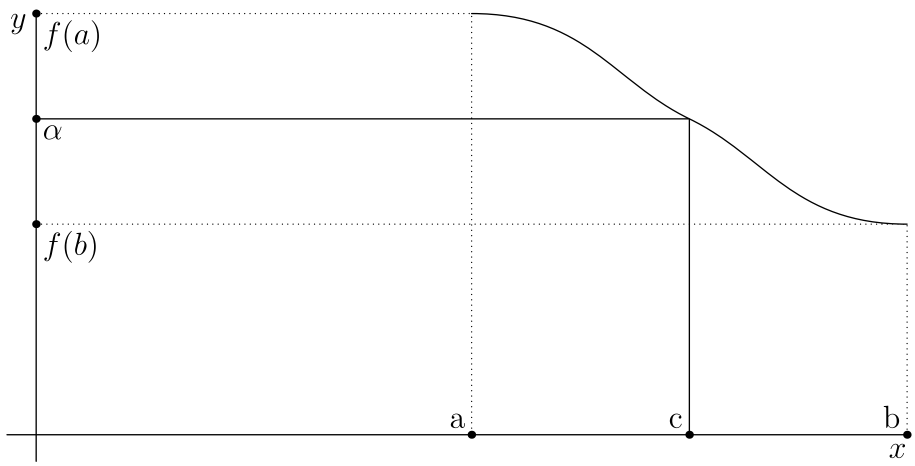


Figure 3.5: Intermediate Value Theorem

3.7.2 Piecewise Functions

We know that all of our familiar elementary functions are continuous on their domains. One might ask why we are worried about continuity at all? A common place where continuity becomes an issue is in piecewise functions. These are functions which have different expressions or definitions over different pieces of their domain. There is a particular notation for piecewise functions. Say we have a function on $\mathbb{R}$ which has two definitions, one for $x < a$ and another $x \geq a$. Then the function is written

$$f(x) = \begin{cases} g(x) & x < a \\ h(x) & x \geq a \end{cases}$$

A very well-known example is the Heaviside function.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

The Heaviside function is very frequently used to model switches: it suddenly changes from off (0) to on (1) at $x = 0$. This is a discontinuous change, as the function jumps from 0 to 1 without going through any intermediate value.

In the definition of a piecewise function, there can be more than two pieces and the conditions on x can be more complicated than inequalities. Here is a piecewise function with three pieces defined on intervals.

$$f(x) = \begin{cases} x^2 - 1 & x \in (-5, 0) \\ x^2 + 1 & x \in [0, 3] \\ 3x - 5 & x \in (3, 7) \end{cases}$$

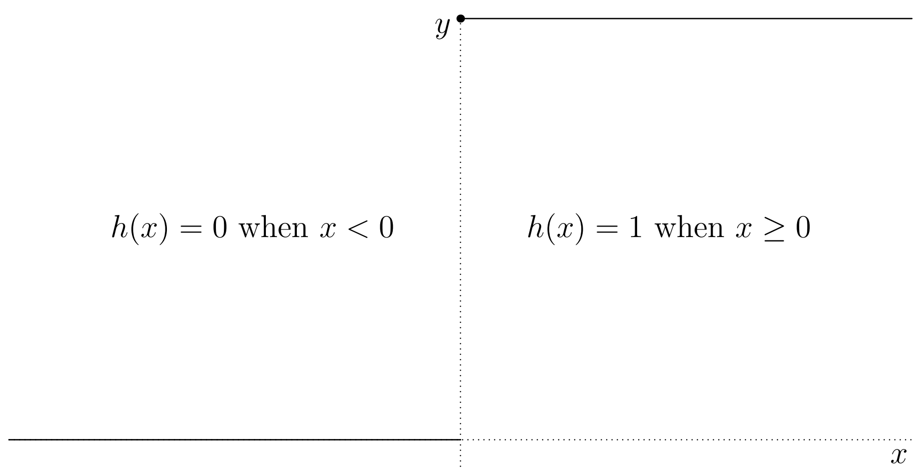


Figure 3.6: The Heaviside Function

The domain of this function is $(-5, 7)$, which is the union of all the intervals of definition in the piecewise expression. It's graph is Figure 3.7.

Piecewise functions can be extremely strange.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This function depends on whether its input is rational or irrational, returning one and zero respectively. It is a horendously discontinuous functions, with ones and zeros everywhere and no intermediate values.

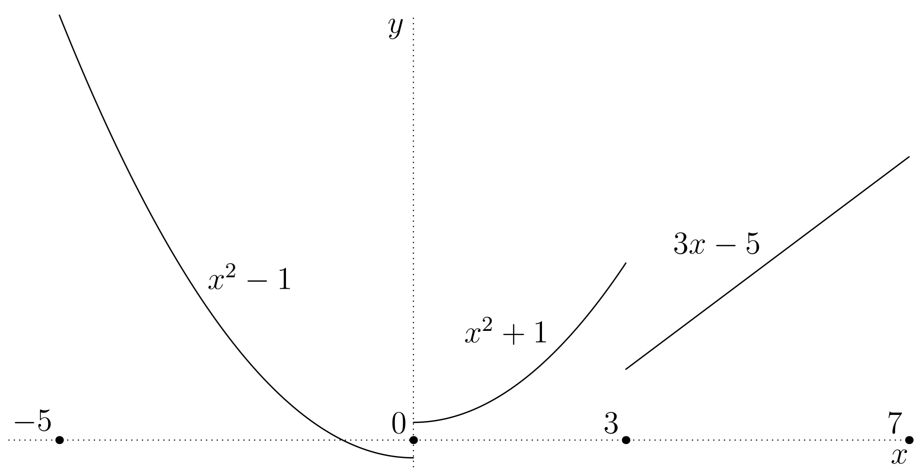


Figure 3.7: A Discontinuous Piecewise Function

3.7.3 Continuity of Piecewise Functions

It is important to be able to check if a piecewise function is continuous at its crossover point. Say we have a piecewise function with a break at $x = a$.

$$f(x) = \begin{cases} g(x) & x < a \\ h(x) & x \geq a \end{cases}$$

It is natural to ask if f is continuous at $x = a$. In order to investigate this, we need to look at the function value and the limits from both sides. The function is continuous if all three are the same.

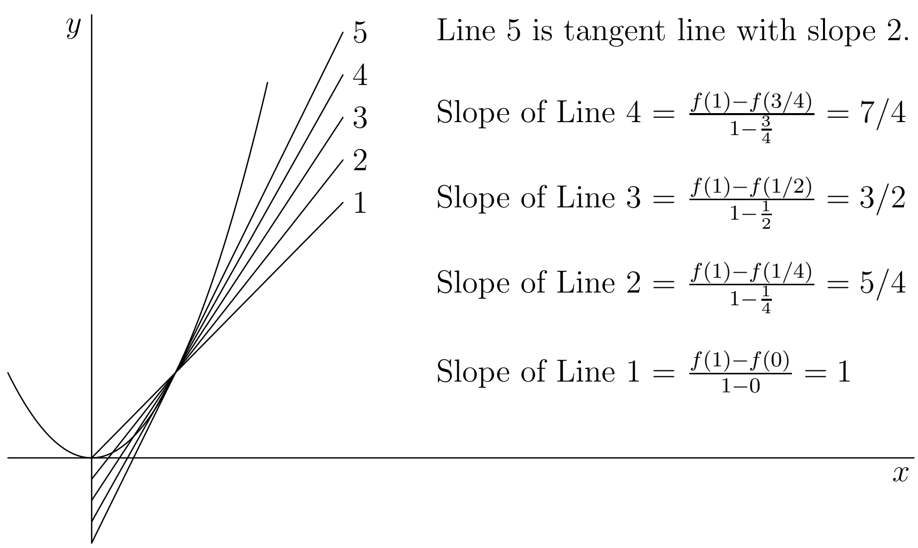
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Chapter 4

Derivatives

4.1 Definition of the Derivative

4.1.1 Limits of Secant Lines



Line 5 is tangent line with slope 2.

Slope of Line 4 $= \frac{f(1)-f(3/4)}{1-\frac{3}{4}} = 7/4$

Slope of Line 3 $= \frac{f(1)-f(1/2)}{1-\frac{1}{2}} = 3/2$

Slope of Line 2 $= \frac{f(1)-f(1/4)}{1-\frac{1}{4}} = 5/4$

Slope of Line 1 $= \frac{f(1)-f(0)}{1-0} = 1$

Figure 4.1: Secant Approximations to the Tangent Line

In Section 2.7, we defined the derivative as the rate of change of a function. In Section 2.9, we connected that definition to the geometry of slopes of tangent lines and constructed a process by which algebra can approximate a tangent line by using secant lines. Now that we have limits, we can ask for the limit of that approximation process.

Let's say we want the slope of the tangent line at a point $(a, f(a))$ on the graph of a function. We can take a as one point to define a secant line and let x be the other point (with $x \neq a$). Then the slope of the secant line is the difference in output $(f(x) - f(a))$ dividing by the difference in input $(x - a)$.

$$\frac{f(x) - f(a)}{x - a}$$

We said that we should get better and better approximations to the tangent line as x gets closer to a. Now we can ask for the limit as $x \to a$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This limit, if it exists, will be the slope of the tangent line. It is called the derivative and is the rate of change of the function at $x = a$.

$$f'(a) = \frac{df}{dx}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

There is a slight alteration of the definition, which is useful for some calculations. If we let $h = x - a$, then we can write the limit as:

$$f'(a) = \frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This second definition allows us to see that the derivative is an entirely new function. At each point x in the domain of f, we can ask for the slope of the tangent line. That gives a new function:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

4.1.2 Differential Operators

We think of the derivative as an operation on functions: it takes a function and gives us a new function which measures the rate of change of the previous function. This solves the velocity problem: if $x(t)$ is a position function, then its derivative $x'(t)$ is the velocity function.

Leibniz notation is useful for thinking of derivatives as operators. If we seperate the notation slightly, we can write

$$\frac{df}{dx} = \frac{d}{dx} f$$

With this notation, we think of $\frac{d}{dx}$ as the operator: the thing that takes derivatives. Having notation for such an operator is extremely convenient.

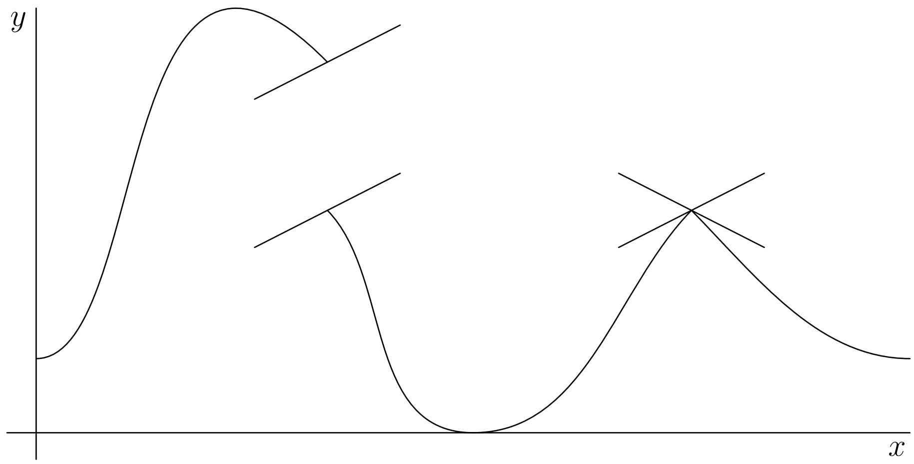


Figure 4.2: Differentiability Problems

4.1.3 Differentiability

The limit defining the derivative may not always exist. If it does exist at a point a in the domain, we say that f is *differentiable* at the point a. If it exists for all points in the domain of f, we say that f is a differentiable function. Differentiability requires continuity: if a function makes a sudden jump, it doesn't make sense to talk about a rate of change and a tangent line cannot be defined. Differentiability also requires a 'smoothness' condition. A function whose graph has sharp corners is not differentiable at the sharp corners, because it doesn't make sense to define a tangent line at a sharp corner. The graph of the function must be smooth. Figure 4.2 shows how a jump or sharp corner makes the choice of a tangent line problematic.

4.1.4 Interpretation

Now that we have a definition of the derivative and differentiability, let's review what this all means. The derivative has two major interpretation, one geometric and one applied.

- The derivative measures the slope of a tangent line to a function.
- The derivative measures the rate of change of a function.

If a function is differentiable on its domain, that means its derivative exists at all points in the domain. In the geometric interpretation, this means that the graph of the function has a well-defined tangent line at all points in its domain. In the applied interpretation, this means that the function has a well-defined rate of change at all points in its domain.

4.2 Rules for Differentiation

Now that we have the definition and an idea of its meaning, we move on to several lectures on the technical details of calculating derivatives.

