Course Notes for Math 434

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Contents

Chapter 1

Preliminaries

This introductory chapter collects a number of definitions from prerequiste courses which are important enough to bear repeating.

1.1 Functions

Definition 1.1.1. Let A be the $\mathbb{R}$ or an interval subset of $\mathbb{R}$.

- $C(A)$ is the set of continuous real-valued function on A.
- $C^1(A)$ is the set of continuously differentiable functions on A.
- $C^n(A)$ is the set of functions on A with n continuous derivatives.
- $C^\infty(A)$ is the set of infinitly differentiable functions on A.

Definition 1.1.2. Let A be a subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is *piecewise-continuous* on A if it continous everywhere on A except for a set of isolated point. Likewise, f is *piecewise-differentiable* on A if it differentiable everywhere on A except for a set of isolated points. Note that A is the domain of the function: these piecewise functions can have jumps or break, but they are defined on all of A. They cannot have, for example, asymptotes at these isolated points.

We almost exclusively use the base e in this course, since it is the easiest base for derivatives and integrals. Since any other base a^t can be written $a^t = e^{\ln at}$, we will very frequently work with functions of the form $e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. In this way, we cover all exponential bases.

1.2 Linear Algebra

Definition 1.2.1. A *vector space* over $\mathbb{R}$ is a set V with addition and scalar multiplication (if $\alpha \in \mathbb{R}$ and $vinV$, when αv is defined and remains in V).

Example 1.2.2. $\mathbb{R}^n$ is a vector space. The scalar multiplication is given by multiplying each component by the real number.

Example 1.2.3. If A is $\mathbb{R}$ or an interval subset, then $C(A), C^n(A)$ and $C^\infty(A)$ are vector spaces. Scalar multiplication is simply multiplying a constant by a scalar.

Definition 1.2.4. Let V be a real vector space and Let $v_1, \ldots, v_k \in V$. A *linear combination* of these vectors is a sum $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$ where $\alpha_i \in \mathbb{R}$. These vectors are called *linearly independent* if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$$

has only the trivial solution, where all $\alpha_i = 0$. Otherwise, the set of vectors is called *linearly dependent.* A maximal linearly independent set in V is called a bases. The dimension of V is the number of vector in any bases.

Example 1.2.5. The vector space $\mathbb{R}^n$ has dimension n. The vectors spaces $C(A), C^n(A)$ and $C^\infty(A)$ are all infinte dimensional.

Definition 1.2.6. If V_1 and V_2 are vector spaces, a *linear transformation* $f : V_1 \to V_2$ is a function which respects addition and scalar multiplication.

$$f(u+v) = f(u) + f(v)$$
$$f(\alpha v) = \alpha f(v)$$

If V_1 and V_2 are finite dimensional, all linear transformations can be encoded by matrcies using matrix multiplication acting on vectors.

Definition 1.2.7. If M is a square matrix, then the *determinant of* M, written $\det M$, is a real number with two properties:

- $|\det M|$ is the effect of the transformation on the appropriate notion of size (length, area, volume, hypervolume, etc).
- If $\det M$ is positive, then M preserves orientation; if $\det M$ is negative, then M reverses orientation.

Please consult my linear algebra note (or any other linear algebra reference) for the details of calculating determinants.

Definition 1.2.8. Let $f : V_1 \to V_2$ be a linear transformation. A vector $v \in V_1$ is an *eigenvector* for f with eigenvalue λ if $f(v) = \lambda v$.

1.3 Taylor Series

Definition 1.3.1. A function is *analytic* if it can be expressed as a *Taylor series*.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-\alpha)^n$$

A Taylor series is centered at a point α; if $\alpha = 0$ we call it a *McLaurin series*. A series defines a function on some domain $(\alpha - R, \alpha + R)$ for some number $R \geq 0$, which is called a *radius of convergence*. If $R = \infty$, the series is defined on all real numbers. If $R = 0$, the series is only defined at $x = \alpha$ and is basically useless.

We use the ratio or root tests to calculate the radius of convergence of a series. After some manipulation of those tests, (if the coefficients are non-zer) the radius of convergence is given by the formula

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

Inside the radius of convergence, the behaviour of a Taylor series is very reasonable. We can add and subtract terms when the indices match. We can multiply series like polynomials, though the calculation gets arduous. We can even divide using long division (though it is an infinite process). With multiplication and division (and with many other uses of series), we often only calculate the first few terms of the series.

There are two important manupulation techniques for series. The first is adjustment on indices.

$$\sum_{n=k}^{\infty} c_n x^n = \sum_{n=k+1}^{\infty} c_{n-1} x^{n-1} = \sum_{n=k-1}^{\infty} c_{n+1} x^{n+1}$$

The second is removal of initial terms.

$$\sum_{n=0}^{\infty} c_n x_n = c_0 + c_1 x + c_2 x^2 + \sum_{n=3}^{\infty} c_n x_n$$

Inside the radius of convergence, the calculus of Taylor series is well behaved. We can integrate and differentiate term-wise.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ f'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\ \int f(x)dx &= \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1} + c \end{aligned}$$

In particular, we know that integrals and derivatives are always defined. This shows that analytic functions are necessarily C^∞ on the domain given by the radius of convergence. (This is, in fact, an equivalence: any C^∞ function has a Taylor series with some radius R).

Evaluating the derivatives of a series at the centre point α gives a list of derivatives.

$$\begin{aligned}
f(\alpha) &= c_0 \\
f'(\alpha) &= c_1 \\
f''(\alpha) &= 2c_2 \\
f^{(3)}(\alpha) &= 2 \cdot 3c_4 \\
f^{(4)}(\alpha) &= 4!c_4 \\
f^{(n)}(\alpha) &= n!c_n \\
c_n &= \frac{f^{(n)}(\alpha)}{n!}
\end{aligned}$$

This is a way to calculate coefficients, if we know the derivatives of a function. Remember that the coefficients totally describe the taylor series, so the derivatives at the centre point give all the information.

1.4 Recurrence Relations

A sequence $\{a_n\}_{n=1}^\infty$ of real numbers is called a *linear recurrence relation* if each term a_n is a linear function of the previous k terms $a_{n-1}, \ldots a_{n-k}$. k is called the *order* of the recurrence relation. Since terms depends on previous terms, the first k terms of the sequence must be explicitly defined.

The most famous example is the Fibonacci sequence, which is a second order linear recurrence relation. Its terms, f_n, start with $f_1 = f_2 = 1$ and then obey the linear recurrence $f_n = f_{n-1} + f_{n-2}$.

If we are given a recurrence relation relation, we don't have a direct way to calculate the term a_n without calculating all the previous terms. *Solving* a recurrence relation is the term for finding a fixed formula $a_n = f(n)$ that describes the nth term of the sequence. These can be easy or difficult to find. The fixed formula for Fibonnaci is

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

The coefficients of a Taylor series are sequences of real numbers. In our use of Taylor series, sequences of coefficients will be presented as recurrence relations. We will want to solve the recurrence relations to give a closed formula for the coefficients, in order to understand the resulting Taylor series.

1.5 Complex Numbers

Definition 1.5.1. We define the number $\imath$ by the property $\imath^2 = -1$. Then a *complex number* is any expression $a + b\imath$ where a and b are real numbers. a is called the *real part* and b is called the *imaginary part*. The set of all complex number is written $\mathbb{C}$.

Addition and multiplication are extended from $\mathbb{R}$ by linearity and the distribution of multiplication. Note in the multiplication we use the defining property that $\imath^2 = -1$.

$$\begin{aligned}(a + b\imath) + (c + d\imath) &= (a + b) + (c + d)\imath \\ (a + b\imath)(c + d\imath) &= ac + bc\imath + ad\imath + bd\imath^2 \\ &= ac + (bc + ad)\imath + bd(-1) = (ac - bd) + (bc + ad)\imath\end{aligned}$$

$\mathbb{C}$ is identified with the cartesian plane, where 1 is $(1, 0)$ and $\imath$ is $(0, 1)$. The horizontal axis is called the real axis and the vertical axis is called the imaginary axis. The number $a + bi$ is treated as the coordinate (a, b) in the plane. The inversion of a complex number (non-zero) is

$$\frac{1}{a + b\imath} = \frac{a - b\imath}{a^2 + b^2}.$$

Definition 1.5.2. The *modulus* of a complex number $a + b\imath$ is its length as a vector (a, b), namely $|a + b\imath| = \sqrt{a^2 + b^2}$.

Definition 1.5.3. The *conjugation* of a complex number $a + b\imath$ is $a - b\imath$. It is indicated with a vertical bar $\overline{a + b\imath} = a - b\imath$. Geometrically, conjugation is reflection over the real axis (x axis) in the plane.

Complex numbers have many intriguing property. The most important for us it the existence of roots of polynomial.

Theorem 1.5.4 (Fundamental Theorem of Algebra)**.** *Let $p(x)$ be a polynomial of degree d with real or complex coefficients. Then $p(x)$ has exactly d roots in the complex numbers (though some may be repeated). Equivalently, $p(x)$ factors completely over the complex numbers: there are complex numbers $\alpha_1, \ldots, \alpha_d$, not necessariliy distinct from each other, such that*

$$p(x) = (x - \alpha_i)(x - \alpha_2) \ldots (x - \alpha_d).$$

Complex exponentials are understood by Euler's formula.

Proposition 1.5.5 (Euler's Formula)**.**

$$e^{\imath\theta} = \cos\theta + i \sin\theta.$$

Proof. Even without a background in complex analysis, Euler's formula can be justified by looking at Taylor series.

$$\begin{aligned}
e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
\sin t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
\cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\
e^{\imath t} &= \sum_{n=0}^{\infty} \frac{(\imath t)^n}{n!} = \sum_{n=0}^{\infty} \frac{\imath^n t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\imath^{2n} t^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{\imath^{2n+1} t^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n!} + \imath \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
&= \cos t + \imath \sin t
\end{aligned}$$

□

1.6 A Note on Variables

In differential equation, the name of the independent variable is almost always either x or t. x is conventional, of course, in calculus; it is also useful for certain geometric interpretation where we want x and y to correspond to the familiar cartesian axes. We will use x as the independnet variable at certain points in the courses, particularly when we use these geometric techniques.

The vast majority of differential equations that we want to solve involve time derivatives; therefore, t is often the most natural choice for the independent variable. We will use t instead of x for the majority of this course. Often in the literature, a derivative in Newton's notation (f') implies that the independent variable is time.

Chapter 2

First Order Differential Equations

2.1 What is a Differential Equation?

2.1.1 First Definitions

Definition 2.1.1. A *differential equation* is any equation involving a function $f(t)$ and its derivatives $f'(t)$, $f''(t)$, etc. The highest derivative in the equation is called the *order* of the equation. A DE which involves partial derivatives is called a *partial differential equation*; otherwise, it is called an *ordinary differtial equation*.

More holistically, a differential equation asks the question: what function satisfies a given relationship? In algebra, we are familiar with this question for numbers. $t+5=7$ implicitly asks which number, when replacing t, satisfies the equation. The answer, of course, is $t=2$; the solution to the equation gives the correct replacement. For differential equations, we ask 'what function' instead of 'what number'.

There are many ways to write a differential equation. The most general way is to think of any algebraic combination of a function y, its derivatives and the independent variable t. In this generality, we could write the equation using a multivariable function F.

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

Often, if we can, we isolate the highest derivative.

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

Definition 2.1.2. A differential equation is called *autonomous* is the independent variable doesn't appear. For a first order equation, this looks like

$$\frac{dy}{dt} = f(y).$$

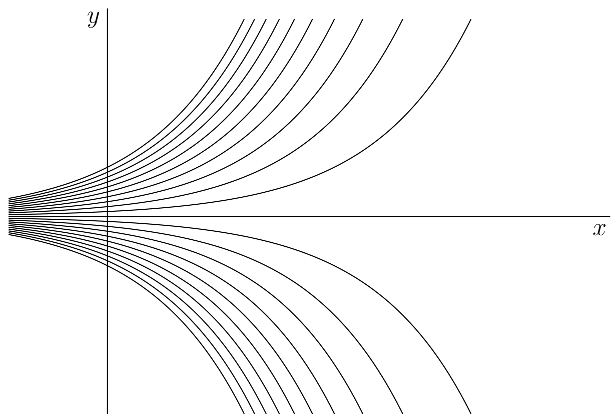


Figure 2.1: The family $f(t) = ce^{\alpha t}$ where $c \in \mathbb{R}$.

Definition 2.1.3. An equation is called *linear* if it has the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = f(t).$$

In addition, a linear equation is called *homogenous* if the $f(t)$ term is simply $f(t) = 0$.

A very useful notational tool for linear DEs is the idea of a linear differential operator.

Definition 2.1.4. A *differentiable operator* is an operator which acts on functions by either multplication by fixed functions or by differentiation. The operator is *linear* if the various pieces of the operation are added together. Fore example, a second order linear equation has this form

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c.$$

We use differentiable operators to simplify notation for DE. A second order linear DE become simply $Ly = f(t)$.

2.1.2 The Most Important Example

Example 2.1.5. The simplest differential equation is

$$\frac{df}{dt} = \alpha f(t).$$

For many differential equations, we seek a translation of their meaning. This simple equation means the rate of change is propotional to the current value. This is a statement of percentage growth; the proportionality constant is the percentage.

The solutions to this equation are are $f(t) = ce^{\alpha t}$. Notice that there is a constant, c, such that there will be a whole infinite family of solutions. A specific solution is specified by a choice of c.

Percentage growth applies to population models. If this is the case, then $f(0) = ce^0 = c$ and we must interpret the constant c as the population when $t = 0$. If this is our starting point, we consider c the *initial value* of the solution. A full solution of a differential equation will usually consist of a function and choice(s) for the initial value(s).

Definition 2.1.6. A differential equation along with a specified initial value is called an *initial value problem* or IVP.

If we don't make a choice, as we said, we get an infinite family of solutions. We can visualize this family as a series of graphs in $\mathbb{R}^2$. Figure 2.1 shows the graphs for $f(t) = ce^{\alpha t}$.

2.1.3 Pure and Applied Perspectives

We will be looking at differential equation both from a pure mathematics and applied mathematics point of view. The pure mathematician is interested in these kinds of questions:

- When does a solution exists?
- Can we prove that a solution exists?
- Is the solution unique?
- Is there a complete family? How many parameters exist, and what are their domains?
- Can we write and prove theorems to answer these questions?

The applied mathematician is interested in these kinds of questions:

- How many solutions fit the data or initial values?
- How do the solutions grow? What is their behaviour?
- Are the solutiosnts stable?
- How difficult are the solutions to calculate? Can we get them exacly, or only approximately?
- Can we answer questions about the model even without a solution?

2.1.4 Interesting Examples

We want to know the behaviour of solutions of DEs. Several examples in this section give the lay of the land.

Example 2.1.7.

$$\frac{d^2 f}{dt^2} + 9f = 0$$

This is solved by $\sin 3t$ and also by $\cos 3t$. Moreover, any linear combination $a \sin 3t + b \cos 3t$ is a solution. In a second order equation, we often expect two linearly independent solutions and the general solution is a linear combination of the two linearly independent solutions.

Example 2.1.8.

$$\frac{dy}{dt} = t\sqrt{y}$$

This is solved by $y = (\frac{t^2}{4} + c)^2$, which is a nice family with one real parameter. However, this is also solved by $y = 0$, even though it isn't in the family.

Definition 2.1.9. An extraneous solutions to a DE, one which fall outside families with parameters, is called *singular solution.*

Example 2.1.10.

$$t\frac{dy}{dt} = 4y$$

This is solved by $y = ct^4$, which is another reasonable family. However, there is a strange, singular solution.

$$y(t) = \begin{cases} -t^4 & t \leq 0 \\ t^4 & t > 0 \end{cases}$$

The derivatives of this function line up at the origin, so the function is of class C^1 (actually C^3, if you work it out).

Example 2.1.11.

$$\frac{dy}{dx} = \frac{-x}{y}$$

The curve $x^2 + y^2 = c$ solves this equation implicitly. We could break this up into two functions, but its much more natural to leave it as an implicit locus, in this case, a circle. This is very typical: often solutions are left in an implicit form as loci, even though in theory we always look for functions $y = f(x)$. Also notice in this example that only non-negative c values are allowed in this family of solutions. There is no guarantee that all values of a parameter will lead to reasonable solutions.

2.1.5 The Pessimistic Outlook

In some sense, a DE is an mathematical application of the scientific method. Often an observation about a phenomenon can be expressed as a relationship between a function and its derivative, such as the observation of percentage growth. The DE, then, is the hypothesis born of observation. If we can find the solution, it gives us a predictive model of the phenomenon, which we can test. If the solution matches the observed behaviour, we conclude the DE model is relatively reliable; if the solution diverges from the observed behaviour, we discard or amend the DE.

In this way, DEs allow the modelling many phenomena: popluations, radioactive decay, cooling, disease, metabolish, newtonian motion with friction, chemical reactions, gravity, predaor-prey models, hamiltonian mechanics, quantum mechanics, interest, bacterial growth, neuron firing, ecology, mixtures, draining, series circuits, suspended cables, and many, many more.

However, for all this power and utility, DEs are terribly difficult to solve. The sad truth is that we can exactly solve only a very small portion of them. Due to this limitation, many techniques are developed to understand approximate solutions or infer information about solutions indirectly.

In addition, many DEs have solutions which are entirely new functions.

Example 2.1.12.

$$\frac{dy}{dt} = e^{-t^2} \implies y = \int e^{-t^2} dt + C$$

This integrand is a C^∞ function, so the anti-derivative exists, but there is no name for it in the elementary functions. This is not uncommon. Often we will 'solve' a differential equation by simply naming the solution with a new name, since it is an entirely new function.

2.2 Qualitative Methods for First Order DEs

As we said before, many DEs are very difficult or impossible to solve directly. Implicit or qualitative methods try to say something about the solutions without actually finding the solutions. Even when we can exactly solve an equation, these methods are great for interpreting the behaviour of the solutions.

2.2.1 Autonomous DEs and Phase-Line Analysis

Consider a population model $p(t)$ with its autonomous DE.

$$\frac{dp}{dt} = f(p)$$

There is a lovely piece of qualitative analysis for autonomous equations called the *phase line analysis*. Phase line analysis looks at the right side of the equation and asks: what values of p set the right side to zero. What does this mean? When the right side of our differential equation is 0, the left side is 0 as well. The left side is the growth rate, so that means the growth rate is zero. This is a value of the population p where there is no growth.

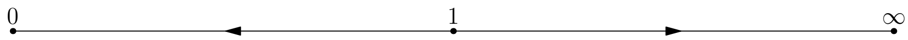


Figure 2.2: A Phase Line Diagram for $\frac{dp}{dt} = p^2 - p$

Definition 2.2.1. In an autonomous DE, a value of the function where the derivative vanishes is called a *steady states*

If the population is exactly at its steady state, it will not change; steady states are constant populations which do no grow or decline. (For population models, we can make the reasonable assumptions that $p \geq 0$ and $p = 0$ is always a steady state. Otherwise, we can have any real values on the phase line.)

Once we have the steady states, we ask what happens between each steady state. Assuming that the DE is reasonable, then between the steady states, the right side will be either positive or negative. When it is positive, we have a positive growth rate and the population increases. When it is negative, we have a negative growth rate and the population decreases.

Definition 2.2.2. In an autonomous DE, the direction of growth negative or positive, called the *trajectory* of the popluation.

Steady states and trajectories give us an remarkably complete understanding of the population.

- If the popluation is at a steady state, it doesn't change.
- If the popluation is not at a steady state, we look at the trajectory.
- If the trajectory is positive, the popluation grows either to the closest larger steady state or to infinity.
- If the trajectory is neagative, the population declines either to the closest smaller steady state or to zero.

We summarize this information is a phase line diagram. We take a ray representing $p \geq 0$ and put dots on the ray for the steady states. In between, we put arrows to show the trajectories. Its best to see the phase line diagrams through examples.

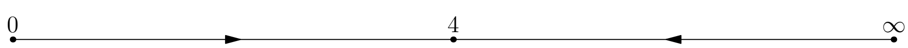


Figure 2.3: The Logistic Phase-Line Diagram

Example 2.2.3.

Example 2.2.4.

$$\frac{dp}{dt} = p^2 - p$$

The right side is zero when $p = 0$ or $p = 1$, so those are the steady states. When $p \in (0, 1)$ the derivative is negative, so the trajectory is decreasing. When $p \in (1, \infty)$, the derivative is positive, so the trajectory is increasing. Figure 2.2 shows the resulting phase-line.

This example is a specific instance of a form known as the logistic equation.

$$\frac{dp}{dt} = 4p - p^2$$

The right side is zero when $p = 0$ or $p = 4$, so those are the steady states. When $p \in (0, 4)$ the derivative is positive, so the trajectory is increasing. When $p \in (4, \infty)$, the derivative is negative, so the trajectory is decreasing. Figure 2.3 shows the resulting phase-line.

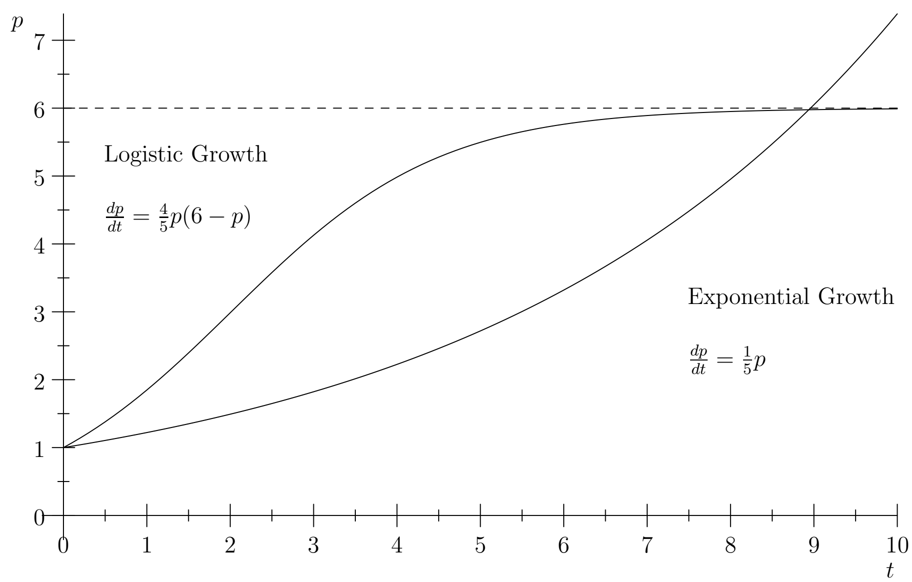


Figure 2.4: Exponential and Logistic Growth

The logistic equation leads to logistic growth. We can see from the phase line diagram that the trajectories all point towards the value $p = 4$. In logistic growth, the values always want to revert to some non-zero steady state. From below, this is growth up to some firm maximum. After exponential growth, logistic growth is the most commonly used model for populations. Figure 2.4 shows both exponential and logistic growth (where the steady state for the logistic model is at $p = 6$.)

Example 2.2.5.

$$\frac{dp}{dt} = p^3 - 7p^2 + 10p$$

The right side factors as $p(p-2)(p-5)$, so it is zero then $p = 0$, $p = 2$ or $p = 5$. Those are the steady states. When $p \in (0, 4)$ the derivative is positive, so the trajectory is increasing. When $f \in (2, 5)$, the derivative is negative, so the trajectory is decreasing. When $p \in (5, \infty)$, the derivative is positive, so the trajectory is increasing. Figure 2.5 shows the phase line.

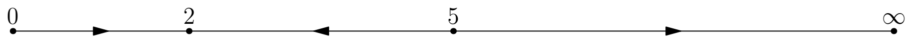


Figure 2.5: A Phase Line Diagram

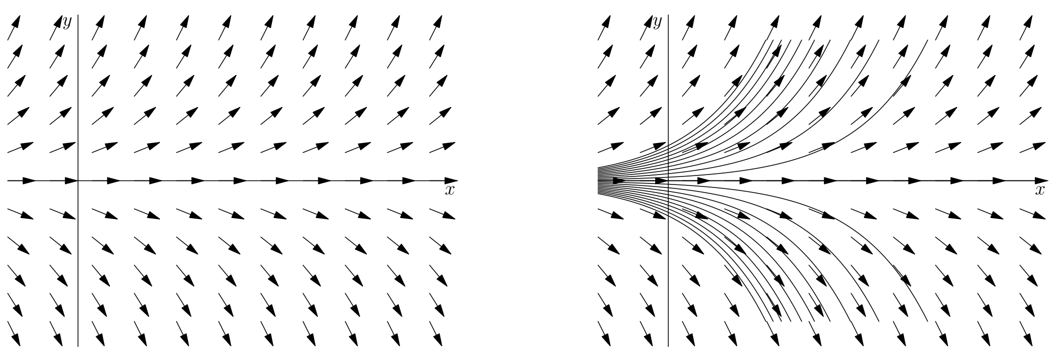
2.2.2 Direction Fields

If the equation is not autonomous, then the phase-line is too simple a tool to capture the details. However, if we can solve for the derivative, we can write a first order DE as

$$\frac{dy}{dx} = F(x, y).$$

This allows a very useful interpretation: the left side is the slope of a graph, the right side is a function on $\mathbb{R}^2$, giving a value at every point in the plane. Together, we determine a slope at every point in the plane.

Definition 2.2.6. A determination of a slope at all point in a subset U of RR^2 is called a *direction field*.



(a) The Slope Directions (b) With Solutions

Figure 2.6: The Direction Field for $\frac{dy}{dx} = y$.

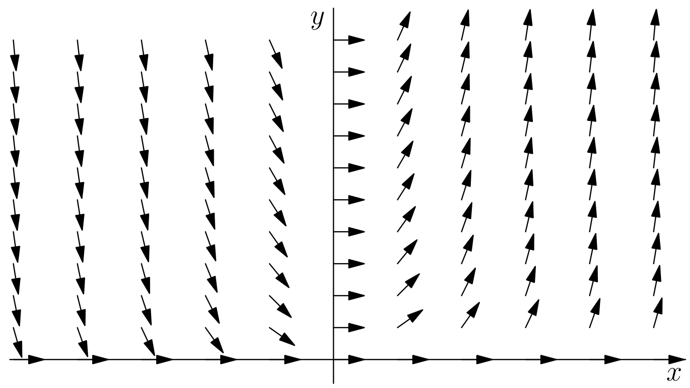
If there are solutions to the DE, then they must be functions which fit these slopes. Therefore, the slopes give us a sense of what the functions look like. Let's go back to the simplest example.

Example 2.2.7.

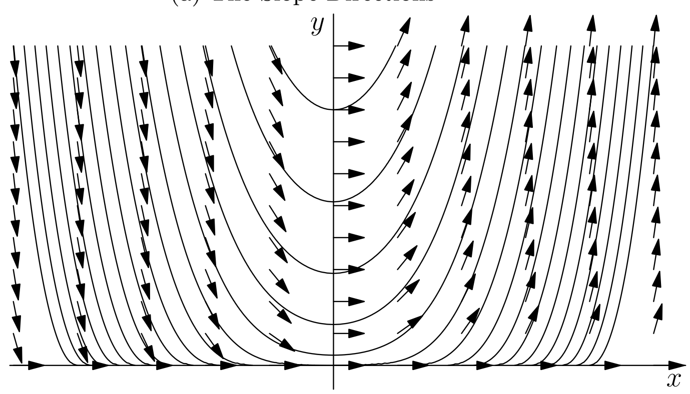
$$\frac{dy}{dx} = y$$

The slope at a point (x, y) is y, so the slope at $(3, 4)$ is 4, the slope at $(-2, -3)$ is -3 and the slope at $(32, 0)$ is 0. Figure 2.6 shows the direction field.

The solutions exactly fit the direction field. Therefore, if we can draw and understand the direction field, we get a sense of the solutions. Notice that since the direction field fills $\mathbb{R}^2$ (or a portion of it), we expect an infinite family of graphs of functions to match all the slopes. Figure 2.6 shows the graphs of the infinite family of solutions.



(a) The Slope Directions



(b) With Solutions

Figure 2.7: The Direction Field for $\frac{dy}{dx} = x\sqrt{y}$.

Let's go back to the strange examples in the previous section and draw their direction fields.

Example 2.2.8.

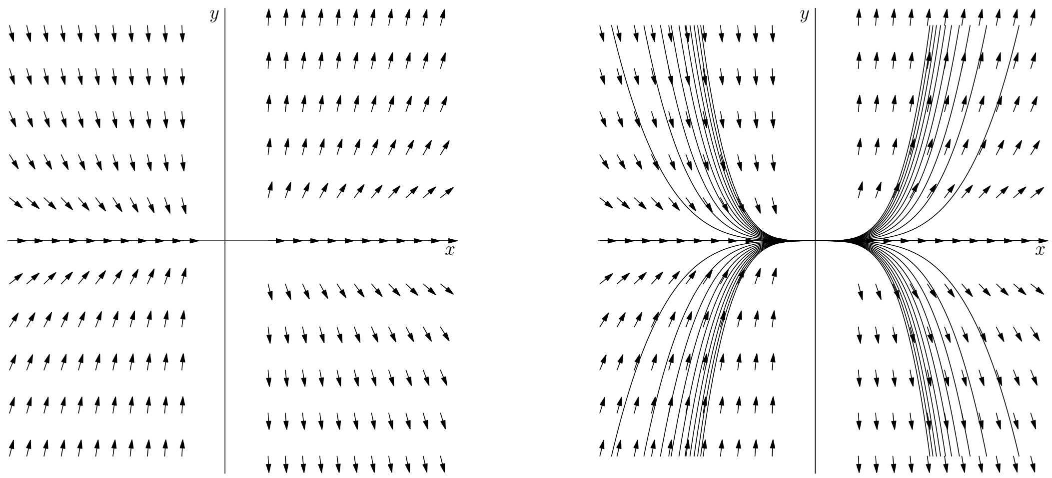
$$\frac{dy}{dx} = x\sqrt{y}$$

Figure 2.7 shows the direction field and the infinite family of solutions. We see the entire family $y = (\frac{x^2}{4} + c)^2$, but we also see the singular solution $y = 0$.

Example 2.2.9.

$$\frac{dy}{dx} = \frac{4y}{x}$$

There family of solutions $y = cx^4$ which fits the direction field. The singular solutions are put together from one positive and one negative piece. Figure 2.8 shows the direction field and the infinite family of solutions.



(a) The Slope Directions (b) With Solutions

Figure 2.8: The Direction Field for $\frac{dy}{dx} = \frac{4y}{x}$.

Example 2.2.10.

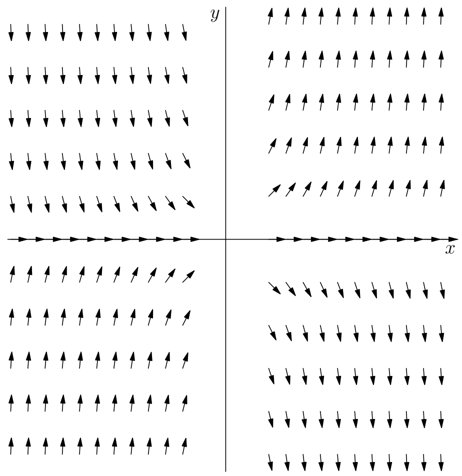
$$\frac{dy}{dx} = xy$$

This is solved by $y = ce^{x^2}$, including $c = 0$ for $y = 0$ as a trivial solution. The direction field also shows the stability behaviour of the function: in this case, the functions grows very quickly away from the origin except for the stable and trivial $y = 0$ solution. Figure 2.9 shows the direction field and the infinite family of solutions.

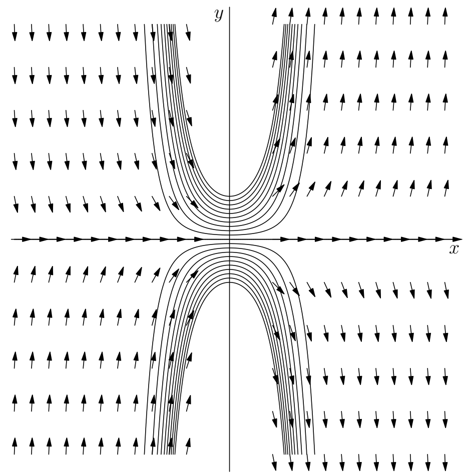
Example 2.2.11.

$$\frac{dy}{dx} = -\frac{x}{y}$$

This is solved by $y = \pm\sqrt{c - x^2}$, which gives a series of circles. The direction field shows the bounded, relatively stable behaviour which is confirmed by the solutions. Notice that these solution have finite domains: we are not guaranteed solutions that apply to all real numbers. Figure 2.10 shows the direction field and the infinite family of solutions.



(a) The Slope Directions



(b) With Solutions

Figure 2.9: The Direction Field for $\frac{dy}{dx} = xy$.

2.3 Stability and Linearization

2.3.1 Stability

Coming from applied mathematics, the language of stability is a very useful language for talking about DEs. We again think of a general autonomous DE as a population model.

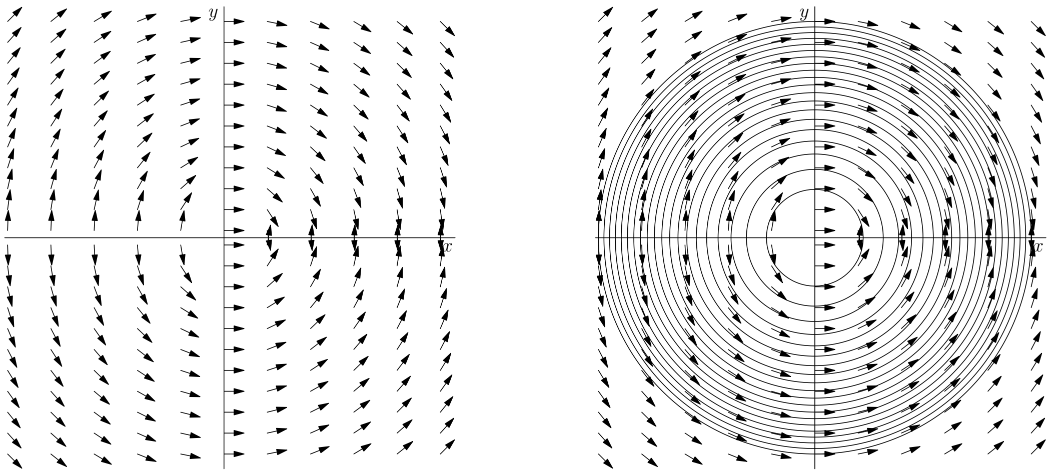
$$\frac{dp}{dt} = f(p)$$

We defined steady states when we did phase-line analysis: these were roots of the function $f(p)$ and hence values of p where the rate of change is zero.

Definition 2.3.1. The steady states of an autonomous DE are classified by their *stability* A steady state P is *stable* if $p(t) \to P$ when the initial value is close to P. We can also call these steady states *attractors*. In the phase line diagram, the trajectories points toward such states. Similar, we can call a steady state *stable or attractive from above or below* if only one of the trajectories points towards the steady state. If both trajectories point away, the steady state is *unstable*.

