Course Notes for Math 434

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Contents

Chapter 1

Preliminaries

This introductory chapter collects a number of definitions from prerequiste courses which are important enough to bear repeating.

1.1 Functions

Definition 1.1.1. Let A be the $\mathbb{R}$ or an interval subset of $\mathbb{R}$.

- $C(A)$ is the set of continuous real-valued function on A.
- $C^1(A)$ is the set of continuously differentiable functions on A.
- $C^n(A)$ is the set of functions on A with n continuous derivatives.
- $C^\infty(A)$ is the set of infinitly differentiable functions on A.

Definition 1.1.2. Let A be a subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is *piecewise-continuous* on A if it continous everywhere on A except for a set of isolated point. Likewise, f is *piecewise-differentiable* on A if it differentiable everywhere on A except for a set of isolated points. Note that A is the domain of the function: these piecewise functions can have jumps or break, but they are defined on all of A. They cannot have, for example, asymptotes at these isolated points.

We almost exclusively use the base e in this course, since it is the easiest base for derivatives and integrals. Since any other base a^t can be written $a^t = e^{\ln a t}$, we will very frequently work with functions of the form $e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. In this way, we cover all exponential bases.

1.2 Linear Algebra

Definition 1.2.1. A *vector space* over $\mathbb{R}$ is a set V with addition and scalar multiplication (if $\alpha \in \mathbb{R}$ and $vinV$, when αv is defined and remains in V).

Example 1.2.2. $\mathbb{R}^n$ is a vector space. The scalar multiplication is given by multiplying each component by the real number.

Example 1.2.3. If A is $\mathbb{R}$ or an interval subset, then $C(A), C^n(A)$ and $C^\infty(A)$ are vector spaces. Scalar multiplication is simply multiplying a constant by a scalar.

Definition 1.2.4. Let V be a real vector space and Let $v_1, \ldots, v_k \in V$. A *linear combination* of these vectors is a sum $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$ where $\alpha_i \in \mathbb{R}$. These vectors are called *linearly independent* if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$$

has only the trivial solution, where all $\alpha_i = 0$. Otherwise, the set of vectors is called *linearly dependent*. A maximal linearly independent set in V is called a bases. The dimension of V is the number of vector in any bases.

Example 1.2.5. The vector space $\mathbb{R}^n$ has dimension n. The vectors spaces $C(A), C^n(A)$ and $C^\infty(A)$ are all infinte dimensional.

Definition 1.2.6. If V_1 and V_2 are vector spaces, a *linear transformation* $f : V_1 \to V_2$ is a function which respects addition and scalar multiplication.

$$f(u+v) = f(u) + f(v)$$
$$f(\alpha v) = \alpha f(v)$$

If V_1 and V_2 are finite dimensional, all linear transformations can be encoded by matrcies using matrix multiplication acting on vectors.

Definition 1.2.7. If M is a square matrix, then the *determinant of* M, written $\det M$, is a real number with two properties:

- $|\det M|$ is the effect of the transformation on the appropriate notion of size (length, area, volume, hypervolume, etc).
- If $\det M$ is positive, then M preserves orientation; if $\det M$ is negative, then M reverses orientation.

Please consult my linear algebra note (or any other linear algebra reference) for the details of calculating determinants.

Definition 1.2.8. Let $f : V_1 \to V_2$ be a linear transformation. A vector $v \in V_1$ is an *eigenvector* for f with eigenvalue λ if $f(v) = \lambda v$.

1.3 Taylor Series

Definition 1.3.1. A function is *analytic* if it can be expressed as a *Taylor series.*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-\alpha)^n$$

A Taylor series is centered at a point α; if $\alpha = 0$ we call it a *McLaurin series.* A series defines a function on some domain $(\alpha - R, \alpha + R)$ for some number $R \geq 0$, which is called a *radius of convergence.* If $R = \infty$, the series is defined on all real numbers. If $R = 0$, the series is only defined at $x = \alpha$ and is basically useless.

We use the ratio or root tests to calculate the radius of convergence of a series. After some manipulation of those tests, (if the coefficients are non-zer) the radius of convergence is given by the formula

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

Inside the radius of convergence, the behaviour of a Taylor series is very reasonable. We can add and subtract terms when the indices match. We can multiply series like polynomials, though the calculation gets arduous. We can even divide using long division (though it is an infinite process). With multiplication and division (and with many other uses of series), we often only calculate the first few terms of the series.