4.2.1 Important Derivatives

We'll start with a number of standard derivatives. If a function is constant, its rate of change is zero, so its must have a zero derivative.

$$\frac{d}{dx}c = 0$$

For the function $f(x) = ax + b$, the graph is a straight line with slope a. Since the derivative measures the slope, it should be constant a.

$$\frac{d}{dx}ax + b = a$$

We have two standard derivatives we start with for trigonometry. There are given here without proof.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

We also have a standard derivative for the exponential function, established using the definition of the derivative.

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x \frac{a^h - 1}{h} == a^x \lim_{h\to 0}\frac{a^h - 1}{h} = a^x\left(\frac{d}{dx}a^x\bigg|_{x=0}\right)\end{aligned}$$

In the last step, we changed the limit into a derivative using the definition of the derivative at the specific point $x = 0$. The vertical line means "evaluated at $x = 0$".

This formula shows that derivative of a^x is formed by multiply by the derivative at $x = 0$. Note that all exponentials go through the point $(0, 1)$. Figure 4.3 shows the the graphs of several exponential function (for bases $a > 1$).

The slope at $(0, 1)$ determines the derivative. As seen in the Figure 4.3, for different choices of the base, we get different slopes going through the point $(0, 1)$. There is one special base which has slope 1 at the point $(0, 1)$. By definition, that base is the number e. (This number was defined earlier by a limit and we in Calculus II that the two definitions coincide.) By definition, the function e^x satisfies an important property.

$$\frac{d}{dx}e^x = e^x$$

This is the main reason that we are so fond of e as an expoenntial base. e^x is the nicest function to differentiate, since it doesn't change at all.

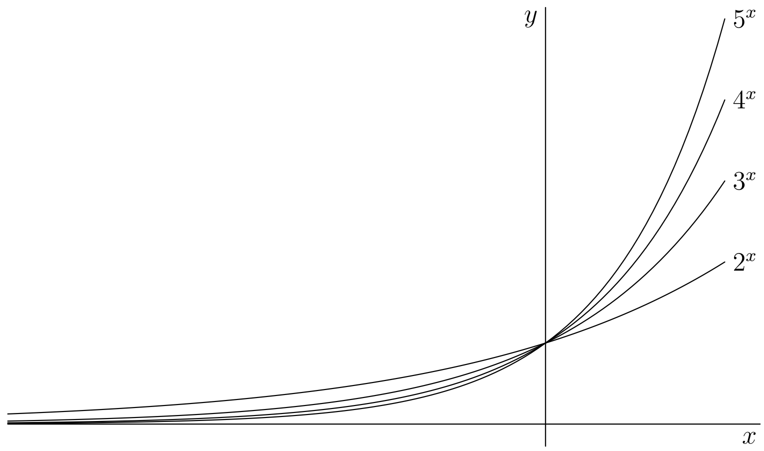


Figure 4.3: Four Exponential Functions

4.2.2 Power Rule

So far, we've looked at specific derivatives and used the definition to work them out. This approach quickly becomes intenable due to complications in the limit calculation. Therefore, we'd like to find some general rules that allow us to calculate derivatives without using the definition. The first rules is called the power rules. Let $n \in \mathbb{R}$ with $n = 0$.

$$\frac{d}{dx}x^n = nx^{n-1}$$

We can give a proof when n is a positive integer, using the binomial theorem.

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \\
&= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

4.2.3 Linearity

The next rules is a set of three rules which are called linearity. (Linearity shows that derivatives addition, subtraction and multiplication by constants. In general, any operation in mathematics that cooperates with those these actions is called a linear property). If f and g are differentiable functions and c is a real constant, then three rules hold.

$$\frac{d}{dx} f + g = \frac{df}{dx} + \frac{dg}{dx}$$
$$\frac{d}{dx} f - g = \frac{df}{dx} - \frac{dg}{dx}$$
$$\frac{d}{dx} cf = c\frac{df}{dx}$$

The proof of the linearity rules is excluded here, though it is relatively easy to construct using the linearity of the limit and the definition of the derivative. Using linearity and the power rule, we can take the derivative of any polynomial.

Example 4.2.1.

$$\frac{d}{dx} x^2 - 3x - 4 = \frac{d}{dx} x^2 - \frac{d}{dx} 3x - \frac{d}{dx} 4 = 2x - 3 - 0 = 2x - 3$$
$$\begin{aligned} \frac{d}{dx} 7x^3 + 9x^2 - 14x - 26 &= 7\frac{d}{dx} x^3 + 9\frac{d}{dx} x^2 - 14\frac{d}{dx} x - \frac{d}{dx} 26 \\ &= 7(3x^2) + 9(2x) - 14 - 0 = 21x^2 + 18x - 14 \end{aligned}$$

4.2.4 Leibniz Rule

Limits worked well with all four arithmetic operations: addition, subtraction, multiplication and division. We might hope the same is true for derivatives, but we would be disappointed. There are rules for multiplication and division, but they are more complicated that what we had for limits. The rule for products is called the product rules or the Leibniz rule. Let f and g be differentiable functions.

$$\frac{d}{dx} fg = g\frac{df}{dx} + f\frac{dg}{dx}$$

The Leibniz rule is, in many ways, the archetypical rule that identifies differentiation. There are many derivative-like operations in mathematics and they all conform to some version of the Leibniz rule.

The following calculation is a proof for the Leibniz rule. Note in the second step, we've added and subtracted the same term in the numerator.

$$\begin{aligned} \frac{d}{dx} fg &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x)\frac{f(x+h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x)\frac{dg}{dx} + f(x)\frac{df}{dx} \end{aligned}$$

Example 4.2.2.

$$\frac{d}{dx}x^2e^x = 2xe^x + x^2e^x$$
$$\frac{d}{dx}x^2\sin x = 2x\sin x + x^2\cos x$$
$$\frac{d}{dx}e^x\sin x = e^x\sin x + e^x\cos x$$

4.2.5 Quotient Rule

Let f be a differentiable function and g be a differentiable function with $g(x) \neq 0$, then the quotient rule states:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{(g(x))^2}$$

Example 4.2.3.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\frac{d\sin x}{dx}\cos x - \sin x\frac{d\cos x}{dx}}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{0 - -\sin x}{\cos^2 x} = \sec x\tan x$$

4.2.6 Chain Rule

The rules we have so far give us the tools to calculate many derivatives. However, they don't address composition. How do we differentiate $\sin x^2$? The rule that addresses derivatives of composed functions is called the chain rule. The chain rule has two notations. Let f and g be differentiable functions and consider the composition $f(g(x))$. It is often useful to label $g(x)$ by a new, temporary variable $u = g(x)$. The vertical line mean evaluate by replaing u with $g(x)$.

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \left.\frac{df(u)}{du}\right|_{u=g(x)}\frac{dg}{dx}$$

The variety of notations is somewhat due to the difficult nature of the rule—it's hard to clearly describe. The key idea is to think of composition in terms of inside and outside functions. The derivative of the composition is the derivative of the outside (f') evaluted at the inside ($f'(g(x))$), then multiplied by the derivative of the inside (g'). There is one other way of expressing the chain rule.

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

In this last version of the chain rule, we can think of the infinitesimals df, dg and dx as something like fractions in Leibniz notation ($\frac{df}{dx}$). These terms are not fractions, but they act a bit like fractions.

The proof of the chain rule is tricky, but the following argument gives the idea. The argument is a little imprecise in the fourth step, where we replace $g(x)$ with the temporary variable u, since we haven't established that such an operation is valid. In the proof, let $F = f \circ g$.

$$\begin{aligned}
\frac{dF}{dx}(a) &= \lim_{x\to a} \frac{f\circ g(x) - f\circ g(a)}{x-a} = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \frac{g(x)-g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= \lim_{u\to g(a)} \frac{f(u) - f(g(a))}{u-g(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= \left.\frac{df}{du}\right|_{u=g(a)} \left.\frac{dg}{dx}\right|_{x=a} = f'(g(a))g'(a)
\end{aligned}$$

Example 4.2.4.

$$\begin{aligned}
\frac{d}{dx}e^{ax} &= ae^{ax} \\
\frac{d}{dx}\sin 3x &= 3\cos 3x \\
\frac{d}{dx}e^{x^2+2} &= (2x)e^{x^2+2} \\
\frac{d}{dx}e^{e^x} &= e^x e^{e^x} \\
\frac{d}{dx}\sin\left(\frac{x^2+1}{x^2-1}\right) &= \cos\left(\frac{x^2+1}{x^2-1}\right)\frac{d}{dx}\frac{x^2+1}{x^2-1} \\
&= \cos\left(\frac{x^2+1}{x^2-1}\right)\frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\
&= \cos\left(\frac{x^2+1}{x^2-1}\right)\frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \\
&= \cos\left(\frac{x^2+1}{x^2-1}\right)\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = -\cos\left(\frac{x^2+1}{x^2-1}\right)\frac{4x}{(x^2-1)^2} \\
\frac{d}{dx}\cos(e^{\sin x}) = -\sin(e^{\sin x})\frac{d}{dx}e^{\sin x} &= -\sin(e^{\sin x})e^{\sin x}\frac{d}{dx}\sin x = -\sin(e^{\sin x})e^{\sin x}\cos x
\end{aligned}$$

Example 4.2.5. We can use the chain rule to differentiate arbitrary exponential functions in more detail than before:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^x \ln a$$

Example 4.2.6. We can also use the chain rule with a neat little exponential trick to differentiate $f(x) = x^x$. (In this calculation, we use $\frac{d}{dx}\ln x = \frac{1}{x}$, which we will prove in the next section.)

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}x\ln x = x^x(\ln x\frac{d}{dx}x + x\frac{d}{dx}\ln x) = x^x(\ln x + 1)$$

Example 4.2.7. We can use the chain rule to prove the quotient rule.

$$\frac{d}{dx}\frac{f}{g} = \frac{d}{dx} fg^{-1} = f\frac{d}{dx}g^{-1} + g^{-1}\frac{d}{dx} = -fg^{-2}\frac{dg}{dx} + g^{-1}\frac{df}{dx} = \frac{gf' - fg'}{g^2}$$

4.2.7 Derivatives of Inverse Functions

We just used the rule $\frac{d}{dx}\ln x = \frac{1}{x}$ without justification. What is this justification? The logarithm is the inverse of the exponential, which leads us to thinking about the derivatives of inverse functions. There is a rule for such derivatives. Let's write $x = f \circ f^{-1}(x)$, differentiate both sides using the chain rule, and rearrange to get an expression for $\frac{d}{dx}f^{-1}(x)$.

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}f \circ f^{-1} = f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) \\ \frac{d}{dx}f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Then we can prove the derivative formula we just used for $\ln x$.

$$\frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

We can also prove derivative rules for the inverse trig functions.

$$\begin{aligned} \frac{d}{dx}\arccos x &= \frac{1}{-\sin(\arccos x)} = \frac{-1}{\sqrt{1-\cos^2(\arccos x)}} = \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\arcsin x &= \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\arctan x &= \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2} \end{aligned}$$

Example 4.2.8.

$$\begin{aligned} \frac{d}{dx}\ln(x^2+1) &= \frac{1}{x^2+1}\frac{d}{dx}x^2+1 = \frac{2x}{x^2+1} \\ \frac{d}{dx}\ln(x+3x\arcsin x) &= \frac{1}{x+3\arcsin x}\left(1+3\arcsin x + \frac{3x}{\sqrt{1-x^2}}\right) \end{aligned}$$

Example 4.2.9. Usually, when we differentiate, we get complicated expressions which we can't simplify into nice forms. Every now and then, we have an example that allows a very nice simplifcation.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\ln(x+\sqrt{x^2-9})\right)\\
&\qquad=\frac{\sqrt{x^2-9}}{2}+\frac{x2x}{4\sqrt{x^2-9}}-\frac{9}{2}\frac{1}{x+\sqrt{x^2-9}}\left(1+\frac{2x}{2\sqrt{x^2-9}}\right)\\
&\qquad=\frac{\sqrt{x^2-9}}{2}+\frac{x^2}{2\sqrt{x^2-9}}-\frac{9}{2}\frac{2\sqrt{x^2-9}+2x}{2\sqrt{x^2-9}(x+\sqrt{x^2-9})}\\
&\qquad=\frac{\sqrt{x^2-9}}{2}+\frac{x^2}{2\sqrt{x^2-9}}-\frac{9}{2}\frac{\sqrt{x^2-9}+x}{\sqrt{x^2-9}(x+\sqrt{x^2-9})}\\
&\qquad=\frac{\sqrt{x^2-9}}{2}+\frac{x^2-9}{2\sqrt{x^2-9}}\\
&\qquad=\frac{\sqrt{x^2-9}}{2}+\frac{\sqrt{x^2-9}}{2}=\sqrt{x^2-9}
\end{aligned}$$

4.3 Tangent Lines and Implicit Derivatives

4.3.1 Equations of Tangent Lines

By definition, a derivative is the slope of a tangent line to a function. Therefore, once we can calculate derivatives using the rules of the previous sections, we can calculate equations of tangent lines. We'll demonstrate this by two examples.