2.3.2 Linearization

Once we have identified a steady state of a DE system, we often are only interested in the behaviour of slight perterbations from the steady state. This is the behaviour that stability capture: whether we



(a) The Slope Directions (b) With Solutions

Figure 2.10: The Direction Field for $\frac{dy}{dx} = \frac{x}{y}$.

approach or diverge from the steady state when we start a small distance away. If P is a steady state, the we can define a new function $q(t) = p(t) - P$. A change of variables results in

$$\frac{dq}{dt} = f(P + q).$$

If we expand f as a taylor series centered at p, this becomes

$$\frac{dq}{dt} = f(P + q) = f(P) + f'(P)(P + q - P) + \ldots = f'(P)q + \ldots.$$

Definition 2.3.2. The *linearization* of the DE at the steady state P is

$$\frac{dq}{dt} = f'(P)q.$$

The solution to the linearized DE is

$$q(t) = q_0 e^{f'(P)t}.$$

In particular, this linearized solution is either exponential growth or decay, depending on the sign of $f'(P)$. Therefore, the sign of $f'(P)$ determines the stability: positive and the solution is unstabe, negative and the solution is stable. This can also be seen in the phase line, since the sign of the derivative can indicate the trajectories on either side of the steady state. If $f'(P) = 0$, then the stability is determined by the higher order terms of the taylor series expansion.

For now, there isn't much more we will do with this linearization. However, it is worth introducing here as a theme since it is so central to applied mathematics. Linear equations are almost always the first kind of DE that we try to use, typically since they have elegant and accessible solutions. Everything else gets simply referred to a ‘non-linear’; in many ways, ‘non-linear’ is a synonmy for annoying and complication. However, linear models only go so far and often the non-linearity holds the key to understanding a model. Even so, we will often try to understand the linear part and then figure out how to add in the non-linearity in a reasonable fashion to add subtleties to various models.

2.4 Seperable Equations

Definition 2.4.1. A *seperable equation* is a DE which has the form

$$\frac{dy}{dt} = f(y)g(t).$$

The method of solving seperable equations treats the dt and dy terms as independent infinitesimals, a strange but historically reasonable treatment. If we allow for these independent infinitesimals, we can seperate (hence the name) the DE into two pieces, one involving each variable.

$$\frac{1}{f(y)}dy = g(t)dt$$

Then, again acting somewhat strangely by modern notational conventions, we integrate both sides with respect to their own variables.

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$

The solution is then left implicit unless we can reasonable solve for y, in which case we can write y as a conventional function of t.

If you are interested in the justification of this splitting procedure, we could think of the operation alternatively, writing $f(y(t))$ to remember the independent variable. If we bring the $f(y)$ to the left side, we get the expression

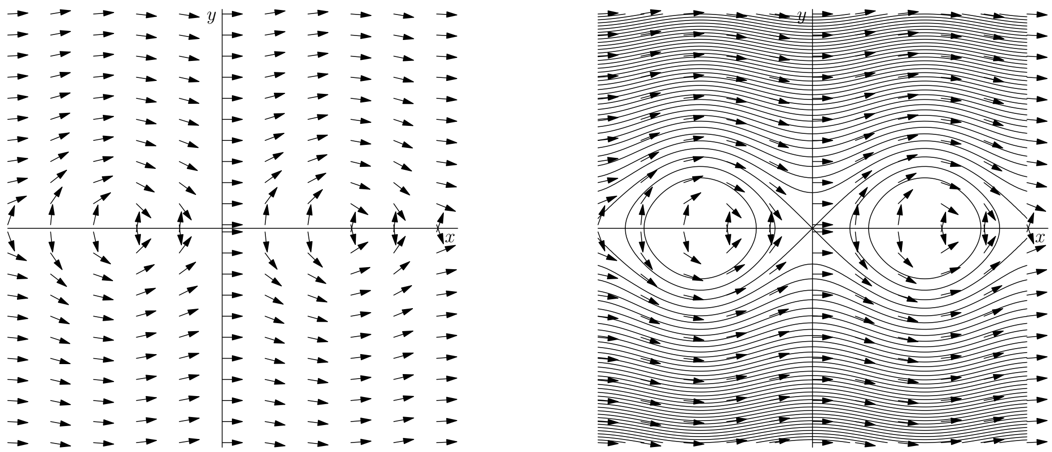
$$\frac{1}{f(y(t))}\frac{dy}{dt} = g(t).$$

Then we can integrate both sides with respect to t, which is a reasonable and justified operation.

$$\int \frac{1}{f(y(t))}\frac{dy}{dt}dt = \int g(t)dt$$

Finally, we change of variables from t to y in the left side integral.

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$



(a) Direction Field for $\frac{dy}{dx} = \frac{\sin x}{y}$ (b) Direction Field with Solutions for $\frac{dy}{dx} = \frac{\sin x}{y}$

Figure 2.11: Two More Direction Fields

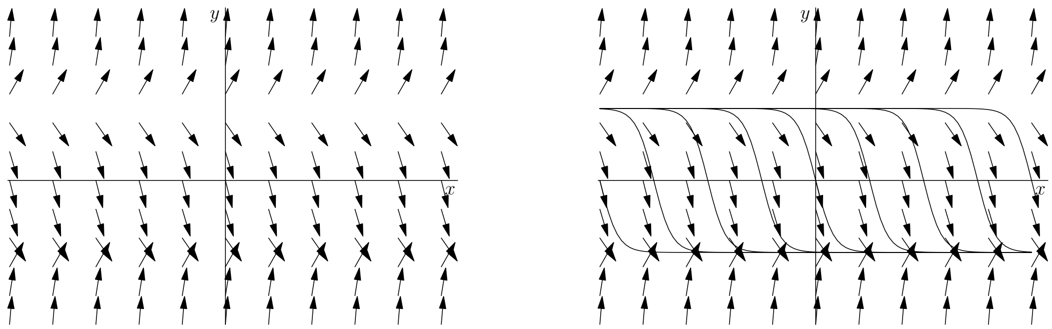
In theory, we should get two constants of integration, one from each side. However, we can move the left side constant to the ride side and have the difference of two arbitrary constants, which is equivalent to one arbitrary constant. Therefore, we will only write on constant of integration for seperable equations.

In general, mathematicians have a practice of being somewhat careless with this constant. Since it doesn't need to be determined until we use an initial condition, we often forgoe various operations on the constant. For example, if we had $2(t+c)$, we would often simplify this to $2t+c$, since whether we figure out the constant from c or from $2c$ later, its value is still determined by the initial condition. It's useful to become accustomed to this carelessness constants of integration.

Example 2.4.2.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x}{y} \\
ydy &= \sin x dx \\
\frac{y^2}{2} &= -\cos x + c \\
y &= \pm\sqrt{c - 2\cos x}
\end{aligned}$$

It is interesting to note where the constant of integration ends up. Since integration isn't the final step (we have to also solve for y), the constant moves around in the resulting algebra. In DE, we can't just add $+c$ at the very end of the process.



(a) The Slope Directions

(b) With Solutions

Figure 2.12: The Direction Field for $\frac{dy}{dx} = y^2 - 4$.

If we impose an initial condition of $y(0) = 1$, we can determined the value of the constant of integration.

$$\begin{aligned}
&1 = \sqrt{c - 2\cos(0)} = \sqrt{c-2} \\
&1 = c - 2 \\
&c = 3 \\
&y = \sqrt{3 - 2\cos x}
\end{aligned}$$

Figure 2.11 shows the direction field and solution for this example. Notice the strange domain issues with this implicit plot. When $|c| \leq 2$, we have restricted domain solutions, represented by the closed curves. There are no solutions at all when $c < -2$. We have solutions with domain $\mathbb{R}$ only for $c \geq 2$. When $c = 2$, we get the strange crossed graph, which is not always differentiable. When $c = -2$. the "solution" is only defined at discrete points.

Example 2.4.3. This is an autonomous example.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 - 4 \\
\int \frac{1}{y^2 - 4} dy &= \int 1 dx \\
\frac{-1}{2} \operatorname{arctanh}\left(\frac{y}{2}\right) &= x + c \\
\operatorname{arctanh}\left(\frac{y}{2}\right) &= -2x + c \\
\frac{y}{2} &= \tanh(-2x + c) \\
y &= 2\tanh(-2x + c)
\end{aligned}$$

Since this is an autonomous equation, we can look for steady (constant) singular solutions when the right side of the equation vanishes. Here, $y = 2$ and $y = -2$ are steady states. Moreover, $y = 2$ is stable and $y = -2$ is unstable. This can be seen in the direction field and implicit plot in Figure 2.12

It is also interesting to note that the output of tanh is only -1 to 1, so it is impossible to get $y \leq -2$ or $y \geq 2$. We should wonder if there are solutions in this range at all. In the implicit plot, we could draw curves with these y values. These other curves are found by doing the integral differently, since both hyperbolic tangent and hyperbolic cotangent have the same anti-derivative. $y = 2\coth(-2x+c)$ is also a solution.

2.5 Existence and Uniqueness of Solutions

Before moving on with other techniques for solving first order equations, this is a nice place to take a pure-mathematical detour and talk about existence and uniqueness of solutions. We're going to deal with first order equations where we can isolate the derivative term, that is, equations of the form

$$\frac{dy}{dx} = F(x,y).$$

The details of existence and uniqueness theorems rely on the properties of F as a function of two variable. In this treatment, we think of y and x as two unrelated, independent variables, even though the DE itself implies that y is a function of x.

2.5.1 Existence

Existence of solutions to first order DES is established by the Peano Existence Theorem.

Theorem 2.5.1. *In a equation of this form above, if F is continuous in both variables in an open set $U \subset \mathbb{R}^2$ and if $(x_0, y_0) \in U$, then there exists $\epsilon > 0$ such that the Initial Value Problem associated to the DE and the initial condition $y(x_0) = y_0$ has a solution with domain at least $[x_0 - \epsilon, x_0 + \epsilon]$.*

This is a very local result: the small positive ϵ only guarantees a tiny piece of a function as a solution. Existence (and later uniqueness) are only guaranteed very close to the initial value of the function. We do know that the function is differentiable in this small inteval, but we don't know anything else: outside the interval, anything could happen with the solution.

It's also useful to note, particularly for students with experience in other senior mathematics classes, that this theorem relies on topological considerations: we need an open subset U where F is continuous.

Example 2.5.2.

$$y' = y^{\frac{2}{3}} \qquad\qquad y(0) = 0$$

This satisfies the conditions of the theorem, so a solutions exists. However, there are two solutions: $y = 0$ and $y = x^3/27$. Peano's theorem doesn't imply the uniqueness of a solution.

Example 2.5.3.

$$y' = \sqrt{|y|} \qquad\qquad y(0) = 0$$

This F is also continuous near $(0, 0)$, so a solution exists by Peano's theorm. This IVP is solved by $y = 0$ and $y = x^2/4$. It's obvious that we need something more to ensure uniqueness.

Before we move on to the next result, we could ask for the proof of this theorem. Unforunately, that proof doesn't fall within the scope of this course. We would need to establish a number of new definitions and techniques from real analysis, as well as struggle through a bunch of tricky ϵ and δ arguments. It's interesting stuff, but very challenging and therefore left aside for this course.

2.5.2 Lipschitz Continuity

In order to state the theorem about uniqueness, we need a new definition of continuity.

Definition 2.5.4. A function from an open set U in $\mathbb{R}$ to $\mathbb{R}$ is called *Lipschitz continuous* if $\exists K > 0$ such that $\forall x_1, x_2 \in U$ we have $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$.

This is a strange kind of continuity. The definition is stronger than normal: Lipschitz continuity implies normal continuity. Moreover, the definition is global over U, not just local like conventional continuity. Therefore, the domain matters: a function might be Lipschitz continuous on a small set, but not on a larger one.

As an interpretation, Lipschitz continuity is a control on the global growth over the designated set. For a Lipschitz continuous function, there is a linear function that bounds the function on the designated set. Unsurprisingly, this means that the defintion usually only works on bounded sets. As a visualization, we can think of the definition as comparing f to a function $g(x) = K|x|$. The graph of g gives a cone in $\mathbb{R}^2$ and the f must stay inside this cone to be Lipschitz continuous. In this way, the definition limits the local growth of f and its derivatives.

Example 2.5.5. Here are some examples to illustrate the idea of Lipschitz continuity.

- $f(x) = x$ is Lipschitz continuous for $K = 1$ on any interval, since it is bounded by itself, a linear function.
- $f(x) = x^2$ is Lipschitz continuous on any *bounded* interval. Specifically, on $(-7, 7)$ we can take $K = 14$. However, $f(x) = x^2$ is not Lipschitz continuous on all of $\mathbb{R}$, since no linear function bounds it.
- $f(x) = x^{\frac{2}{3}}$ is not Lipschitz continuous on $(-1, 1)$, since its slope gets arbitrarily steep near the origin. This means that very close to 0, it cannot be bounded by a linear function through $(0, 0)$.

The previous example was not Lipschitz continuous at 0, and it also failed to be differentiable at that point. We might wonder if differentiability is a sufficient condition. That would be convenient, since we know how to check for differentiability. However, consider a strange example.

Example 2.5.6.

$$f(X) = \begin{cases} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

This isn't Lipschitz continuous, but it is differentiable near zero, showing that differentiability isn't sufficient. However, there is some good news: this example is a strange aberation.

Proposition 2.5.7. *A function which is C^1 at a point a in its domain is also Lipschitz continuous at a.*

$f \in C^1$ is roughly equivalent to saying that $\frac{\partial f}{\partial y}$ must exist and be bounded. This last criterion is the one we will use in practice: we check if the derivatives exists and if it is bounded.

Consider the $f(x) = x^2$. The derivaitve is $2x$, which is bounded locally near 0. $f(x) = x^{\frac{2}{3}}$ has derivative $\frac{2}{3}x^{\frac{-1}{3}}$ which is unbounded near 0. The former function is Lipschitz continuous and the later is not.

2.5.3 Uniqueness

The theorem for Uniqueness is called the Picard-Lindelöf thereom. It is a small improvement and adjustment of the Peano existence theorem.

Theorem 2.5.8. *If F is Lipschitz continuous in y and continuous is x on an open set U in $\mathbb{R}^2$ and if $(x_0, y_0) \in U$ then the initial value problem*

$$\frac{dy}{dx} = F(x,y) \qquad\qquad y(x_0) = y_0$$

has a unique solution $y = f(x)$ defined on the domain $[x_0 - \epsilon, x_0 + \epsilon]$ for some $\epsilon > 0$.

All of the comments from the previous theorem apply here. The result is very local, relies on topology, and the proof is beyond the scope of the course.

Example 2.5.9.

$$\frac{dy}{dx} = x\sqrt{y}$$

We considered this differential equation earlier. It had multiple solutions for initial value $y(0) = 0$. The function $F(x,y) = x\sqrt{y}$ has y deriavtive $\frac{\partial F}{\partial y} = \frac{x}{2\sqrt{y}}$, which is unbounded near 0. Therefore, it is not Lipschitz continuous and doesn't satisfy the continious of the Picard-Lidenlöf theorem, meaning that we don't expect a unique solution.

Lastly, anticipating the next section, consider the general first-order linear DE where P and Q are continuous functions.

$$\frac{dy}{dx} = Q(x) - P(x)y$$

In this case, $F(x,y) = Q(x) - P(x)y$ and the y derivative is $-P(x)$, which is bounded assuming that F is continuous. We can then apply the Picard-Lindelöf theorem to conclude that linear equations have unique solutions whenever their coefficient functions P and Q are continuous.

2.6 Linear Equations and Integrating Factors

First order linear equation have the general form

$$a(t)\frac{dy}{dt} + b(t)y = f(t).$$

Here $a(t)$, $b(t)$ and $f(t)$ are various (usually continuous) functions. If we avoid the values where $a(t) = 0$, we can divide by $a(t)$ to isolate the derivative, giving the following more typical form. (Remember the denominators! We have to pay attention to the roots of $a(t)$ throughout the solution.)

$$\frac{dy}{dt} + P(t)y = Q(t)$$

2.6.1 Homogeneous Solutions

The linear DE is called homogeneous if $Q(t) = 0$. In the homogeneous case, the DE is relatively easy to solve as a seperable equation.

$$\begin{aligned}
\frac{dy}{dt} &= -P(t)y \\
\frac{1}{y}dy &= -P(t)dt \\
\int \frac{1}{y}dy &= -\int P(t)dt \\
\ln|y| &= -\int P(t)dt + c \\
y &= ce^{-\int P(t)}dt
\end{aligned}$$

I should make a couple of notes about this calculation.

- We are informal with the constant. When we take exponents of each side of the equation, we should have multiplication by e^c. However, since this is still an undetermined constant, we simply write c instead of e^c. Also, when we drop the absolute value bars from y, we should have a $\pm$ term. Again, since c can be either positive or negative, we don't worry about that $\pm$.
- We had a y in the denominator for this process, which means that we have to be careful at points where $y = 0$. We use limits to figure out behaviour when y gets close to zero.
- If P is constant, this is our most basic exponential equation with solution $y = ce^{-\alpha t}$.

2.6.2 Linear Operators and Superposition

Recall from Definition 2.1.4 the idea of linear operators. If we write $L = \frac{d}{dt} + P(t)$, we can write a second order DE as

$$Ly = Q(t).$$

This is a *linear* differential operator, so it behaves linearly, that is, it has the two linearity properties.

$$\begin{aligned} L(y_1 + y_2) &= Ly_1 + Ly_2 \\ L(cy) &= cL(y) \end{aligned}$$

Then consider both the homoegenous equation $Ly = 0$ and a non-homogenous equation $Ly = Q(t)$. If f is a solution to the non-homogenous equation, then $Lf = Q(t)$, and if g is a solution to the homogenous equation, then $Lg = 0$. By linearity, we can conclude that

$$L(f + \alpha g) = Lf + \alpha Lg = Q(t) + \alpha \cdot 0 = Q(t).$$

Definition 2.6.1. If we have a solution to $Ly = Q(t)$, we can get other solutions by adding multiples of the solution to the homogeneous equation $Ly = 0$. The solution to $Ly = Q(t)$ is called the *particular solution* and this process is called *superposition of solutions.*

For those who remember their linear algebra, we can think of the solutions of a linear equation as a linear subspace of the vector space of differentiable functions (on an appropriate domain). In this sense the solution set of a linear equation is an offset span: the non-homogenous solution is the offset and the basis of the span is any homogenous solution. Superposition gives us the family-structure of solutions to linear equations. Any multiple of a homogenous solutions can be added to the particular solution, so the family is the particular solution plus any multiple of a homogeneous solutions: $y_p + \alpha y_h$. This α is the parameter of the family.

2.6.3 Integrating Factors

We know how to solve homogenous linear equations, since they are seperable. Finding non-homogenous solutions is somewhat more difficult; since these are not seperable equations, we need a new technique.

Let $L = \frac{d}{dt} + Q(t)$ be as before, and let y_h be a homogeneous solutions (a solution to $Ly = 0$). The technique we will use is called *variation of parameters.* This technique says we should look for a particular solution of the type $y_p = g(t)y_h$. Assuming that our solution has this special form, we have to try to find this $g(t)$. In order to do that, we put $g(t)y_h$ into the equation.

$$\begin{aligned} Ly_p(t) = Lg(t)y(t) &= Q(t) \\ g'(t)y(t) + g(t)y'(t) + g(t)P(t)y(t) &= Q(t) \\ g'(t)y(t) + g(t)\Big(y'(t) + P(t)y(t)\Big) = g'(t) + g(t)Ly &= Q(t) \\ g'(t)y(t) + g(t)0 &= Q(t) \\ g'(t) &= \frac{Q(t)}{y(t)} \\ g(t) &= \int \frac{Q(t)}{y(t)} \end{aligned}$$

We can put this back into our special form and use the fact that the homogeneous solution is $y = e^{-\int P(t)dt}$.

$$y_p = g(t)y = \left(\int \frac{Q(t)}{e^{-\int P(t)dt}}dt\right) e^{-\int P(t)dt}$$
$$y_p = e^{-\int P(t)dt}\int e^{\int P(t)dt}Q(t)dt$$

Definition 2.6.2. In solving a linear equation of the for $\frac{dy}{dt} + P(t)y = Q(t)$, the expression $e^{\int P(t)dt}$ is called an *integrating factor* and it is typically written $\mu(t)$.

We rearrange the expression using the integrating factor.

$$e^{\int P(t)y}y_p = \int e^{\int P(t)dt}Q(t)dt$$
$$\frac{d}{dt}\left(e^{\int P(t)dt}y_p\right) = e^{\int P(t)dt}Q(t)$$
$$\frac{d}{dt}(\mu(t)y_p(t)) = \mu(t)Q(t)$$
$$y_p = \frac{\int \mu(t)Q(t)dt + c}{\mu(t)}$$

Multiplying by the integrating factor turns the $y_p'(t) + P(t)y_p(t)$ side of the equation into a product rule derivative, which we can just integrate to solve. The integrating factor turns a linear equation into something that can be directly solved by integration, hence the name. It is best to remember the process this way: the original DE becomes a product rule derivative problem by multiplying both sides of the original DE by the integrating factor and then isolating y_p.

This variation of parameters is the first instance of a very common approach to solving DEs. Often, working with no idea of what kind of function we are looking for is simply too open-ended and too difficult. Therefore, we make a reasonable guess regarding the form of the solution. Here, that guess was $y_p = g(t)y_h$ where $g(t)$ was some unknown function. Then we put out special form back into the DE and try to find specific information about the parameter or unknown involved.

2.6.4 Examples

Example 2.6.3.

$$\frac{dy}{dt} + \frac{y}{t} = 2e^t$$

Since there is a t in the denominator, we must avoid $t = 0$ in the domain of solutions. We first look for the homogeneous solution.

$$y_h(t) = ce^{-\int \frac{1}{t}dt} = ce^{-\ln|t|} = \frac{c}{t}$$

Then the integrating factor is

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln |t|} = t.$$

We use the integrating factor to get a new DE.

$$\begin{aligned}
\mu(t)\frac{dy}{dt} + \mu(t)\frac{y}{t} &= \mu(t)2e^t \\
t\frac{dy}{dt} + t\frac{y}{t} &= t2e^t \\
\frac{d}{dt}(ty) &= 2te^t \\
ty &= \int (2te^t)dt \\
ty &= 2(te^t - e^t) + c \\
y &= 2\left(e^t - \frac{e^t}{t}\right) + \frac{c}{t}
\end{aligned}$$

Notice that we actually get the the homogeneous pieces here from the constant of integration, getting the whole linear family. Also notice that $t = 0$ is excluded from the domain.

Example 2.6.4.

$$(t^2 - 9)\frac{dy}{dt} + ty = 0$$

This is just a homogeneous DE. We note that $t \neq \pm 3$ in the domain. The solution to the homogeneous case is

$$y = e^{-\int P(t)dt} = e^{-\int \frac{t}{t^2-9} dt} = e^{-\frac{1}{2}\ln|t^2-9|+c} = \frac{c}{\sqrt{t^2-9}}.$$

We should be careful with the absolute value in $\ln |f(t)|$. The calculation can be done in two pieces, when $f(t) > 0$ and when $f(t) < 0$. In the previous calculation, we were a little careless with this detail.

Example 2.6.5.

$$\frac{dy}{dt} + y = f(t)$$

To add some complication, the non-homogeneous function here is going to be a step function.

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

This is still allowed: often the coefficients and functions involved in DEs are only piecewise continuous and/or piecewise differentiable. We can still work with them. The integrating factor is $\mu(t) = e^{\int 1 dt} = e^t$. We have to work with two different intervals. First look at $[0, 1]$.

$$\frac{d}{dt}e^t y = e^t \implies e^t y = e^t + c_1 \implies y = 1 + c_1 e^{-t}$$

Alternatively, look at $(1, \infty)$.

$$\frac{d}{dt}e^t y = 0 \implies e^t y = t + c_2 \implies y = c_2 e^{-t}$$

We have two constants, but we want a continuous solution. (It will fail to be differentiable at $t = 1$, but that's alright. There is a sudden change in the situation, so that's expected.) Continuity at 1 means that $e - c_1 = c_2$. The final solution is a continuous piecewise function.

$$y = \begin{cases} 1 - ce^{-t} & t \in [0, 1] \\ (e - c)e^{-t} & t \in (1, \infty) \end{cases}$$

Even with this piecewise function, we can still do initial value problems. If $y(0) = 0$, we find that $c = 1$ and we get a specific solution.

2.7 Substitutions

The third and final technique we will study for first order DEs is substitution. At some level, solving DEs is a more complicated and involved version of doing integrals, since we are trying to undo the results of differentiation. Substitution is the most common and important technique for solving integrals. It takes complicated integrals and changes their setup to make them more approachable. The technique is exactly the same here: we change the DE with some substitution operation to turn it into something we already know how to do. As with substitution for integrals, we can recognize some typical forms, but many others require creativity and ingenuity to solve. In this section, we'll introduce two forms which use substitution.

2.7.1 Homogeneous DEs

The first substitution is for a class of terribly named DEs called homogenous equations. Please note, this homogeneous has *nothing* to do with the previous definition for linear equations. The word here comes from a different use of the term in algebra. In any case, a homogenous DE has the form

$$\frac{dy}{dt} = f\left(\frac{y}{t}\right).$$

The substitution is the relatively obvious replacement $v = \frac{y}{t}$. The right side of the equation easily turns into $f(v)$, but the transformation of the left side is trickier.

$$\begin{aligned} y &= tv \\ f(v) = \frac{dy}{dt} &= v + t\frac{dv}{dt} \\ \frac{dv}{dt} &= \frac{f(v) - v}{t} \\ \int \frac{dv}{f(v) - v} &= \int \frac{dt}{t} \end{aligned}$$

In the last step, we've already started to solve the homogeneous equation as a seperable equation after the substitution is complete.

Example 2.7.1. This example is from Roberts and Marion. It is technically a seperable equation, but it is still useful to see how the substitution technique works.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{t}{y} \\
v &= \frac{y}{t} \\
f(v) &= \frac{1}{v} \\
\frac{dv}{dt} &= \frac{\frac{1}{v} - v}{t} \\
\int \frac{dv}{\frac{1}{v} - v} &= \int \frac{dt}{t} \\
\int \frac{v}{1 - v^2} dv &= \ln|t| + c \\
\frac{-1}{2} \ln|1 - v^2| &= \ln|t| + C \\
\frac{1}{\sqrt{1 - v^2}} &= ct \\
\sqrt{1 - v^2} &= \frac{c}{t} \\
1 - v^2 &= \frac{c}{t^2} \\
v^2 &= 1 - \frac{c}{t^2} \\
v &= \pm\sqrt{1 - \frac{c}{t^2}} \\
\frac{y}{t} &= \pm\sqrt{1 - \frac{c}{t^2}} \\
y &= \pm t\sqrt{1 - \frac{c}{t^2}} = \pm\sqrt{t^2 - c}
\end{aligned}$$

Notice that we reverse the substitution at the end, so that we end with the same variables as we started with.

Example 2.7.2.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{-y^2 - yt}{t^2} = \frac{-y^2}{t^2} - \frac{y}{t} \\
v &= \frac{y}{t} \\
f(v) &= -v^2 - v \\
\frac{dv}{dt} &= \frac{f(v) - v}{t} = \frac{-v^2 - v - v}{t} = \frac{-v^2 - 2v}{t} \\
\int \frac{dv}{-v^2 - 2v} &= \int \frac{dt}{t} = \ln|t| + c \\
\frac{1}{-v^2 - 2v} &= \frac{-1}{v(2+v)} = \frac{1}{2}\frac{1}{v+2} + \frac{-1}{2}\frac{1}{v} && \text{(Partial Fractions)} \\
\frac{1}{2}\int \frac{dv}{v+2} + \frac{-1}{2}\int \frac{dv}{v} &= \ln|t| + c \\
\frac{1}{2}\ln|v+2| + \frac{-1}{2}\ln|v| &= \ln|t| + c \\
\ln|v+2|^{\frac{1}{2}} + \ln|v|^{\frac{-1}{2}} &= \ln|t| + c \\
\sqrt{1 + \frac{2}{v}} &= ct \\
\frac{1+\sqrt{2}}{v} &= ct \\
1 + \frac{2}{v} &= ct^2 \\
\frac{2}{v} &= ct^2 - 1 \\
v &= \frac{2}{ct^2 - 1} \\
\frac{y}{t} &= \frac{2}{ct^2 - 1} \\
y &= \frac{2t}{ct^2 - 1}
\end{aligned}$$

2.7.2 Bernoulli Equations

The other substitution is for a class of DEs called Bernoulli equations.

These equations are almost linear, but they have an extra y^n term. The most general form is

$$\frac{dy}{dt} + P(t)y = Q(t)y^n.$$

The substitution is $v = y^{1-n}$. Be careful: this is y^{1-n}, not y^{n-1}. It's a very easy mistake to get this exponent wrong. We transform the DE by looking at the following derivations.

$$\begin{aligned}
\frac{dv}{dt} &= (1-n)y^{1-n-1}\frac{dy}{dt} = (1-n)y^{-n}\frac{dy}{dt} \\
&= (1-n)y^{-n}\left(Q(t)y^n - P(t)y\right) = (1-n)Q(t) - (1-n)P(t)y^{1-n} \\
&= (1-n)Q(t) - (1-n)P(t)v \\
\frac{dv}{dt} + (1-n)vP(t) &= (1-n)Q(t)
\end{aligned}$$

This is now a linear equation in v. We can solve it as a linear equation in v and then use the reverse substitution to get back to y.

Example 2.7.3. This example is from Roberts and Marion.

$$\begin{aligned}
\frac{dy}{dt} - \frac{1}{2}\frac{y}{t} &= -e^t y^3 \\
n &= 3 \\
v &= y^{-2} \\
\frac{dv}{dt} - (-2)v\frac{1}{2t} &= -2(-e^t) = 2e^t \\
\frac{dv}{dt} + \frac{v}{t} &= 2e^t
\end{aligned}$$

This is a familiar linear equation which was solved in Example 2.6.3. We recall the solution from before.

$$\begin{aligned}
v &= 2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t} \\
y^{-2} &= 2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t} \\
y &= \left(2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t}\right)^{\frac{-1}{2}} \\
y &= \frac{\pm 1}{\sqrt{2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t}}}
\end{aligned}$$

Example 2.7.4.

$$
\begin{aligned}
\frac{dy}{dt} &= y(ty^3 - 1) = ty^4 - y \\
\frac{dy}{dt} + y &= ty^4 \\
v &= y^{-3} \\
\frac{dv}{dt} - 3v &= -3t \\
P(t) &= -3 \\
Q(t) &= -3t \\
\mu(t) &= e^{\int P(t)dt} = e^{-3t} \\
\frac{d}{dt} e^{-3t} v &= -3te^{-3t} \\
e^{-3t} v &= \int -3te^{-3t} dt = -3 \left(\frac{te^{-3t}}{-3} - \int \frac{e^{-3t} - 3}{d} t \right) \\
e^{-3t} v &= te^{-3t} - \int e^{-3t} dt = te^{-3t} + \frac{e^{-3t}}{3} + c \\
v &= t + \frac{1}{3} + ce^{3t} = \frac{3t + 1 + ce^{3t}}{3} \\
y^{-3} &= \frac{3t + 1 + ce^{3t}}{3} \\
y^3 &= \frac{3}{3t + 1 + ce^{3t}} \\
y &= \sqrt[3]{\frac{3}{3t + 1 + ce^{3t}}}
\end{aligned}
$$

2.8 Approximation Methods and Applications for First Order Equations

As this point, many DE courses might include a section on approximation methods or a section on applications, or both. We will be not spending any time on either of these.

The applications are useful for motivation. The approximation methods are interesting due to the fact, observed previously, that many DEs are terribly difficult to solve. The techniques we included above for first order equations cover only a small portion of all the possible equations and even then, we have to rely on integrals for seperable and linear equations. Integrals themselves are difficult and often only possible up to approximation. Therefore, a huge part of the mathematics of differential equations is the study of approximation techniques. Getting a sense of how these approximations are set up is an important insight into the field.

Chapter 3

Second Order Linear Differential Equations

A second order (ordinary) differential equation involves the second derivative of a function of one variable. In full generality, this could be any ridiculously complicated expression involving the independent variable, the function, its first derivative, and its second derivative.

Example 3.0.1.

$$\left(\frac{d^2 f}{dt^2}\right)^2 + \frac{\frac{df}{dt} - t^2 f^2}{\frac{df}{dt} - \frac{d^2 f}{dt^2}} = \frac{t^2 - \sin t}{f}$$

We have basiclaly no idea and no hope for approaching such complicated general cases. In this chapter, even more than with first order equations, we will restrict to very particular cases where there are useful techniques of solutions. The general restriction will be to linear equations, but even inside that restriction, we will make further assumptions about the structure.

As with first order equations, we solve initial value problems. However, since we have a second derivative involved, we expect (at least implicitly) to have to integrate twice. This results in two constants of integration. Therefore, we will require two initial values. Typically, for an equation about a function $f(t)$, there will be an initial value for the function $f(t_0) = a$ and an initial value for the derivative $f'(t_1) = b$. In most cases, a specific solution can only be found if both initial values are specified.

3.1 Linear DEs with Constant Coefficients

Definition 3.1.1. A *second order linear differential equation* is an equation of the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t).$$

As with the first order case, this is called homogeneous if $f(t) = 0$.

Definition 3.1.2. A *second order linear differential operator* is an operator of the form.

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

In operator notation, a second order linear differential equation is $Ly = f(t)$. The principles of linearity and superposition defined for first order equation in Section 2.6 still hold for higher-order linear equations.

Even though we look to linear equations for solvable second order DEs, linear equations are still very tricky, particularly if the coefficient functions are complicated. We need to restrict even further, to the simplest case where the coefficient functions are constant.

Definition 3.1.3. A *second order constant coefficient linear operator* has the form

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c.$$

The corresponding homogeneous equation is $Ly = 0$ and the corresponding non-homogeneous equation is $Ly = f(t)$. To avoid writing a lengthy name over and over again, I'll write SOCCLDE for a second order constant coefficient linear differential equation.

Second order constant coefficient linear differential equations, even with their very restricted form, are quite important for applied mathematics. They give models to understand both harmonic motion and alternating current electric curcuits. The harmonic motion interpretation is our starting point.

3.1.1 Harmonic Motion

Hooke's Law for a position function $x(t)$ in a harmonic system is the equation $F = -kx(t)$. We interpret this equation in terms of Newton's law of motion.

$$\begin{aligned} F &= -kx(t) \\ ma(t) + kx(t) &= 0 \\ m\frac{d^2x}{dt^2} + kx(t) &= 0 \end{aligned}$$

This gives us a homogeneous SOCCLDE with $a = m$, $b = 0$ and $c = k$. This is the simplest possible spring and there are two solutions.

$$x_1(t) = \sin\left(\sqrt{\frac{k}{m}}x\right) \qquad x_2(t) = \cos\left(\sqrt{\frac{k}{m}}x\right)$$

Since we can solve the homogeneous case, superposition applies and we can get a general solution.