There are two important manupulation techniques for series. The first is adjustment on indices.

$$\sum_{n=k}^{\infty} c_n x^n = \sum_{n=k+1}^{\infty} c_{n-1} x^{n-1} = \sum_{n=k-1}^{\infty} c_{n+1} x^{n+1}$$

The second is removal of initial terms.

$$\sum_{n=0}^{\infty} c_n x_n = c_0 + c_1 x + c_2 x^2 + \sum_{n=3}^{\infty} c_n x_n$$

Inside the radius of convergence, the calculus of Taylor series is well behaved. We can integrate and differentiate term-wise.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ f'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\ \int f(x)dx &= \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1} + c \end{aligned}$$

In particular, we know that integrals and derivatives are always defined. This shows that analytic functions are necessarily C^∞ on the domain given by the radius of convergence. (This is, in fact, an equivalence: any C^∞ function has a Taylor series with some radius R).

Evaluating the derivatives of a series at the centre point α gives a list of derivatives.

$$\begin{aligned}
f(\alpha) &= c_0 \\
f'(\alpha) &= c_1 \\
f''(\alpha) &= 2c_2 \\
f^{(3)}(\alpha) &= 2 \cdot 3c_4 \\
f^{(4)}(\alpha) &= 4!c_4 \\
f^{(n)}(\alpha) &= n!c_n \\
c_n &= \frac{f^{(n)}(\alpha)}{n!}
\end{aligned}$$

This is a way to calculate coefficients, if we know the derivatives of a function. Remember that the coefficients totally describe the taylor series, so the derivatives at the centre point give all the information.

1.4 Recurrence Relations

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called a *linear recurrence relation* if each term a_n is a linear function of the previous k terms $a_{n-1}, \ldots a_{n-k}$. k is called the *order* of the recurrence relation. Since terms depends on previous terms, the first k terms of the sequence must be explicitly defined.

The most famous example is the Fibonacci sequence, which is a second order linear recurrence relation. Its terms, f_n, start with $f_1 = f_2 = 1$ and then obey the linear recurrence $f_n = f_{n-1} + f_{n-2}$.

If we are given a recurrence relation relation, we don't have a direct way to calculate the term a_n without calculating all the previous terms. *Solving* a recurrence relation is the term for finding a fixed formula $a_n = f(n)$ that describes the nth term of the sequence. These can be easy or difficult to find. The fixed formula for Fibonnaci is

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

The coefficients of a Taylor series are sequences of real numbers. In our use of Taylor series, sequences of coefficients will be presented as recurrence relations. We will want to solve the recurrence relations to give a closed formula for the coefficients, in order to understand the resulting Taylor series.

1.5 Complex Numbers

Definition 1.5.1. We define the number $\imath$ by the property $\imath^2 = -1$. Then a *complex number* is any expression $a + b\imath$ where a and b are real numbers. a is called the *real part* and b is called the *imaginary part*. The set of all complex number is written $\mathbb{C}$.

Addition and multiplication are extended from $\mathbb{R}$ by linearity and the distribution of multiplication. Note in the multiplication we use the defining property that $\imath^2 = -1$.

$$\begin{aligned}
(a + b\imath) + (c + d\imath) &= (a + b) + (c + d)\imath \\
(a + b\imath)(c + d\imath) &= ac + bc\imath + ad\imath + bd\imath^2 \\
&= ac + (bc + ad)\imath + bd(-1) = (ac - bd) + (bc + ad)\imath
\end{aligned}$$

$\mathbb{C}$ is identified with the cartesian plane, where 1 is $(1, 0)$ and $\imath$ is $(0, 1)$. The horizontal axis is called the real axis and the vertical axis is called the imaginary axis. The number $a + bi$ is treated as the coordinate (a, b) in the plane. The inversion of a complex number (non-zero) is

$$\frac{1}{a + b\imath} = \frac{a - b\imath}{a^2 + b^2}.$$

Definition 1.5.2. The *modulus* of a complex number $a + b\imath$ is its length as a vector (a, b), namely $|a + b\imath| = \sqrt{a^2 + b^2}$.

Definition 1.5.3. The *conjugation* of a complex number $a + b\imath$ is $a - b\imath$. It is indicated with a vertical bar $\overline{a + b\imath} = a - b\imath$. Geometrically, conjugation is reflection over the real axis (x axis) in the plane.