Example 4.3.1. The point $(2,4)$ is on the parabola $y=x^2$, which is the graph of the function $f(x)=x^2$. The derivative of the function is $f'(x)=2x$. The derivative evaluated at $x=2$ is $f'^{(2)}=2(2)=4$, which tells us the slope of the tangent line at the point $(2,4)$ is 4. Now we have a point and slope; by the review we did at the very start of these notes, we can determine the equation of the tangent line. All we have left to do is find the y intercept.

$$\begin{aligned}
&y=mx+b\\
&4=(4)(2)+b \implies b=4-8=-4\\
&y=4x-8
\end{aligned}$$

The tangent line to the function $f(x)=x^2$ at the point $(2,4)$ is the line $y=4x-8$.

Example 4.3.2. The point $(9,3)$ is on the graph of the function $f(x)=\sqrt{x}$. The derivative of the function is $f'(x)=\frac{1}{2\sqrt{x}}$. If we input the value $x=9$, we get $f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$. Now we have a slope and a point, so we can calculate the equation of the tangent line by finding the y intercept.

$$\begin{aligned}
&y=mx+b\\
&3=\frac{1}{6}9+b \implies b=3-\frac{9}{6}=3-\frac{3}{2}=\frac{3}{2}\\
&y=\frac{x}{6}+\frac{3}{2}
\end{aligned}$$

The tangent lien to the function $f(x) = \sqrt{x}$ at the point $(9, 3)$ is the line $y = \frac{x}{6} + \frac{3}{2}$.

4.3.2 Implicit Derivatives

So far, we've looked for the slopes of tangent lines to the graphs of functions. We could consider a broader category of geometric loci in $\mathbb{R}^2$ where we want to find tangents lines.

As an archetypical example, consider the unit circle $x^2 + y^2 = 1$. Obviously, this smooth curve should have tangent lines with well-defined slopes and equations How do we find them?

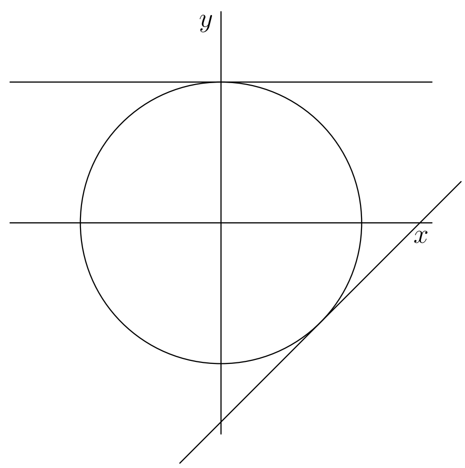


Figure 4.4: Tangent Lines to the Circle

The technique for finding these slopes is a technique called implicit differentiation. The circle $x^2+y^2 = 1$ isn't the graph of a function. Implicit differentiation pretends (at least, locally) that it is the graph of a function where y depends on x.. With that pretense, differentiating any expression involving x happens as usual. However, differentiating any expression involving y requires the chain rule. If y is (implicity) a function of x, then the expression $g(y)$ uses the chain rule.

$$\frac{d}{dx} g(y) = g'(y) \frac{dy}{dx}$$

Now we take the equation $x^2 + y^2 = 1$ and differentiate both sides. We will get a $\frac{dy}{dx}$ term our of the implicit differentiation of y, which we will try to isolate.

$$\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}x^2 + y^2 &= \frac{d}{dx}1 \\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-x}{y}
\end{aligned}$$

This gives us an expression for the slope of the tangent line to a circle. Notice that the expression involves both x and y: that is to be expected. We can evaluate it on any point (x, y) which lies on the circle. To get the lines in the previous diagram, we evaluate at $(0, 1)$ to get a slope of 0 and at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ to get a slope of 1.

This works for most of the circle, but the expression is undefined when $y = 0$. We can't find the slope of the tangent line at $(1, 0)$ or $(-1, 0)$. By looking at the shape, this makes sense for two reasons. First, the tangent lines are becoming steeper and steeper approaching this point: it's reasonable to conclude that they are actually vertical at $(\pm 1, 0)$. A vertical line doesn't have a well defined slope. The second reason is equally as important. Recall we assumed that (locally) y was a function of x. This is a reasonable assumption everywhere except near $(\pm 1, 0)$. At these points, the circle doubles-back on itself. If we zoom in on this point of the circle, we always get a locus that would fail a vertical line test, thus cannot be the graph of a function.

In general, there are three ways in which implicit differentiation can fail. The first is for discontinuities or sharp corners, as with ordinary derivatives. The second is where we have vertical tangents and the slope is undefined. The third is due to places where the assumption that y is a function of x breaks down. However, even with these restrictions, implicit differentiation is a powerful technique.

4.4 Higher Derivatives

Starting with a differentiable function $f(x)$, we used derivatives to get a whole new function $f'(x)$ which measured the slope of the previous function. This solves the velocity problem: if $p(t)$ was position of an object, then $p'(t)$ was the velocity of the object.

We can continue this process. If $f'(x)$ is still a differentiable function itself, we can take another derivative to get $f''(x)$. This is called the second derivative. The process is exactly the same: we use the same limit definition and the same differentiation rules to find this second derivative. $f''(t)$ is Newton's notation for the second derivative; the Leibniz notation is $\frac{d^2 f}{dx^2}$. If $p(t)$ was a position function and $p'(t)$ was its velocity (the rate of change of position), the $p''(t)$ would be the rate of change of velocity. That woud be acceleration. Acceleration is the second derivative of position.

This has important implications for physics. Newton's first law of motion, $F = ma$, says that force equals mass times acceleration. If force depends on position (as it often does), then we can write Newton's first law as a differential equation.

$$F(p) = m\frac{d^2p}{dt^2}$$

As with most of the fundanmental equations of physics, Newtonian motion is determined by the solution of a differential equation. Let's take a more specific example. On the surface of the earth, it is assumed that the acceleration due to gravity is constant at rougly $9.8m/s^2$. We can observe that the flight of projectiles is roughly parabolic, that is, height is described by a parabola $h(t) = at^2 + bt + c$. If we differentiate once, vertical velocity is $h'(t) = 2at + b$. If we differentiate twice, vertical acceleration is $h''(t) = 2a$, which is constant. So parabolic paths fit with the notion of a constant acceleration due to gravity. Even more, we know that the leading coefficient of those quadratics should be roughly $a = 4.9$ to get the constant acceleration $2a = 9.8m/s^2$.

Another specific example is Hooke's law. This law describes the force caused by a spring; the law states that the force is (negatively) proportional to the distance from equalibrium. If equalibrium is at $p = 0$, then $F = -kp$. That gives this differential equation:

$$-kp = m\frac{d^2p}{dt^2}$$

The behaviour of an object on a (perfect, frictionless) spring is determined by this differential equation. What is that behaviour? We simply have to figure out which function $p(t)$ matches this differential equation. In this case, there are two solutions:

$$p(t) = \sin\left(\sqrt{\frac{m}{k}}t\right) \qquad \text{or} \qquad p(t) = \cos\left(\sqrt{\frac{m}{k}}t\right)$$

Since the sinusoidal functions solve the differential equation, we can conclude that an object on a spring should act with a sinusoidal behaviour.

Example 4.4.1. There are other interesting properties of the higher derivatvies of trigonometric function.

$$\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f'''(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{aligned}$$

If we take four derivatives, we get back to the original function. The same is true for $\cos x$, but those two functions are unique among functions for this property.

Example 4.4.2. Let's see what happens with higher derivatives of a polynomial.

$$\begin{aligned}
f(x) &= x^5 \\
\frac{df}{dx} &= 5x^4 \\
\frac{d^2 f}{dx^2} &= 20x^3 \\
\frac{d^3 f}{dx^3} &= 60x^2 \\
\frac{d^4 f}{dx^4} &= 120x \\
\frac{d^5 f}{dx^5} &= 120 \\
\frac{d^6 f}{dx^6} &= 0
\end{aligned}$$

Each differentiation reduces the degree of the polynomial by one, until we eventually get to zero. If we take enough derivatives of any polynomial, we will eventually reach zero.

4.5 Linear Approximation

In this last section on differentiation, we discuss a third interpretation of the derivative. So far, we have viewed the derivative as the rate of change or the slope of a tangent line. The third interpretation is as a linear approximation to the function.

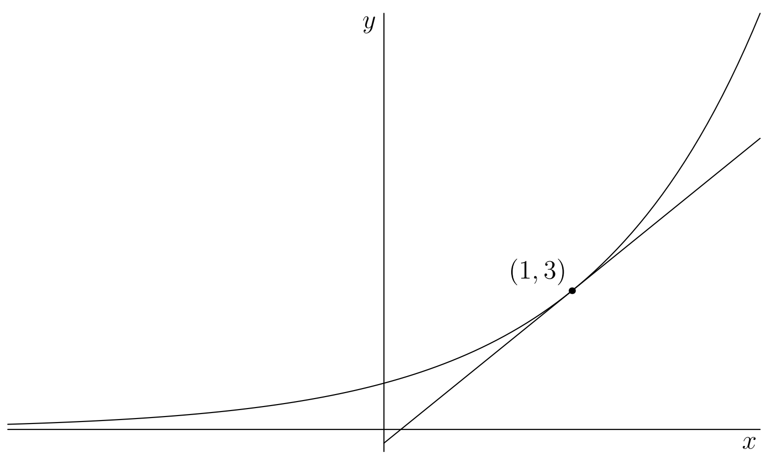


Figure 4.5: Tangent Lines and Linear Approximation

In Figure 4.5, we've drawn the function $f(x) = 3^x$ and the tangent line at the point $(1, 3)$. The derivative is $f'(x) = 3^x \ln 3$, so the slope at $(1, 3)$ is $3 \ln 3$, which is approximatly 3.296. The equation of the tangent line is $y = (3 \ln 3)x + (3 - 3 \ln 3)$.

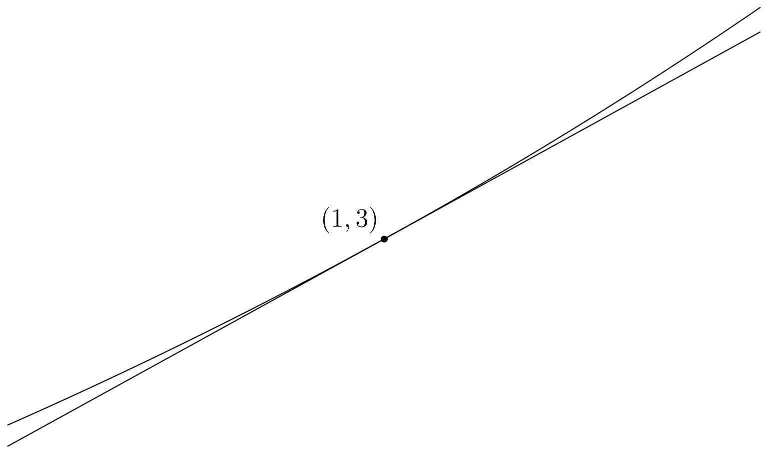


Figure 4.6: Close Zoom of a Tangent Line

In Figure 4.6 we zoom in very close on $(1, 3)$. In this zoom, the graph of the function and the graph of the tangent line are very close. That means that value of the function is very close to the value of the linear function described by the tangent line. Therefore, we can use the tangent line as a linear approximation of the function.

$$3^x \cong (3 \ln 3)x + (3 - 3 \ln 3) = 3 + (3 \ln 3)(x - 1)$$

In more generality, if $f(x)$ is differentiable at $(a, f(a))$, then near that point we can use the derivative to produce a good linear approximation.

$$f(x) \cong f(a) + f'(a)(x - a)$$

Obviously, a linear approximation is only useful near the chosen point. However, that use is still significant. Consider the functions $\sin x$ and $\cos x$. At $a = 0$, we have these linear approximations:

$$\sin x \cong \sin(0) + \cos(0)x = x$$
$$\cos x \cong \cos(0) + \sin(0)x = 1$$

In trigonometry, we use these identities frequently. For small angles, we often replace $\sin x \cong x$ and $\cos x \cong 1$.

Chapter 5

Integrals

5.1 Sigma Notation

Sigma notation is a convenient way to write large and complicated sums. The name comes from the fact that is uses the upper case greek letter sigma: Σ. Sigma notation relies on an index and expresses the terms of the sum as expression of the index. In the following, k is the index. The term below the sigma tells us where the index starts and the term above the sigma tells us where the index ends. The term following the sigma is the expression of the sum: for each index from the start to the end (in integer s.eps) we evaluate the expression.

$$\sum_{k=1}^{5} 5k = 5(1) + 5(2) + 5(3) + 5(4) + 5(5) = 5 + 10 + 15 + 20 + 25 = 75$$

$$\begin{aligned}
\sum_{k=1}^{8} \sin\left(\frac{k\pi}{2}\right) &= \sin\left(\frac{(1)\pi}{2}\right) + \sin\left(\frac{(2)\pi}{2}\right) + \sin\left(\frac{(3)\pi}{2}\right) + \sin\left(\frac{(4)\pi}{2}\right) + \sin\left(\frac{(5)\pi}{2}\right) \\
&\quad + \sin\left(\frac{(6)\pi}{2}\right) + \sin\left(\frac{(7)\pi}{2}\right) + \sin\left(\frac{(8)\pi}{2}\right) \\
&= 1 + 0 + -1 + 0 + 1 + 0 + -1 + 0 = 0
\end{aligned}$$

$$\sum_{k=1}^{36} k = 1 + 2 + 3 + 4 + \ldots + 35 + 36 = 666$$

There are some important rules for manipulating sigma notation. We can factor out constants from a sum.