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Note there are two parameters and the solution is a *linear combination* of the two independent solutions. That's expected for a second order equation. We expect two linearly independent solutiolns (neither is a constant multiple of the other) and the full family of homogeneous solutions is all linear combinations of the two solutions.

As an aside: the sums of sin and cos can be expressed as a single sinusoidal wave. We are helped by a relatively obscure but useful identity (if $a \geq 0$).

$$a\sin t + b\cos t = \sqrt{a^2 + b^2}\sin(t + \phi) \qquad \text{where} \qquad \phi = \arcsin\left(\frac{b}{\sqrt{a^2 + b^2}}\right)$$

This identity lets us see linear combinations of sine and cosine as a single wave. In particular, the amplitude of a linear combination is the pythagorian combination $\sqrt{a^2 + b^2}$ of the original amplitudes and frequency is altered, though the wave is shifted.

Hooke's Law is a nice starting point for understanding harmonic motion. However, these SOCCLDEs have no b term. We can wonder: what should b be? This b is a coefficient of the first derivative, the velocity of the system. Velocity causes friction in the system, and the greater the velocity the greater the friction. Hooke's law was an idealization which ignored friction; by adding in a non-zero b coefficient, we account for friction in the harmonic system. Again, we set up Hooke's law as a force quation using Newton's laws of motion.

$$F = -kx(t) - b\frac{dx}{dt}$$

$$ma(t) + b\frac{dx}{dt} + kx(t) = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0$$

Harmonic systems with friction are called *damped harmonic systems*. What do we expect out of such systems? The undamped system is a sine or cosine wave, which just oscillates forever. We expect that the friction should cause the oscillations to eventually slow down. Therefore, we expect decay in the amplitude of the sine way.

Now consider the non-homogeneous system. If we have $L = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$ where m is mass, b is the coefficient of friction and k is the spring constant, then what is the interpretation of $f(t)$ in $Lx(t) = f(t)$? The term $Lx(t)$ represents the force, since we derived it using Newton's law. Therefore, this must be an equation of forces and $f(t)$ must be some external force on the system. The homogenous equation gives the behaviour of the system on its own, and the non-homogenous equation add the complication of external forces.

3.1.2 Alternating Current Circuits

The second major interpretations of SOCCLDEs is as alternating current circuits. The DE that will result is exactly the same, but the interpretation of each of the coefficients is quite different. Instead of a position function $x(t)$, we will have a charge function $q(t)$ and the movement of that charge will constitute current. For our purposes, we will have four components to a circuit: resistors, capacitors, inductors and an external electro-motive force. Let's quickly given an account.

- Resistors allow for energy leaving the system and they represent the resistance to energy flow. The resistance is written R and measured in ohms. They act like friction in the mechanical system in that they want to slow down the flow of current. They will result in a decrease in current over time if there is no external forcing. Resistors represent the devices powered by the circuit, whatever those devices are.
- Capacitors are storage devices for electrical energy in electric fields. They have a measurement c called capacitance, which has units of farads (coloumbs per volt). They stabilize alternating current flow; as such, they can be see as controlling the natural way in which current flows. They (up to a reciprocal) align with the spring constant, which controlled the natural behaviour of the harmonic system (before friction and external forces).
- Inductors are storage devices for electrical energy in a magnetic field. They have a measurement L called inductance, which has units of henrys. Inductors block alternating current; as such, they represent the difficulty of moving charge through the system. In the harmonic system, the difficult of moving the object was its mass. Inductance takes the place of the mass term.
- Electromotive forces are external forces to the system, from batteries or generators. They are written $E(t)$ and have units of volts. Like the forces that add movement to a harmonic system, these electromotive forces add charge to a circuit.
- To give a complete account of the units, charge is written q and measured in coloumbs. The movement of charge is current, represented by the derivative $\frac{dq}{dt}$.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E(t)$$

3.1.3 Solving SOCCLDEs

Now that we have the two classic interpretations in place, we need to move on to actually solving SOCCLDEs. We'll use the operator

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c.$$

In the homogeneous constant-coefficient first order case, $\frac{dy}{dt} = at$, the solution was exponential $y = e^{at}$. We look for the same kind of solution here by assuming that $Ly = 0$ has solution $y = e^{rt}$. We put this potential solution in the DE.

$$\begin{aligned} \frac{d}{dt}e^{rt} &= re^{rt} \\ \frac{d^2}{dt^2}e^{rt} &= r^2e^{rt} \\ Le^{rt} &= ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) \end{aligned}$$

If $Ly = 0$, then this equation is only satisfied if $ar^2 + br + c = 0$.

Definition 3.1.4. This equation is called the *characteristic equation* of the SOCCLDE.

The characteristic equation has two roots (with a possible repeated root).

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall the term $b^2 - 4ac$ is called the discriminant. In terms of the discriminat, there are three cases. We are making use here of the fundamental theorem of arithmetic, which says that polynoials always have roots over $\mathbb{C}$. See Section 1.5 for reference.

(a) $b^2 - 4ac > 0 \implies$ two real roots.
(b) $b^2 - 4ac = 0 \implies$ one real repeated root.
(c) $b^2 - 4ac < 0 \implies$ two complex roots.

3.1.4 Case 1: Two Real Roots

In this case, the solutions are normal exponential functions $e^{r_1 t}$ and $e^{r_1 t}$. Since the roots are distinct, these are linearly independent solutions (neither exponential function is a multiple of the other). We can simply check that they satisfy the DE.

$$Le^{rt} = e^{rt}(ar^2 + br + c) = 0$$

Since we expect two linearly indepedent solutions to a linear equation, the are no other solutions. The general solution is a superposition (linear combination) of the two solutions.

$$y = Ae^{r_1 t} + Be^{r_2 t}$$

The behaviour is this case exponential. If a, b and c are positive (as they are in both the major interpretations), then the roots are both negative, which means that both solutions are exponential decay. If we are modelling harmonic motion, the harmonic system decays to equilibrium with no oscillations. The discriminant condition is $b^2 > 4ac$ and recall that b was the coefficient of friction. The case of two real roots only happens if b is large enough. Exponential decay is the behaviour that results from a surplus of friction. There is too much friction even to have oscillations: we only have exponential decay to equilibrium. We called these situations *overamped* harmonic systems.

Example 3.1.5. We start with a classic example.

$$y'' - y = 0$$

The characteristic equation is is $r^2 - 1 = 0$ which has roots $r = \pm 1$. The solutions are $y = e^t$ and $y = e^{-t}$. The general solution is $y = Ae^t + Be^{-t}$. If we have initial conditions of $y(0) = 1$ and $y'(0) = 0$ then we get a system of equations for A and B: $A + B = 1$ and $A - B = 0$, which is solved by $A = B = \frac{1}{2}$. The final solution is $\frac{1}{2}e^t + \frac{1}{2}e^{-t}$. We should notice that this is just $\cosh t$, and then realize that we should have predicted this solution. The DE asks: what function returns to itself after two derivatives? Only the hyperbolics have this behaviour.

We could have made different choices for the initial linearly independent solutions by taking $y_1 = \cosh t$ and $y_2 = \sinh t$. The linearly independent solutions are not unique. For those who know linear algebra, these functions form a basis for the space of solutions and it is well known that a linear space has many different bases.

3.1.5 Case 2: Repeated Real Roots

If the characteristic equation factors as $(x - r)^2$, then r is a repeated real root (with value $\frac{-b}{2a}$). This gives us only one exponential solution: e^{rt}. This is a problem, since we expect two linearly independent solutions.

The solution to this problem is a bit strange, but turns out to be a common trick for linear equations: multiply by the independent variable. The second solution is te^{rt}.

$$\begin{aligned}
y &= te^{rt} \\
y' &= rte^{rt} + e^{rt} \\
y'' &= r^2te^{rt} + 2re^{rt} \\
ay'' + ay' + yc &= a(r^2te^{rt} + 2re^{rt}) + b(rte^{rt} + e^{rt}) + cte^{rt} \\
&= e^{rt}(ar^2t + 2ar + brt + b + ct) \\
&= e^{rt}(t(ar^2 + br + c) + 2ar + b) \\
&= e^{rt}(t \cdot 0 + 2a\frac{-b}{2a} + b) = 0 \qquad \text{Using } r = \frac{-b}{2a}
\end{aligned}$$

So we have two linearly indepedent solutions: $y_1 = e^{rt}$ and $y_2 = te^{rt}$. We already understand the first, since it is similar to the case of two distinct real roots. For harmonic systems, $r < 0$, the first solution is exponential decay and it corresponds to sufficient friction. The second solution is also exponential decay, since asymptotically the e^{-rt} dominates over t. However, it allows for one oscillation before decay.

The situation of repeated roots happens when $b^2 = 4ac$, so that the $\pm$ disappears from the solutions to the quadratic. For harmonic systems, this only happens if the friction is exactly as this special value $b = \sqrt{4ac}$. This is the exact friction that allows for this one oscillation before exponential decay. We call these systems *critically damped*. This is the tipping point for friction: there is exactly enough friction to result in exponential decay.

Example 3.1.6.

$$y'' - 2y' + y = 0$$

The characteristic equation is $r^2 = 2r + 1 = (r-1)^2$. The solutions are $y = e^t$ and $y = te^t$, so the general solution is $y = Ae^t + Bte^t$. If $y(0) = 1$ and $y'(0) = 0$ then we substitute into the equations to get $A = 1$ and $B = -1$ for a solution of $e^t - te^t$.

3.1.6 Case 3: Complex Roots

We have complex roots when $b^2 - 4ac < 0$. In that situation, we can factor $\imath$ out of the square root to get a real root.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \imath \frac{\sqrt{4ac - b^2}}{2a}$$

This gives a pair of complex numbers $x \pm y\imath$ with the real part $x = \frac{-b}{2a}$ and imaginary part $y = \frac{\sqrt{4ac-b^2}}{2a}$. They are a conjugate pair (each is the conjugate of the other). This is expected behaviour: the complex roots to a real quadratic always come as a conjugate pair. To write this more succinctly, we define two new constants.

$$\alpha = \frac{-b}{2a} \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

Then the complex roots can be written as $\alpha \pm \imath\beta$. The solutions to the DE are

$$e^{(\alpha \pm \imath\beta)t} = e^{\alpha t} e^{\imath\beta t}.$$

What do these complex functions means? The e^α term is fine, it is just a real exponential. The complex exponential is understood through Euler's formula (Proposition 1.5.5 in Section 1.5).

$$e^{\alpha t} e^{\pm \imath\beta t} = e^{\alpha t}(\cos(\beta t) \pm \imath \sin(\beta t))$$

These solutions are problematic because they involve complex numbers. We are trying to solve a real system with real coefficient and we want real solutions. To find them, we need to take clever linear combinations (over $\mathbb{C}$!) of the two solutions.

$$\frac{1}{2}e^{\alpha t}(\cos\beta t + \imath\sin\beta t) + \frac{1}{2}e^{\alpha t}(\cos\beta t - \imath\sin\beta t) = e^{\alpha t}\cos\beta t$$
$$\frac{1}{2\imath}e^{\alpha t}(\cos\beta t + \imath\sin\beta t) - \frac{1}{2\imath}e^{\alpha t}(\cos\beta t - \imath\sin\beta t) = e^{\alpha t}\sin\beta$$

On the basis of these linear combinations, we can take the following as our linearly independent real solutions.

$$y_1 = e^{\alpha t}\sin\beta t \qquad y_2 = e^{\alpha t}\cos\beta t$$

The general, real-valued solutions are linear combinations of the two linearly independent solutions.

$$y = Ae^{\alpha t}\sin\beta t + Be^{\alpha t}\cos\beta t$$

Example 3.1.7.

$$y'' + y = 0$$

The characteristic equation is $r^2 + 1$ which has roots $\pm\imath$. Therefore $\alpha = 0$ and $\beta = 1$, so the solutions are $\cos t$ and $\sin t$, with the general solution of $A\cos t + B\sin t$. If $y(0) = 1$ and $y'(0) = 0$, substitution into the equation gives $A = 1$ and $B = 0$ for $y = \cos t$ as the unique solution.

Example 3.1.8.

$$\begin{aligned}
y'' - 2y' + 5y &= 0 \\
r^2 + 2r + 5 &= 0 && \text{(Characteristic Equation)} \\
r &= \frac{2}{2} \pm \frac{\sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm \imath 2 \\
\alpha &= 1 \qquad \beta = 2 \\
y &= Ae^t \cos 2t + Be^t \sin 2t \\
y(0) &= 4 && \text{(Initial Conditions)} \\
y'(0) &= 6 \\
A &= 4 \qquad B = 1 \\
y &= 4e^t \cos 2t + e^t \sin 2t
\end{aligned}$$

Example 3.1.9.

$$\begin{aligned}
y'' + 3y' + 4y &= 0 \\
r^2 + 3r + 4 &= 0 && \text{(Characteristic Equation)} \\
r &= \frac{-3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{-3}{2} \pm \frac{\sqrt{-7}}{2} = \frac{-3}{2} \pm \imath\sqrt{7} \\
\alpha &= \frac{-3}{2} \qquad \beta = \sqrt{7} \\
y &= Ae^{-\frac{3t}{2}} \cos \sqrt{7}t + Be^{-\frac{3t}{2}} \sin \sqrt{7}t \\
y(0) &= 2 && \text{(Initial Conditions)} \\
y'(0) &= 2 \\
A &= 2 \qquad B = \frac{5}{\sqrt{7}} \\
y &= 2e^{-\frac{3t}{2}} \cos \sqrt{7}t + \frac{5}{\sqrt{7}} e^{-\frac{3t}{2}} \sin \sqrt{7}t
\end{aligned}$$

The previous example could be a harmonic system, since its coefficients are all positive. The results are sinusoidal functions with exponentially decaying amplitude. This fits our expectations for harmonic systems. We expect sinusoidal behaviour, but with decreasing amplitude. The complex roots happen when $b^2 < 4ac$, meaning that friction is small enough to allow for sinusoidal behaviour. We call such harmonic systems *underdamped*.

3.2 The Method of Undetermined Coefficients

Undetermined coefficients is the first of two methods we use to solve non-homogeneous SOCCLDEs. The second method, variation of parameters, is more general, but undetermined coefficients is easier and faster for particular types of non-homogemeous terms. In the spirit of applications to harmonic motion, I will often refer to the non-homogenous part of the SOCCLDE as a forcing term.

Recall, before we start, that if $Ly = f(t)$ is a non-homogeneous SOCCLDE, we know the structure of the general family of solutions. We expect two linearly independent solutions, y_1 and y_2, for the homogenous equaiton $Ly = 0$. We look for only one particular solution y_p of $Ly = f(t)$. The general solution has the form (where α and β are real parameter)

$$y = y_p + \alpha y_1 + \beta y_2.$$

Undetermined coefficents and variation of parameters try to find this particular solution y_p, and then we use the homogenous solutions to write the complete family.

The idea of undetermined coefficients is relatively simple: we try to guess a particular solution which is the same type of function as the forcing. If the forcing is polynomial, we look for a polynomial solution; if the forcing is exponential, we look for an exponential solution; and if the foring is sinusoidal, we look for a sinusoidal solution. Undertermined coefficients is going to work well for these three types of forcing terms or forcing terms given by products of functions of these types. However, it doesn't apply to other types of functions, where we can't reasonably expect the solution to look like the forcing term.

There is one possible pitfall to the process: sometimes the forcing terms is similar to the homogeneous solutions. In this case, the same type of function will only solve the homogeneous equation, not the non-homogeneous case. Our solution here is reminiscent of the case of repeated roots: we multiply by the independent variable t until we get something new. We'll see how this works out in examples.

Before we get into examples, here is a useful chart. As I said, we want to guess a similar type of functions to the forcing. What is similar? This chart gives the guesses. The constants C_i and D_j need to be determined: these constants are the undetermined coefficients which give the process its name.

$f(t)$	y_p
ke^{at}	Ce^{at}
kt^n	$C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0$
$k\cos(at)$ or $k\sin(at)$	$C\cos(at) + D\sin(at)$
$kt^n e^{at}$	$e^{at}\left(C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0\right)$
$kt^n\cos(at)$ or $kt^n\sin(at)$	$(C_n t^n + \cdots + C_0)\cos(at) + (D_n t^n + \cdots + D_0)\sin(at)$
$ke^{at}\cos(bt)$ or $ke^{at}\sin(bt)$	$e^{at}\left(C\cos(at) + D\sin(at)\right)$
$kt^n e^{at}\cos(bt)$ or $kt^n e^{at}\sin(bt)$	$(C_n t^n + \cdots + C_0)\, e^{at}\cos(bt) + (D_n t^n + \cdots + D_0)\, e^{bt}\sin(bt)$

With this chart, the process is simple. We find the right guess, put it into the DE, and try to work out the unknown coefficients. Let's see this in examples.

Example 3.2.1. This example is from Roberts and Marion.

$$y'' - 3y' + 2y = t \qquad y(0) = \frac{3}{4} \qquad y'(0) = \frac{3}{2}$$

$$L = \frac{d^2}{dt^2} + 3\frac{d}{dt} + 2$$

Solve the Homogeneous Case:

$$Ly = 0 \implies r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0 \implies r = 1, 2$$

$$y_h = Ae^{2t} + Be^t$$

Our Guess for Undetermined Coefficients:

$$y_p = Ct + D$$

$$y_p' = C$$

$$y_p'' = 0$$

Solve for the Coefficients:

$$Ly_p = 0 - 3C + 2(Ct + D) = t$$

$$2C = 1 \implies C = \frac{1}{2}$$

$$-3C + 2D = 0$$

$$-3\frac{1}{2} + 2D = 0 \implies D = \frac{3}{4}$$

$$y_p = \frac{t}{2} + \frac{3}{4}$$

$$y = \frac{t}{2} + \frac{3}{4} + Ae^{2t} + Be^t$$

Use the Initial Values:

$$y(0) = \frac{3}{4} \qquad y'(0) = \frac{3}{2}$$

$$y' = 2Ae^{2t} + Be^t + \frac{1}{2}$$

$$y(0) = A + B + \frac{3}{4} = \frac{3}{4} \implies A = -B$$

$$y'(0) = 2A + B + \frac{1}{2} = \frac{3}{2}$$

$$2A + B = 1$$

$$2A - B = 1 \implies A = 1 \implies B = -1$$

$$y = \frac{t}{2} + \frac{3}{4} + e^{2t} - e^t$$

Example 3.2.2. This example lacks initial values. I've skipped over some of the algebra for the sake of brevity.

$$
\begin{aligned}
y'' + 8y' + 3y &= e^{-2t}\cos 2t \\
L &= \frac{d^2}{dt^2} + 8\frac{d}{dt} + 3 \\
&\text{Solve the Homogeneous Case:} \\
Ly = 0 &\implies r^2 + 8r + 3 = 0 \\
r &= -4 \pm \sqrt{13} \\
y_h &= Ae^{(-4+\sqrt{13})t} + Be^{(-4-\sqrt{13})t} \\
&\text{Our Guess for Undetermined Coefficients:} \\
y_p &= e^{-2t}(C\cos 2t + D\sin 2t) \\
y' &= e^{-2t}((-2C+2D)\cos 2t + (-2D-2C)\sin 2t) \\
y'' &= e^{-2t}((-8D)\cos 2t + (8C)\sin 2t) \\
&\text{Solve for the Coefficients:} \\
Ly_p &= e^{-2t}((8D-13C)\cos 2t + (-8C-13D)\sin 2t) = e^{-2t}\cos 2t \\
8D - 13C &= 1 \\
-8C - 13D &= 0 \\
D &= \frac{-8C}{13} \\
C &= \frac{-13}{233} \implies D = \frac{8}{233} \\
y_p &= e^{-2t}\left(\frac{-13}{233}\cos 2t + \frac{8}{233}\sin 2t\right) \\
y &= Ae^{(-4+\sqrt{13})t} + Be^{(-4-\sqrt{13})t} + e^{-2t}\left(\frac{-13}{233}\cos 2t + \frac{8}{233}\sin 2t\right)
\end{aligned}
$$

It's useful to go back to our harmonic system interpretation to understand these solutions. There are no oscillations in the homogeneous case here: sufficient friction gives exponential decay. However, there is oscillating forcing, though the forcing is also undering going exponential decay. This forcing is enough to add sinusoidal behaviour to the full solutions, but the decaying forcing terms means that the sinusoidal term will also decay over time. The amplitude of this combination of waves is given by the pythagorean combination: $\frac{\sqrt{13^2+8^2}}{233} = \frac{1}{\sqrt{233}}$.

Example 3.2.3. In this example, the forcing term is similar to one of the homogeneous solutions, so you will notice that we have multiplied by t in our guess for the particular solution. Again, I've skipped more of the algebra in this example, particularly omitting the long derivatives and calculation of Ly_p.

$$
\begin{aligned}
y'' + 2y' + 2y &= f(t) = 4e^{-t}\sin t \\
&\text{Solve the Homogeneous Case:} \\
r^2 + 2r + 2 = 0 &\implies r = -1 \pm \imath \\
y_h &= e^{-t}(A\sin t + B\cos t) \\
&\text{Our Guess for Undetermined Coefficients:} \\
y_p &= te^{-t}(C\sin t + D\cos t) \\
y' &= e^{-t}(C\sin t + D\cos t) + te^{-t}((-C-D)\sin t + (-D+C)\cos t) \\
y'' &= e^{-t}((-2C-2D)\sin t + (-2D+2C)\cos t) + te^{-t}((2D)\sin t + (-2C)\cos t) \\
&\text{Solve for the Coefficients:} \\
Ly_p &= e^{-t}((-2D)\sin t + (2C)\cos t) = 4e^{-t}\sin t \\
-2D = 4 &\implies D = -2 \\
2C = 0 &\implies C = 0 \\
y_p &= -2te^{-t}\cos t \\
y &= -2te^{-t}\cos t + e^{-t}(A\sin t + B\cos t)
\end{aligned}
$$

In this result, we still have exponential decay, since the exponential is asymptotically dominant in the te^{-t} term. However, the trajectory and behaviour of the decay differs, particular for small t, from the homogeneous solutions.

3.3 Resonance

The discussion in the previous section about the similarity between forcing terms and the homogeneous solutions leads us into the subject of resonance in harmonic sequences. The question of resonance is this: is there a particular frequency for an external force on a harmonic system which produces the strongest effect?

This is an important question in a number of situations. In music and acoustics, we may want to design explicitly for resonance. In the safety of structures, we would like to ensure that resonant behaviour is impossible.

Let's start with the SOCCLDE describing an underdamped harmonic system. (Underdamped is necessary to allow for the possibility of resonance, as we will see in the calculations). Recall that for harmonic systems, the coefficients can be identified as mass m, spring constant k, coefficient of friction b and forcing $f(t)$.

$$my'' + by' + ky = f(t)$$

Look at the characterstic equation $mr^2 + br + k = 0$. It has solutions

$$r = \frac{-b}{2m} \pm \frac{\sqrt{b^2 - 4km}}{2m}.$$

We can define a new constant to keep track of the behaviour. This constant is called the damping constant.

$$\zeta := \frac{b}{2\sqrt{km}}$$

The damping constant gives a nice measure of the friction. If $\zeta < 1$ then the system is underdamped and we have sinusoidal behaviour. If $\zeta = 1$ the situation is critically damped and if $\zeta > 1$, the situation is overdamped; in both cases, we just have exponential decay. The frictionless case is $\zeta = 0$. Let's go back to the frictionless case for a moment. The solutions when $\zeta = 0$ are

$$y = A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t.$$

We're going to define another useful constant.

$$\omega := \sqrt{\frac{k}{m}}$$

This constant is called the natural frequency. It represents the frequency of sinusoidal oscillation in a perfect system without friction. (Note that all of the frequencies in this section are not true frequencies: they are off by a factor of $\frac{1}{2\pi}$. However, we'll ignore this fact and keep referring to them as frequencies.)

Finally, we're going to define one more constant.

$$\lambda := \frac{b}{2m}$$

If we look back to the underdamped case, the exponential decay term can be written $e^{-\lambda t}$. Therefore, λ is called the decay coefficient. With these new constants, the roots of the characteristic equation become $r = -\lambda + \pm\imath\sqrt{\omega^2 - \lambda^2}$ and we can rewrite the homogeneous differential equation as

$$y'' + 2\lambda y' + \omega^2 y = 0.$$

Now let's return to the idea of a sinusoidal forcing term $f(t) = F \sin \gamma t$ with some frequency γ. We want to look at four situation to understand the effect of this force and the possibility of resonance.

3.3.1 Situation 1: No Friction, No Forcing

This is the trivial base case, where the system just oscillates forever with frequency ω. This frequency is called the natural frequency because it describes the dynamics of this trivla base case, even if the ideal frictionless situation isn't particularly natural.

3.3.2 Situation 2: No Friction, Forcing

The differential equation currently has the form

$$y'' + \omega^2 y = F \sin \gamma t.$$

This is something that we can solve with undetermined coefficients. If we assume that $\gamma \neq \omega$, then the forcing term is unlike the homogeneous solutions. We use undetermined coefficients.

$$\begin{aligned}
y_p &= C \sin \gamma t + D \cos \gamma C \\
y_p' &= \gamma C \cos \gamma t - \gamma D \sin \gamma C \\
y_p'' &= -\gamma^2 C \sin \gamma t - \gamma^2 D \cos \gamma C \\
L y_p &= (-\gamma^2 C + \omega^2 C) \sin \gamma t + (-\gamma^2 D + \omega^2 D) \cos \gamma t = F \sin \gamma t \\
-\gamma^2 C + \omega^2 C = F &\implies C = \frac{F}{\omega^2 - \gamma^2} \\
-\gamma^2 D + \omega^2 D = 0 &\implies D = 0 \\
y_p &= \frac{F}{\omega^2 - \gamma^2} \sin \gamma t \\
y &= A \sin \omega t + B \cos \omega t + \frac{F}{\omega^2 - \gamma^2} \sin \gamma t
\end{aligned}$$

We see that the particular solution is a sine wave with amplitutde $\frac{F}{\omega^2-\gamma^2}$. The closer the forcing frequency is to the natural frequency, the greater the amplitude of the resulting oscillation. If we impose the initial conditions $y(0) = 0$ and $y'(0) = 0$, then the system is initially at rest and the only energy in the system comes from the external forcing. In this case, we easily get $B = 0$ from the first initial condition and we calculate for the second.

$$\begin{aligned}
y' &= \omega A \cos \omega t + \frac{\gamma F}{\omega^2 - \gamma^2} \cos \gamma t \\
y'(0) &= A\omega + \frac{\gamma F}{\omega^2 - \gamma^2} = 0 \implies A = \frac{-\gamma F}{\omega(\omega^2 - \gamma^2)} \\
y &= \frac{-\gamma F}{\omega(\omega^2 - \gamma^2)} \sin \omega t + \frac{F}{\omega^2 - \gamma^2} \sin \gamma t = \frac{-\gamma F \sin \omega t + \omega F \sin \gamma t}{\omega(\omega^2 - \gamma^2)}
\end{aligned}$$

Now, to answer the question of resonance, let's take the limit of this solution as $\gamma \to \omega$. The denominator is undefined, but the numerator also evaluates to 0, so we can apply L'Hôpital's rule to calculate the limit.

$$\begin{aligned}
\lim_{\gamma \to \omega} \frac{-\gamma F \sin \omega t + \omega F \sin \gamma t}{\omega(\omega^2 - \gamma^2)} &= \lim_{\gamma \to \omega} \frac{-F \sin \omega t + \omega t F \cos \gamma t}{-2\gamma\omega} \\
&= \frac{-F \sin \omega t + \omega t F \cos \omega t}{-2\omega^2} \\
&= \frac{F}{2\omega^2} \sin \omega t - \frac{F t \cos \omega t}{2\omega}
\end{aligned}$$

The result has a $t \cos \omega t$ term, which is a sinusoidal wave with linear growth in amplitude. The amplidute of the system will grow linearly without bound. This is the ideal (frictionless) understanding of resonance, where the oscillations of the system continue to grow.

This is a justification, if you want, of why multiplying by t gives the particular solutions when you have a forcing term similar to the homogeneous solutions. When $\gamma = \omega$, the forcing is already a homogeneous solution, so expect a solution multiplied by t. That's precisely what we have here. It is also what we would have found if we with $\omega = \gamma$ and used undetermined coefficients, instead of using the limit process.

3.3.3 Section 3: Friction, No Forcing

The homogeneous SOCCLDE with our choice of constants is

$$y'' + 2\lambda y' + \omega^2 y = 0.$$

The characteristic equation is $r^2 + 2\lambda r + \omega^2$ with roots $-\lambda \pm \imath\sqrt{\omega^2 - \lambda^2}$ If we let $\omega_d = \sqrt{\omega^2 - \lambda^2}$, then the homogeneous solutions are

$$y = e^{-\lambda t}(A \cos \omega_d t + B \sin \omega_d t).$$

The new constant ω_d is called the damped frequency. It is not the same as the natural frequency, since friction changes the desire frequency of th system. We can observe from the previous definition that.

$$\lambda^2 = \frac{\zeta^2}{\omega^2} \qquad \text{or equivalently} \qquad \lambda^2\omega^2 = \zeta^2.$$

Then we can also observe that

$$\omega_d = \omega\sqrt{1 - \zeta^2}.$$

With friction involved and no forcing, we have sinusoidally decaying oscillations where the frequency is this ω_d. As an interesting aside, as ζ approaches 1 and we approach the critical damped situation with simple exponential decay, this damped frequency approaches zero. This explains the transition from sinusoidal to exponential behaviour: as we approach the critically damped situation, the wavelength of the sine wave (which is a reciprocal of frequency) grows to ∞. The wave stretches out further and further until there isn't any wave left at all, just an exponential decay.

3.3.4 Situation 4: Friction and Forcing

The differential equation is now

$$y'' + 2\lambda y' + \omega^2 y = F \sin \gamma t.$$

The homogeneous solutions are known from the third situation.

$$y_h = e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t)$$

As in the second situation, our forcing term is $F\sin\gamma t$ and has a particular frequency γ. Since the forcing lacks the exponential term, it is not the same as the homogeneous solutions. We can use undetermined coefficients without any alteration. (What does this already imply?) Here is the solution using undetermined coefficients, with some matrix algebra in the middle to solve the system of equations.

$$\begin{aligned}
&\text{Our Guess for Undetermined Coefficients:}\\
y_p &= C\sin\gamma t + D\cos\gamma t\\
y_p' &= C\gamma\cos\gamma t - D\gamma\sin\gamma t\\
y_p'' &= -C\gamma^2\cos\gamma t - D\gamma^2\sin\gamma t\\
&\text{Solve for the Coefficients:}\\
Ly_p &= (-C\gamma^2 - 2\lambda D\gamma + C\omega^2)\sin\gamma t + (-D\gamma^2 + 2\lambda C\gamma + D\omega^2)\cos\gamma t\\
-C\gamma^2 - 2\lambda D\gamma + C\omega^2 &= F\\
-D\gamma^2 + 2\lambda C\gamma + D\omega^2 &= 0\\
\begin{bmatrix}\omega^2-\gamma^2 & -2\lambda\gamma\\ 2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix} &= \begin{bmatrix}F\\ 0\end{bmatrix}\\
M &= \begin{bmatrix}\omega^2-\gamma^2 & -2\lambda\gamma\\ 2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\\
\det M &= (\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2\\
M^{-1} &= \frac{1}{\det M}\begin{bmatrix}\omega^2-\gamma^2 & 2\lambda\gamma\\ -2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\\
\begin{bmatrix}C\\ D\end{bmatrix} &= \frac{1}{\det M}\begin{bmatrix}\omega^2-\gamma^2 & 2\lambda\gamma\\ -2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\begin{bmatrix}F\\ 0\end{bmatrix}\\
C &= \frac{(\omega^2-\gamma^2)F}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
D &= \frac{-2\gamma\lambda F}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
y_p &= \frac{(\omega^2-\gamma^2)F\sin\gamma t - 2\gamma\lambda F\cos\gamma t}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
y &= e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{(\omega^2-\gamma^2)F\sin\gamma t - 2\gamma\lambda F\cos\gamma t}{(\omega^2-\gamma^2) + 4\lambda^2\gamma^2}
\end{aligned}$$

As time passes, the homogeneous solutions fall out and only the term with the forcing frequency γ remains. What is its amplitude? The term is a linear combination of a sine and cosine wave, so we can use the identity mentioned previously that told us the amplitude of the combined wave was a pythagorean combination of the two amplitudes.

$$a = F\frac{\sqrt{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2}}{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2} = \frac{F}{\sqrt{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2}}$$

The question of resonance now becomes a question of amplitude. If F is fixed, what is the maximum amplitude we can achieve with a fixed force? Note this amplitude is always finite – with friction, there is no infinite growth of amplitude. (As predicted, since we didn't need to multiply by t when we setup the undetermined coefficients.) However, the amplitude can be quite large. This is a optimization problem, so we differentiate the expression for amplitude and find its critical points.

$$\begin{aligned}
a(\gamma) &= \frac{F}{\sqrt{(\omega^2-\gamma^2)^2+4\gamma^2\lambda^2}} \\
a'(\gamma) &= \frac{-F(2(\omega^2-\gamma^2)(-2\gamma)+8\gamma\lambda^2)}{2((\omega^2-\gamma^2)^2+4\gamma^2\lambda^2)^{\frac{3}{2}}} = 0 \\
4\gamma(\omega^2-\gamma^2) &= 8\gamma\lambda^2 \\
\omega^2-\gamma^2 &= 2\lambda^2 \\
\gamma^2 &= \omega^2-2\lambda^2 \\
\gamma &= \sqrt{\omega^2-2\lambda^2}
\end{aligned}$$

This is almost in the form we want. Recall that $\lambda^2 = \omega^2\zeta^2$ which allows the form

$$\gamma = \sqrt{\omega^2 - 2\zeta^2\omega^2} = \omega\sqrt{1-2\zeta^2}.$$

This is the critical point, and it is indeed a maximum for the amplitude. It is the resonant frequency. However, are we certain that it exists? We need $1-2\zeta^2$ to be positive to define this square root. This implies, certainly, that $\zeta < 1$, which we assumed when we decided to work with the underdamped case (that assumption is now justified). However, the inequality is stricter.

$$\zeta^2 \leq \frac{1}{2} \implies \zeta \leq \frac{1}{\sqrt{2}}$$

The bound for the damping coefficient is smaller than simply for the underdamped case. The conclusion we reach is that a resonant frequency only exists is the friction is minimal enough, measured by this $\zeta < \frac{1}{\sqrt{2}}$. In particular, if we are concerned about safety and want to avoid the situation of resonant frequency, we know how much friction we need to build into the system.