Complex numbers have many intriguing property. The most important for us it the existence of roots of polynomial.

Theorem 1.5.4 (Fundamental Theorem of Algebra)**.** *Let $p(x)$ be a polynomial of degree d with real or complex coefficients. Then $p(x)$ has exactly d roots in the complex numbers (though some may be repeated). Equivalently, $p(x)$ factors completely over the complex numbers: there are complex numbers $\alpha_1, \ldots, \alpha_d$, not necessariliy distinct from each other, such that*

$$p(x) = (x - \alpha_i)(x - \alpha_2) \ldots (x - \alpha_d).$$

Complex exponentials are understood by Euler's formula.

Proposition 1.5.5 (Euler's Formula)**.**

$$e^{\imath\theta} = \cos\theta + i \sin\theta.$$

Proof. Even without a background in complex analysis, Euler's formula can be justified by looking at Taylor series.

$$
\begin{aligned}
e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
\sin t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
\cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\
e^{\imath t} &= \sum_{n=0}^{\infty} \frac{(\imath t)^n}{n!} = \sum_{n=0}^{\infty} \frac{\imath^n t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\imath^{2n} t^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{\imath^{2n+1} t^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n!} + \imath \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
&= \cos t + \imath \sin t
\end{aligned}
$$

□

1.6 A Note on Variables

In differential equation, the name of the independent variable is almost always either x or t. x is conventional, of course, in calculus; it is also useful for certain geometric interpretation where we want x and y to correspond to the familiar cartesian axes. We will use x as the independnet variable at certain points in the courses, particularly when we use these geometric techniques.

The vast majority of differential equations that we want to solve involve time derivatives; therefore, t is often the most natural choice for the independent variable. We will use t instead of x for the majority of this course. Often in the literature, a derivative in Newton's notation (f') implies that the independent variable is time.

Chapter 2

First Order Differential Equations

2.1 What is a Differential Equation?

2.1.1 First Definitions

Definition 2.1.1. A *differential equation* is any equation involving a function $f(t)$ and its derivatives $f'(t)$, $f''(t)$, etc. The highest derivative in the equation is called the *order* of the equation. A DE which involves partial derivatives is called a *partial differential equation*; otherwise, it is called an *ordinary differtial equation*.

More holistically, a differential equation asks the question: what function satisfies a given relationship? In algebra, we are familiar with this question for numbers. $t+5=7$ implicitly asks which number, when replacing t, satisfies the equation. The answer, of course, is $t=2$; the solution to the equation gives the correct replacement. For differential equations, we ask 'what function' instead of 'what number'.

There are many ways to write a differential equation. The most general way is to think of any algebraic combination of a function y, its derivatives and the independent variable t. In this generality, we could write the equation using a multivariable function F.

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

Often, if we can, we isolate the highest derivative.

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

Definition 2.1.2. A differential equation is called *autonomous* is the independent variable doesn't appear. For a first order equation, this looks like

$$\frac{dy}{dt} = f(y).$$

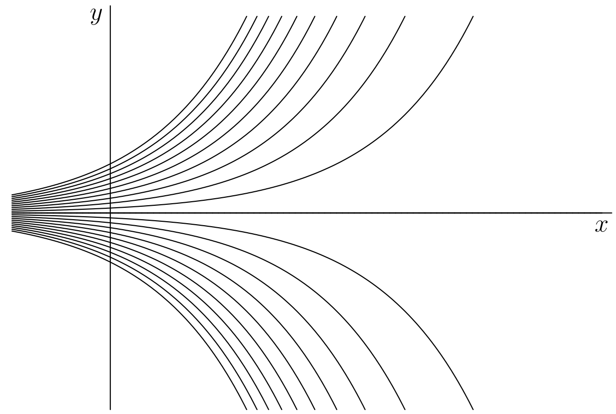

Figure 2.1: The family $f(t) = ce^{\alpha t}$ where $c \in \mathbb{R}$.

Definition 2.1.3. An equation is called *linear* if it has the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = f(t).$$

In addition, a linear equation is called *homogenous* if the $f(t)$ term is simply $f(t) = 0$.

A very useful notational tool for linear DEs is the idea of a linear differential operator.