Example 5.1.1.

$$\sum_{k=1}^{n} 6k = 6\sum_{k=1}^{n} k$$
$$\sum_{k=1}^{n} 3k^2 + 9k = 3\sum_{k=1}^{n} k^2 + 3k$$

We can combine sums if the indices match.

Example 5.1.2.

$$\sum_{k=1}^{15} k^2 + \sum_{k=1}^{15} 4k = \sum_{k=1}^{15}(k^2 + 4k)$$

If we want to combine sums when the indices don't match, we have to adjust the sums. There are two main methods of adjusting the sum. First, we can take out terms.

Example 5.1.3.

$$\sum_{k=1}^{15} k^2 + \sum_{k=3}^{15} 4k = 1^2 + 2^2 + \sum_{k=3}^{15} k^2 + \sum_{k=3}^{15} 4k = 5 + \sum_{k=3}^{15}(k^2 + 4k)$$

We can also shift the indices. This might seem tricky, but it's easy to remember if you think of balance: whatever we do to the index, we have to do the opposite to the expression.

Example 5.1.4. Here we shift the index by $+2$, so the expression is shifted by -2.

$$\sum_{k=1}^{20} 4k = \sum_{k=3}^{22} 4(k-2)$$

Example 5.1.5. We can use these two manipulations to combine sums if the indices don't initially line up.

$$\sum_{k=1}^{15} k^2 + \sum_{k=3}^{17} 4k = \sum_{k=1}^{15} k^2 + \sum_{k=1}^{15} 4(k+2) = \sum_{k=1}^{15}(k^2 + 4k + 8)$$

Finally, we have evaluation for some common sums. These expressions will be used, along with the previously stated rules, to calculate integrals in the next section.

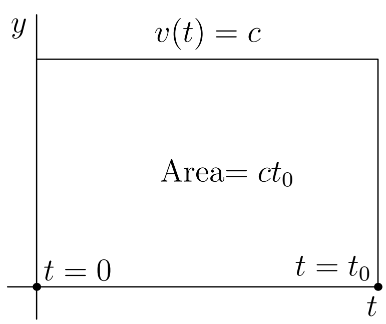
$$\sum_{k=1}^{n} 1 = n$$
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

The last sum establishes a interesting result: the sum of the first n cubes is always a square. You are welcome to check for yourself: the first few sums are 1, 9, 36, 100 and 225, all of which are square.

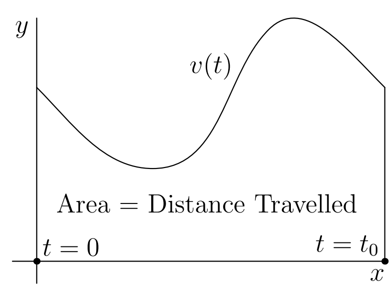
5.2 The Riemann Integral

5.2.1 Definition

Recall the distance travelled problem, where we tried to understand the distance travelled by an object when we knew its velocity function $v(t)$. Geometrically, this was equivalent to finding the area under the graph of the function, as in Figure 5.1.



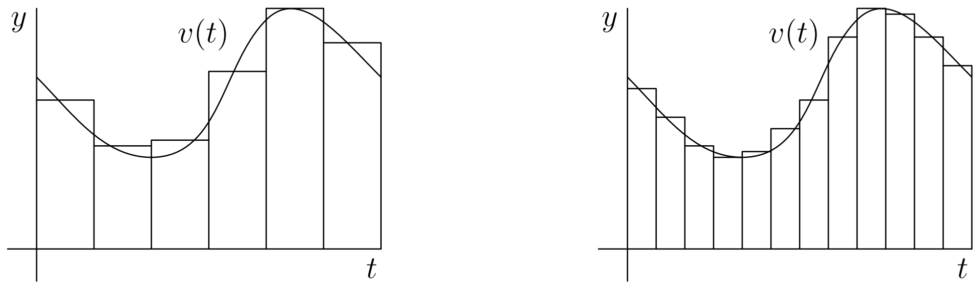
(a) Constant Velocity



(b) Non-Constant Velocity

Figure 5.1: Distance Travelled

We set up a process to solve this problem, at least approximately. That process involved making rectangles under the curve and adding up the area of the rectangles.



(a) Approximation of Area by Rectangles (b) Improved Approximation with More Rectangles

Figure 5.2: Approximating Areas under Curves

Integration will be the limit of this approximation process, just as differentiation was the limit of the approximation process of secant lines approaching the tangent line.

Sigma notation gives us a concice notation for the approximation of areas by rectangles. Let's say we are trying to calculate the area under the curve of $f(x)$, a positive function, defined on an interval $[a, b]$. We are dividing the area into n rectangles; if we divide equally, the width of each rectangle will be $\frac{b-a}{n}$. The height of the rectangle will be the function value $f(x^*)$ where x^* is some particular x in the subinterval.

With these notations, the area of a rectangle is width times height, that is $\frac{b-a}{n} f(x^*)$. To be more specific, is we use k as an index to refer to various rectganles and x_k^* is in the kth interval, the are of the kth rectangle is $\frac{b-a}{n} f(x_k^*)$. To get the complete approximation, we add up all these rectangles with sigma notation.

$$A \cong \sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*)$$

This sum is called a Riemann Sum approximation for area. If the function is well behaved (for us, continuous), then we can keep taking finer and finer partitions for better and better approximations. This process can be extended into some kind of limit process. In the limit, we expect a perfect calculation of area.

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \frac{b-a}{n}$$

This limit, if it exists, is called a *definite integal.* The area under the curve is calculated by a definite integral. Since this definition uses Riemann sums, we call this the Riemann definiton of the integral, or simply the Riemann integral. It has a new notation.

$$\int_a^b f(x)dx := \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \frac{b-a}{n}$$

We can explain the notation. The sigma of sigma notation gets replaced with $\int$, which is the integral sign. The bounds $k = 1$ and n get replaced with the endpoint of the interval, a and b respectively. The

$f(x_k^*)$ just becomes the function $f(x)$. Finally, the term $\frac{b-a}{n}$ gets infinitesimally small as $n \to \infty$, so it becomes the infinitesimal term dx.

The first problem with limit definition is the existence of the limit. Unfortunately, this is an extremely difficult question to solve in general, particular since it has to work for any possible choices of the values x_k^*. Fortunately, this limit will always exist if $f(x)$ is continuous, a fact that is presented here without proof.

We should point out some important facts. First, this integral returns a number: given a function and endpoints, it calculates a fixed area. It isn't (yet) a new function. Second, we need the bounds: the Riemann integral doesn't make any sense without the bounds of the interval $[a, b]$. Third, we defined this for positive functions. It also works for functions which have negative values, but areas below the x axis are interpreted as negative. If a function has both positive and negative values, the integral will give the difference between the area above the axis (the positive area) and the area below the (the negative area).

Recall when we talked about the derivatives, we mentioned that the definition was correct and rigorous but difficult to use. That comment is also true here, but even more severe. The definition of the integral is nearly impossible to use for most functions. We will only try to use the definition for small order polynomials.

Example 5.2.1. We choose $x_k^* = a + k\frac{b-a}{n}$.

$$\begin{aligned}
\int_0^3 x^2 &= \lim_{n\to\infty} \sum_{k=1}^{n} (x_k^*)^2 \frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^2 \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{k=1}^{n} k^2 \\
&= \lim_{n\to\infty} \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{9}{2} \frac{2n^3+3n^2+n}{n^3} = 9
\end{aligned}$$

Example 5.2.2.

$$
\begin{aligned}
\int_2^5 (x^3 - x) &= \lim_{n\to\infty} \sum_{k=1}^{n} \left[(x_k^*)^3 - x_k^*\right] \frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\left(2 + \frac{3k}{n}\right)^3 - \left(2 + \frac{3k}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[8 + \frac{36k}{n} + \frac{54k^2}{n^2} + \frac{27k^3}{n^3} - 2 - \frac{3k}{n}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[6 + \frac{33k}{n} + \frac{54k^2}{n^2} + \frac{27k^3}{n^3}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left(6n + \frac{33}{n}\frac{n(n+1)}{2} + \frac{54}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^3}\left(\frac{n(n+1)}{2}\right)^2\right) \\
&= 18 + \frac{99}{2} + 54 + \frac{81}{4} = \frac{72 + 199 + 216 + 81}{4} = \frac{567}{4}
\end{aligned}
$$

5.2.2 Properties of the Definite Integral

The definite integral is linear.

$$
\begin{aligned}
\int_a^b (f \pm g)dx &= \int_a^b f dx + \int_a^b g dx \\
\int_a^b cf(x)dx &= c\int_a^b f(x)dx
\end{aligned}
$$

To integrate a constant, we just get the area of a rectangle.

$$\int_a^b cdx = c(b-a)$$

If, for some reason, we end up with reversed bounds, we can change them by introducing a negative sign.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

If f is continuous on $[a,b]$ and $[b,c]$, then the integral over the two pieces is equivalent to the integral over the whole interval $[a,c]$.

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

5.3 The Fundamental Theorem

As mentioned above, we need a way to avoid using the definition. With derivative, we just developed a series of rules for various types and combinations of functions. That doesn't work as easily or cleanly for integrals. However, there is a very powerful theorem that gives us an approach to solving integrals.

To state the theorem, we need to consider a strange new function. For $g(t)$ continuous on $[a, b]$ and $x \in [a, b)$, define a new function f.

$$f(x) := \int_a^x g(t)dt$$

Let's give some interpretation: $f(x)$ is the area under the function $g(t)$ along the t axis between a fixed point $t = a$ and a varying endpoint x. Be careful to keep the variables straight! The variable t is only inside the integral and the variable x is only outside. x is the endpoint of the interval and t is the variable that goes between a and x on the interval.

Though very strange, this definition is a rich source of new and interesting functions in mathematics.

Example 5.3.1.

$$\begin{aligned}
\text{Fresnel} \quad S(x) &:= \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \\
\text{Logarithmic Integral} \quad \text{li}(x) &= \int_0^x \frac{dt}{\ln t} \\
\text{Sine Integral} \quad Si(x) &= \int_0^x \frac{\sin t}{t} dt
\end{aligned}$$

Now that we have this function $f(x)$, we can look at its derivative. Figure 5.3 gives a useful visualization. In the figure, we've labelled a point x and a small increase $x + h$.

We use the limit definition of the derivative.

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since f measures area, the difference $f(x + h) - f(x)$ is just the area of the very thin rectangle in Figure 5.3. When h is small, this area is roughly the height $g(x)$ times the width h.

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{g(x)h}{h} = g(x)$$

All this work can be summarized in the following statement, which is called the *Fundamental Theorem of Calculus*.

$$\frac{d}{dx} f(x) = \frac{d}{dx} \int_a^x g(t)dt = g(x)$$

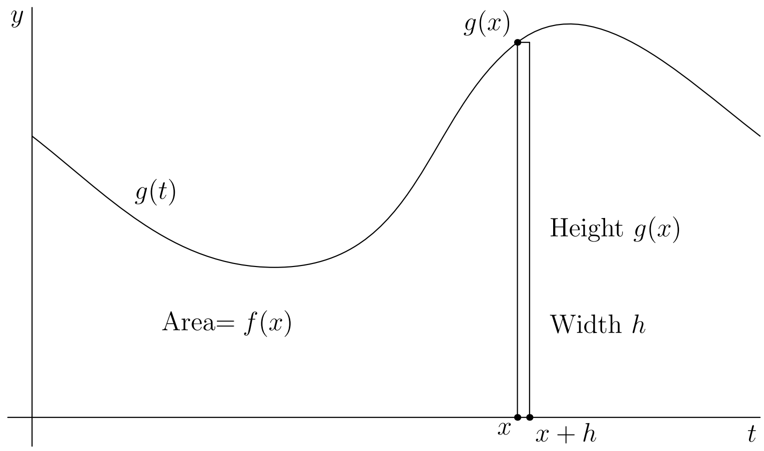


Figure 5.3: The Area Function $f(x)$

The important implication of the fundamental theorem is this: integral and derivatives are opposite operations. In the previous statement, we start with a function g, take and integral to get f and then take a derivative to get back to g. Derivatives and integrals are reverse processes.

There are several versions of the fundamental theorem, but they all capture this basic idea of reserve processes. The derivative can be inside the integral.

$$\int_a^b f'(t)dt = f(b) - f(a)$$

This version of the fundamental theorem gives us a wonderful way to calculate definite integrals. Let F be any function such that $F'(x) = f(x)$. Such a function F is called an *antiderivative* for f. We use antiderivatives to evaluated integrals.

$$\int_a^b f(t)dt = F(b) - F(a)$$

To calculate antiderivatives, we need to do deriavitves backwards. Fortunately, this is much easier than using the definition of the integral; it will become our main strategy for integrals. Unfortunately, it is still reasonably difficult.