3.4 Variation of Parameters

Undetermined coefficients is an effective and efficient (relatively speaking!) way to solve for a particular non-homogeneous solution of a SOLCCDE. It's restriction is its scope: it only works for specific forcing terms. Variation of parameters is a much more general method of finding particular solutions. It's weakness is the reliance on difficult integrals – often we'll have to leave the solutions in a form which involves unfinished integrals. In the examples, we'll often stick to forcing terms which could have used undetermined coefficients, just to make the integrals reasonable.

3.4.1 Wronskians

Definition 3.4.1. Let $f_1, f_2, \dots f_n \in C^{n-1}(\mathbb{R})$. The *Wronskian* of this set of functions is defined to be the determinant of a matrix involving the f_i and their derivatives.

$$W(f_1, f_2, \dots, f_n) := \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ f_1'' & \dots & f_n'' \\ \dots & \dots & \dots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

The Wronskian of a set of functions has a number of different uses. Its first use is checking for linear independence. The set of functions linearly independent if and only if the Wronskian is never zero. Most of the time, this is easy to satisfy. In addition, if we have linearly independent functions, we know the Wronskian is never zero and we can divide by if with impunity.

For variation of parameters, we only need to the Wronskain of two functions. The definition here is relatively simple.

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$$

3.4.2 The Technique of Variations of Parameters

Let L be the second order operator

$$L = \frac{d^2}{dt^2} + P\frac{d}{dt} + Q.$$

Let y_1 and y_2 be solutions to the homogeneous equation $Ly = 0$. The general solution to the homogeneous equation is $Ay_1 + By_2$. As with the first-order case, variation of parameters replaces the constants A and B with functions. That is, we look for a solution which has the form (where $u_1(t)$ and $u_2(t)$ are unknown functions)

$$y_p = u_1 y_1 + u_2 y_2.$$

Then we put this y_p into the differential equation. A long and tedious calculation ensues.

$$\begin{aligned} Ly_p &= u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' + P\left(u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right) \\ &\qquad + Q\left(u_1 y_1 + u_2 y_2\right) = f \\ &= u_1\left(y_1'' + Py_1' + Qy_1\right) + u_2\left(y_2'' + Py_2' + Qy_2\right) + \left(y_1 u_1'' + u_1' y_1'\right) + \left(y_2 u_2'' + u_2' y_2'\right) \\ &\qquad + P\left(y_1 u_1' + y_2 u_2'\right) + y_1' u_1' + y_2' u_2' = f \\ &= u_1(Ly_1) + u_2(Ly_2) + \frac{d}{dt}\left(y_1 u_1'\right) + \frac{d}{dt}\left(y_2 u_2'\right) + P\left(y_1 u_1' + y_2 u_2'\right) + \left(y_1' u_1' + y_2' u_2'\right) = f \\ &= 0 + 0 + \frac{d}{dt}\left(y_1 u_1' + y_2 u_2'\right) + P\left(y_1 u_1' + y_2 u_2'\right) + \left(y_1' u_1' + y_2' u_2'\right) = f \end{aligned}$$

So far, this doesn't look particularly useful. Part of the problem is that we've given ourselves too much freedom. Choosing two entirely unknown functions is not limiting enough. The advantage of this situation is that we can impose a reasonable restriction while still allowing for general solutions. We impose the restriction $y_1 u_1' + y_2 u_2' = 0$. (Why this restriction? Because it works, and somewhere in matheamtical history, some clever mathematician figured out that it works.) This restriction removes two terms from the previous equation and leaves us with a simple equation.

$$(y_1' u_1' + y_2' u_2') = f$$

If we remember the restriction, we now have a system of two equations.

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= f \end{aligned}$$

We can express this system in matrix terminology.

$$\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$$

Linear algebra then solves the system. Conveniently, the determinant of the system is the Wronskian $W(y_1, y_2)$, which we will simply write W for the remainder of this derivation.

$$\begin{aligned} u_1' &= \frac{-y_2 f}{W} \\ u_2' &= \frac{y_1 f}{W} \end{aligned}$$

The solution gives us the derivatives of u_1 and u_2, so we can integrate.

$$\begin{aligned} u_1 &= -\int \frac{y_2 f}{W} dt \\ u_2 &= \int \frac{y_1 f}{W} dt \end{aligned}$$

We insert these u_i in the original expression.

$$y_p = \left(-\int \frac{y_2 f}{W} dt \right) y_1 + \left(\int \frac{y_1 f}{W} dt \right) y_2$$

Therefore, the general solution to the original equation $Ly = f$ is

$$y_p = \left(-\int \frac{y_2 f}{W} dx \right) y_1 + \left(\int \frac{y_1 f}{W} dt \right) y_2 + Ay_1 + By_2.$$

Note that this general form implies, conveniently, that we don't have to worry about constants of integration in either of the two integrals. If we had a constant of integration, it would lead to a multiple of one of the homogeneous solutions. Those multiples are already accounted for in the general homogeneous solution.

Example 3.4.2. This example is from Roberts and Marion.

$$
\begin{aligned}
y'' - 2y' + 2y &= 2e^t \\
r^2 - 2r + 2 &= 0 \\
r &= 1 \pm \imath \\
y_1 &= e^t \cos t \\
y_2 &= e^t \sin t \\
y_1' &= e^t \cos t - e^t \sin t \\
y_2' &= e^t \sin t + e^t \cos t \\
W &= e^t \cos t(e^t \sin t + e^t \cos t) - e^t \sin t(e^t \cos t - e^t \sin t) \\
&= e^{2t}(\cos t \sin t + \cos^2 t - \sin t \cos t + \sin^2 t) = e^{2t} \\
u_1 &= \int \frac{-y_2 f}{W} dt = -\int \frac{e^t \sin t 2e^t}{e^{2t}} dt \\
&= -2 \int \sin t dt = 2 \cos t \\
u_2 &= \int \frac{y_1 f}{W} dt = \int \frac{e^t \cos t 2e^t}{e^{2t}} dt \\
&= 2 \int \cos t dt = 2 \sin t \\
y_p &= 2 \cos t e^t \cos t + 2 \sin t e^t \sin t = 2e^t \\
y &= 2e^t + Ae^t \cos t + Be^t \sin t
\end{aligned}
$$

Example 3.4.3.

$$\begin{aligned}
y'' - 4y' + 4y &= (t+1)e^{2t} \\
r^2 - 4r + 4 &= 0 \\
r &= 2 \\
y_1 &= e^{2t} \\
y_2 &= te^{2t} \\
y_1' &= 2e^{2t} \\
y_2' &= 2te^{2t} + e^{2t} \\
W &= e^{2t}(e^{2t} + 2te^{2t}) - te^{2t}2e^{2t} = e^{4t} \\
u_1 &= \int \frac{-y_2 f}{W} dt = -\int \frac{te^{2t}(t+1)e^{2t}}{e^{4t}} dt \\
&= -\int t^2 + t = \frac{-t^3}{3} - \frac{t^2}{2} \\
u_2 &= \int \frac{y_1 f}{W} dt = \int \frac{e^{2t}(t+1)e^{2t}}{e^{4t}} dt \\
&= \int t + 1 dt = \frac{t^2}{2} + t \\
y_p &= \left(\frac{-t^3}{3} - \frac{t^2}{2}\right) e^{2t} + \left(\frac{t^2}{2} + t\right) te^{2t} \\
&= \frac{t^3 e^{2t}}{6} + \frac{t^2 e^2 t}{2} \\
y &= \frac{t^3 e^{2t}}{6} + \frac{t^2 e^2 t}{2} + Ae^{2t} + Bte^{2t}
\end{aligned}$$

Example 3.4.4. This is (finally) an example that couldn't have used undetermined coefficients.

$$
\begin{aligned}
y'' + 9y &= \csc 3t \\
r^2 + 9 &= 0 \\
r &= \pm 3\imath \\
y_1 &= \cos 3t \\
y_2 &= \sin 3t \\
y_1' &= -3\sin 3t \\
y_2' &= 3\cos 3t \\
W &= 3\cos^2 3t + 3\sin^2 3t = 3 \\
u_1 &= \int \frac{-y_2 f}{W} dt = -\int \frac{\sin 3t \csc 3t}{3} dt = \frac{-1}{3}\int dt = \frac{-t}{3} \\
u_2 &= \int \frac{y_1 f}{W} dt = \int \frac{\cos 3t \csc 3t}{3} dt = \frac{1}{3}\int \cot 3t = \frac{1}{9}\ln|\sin 3t| \\
y_p &= \frac{-t}{3}\cos 3t + \frac{\sin 3t \ln|\sin 3t|}{9} \\
y &= \frac{-t}{3}\cos 3t + \frac{\sin 3t \ln|\sin 3t|}{9} + A\cos 3t + B\sin 3t
\end{aligned}
$$

One of the reasons that undetermined coefficients wouldn't have worked is that the form of the particular solution, $\ln|\sin 3t|$, is not a function likely to be guessed.

Example 3.4.5. In this example, the resulting integrals are not expressable by elementary functions. In these cases, we just leave the integrals in the final solution. We just have to live with these unexpressible function (or, I suppose, come up with new names for them.)

$$\begin{aligned}
y'' - y &= \frac{1}{t} \\
r^2 - 1 &= 0 \\
r &= \pm 1 \\
y_1 &= e^t \\
y_2 &= e^{-t} \\
y_1' &= e^t \\
y_2' &= -e^{-t} \\
W &= -1 - 1 = -2 \\
u_1 &= \int \frac{-y_2 f}{W} dt = -\int \frac{e^{-t}\frac{1}{t}}{-2} = \frac{1}{2}\int \frac{e^{-t}}{t} dt \\
u_2 &= \int \frac{y_1 f}{W} dt = \int \frac{e^{t}\frac{1}{t}}{-2} = \frac{-1}{2}\int \frac{e^{t}}{t} dt \\
y_p &= \frac{e^t}{2}\int \frac{e^{-t}}{t} dt - \frac{e^{-t}}{2}\int \frac{e^t}{t} dt \\
y &= \frac{e^t}{2}\int \frac{e^{-t}}{t} dt - \frac{e^{-t}}{2}\int \frac{e^t}{t} dt + Ae^t + Be^{-t}
\end{aligned}$$

Chapter 4

Series Solutions to Linear Differential Equations

There is a common approach to DEs: we assume the solution has a certain form, put that form into the differential equation, and solve for the parameters of the chosen form. This chapter takes the same approach, using Taylor series as the certain form. See 1.3 for a review of Taylor series.

We assume that the solution to a differential equation, a function f, can be written as a series centered at some α (usually $\alpha = 0$).

$$f = \sum_{n=0}^{\infty} c_n (t-\alpha)^n$$

The series is entirely determined by the coefficients c_n. Therefore, we use the differential equation to find information about the coefficients c_n. In that way, we can find a description of a Taylor series, and hence a function, which solves the differential equation.

There is one major caution for this approach. While many functions are analytic, is it a relativley restrictive condition. The method of this chapter only finds analytic solutions; it may miss many non-ananlytic functions which also solve the differential equation. This is the risk in any method which imposes a form, since the solution may not have the desired form.

4.1 Ordinary Points

4.1.1 Existence of Solutions

Consider a homogeneous linear second order differential equation.

$$y'' + P(t)y' + Q(t)y = 0$$

In a departure from the Chapter 3, we will now allow P and Q to be functions instead of just constants. Before we jump into the method of series solutions, we need some theory about the existence of solutions.

Definition 4.1.1. A point α is called an *ordinary point* for the differential equation if P and Q are both analytic at α. Otherwise it is called a *singular point*.

Theorem 4.1.2. *If α is an ordinary point of a linear homogeneous second order differential equation, the there exists two linearly independent analytic solutions centered at α. These analytic solutions will have radii of convergence at least as large as the distance from α to the nearest singular point of the DE.*

This is a lovley theorem, but these is one catch. The 'distance' mentioned to the nearest singular point is actually distance in $\mathbb{C}$ to a possibly-complex singular point. In practice, this isn't too much of a worry, but we should be careful with our assumptions about the radii of convergence.

4.1.2 The Method for Solutions at Ordinary Points

Example 4.1.3. Let's start with a simple and known DE, to see how the method works.

$$y'' + y = 0$$

Both P and Q are constant, so they analytic everywhere, even in $\mathbb{C}$. Since there are no singular points, we expect solutions which have Taylor series defined for all of $\mathbb{R}$. We can centre them at 0 for convenience. We assume the solution has a Taylor series.

$$y = \sum_{n=0}^{\infty} c_n t^n$$

We put this into the DE and calcluate, keeping in mind our tools to manipulate the inidices of series.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n t^n \\
y' &= \sum_{n=1}^{\infty} c_n n t^{n-1} \\
y'' &= \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} \\
y'' + y &= \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} c_n t^n = 0 \\
&= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) t^n + \sum_{n=0}^{\infty} c_n t^n = 0 \\
&= \sum_{n=0}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_n\right] t^n = 0
\end{aligned}$$

Let's rewrite this last equation in a slightly different form.

$$\sum_{n=0}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_n\right] t^n = \sum_{n=0}^{\infty} 0 t^n$$

We've just written the constant function 0 as a series. Now, two series are only equal is every coefficient is the same. Therefore, we can take this equation and turn it into an equation of coefficients. For every $n \in \mathbb{N}$, we have a recurrence relation.

$$(n+1)(n+2)c_{n+2} + c_n = 0 \implies c_{n+2} = \frac{-c_n}{(n+1)(n+2)}$$

We'd like to solve this recurrence relation to get a easier decription of the series coefficients. Without any heavy-duty techniques for solving recurrence relations, we work by inspection.

Before we start, we need two starting terms, c_0 and c_1. This works out very well, since $c_0 = y(0)$ and $c_1 = y'(0)$. The initial seed terms for our recurrence relation are exactly initial conditions for the DE. We can either leave them as variables a and b, or set them to specific values.

Let's set $c_0 = 1$ and $c_1 = 0$. Then we use the recurrence relation to calculate some terms.

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= 0 \\
c_2 &= \frac{-c_0}{(1)(2)} = \frac{-1}{2} \\
c_3 &= \frac{-c_1}{(2)(3)} = 0 \\
c_4 &= \frac{-c_2}{(3)(4)} = \frac{1}{2 \cdot 3 \cdot 4} \\
c_5 &= \frac{-c_3}{(4)(5)} = 0 \\
c_6 &= \frac{-c_4}{(5)(6)} = \frac{-1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\
c_7 &= \frac{-c_5}{(6)(7)} = 0 \\
c_8 &= \frac{-c_6}{(7)(8)} = \frac{1}{8!} \\
c_9 &= \frac{-c_7}{(8)(9)} = 0 \\
c_{2n} &= \frac{(-1)^n}{(2n)!} \\
c_{2n+1} &= 0
\end{aligned}$$

After some of these terms, we can intuite the closed form. We put these terms into the series.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

This is exactly the Taylor series for cosine, which makes perfect sense, since cosine solves the original DE and matches the initial conditions we imposed.

Example 4.1.4. This is our first example which has non-constant coefficients, since $Q = t$.

$$y'' + ty = 0$$

$P = 0$ and $Q = t$, both of which are analytic everywhere, so we expect a series solution with infinite radius of convergence. We repeat the calculating from the previous example.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n t^n \\
y' &= \sum_{n=1}^{\infty} c_n n t^{n-1} \\
y'' &= \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} \\
y'' + ty &= \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} + t \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) t^n + \sum_{n=0}^{\infty} c_n t^{n+1} \\
&= 2c_2 + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1) t^n + \sum_{n=1}^{\infty} c_{n-1} t^n \\
&= 2c_2 + \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_{n-1} \right] t^n = 0
\end{aligned}$$

As before, to match the homogeneous $= 0$, we need all these coefficients to be zero. The second coefficient is isolated, so we have $c_2 = 0$. For the rest, we have c_0 and c_1 unknown and a recurrence relation.

$$\begin{aligned}
(n+2)(n+1)c_{n+2} + c_{n-1} = 0 \\
c_{n+2} &= \frac{-c_{n-1}}{(n+2)(n+1)} \\
c_{n+3} &= \frac{-c_n}{(n+3)(n+2)}
\end{aligned}$$

This is a third order recurrence relation. However, we still have only two parameters, due to the condition that $c_2 = 0$. We calculate the coefficients.

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= c_1 \\
c_2 &= 0 \\
c_3 &= \frac{-c_0}{(3)(2)} \\
c_4 &= \frac{-c_1}{(4)(3)} \\
c_5 &= \frac{-c_2}{(5)(4)} = 0 \\
c_6 &= \frac{-c_3}{(6)(5)} = \frac{c_0}{(6)(5)(3)(2)} \\
c_7 &= \frac{-c_4}{(7)(6)} = \frac{c_1}{(7)(6)(4)(3)} \\
c_8 &= \frac{-c_5}{(8)(7)} = 0 \\
c_9 &= \frac{-c_6}{(9)(8)} = \frac{-c_0}{(9)(8)(6)(5)(3)(2)} \\
c_{10} &= \frac{-c_7}{(10)(9)} = \frac{-c_1}{(10)(9)(7)(6)(4)(3)} \\
c_{11} &= \frac{-c_8}{(11)(10)} = 0
\end{aligned}$$

We see that we have three groups of terms.

- Terms of the form c_{3n+2} are all zero, since they all relate back to c_2.
- Terms of the form c_{3n} all involve c_0.
- Terms of the form c_{3n+1} all involve c_1.

Expressing the coefficients in closed form is more difficult than before, but we still can intuite the general form. We use some nice factorial tricks to express the coefficients.

$$\begin{aligned}
c_{3n} &= \frac{(-1)^n(1)(4)(7)\dots(3n-2)}{(3n)!}c_0 \\
c_{3n+1} &= \frac{(-1)^n(2)(5)(8)\dots(3n-1)}{(3n+1)!}c_1 \\
c_{3n+2} &= 0
\end{aligned}$$

Then we group the c_1 terms into a series and the c_2 terms into a series, to get a general solution.

$$y = c_0\left[1 + \sum_{n=n}^{\infty}\frac{(-1)^n(1)(4)(7)\dots(3n-2)}{(3n)!}t^{3n}\right] + c_1\left[t + \sum_{n=0}^{\infty}\frac{(-1)^n(2)(5)(8)\dots(n+2)}{(3n-1)!}t^{3n+1}\right]$$

If we need to, we can easily check that each of these series has infinite radius of convergence.

We might wonder: what are these functions? These are two new, strange and unfamiliar functions. Unless we have the good fortune (as in the previous example) to recognize the resulting Taylor series, we simply treat the solutions as new functions. However, we still consider the DE solved, since we have Taylor series to define new functions. We know a great deal about a function based on its Taylor series, so this is a sufficient threshold of information to consider the DE solved.

Example 4.1.5.

$$\begin{aligned}(t^2+1)y''+ty'-y&=0\\ y''+\left(\frac{t}{t^2+1}\right)y'-\left(\frac{1}{t^2+1}\right)y&=0\end{aligned}$$

$$P(t)=\frac{t}{t^2+1}$$
$$Q(t)=\frac{1}{t^2+1}$$

0 is an ordinary point, but what is the distance to the nearest singular point? Here, we need to remember to look for singularities in $\mathbb{C}$. The denominators are undefined at $\pm\imath$, which is 1 unit away from the origin. Therefore, centered at 0, we expect a radius of $R=1$.

(As an aside, note that we can centre the series wherever we wish; we default to 0 simply because it is convenient and familiar. Each choice of a center point gives a different series solution, with a different radius of convergence. At $t=1$, we would have $R=\sqrt{2}$ (the distance to $\imath$ in $\mathbb{C}$). At $t=4$, we would have $R=\sqrt{17}$.)

This series solutions starts with a lengthly calculation.

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n t^n \\
y' &= \sum_{n=1}^{\infty} c_n n t^{n-1} \\
y'' &= \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2} \\
(t^2+1)y'' + ty' - y &= (t^2+1)\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} + t\sum_{n=1}^{\infty} c_n n t^{n-1} - \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n t^n + \sum_{n=2}^{\infty} n(n-1)c_n t^{n-2} \\
&\qquad + \sum_{n=1}^{\infty} n c_n t^n - \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n t^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}t^n \\
&\qquad + \sum_{n=1}^{\infty} n c_n t^n - \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n t^n + 2c_2 + 6c_3 t + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}t^n \\
&\qquad + c_1 t + \sum_{n=2}^{\infty} n c_n t^n - c_0 - c_1 t - \sum_{n=2}^{\infty} c_n t^n \\
&= (2c_2 - c_0) + (6c_3 + c_1 - c_1)t \\
&\qquad + \sum_{n=2}^{\infty} \left[n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_n\right] t^n = 0
\end{aligned}
$$

We proceed to find the recurrence relation by setting all the coefficients to 0. We leave c_0 and c_1 as unknowns.

$$
\begin{aligned}
n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_n &= 0 \\
(n+1)(n+2)c_{n+2} &= (1 - n - n(n-1))c_n \\
c_{n+2} &= \frac{-c_n(n^2-1)}{(n+2)(n+1)} = \frac{-c_n(n+1)(n-1)}{(n+2)(n+1)} \\
&= \frac{-c_n(n-1)}{n+2}
\end{aligned}
$$

This is a second order recurrence relation. Here are the first few terms. (We must be careful to use the isolated expressions to find c_2 and c_3, and then the standard recurrence relation for c_n where $n \geq 4$).

$$
\begin{aligned}
c_0 &= c_0 \\
c_1 &= c_1 \\
c_2 &= \frac{c_0}{2} \\
c_3 &= 0 \\
c_4 &= \frac{-c_2(1)}{4} = \frac{c_0}{(2)(4)} \\
c_5 &= 0 \\
c_6 &= \frac{-c_4(3)}{6} = \frac{-c_0(3)}{(6)(4)(2)} \\
c_7 &= 0 \\
c_8 &= \frac{-c_6(5)}{8} = \frac{c_0(5)(3)}{(8)(6)(4)(2)} \\
c_9 &= 0 \\
c_{10} &= \frac{-c_8(7)}{10} = \frac{-c_0(7)(5)(3)}{(10)(8)(6)(4)(2)} \\
c_{2n+1} &= 0 \\
c_{2n} &= \frac{(-1)^n(3)(5)(7)\ldots(2n-3)}{(2)(4)(6)(8)\ldots(2n)}c_0 = \frac{(-1)^n(2n-3)!}{(2)(4)(6)\ldots(2n-4)2^n n!}c_0 \\
&= \frac{((-1)^n(2n-3)!}{2^{n-2}(n-2)!2^n n!}c_0 = \frac{((-1)^n(2n-3)!}{2^{2n-2}n!(n-2)!}c_0
\end{aligned}
$$

We have two cases. The odd terms all vanish, so the only place where we see c_1 is the isolated term $c_1 t$. That implies that the linear polynomial $c_1 t$ is a solution for any c_1. For the c_0 terms, we get a series. Putting this together gives the general solution. Note that we have to start the series at $n = 2$, since the pattern doesn't hold before that point.

$$
y = c_0 \left[1 + \frac{t^2}{2} + \sum_{n=2}^{\infty} \frac{((-1)^n(2n-3)!}{2^{2n-2}n!(n-2)!} \right] + c_1 t
$$

If we look at the radius of convergence (due to the factorials on numerator and denominators) we can calculate the expected value of $R = 1$. Note that the second term, the polynomial $c_1 t$ doesn't have a domain restriction. It's possible to find solutions that exceed the limitation of the radius of convergence; Theorem 4.1.2 guaranteed a solution with a minimum radius of convergence but did not stipulate a maximum.

4.2 Legendre Functions

There is a nice example of series solutions which deserves its own section, since it leads to an important class of functions. Let $k \in \mathbb{N}$ and consider the differential equation

$$(1-t^2)y'' - 2ty' + k(k+1)y = 0.$$

We need to divide by $1-t^2$ to get this DE in standard form, implying that $t = \pm 1$ are singular points. A solution centered at 0 should have radius of convergence at least $R = 1$. We then apply the process of series solution.

$$\begin{aligned}
(1-t^2)y'' - 2ty' + k(k+1)y &= (1-t^2)\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} \\
&\qquad - 2t\sum_{n=1}^{\infty} c_n n t^{n-1} + k(k+1)\sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} - \sum_{n=2}^{\infty} c_n n(n-1)t^n \\
&\qquad - \sum_{n=1}^{\infty} 2c_n n t^n + \sum_{n=0}^{\infty} k(k+1)c_n t^n \\
&= \sum_{n=0}^{\infty} c_{n_2}(n+2)(n+1)t^n - \sum_{n=2}^{\infty} c_n n(n-1)t^n \\
&\qquad - \sum_{n=1}^{\infty} 2c_n n t^n + \sum_{n=0}^{\infty} k(k+1)c_n t^n \\
&= 2c_2 + 6c_3 t + \sum_{n=2}^{\infty} c_{n_2}(n+2)(n+1)t^n \\
&\qquad - \sum_{n=2}^{\infty} c_n n(n-1)t^n - 2c_1 t - \sum_{n=2}^{\infty} 2c_n n t^n \\
&\qquad + k(k+1)c_0 + k(k+1)c_1 t + \sum_{n=2}^{\infty} k(k+1)c_n t^n \\
&= [2c_2 + k(k+1)c_0] + [6c_3 - 2c_1 + k(k+1)c_1]\, t \\
&+ \sum_{n=2}^{\infty} [c_{n+2}(n+2)(n+1) - c_n n(n-1) - 2c_n n + k(k+1)c_n]\, t^n
\end{aligned}$$

We need all these coefficients to be zero. We leave c_0 and c_1 as unknown parameters, since we haven't imposed initial conditions. The first two isolated terms give special relationships.

$$\begin{aligned}
[2c_2 + k(k+1)c_0] &= 0 \\
c_2 &= \frac{-c_0 k(k+1)}{2!} \\
[6c_3 - 2c_1 + k(k+1)c_1] &= 0 \\
6c_3 + (k+2)(k-1)c_1 &= 0 \\
c_3 &= \frac{-c_1(k+2)(k-1)}{3!}
\end{aligned}$$

The remaining terms are determined by the recurrence relation.

$$\begin{aligned}
c_{n_2}(n+2)(n+1) - c_n n(n-1) - 2c_n n + k(k+1)c_n &= 0 \\
c_{n_2}(n+2)(n+1) + (-n^2 + n - 2n + k(k+1))c_n &= 0 \\
c_{n_2}(n+2)(n+1) + (-n^2 - n + k^2 + k)c_n &= 0 \\
c_{n_2}(n+2)(n+1) + (k-n)(k+n+1)c_n &= 0 \\
c_{n+2} &= \frac{-c_n(k-n)(k+n+1)}{(n+1)(n+2)}
\end{aligned}$$

This is an order two recurrence relation. We calculate coefficients using this relation.

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= c_1 \\
c_2 &= \frac{-c_0 k(k+1)}{2!} \\
c_3 &= \frac{-c_1(k+2)(k-1)}{3!} \\
c_4 &= \frac{-c_2(k-2)(k+3)}{(3)(4)} = \frac{c_0(k)(k+1)(k-2)(k+3)}{4!} \\
c_5 &= \frac{-c_3(k-3)(k+4)}{(4)(5)} = \frac{c_1(k-1)(k+2)(k-3)(k+4)}{5!} \\
c_6 &= \frac{-c_4(k-4)(k+5)}{(5)(6)} = \frac{-c_0(k)(k+1)(k-2)(k+3)(k-4)(k+5)}{6!} \\
c_7 &= \frac{-c_5(k-5)(k+6)}{(6)(7)} = \frac{-c_1(k-1)(k+2)(k-3)(k+4)(k-5)(k+6)}{7!}
\end{aligned}$$

The pattern continues and gives us two series, one for the c_0 terms and one for the c_1 terms.

We assumed that $k \in \mathbb{N}$. Let's consider two cases: k even or k odd. If k is odd, then the term involving c_1 eventually has $(k-k)$ in the numerator. That means the numerator will be zero: this recurrence relationship eventually stop and we get a polynomial solution. If k is even, then terms involving c_0

eventually have $(k-k)$ in the numerator, and again we get a polynomial solution. So, for any k, we find a polynomial solution. (Note this extends the expected radius of convergence. We were only guaranteed a radius of $R = 1$, but polynomials converge everywhere, so we get $R = \infty$).

These special polynomial solutions are the solutions we're interested in. They are called Legendre Polynomials for $k \in \mathbb{N}$. They are historically important special functions with several other important properties in addition to the property of solving the Legendre DE. We'll calculate the first few. We have the unknown c_0 or c_1 to consider; we'll always choice c_0 or c_1 so that the polynomial goes through the point $(1,1)$. (Which is equivalent to the initial condition $y(1) = 1$.) This convention leads to one of the important properties of the Legendre polynomials.

$$\begin{aligned}
k = 0 \implies P_0(t) &= c_0 \qquad \text{choose } c_0 = 1 \\
P_0(t) &= 1 \\
k = 1 \implies P_1(t) &= c_1 t \qquad \text{choose } c_1 = 1 \\
P_1(t) &= t \\
k = 2 \implies P_2(t) &= c_0(1 - 3t^2) \qquad \text{choose } c_0 = \frac{-1}{2} \\
P_2(t) &= \frac{1}{2}(3t^2 - 1) \\
k = 3 \implies P_3(t) &= c_1(t - \frac{5}{3}t^3) \qquad \text{choose } c_1 = \frac{-3}{2} \\
P_3(t) &= \frac{1}{2}(5t^3 - 3t) \\
P_4(t) &= \frac{1}{8}(35t^4 - 30t^2 + 3) \\
P_5(t) &= \frac{1}{8}(63t^5 - 70t^3 + 15t) \\
P_6(t) &= \frac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5) \\
P_7(t) &= \frac{1}{16}(429t^7 - 693t^5 + 315t^3 - 35t) \\
P_8(t) &= \frac{1}{128}(6435t^8 - 12012t^6 + 6903t^4 - 1260 + 35) \\
P_9(t) &= \frac{1}{128}(12155t^9 - 25740t^7 + 18018t^5 - 4620t^3 + 315t) \\
P_{10}(t) &= \frac{1}{256}(46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63)
\end{aligned}$$

We can notice a number of interesting properties. First, all the even P_k are even functions, all the odd are odd. The even one all pass through $(-1,1)$ and $(1,1)$. All the odd ones pass through $(-1,-1)$ and $(1,1)$. We usually only work with the Legendre polynomials on the domain $[-1,1]$. We can notice interesting patterns in the coefficients. We have powers of 2 in the denominators, but some are skipped. Interesting patterns are also found in the prime divisors of the numerators.

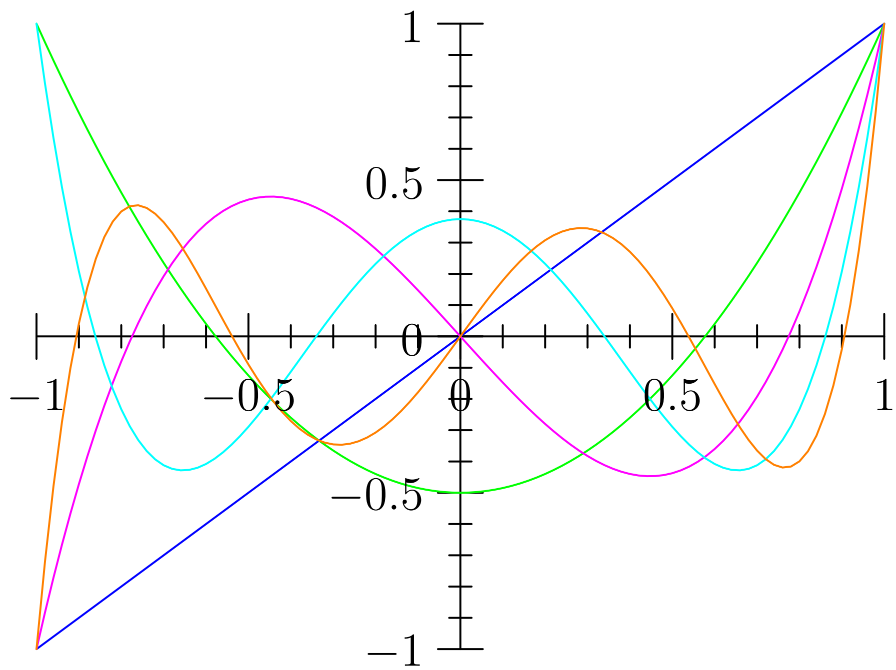


Figure 4.1: Legendre Polynomials

Here's a list of further interesting properties.

$$
\begin{aligned}
P_k(-t) &= (-1)^k P_k(t) \\
\int_{-1}^{1} P_k(t)P_l(t) &= \frac{2}{2l+1}\delta_{kl} \\
(k+1)P_{k+1}(t) - (2k+1)tP_k(t) + kP_{k-1}(t) &= 0 \\
(2k+1)P_k(t) &= \frac{d}{dt}\left(P_{k+1}(t) - P_{k-1}(t)\right) \\
P_k(t) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j}^2 \left(\frac{1+t}{2}\right)^{k-j} \left(\frac{1-t}{2}\right)^j
\end{aligned}
$$

The δ_{kl} in the second property is the Kronecker delta. It evalutes to 1 if $k = l$ and to 0 otherwise. It's a very useful piece of notation. This second property is an orthogonality property: if we define an inner product of two functions as the integral of their product over $[-1, 1]$, then the Legendre polynomials are all orthogonal to each other. In linear algebra language, they form an orthogonal basis for the infinite dimensional linear space of polynomials (with this particular inner product). Orthogonal bases are nice to work with, another fact that recommends the Legendre polynomials.

The third property is called the functional equation. This section is the start of a whole branch of mathematicals on so-called 'special functions'. The existence of a functional equation which relates members of the family to each other is quite typical for special functions.

As an interesting aside, let's very briefly detour to talk about generating functions.

Definition 4.2.1. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, a generating function for the sequence is a Taylor series (or other series in some contexts) where the a_n are the series coefficients.

In some way, the generating function 'knows' the properties of the sequence. I will share two interesting examples.

Example 4.2.2. In some way, the function on the left relates to and accounts for this sequences of integer squares.

$$\frac{t(t+1)}{(1-t)^3} = \sum_{n=0}^{\infty} n^2 t^n \qquad |t| < 1$$

Example 4.2.3. This is the generativing function for squares with factorial denominators.

$$t(t+1)e^t = \sum_{n=0}^{\infty} \frac{n^2}{n!} t^n \qquad t \in \mathbb{R}$$

It turns out, if we briefly allow ourselves to consider functions of two variables, that there are generating functions for Legendre polynomials. This relevant identity is

$$\frac{1}{\sqrt{1-2tx+x^2}} = \sum_{k=0}^{\infty} P_k(t)x^n.$$

Somehome, this strange two-variables square root functions 'knows' all the Legendre polynomials. There are all there, encoded in the Taylor series coefficients.

4.3 Issues and Next Directions

After these initial examples, two of questions seem natural. First, we might observe that P and Q were, so far, only rational functions, which means that we only need to deal with polynomials in the calculation. What if P and Q were other analytic functions? In this cases, we would have to expand P and Q as series about the ordinary point and then multiply them by the series for y in the calculation. This is possible, but miserable.