Definition 2.1.4. A *differentiable operator* is an operator which acts on functions by either multplication by fixed functions or by differentiation. The operator is *linear* if the various pieces of the operation are added together. Fore example, a second order linear equation has this form

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c.$$

We use differentiable operators to simplify notation for DE. A second order linear DE become simply $Ly = f(t)$.

2.1.2 The Most Important Example

Example 2.1.5. The simplest differential equation is

$$\frac{df}{dt} = \alpha f(t).$$

For many differential equations, we seek a translation of their meaning. This simple equation means the rate of change is propotional to the current value. This is a statement of percentage growth; the proportionality constant is the percentage.

The solutions to this equation are are $f(t) = ce^{\alpha t}$. Notice that there is a constant, c, such that there will be a whole infinite family of solutions. A specific solution is specified by a choice of c.

Percentage growth applies to population models. If this is the case, then $f(0) = ce^0 = c$ and we must interpret the constant c as the population when $t = 0$. If this is our starting point, we consider c the *initial value* of the solution. A full solution of a differential equation will usually consist of a function and choice(s) for the initial value(s).

Definition 2.1.6. A differential equation along with a specified initial value is called an *initial value problem* or IVP.

If we don't make a choice, as we said, we get an infinite family of solutions. We can visualize this family as a series of graphs in $\mathbb{R}^2$. Figure 2.1 shows the graphs for $f(t) = ce^{\alpha t}$.

2.1.3 Pure and Applied Perspectives

We will be looking at differential equation both from a pure mathematics and applied mathematics point of view. The pure mathematician is interested in these kinds of questions:

- When does a solution exists?
- Can we prove that a solution exists?
- Is the solution unique?
- Is there a complete family? How many parameters exist, and what are their domains?
- Can we write and prove theorems to answer these questions?

The applied mathematician is interested in these kinds of questions:

- How many solutions fit the data or initial values?
- How do the solutions grow? What is their behaviour?
- Are the solutiosnts stable?
- How difficult are the solutions to calculate? Can we get them exacly, or only approximately?
- Can we answer questions about the model even without a solution?

2.1.4 Interesting Examples

We want to know the behaviour of solutions of DEs. Several examples in this section give the lay of the land.

Example 2.1.7.

$$\frac{d^2 f}{dt^2} + 9f = 0$$

This is solved by $\sin 3t$ and also by $\cos 3t$. Moreover, any linear combination $a \sin 3t + b \cos 3t$ is a solution. In a second order equation, we often expect two linearly independent solutions and the general solution is a linear combination of the two linearly independent solutions.

Example 2.1.8.

$$\frac{dy}{dt} = t\sqrt{y}$$

This is solved by $y = (\frac{t^2}{4} + c)^2$, which is a nice family with one real parameter. However, this is also solved by $y = 0$, even though it isn't in the family.

Definition 2.1.9. An extraneous solutions to a DE, one which fall outside families with parameters, is called *singular solution.*

Example 2.1.10.

$$t\frac{dy}{dt} = 4y$$

This is solved by $y = ct^4$, which is another reasonable family. However, there is a strange, singular solution.

$$y(t) = \begin{cases} -t^4 & t \leq 0 \\ t^4 & t > 0 \end{cases}$$

The derivatives of this function line up at the origin, so the function is of class C^1 (actually C^3, if you work it out).

Example 2.1.11.

$$\frac{dy}{dx} = \frac{-x}{y}$$

The curve $x^2 + y^2 = c$ solves this equation implicitly. We could break this up into two functions, but its much more natural to leave it as an implicit locus, in this case, a circle. This is very typical: often solutions are left in an implicit form as loci, even though in theory we always look for functions $y = f(x)$. Also notice in this example that only non-negative c values are allowed in this family of solutions. There is no guarantee that all values of a parameter will lead to reasonable solutions.

2.1.5 The Pessimistic Outlook

In some sense, a DE is an mathematical application of the scientific method. Often an observation about a phenomenon can be expressed as a relationship between a function and its derivative, such as the observation of percentage growth. The DE, then, is the hypothesis born of observation. If we can find the solution, it gives us a predictive model of the phenomenon, which we can test. If the solution matches the observed behaviour, we conclude the DE model is relatively reliable; if the solution diverges from the observed behaviour, we discard or amend the DE.