It's useful to have a notation for antiderivatives. Since integrals are, in some way, the inverse operation, it seems natural to use an integral symbol to indicate an antiderivative. We drop the bounds that we used for definition integrals.

$$\text{Any antiderivative of f is written } \int f(x)dx$$

This is called an indefinite integral. Note that the notation means *all* anti-derivatives, since there may be more than one.

Example 5.3.2. In these examples, I'm using what I know about derivatives to do the operation backwards. The sixth example is just the power rules done backwards. The proofs of all of these are easy: just differentiate the right side of the equation. Since they are anti-derivatives, the derivative of the right side should give the original function on the left (inside the integral sign). The $+C$ is an arbitrary constant which we must add. It shows up because it would disappear in differentiation.

$$\begin{aligned}
\int e^x dx &= e^x + c \\
\int \sin x dx &= -\cos x + c \\
\int \cos x dx &= \sin x + c \\
\int a^x dx &= \frac{a^x}{\ln a} + c \\
\int \frac{1}{1+x^2} dx &= \arctan x + c \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + c \\
\int \frac{1}{x} &= \ln |x| \\
\int \frac{1}{\sqrt{1-x^2}} dx &= \arcsin x + c \\
\int \sec^2 x dx &= \tan x + c \\
\int \csc^2 x dx &= \cot x + c
\end{aligned}$$

Example 5.3.3. We can use the fundamental theorem to calculate definite integrals as well. There is some special notation that is useful here: when we write $f(x)|_a^b$ we mean to evaluate f at a and b and subtract.

$$f(x)|_a^b = f(b) - f(a)$$

The antiderivative of x^2 is $\frac{x^3}{3}$.

$$\int_2^4 x^2 dx = \left.\frac{x^3}{3}\right|_2^4 = \frac{4^3}{3} - \frac{2^3}{3} = \frac{64-8}{3} = \frac{56}{3}$$

5.4 The Substitution Rule

Since doing integrals is doing derivatives backwards, we might try to start reversing all the differentiation rules. Linearity works exactly the same in reverse. We inverted the power rule in the previous list of examples. Inverting the product rule starts to become strange; we postpone that to integration techniques covered in Calculus II. Arguably the most important differentiation rule is the chain rule. We will try to reverse it here.

If we have $f(g(x))$, then $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. Therefore, we can simply reverse the identity to get:

$$\int f'(g(x))g'(x) = f(g(x)) + c$$

When we covered the chain rule, I recommended labelling the inside function with a new variable $u = g(x)$. That becomes even more important here. It's easiest to explain the process by example.

Example 5.4.1.

$$\int 2x(x^2+1)^4 dx$$

This integral involves a composition. Label the inside function $u = x^2 + 1$. Then we change the entire integral from the variable x to the varaibles u. This is a substitution, hence the rule is called the substitution rule. We also need to change the differential term dx. This term is strange and confusing, and we really don't have the room and energy to go into all the historical subtleties of differentials. If $u = u(x)$ is the relationthip between u and x, then $du = u'(x)dx$ is the relationship between dx and du. Here $u = x^2 + 1$, so $du = (2x)dx$. Let's rewrite the original integral.

$$\int (x^2+1)^4(2x)dx$$

We can see that substitution works well here: we can replace $x^2 + 1$ with u and $(2x)dx$ with du.

$$\int u^4 du$$

We can find the anti-derivative easily by reversing the power rule.

$$\int u^4 du = \frac{u^5}{5} + c$$

Then we undo the substitution, by replacing u with $x^2 + 4$.

$$\int 2x(x^2+1)^4 dx = \int u^4 du = \frac{u^5}{5} + c = \frac{(x^2+4)^5}{5} + c$$

Example 5.4.2.

$$\int xe^{x^2}dx \qquad u = x^2 \quad du = 2xdx$$

$$\int e^u \frac{du}{2} = \frac{e^u}{2} + c = \frac{e^{x^2}}{2} + C$$

$$\int \frac{x}{x-2}dx \qquad u = x-2 \quad du = dx$$

$$\int \frac{u+2}{u}du = \int 1 + \frac{2}{u}du$$

$$= u + 2\ln|u| + c = x - 2 + 2\ln|x-2| + c$$

$$\int \frac{1}{10x-3}dx \qquad u = 10x-3 \quad du = 10dx$$

$$\int \frac{1}{u}\frac{du}{10} = \frac{\ln|u|}{10} + c = \frac{\ln|10x-3|}{10}$$

When we do substitution with definite integrals, we also need to change the bounds. If a and b are the bounds in x and $u = u(x)$ is the relationship between u and x, then $u(a)$ and $u(b)$ will be the bounds in u. One nice thing about definite integrals is that we can use these new bounds to evaluate the integral. We don't have to substitute back after we finish.

Example 5.4.3.

$$\int_0^2 \frac{2x}{(x^2+1)^2}dx \qquad ux^2+1 \quad du = 2xdx \quad u(0) = 1 \quad u(2) = 5$$

$$\int_1^5 \frac{du}{u^2} = -\frac{1}{u}\Big|_2^5 = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\int_{-1}^2 x^2e^{x^3+1}dx \qquad u = x^3+1 \quad du = 3x^2dx \quad u(-1) = 0 \quad u(2) = 9$$

$$\int_0^9 e^u\frac{du}{3} = \frac{e^u}{3}\Big|_0^9 = \frac{e^9}{3} - \frac{1}{3} = \frac{e^9-1}{3}$$

$$\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}dx \qquad u = \cos x \quad du = -\sin xdx \quad u(0) = 1 \quad u(\pi/4) = \frac{\sqrt{2}}{2}$$

$$\int_1^{\frac{\sqrt{2}}{2}} \frac{-du}{u^3} = \frac{2}{u^2}\Big|_1^{\frac{\sqrt{2}}{2}} = 4 - 2 = 2$$

5.5 Solving Differential Equations

5.5.1 Solving By Direct Integration

Early in the course, we talked about differential equations in order to understand derivatives as rates of changes. We looked at autonomous differential equations, particularly for populations models, and we used the qualitative technique of phase-line analysis to understand them.

Now that we can calculate derivatives and integrals in some detail, we can return to differential equations. With these techniques, we'll actually try to solve the DEs, instead of using a qualitative approach. In the easiest case, the right side of a DE only involves the independant variable.

$$\frac{df}{dx} = g(x)$$

Since only the independant variable x appears on the right side, we can simply integrate both sides in x.

$$\int \frac{df}{dx} dx = \int g(x) dx$$

On the left, the integral of the derivative is the original funciton f.

$$f(x) = \int g(x) dx$$

5.5.2 Initial Value Problems

When we integrate to solve a DE, we will get a constant of integration. In order to determine that constant, we are often given an initial condition, such as $f(0) = 0$. Differential equations problems with initial conditions are often called *initial value problems* (IVPs).

Example 5.5.1.

$$\begin{aligned} \frac{df}{dx} &= x^3 \qquad f(0) = 7 \\ \int \frac{df}{dx} &= \int x^3 dx \\ f &= \frac{x^4}{4} + c \\ 7 &= \frac{0^4}{4} + c \implies c = 7 \\ f(x) &= \frac{x^4}{4} + 7 \end{aligned}$$

Example 5.5.2.

$$\begin{aligned} \frac{df}{dx} &= 6 \sin 3t \qquad f\left(\frac{\pi}{6}\right) = 6 \\ f(x) &= \int 6 \sin 3t dt = -2 \cos 3x + c \\ 6 &= -2 \cos \frac{\pi}{2} + c = 0 + c \implies c = 6 \\ f(x) &= -2 \cos 3x + 6 \end{aligned}$$

Example 5.5.3.

$$\frac{df}{dx} = \frac{1}{x^2+1} \qquad f(1) = \pi$$
$$f = \int \frac{1}{x^2+1} dx = \arctan x + c$$
$$\pi = \arctan(1) + c = \frac{\pi}{4} + c \implies c = \frac{3\pi}{4}$$
$$f(x) = \arctan x + \frac{3\pi}{4}$$

5.5.3 Seperable Differential Equations

Solving DEs is a very difficult task is general; the previous examples were all artificially simple. For this course, we're going to restrict to a specific type of DE called a seperable equation. Let $f(x)$ and $g(y)$ be continuous functions in x and y, respectively. Seperable differential equations are DEs where the dependent and independent variable can be seperated.

$$\frac{dy}{dx} = f(x)g(y)$$

In particular, this includes the autonomous equations we previously studied; if we set $f = 1$ then we get an autonomous equation.

To solve these equations, take $g(y)$ to the left side of the equation and then integrate in x.

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$
$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = f(x)dx$$

With a substitution, we can change the left side of the integral into an integral in y.

$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

To remember this set up, often we abuse Leibniz notation and pretend that $\frac{dy}{dx}$ is a fraction of infinitesimals. If we do this, all we need to do is isolate $g(y)$ with y on the left and $f(x)$ with dx on the right, then integrate.

$$\frac{dy}{dx} = f(x)g(y)$$
$$\frac{1}{g(y)} dy = f(x)dx$$
$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

Finally, after the integration is complete, we try to solve for y as a function of x. If we can, that function is the solution. Often we can't, and we leave the result as an implicit expression in x and y.

Example 5.5.4.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x}{y} \\
y dy &= \sin x dx \\
\frac{y^2}{2} &= -\cos x + c \\
y &= \pm\sqrt{c - 2\cos x} \\
\text{Initial condition: } y(0) &= 1 \\
1 &= \sqrt{c - 2\cos(0)} = \sqrt{c-2} \\
1 &= c - 2 \\
c &= 3 \\
y &= \sqrt{3 - 2\cos x}
\end{aligned}$$

Example 5.5.5.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-x}{y} \\
\int y dy &= -\int x dx \\
\frac{y^2}{2} &= \frac{-x^2}{2} + c \\
y^2 + x^2 &= c \\
\text{Initial condition: } y(4) &= 3 \\
4^2 + 3^2 &= c \\
c &= 25 \\
y^2 + x^2 &= 25
\end{aligned}$$

There is no way to solve for y as a function of x, so we leave this as an implicit expression in both variables.

Example 5.5.6.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 - 4 \\
\int \frac{1}{y^2-4} dy &= \int 1 dx \\
\frac{-1}{2} \operatorname{arctanh}\left(\frac{y}{2}\right) &= x + c \\
\operatorname{arctanh}\left(\frac{y}{2}\right) &= -2x + c \\
\frac{y}{2} &= \tanh(-2x + c) \\
y &= 2\tanh(-2x + c) \\
\text{Initial condition: } y(0) &= 0 \\
0 &= 2\tanh(0 + c) \\
c &= 0 \\
y &= 2\tanh(-2x)
\end{aligned}$$

Chapter 6

Applications of Derivatives

6.1 L'Hôpital's Rule

6.1.1 Indeterminate Forms

Recall that a limit is called an *indeterminate form* if it cannot be evaluated or logically determined. We can classify indeterminate forms by their type. For now, we look at three types. The first two types involve quotient limits.

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

If $f(x)$ and $g(x)$ approach $\pm\infty$, the limit is called an indeterminate form of type $\frac{\infty}{\infty}$. If instead both $f(x)$ and $g(x)$ approach 0, then it is an indeterminate form of type $\frac{0}{0}$. In both cases, we want to use algebra to factor out and cancel off the pieces of the quotients which tends to ∞ or 0 to solve the limit.

The third type involves a difference limit.

$$\lim_{x \to a} f(x) - g(x)$$

If both $f(x)$ and $g(x)$ approach $\pm\infty$, this limit is called an indeterminate form of type $\infty - \infty$. For this type, we want to use common denominator, conjugates or other algegraic tricks to reduce it to a limit of type $\frac{\infty}{\infty}$ or type $\frac{0}{0}$.

In all these definitions, we could replace $x \to a$ with the one-sided limit or $x \to \pm\infty$; the indeterminate forms are classified the same way for any type of limit.

6.1.2 L'Hôpital's Rule

The first application of derivatives is a strange reversal: after using limits to define derivatives, we are going to use derivatives to calculate limits. L'Hôpital's rule is a method that applies to limits of type $\frac{\infty}{\infty}$ or type $\frac{0}{0}$. In this case, if f and g are differentiable functions, L'Hôpital's rule states that the limit is preserved if we differentiate both numerator and denominator.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

This is particular useful in asymptotic analysis, where many limits (as $x \to \infty$) have type $\frac{\infty}{\infty}$.

Example 6.1.1. Consider the claim that $\ln x$ is asymptotically slower than $\sqrt{x}$. We use L'Hôpital's rule to prove the claim.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

A limit of 0 means that the numerator has lower asymptotic order than the denominator; we were justified in saying that the logarithm $\ln x$ is asymptotically slower that $\sqrt{x}$.