We might also wonder why we restricted to homogeneous cases. What happens if we add a forcing term $f(t)$? We can certainly do this if f is analytic on the same domain as P and Q. In all our examples so far, we compared the coefficients to 0. If we have a forcing terms, we would instead compare the coefficients to the coefficient of f instead of 0. This leads to more complications, since the relationship between the coefficients may no longer be a linear recurrence relation.

In both of these situations (non-polynomial P and Q or non-homogeneous DE with forcing), we can easily end up in a situation were the challenges of computation prevent us from finding a nice form for the coefficients of y. Quite frequently, we will only decide to calculate the first few terms. The result is a Taylor polynomial approximation to the solution, instead of a complete Taylor series solution. However, approximate solutions (if they have sufficient precision) are often sufficient.

4.4 Regular Singular Points

Throughout this section, we are considering this second order linear homogeneous differential equation.

$$y'' + P(t)y' + Q(t)y = 0$$

Recall that this differential equation has a regular point if $P(t)$ and $Q(t)$ are analytic at that point. We've dealt with solutions at ordinary points; now we will consider solutions at singular points. There are particular types of singular points which are reasonable to deal with, since the functions P and Q are quite close to being analytic. Here is the definition:

Definition 4.4.1. A singular point t_0 of this DE is called a *regular singular point* if the two functions $(t - t_0)P(t)$ and $(t - t_0)^2Q(t)$ are analytic at $t - 0$. Equivalently, t_0 is a regular singular point if the limits of $(t-t_0)P(t)$ and $(t-t_0)^2Q(t)$ as $t \to t_0$ exist and are finite. (In complex variables terminology, $P(t)$ and $Q(t)$ have to have poles, not essential singaularities, the pole of P has to be order 1, and the pole of Q has to be order 1 or 2.) Note that P and Q don't have to be rational functions for this definition.

4.4.1 Examples of Regular Singular Points

Example 4.4.2.

$$y'' + \frac{y'}{t^3(t-1)^2(t-2)(t-3)(t-4)} + \frac{y}{t^2(t-1)(t-2)^2(t-3)^2(t-4)^3} = 0$$

All points other than $t = 0, 1, 2, 3, 4$ are ordinary points. Of the singular points, only $t = 2, 3$ are regular singular points. The singular points $t = 0, 1$ have $P(t)$ with a pole of order 2 or higher (a square term or worse in the denominator), which is not allowed. The singular point $t = 4$ has $Q(t)$ with a pole of order 3 (a cubic term in the denominator), which is also not allowed.

Example 4.4.3.

$$y'' + \cot t y' + y = 0$$

We use a limit to find the singular points.

$$\lim_{t\to 0} t \cot t = \lim_{t\to 0} \frac{t \cos t}{\sin t} = 1$$

The limit shows that $t \cot t$ is analytic at $t = 0$, so $t = 0$ is a regular singular point.

4.4.2 The Method of Frobenius

The method of Frobenius is a method of constructing solutions at regular singular points. It relies on this existence theorem.

Theorem 4.4.4. *Consider a second order homogeneous linear differential equation.*

$$y'' + P(t)y' + Q(t) = 0.$$

If t_0 is a regular singular point of this equation, then there exists at least one (but possibly two) $r \in \mathbb{R}$ such that we have a series solution of this form (with $c_0 \neq 0$)).

$$y = (t - t_0)^r \sum_{n=0}^{\infty} c_n (t - t_0)^n$$

This is an extension of the idea of analytic solutions. This solution is analytic if $r \in \mathbb{N}$. Otherwise, it is very close, only differing by this multiple $(t - t_0)^r$. If r is any negative real number, we have an asymptote at $t = t_0$, so we can't evaluate there. However, we expect convergence on $0 < |t - t_0| < R$. If r is a negative integer, we have a new name for these series.

Definition 4.4.5. A *Laurent series* is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n (t - t_0)^n$$

where $c_n \in \mathbb{R}$. This is exactly the form of a Taylor series, but we now allow negative exponents.

Now that we have this theorem, and the form of the solution, we proceed as usual: we throw the form into the differential equation and see what happens. We expect to have a similar process for finding the coefficient c_n of the series. However, we also need a process for finding the number r. Let's assume, for convenience in this derivation, that $t_0 = 0$ is the regular singular point.

We know that $tP(t)$ and $t^2Q(t)$ are analytic, from the definition of a regular singular point. This means that $P(t)$ and $Q(t)$ can be written in the form

$$P(t) = \frac{p_{-1}}{t} + \sum_{n=0}^{\infty} p_n t^n \qquad\qquad Q(t) = \frac{q_{-2}}{t^2} + \frac{q_{-1}}{t} + \sum_{n=0}^{\infty} q_n t^n.$$

We will call these Laurent forms for P and Q. The coefficients p_1 and q_2 will be useful a bit later, so here are two useful identities to calculate p_1 and q_2.

$$p_{-1} = \lim_{t \to 0} tP(t) \qquad\qquad q_{-2} = \lim_{t \to 0} t^2 Q(t)$$

Now we start the method of Frobenious by calculating the derivatives of y in this new series form.

$$\begin{aligned}
y &= t^r \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+r} \\
y' &= \sum_{n=0}^{\infty} c_n (n+r) t^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) t^{n+r-2}
\end{aligned}$$

(Notice that we don't necessarily lose terms when taking derivatives; if r is not an integer, there are no constant terms in the series which go to zero under differentiation. If r is an integer, we should make a note to worry about derivatives setting constant terms to zero.)

With these expressions for y, P and Q, we put it all together into the original DE.

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) t^{n+r-2} + P(t) \sum_{n=0}^{\infty} c_n (n+r) t^{n+r-1} + Q(t) \sum_{n=0}^{\infty} c_n t^{n+r} = 0 \\
\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) t^{n+r-2} + \left(\frac{p_{-1}}{t} + \sum_{n=0}^{\infty} p_n t^n \right) \sum_{n=0}^{\infty} c_n (n+r) t^{n+r-1} \\
+ \left(\frac{q_{-2}}{t^2} + \frac{q_{-1}}{t} + \sum_{n=0}^{\infty} q_n t^n \right) \sum_{n=0}^{\infty} c_n t^{n+r} = 0
\end{aligned}$$

This is quite a mess: we have r to determine as well as the series coefficients. However, we can focus on the coefficient of the leading term (t^{r-2}).

$$\begin{aligned}
r(r-1)c_0 t^{r-2} + \frac{p_{-1}}{t} r c_0 t^{r-1} + \frac{q_{-2}}{t^2} c_0 t^r &= 0 \\
(r(r-1) + p_{-1} r + q_{-2}) c_0 t^{r-2} &= 0 \\
r(r-1) + p_{-1} r + q_{-2} &= 0 \qquad \text{using the fact that } c_0 \neq 0
\end{aligned}$$

Definition 4.4.6. This quadratic determines r. It is called the *indicial equation.*

So, before we proceed to find the recurrence relations and the series coefficients, we use this equation to determine r. After finding r, the method looks very similar to solutions at ordinary points. If there are two real roots of the indicial equation, is seems we'll have to repeat the same process for each root. However, in practice, we can leave r undetermined for most of the procees.

4.4.3 Examples

Example 4.4.7.

$$\begin{aligned}
3ty'' + y' - y &= 0 \\
P &= \frac{1}{3t} \\
Q &= \frac{-1}{3t} \\
tP &= \frac{1}{3} \implies p_{-1} = \frac{1}{3} \\
t^2 Q &= \frac{-t}{3} \implies q_{-2} = 0 \\
r(r-1) + \frac{r}{3} &= 0 \\
3r^2 - 2r = 0 \implies r &= 0 \text{ or } r = \frac{2}{3}
\end{aligned}$$

We'll deal with the $r = 0$ case first. (Note that $r = 0$ means we get a conventional Taylor series solution.)

$$\begin{aligned}
3t\sum_{n=2}^{\infty} c_n(n)(n-1)t^{n-2} + \sum_{n=1}^{\infty} c_n(n)t^{n-1} - \sum_{n=0}^{\infty} c_n t^n &= 0 \\
\sum_{n=2}^{\infty} 3c_n(n)(n-1)t^{n-1} + \sum_{n=1}^{\infty} c_n(n)t^{n-1} - \sum_{n=0}^{\infty} c_n t^n &= 0 \\
\sum_{n=1}^{\infty} 3c_{n+1}(n+1)nt^n + \sum_{n=1}^{\infty} c_{n+1}(n+1)t^n - \sum_{n=0}^{\infty} c_n t^n &= 0 \\
c_1 - c_0 + \sum_{n=1}^{\infty} \left[3(n+1)nc_{n+1} + (n+1)c_{n+1} - c_n\right] t^n &= 0 \\
c_{n+1} = \frac{c_n}{3(n^2+n)+n+1} = \frac{c_n}{3n^2+4n+1} &= \frac{c_n}{(3n+1)(n+1)}
\end{aligned}$$

We have the recurrence relation (which is first order this time), so we start calculating terms.

$$\begin{aligned}
c_0 &= c_0 \\
c_1 &= c_0 \\
c_2 &= \frac{c_1}{(4)(2)} = \frac{c_0}{(4)(2)} \\
c_3 &= \frac{c_2}{(7)(3)} = \frac{c_0}{(7)(4)(3)(2)} \\
c_4 &= \frac{c_3}{(10)(4)} = \frac{c_0}{(10)(7)(4)(4)(3)(2)} \\
c_5 &= \frac{c_4}{(13)(5)} = \frac{c_0}{(13)(10)(7)(4)(5)(4)(3)(2)} \\
c_n &= \frac{c_0}{n!(4)(7)(10)\dots(3n-2)} \\
y_1 &= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!(4)(7)(10)\dots(3n-2)}
\end{aligned}$$

This is the solution for $r = 0$. Now we proceed to the $r = \frac{2}{3}$ case. (This is a non-integer exponent, so the solution is not a conventional Taylor series.)

$$\begin{aligned}
3t\sum_{n=0}^{\infty} c_n\left(n+\frac{2}{3}\right)\left(n+\frac{2}{3}-1\right)t^{n+\frac{2}{3}-2} + \sum_{n=0}^{\infty} c_n\left(n+\frac{2}{3}\right)t^{n+\frac{2}{3}-1} - \sum_{n=0}^{\infty} c_n t^{n+\frac{2}{3}} &= 0 \\
t^{\frac{2}{3}}\left[\sum_{n=0}^{\infty} 3c_n\left(n+\frac{2}{3}\right)\left(n-\frac{1}{3}\right)t^{n-1} + \sum_{n=0}^{\infty} c_n\left(n+\frac{2}{3}\right)t^{n-1} - \sum_{n=0}^{\infty} c_n t^{n}\right] &= 0 \\
t^{\frac{2}{3}}\left[\sum_{n=0}^{\infty} 3c_n\left(n+\frac{2}{3}\right)\left(n-\frac{1}{3}\right)t^{n-1} + \sum_{n=0}^{\infty} c_n\left(n+\frac{2}{3}\right)t^{n-1} - \sum_{n=1}^{\infty} c_{n-1} t^{n-1}\right] &= 0 \\
t^{\frac{2}{3}}\left[\left(3\frac{2}{3}\frac{-1}{3}+\frac{2}{3}\right)c_0 t^{-1} + \sum_{n=1}^{\infty}\left[3c_n\left(n+\frac{2}{3}\right)\left(n-\frac{1}{3}\right) + c_n\left(n+\frac{2}{3}\right) - c_{n-1}\right]t^{n-1}\right] &= 0 \\
\left[\left(n+\frac{2}{3}\right)(3n-1+1)\right]c_n - c_{n-1} &= 0 \\
c_n = \frac{c_{n-1}}{\left(n+\frac{2}{3}\right)(3n)} &= \frac{c_{n-1}}{(3n+2)n} \\
c_{n+1} &= \frac{c_n}{(3n+5)(n+1)}
\end{aligned}$$

The term in front of c_0 reduces to 0, so c_0 is free. After the first term, we have a recurrence relation, so we start calculating terms.

$$
\begin{aligned}
c_0 &= c_0 \\
c_1 &= \frac{c_0}{5} \\
c_2 &= \frac{c_1}{(8)(2)} = \frac{c_0}{(8)(5)(2)} \\
c_3 &= \frac{c_2}{(11)(3)} = \frac{c_0}{(5)(8)(11)(2)(3)} \\
c_4 &= \frac{c_3}{(14)(4)} = \frac{c_0}{(5)(8)(11)(14)(2)(3)(4)} \\
c_5 &= \frac{c_4}{(17)(5)} = \frac{c_0}{(5)(8)(11)(14)(17)(2)(3)(4)(5)} \\
c_n &= \frac{c_0}{n!(5)(8)(11)\ldots(3n+2)} \\
y_2 &= 1 + \sum_{n=1}^{\infty} \frac{t^{n+\frac{2}{3}}}{n!(5)(8)\ldots(3n+2)} \\
y &= A\left(1 + \sum_{n=1}^{\infty} \frac{t^n}{n!(4)(7)(10)\ldots(3n-2)}\right) + B\left(1 + \sum_{n=1}^{\infty} \frac{t^{n+\frac{2}{3}}}{n!(5)(8)\ldots(3n+2)}\right)
\end{aligned}
$$

The radius of convergence here is $R = \infty$, which we can find by ratio test. It is good to note, though, that there are no guarantees about the radius of convergence with the method of Frobenius.

In this example, we could have gone as far as the recurrence relation using an arbitrary r to avoid repetition. We do need to specify r once we get to the recurrence relation.

Example 4.4.8.

$$ty'' + y = 0$$

Here $P = 0$ and $Q = \frac{1}{t}$, so $p_{-1} = 0$ and $q_{-2} = 0$. That means the indicial equation is $r(r-1) = 0$ with roots $r = 0$ and $r = 1$. We start with the case $r = 0$.

$$\begin{aligned}
t\sum_{n=2}^{\infty} n(n-1)c_n t^{n-2} + \sum_{n=0}^{\infty} c_n t^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)c_n t^{n-1} + \sum_{n=0}^{\infty} c_n t^n &= 0 \\
\sum_{n=1}^{\infty} (n+1)nc_n t^n + \sum_{n=0}^{\infty} c_n t^n &= 0 \\
c_0 + \sum_{n=1}^{\infty} \left[(n+1)nc_n + c_n\right] t^n &= 0
\end{aligned}$$

$$\begin{aligned}
c_{n+1} &= \frac{-c_n}{(n+1)(n)} \\
c_0 &= 0 \\
c_1 &= c_1 \\
c_2 &= \frac{-c_1}{2} \\
c_3 &= \frac{-c_2}{(3)(2)} = \frac{c_1}{(3)(2)} \\
c_4 &= \frac{-c_3}{(4)(3)} = \frac{-c_1}{(4)(3)(3)(2)(2)} \\
c_5 &= \frac{-c_4}{(5)(4)} = \frac{c_1}{(5)(4)(4)(3)(3)(2)(2)} \\
c_n &= \frac{(-1)^{n+1}c_1}{n!(n-1)!} \\
y_1 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}t^n}{n!(n-1)!}
\end{aligned}$$

Then we calculate with $r = 1$. Note the bounds in this case: since the series doesn't have a constant term, we only loose one term in the second derivative.

$$
\begin{aligned}
t\sum_{n=1}^{\infty}(n+1)nc_nt^{n-1} + \sum_{n=0}^{\infty}c_nt^{n+1} &= 0\\
\sum_{n=1}^{\infty}(n+1)nc_nt^{n} + \sum_{n=0}^{\infty}c_nt^{n+1} &= 0\\
\sum_{n=1}^{\infty}(n+1)nc_nt^{n} + \sum_{n=1}^{\infty}c_{n-1}t^{n} + &= 0\\
(n+1)nc_n + c_{n-1} &= 0\\
c_n &= \frac{-c_{n-1}}{(n+1)(n)}\\
c_{n+1} &= \frac{-c_n}{(n+2)(n+1)}\\
c_0 &= c_0\\
c_1 &= \frac{-c_0}{2}\\
c_2 &= \frac{-c_1}{(3)(2)} = \frac{c_0}{(3)(2)}\\
c_3 &= \frac{-c_2}{(4)(3)} = \frac{-c_0}{(4)(3)(3)(2)(2)}\\
c_4 &= \frac{-c_3}{(5)(4)} = \frac{c_0}{(5)(4)(4)(3)(3)(2)(2)}\\
c_n &= \frac{(-1)^nc_0}{(n+1)!n!}\\
y_1 &= \sum_{n=0}^{\infty}\frac{(-1)^nt^{n+1}}{(n+1)!n!}\\
y_1 &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^{n}}{n!(n-1)!}
\end{aligned}
$$

Very curiously, we get the same series. The two roots don't lead to two independent series, but to the same series. We would need other information to get the second solutions. In general, finding another solution can be quite difficult. With the method of Frobenius, we are not guaranteed to find both solutions.

4.4.4 Multiple Solutions in the Method of Frobenius

We have a theorem which deals with the situation in the previous example, where both roots gave the same series.

Theorem 4.4.9. *In the setting of the method of Frobenius, assume that r_1 and r_2 are two real roots of the indicial equation. Without loss of generalization, assume that $r_1 \geq r_2$. Then there are three cases.*

Case 1: *If $r_1 - r_2 \notin \mathbb{N}$ then each root will derive a linearly independent series solutions. (The idea here is that different fractional or irrational exponents lead to radically different behaviour.)*

Case 2: *If $r_1 - r_2 \in \mathbb{N}$ but $r_1 \neq r_2$, then the second root may not produce a new linearly independent solution. If it doesn't, and both roots produce the solution y_1, then a second solution exists and has the form:*

$$y_2 = ay_1 \ln t + \sum_{n=0}^{\infty} b_n t^{n+r_2}.$$

In this case, we would need to do extra work to find the constant a and the series coefficients b_n.

Case 3 *If the two roots are the same, then (obviously) only one series solution is generated. A second solution exists and has the same form as the previous case.*

The proof of this theorem relies on another differential equation technique called *reduction of order*. I've chosen not to cover that technique in this courses, but it is good to be aware that it exists. In general, if we have one solution to a second order linear DE, then reduction of order is a method for using that solution to change the second order DE into a first order DE, which can then be solved by first order methods.

Theorem 4.4.10 (Reduction of Order)**.** *Assume that y_1 is a solution of homogeneous linear second order DE of the form*

$$y'' + P(t)y' + Q(t) = 0.$$

Then a second linearly independent is given by the formula

$$y_2 = y_1(t) \int \frac{e^{-\int P(t)dt}}{y_1^2(t)} dt.$$

This reduction of order technique is fairly impractical for series solution, since dividing by y_1^2 and then integrating is a miserable calculation.

4.5 Bessel Functions

4.5.1 The Γ Function

Before we get to a very interesting example of the method of Frobenious, we need to define the Γ (gamma)function. This is a ubiquitous and useful function which generalizes the factorial.

Definition 4.5.1. The Γ function is defined by this integral

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

The Γ function is defined on all $\mathbb{R}$ except 0 and negative integers. It is continuous and differentiable on $(0, \infty)$.

There are several important properties of $\Gamma(t)$: First, (as mentioned) it generalizes the factorial.

$$\begin{aligned}\Gamma(a+1) &= a\Gamma(a) \\ \Gamma(1) &= 1 \\ \Gamma(n) &= (n-1)! \text{ for } n \in \mathbb{N}\end{aligned}$$

Asymptotically, the factorial nature of the Γ functions means it grows faster than e^t.

If the positive integer values of the Γ function generalize the factorial, what can be expect for other values? The results are surprising. Look at the value of $\Gamma(\frac{1}{2})$ (in this calculation, we use a well-known result for the integral e^{-u^2}).

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\ x &= u^2 \\ dx &= 2udu \\ x^{-\frac{1}{2}} &= \frac{1}{u} \\ &= 2\int_0^\infty e^{-u^2} u \frac{1}{u} du \\ &= 2\int_0^\infty e^{-u^2} \\ &= 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\end{aligned}$$

Then, from the factorial-like nature of Γ, all other half-integer values are multiples of $\sqrt{\pi}$.

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(3)(5)(7)\ldots(2n-1)\sqrt{\pi}}{2^n}$$

4.5.2 Bessel's Equation

The method of Frobenius is used to solve a historically interesting equation: Bessel's Equation. This equation shows up in harmonic problems with certain cirucular boundary conditions, such as the vibrations on a drum. It is also useful for springs with fatigue (where the spring constant gets weaker over time), the quantum model of the hydrogen atom, and various electical/gravitational potentials (particular those which are circularly or spherically symmetric). Its solutions are another piece of the mathematics of special functions.

Let $\nu \in \mathbb{R}$. Bessel's equation of order ν is the equation

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0.$$

Written in standard form, 0 is a regular singular point of Bessel's equation.

$$\begin{aligned} P(t) &= \frac{1}{t} \implies p_{-1} = 1 \\ Q(t) &= 1 - \frac{\nu^2}{t^2} \implies q_{-2} = -\nu^2 \end{aligned}$$

We calculate the indicial equation.

$$\begin{aligned} r(r-1) + r - \nu^2 &= 0 \\ r^2 - r + r - \nu^2 &= 0 \\ r^2 &= \nu^2 \\ r &= \pm\nu \end{aligned}$$

We start with $r = \nu$ first and assume that $\nu \geq 0$ without loss of generality. We calculate the derivatives of y.

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n t^{n+\nu} \\ y' &= \sum_{n=0}^{\infty} c_n (n+\nu) t^{n+\nu-1} \\ y'' &= \sum_{n=0}^{\infty} c_n (n+\nu)(n+\nu-1) t^{n+\nu-2} \end{aligned}$$

Then we proceed with a lengthly calculation.

$$
\begin{aligned}
t^2y'' + ty' + (t^2-\nu^2)y &= 0 \\
t^2\sum_{n=0}^{\infty} c_n(n+\nu)(n+\nu-1) + t\sum_{n=0}^{\infty} c_n(n+\nu)t^{n+\nu-1} + t^2\sum_{n=0}^{\infty} c_n t^{n+\nu} - \nu^2\sum_{n=0}^{\infty} c_n t^{n+\nu} &= 0 \\
\sum_{n=0}^{\infty} c_n(n+\nu)(n+\nu-1)t^{n+\nu} + \sum_{n=0}^{\infty} c_n(n+\nu)t^{n+\nu} + \sum_{n=0}^{\infty} c_n t^{n+\nu+2} - \sum_{n=0}^{\infty} \nu^2 c_n t^{n+\nu} &= 0 \\
\sum_{n=0}^{\infty} c_n(n+\nu)(n+\nu-1)t^{n+\nu} + \sum_{n=0}^{\infty} c_n(n+\nu)t^{n+\nu} + \sum_{n=2}^{\infty} c_{n-2} t^{n+\nu} - \sum_{n=0}^{\infty} \nu^2 c_n t^{n+\nu} &= 0 \\
\nu(\nu-1)c_0t^{\nu} + (\nu+1)\nu c_1 t^{\nu+1} + \nu c_0 t^{\nu} + (\nu+1)c_1t^{\nu+1} - \nu^2c_0t^{\nu} - \nu^2c_1t^{\nu+1} \\
+\sum_{n=2}^{\infty} c_n(n+\nu)(n+\nu-1)t^{n+\nu} + \sum_{n=2}^{\infty} c_n(n+\nu)t^{n+\nu} + \sum_{n=2}^{\infty} c_{n-2} t^{n+\nu} - \sum_{n=2}^{\infty} \nu^2 c_n t^{n+\nu} &= 0 \\
\nu(\nu-1)c_0t^{\nu} + (\nu+1)\nu c_1 t^{\nu+1} + \nu c_0 t^{\nu} + (\nu+1)c_1t^{\nu+1} - \nu^2c_0t^{\nu} - \nu^2c_1t^{\nu+1} \\
+\sum_{n=2}^{\infty}\left[c_n(n+\nu)(n+\nu-1) + c_n(n+\nu) + c_{n-2} - \nu^2 c_n\right]t^{n+\nu} &= 0
\end{aligned}
$$

Look at the coefficient of t^{ν}.

$$
\begin{aligned}
\left((\nu^2-\nu)c_0 + \nu c_0 - \nu^2 c_0\right) &= 0 \\
\left(\nu^2 - \nu + \nu - \nu^2\right) c_0 &= 0 \\
0c_0 &= 0
\end{aligned}
$$

Therefore, c_0 is free. Look at the coefficient of $t^{\nu+1}$.

$$
\begin{aligned}
\left(\nu(\nu+1)c_1 + (\nu+1)c_1 - \nu^2 c_1\right) &= 0 \\
\left(\nu^2 + \nu + \nu + 1 - \nu^2\right) c_1 &= 0 \\
(2\nu+1)\, c_1 &= 0
\end{aligned}
$$

So $c_1 = 0$ unless $\nu = \frac{-1}{2}$. However, we assumed that we were dealing with ν positive, so we conclude that $c_1 = 0$. (We'll pay some special attention to $\nu = \frac{-1}{2}$ when we look at negative ν, and to half-integer values of ν in general.)

Then we have the general recurrence relation, for $n \geq 2$.

$$
\begin{aligned}
((n+\nu)(n+\nu-1) + n + \nu - \nu^2)c_n + c_{n-2} &= 0 \\
(n^2 + n\nu + n\nu + \nu^2 - n - \nu + n + \nu - \nu^2)c_n &= -c_{n-2} \\
(n^2 + 2n\nu)c_n &= -c_{n-2} \\
c_n &= \frac{-c_{n-2}}{n(n+2\nu)} \\
c_{n+2} &= \frac{-c_n}{(n+2)(n+2+2\nu)}
\end{aligned}
$$

These are the equations that calculate coefficients, so we start calculating.

$$\begin{aligned}
c_0 &= c_0 \\
(2\nu+1)c_1 &= 0 \implies c_1 = 0 \implies c_{2n+1} = 0 \ \forall n \in \mathbb{N} \\
c_2 &= \frac{-c_0}{2(2+2\nu)} \\
c_4 &= \frac{c_0}{(2)(4)(2+2\nu)(4+2\nu)} \\
c_6 &= \frac{-c_0}{(2)(4)(6)(2+2\nu)(4+2\nu)(6+2\nu)} \\
c_8 &= \frac{c_0}{(2)(4)(6)(8)(2+2\nu)(4+2\nu)(6+2\nu)(8+2\nu)} \\
c_{2n} &= \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\ldots(n+\nu)}
\end{aligned}$$

The constant c_0 is still undetermined. By convention, the choice for c_0 is this strange value

$$c_0 = \frac{1}{2^\nu \Gamma(1+\nu)}.$$

The properties of the Γ function allow us to simplify the denominator of the c_n.

$$c_{2n} = \frac{(-1)^n}{2^\nu \Gamma(1+\nu)} \frac{1}{2^{2n} n!(\nu+1)\ldots(\nu+n)} = \frac{(-1)^n}{2^{2n+v} n!\Gamma(1+n+\nu)}$$

With this simplification, we can write the final solution for positive ν.

Definition 4.5.2. The solution constructed above is written J_ν and called the Bessel function of the first kind of order ν.

$$J_\nu = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+\nu)} \left(\frac{t}{2}\right)^{2n+\nu}$$

It converges on $(0,\infty)$. It may not be defined at 0 or negative numbers. In general, we are only interested in the Bessel function on the domain $(0,\infty)$. The Bessel functions of the first kind for $\nu \in \mathbb{Z}$ are show in Figure 4.2.

We could redo the same work for negative ν, but it is very similar (for $n = \frac{-1}{2}$, we would get a term $(2\nu-1)c_1$, so we would still conclude that $c_1 = 0$, hence all odd coefficients are 0). In this way, we find the rest of the Bessel functions of the first kind.

$$J_{-\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n-\nu)} \left(\frac{t}{2}\right)^{2n-\nu}$$

One caution should be noted. If $n \in \mathbb{N}$, this J_{-n} gives an undefined denominator, hence doesn't define a series. In these cases, the method of Frobenius gives $J_{-n} = (-1)^n J_n$. This is not necessarily surprising: in the method of Frobenius, is the roots of the indicial equation differ by an integer, we may not have linearly independent solution. If $\nu \notin \mathbb{Z}$, then we have two linearly independent solutions, as we would expect with the method of Frobenius.

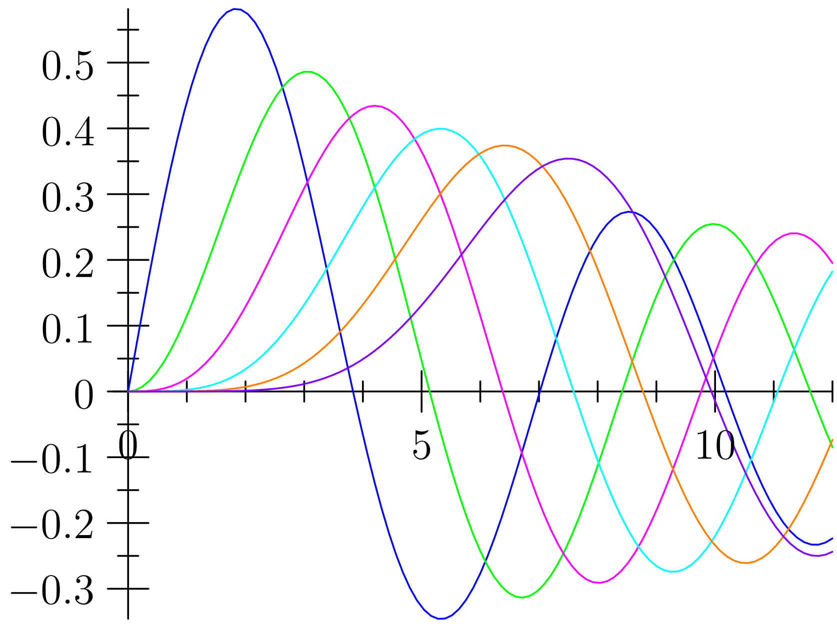


Figure 4.2: Integer-Order Bessel Functions of the First Kind

Definition 4.5.3. Let $n \notin \mathbb{Z}$. The *Bessel function of the second kind* of order ν are written Y_ν and defined by this expression

$$Y_\nu = \frac{\cos(\nu\pi)J_\nu(t) - J_{-\nu}(t)}{\sin \nu\pi}.$$

This definition is a linear combination of Bessel functions of the first kind. For $\nu \in \mathbb{Z}$, this linear combination have been just a multiple of the one solution we know. To get around this, for $\nu \in \mathbb{Z}$, we define the Bessel function of the second kind as a limit.

$$Y_\nu(t) = \lim_{\alpha \to \nu} Y_\alpha(t)$$

L'Hopital's rule shows that this limit exists. The Bessel functions of the second kind are shown in Figure 4.3.

To summarize, for any $\nu \geq 0$ we have J_ν and Y_ν, two linearly independent solutions to the differential equation. The general solution is

$$y = AJ_\nu + BY_\nu.$$

4.5.3 The Aging Spring

Consider the spring equation from before.

$$my'' + by' + ky = 0$$

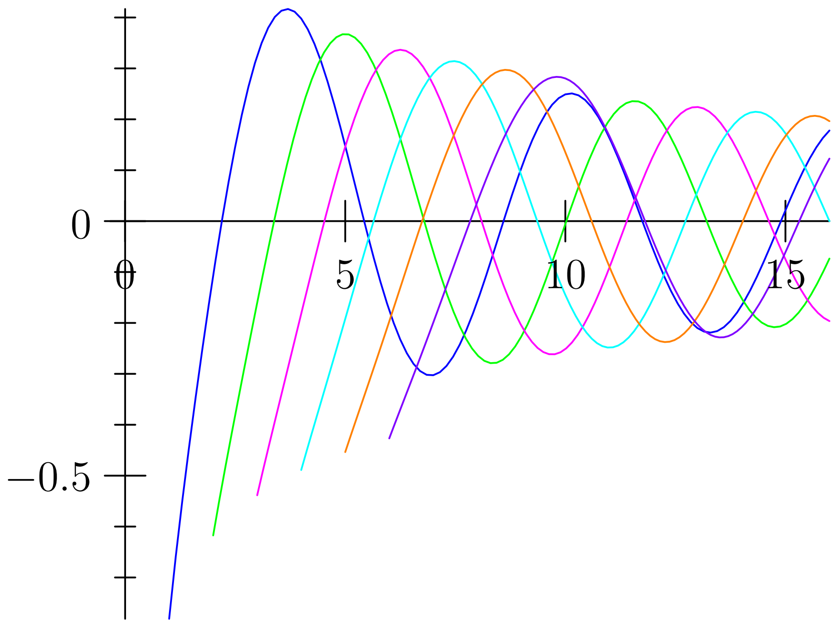


Figure 4.3: Integer-Order Bessel Functions of the Second Kind

Now, instead of all constants, lets assume this is an aging spring. That is, at time passes, the spring constant decreases. One model is exponential decay, so let $k(t) = ke^{-\alpha t}$ for $\alpha > 0$. Let's see what happens without friction.

$$my'' + ke^{-\alpha t} = 0$$

Here's a strange change of variables.

$$
\begin{aligned}
s &= \frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{\frac{-\alpha t}{2}} \\
\frac{ds}{dt} &= \frac{2}{\alpha}\frac{-\alpha}{2}\sqrt{\frac{4k}{m\alpha^2}}e^{\frac{-\alpha t}{2}} = -\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}} \\
\frac{d^2s}{dt^2} &= \frac{\alpha}{2}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}} \\
\frac{dy}{dt} &= \frac{dy}{ds}\frac{ds}{dt} \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{ds}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\frac{dy}{ds} + \frac{ds}{dt}\frac{d}{dt}\frac{dy}{ds} \\
&= \frac{d^2s}{dt^2}\frac{dy}{ds} + \frac{ds}{dt}\frac{d^2y}{ds^2}\frac{ds}{dt} \\
&= \frac{d^2s}{dt^2}\frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2y}{ds^2} \\
&= \frac{\alpha}{2}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}}\frac{dy}{ds} + \frac{k}{m}e^{-\alpha t}\frac{d^2y}{ds^2}
\end{aligned}
$$

All this nonsense allows us to alter the original equation as follows. (The change from the third to the fourth line involves dividing by m, then dividing by α^2, then multiplying by 4.)

$$
\begin{aligned}
my'' + ke^{-\alpha t} &= 0 \\
m\left(\frac{d^2s}{dt^2}\frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2y}{ds^2}\right) + ke^{-\alpha t} &= 0 \\
\frac{m\alpha}{2}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}}\frac{dy}{ds} + m\frac{k}{m}e^{-\alpha t} + ke^{-\alpha t}y &= 0 \\
\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}}\frac{dy}{ds} + \frac{4}{\alpha^2}\frac{k}{m}e^{-\alpha t}\frac{d^y}{ds^2} + \frac{4k}{m\alpha^2}e^{-\alpha t}y &= 0 \\
s^2\frac{d^2y}{ds^2} + s\frac{dy}{ds} + s^2y &= 0
\end{aligned}
$$

This is Bessel's equation with $\nu = 0$. It is solved by Bessel's functions (of the first and second kind) for $\nu = 0$.

$$
\begin{aligned}
y &= AJ_0(s) + BY_0(s) \\
&= AJ_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}}\right) + BY_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\frac{\alpha t}{2}}\right)
\end{aligned}
$$

These functions completely describe the behaviour of an aging spring with exponential decay of the spring constant.