In this way, DEs allow the modelling many phenomena: popluations, radioactive decay, cooling, disease, metabolish, newtonian motion with friction, chemical reactions, gravity, predaor-prey models, hamiltonian mechanics, quantum mechanics, interest, bacterial growth, neuron firing, ecology, mixtures, draining, series circuits, suspended cables, and many, many more.

However, for all this power and utility, DEs are terribly difficult to solve. The sad truth is that we can exactly solve only a very small portion of them. Due to this limitation, many techniques are developed to understand approximate solutions or infer information about solutions indirectly.

In addition, many DEs have solutions which are entirely new functions.

Example 2.1.12.

$$\frac{dy}{dt} = e^{-t^2} \implies y = \int e^{-t^2} dt + C$$

This integrand is a C^∞ function, so the anti-derivative exists, but there is no name for it in the elementary functions. This is not uncommon. Often we will 'solve' a differential equation by simply naming the solution with a new name, since it is an entirely new function.

2.2 Qualitative Methods for First Order DEs

As we said before, many DEs are very difficult or impossible to solve directly. Implicit or qualitative methods try to say something about the solutions without actually finding the solutions. Even when we can exactly solve an equation, these methods are great for interpreting the behaviour of the solutions.

2.2.1 Autonomous DEs and Phase-Line Analysis

Consider a population model $p(t)$ with its autonomous DE.

$$\frac{dp}{dt} = f(p)$$

There is a lovely piece of qualitative analysis for autonomous equations called the *phase line analysis*. Phase line analysis looks at the right side of the equation and asks: what values of p set the right side to zero. What does this mean? When the right side of our differential equation is 0, the left side is 0 as well. The left side is the growth rate, so that means the growth rate is zero. This is a value of the population p where there is no growth.

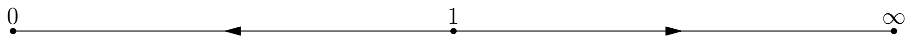

Figure 2.2: A Phase Line Diagram for $\frac{dp}{dt} = p^2 - p$

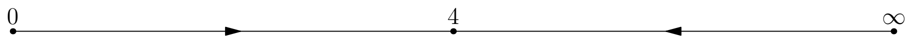

Figure 2.3: The Logistic Phase-Line Diagram

Definition 2.2.1. In an autonomous DE, a value of the function where the derivative vanishes is called a *steady states*

If the population is exactly at its steady state, it will not change; steady states are constant populations which do no grow or decline. (For population models, we can make the reasonable assumptions that $p \geq 0$ and $p = 0$ is always a steady state. Otherwise, we can have any real values on the phase line.)

Once we have the steady states, we ask what happens between each steady state. Assuming that the DE is reasonable, then between the steady states, the right side will be either positive or negative. When it is positive, we have a positive growth rate and the population increases. When it is negative, we have a negative growth rate and the population decreases.

Definition 2.2.2. In an autonomous DE, the direction of growth negative or positive, called the *trajectory* of the popluation.

Steady states and trajectories give us an remarkably complete understanding of the population.

- If the popluation is at a steady state, it doesn't change.
- If the popluation is not at a steady state, we look at the trajectory.
- If the trajectory is positive, the popluation grows either to the closest larger steady state or to infinity.
- If the trajectory is neagative, the population declines either to the closest smaller steady state or to zero.

We summarize this information is a phase line diagram. We take a ray representing $p \geq 0$ and put dots on the ray for the steady states. In between, we put arrows to show the trajectories. Its best to see the phase line diagrams through examples.

Example 2.2.3.

Example 2.2.4.

$$\frac{dp}{dt} = p^2 - p$$

The right side is zero when $p = 0$ or $p = 1$, so those are the steady states. When $p \in (0, 1)$ the derivative is negative, so the trajectory is decreasing. When $p \in (1, \infty)$, the derivative is positive, so the trajectory is increasing. Figure 2.2 shows the resulting phase-line.