6.2 Extrema

Let $f : [a, b] \to \mathbb{R}$ be a function.

- A point $(c, f(c))$ is called a *local maximum* for f if $f(c) < f(x)$ for all x close to c.
- A point is a *local minimum* if $f(c) < f(x)$ for all x close to c.

The plurals of these terms are local minima and local maxima; if we want to refer to both, we will simply say local extrema. We can also define global extrema.

- The point $(c, f(c))$ is a global maximum if $f(c) > f(x)$ for all x in the domain of the function.
- The point is a global minimum if $f(c) < f(x)$ for all x in the domain.

One of the most important uses of derivatives is finding local extrema. The reason that derivatives are useful is that local extrema (for differentiable functions) are identified by their tangent lines. Look at the tangent lines Figure 6.1.

The tangent lines are horizontal at maximum or minimum values. We can make use of this fact to try to find these extrema. Horizontal tangent lines have zero slope, to they can only occur when $f'(x) = 0$. Therefore, we look for points x where the derivative vanishes. Such points will be called the *critical points* of the function.

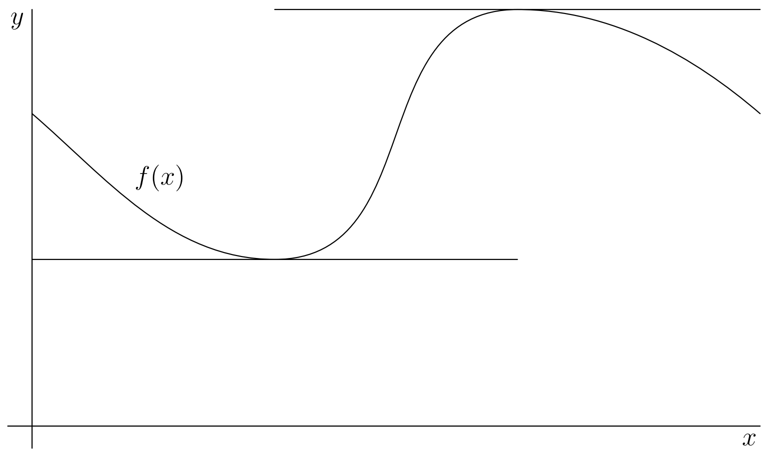


Figure 6.1: Tangent Lines at Extrema

Example 6.2.1. Consider the quadratic $f(x) = x^2 + 2$. The derivative is $f'(x) = 2x$. To find the critical points, we set the derivative equal to zero.

$$\begin{aligned} f(x) &= x^2 + 2 \\ f'(x) &= 2x \\ 2x &= 0 \implies x = 0 \end{aligned}$$

So $x = 0$ is the only critical point. After we have found a critical point, we need to determine if it is a maximum, minimum, or neither. We do this by looking at the sign of the derivative near the critical point. The easiest way to record this information is in intevals of increase and decrease. We divide the *domain* of the function into pieces seperated by the critical points.

In this example, the domain is $\mathbb{R}$ and the only critical point is $x = 0$. Splitting the domain by that critical point gives us the intervals $(-\infty, 0)$ and $(0, \infty)$. On these intervals, the derivative will either be positive or negative, so the function will either be increasing or decreasing respectively. For this example, $f'(x) = 2x$, which is negative on $(-\infty, 0)$ and positive on $(0, \infty)$; therefore, the function is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

This lets us classify the critical point. At $x = 0$, the function switches from decreasing to increasing. This means that x must be a minimum. We can conclude that $(0, 2)$ is a local minimum of the function.

Example 6.2.2. If $f(x) = 2x^3 + 3x^2 - 36x + 4$ then $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6)$. We set the derivative equal to zero to find the critical points.

$$\begin{aligned} 6(x^2 + x - 6) &= 0 \\ x^2 + x - y &= 0 \\ (x - 2)(x + 3) &= 0 \\ x &= 2 \\ x &= -3 \end{aligned}$$

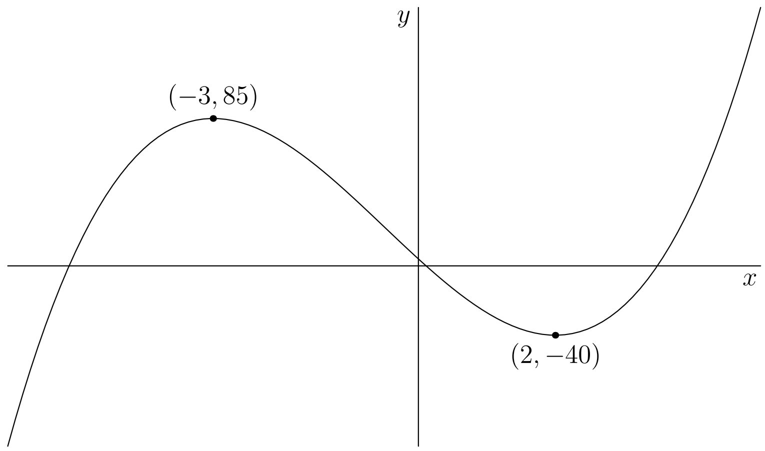


Figure 6.2: Extrema of a Cubic

We have two critical points, $x = 2$ and $x = -3$. The domain of the function is $\mathbb{R}$, so the intervals are $(-\infty, -3)$, $(-3, 2)$ and $(2, \infty)$. We check the sign of the derivative on these intevals; we do this by simply choosing any point in the interval and evaluated. The following table shows the intervals and the behaviour of the derivative on each interval.

$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
$f'(-4) = 36$	$f'(0) = -36$	$f'(3) = 36$
$f'(-4) > 0$	$f'(0) < 0$	$f'(3) > 0$
f is increasing.	f is decreasing.	f is increasing.

Then at $x = -3$, the function switches from increasing to decreasing, so $(-3, 85)$ is a local maximum. At $x = 2$, the function switches from decreasing to increasing, so $(2, -48)$ is a local minimum. Figure 6.2 shows the behaviour.

6.3 Optimization

6.3.1 Extreme Values of Models

Now that we understand how to find extrema using derivatives, we can use this technique to solve optimization problems. An optimization problem is any problem in applied mathematics where the goal is the optimal value of function, either expressed as a minimum or a maximum. The method for finding extrema is unchagned; most of the challenge in optimization problems is translating the problem into an appropriate function so that we can use derivatives.

Example 6.3.1. A very classic (if somewhat contrived) example of an optmization problem is maximing the area of a rectangle with fixed perimeter. Let's say that P is the fixed perimeter and the rectangle has height h and length l, as in Figure 6.3.

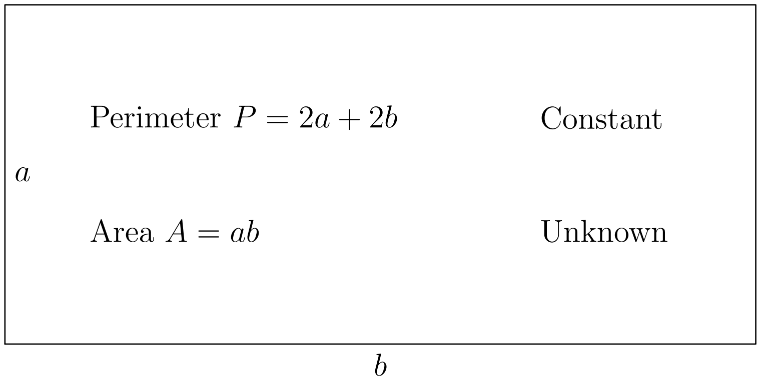


Figure 6.3: A Fixed-Perimeter Rectangle

We want to maximize area, so we will eventually be differentiating an area function. However, the area function is $A = hl$, which has two variables. We need to use the perimeter restriction to eliminate one of the variables. We know $P = 2h + 2l$ so $h = \frac{P}{2} - l$. If we substitute for h in the area function, we get a single variable area function $A(l)$.

$$A(l) = l\left(\frac{P}{2} - l\right)$$

Then we can optimize. The derivative is $A'(l) = \frac{P}{2} - 2l$. This vanishes when $l = \frac{P}{4}$. We can test that the critical point is a maximum. Unsurprisingly, the result shows that a square (where both l and h are exactly one quarter of the perimeter) maximizes area.

6.3.2 Optimized Distances

Many applications of optmization are found in geometry, both for purely mathematics and very practical reasons. As shown in Figure 6.4, the distance between two points (a, b) and (c, d) is determined by Pythagorus: $\sqrt{(c-a)^2 + (b-d)^2}$. This function, with squares and a square root, is not the nicest funtion to work with and differentiate. However, if two points are at an optimized distance (closest or most distance in some situation), the square of the distance will also be smallest or largest. Obviously, the square of the distance is a different number, but it is optimized at the same place that distance is optmized. For this reason, we usually optimize the square of distance: $(c-a)^2 + (b-d)^2$. Having removed the square root, this is a much easier function to work with.

Example 6.3.2. As an example, let's ask which point on the parabola $y = \frac{x^2}{4}$ is closest to the point $(4, 2)$. The distance squared from a point (x, y) on the parabola to $(4, 2)$ is given by the Pythagorean sum we just derived.

$$h = (4-x)^2 + (2-y)^2$$

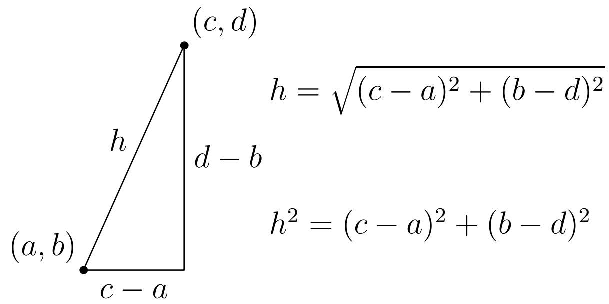


Figure 6.4: Distance Between Points

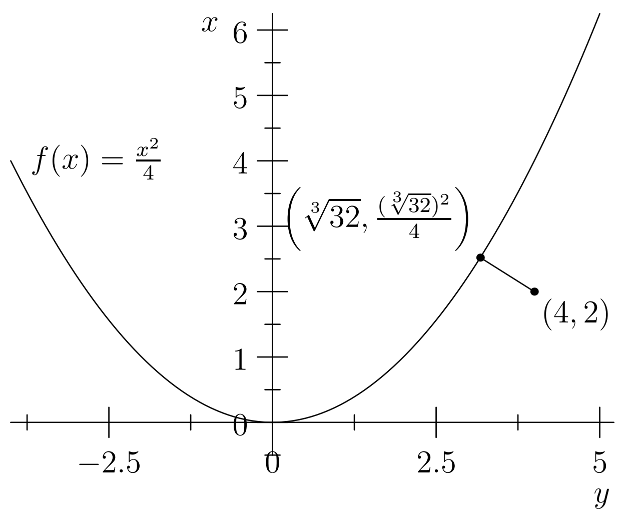


Figure 6.5: A Distance Optimization

We use the function to replace y with $\frac{x^2}{4}$, so that we have only one variable.

$$h(x) = (4 - x)^2 + \left(2 - \frac{x^2}{4}\right)^2$$
$$h(x) = 16 - 8x + x^2 + 4 - x^2 + \frac{x^4}{16} = 20 - 8x + \frac{x^4}{16}$$

We then differentiate to find the critical points.

$$h'(x) = -8 + \frac{x^3}{4}$$
$$h'(x) = 0 \implies \frac{x^3}{4} = 8 \implies x^3 = 32 \implies x = \sqrt[3]{32}$$

Once we have the critical point, we break up the domain (which is all $\mathbb{R}$ here, since any x value is possible) and look at the intervals.

$$\begin{array}{cc} \left(-\infty, \sqrt[3]{32}\right) & \left(\sqrt[3]{32}, \infty\right) \\ h'(0) = -8 & h'(4) = 8 \\ h'(x) < 0 & h'(x) > 0 \\ h \text{ is decreasing} & h \text{ is increasing} \end{array}$$

We conclude that there is a minimum at $x = \sqrt[3]{32}$. Since $y = \frac{x^2}{4}$, the corresponding y value is $\frac{(\sqrt[3]{32})^2}{4}$. The closest point on the parabola to $(4, 2)$ is $P = \left(\sqrt[3]{32}, \frac{(\sqrt[3]{32})^2}{4}\right)$. Figure 6.5 shows the outcome of the optimization.

6.3.3 Marginal Analysis

A nice use of optimization happens in economic and finance through a technique called marginal analysis. Naively, marginal analysis is frequently presented as a sales and production question: $C(x)$ is the cost of producing x units of a product and $B(x)$ is the revenue from selling x units of a product. We'll take a slightly more general interpretation, where we measure cost $C(x)$ and benefit $B(x)$ without specifying exactly what form those costs and benefits can take.

In this context, the derivative $C'(x)$ is called the *marginal cost* and the derivative $B'(x)$ is called the *marginal benefit*.

We can ask three questions.