4.5.4 Properties of the Bessel Functions

We have some nice symmetry properties for integer orders. Let $m \in \mathbb{Z}$.

$$J_{-m}(t) = (-1)^m J_m(t)$$
$$J_m(-t) = (-1)^m J_m(t)$$

We mentioned a functional equation for the Legendre polynomials. We have one here as well; this one is true for arbitrary ν.

$$tJ'_\nu(t) = \nu J_\nu(t) - tJ_{\nu+1}(t)$$

There are some interesting properties of Bessel functions with half-integer orders. Look closely again at $J_{\frac{1}{2}}$.

$$\begin{aligned}
J_{\frac{1}{2}}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\frac{3}{2}+n))} \left(\frac{t}{2}\right)^{2n+\frac{1}{2}} \\
\Gamma\left(\frac{3}{2}+n\right) &= \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi} \\
J_{\frac{1}{2}}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}} \left(\frac{t}{2}\right)^{2n+\frac{1}{2}} \\
&= \sqrt{\frac{2}{\pi t}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \\
&= \sqrt{\frac{2}{\pi t}} \sin t \\
J_{-\frac{1}{2}} &= \sqrt{\frac{2}{\pi t}} \cos t
\end{aligned}$$

These special half-integer function can actually be expressed as elementary functions. The decay of the amplitude of these waves is $\frac{1}{\sqrt{t}}$ not e^{-t}; this decay is much slower than exponential decay. The half-integer Bessel functions are called *spherical Bessel functions*. The reason for that name is that they solve the wave equation in spherical coordinates. The wave equation in $\mathbb{R}^3$ is $\nabla^2 A + k^2 A = 0$ for ∇^2 the Laplacian. Changing to spherical coordinates and restricting to the radial terms gives Bessel's equation with half-integer order.

We can think of these half-integer Bessel functions as the standing waves of spherical harmonic systems. $J_{\frac{1}{2}}$ is the first harmoinc, then we get higher harmonics as we go. The typical image of these spherical waves are the decaying amplitude ripples radiating from a stone dropped in a pond.

Chapter 5

Laplace Transforms

5.1 Definitions

In functional analysis, we have a group of operations on functions called transforms. These operations act on a certain set of functions and transform then into new functions: through the transforms, we see behaviour that was previously hidden.

The changes due to transforms are often radical: the resulting functions do not look anything like the originals. We can think of transforms as radically altering the environment of the function, so that everything changes into a surprisingly different form.

For those who like to think in linear algebra terms, we can think of the space of functions defined on some interval in $\mathbb{R}$ as a vector space. (The specific vector space depends on the interval and the class of function: continuous, piecewise-continuous, differentiable, etc). In this language, transforms are nothing but interesting linear transformations between vector spaces of functions.

In this section, we study the Laplace transform. It applies to a functions with certain controls on their asymptotic growth, which we now define.

Definition 5.1.1. A function $f(t)$ defined on $[0, \infty)$ is of *exponential order* c if $\exists M > 0 \;\; \exists T > 0$ such that $\forall t > T \quad |f(t)| < Me^{ct}$. Asymptotically, this is equivalent to $f \in \mathcal{O}(e^{ct})$ for some real positive c.

If $f(t)$ is a piecewise-continuous function on $[0, \infty)$ of exponential order c, then its Laplace transform is

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

The Laplace transform is defined on the domain (c, ∞). We can check that this improper integral converges for all $s \in (c, \infty)$.

The restriction of exponential order is a fairly reasonable one: in differential equations and applied mathematics, we are rarely concerned with functions which grow faster than the exponential.

The choice of variables is standard for Lapalce transforms. We will often refer to the original functions as functions in the t-domain, and to their transforms as functions in the s-domain. In addition, if $f(t)$ and $g(t)$ are function in the t-domain, we will also often write $F(s)$ and $G(s)$ for their Laplace transforms. By convention, we use lower case for the t-domain and uppercase for the matching function in the s-domain.

Most of the following examples are sub-exponential, so we expect a transform on $(0, \infty)$. Note that we are never guaranteed convergence at 0.

Example 5.1.2.

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st} dt = \left. \frac{-e^{-st}}{s} \right|_0^\infty = \lim_{a \to \infty} \frac{-e^{-sa} + 1}{s} = \frac{1}{s} \\
\mathcal{L}\{t\} &= \int_0^\infty t e^{-st} dt = \lim_{a \to \infty} \left[\left. \frac{-te^{-st}}{s} \right|_0^a + \int_0^a \frac{e^{-st}}{s} dt \right] \\
&= \lim_{a \to \infty} \left[\frac{0 - ae^{-sa}}{s} + \left. \frac{-e^{-st}}{s^2} \right|_0^a \right] = \frac{1}{s^2} \\
\mathcal{L}\{t^n\} &= \int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}} \\
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{at-st} dt = \left. \frac{e^{at-st}}{a-s} \right|_0^\infty = \frac{-1}{a-s} = \frac{1}{s-a} \qquad s \in (a, \infty)
\end{aligned}$$

Our first observation about Laplace transforms is simply their strangeness. Powers of t turn into inverse powers of s, but exponentials (which are very different) turn into very similar reciprocals.

Example 5.1.3. We said that Lapalce transforms exist for piecewise-continuous functions with a certain exponential order. Here is an example for a piecewise continuous function and its transform.

$$\begin{aligned}
f(t) &= \begin{cases} 0 & t \in [0,1] \\ t & t \in [1, \infty) \end{cases} \\
\mathcal{L}\{f(t)\} &= \int_0^\infty f(t) e^{-st} dt \\
&= \int_1^\infty te{-}stdt = \left. \frac{-te^{-st}}{s} \right|_1^\infty + \int_1^\infty \frac{e^{-st}}{s} dt \\
&= \frac{-e^{-s}}{s} + 0 + \left. \frac{-e^{-st}}{s^2} \right|_1^\infty = \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} = \frac{-(s+1)e^{-s}}{s^2}
\end{aligned}$$

The Laplace transform is not a piecewise function, even thought the original was.

Example 5.1.4. The Laplace transform of t^n for $n \in \mathbb{Z}$ involved factorials, so it is not surprising that t^α for non-integer α involves the extension of the factorial: the Γ function.

$$\begin{aligned}
\mathcal{L}\{t^\alpha\} &= \int_0^\infty e^{-st} t^\alpha dt \\
&= \int_0^\infty t^{(\alpha+a)-1} e^{-st} \\
u &= st \\
&= \int_0^\infty \left(\frac{u}{s}\right)^{(\alpha+1)-1} e^{-u} \frac{du}{s} \\
&= \frac{1}{s^{\alpha+1}} \int_0^\infty u^{(\alpha+1)-1} e^{-u} du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}
\end{aligned}$$

Example 5.1.5. Finally, Laplace transforms are defined for some important oscillating functions, such as the trigonometric functions and the Bessel functions.

$$\begin{aligned}
\mathcal{L}\{\sin t\} &= \int_0^\infty e^{-st} \sin t dt = \frac{1}{s^2+1} \\
\mathcal{L}\{\cos t\} &= \frac{s}{s^2+1} \\
\mathcal{L}\{J_0(kt)\} &= \frac{1}{\sqrt{s^2+k^2}}
\end{aligned}$$

Again, note the strangeness: even starting with transcendental and non-elementary functions like these, the Lapalce transforms are rational and algebraic functions.

As an aside, I mentioned at the start of this chapter that the Laplace transform is not the only transform in mathematics. The most well-known and well-used transform is the Fourier transform. It uses the same conventions about the s and the t domains, but it applies to complex valued functions. In place of the e^{-st} term in the Laplace transform, the Fourier transform uses the complex $e^{-2\pi \imath st}$.

Definition 5.1.6. The *Fourier transform* of a function $f(s)$ is given by this integral.

$$\hat{f}(s) = \int_{-\infty}^\infty f(t) e^{-2\pi \imath ts} dt$$

5.2 Properties of the Laplace Transform

5.2.1 First Properties

Proposition 5.2.1. *The Laplace transform (like all reasonable operations in analysis) is linear.*

$$\begin{aligned}
\mathcal{L}\{f+g\} &= \mathcal{L}\{f\} + \mathcal{L}\{g\} \\
\mathcal{L}\{cf(t)\} &= c\mathcal{L}\{f\}
\end{aligned}$$

As we will discover, the Laplace transform is not multiplicative. The question of how we deal with products ($\mathcal{L}\{fg\}$) will come later.

Second, the asymptotic order of f was important to allowing the definition of the Laplace transform. However, all the transforms we've seen so far are decaying functions. This is a general statment: for any function $f(t)$ of exponential order $c > 0$, we have

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\}(s) = 0.$$

This property is a useful check: if you ever get a Laplace transform which doesn't have this limit, you've made a mistake.

For the same students who like linear algebra, this gives us a specific understanding of the Laplace transform. Let $EOF(0,\infty)$ be the space of piecewise-continuous functions of exponential order defined on $(0,\infty)$ (EOF for exponential order functions). Let DF^+ be the set of continuous functions which are defined on (a,∞) for some $a > 0$ and have limit 0 as $s \to \infty$ (DF for decay functions). These are both linear spaces, since addition, subtraction and multiplication by constants preserve the properties of definition. The the Laplace transform is a linear map

$$\mathcal{L} : EOF(0,\infty) \to DF^+.$$

5.2.2 Shifts

We said we'd return to Laplace transforms of products. We still can't handle the general case, but there is a nice property for multiplication by an exponential function.

Proposition 5.2.2.

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\}(s) &= \int_0^\infty e^{at}f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{at-st}dt = \int_0^\infty f(t)e^{(a-s)t}dt \\ &= \mathcal{L}\{f(t)\}(s-a)\end{aligned}$$

Multiplication by e^{at} in the t domain results in a shift by a in the s domain. This also adjust the domain of s, reflected the fact that multiplication by e^{at} adjusts the exponential order.

Shifts are quite important in the whole theory. We can also ask what happens when we shift in the t domain.

Definition 5.2.3. The *unit step function* $u_a(t)$ if defined as

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

The unit step function $u_a(t)f(t)$ cancels any part of f for $t < a$, but preserves f for $t \geq a$. To define a shift of f, we can write $u_a(t)f(t-a)$ to remove anything before $t = 0$ from f. (Note that we're allowed piecewise continuous function for Laplace transforms.) We calculate to find the result of this shift in the t-domain.

$$\begin{aligned}
\mathcal{L}\{u_a(t)\}(s) &= \int_0^\infty u_a(t)e^{-st}dt = \int_a^\infty e^{-st}dt \\
&= \frac{-e^{-st}}{s}\bigg|_a^\infty = \frac{e^{-as}}{s} \\
\mathcal{L}\{u_a(t)f(t-a)\}(s) &= \int_a^\infty e^{-st}f(t-a)dt \\
v &= t-a \\
&= \int_0^\infty e^{-s(v+a)}f(v)dv = e^{-sa}\int_0^\infty e^{-sv}f(v)dv \\
&= e^{-sa}\mathcal{L}\{f(t)\}(s) = e^{-sa}F(s)
\end{aligned}$$

There is a nice parallel here, though it's not a perfect symmetry. Shifts in one domain correspond to multiplication by exponentials in the other domain.

5.3 Distributions

In the world of functional analysis and the use of transforms, it turns out we can extend our notion of a function in strange and novel ways. While I'm not going to give a formal definition, these extensions are called distributions. The basic idea is that a distribution may not be a well-defined function, but it is something that behaves well in integration. (If the word 'distribution' reminds you of probability and statistics, that's a good intuition. These distributions are very similar to distributions used in statistics).

The distribution we will be using in this section is the δ-function. (The name is terrible, since it is most certainly *not* a function.)

Definition 5.3.1. The δ-function is a distribution with values

$$\delta_a(t) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

This is not a function, since ∞ is not a valid output for a function. However, this does give us an intuition. We can think of $\delta_a(t)$ as a thing which is zero away from $t = a$ and has an infinitely tall spike at $t = a$. Often, this is defined by a limit.

Let $b > 0$ and consider the bell curve function $\frac{\sqrt{b}}{\sqrt{\pi}}e^{-b(t-a)^2}$. All of these functions have integral 1, by design of the choice of the coefficient.

$$\int_{-\infty}^\infty \frac{\sqrt{b}}{\sqrt{\pi}}e^{-b(t-a)^2}dt = \frac{\sqrt{b}}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-b(t-a)^2}dt = \frac{\sqrt{b}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{b}} = 1$$

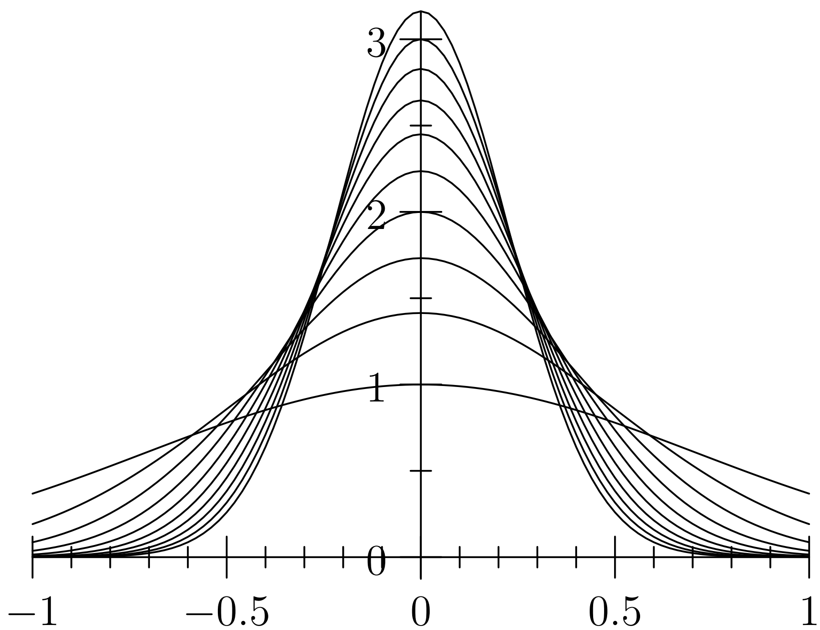


Figure 5.1: Narrower and Narrower Bell Curves

These functions are all bell curves, but they become taller and narrower at b increases. Figure 5.1 shows the progression of these bell curves.

Then we could define $\delta_a(t)$ as

$$\delta_a(t) = \lim_{b\to\infty} \frac{\sqrt{b}}{\sqrt{\pi}} e^{b(t-a)^2}.$$

The delta function is the limit of these functions. The intuition before, of an infinitely narrow and infinitely tall spike, fits this limit process, since the bell curves are becoming taller and narrower with each step. The value of the integral is unchanged for the entire limit process. Since integration works well with limits, the integral of the delta function should be 1.

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1$$

We said that distribution worked well with integration and we just defined the integral of δ_a over $\mathbb{R}$. Now we can think about integrating products $f(t)\delta_a(t)$.

If we integrate $f(t)\frac{\sqrt{a}}{\sqrt{\pi}}e^{-at^2}$, we get a weighted average of $f(t)$ values near a. In the limit, though, only the value at $f(a)$ matters, since we multiply by zero everywhere else. This gives the most convenient proprety of δ-function.

$$\int_{-\infty}^{\infty} \delta_a(t)f(t)dt = f(a)$$

In some sense, the δ-functions are the nicest possible functions to integrate, since their integrals are just evaluations of functions.

Since the δ-function, and distributions in general, work well with integration, we can take their Laplace transforms. The Laplace transform for the δ-function is very easy to calculate.

$$\mathcal{L}\{\delta_a(t)\}(s) = \int_0^\infty \delta_a(t)e^{-st}dt = \begin{cases} 0 & a < 0 \\ 1 & a = 0 \\ e^{-as} & a \geq 0 \end{cases}$$

Again, this is quite an odd result. We started with a distribution which wasn't even a proper function, but it's Laplace transform is a proper, well-behaved differentiable function.

Before we end this section, we can ask why we would define such a strange function. Let's think about harmonic systems and forcing terms again. The δ-function can act as a forcing term; if it does, it represents an instantaneous jolt to the system. The standard image of a harmonic system is a mass on a spring. In this image, a δ-function represents hitting the mass with a hammer at one moment in time. The force only acts for an instant, but it transfers some finite energy and causes a change in the system.

5.4 Laplace Transforms and Derivatives

This is a course on differential equations; if Laplace transforms are useful, we will need them to relate to derivatives. Let's calculate what happens to a derivative of a function (of exponential order) in a Laplace transform.

$$\begin{aligned} \mathcal{L}\left\{\frac{df}{dt}\right\} &= \int_0^\infty \frac{df}{dt}e^{-st}dt \\ &= fe^{-st}\Big|_0^\infty - \int_0^\infty (-s)f(t)e^{-st}dt \\ &= \lim_{a\to\infty} f(a)e^{-sa} - f(0) + s\mathcal{L}\{f(t)\}(s) = -f(0) + sF(s) \end{aligned}$$

We can sumarize the result in a rule.

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + sF(s).$$

This is a lovely and convenient property. In the s-domain, there is no more differentiation. We just have multiplication by the s term and evaluation of the function at a point.

We can do a similar calculation for second derivatives. The calculation will involve two uses of integration by parts, though we can make use of the previous calcullation to simplify the work.

$$\begin{aligned} \mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} &= \int_0^\infty \frac{d^2f}{dt^2}e^{-st}dt \\ &= \frac{df}{dt}e^{-st}\Big|_0^\infty + s\int_0^\infty \frac{df}{dt}e^{-st}dt \\ &= -f'(0) - sf(0) + s^2F(s) \end{aligned}$$

Again, this is very convenient. The derivatives are completely removed in the s-domain, replaced with simple algebraic operations. The general result for any order of differentiation is

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0).$$

This leads to the amazing use of Laplace transforms in solving DEs. A Laplace transform change differentiation into simple algebraic operations; therefore, it should change some DEs into algebraic equations. Since algebraic equations are much easier to solve than DEs, this is a huge advantage. However, we have one problem. If we want to solve a DE, we want to solve it in the t-domain. We can transform to the s-domain and solve the algebraic equation, but we need to get back to the t-domain to finish. For this, we need to invert the transform, which we will defined in the next section.

Before we move on, here are two other properties of the Laplace transform which relate to derivatives. The first shows that integrals are also changed into something algebraic.

$$\mathcal{L}\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

The second is a parallel identity which shows how differentiation in the s-domain also comes from an algebraic operation on the t-domain.

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

5.5 Inverse Laplace Transforms

We would like an operation $\mathcal{L}^{-1}$ which undoes the Laplace transform.

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}\right\} = f(t)$$

This is very difficult to calculate directly, and the construction of the inverse transform is beyond the scope of this course. (It relies on complex ananlysis.) I'll state the result for completeness.

Theorem 5.5.1. *Let α be a real number which is larger than all real singularities of $F(s)$. Let γ be a path in $\mathbb{C}$ which goes in a straight line from $\alpha - \imath T$ to $\alpha + \imath T$.*

$$\mathcal{L}^{-1}(F(s)\} = \frac{1}{2\pi\imath}\lim_{T\to\infty}\int_\gamma e^{st}F(s)ds$$

Note this is a contour integral in $\mathbb{C}$, a common object in the study of complex variables. The use of the distance to singularities makes sense in that discipline, since contour integrals depend on the location of singularities. For those with a background in complex variable, note that this is not a closed curve and the homotopy implications depend a great deal on the function F.

For our purposes, we'll just use tables to find inverse Laplace transforms, using the functions we already know. We'll try to turn the s domain answers into familiar forms and invert known functions. This is a bit tricky, but can often be done. The use of shifts will be particularly inportant. We should note that $\mathcal{L}^{-1}$ is also linear.

We can make a table of the most important inverse transforms.

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$
$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$
$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$
$$\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$$
$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$
$$\mathcal{L}^{-1}\left\{e^{-sa}\right\} = \delta_a(t)$$

The shift in an inverse transform is captured by the rule

$$\mathcal{L}^{-1}\left\{e^{-sa}F(s)\right\} = u_a(t)f(t-a).$$

This shifts of trigonometric functions are common enough that it is useful to mention them here.

$$\mathcal{L}^{-1}\left\{\frac{\beta}{(s+\alpha)^2+\beta^2}\right\} = e^{-\alpha t}\sin\beta t$$
$$\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\beta^2}\right\} = e^{-\alpha t}\cos\beta t$$

5.6 Using Laplace Transforms to Solve Linear DEs

As we've seen, Laplace transforms turn derivatives into algebraic operations. Therefore, particularly for certain linear equations, we can expect Laplace transforms to turn DEs into algebraic equations. We'll start with a very well known case: homogeneous second order constant coefficient linear equations.

Example 5.6.1.

$$y'' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

We apply the Laplace transform, making use of the initial values when we transform the derivative.

$$\begin{aligned}
\mathcal{L}\{y'' + y\} &= 0 \\
(s^2Y - sy(0) - y'(0)) + Y &= 0 \\
s^2Y - s + Y &= 0 \\
(s^2+1)Y &= s \\
Y &= \frac{s}{s^2+1} \\
y &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t
\end{aligned}$$

We recover the expected $y(t) = \cos t$, but without any calculation of the characteristic equation or interpretation of complex roots. We also didn't have to get the complete solution and solve for unknown constants, since we made use of the initial values in the process. If initial values were not given, we would have to use unknown constants in their place in the calculation.

Example 5.6.2. Let's move on to a move involved harmonic system and assume that $b^2 - 4mk < 0$, so that we know to expect sinusoidal solutions.

$$my'' + by' + ky = 0 \qquad y(0) = 1 \qquad y'(0) = \frac{-b}{2m}$$

Then we can apply the Laplace transform to the entire equation.

$$\begin{aligned}
\mathcal{L}\{my'' + by' + ky\} &= \mathcal{L}\{0\} \\
m(s^2Y - sy(0) - y'(0)) + b(sY - y(0)) + kY &= 0 \\
Y(ms^2 + bs + k) - ms - b + \frac{b}{2} &= 0 \\
Y &= \frac{ms + \frac{b}{2}}{ms^2 + bs + k} = \frac{s + \frac{b}{2m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \\
&= \frac{s + \frac{b}{2m}}{\left(s^2 + \frac{b}{m}s + \frac{b^2}{4m^2}\right) + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right)} \\
&= \frac{s + \frac{b}{2m}}{\left(s + \frac{b}{2m}\right)^2 + \left(\frac{4km - b^2}{4m^2}\right)} \\
y &= \mathcal{L}^{-1}\left\{\frac{s + \frac{b}{2m}}{\left(s + \frac{b}{2m}\right)^2 + \left(\frac{4mk - b^2}{4m^2}\right)}\right\} \\
&= e^{-\frac{b}{2m}t}\cos\left(\frac{\sqrt{4mk - b^2}}{2m}t\right)
\end{aligned}$$

Now let's add a forcing term. Since we can take a Laplace transform of a delta function, we'll use that for the forcing term. $F\delta_0(t)$ is a sudden impact with force F (in appropriate units) at time t. We'll use initial values of $y(0) = y'(0) = 0$, so that the system is initially at rest. Again, we'll assume that $b^2 - 4km < 0$ for harmonic motion.

$$
\begin{aligned}
my'' + by' + ky &= F\delta_0(t) \\
\mathcal{L}\{my'' + by' + ky\} &= \mathcal{L}\{F\delta_0(t)\} \\
m(s^2Y - sy(0) - y'(0)) + b(sY - y(0)) + kY &= F \\
(ms^2 + bs + k)Y &= F \\
Y &= \frac{F}{ms^2 + bs + k} \\
Y &= \frac{\frac{F}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \\
Y &= \frac{\frac{F}{m}}{\left(s^2 + \frac{bs}{m} + \frac{b^2}{4m^2}\right) + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right)} \\
Y &= \frac{F}{m}\frac{2m}{\sqrt{4km - b^2}}\frac{\frac{\sqrt{4km-b^2}}{2m}}{\left(s + \frac{b}{2m}\right)^2 + \left(\frac{4km-b^2}{4m^2}\right)} \\
y(t) &= \frac{2m}{4km - b^2}\frac{F}{m}e^{\frac{-b}{2m}t}\sin\left(\frac{\sqrt{4km - b^2}}{2m}t\right) \\
y(t) &= \frac{2F}{4km - b^2}e^{\frac{-b}{2m}t}\sin\left(\frac{\sqrt{4km - b^2}}{2m}t\right)
\end{aligned}
$$

We could ask: what changes if we move the impact to a later time? If the impact is at $t = 4$, then the forcing term is $\delta_4(t)$. We again proceed with the Laplace transform.

$$\begin{aligned}
my'' + by' + ky &= F\delta_4(t) \\
\mathcal{L}\{my'' + by' + ky\} &= \mathcal{L}\{F\delta_0(t)e^{-4s}\} \\
m(s^2Y - sy(0) - y'(0)) + b(sY - y(0)) + kY &= Fe^{-4s} \\
(ms^2 + bs + k)Y &= Fe^{-4s} \\
Y &= \frac{Fe^{-4s}}{ms^2 + bs + k} \\
Y &= \frac{\frac{Fe^{-4s}}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \\
Y &= e^{-4s}\frac{\frac{F}{m}}{\left(s^2 + \frac{bs}{m} + \frac{b^2}{4m^2}\right) + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right)} \\
Y &= e^{-4s}\frac{F}{m}\frac{2m}{\sqrt{4km - b^2}}\frac{\frac{\sqrt{4km-b^2}}{2m}}{\left(s + \frac{b}{2m}\right)^2 + \left(\frac{4km-b^2}{4m^2}\right)} \\
y(t) &= u_4(t)\frac{2m}{4km - b^2}\frac{F}{m}e^{\frac{-b}{2m}(t-4)}\sin\left(\frac{\sqrt{4km - b^2}}{2m}(t-4)\right) \\
y(t) &= u_4(t)\frac{2F}{4km - b^2}e^{\frac{-b}{2m}(t-4)}\sin\left(\frac{\sqrt{4km - b^2}}{2m}(t-4)\right)
\end{aligned}$$

Unsurprinsingly, we get precisely the same wave, just shifted 4 units to the right.

Example 5.6.3. This example has an exponential forcing term.

$$\begin{aligned}
y'' + 4y' + 4y &= e^{-2t} \\
y(0) &= 0 \qquad\qquad y'(0) = 2 \\
s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 4Y &= \frac{1}{s+2} \\
s^2Y - 2 + 4sY + 4Y &= \frac{1}{s+2} \\
(s^2 + 4s + 4)Y &= \frac{1}{s+1} + 2 = \frac{2s+3}{s+2} \\
Y &= \frac{2s+3}{(s+2)(s+2)^2} = \frac{2s+3}{(s+2)^3} \\
Y &= \frac{2}{(s+2)^2} + \frac{1}{(s+2)^3} \\
y &= 2te^{-2t} + t^2e^{-2t} = e^{-2t}(2t + t^2)
\end{aligned}$$

Example 5.6.4. This example has a linear forcing term.

$$\begin{aligned}
y'' + 4y &= 4t^2 - 4t + 10 \\
y(0) &= 0 \qquad\qquad y'(0) = 3 \\
s^2Y - sy(0) - y'(0) + 4Y &= \mathcal{L}\{4t^2 - 4t + 10\} \\
s^2Y + 4sY - 3 &= \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} \\
Y &= \frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)} \\
Y &= \frac{2s^2 - s + 2}{s^3} + \frac{-2s+4}{s^2+4} \\
&= \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} - \frac{2s}{s^2+4} + \frac{4}{s^2+4} \\
y &= 2 - t + t^2 - 2\cos 2t + 2\sin 2t
\end{aligned}$$

Example 5.6.5. Since we can take Laplace transforms of piecewise-continuous functions, here is an example with a piecewise forcing term.

$$f(t) = \begin{cases} 0 & t \leq \pi \\ 3\cos t & t > \pi \end{cases}$$

This represents a sinusoidal forcing term which is turned of at time $t = \pi$. We calculate the Lapalce transform of this piecewise function.

$$\begin{aligned}
y'' + y &= f(t) \qquad\qquad y(0) = 0 \qquad y'(0) = 0 \\
s^2Y + Y &= 3\mathcal{L}\{u_\pi(t)\cos t\} = 3\mathcal{L}\{-u_\pi(t)\cos(t-\pi)\} \\
&= -3e^{-\pi s}\frac{s}{s^2+1} \\
(s^2+1)Y &= \frac{-3se^{-\pi s}}{s^2+1} \\
Y &= \frac{-3se^{-\pi s}}{(s^2+1)^2} \\
Y &= -3e^{-\pi s}\frac{s}{(s^2+1)^2} \\
Y &= -3e^{-\pi s}\frac{1}{2}\frac{d}{dt}\frac{-1}{s^2+1} \\
y &= \frac{-3}{2}u_\pi(t)(t-\pi)\sin(t-\pi)
\end{aligned}$$

Notice that the forcing term is discontinuous, representing a sudden change, but the resulting solution is still continuous. The force suddenly changes, but the system still responds continuously.

5.7 Laplace Transforms of Periodic Functions

Laplace transforms cooperate well with periodic functions. We've already see this for sine and cosine, but it is also true for arbitrary, even discontinuous periodic functions.

Example 5.7.1. This is the square wave.

$$f(t) = \begin{cases} 0 & t \in [2n+1, 2n+2] \\ 1 & 1 \in (2n, 2n+1) \end{cases}$$

Example 5.7.2. This is the sawtooth wave.

$$f(t) = \{t - n \qquad t \in [n, n+1) \qquad \forall n \in \mathbb{N}$$

For each of these functions, doing an integration over $(0, \infty)$ is problematric. Instead, we have a convenient theorem for Laplace transforms of such waves.

Proposition 5.7.3. *If $f(t)$ is periodic with period T, and of exponential order, then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

Example 5.7.4. Let $f(t)$ be the square wave, which has period 2. Then we can use f as a forcing term and solve the following DE with a Laplace transform (we factor $1 - e^{-2s}$ as a difference of squares in this calculation).

$$\begin{aligned}
y' + by &= f(t) \\
sY + bY &= \frac{1}{1 - e^{-2s}} \int_0^2 f(t) e^{-st} dt \\
&= \frac{1}{1 - e^{-2s}} \int_0^1 e^{-st} dt \\
&= \frac{1}{1 - e^{-2s}} \left. \frac{e^{-st}}{-s} \right|_0^1 = \frac{-(e^{-s} - 1)}{s(1 - e^{-2s})} \\
&= \frac{1}{s(1 + e^{-s})} \\
Y &= \frac{1}{s(s+b)(1 - (-e^{-s}))}
\end{aligned}$$

This expression is problematic for an inverse transform. Specficially, the $\frac{1}{1-e^{-s}}$ term is the problem.

We solve the problem by expressing it as a geometric series.

$$
\begin{aligned}
Y &= \frac{1}{s(s+b)(1-(-e^{-s})} \\
&= \frac{1}{s(s+b)}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots\right) \\
&= \frac{1}{b}\left(\frac{1}{s} - \frac{1}{s+b}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots\right) \\
&= \frac{1}{b}\left(\frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \ldots\right) \\
&\qquad - \frac{1}{b}\left(\frac{1}{s+b} - \frac{e^{-s}}{s+b} + \frac{e^{-2}}{s+b} - \frac{e^{-3}}{s+b} + \ldots\right) \\
y &= \frac{1}{b}\left(1 - u_1(t) + u_2(t) - u_3(t) + \ldots\right)) \\
&\qquad - \frac{1}{b}\left(e^{-bt} - u_1(t)e^{-b(t-1)} + u_2(t)e^{-b(t-2)} - u_3(t)e^{-b(t-3)} + \ldots\right) \\
y &= \frac{1}{b}\left(1 - e^{-bt}\right) + \frac{1}{b}\sum_{n=1}^{\infty}(-1)^n u_n(t)(1 - e^{-b(t-n)})
\end{aligned}
$$

We might ask about convergence, since this is an infinite series (though not a taylor series, since its terms are exponentials). We'll leave the technical details of convergence for now, and trust that the solution of the DE is attainable and converges for some reasonable domain.

5.8 Algebraic Properties of the Laplace Transform

We know that the Laplace transform is linear, but we said we would return to the issue of product. To start, it is easier to think in reverse: if we have a product FG in the s-domain, where did it come from in the t-domain? Here is the relevant calculation (the change in the order of integration near the end is jutified by theorems in multivariable analysis).

$$
\begin{aligned}
FG &= \int_0^\infty e^{-st} f(t) df \int_0^\infty e^{-su} g(t) du \\
&= \int_0^\infty \int_0^\infty e^{-s(u+t)} f(t) g(u) dt du \\
&= \int_0^\infty f(t) \left[\int_0^\infty e^{-s(u+t)} g(u) du \right] du \\
v &= u + t \\
&= \int_0^\infty f(t) \left[\int_t^\infty e^{-sv} g(v-t) dv \right) dt \\
&= \int_0^\infty e^{-sv} \left[\int_0^t f(t) g(v-t) dt \right] dv \\
&= \mathcal{L} \left\{ \int_0^t f(t) g(v-t) dt \right\}
\end{aligned}
$$

The product FG in the s-domain turns into this strange integral-based combination of the function f and g. This is a new 'product'; it is called a convolution.

Definition 5.8.1. Let f, g be integrable function on $[0, \infty)$. Their convoluiton is defined by this integral.

$$
f \star g(t) = \int_0^t f(\tau) g(t - \tau) d\tau.
$$

The convolution takes two functions and produces a new function, so it is a product. However, it is a strange product with new properties.

Proposition 5.8.2.