- First, when do we have maximum net benefit? That is, when is the difference $B(x) - C(x)$ maximized? The derivative must be zero, so $B'(x) - C'(x) = 0$, which is $B'(x) = C'(x)$. We optimize net benefit when the marginal costs are equal. After finding such a point, we will still have to investigate to see if the point is a minimum, maximum or neither.
- The second question is a local variant of the first question. If we are currently at a production rate of x_0 units, would more production increase the net benefit? That is, is the net benefit currently an increasing function? Increasing means positive derivative, so we look for $B'(x) - C'(x) > 0$, which is $B'(x) > C'(x)$. Increasing production increases the net benefit if the marginal benefit is larger than the marginal cost. This leads to a reasonable interpretation of these derivatives: marginal cost is (roughly) the cost to produce *the next* unit, and marginal benefit is the benefit due to *the next* unit. We can increase net benefit if the next unit has greater benefit than cost.
- Lastly, we can ask a different strategic question. Instead of maximizing net benefit (like profit for a company), what if our motivation is just maximum benefit while still breaking even (more like a non-profit). This is still an optimization question, but with a different approach. Now we want to break even, which mathematically is $B(x) = C(x)$. The derivatives no longer come into play; we just have to solve this equality of functions and find the solution x with the highest gross benefit.

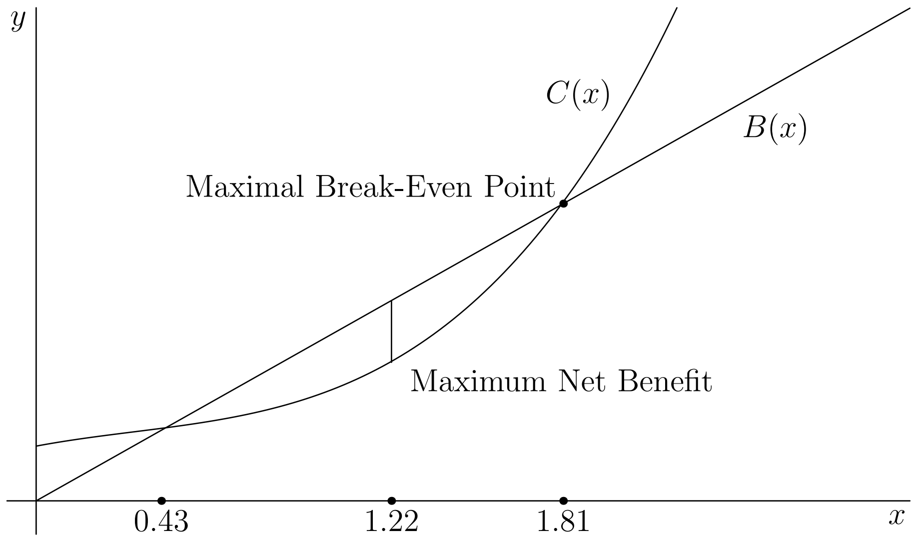


Figure 6.6: Example of Marginal Analysis

This certainly isn't an exhaustive list of all possible questions; strategically, there could be many considerations leading to many questions. Perhaps we have a fixed budget, so the cost $C(x)$ cannot, under any circumstances, cross a fixed maximum. If our product is a service, our goal might be maximum usage instead of maximum net benefit. Perhaps we need to minimize average cost of production instead of net benefit. Whenever we use mathematics for strategic reasons in a applied situation, it is important to remember that the mathematics only answers the questions we asked. It doesn't tell us which question we actually want to ask, nor how to compare between the various questions.

Example 6.3.3. Take the cost function $C(x) = x^3 - x^2 + x + 1$ and the benefit function $B(x) = 3x$. Assume that we are producing some unit for sale, that x is measured in thousdands of units, and that B and C are measured in millions of dollars. Notice that $C(0) = 1$, which could represent an initial start-up cost before the production of a single unit.

We calculate the derivatives: $C'(x) = 3x^2 - 2x + 1$ and $B'(x) = 3$. These are equal when $x = \frac{1}{3}(1 + \pm\sqrt{7})$. Discarding the negative, the other root is approximately 1.215, so a production of 1215 units gives the maximum net benefit. Below that point, marginal benefit exceeds cost, so we should increase production. Above that point, marginal cost will exceed benefit, so we should decrease production.

The break even points are found by solving the cubic: $x^3 - x^2 - 2x + 1 - 3x = 0$. The cubic doesn't factor nicely, but a computer can give us the approximate even x values: -1.25, 0.45 and 1.81. We discard the negative value again. The largest break-even point is at approximately 1.81.

6.4 Curve Sketching

In the last two lectures of the course, we want to pull together all of the techniques we have learned in order to develop a holistic way of describing functions. The first holistic problem is drawing graphs of function. Curve sketching is attempting to draw functions (without computer assistance) based on the properties we can derive using algebra and calculus.

Before we get to the general process of curve sketching, we need to talk about interpretation of derivatives. We have seen in the previous lectures that the first derivative gives us information about increase and decrease. If $f'(x)$ is positive, the function is increasing, and if $f'(x)$ is negative, the function is decreasing. If $f'(x) = 0$, then we may have a maximum or minimum.

We have a similar interpretation for the second derivative. The first derivative is slope, so the second derivative is the rate of change of slope. That measures the concavity of the graph: whether it is curving upwards or downwards. These shapes are called concave up and concave down, respectively. If $f''(x)$ is positive, then the function is concave up (curving updwards) and if $f''(x)$ is negative, the function is concave down (curving downards). Where $f''(x) = 0$ we might have a change in concavity; we call these points inflection points.

When we sketch a curve, we will consider all of the following.

(a) Domain. We determine domain by looking for domain restrctions such as division by zero.
(b) Range. Range is still difficult, even with calculus, However, once we know minima, maxima, increase and decrease, we can often figure out the range.
(c) Continuity. For standard functions, we get continuity for free on the domain. For piecewise functions, we have to do some work to investigate continuity at the cross-over points.
(d) Intercepts. We evaluate $f(0)$ for the y-intercept, provided that $x = 0$ is in the domain. For the x-intercepts, we try to solve $f(x) = 0$.
(e) Symmetry. We look for odd ($f(-x) = -f(x)$) or even ($f(-x) = f(x)$) symmetry, as well as periodicity ($f(x+p) = f(x)$).
(f) Limits and Asymptotes. We look at limits near undefined points. If these limits are infinite, we get vertical asymptotes. We also look at limits at $\pm\infty$. If these limits are finite, we get a horizontal asymptotes.
(g) First Derivative. We calculate the first derivative and solve $f'(x) = 0$ to find the critical points. We classify these points to find the minima or maxima. We also get the intervals of increase or decrease (which often let us determine the range, as mentioned before).
(h) Second Derivative. We calculate the second derivative and solve $f''(x) = 0$ to find the inflection points. We also get the intervals of concavity from the sign of the second derivative.

Example 6.4.1. Consider this function: $f(x) = x \ln x$.

(a) The domain is $x > 0$.
(b) The range is unclear from first impressions.
(c) The function is continuous.
(d) There is no symmetry.
(e) There is no y intercept. The x intercept is $(1, 0)$.

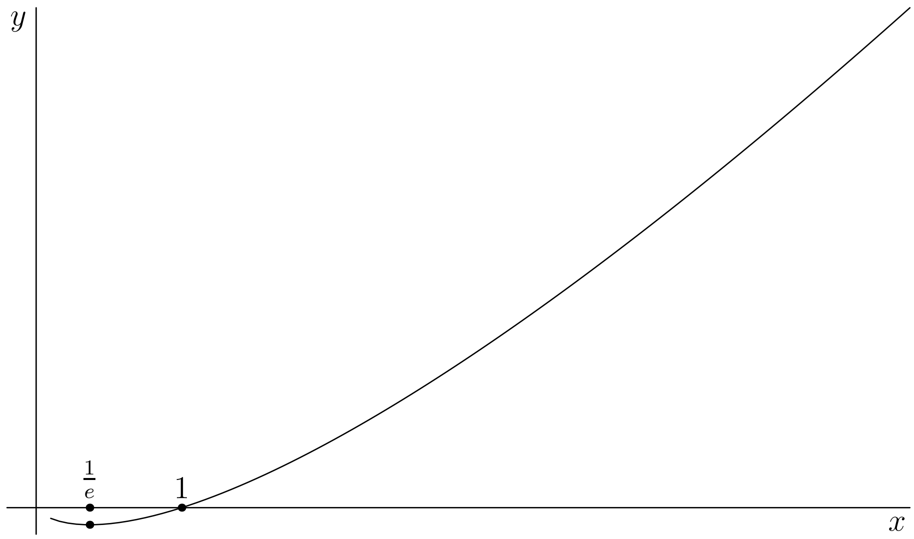


Figure 6.7: $f(x) = x \ln x$

(f) The only boundary point of the domain is $x = 0$. The limit as $x = 0$ is

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0_+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

Therefore, there are no vertical asymptotes. The limit as $x \to \infty$ is ∞, so there are no horizontal asymptotes.

(g) We look at the first derivative: $f'(x) = \ln x + 1$, so $f'(x) = 0$ implies $x = \frac{1}{e}$. The point $\left(\frac{1}{e}, \frac{-1}{e}\right)$ is a potential extrema.

$\left(0, \frac{1}{e}\right)$	$\left(\frac{1}{e}, \infty\right)$
$f'\left(\frac{1}{e^2}\right) = -1$	$f'(e) = 2$
$f'(x) < 0$	$f'(x) > 0$
$f(x)$ is decreasing	$f(x)$ is increasing

The point $\left(\frac{1}{e}, \frac{-1}{e}\right)$ is a local minimum.

(h) We look at the second derivative: $f''(x) = \frac{1}{x}$. The second derivative is always positive, so the function is concave up on $(0, \infty)$, which is the entire domain.

Example 6.4.2. Consider this function: $f(x) = xe^{-x^2}$.

(a) The domain is all of $\mathbb{R}$.

(b) The range is uncertain at first glance.

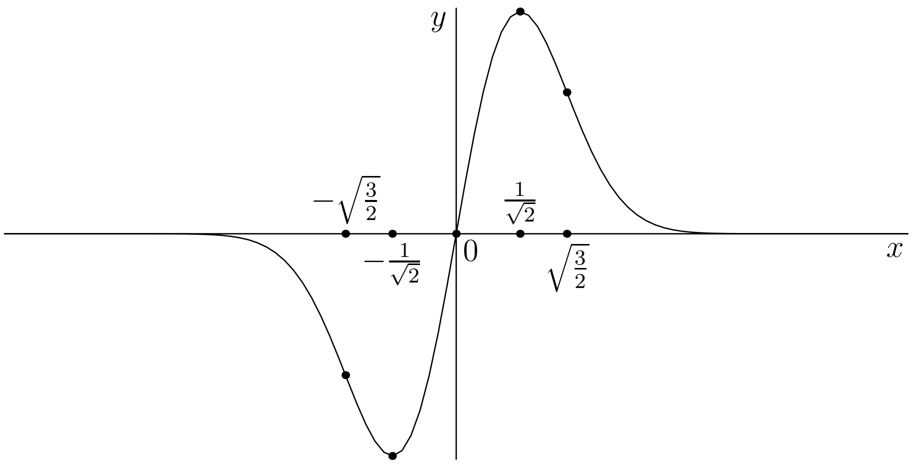


Figure 6.8: $f(x) = xe^{-x^2}$

(c) The function is continuous.
(d) $f(0) = 0$ and that is the only intercept.
(e) The function is odd.
(f) There are no undefined points, so there are no vertical asymptotoves. The limits as $x \to \pm\infty$ are both 0, so there are horizontal asymptotes $y = 0$ in both the positive and negative directions.
(g) The first derivative is $f' = e^{-x^2} - 2x^2e^{-x^2}$. This vanishes when $x = \pm\frac{1}{\sqrt{2}}$.

$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$	$(\sqrt{2}, \infty)$
$f'(-2) = -7e^{-4}$	$f'(0) = 1$	$f'(2) = -7e^{-4}$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
$f(x)$ is decreasing	$f(x)$ is increasing	$f(x)$ is decreasing

The point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ is a local minimum and the point $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2e}}\right)$ is a local maximum. This calculation also gives the range, since these extrema are also global extrema. The range is $\left[-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right]$.

(h) The second derivative is $f'' = -6xe^{-x^2} + 4x^3e^{-x^2}$. This vanishes when $x = \pm\sqrt{\frac{3}{2}}$. which gives inflection points $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2e^3}}\right)$ and $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2e^3}}\right)$.