- *The convolution is commutative:* $f \star g = g \star f$.
- *The convolution is associative:* $(f \star g) \star h = f \star (g \star h)$.
- *The convolution is distributive:* $f \star (g \pm h) = f \star g \pm f \star h$.
- *The convolution respect constants.* $a(f \star g) = (af) \star g = f \star (ag)$
- *The Laplace transform turns convolutions into products:*

$$
\mathcal{L} \{f \star g\} = F(s) G(s)
$$

Here is an interesting question: if this is a product, what is the identity? That is, what function g satisfies

$$
f \star g = \int_0^t f(\tau) g(t - \tau) d\tau = f(t).
$$

In this question, the integral needs to evaluate $f(\tau)$ at $\tau = t$. We know a 'function' that does this: δ_0.

$$
f \star \delta_0 = \int_0^t f(\tau) \delta_0(t - \tau) d\tau = f(t)
$$

Proposition 5.8.3. *The δ-function at 0 is the identity for the convolution.*

The convolution behaves well with differentiation.

$$\frac{d}{dt}(f \star g) = \frac{df}{dt} \star g = g \star \frac{dg}{dt}$$

It also behaves well with integration.

$$\int_0^\infty f \star g dt = \int_0^\infty f(t)dt \int_0^\infty g(t)dt$$

Finally, it lets us understand inverse Lapalce transforms of products.

$$\begin{aligned}
\mathcal{L}^{-t}\left\{\frac{k^2}{(s^2+k^2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\frac{k}{s^2+k^2}\right\} \\
&= \sin kt \star \sin kt \\
&= \int_0^t \sin ku \sin(kt-ku)du \\
&\text{Using } \left[\sin A \sin B = \frac{1}{2}\cos(A-B) - \cos(A+B)\right] \\
\mathcal{L}^{-t}\left\{\frac{k^2}{(s^2+k^2)^2}\right\} &= \frac{1}{2}\int_0^t \cos(ku-kt+ku) - \cos(ku+kt-ku)du \\
&= \frac{1}{2}\int_0^t \cos(2ku-2t) - \cos(kt)du \\
&= \frac{1}{2}\left.\frac{\sin(2ku-kt)}{2k}\right|_0^t - \left.\frac{1}{2}u\cos(kt)\right|_0^t \\
&= \frac{\sin(2kt-kt)-\sin(-kt)}{4k} - \frac{t\cos kt}{2} \\
&= \frac{\sin(kt)}{2k} - \frac{t\cos kt}{2} = \frac{\sin kt - kt\cos kt}{2k}
\end{aligned}$$

In addition to differential equations, sometimes we get integral equations in mathematics. Laplace transforms and convolutions can help solve certain types of integral equations.

$$
\begin{aligned}
3t^2 - e^{-t} - \int_0^t f(u)e^{t-u}du &= f(t) \\
3t^2 - e^{-t} - f \star e^t &= f(t) \\
\frac{6}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1} &= F(s) \\
\frac{6}{s^3} - \frac{1}{s+1} &= F(s)\left(1 + \frac{1}{s+1}\right) \\
\frac{6}{s^3} - \frac{1}{s+1} &= F(s)\left(\frac{s}{s-1}\right) \\
F(s) &= \frac{6(s-1)}{s^4} - \frac{(s-1)}{s(s+1)} \\
&= \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1} \\
f &= 3t^2 - t^3 + 1 - 2e^{-t}
\end{aligned}
$$

5.9 More Examples of Solving DEs with Laplace Transforms

Example 5.9.1. Consider a slightly different version of the square wave.

$$
f(t) = \begin{cases} 1 & t \in [2n, 2n+1) \\ -1 & t \in [2n+1, 2n+2) \end{cases}
$$

Using initial conditions $y(0) = y'(0) = 0$ we solve the equation $y'' + y = f(t)$.

$$\begin{aligned}
(s^2+1)Y &= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) dt \\
&= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt \right] \\
&= \frac{1}{1-e^{-2s}} \left[\left. \frac{e^{-st}}{-s} \right|_0^1 + \left. \frac{e^{-st}}{s} \right|_1^2 \right] \\
&= \frac{1}{1-e^{2s}} \left[\frac{-e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} \right] \\
&= \frac{1-2s^{-s}+e^{-2s}}{s(s^2+1)} \frac{1}{1-e^{-2s}} \\
&= \frac{1}{s(s^2+1)} (1-2e^{-s}+e^{-2s})(1-e^{-2s}+e^{-4s}-e^{-6s}+e^{-8s}-\ldots) \\
&= \left(\frac{-s}{s+1} + \frac{1}{s} \right) (1-2e^{-s}+2e^{-3s}-2e^{-5s}+2e^{-7s}-\ldots) \\
y &= 1 - \cos t - 2u_1(t)(1-\cos(t-1)) + 2u_3(t)(1-\cos(t-3)) \\
&\qquad - 2u_5(t)(1-\cos(t-5)) + 2u_7(t)(1-\cos(t-7)) - \ldots \\
&= 1 - \cos t + 2 \sum_{k=0}^{\infty} (-1)^{k+1} u_{2k+1}(t)(1-\cos(t-(2k+1)))
\end{aligned}$$

As with the previous square wave solution in Example 5.7.4, this solution involves an infinite series of shifts, each one slightly further along. Periodic functions always lead to an infinite series of shifts.

Example 5.9.2. This examples also involves a step function.

$$
\begin{aligned}
y'' + 6y' + 5y &= t - tu_2(t) \qquad y(0) = 1 \qquad y'(0) = 0 \\
s^2Y - s + 6sY - 6 + 5Y &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) \\
(s+5)(s+1)Y &= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s} + s + 6 \\
Y &= \frac{1}{s^2(s+1)(s+5)} - \frac{e^{-2s}}{s^2(s+1)(s+5)} + \frac{2e^{-2s}}{s(s+1)(s+5)} \\
&\quad + \frac{s}{(s+5)(s+1)} + \frac{6}{(s+5)(s+1)} \\
&= \frac{1}{5s^2} + \frac{1}{4(s+1)} - \frac{1}{100(s+5)} - \frac{6}{25s} - \frac{e^{-2s}}{5s^2} - \frac{e^{-2s}}{4(s+1)} + \frac{e^{-2s}}{100(s+5)} \\
&\quad + \frac{6e^{-2s}}{25s} - \frac{e^{-2s}}{2(s+1)} + \frac{e^{-2s}}{10(s+5)} + \frac{2e^{-2s}}{5s} + \frac{5}{4(s+5)} - \frac{1}{4(s+1)} \\
&\quad - \frac{3}{2(s+1)} + \frac{3}{2(s+5)} \\
Y &= \frac{1}{5s^2} - \frac{6}{25s} + \frac{3}{20(s+1)} + \left(\frac{-1}{100} + \frac{5}{4} + \frac{3}{2}\right)\frac{1}{s+5} \\
&\quad + e^{-2s}\left(\frac{1}{5s^2} + \left(\frac{6}{25} + \frac{2}{5}\right)\frac{1}{s} + \left(\frac{-1}{4} - \frac{1}{2}\right)\frac{1}{s+1} + \left(\frac{1}{100} + \frac{1}{10}\right)\frac{1}{s+5}\right) \\
&= \frac{1}{5s^2} - \frac{6}{25s} - \frac{3}{2(s+1)} + \frac{137}{50(s+5)} \\
&\quad + e^{-2s}\left(\frac{1}{5s^2} + \frac{16}{25s} - \frac{3}{4(s+1)} + \frac{11}{100(s+5)}\right) \\
y &= \frac{t}{5} - \frac{6}{25} - \frac{3e^{-t}}{2} + \frac{137e^{-5t}}{50} + u_2(t)\left[\frac{(t-2)}{5} + \frac{16}{25} - \frac{3e^{2-t}}{4} + \frac{11e^{10-5t}}{100}\right] \\
y &= \frac{t}{5} - \frac{6}{25} - \frac{3e^{-t}}{2} + \frac{137e^{-5t}}{50} + u_2(t)\left[\frac{t}{5} + \frac{6}{25} - \frac{3e^{2-t}}{4} + \frac{11e^{10-5t}}{100}\right]
\end{aligned}
$$

Example 5.9.3. This example uses two δ-functions, representing two sudden impacts.

$$\begin{aligned}
y'' - 7y' + 6y &= e^t + \delta_2(t) + \delta_4(t) \qquad\qquad y(0) = y'(0) = 0 \\
s^2Y - 7sY + 6Y &= \frac{1}{s-1} + e^{-2s} + e^{-4s} \\
Y &= \frac{1}{(s-1)(s-6)(s-1)} + \frac{e^{-2s}}{(s-6)(s-1)} + \frac{e^{-4s}}{(s-6)(s-1)} \\
&= \frac{-1}{25(s-1)} - \frac{1}{5(s-1)^2} + \frac{1}{25(s-6)} \\
&\quad + e^{-2t}\left(\frac{1}{5(s-6)} - \frac{1}{5(s-1)}\right) \\
&\quad + e^{-4t}\left(\frac{1}{5(s-6)} - \frac{1}{5(s-1)}\right) \\
y &= \frac{e^{-t}}{25} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{d}{dt}\frac{-1}{s-1}\right\} + \frac{1}{25}e^{6t} \\
&\quad \frac{u_2(t)}{5}\left(e^{6(t-2)} - e^{t-2}\right) + \frac{u_4(t)}{5}\left(e^{6(t-4)} - e^{t-4}\right) \\
y &= \frac{e^{-t}}{25} - \frac{te^t}{5} + \frac{1}{25}e^{6t} \\
&\quad \frac{u_2(t)}{5}\left(e^{6t-12} - e^{t-2}\right) + \frac{u_4(t)}{5}\left(e^{6t-24} - e^{t-4}\right)
\end{aligned}$$

Example 5.9.4. This is a ridiculous example with infinitely many δ-functions, representing a new impact at every unit of time.

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} \delta_n(t) \\
y'' + 3y' + 2y &= f(t) \qquad\qquad y(0) = y'(0) = 0 \\
s^2Y + 3sY + 2Y &= \frac{1}{1-e^{-s}} \int_0^1 e{-}st\delta_0(t)dt = \frac{1}{1-e^{-s}} \\
Y &= \frac{1}{1-e^{-s}} \frac{1}{(s+2)(s+1)} \\
Y &= \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\left(1_e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + \ldots\right) \\
y &= e^{-t} - e^{-2t} + u_1(t)\left(e^{-(t-1)} - e^{-(2t-2)}\right) + u_2(t)\left(e^{-(t-2)} - e^{-(2t-4)}\right) \\
&\quad + u_3(t)\left(e^{-(t-3)} - e^{-(2t-6)}\right) + u_4(t)\left(e^{-(t-4)} - e^{-(2t-8)}\right) \\
y &= e^{-t} - e^{-2t} + u_1(t)\left(e^{1-t} - e^{2-2t}\right) + u_2(t)\left(e^{2-t} - e^{4-2t}\right) \\
&\quad + u_3(t)\left(e^{3-t} - e^{6-2t}\right) + u_4(t)\left(e^{4-t} - e^{8-2t}\right) \\
&= \sum_{k=0}^{\infty} u_k(t)\left(e^{k-t} - e^{2(k-t)}\right)
\end{aligned}$$

This is an interesting superposition of decay functions. They all have the same shape with a slightly higher peak. Adding them all up gives something which slowly rises higher and higher, while still trying to decay. Eventually the system does grow beyond any bounds, even with all the decay functions.

Example 5.9.5. This is an example with an integral.

$$\begin{aligned}
\frac{dy}{dt} + 6y(t) + 9\int_0^t y(u)du &= -1 \qquad y(0) = 0 \\
sY + 6Y + 9\frac{Y}{s} &= \frac{-1}{s} \\
s^2 + 6sY + 9Y &= -1 \\
Y &= \frac{-1}{(s+3)^2} = \frac{d}{ds}\frac{1}{s+3} \\
y &= te^{-3t}
\end{aligned}$$

We could try the previous example another way, by first differentiating both sides.

$$\begin{aligned}
\frac{dy}{dt} + 6y(t) + 9\int_0^t y(u)du &= -1 \qquad y(0) = 0 \\
\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y &= 0 \\
r^2 + 6r + 9 &= 0 \implies r = 3 \\
y &= ae^{-3t} + bte^{-3t} \\
y(0) &= 0 \implies a = 0 \\
y &= bte^{-3t}
\end{aligned}$$

In this previous example, we would ask: what about b? In the original method, to match with the -1, we need $b = 1$. In th second method, since we differentiated both sides of the equation, we lost the information that determined the constant b.

Chapter 6

Systems of Differential Equations

So far, our differential equations involves one dependent variable and one independent variable (typically time). It's very common to have two or more variables which all depend on the same independent variable and whose derivatives interact with each other. This interaction produces systems of differential equations involving those functions and their derivatives. This chapter introduces qualitative analysis of systems of DEs and solutions to linear systems using Laplace transforms.

6.1 Key Examples

There are a number of important motiavting examples. We first look to biology, where the interaction of species produces systems of linear equations. The Lokta-Volterra equations model predator-prey relationships. Let $p(t)$ be the population of prey and $q(t)$ be the population of predators. These equations describe the interactions between the two derivatives.

$$\frac{dp}{dt} = ap - bpq$$
$$\frac{dq}{dt} = cpq - dq$$

The coefficients a, b, c and d are all positive coefficients. a is a natural birth rate for the prey, b is the death of prey due to predation, c is the growth rate of predators due to predation, and d is the natural death rate of predators. The product pq represents the number of interactions between predators and prey. The Lokta-Volterra system is a classic system in mathematical biology; however, its nonlinearity makes it difficult to solve and the solutions are not expressible by elementary functions. To help understanding the system without having to actually calculate difficult solutions, we will develop qualitative methods of analysis.

Mathematical biology has many systems similar to Lokta-Volterra. If p and q are two species competing for the same resources, we have a similar two-equation DE model.

$$\frac{dp}{dt} = a_p p \left(1 - \frac{p + b_p q}{c_p}\right)$$
$$\frac{dq}{dt} = a_q p \left(1 - \frac{q + b_q p}{c_q}\right)$$

The coefficients a_p and a_q are the natural growth rates of each species, c_p and c_q are the carrying capacities of the environment for each species, and b_p and b_q measure the effects of competition.

Infection disease models are also systems of differential equations. The SIR model is a good example. In a population exposed to an infection disease, let S be the susceptible population, I the infected population, and R the recovered population. The model is this system of three differential equations.

$$\frac{dS}{dt} = -aIS$$
$$\frac{dI}{dt} = aIS - bI$$
$$\frac{dR}{dt} = bI$$

Here a measures the increase in infection due to interactions between the infected and susceptible population and b measure the natural recovery of the infected population. Many similar models exist to model infection disease.

Mathematical biology only gets more complex and involved from here. The following link shows the system of DEs in the 2004 Molecular Biology of the Cell Paper "Integrative Analysis of Cell Cycle Control in Budding Yeast" by K.C. Chen et. al.

`http://www.ncbi.nlm.nih.gov/pmc/articles/PMC491841/table/tbl1/`

Examples exist in other disciplines as well. If we go back to physics, we can look at the physics of the coupled spring. In this sytem, we have a mass m_1 on a spring with spring constant k_1 attached to a fixed object. Then we have a second mass m_2 attached to the first by a spring with spring constant k_2. Analyzing the forces and using Newton's law of motion results in a system of DEs.

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1)$$

6.2 Qualitative Methods

For first order autonomous equations, we used a phase-line analysis to determine the steady states and the trajectories of movement between the steady states. We analyzed the stability of the steady states, and the phase line gave a fairly complete picture of the behaviour of the system depending on an initial value. For systems of DEs, the same analysis applies; however, for each function involved in the system, we get an additional dimension of analysis. In this section, for ease of drawing the analysis, we restrict to systems with two functions and two equations.

Let's write $p(t)$ and $q(t)$ for the two functions. As with phase-line analysis, we need autonomous equations. An autonomous system does not involve the independent variable explicitly, so it can be written in the form

$$\frac{dp}{dt} = f(p, q)$$
$$\frac{dq}{dt} = g(p, q).$$

The qualititative analysis takes place in the plane, which we call the phase-plane. The axes are the values of the functions p and q, so that an isolated point represents a starting value for p and q. We would like to identify the steady states, look at the trajectories, and analyze the stability of the steady states. We lay out the process.

- Draw the locus $f = 0$ (this is called the *nullcline* of f). Do likewise for g (the nullcline of g). Label where $f > 0$ and $f < 0$ on either side of the nullcines, and likewise for g.
- The steady states exists at the intersections of the nullclines, where for f and g are zero.
- Regions where $f > 0$ and $g > 0$ have growth in both variables. Label these with an arrow pointing up and right. Do likewise for the other three cases: $f > 0$ and $g < 0$ has an arrow pointing up and left; $f < 0$ and $g > 0$ has an arrow pointing down and right; and $f < 0$ and $g < 0$ has an arrow pointing down and left.
- On the nullclines of f away from the steady states, use the value of g to determine the movement along the nullclines and label that with an arrow. Do the same for the nullclines of g.
- The solution trajectories are paths in the plane which roughly follow the arrows.

We want to know the stability of the steady states in the model. Recall for phase-linea analysis that the stability of the steady states was a relatively simple situation. There were only three cases: stable, unstable, and stable only from one side. For the phase-plane, there are many different behaviours for steady states. However, there are six frequent behaviours of steady states which we classify in the following list.

- The steady state can be entirely attractive, where all the nearby trajectories tend towards the steady state. If the trajectories tend towards the steady state with relatively little rotation, this is called a stable focus.
- The steady state can be entirevly repulsive, where all the nearby trajectories tend away from the steady state. If the trajetories tend directly away from the steady state with relatively little rotation, this is called an unstable focus.
- The steady state can be partially stable, where there is an axis which is attractive and an axis which is respulsive. This is called a saddle point.

- If the steady state is entirely attractive but the trajectories spiral inwards, then the steady state is a stable node.
- Similarly, if the steady state is entirely repulsive and the trajectories spiral outwards, then the state state is an unstable node.
- Finally, the steady state can be neither attracitve of repuslive and the nearby trajectories simply form periodic loops around the steady state. This is called a centre.

Figure 6.1 to 6.6 show the six cases. In each figure, vertical and horizontal lines are the nullclines and the intersection is the steady state. The directions of movement are shown on the left and the trajectories are shown on the right.

These six cases, while the most common and the most important, are not the only possible behaviours. Many other strange behaviours and stability situations are possible. However, we will restrict our attention to systems that display one of these six behaviours.

(a) The Nullclines and Directions (b) The Trajectories

Figure 6.1: The Steady State is a Stable Focus

(a) The Nullclines and Directions (b) The Trajectories

Figure 6.2: The Steady State is an Unstable Focus

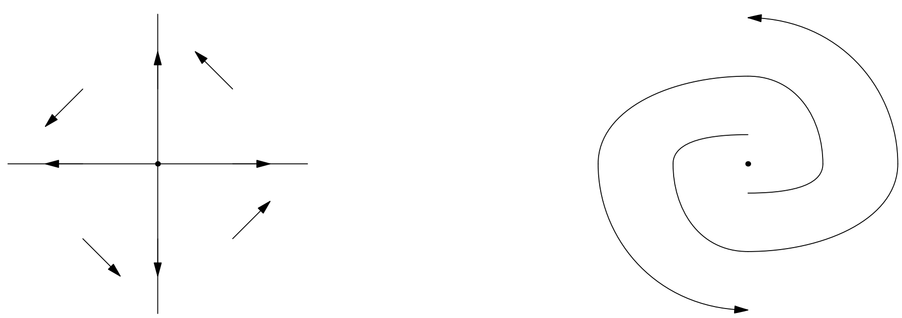
Example 6.2.1. Let's start with Lokta-Volterra. In the above notation, Lokta-Volterra is described

(a) The Nullclines and Directions (b) The Trajectories

Figure 6.3: The Steady State is a Saddle Point

(a) The Nullclines and Directions (b) The Trajectories

Figure 6.4: The Steady State is a Stable Node



(a) The Nullclines and Directions (b) The Trajectories

Figure 6.5: The Steady State is an Unstable Node

by two DEs, with these functions.

$$f(p,q) = ap - bpq$$
$$g(p,q) = cpq - dq$$

(a) The Nullclines and Directions　　(b) The Trajectories

Figure 6.6: The Steady State is a Centre

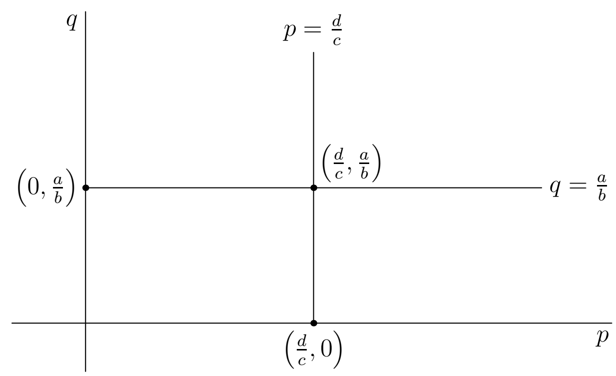


Figure 6.7: The Nullclines for Lokta-Voltera

We need to graph the zero-loci of the right sides.

$$ap - bpq = 0 \implies p(a - bq) = 0$$
$$cpq - dq = 0 \implies q(cp - d) = 0$$

The loci we get are the two axes $p = 0$ and $q = 0$, the horizontal line $q = \frac{a}{b}$ and the vertical line $p = \frac{d}{c}$. There are four intersections points, $(0,0)$, $\left(\frac{d}{c}, 0\right)$, $\left(0, \frac{a}{b}\right)$ and $\left(\frac{d}{c}, \frac{a}{b}\right)$. The later is the only steady state with non-zero values, so the only possibility for a steady state which actually involves both species. The nullcines are shown in Figure 6.7 and the directions in Figure 6.8.

The signs of $f(p, q)$ and $g(p, q)$ give the trajectory directions in each portion of the phase plane and on the nullclines. We can use those directions to sketch an idea of the trajectories of the system. For Lokta-Volterra, these directions show something which is vaguely circular or elliptical, as in Figure 6.9.

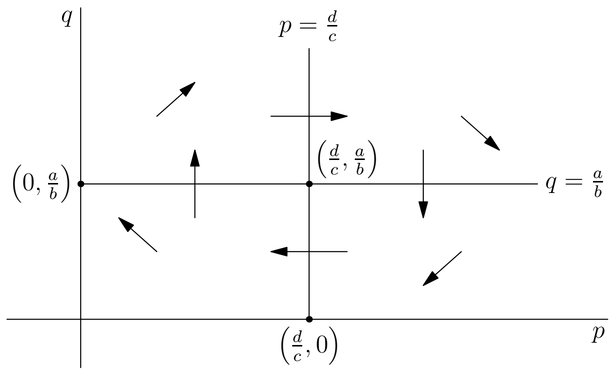


Figure 6.8: The Directions for Lokta-Voltera

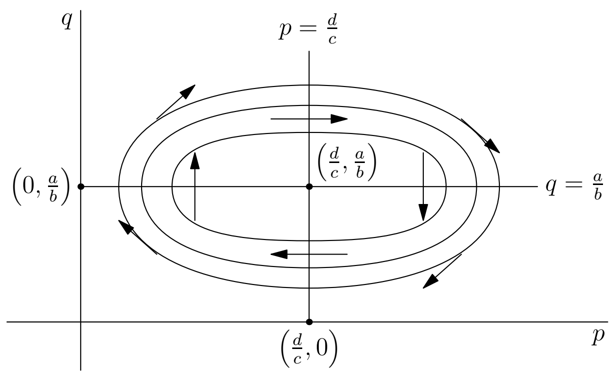


Figure 6.9: The Trajectories for Lokta-Voltera

Example 6.2.2. This is a competition example.

$$\frac{dp}{dt} = \frac{1}{10}p\left(1 - \frac{p + \frac{1}{10}q}{100}\right)$$
$$\frac{dq}{dt} = \frac{1}{5}\left(1 - \frac{q + \frac{1}{5}p}{200}\right)$$

The zero loci of the right sides determine the nullclines.

$$\frac{1}{10}p\left(1 - \frac{p + \frac{1}{10}q}{100}\right) = 0$$
$$\frac{1}{5}\left(1 - \frac{q + \frac{1}{5}p}{200}\right) = 0$$

Again, the axes $p = 0$ and $q = 0$ are nullclines. In addition, we have the lines $p = \frac{-q}{10} + 100$ an $p - \frac{-5q}{+}1000$. Figure 6.10 shows the graph of the nullclines with the directions of movement added.

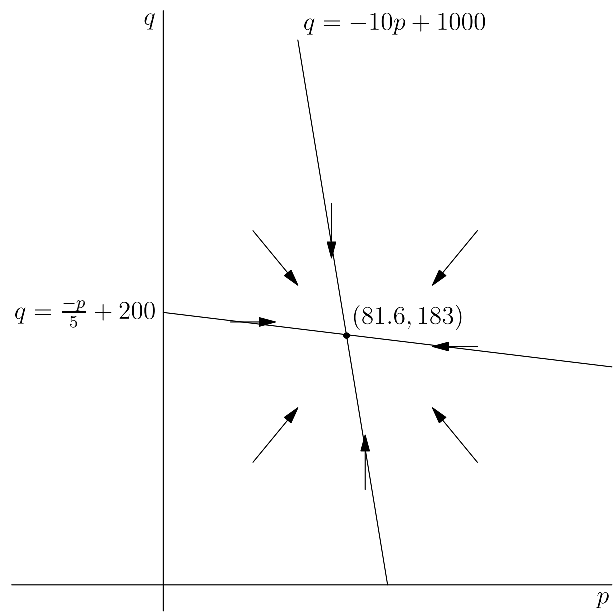


Figure 6.10: The Nullclines and Directions for a Competition Example

The steady state is a stable focus. The trajectories are all inwards towards the steady state. Therefore, we can conclude that there is a stable equilibrium between the two competing species and that they populations will approach this equilibrium over time.

6.3 Linear Systems

6.3.1 Qualitative Analysis

A linear system of two differential equations has the form

$$x' = ax + by$$
$$y' = cx + dx.$$

The nullclines of a linear system are the lines $y = \frac{-a}{b}x$ and $y = \frac{-c}{d}$. The only steady state is $(0, 0)$. The six behaviours listed in the previous section (stable/unstable node, stable/unstable focus, saddle, centre) are all possible. However, for linear systems, these six cases are the *only* possible cases.

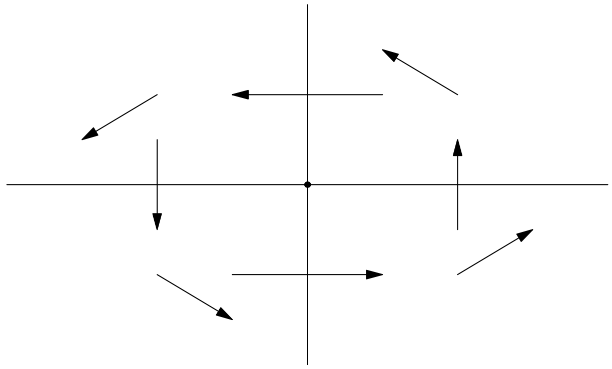


Figure 6.11: First Linear System Example

6.3.2 Laplace Transforms for Linear Systems

Example 6.3.1. Let's start with a simple example. We can solve linear systems with Laplace transforms, since the Laplace transform will be a linear system of algebraic (not differential) equations. We solve these with basic techniques, such as isolating and replacing, or with linear algebra.

$$\begin{aligned}
x' &= -y \\
y' &= x \\
x(0) &= 0 \qquad y(0) = 1 \\
sX &= -Y \implies X = \frac{-Y}{s} \\
sY - 1 &= X \implies sY - 1 = \frac{-Y}{S} \\
sY + \frac{Y}{s} &= 1 \\
Y &= \frac{s}{s^2+1} \\
X &= \frac{-Y}{s} = \frac{-1}{s^2+1} \\
y &= \cos t \\
x &= -\sin t
\end{aligned}$$

In this case, the nullclines are simply the axes. The steady state $(0,0)$ is a centre. Figure 6.11 shows the nullclines and directions.

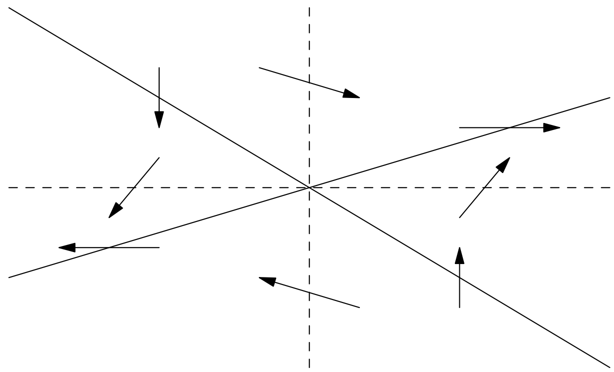


Figure 6.12: Second Linear System Example

Example 6.3.2.

$$
\begin{aligned}
x' &= 2x + 2y & x(0) &= 1 \\
y' &= 2x - y & y(0) &= 1 \\
sX - 1 &= 2X + 2Y \\
sY - 1 &= 2X - Y \implies X = \frac{2Y+1}{s-2}
\end{aligned}
$$

$$
\begin{aligned}
sY - 2\left(\frac{2Y+1}{s-2}\right) + Y &= 1 \\
(s+1)Y - \frac{4Y}{s-2} &= 1 + \frac{2}{s-2} \\
\left(\frac{s^2-s-6}{s-2}\right) Y &= 1 + \frac{2}{s-2} \\
Y &= \frac{s-2}{(s-3)(s+2)} + \frac{2}{(s-3)(s+2)} = \frac{s}{(s-3)(s+2)} \\
&= \frac{\frac{2}{5}}{s+2} + \frac{\frac{3}{5}}{s-3} \\
X &= \frac{2Y+1}{s-2} = \frac{2\left(\frac{s}{(s-3)(s+2)}\right) + 1}{s-2} = \frac{2s + s^2 - s - 6}{(s-3)(s-2)(s+2)} \\
&= \frac{(s+3)(s-2)}{(s-3)(s-2)(s+2)} = \frac{s+3}{(s-3)(s+2)} = \frac{\frac{6}{5}}{s-3} - \frac{\frac{1}{5}}{s+2} \\
x &= \frac{6}{5}e^{3t} - \frac{1}{5}e^{-2t} \\
y &= \frac{2}{5}e^{-2t} + \frac{3}{5}e^{3t}
\end{aligned}
$$

In this case, the nullclines are $y = -x$ and $y = 2x$. The steady state is a saddle point. The solutions are a mix of exponential growth and exponential decay; this mix of growth and decay fits the expected behaviour of a saddle point. Figure 6.12 shows the nullclines and directions.

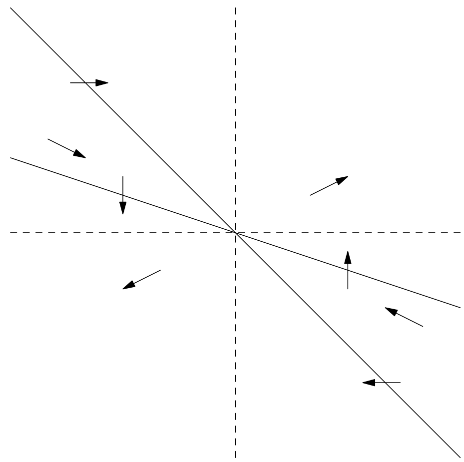


Figure 6.13: Third Linear System Example

Example 6.3.3.

$$
\begin{aligned}
x' &= 2x + 3y \qquad x(0) = 1\\
y' &= 2x + y \qquad y(0) = 1\\
sX - 1 &= 2X + 3Y\\
sY - 1 &= 2X + Y \implies X = \frac{3Y+1}{s-2}\\
sY - Y - 2\left(\frac{3Y+1}{s-2}\right) &= 1\\
\left(\frac{s^2-3s-4}{s-2}\right) Y &= 1 + \frac{2}{s-2}\\
Y &= \frac{s-2}{(s-4)(s+1)} + \frac{2}{(s-4)(s+1)} = \frac{s}{(s-4)(s+1)} = \frac{\frac{4}{5}}{s-4} + \frac{\frac{-1}{5}}{s+1}\\
X &= \frac{3\frac{s}{(s-4)(s+1)} + 1}{s-2} = \frac{3s + s^2 - 3s - 4}{(s+1)(s-4)(s-2)} = \frac{s^2-4}{(s+1)(s-4)(s-2)}\\
&= \frac{s+2}{(s+1)(s-4)} = \frac{\frac{6}{5}}{s-4} - \frac{\frac{1}{5}}{s+1}\\
x &= \frac{6}{5}e^{4t} - \frac{1}{5}e^{-t}\\
y &= \frac{4}{5}e^{4t} + \frac{1}{5}e^{-t}
\end{aligned}
$$

In this case, the nullclines are $y = \frac{-2}{3}x$ and $y = -2x$. The behaviour is again a saddle point and again we have a mix of positive and negative exponential terms. Figure 6.13 shows the nullclines and directions.

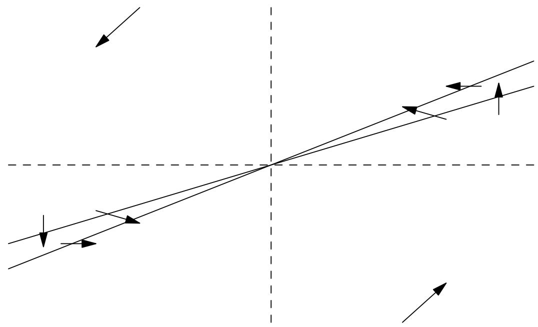


Figure 6.14: Fourth Linear System Example

Example 6.3.4.

$$\begin{aligned}
x' &= 3x - 18y \qquad & x(0) &= 1 \\
y' &= 2x - 9y \qquad & y(0) &= 1 \\
sX - 1 &= 3X - 18Y \\
sY - 1 &= 2X - 9Y \implies X = \frac{1-18Y}{s-3}
\end{aligned}$$

$$\begin{aligned}
sY + 9Y - 2\left(\frac{1-18Y}{s-3}\right) &= 1 \\
\left(\frac{s^2+6s+9}{s-3}\right) Y &= 1 + \frac{2}{s-3} \\
Y &= \frac{s-3}{(s+3)^2} + \frac{2}{(s+3)^2} = \frac{s-1}{(s+1)^2} \\
&= \frac{s}{(s+3)^2} - \frac{1}{(s+3)^2} = \frac{1}{s+3} - \frac{4}{(s+3)^2} \\
X &= \frac{1 - 19\frac{s-1}{(s+3)^2}}{s-3} = \frac{s^2-12s+27}{(s-3)(s+3)^2} \frac{(s-9)(s-3)}{(s-3)(s+3)^2} \\
&= \frac{s}{(s+3)^2} - \frac{1}{(s-3)^2} = \frac{1}{s+3} - \frac{12}{(s+3)^2} \\
x &= e^{-3t} + 12te^{-3t} \\
y &= e^{-3t} + 4te^{-3t}
\end{aligned}$$

In this case, the nullclines are $y = \frac{1}{6}x$ and $y = \frac{2}{9}x$. The steady state is a stable focus. The solutions are all decay terms, so they all tend to the steady state. This doesn't really match the expected behaviour, which looks like rotation. This is an interesting example: the repeated factor leads to the te^{-3t} terms, which do have more spinning motion that the general exponential terms. This is like the critically damped case for harmonic motion: it is very close to sinusoidal behaviour but not quite there. It would be very easy to interpret the trajectories as rotational; in this case, the qualitative analysis is misleading. Figure 6.14 shows the nullclines and directions.

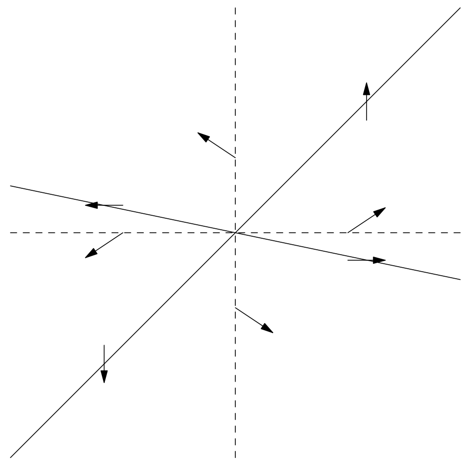


Figure 6.15: Fifth Linear System Example

Example 6.3.5.

$$\begin{aligned}
x' &= 6x - y \qquad\qquad x(0) = 1 \\
y' &= 5x + 4y \qquad\quad\; y(0) = 1 \\
sX - 1 &= 6X - Y \\
sY - 1 &= 5X + 4Y \implies X = \frac{1-Y}{s-6} \\
(s-4)Y - 5\left(\frac{1-y}{s-6}\right) &= 1 \\
\left(\frac{s^2-10s+29}{s-6}\right)Y &= 1 - \frac{5}{s-6} = \frac{s-11}{s-6} \\
Y &= \frac{s-11}{s^2-10s_29} = \frac{s}{(s-5)^2+4} - \frac{11}{(s-5)^2+4} \\
&= \frac{s-5}{(s-5)^2+4} - \frac{6}{(s-5)^2+4} \\
X &= \frac{1-\frac{s-11}{s^2+10s+29}}{s-6} = \frac{s^2-11s+40}{(s-6)(s^2-10s+29)} = \frac{(s-6)(s-5)}{(s-6)(s^2-10s+19)} \\
&= \frac{s-5}{s^2-10s+29} = \frac{s-5}{(s-5)^2+4} - \frac{1}{(s-5)^2+4} \\
x &= e^{5t}\cos 2t - 3e^{5t}\sin 2t \\
y &= e^{5t}\cos 2t - \frac{1}{2}e^{5t}\sin 2t
\end{aligned}$$

In this case, the nullclines are $y = 6x$ and $y = \frac{-5}{4}x$. The steady state is unstable node. The solutions have rotation in the sinusoidal terms and exponential growth, showing the outward spiral expected from an unstable node. Figure 6.15 shows the nullclines and trajectories for this example.

6.3.3 Second Order Systems

We used coupled spring as a motivating example is a linear system. It is a second order system–in fact, it will be the only example we'll consider of a second order system.

$$
\begin{aligned}
m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\
m_2 x_2'' &= -k_2(x_2 - x_1)
\end{aligned}
$$

Let's set the constants $m_1 = m_2 = 1$, $k_1 = 8$ and $k_2 = 3$. Let's also set the initial conditions $x_1(0) = x_1'(0) = 0$ and $x_2(0) = 1$ and $x_2'(0) = 0$. These initial condition implies that we've pushed x_2 away from equilibrium. We don't have a phase-plane diagram here, since this is a second order system. (Our phase-plane diagram intepretation in the previous section was specific to first-order systems.)

We solve using Laplace transforms.

$$
\begin{aligned}
x_1'' + 11x_1 - 3x_2 &= 0 \\
x_2'' + 3x_2 - 3x_1 &= 0 \\
s^2 X_1 + 11X_1 - 3X_2 &= 0 \\
s^2 X_2 - s + 3X_2 - 3X_1 &= 0 \\
X_2 &= \frac{(s^2+11)X_1}{3} \\
(s^2+3)X_2 - 3X_1 &= s \\
(s^2+3)\left(\frac{s^2+11}{3}\right) - 3X_1 &= s \\
\frac{s^4+14s^2+33-9}{3} X_1 &= s \\
\frac{s^4+14s^2+24}{3} X_1 &= s \\
X_1 &= \frac{3x}{s^4+14s^2+24} = \frac{3s}{(s^2+2)(s^2+12)} \\
X_2 &= \frac{s^2+11}{3}\frac{3s}{(s^2+2)(s^2+12)} = \frac{s^3+11s}{(s^2+1)(s^2+12)} \\
X_1 &= \frac{3s}{10(s^2+2)} - \frac{3s}{10(s^2+12)} \\
X_2 &= \frac{9s}{10(s^2+2)} + \frac{s}{10(s^2+12)} \\
x_1 &= \frac{3}{10}\cos\sqrt{2}t - \frac{3}{10}\cos\sqrt{12}t \\
x_2 &= \frac{9}{10}\cos\sqrt{2}t + \frac{1}{10}\cos\sqrt{12}t
\end{aligned}
$$

Here is an adjustment of the coupled spring, where each mass is attached to a fixed object. Then there are three springs in total: one attaching each mass to opposite walls, and one between them. We have a system given by Newton's laws of motion.

$$\begin{aligned} m_2 x'' &= -k_1 x + k_2(y-x) \\ m_2 y'' &= -k_3 y + k_2(x-y) \end{aligned}$$

We set the constants $k_1 = k_3 = 2$, $k_2 = 1$, and $m_1 = m_2 = 1$. For initial conditions, we set $x(0) = y(0) = 0$ and $x'(0) = -1$, $y'(0) = 1$. These initial contidions show that, the masses are pushed apart equal distances to start the system.

$$\begin{aligned} x'' + 3x - y &= 0 \\ y'' + 3y - x &= 0 \\ s^2 X + 1 + 3X - Y &= 0 \\ s^2 Y - 1 + 3Y - X0 & \\ Y &= \frac{1 + +X}{s^2+3} \\ (s^2+3)X - \left(\frac{1+X}{s^2+3}\right) &= -1 \\ \left(\frac{(s^2+3)(s^2+3)-1}{s^2+3)}\right) X &= \frac{1}{s^2+3} - 1 \\ (s^4 + 6s^2 + 8)X &= -s^2 - 2 \\ X &= \frac{-s^2-2}{(s^2+4)(s^2+2)} = \frac{-1}{s^2+4} \\ Y &= \frac{1 + \frac{-1}{s^2+4}}{s^2+3} \\ &= \frac{s^2+4-1}{(s^2+4)(s^2+3)} \\ &= \frac{s^2+3}{(s^2+4)(s^2+3)} = \frac{1}{s^2+4} \\ x &= -\sin 2t \\ y &= \sin 2t \end{aligned}$$

6.3.4 General Theory via Eigenvalues

The general linear system with two equations has the form

$$x' = ax + by \qquad y' = cx + dy.$$

We can use a 2 by 2 matrix with coefficients a, b, c, d to analyze the behaviour. We were x and y as the coordinates of a vector.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let v be an eigenvectors of the matrix and let λ be the corresponding eigenvectors.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = e^{\lambda t} v = \begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_1 \end{bmatrix}$$

To see why this is a solution, see what happens when the matrix acts on the vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^{\lambda t} \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{d}{dt} e^{\lambda t} v_1 \\ \dfrac{d}{dt} e^{\lambda t} v_2 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \end{bmatrix}$$

If we allow for complex eigenvalues, the matrix has two eigenvalues, wth two linearly independent eigenvectors, giving two linearly independent solutions of this type. If the eigenvalues are real, there are exponential growth/decay solutions. If the eigenvalues are complex, these are sinusoidal solutions.

6.3.5 Stability via Eigenvalues

Everything about a linear system can be determined by the eigenvalues of the associated matrix. For a linear system, the nullclines are both lines through the origin. Assuming they are not the same line (which would make the system redundant), the only steady state for a linear system in two equations is $(0, 0)$. As we've stated previously, there are six behaviours at the steady state for two-dimensional linear systems: stable and unstable foci, stable and unstable notes, centres and saddles. We can identify each case simply by the eigenvalues.

- If both eigenvalues are real and negative, then $(0, 0)$ is a stable node.
- If both eigenvalues are real and positive, then $(0, 0)$ is a unstable node.
- If both eigenvalues are real, but one is positive and one is negative, then $(0, 0)$ is a saddle point.
- If both eigenvalues are complex with negative real part, then $(0, 0)$ is a stable focus.
- If both eigenvalues are complex with positive real part, then $(0, 0)$ is an unstable focus.
- If both eigenvalues are complex and purely imaginary, then $(0, 0)$ is a centre.

The eigenvalues become the coefficients of the exponential solutions, which explains this list of behaviours. Only when we have complex coefficients do we get sinusoidal behaviuors, as in the spiral foci and the circular centre. When we have real coefficients, we only get growth and decay. The real part, in either case, is the growth or decay term. When the real part is negative, we have decay, which is stability. When the real part is positive, we have growth, which is instability.

Let M be the matrix and let Let $\alpha = \text{tr} M$ and $\beta = \det M$. Then the characteristic equation of the matrix can be written $\lambda^2 - \alpha\lambda + \beta = 0$. In addition, let δ be the discriminant of the characteristic polynomial, $\delta = \alpha^2 - 4\beta$. Then we can redo the previous list.

- If $\beta > 0$, $\alpha < 0$ and $\delta > 0$ then $(0,0)$ is a stable node.
- If $\beta > 0$, $\alpha > 0$ and $\delta > 0$, then $(0,0)$ is an unstable node.
- If $\beta < 0$, then $(0,0)$ is a saddle point.
- If $\beta > 0$, $\alpha < 0$ and $\delta < 0$, then $(0,0)$ is a stable focus.
- If $\beta > 0$, $\alpha > 0$ and $\delta < 0$ then $(0,0)$ is an unstable focus.
- If $\beta > 0$, $\alpha = 0$ and $\delta < 0$ then $(0,0)$ is a centre.

Chapter 7

Partial Differential Equations

So far, this has been a course in ordinary differential equations, where our functions depended on one variable. The theory of differential equations gets much more involved and interesting when we move to functions of several variables. The equations also get more difficult to solve; solving a single partial differential equation (PDE) can be the work of a whole book, a whole thesis or a whole career. For some examples, the solutions to a particular PDE are a whole branch of mathematics.

Many examples involve functions which depend on both position and time. The root of modern physics is found in differential equations that relate time derivatives to position derivatives. There are two particularly celebrated examples.

Example 7.0.1. The Navier-Stokes equation is the fundamental equation of fluid dynamics. There are several forms, but we'll state the form for incompressible flows. In this equation, u is the field describing the fluid flow as a function on x, y, z, t (position and time), ∇ is a differential operator, P is the pressure of the fluid, ρ is the density of the fluid, and ν is the viscosity of the fluid.

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla P}{\rho} + \nu \nabla^2 u.$$

Example 7.0.2. The Schrodinger equation is the fundamental PDE in quantum mechanics (at least in its early versions). In the Schrodinger equation, Ψ is the wave function which describes the probabilities of measurements in a physical system, $\imath$ is the familiar complex number, $\hbar$ is a constant, m is mass and V is a potential. This is the one-dimensional version, so Ψ depends on x and t.

$$\imath\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$$

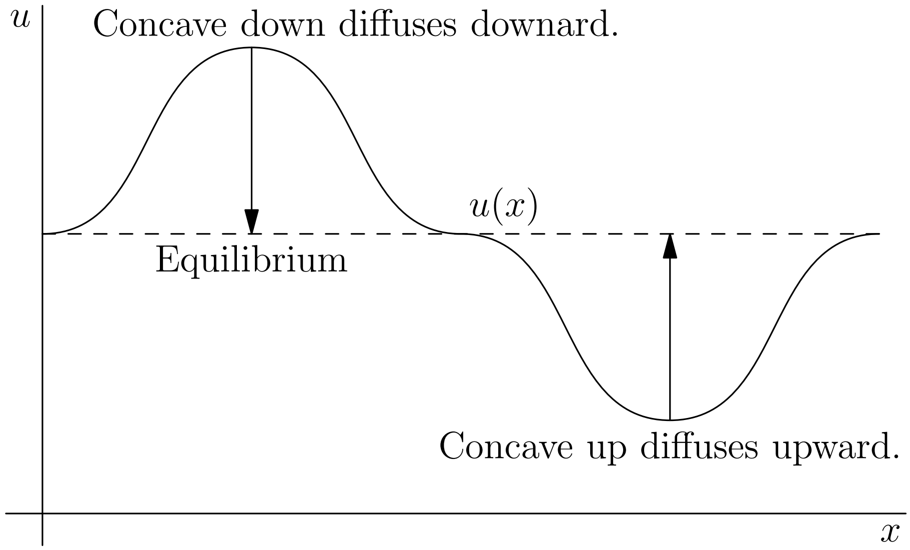


Figure 7.1: Concavity and Heat Diffusion

7.1 The Heat Equation

In our very short investigation of PDEs, we will look at two equations. The first is the heat equation. It concerns a function $u(x,t)$ which depends on position and time. $u(x,t)$ measures the heat in a one-dimensional object (usually a rod, rail, wire, stick or something similar). The heat varies in position along the object as well as in time. We let $x \in [0,l]$ where l is the length of the rod. For convenience (by choosing appropriate units), we will set $l = \pi$.

The question we want to ask is this: how does the heat distribution along the rod develop over time? What happens to the heat at various positions and times? Where does the heat flow, increase, decrease and diffuse? The heat equation tries to answer this question.

Thermodynamics says that heat wants to diffuse. Diffusion is the equalizing of differences, so the greater the difference in heat, the greater the inclination for difussions. How do we measure this? The measurement must be local, since heat cannot jump discontinuously. The local description that measures this heat difference is the concavity of the function. Moreover, concave down regions want to equalize downward and concave up regions want to equalize upwards, as in Figure 7.1. Therefore, the heat diffusion should be proportional to the concavity with a positive proportionality constant.

We translate these thermodynamic realities into a partial differential equation. The second derivative in position measure concavity and the first derivative in time measures the change in heat resulting from the diffusion. The above set-up shows that these should be proportional to each other. Let k be a positive number.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

This is called the heat equation. Though we've defined it by talking about heat, it is a very general equation and applies to any system that involves diffusion. It is sometimes also called the diffusion equation.

7.1.1 Initial and Boundary Conditions

With ordinary differential equations, we needed initial conditions to get a particular solution. The same it true with PDEs, but the initial assumptions are more complicated. At a specific time, say $t = 0$, the function $u(x, 0)$ is still a function of x. Our initial condition, then, is an entire function: $u(x, 0) = f(x)$.

The initial condition determines the situation at the starting time. However, we also need to control the behaviour of position. For position, instead of initial conditions, we have boundary conditions. These tell us the behaviour of the system at its boundaries. For the heat equation on a rod, we have $x \in [0, \pi]$, so the boundary contitions are determination of $u(0, t)$ and $u(\pi, t)$. These themselves can be functions of t. However, for our purposes, we'll set the boundary contitions to be constant. If we set the units of temperature reasonably, we can have $u(0, t) = u(\pi, t) = 0$. These boundary condition mean that we have a constant temperature at both ends of the rod at all moments in time. (In physical terms, there are perfect instantaneous heat-sinks at each end of the rod which equalize it to the ambient temperature of zero).

7.1.2 Solving the Heat Equation

The method of approach we use to solve the equation is a common one for PDE and is called *seperation of variables*. We make the assumption (eventually justified!) that we can pull apart the two variables, x and t, and deal with the seperately. We assume that u is a product of two single variables functions.

$$u(x, t) = T(t)X(x).$$

Like all of our assumptions about DEs, we simply put this back into the equation.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= X''(x)T(t) \\
\frac{\partial u}{\partial t} &= T'(t)X(x) \\
kX''T(t) &= X(t)T'(t) \\
\frac{X''(x)}{X(x)} &= \frac{1}{k}\frac{T'(t)}{T(t)}
\end{aligned}$$

We bring all the terms involving x to one side and all those involving t to the other. Since the left is equal to the right and they involve different variables, both sides must be constant. Let's call this constant α. This completes the seperation of variables, giving us two ordinary differential equations.

$$\begin{aligned}
\frac{X''(x)}{X(x)} &= \alpha \\
\frac{1}{k}\frac{T'(t)}{T(t)} &= \alpha
\end{aligned}$$

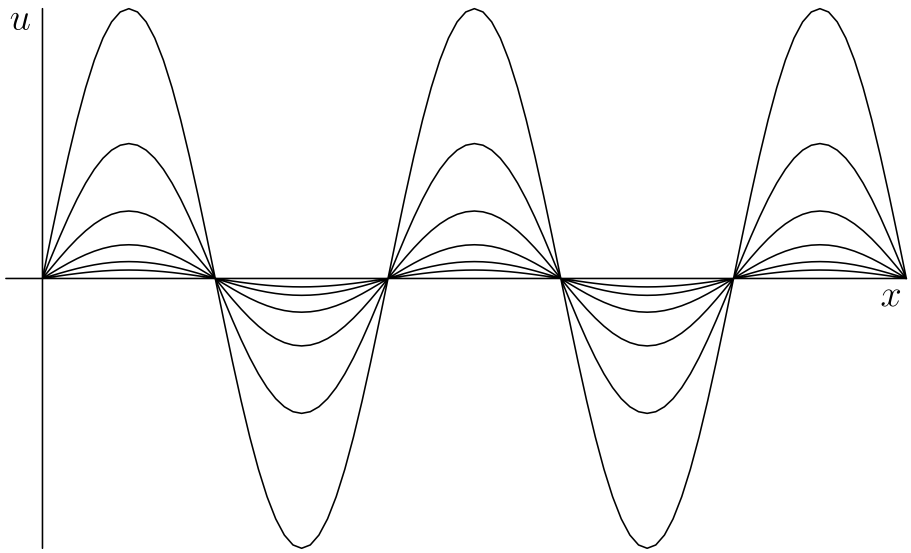


Figure 7.2: Seperated Solution with $n = 5$ at Various Points in Time.

Now, to solve the heat equation with seperation of variables, we solve these two ODEs seperately. The T solutions are easy.

$$T'(t) = \alpha k T(t) \implies T(t) = T(0)e^{\alpha k t}$$

We have written the initial constant as $T(0)$. The time function is just exponential growth/decay, but X solutions depend more carefully on the boundary conditions.

The X equation is a second order linear equation: $X''(x) - \alpha X(x) = 0$. If $\alpha > 0$ then we expect exponential solutions $X(x) = c_1 e^{\alpha x} + c_2 e^{\alpha x}$. However, the boundary conditions apply. $X(0) = X(\pi) = 0$ implies that $c_1 = c_2 = 0$; the only solution here is the trivial solution $X(x) = 0$. Therefore, to get solutions that actually mean somthing, α cannot be positive. If $\alpha = 0$ then we get $X = Ax + B$. Again, the boundary conditions give $A = B = 0$, so we only get the zero solution. Therefore, only $\alpha < 0$ gives interesting solutions.

(This $\alpha < 0$ also effects the T solutions, which relied on the same α. The fact that $\alpha < 0$ means that the T solutions must be decaying exponentials instead of growing exponentials. This is physically expected, since over time the heat should diffuse.)

Let $\alpha = -n^2$ for $n > 0$ a real number. From our study of second order linear equations, the solutions are

$$X(x) = A \cos nx + B \sin nx.$$

Again, we go back to the boundary conditions. $X(0) = 0$ implies $A = 0$, so we are left with $X(x) = B \sin nx$. Then $X(\pi) = 0$ implies $B \sin n\pi = 0$ which is only satisfied if $n \in \mathbb{N}$. In conclusion, matching

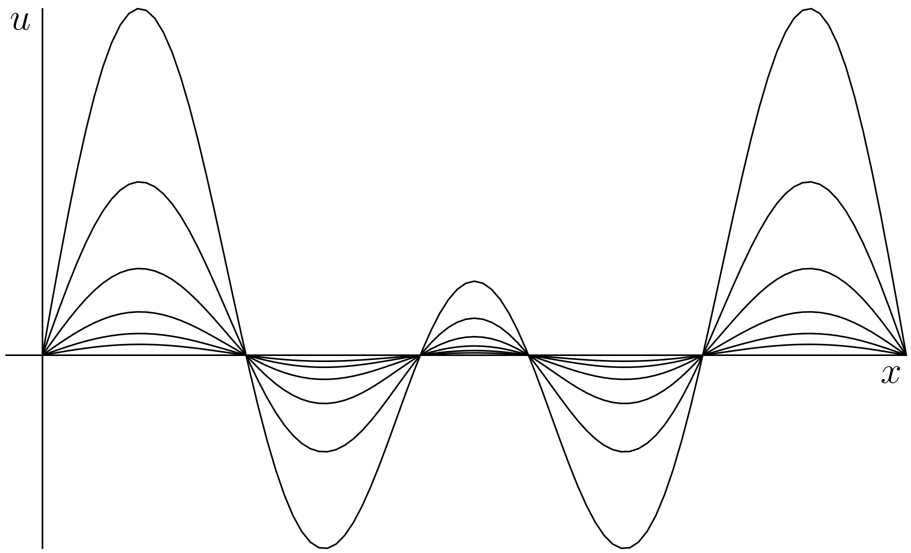


Figure 7.3: Decay of the Solution with Initial Condition $f(x) = 4\sin 3t + 6\sin 5t$

the boundary conditions means that the only possible X solutions are $X(x) = B\sin nx$ for $n \in \mathbb{N}$ (and the zero solution).

Then we put the solutions together.

$$u(x,t) = T(0)e^{-n^2kt}\sin(nx)$$

In these solutions, we see sine waves decaying over time. The integer n gives the number of osscilations of the wave in on the rod, and the exponential terms give a decay in amplitude over time. Figure 7.2 shows the decay of a simple sine wave with $n = 5$.

Now we consider the initial condition $u(x,0) = f(x)$. For each n, the seperable solution is a sine wave $\sin(nx)$. This is great if $f(x)$ happened to be such a sine wave, but we would like the theory to work with more general initial conditions. The key observation here is that the heat equation is linear: if we have two solutions to the heat equation, their sum is also a solution. Therefore, we could add up these seperable solutions. For example, a possible solution could be

$$u(x,t) = 4e^{-9kt}\sin 3t + 6e^{-25kt}\sin 5t.$$

This isn't strictly seperable anymore, but it is the sum of seperable pieces. (This justifies the original assumption! We don't get all possibility as seperable solutions, but we will get all reasonable solutions as the *sum* of various seperable pieces.) The initial value $u(x,0)$ also isn't a simple sine wave anymore, since it is the function $f(x) = 4\sin 3t + .6\sin 5t$. Figure 7.3 shows the decay of this initial condition.

In theory, we could add any number of the seperable solutions together. Therefore, in full generality, a solution could look like an arbitrary sum.

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0) e^{-n^2 kt} \sin(nx)$$

This is a series solution, but the series has $\sin nx$ instead of x^n as the series terms. We try to match our initial condition $u(x,0) = f(x)$ with this series.

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin(nx) = f(x)$$

As long as we can find coefficients $T_n(0)$ that make this series work, we can solve the heat equation. That leads to this question: which functions $f(x)$ can be expressed as a series in $\sin(nx)$ in this way? Once we know that, we can have a full, general solution to the heat equation (with constant zero boundary conditions).

7.2 Fourier Series

The answer to the question at the end of the previous section is very positive: most functions can, in fact, be approximated by a series with term $\sin nx$. These series are called Fourier series and we'll spend the current section defining and investigating them.

7.2.1 Series and Bases for Functions

The series in the previous section only involved sine functions, but the general Fourier series involves both sine and cosine.

Definition 7.2.1. Let a_n and b_n be real numbers. A *Fourier series* is a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

In general, function as series can be consider a sums over a particular basis. For Taylor series, the basis was the set of monomials: x^n for $n \in \mathbb{N}$. A Taylor series is an attempt to write a general function as a sum of these monomials. For Fourier series, the is $\{1, \cos nx, \sin nx\}$ for $n \in \mathbb{N}$.

In a Fourier series, we can split the basis into two pieces: the sine pieces and the cosine pieces (including the constant 1 in the later). If we restrict to only one piece, we could form a Fourier sine series or Fourier cosine series. The sine pieces are all odd functions; therefore, a Fourier sine series must be an odd function. Likewise, the cosine pieces (including the constant) are all even functions, so a Fourier cosines series must be an even function.

If we have a general function f, we can write $f = f_+ + f_-$ where f_+ is even and f_- is odd. The pieces are easy to define: $f_+ = \frac{1}{2}(f(x) + f(-x))$ and $f_- = \frac{1}{2}(f(x) - f(-x))$. If f has a Fourier series, then f_-, the odd piece, will have a Fourier sine series and f_+, the even piece, will have a Fourier cosine series. The heat equation, since we insisted on starting at $(0,0)$ and only considering positive values, only needs one of the two types. Since sines arose in the solution, we'll assume the solution to the heat equation is odd and rely on Fourier sine series.

7.2.2 Scope of Fourier Series

The good news about Fourier series is that they cover a large family of functions.

Theorem 7.2.2. *If f is periodic and piecewise continuous on $\mathbb{R}$ with period $2T$, the there exists a_n and b_n such that*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right).$$

If we don't want to work with periodic functions, we can simply look at any piecewise-continuous function defined on $[0, 2T)$, thinking of it as one period, and find a Fourier series for it. (This method works for any finite intervals.)

Theorem 7.2.3. *If f is piecewise continuous on $[0, 2T)$ then there exists a_n and b_n such that*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right).$$

Therefore, we can use Fourier series to approximate any piecewise continuous function on a bounded interval, which is a very large class of functions.

7.2.3 Properties of Fourier Series

When we talked about the Lagrange polynomials, we derived a very nice orthogonality property for them. As opposed to the ordinary monomials x^n, the Lagrange polynomials were pair-wise orthogonal, using integration on $[-1, 1]$ as the inner product. When we find a basis for a type ofseries, it is very convenient for integration if the basis functions are orthogonal. (Like in linear algebra, an orthogonal basis is always easy to use).

Fourier series terms have wonderful orthogonality properties. For convenience, these properties are stated for the period $T = 2\pi$.

Proposition 7.2.4. *Let $n, m \in \mathbb{N}$, with the possibility that $m = 0$ to account for the constant term.*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$$

Therefore, all the Fourier basis functions are orthogonal under the scalar product given by integration on $[-\pi, \pi]$.

Proof. For the proof, we use the trig identities for products of sin and cos, such as

$$\cos A \cos A = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right).$$

We only show the proof of the first of the three statements.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x) + \cos((m+x)x)dx \\
&= \frac{1}{2}\left[\frac{-\sin((m-n)x)}{m-n} - \frac{-\sin((m+n)x)}{m+n}\right]\Bigg|_{-\pi}^{\pi} = 0 \qquad \text{if } m \neq n \\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2mdx \\
&= \frac{1}{4}\left(\frac{-\sin mx}{m}\right) + \frac{x}{2}\Bigg|_{\pi}^{\pi} = 0 + \frac{2\pi}{2} = \pi \qquad \text{if } m = n
\end{aligned}$$

□

For Taylor series, we had convenient properties for the calculus of the series and the calculation of coefficients. Similar properties exist for Fourier series. If we know a series is convergent, we can integrate and differentiate term-wise. (Not that differentiation and integration change a Fourier sine series into a Fourier cosine series and vice-versa.)

A method for the the calculation of coefficients comes from the orthogonality property. Assume that $f(x)$ is expressed as a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Then we can calculate an integral.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos mx dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx) dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin(nx) dx \\
&= 0 + \sum_{n=1}^{\infty} a_n \pi \delta_{mn} + \sum_{n=1}^{\infty} 0 \\
&= a_n \pi
\end{aligned}$$

The integral calculates the coefficient a_n. Similarly, the constant coefficient a_0 and the sine coefficients are calculated by integrals. This gives a general result for calculating Fourier coefficients, for $n \in \mathbb{N}$.

Proposition 7.2.5. *If f is expressed as a Fourier series, then the Fourier coefficient a_n and b_n satisfy these equations.*

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx
\end{aligned}$$

Example 7.2.6. Let's express $f(x) = x$ on $(-\pi, \pi)$ as a Fourier series. Notice that $f(x)$ is odd, so only sine terms are expected. We calculate the coefficients of the sine terms with integrals.

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx \\
&= \frac{1}{\pi} \left(\frac{-x \cos nx}{n} \right) \Bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} dx \\
&= \frac{1}{\pi} \left(\frac{-\pi \cos n\pi}{n} + \frac{\pi \cos(-n\pi}{n}) + 0 \right) \\
&= \frac{1}{\pi} \left(\frac{-2\pi}{n} \right) \cos n\pi = \frac{-2(-1)^n}{n} = (-1)^{n+1} \ \frac{2}{n} \\
x &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx
\end{aligned}$$

7.2.4 Finishing the Heat Equation Solution

If we go back to the heat equation, the initial conditions where $u(x,0) = f(x)$ for some given function f on $[0,\pi]$ with $f(0) = f(\pi) = 0$. When $t = 0$, we wrote that

$$f(x) = \sum_{n=1}^{\infty} T_n \sin(nx).$$

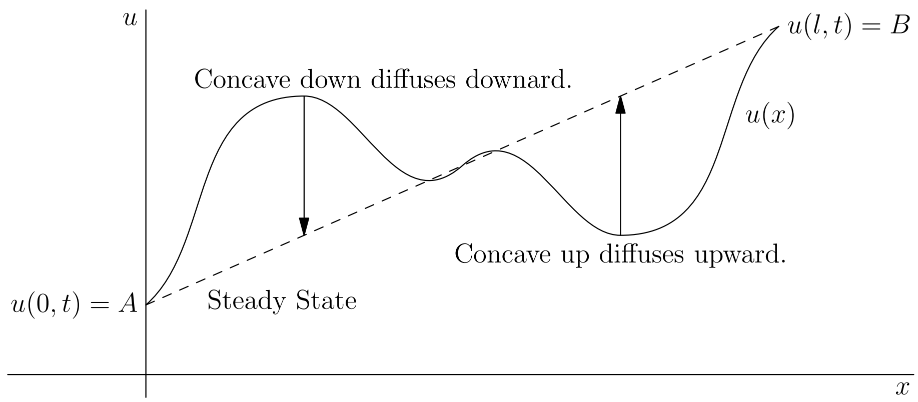


Figure 7.4: Diffusion to a Linear Steady State

The coefficients T_n in the full solution to the heat equation must be the Fourier coefficients of the initial function $f(x)$. Now we know how to calculate these coefficients. Since this is a sine series, we expect that f is odd, and we extend it to $(-\pi, \pi)$ as an odd function. Then we integrate to find the coefficients.

$$T_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$$

Therefore, the full solution to the heat equation, with initial conditions $f(t)$ and boundary conditions $u(0, t) = u(\pi, t) = 0$, is

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\int_0^\pi f(x) \sin nx) dx \right] e^{-kn^2 t} \sin(nx).$$

7.3 Boundary Conditions for the Heat Equation

The general solution in this case is the sum of this steady There are many pieces to the setup and solution of a PDE. In what we've done so far for the heat equation, we allowed a nearly-arbitrary initial conditions, since any function which has a Fourier series satisfied. However, our boundary conditions were very restrictive. We can get more complicated situations if we relax the boundary conditions. A small increase in complexity comes from constant but non-equal boundary conditions: $u(0, t) = A$ and $u(\pi, t) = B$ for some constants A and B. This is a situation where we will keep the ends of the rod at constant temperature, but those temperatures need not be the same.

In this case, seperation of variables when $\alpha = 0$ implies that $X(x)$ must be a linear function. Matching the boundary contions (with a rod of length π) means that $X(x) = \frac{B-A}{\pi} x + A$. This is called the steady-state solution and it is the base case of the situation. (For the previous case with zero boundary conditions, the steady state solution was just $X(x) = 0$.) The steady-state solution is the solution to

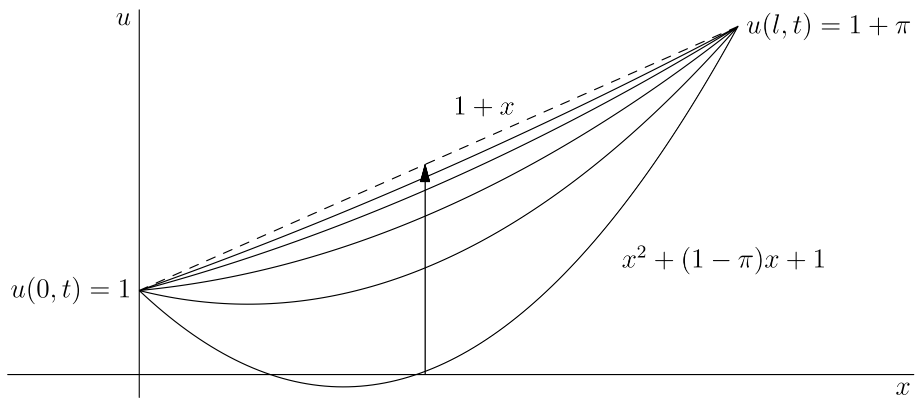


Figure 7.5: Heat Equation Examples with Initial Conditions $f(x) = x^2 + (1-\pi)x + 1$

which we want the situaiton to diffuse; it is the resting case. With non-equal boundary conditions, the steady state involves a heat flow along the rod, from the warmer end to the cooler. A complicated initial heat-situation will diffuse and reduce to a straight-line distribution between the tempeartures at each end as show in Figure 7.4.

state with our previous solutions. The linearity of the system means that the solution with both boundary conditions set to 0 can simply be added to this steady state to give the general solution.

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} T_n e^{-kn^2 t}\sin(nx)$$

The calculation of the Fourier coefficients T_n, however, needs an adjustment. We need to shift the function back to a 0-equilibrium situation to repeat the result of the previous section. With that shift, the integral that calculates the coefficient is

$$T_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) - \frac{B-A}{\pi}x - A\right)\sin nx dx.$$

Example 7.3.1. We use an example to see what this looks like with explicit numbers. This example is from Roberts and Marion. Set the boundary contidions to be $u(0,t) = 1$ and $u(\pi,t) = 1+\pi$ and the initial condition $u(x,0) = x^2 + (1-\pi)x + 1$. The steady state solution is the line $u = x + 1$. We calculate the Fourier coefficients.

$$
\begin{aligned}
T_n &= \frac{2}{\pi}\int_0^\pi (x^2-\pi x)\sin nx = \frac{2}{\pi}\int_0^\pi x^2 \sin nx - 2\int_0^\pi x\sin nx \\
&= \frac{2}{\pi}\left(\frac{-x^2\cos nx}{n}\bigg|_0^\pi + \frac{2}{n}\int_0^\pi x\cos nx\right) - 2\left(\frac{-x\cos nx}{n}\bigg|_0^\pi + \int_0^\pi \cos nx dx\right) \\
&= \frac{2}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n}\frac{x\sin nx}{n}\bigg|_0^\pi - \frac{2}{n^2}\int_0^\pi \sin nx\right) - 2\left(\frac{2\pi}{n} + \frac{\sin nx}{n}\bigg|_0^\pi\right) \\
&= \frac{2}{\pi}\left(\frac{\pi^2(-1)^{n+1}}{n} + 0 + \frac{2\cos nx}{n^3}\bigg|_0^\pi\right) - \frac{4\pi}{n} - 0 \\
&= \frac{2}{\pi}\left(\frac{\pi^2(-1)^{n+1}}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3} - \frac{2\pi^2}{n}\right) \\
&= \frac{2n^2\pi^2(-1)^{n+1} + 2\pi(-1)^n - 2\pi - 4\pi n^2}{\pi n^3}
\end{aligned}
$$

The general solution is

$$u(x,t) = 1 + x + \sum_{n=1}^{\infty}\left[\frac{2n^2\pi^2(-1)^{n+1} + 2\pi(-1)^n - 2\pi - 4\pi n^2}{\pi n^3}\right] e^{-kn^2 t}\sin(nx).$$

Figure 7.5 shows this situation.