$\left(-\infty, -\sqrt{\frac{3}{2}}\right)$	$\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$	$\left(\sqrt{\frac{3}{2}}, \infty\right)$
$f''(-2) = 44e^{-4}$	$f''(0) = -2$	$f''(2) = 44e^{-4}$
$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
$f$is concave up	$f$is concave down	$f$is concave up

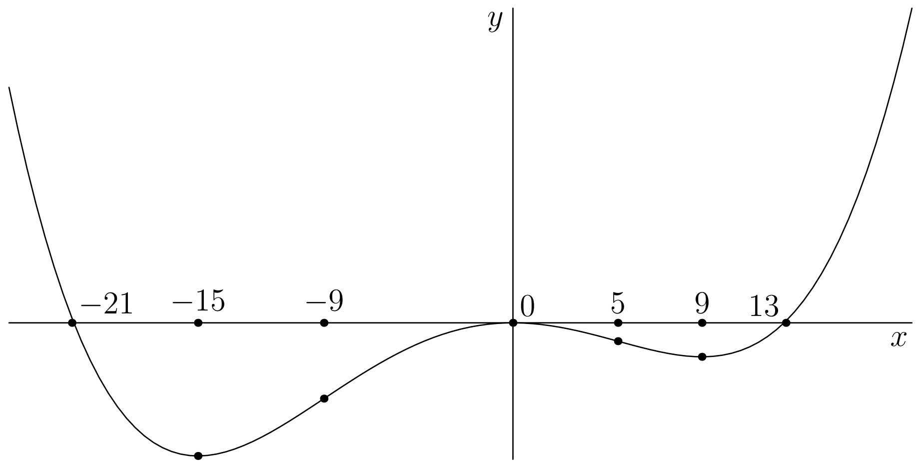


Figure 6.9: $f(x) = x^4 + 8x^3 - 270x^2 + 10$

Example 6.4.3. Take the quartic polynomial $f(x) = x^4 + 8x^3 - 270x^2 + 10$.

(a) The domain is all of $\mathbb{R}$.
(b) The range is bounded below, since is it a quartic with a positive leading coefficient. However, that minimum is not obvious at first glance.
(c) The function is continuous.
(d) $(0,1)$ is the y-intercept. f has four roots, but calculating them is difficult. A computer approximation shows roots near x values of 0.2, -0.2, -21 and 13.
(e) There are no undefined points, so there are no vertical asymptotes. The limit at $\pm\infty$ is ∞, so there are no horizontal asymptotes.
(f) The first derivative is $f'(x) = 4x^3 + 24x^2 - 540x$, which has roots of 0, 9 and -15. The critical points are $(0,1)$, $(9,-9476)$ and $(-15,-37124)$.

$(-\infty,-15)$	$(-15,0)$	$(0,9)$	$(9,\infty)$
$f'(-16) = -18880$	$f'(-1) = 560$	$f'(1) = -512$	$f'(10) = 1000$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
decreasing	increasing	decreasing	increasing

$(0,1)$ is the only maximum and the points $(-15,-37124)$ and $(9,-9476)$ are mimima.
(g) The second derivative is $f''(x) = 12x^2 + 48x - 540$, which has roots at x values 5 and -9. That

gives possible inflection points at $(5, -5124)$ and $(-9, -21140)$.

$(-\infty, -9)$	$(-9, 5)$	$(5, \infty)$
$f''(-10) = 180$	$f''(0) = -540$	$f''(6) = 180$
$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
concave up	concave down	concave up

6.5 Model Interpretation

In the previous section, we combined many techniques from the whole course together in order to sketch curves. In this section, we will also be combining many areas of the course. Our goal here is not visualization but interpretation: if we have a function which is a model of some real world phenomenon, we can use all the tools of calculus to understand what that model is saying. All of the information in curve sketching applies here, but we can also ask a number of questions that relate to the model.

- In addition to the mathematical domain, are there domain restrictions due to the model interpretation. (Very commonly, for example, the model measures only positive input values, so we restrict the domain to positive numbers.)
- Questions of intercepts are often questions of the starting value (y-intercept) or places where the model reaches zero values (x-intercepts).
- Questions about vertical asymptotes tell us where the model reaches unreasonable values; perhaps this is where the model breaks down.
- Questions about limits at ∞ and horizontal asymptotes give us information about the long term behaviour of the model.
- The derivative gives us the growth rate of the model.
- We can ask for an interpretation of the constants in the model.
- We can ask for a narrative: globally, qualitatively, what is the model saying? How can we holistically describe the behaviour?
- Finally, we can critique the model. Are there mathematical reasons (such as vertical asymptotes) where we expect the model may no longer fit the real world phenomenon?

It's best to see this through examples. Unlike curve sketching, where we went through the same list deliberatively one item at a time, here we'll use which ever tools seem germaine to the model at hand.

Example 6.5.1. Conisder a model of radioactivity on a contaminated site, where radioactivity is measured as r in grays and t is time in years. This is the model:

$$r(t) = \begin{cases} 3 + \frac{5t^2}{10000} & t \leq 100 \\ 2^{\left(-\frac{t}{100}+4\right)} & t > 100 \end{cases}$$

- The domain of the model is $t \geq 0$, showing that we start observing at year 0. The starting value at year 0 is $r(0) = 3$ grays.
- The function is piecewise, but we can check that it is continuous at its cross-over point.
- There are no vertical asymptotes. The limit at ∞ is 0, so $y = 0$ is a horizontal asymptote. That means the long term behaviour is a decay to negligable radiation.
- Derivative is positive on the first section ($0 \leq t \leq 100$) and negative on the second section ($t > 100$). Therefore, we expect a maximum at $t = 100$. The radioactivity of the site is increasing for 100 years and decreasing at all times afterward.
- If safe level of radioactivity are under 5 grays, we can ask when the site is safe. To do so, we simply solve $r(t) = 5$. In approximate values, the site become unsafe at $t = 63.2$ years and becomes safe again after $t = 167.8$ years.
- As a narrative, qualitative summary, it looks like there is contamination that slowly increases the radioactivity over the first 100 years. At 100 years, something suddenly changes: either the contamination source is removed or some kind of cleaning process exists that removes contamination faster than it is added. From that point on, contamination decays, eventually dropping to near-zero levels.

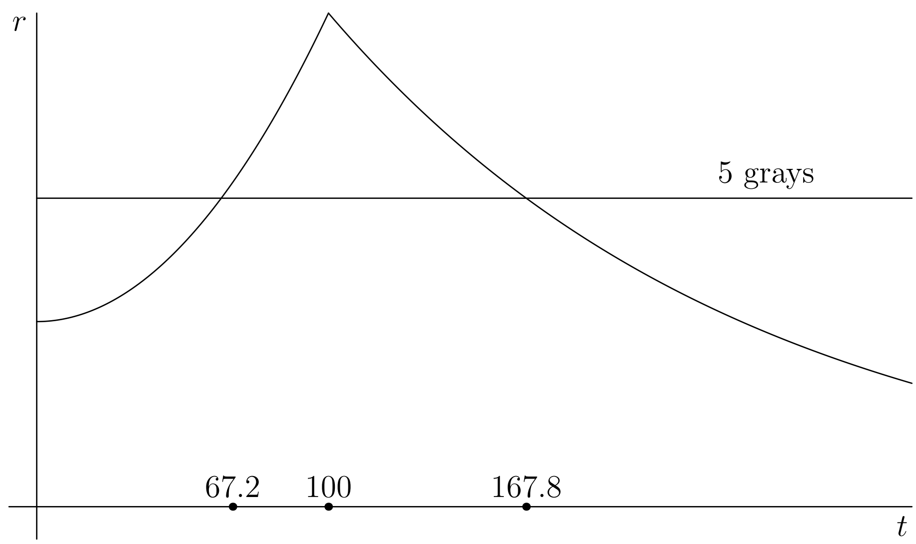


Figure 6.10: A Model of Radioactivity

Example 6.5.2. Conisder a population model, where p is population in thousands and t is time in years.

$$p(t) = e^{-t/10}(100 + 10\sin(2\pi t))$$

- A reasonable domain would be $t \geq 0$, assuming we start at year 0.
- The function has exponential and sinusoidal pieces. It might be useful to look at them separately. The exponential piece is

$$p(t) = 100e^{-t/10}$$

This is exponential decay which starts at 100 when $t = 0$.

- When we add the sine term, the coefficient infront of the exponential function varies between 90 and 110. This will effect the trajectory of the exponential decay. However, since 10 is smaller than 100, the effect is minimal. We would expect an exponential decay curve with small sinusoidal osscilation along its trajectory.
- The starting value is $p = 100$. The longer term behaviour, in the limit, is $p = 0$.
- The population is globally decreasing and decaying. However, due to the sinusoidal behaviour, there may be local small time frames where the population is briefly increasing. Since t is in years, perhaps this sinusoidal term measure seasonal variation in growth.

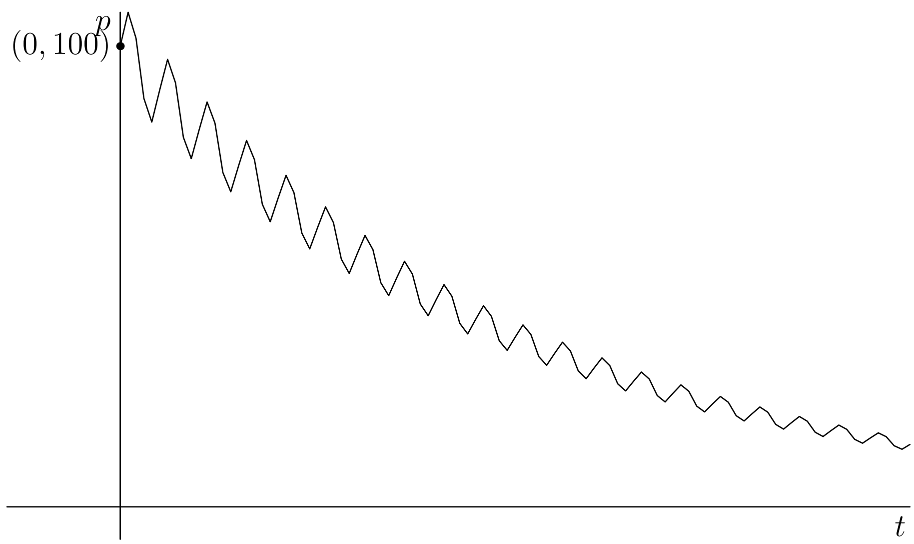


Figure 6.11: A Population Model

Example 6.5.3. Consider a model of temperature in a chemical reaction where T is in degree celcius and t is in minutes. This model is given by a differential equation. This is a relatively common situation: often in our observations of the world, we can observe a differential equation instead of directly observing the function.

$$\frac{dT}{dt} = 6 - 3t$$

For this differential equation, we can just integrate to solve. That gives the function $T = -3t^2+6t+c$. To specify the function completely, we would need an initial temperature. Let's say that initial temperature is 12 degrees: $T = -3t^2 + 6t + 12$

- This is a downward quadratic. If we complete the square, we get the vertex form: $T = -3(t^2 - 2t + 1) + 15 = -3(t-1)^2 + 15$.
- Therefore, we have a maximum at $t = 1$ minutes. That maximum is $T(1) = 15$ degrees. The function rises in temperature of a minute, then start to drop in a parabolic shape.

- The starting temperature, as given, is $T(0) = 12$ degrees.
- The long term behaviour is $T \to -\infty$. However, this is obviously unreasonable. First, mostly likely there is an ambient temperature which the system will eventually reduce to. Second, even if this is happening in the vacuum of space, temperature is still bounded below at -273 degrees. We conclude that this model is only meant to operate for relatively small t.

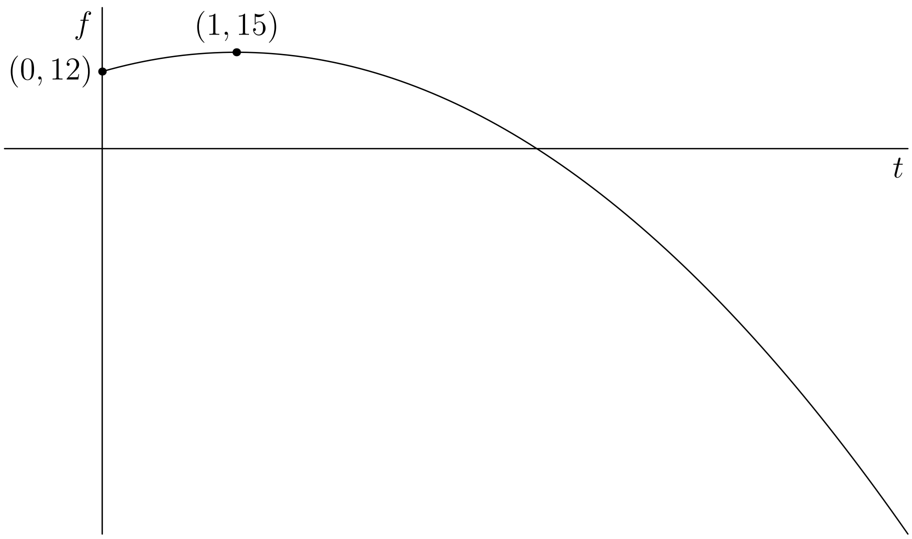


Figure 6.12: A Temperature Model

The description of models is, in many ways, the culmulation of this course. For the majority of students in this calculus class, you will only use the techniques of calculus sparingly throughout your degree and in your future work. However, when you use these techniques, you are likely to be presented with functions describing some phenomena. I hope you now have some skills to look at those function and analyze their expected behaviour.