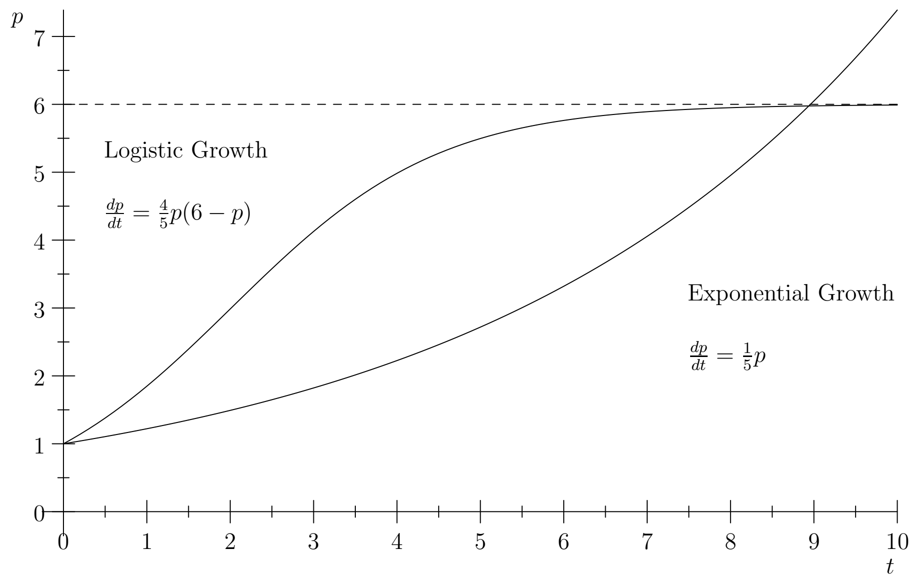

Figure 2.4: Exponential and Logistic Growth

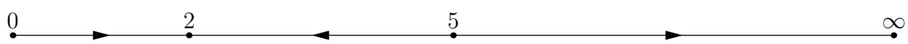

Figure 2.5: A Phase Line Diagram

This example is a specific instance of a form known as the logistic equation.

$$\frac{dp}{dt} = 4p - p^2$$

The right side is zero when $p = 0$ or $p = 4$, so those are the steady states. When $p \in (0, 4)$ the derivative is positive, so the trajectory is increasing. When $p \in (4, \infty)$, the derivative is negative, so the trajectory is decreasing. Figure 2.3 shows the resulting phase-line.

The logistic equation leads to logistic growth. We can see from the phase line diagram that the trajectories all point towards the value $p = 4$. In logistic growth, the values always want to revert to some non-zero steady state. From below, this is growth up to some firm maximum. After exponential growth, logistic growth is the most commonly used model for populations. Figure 2.4 shows both exponential and logistic growth (where the steady state for the logistic model is at $p = 6$.)

Example 2.2.5.

$$\frac{dp}{dt} = p^3 - 7p^2 + 10p$$

The right side factors as $p(p-2)(p-5)$, so it is zero then $p = 0$, $p = 2$ or $p = 5$. Those are the steady states. When $p \in (0, 4)$ the derivative is positive, so the trajectory is increasing. When $f \in (2, 5)$, the derivative is negative, so the trajectory is decreasing. When $p \in (5, \infty)$, the derivative is positive, so the trajectory is increasing. Figure 2.5 shows the phase line.

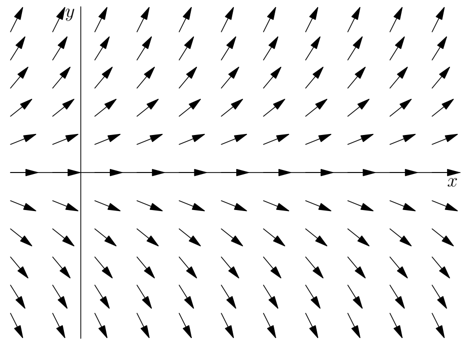

Figure 2.6: The Direction Field for $\frac{dy}{dx} = y$.

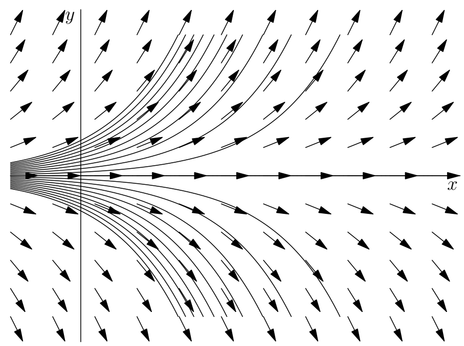

Figure 2.7: The Integral Curves for $\frac{dy}{dx} = y$.

2.2.2 Direction Fields

If the equation is not autonomous, then the phase-line is too simple a tool to capture the details. However, if we can solve for the derivative, we can write a first order DE as

$$\frac{dy}{dx} = F(x, y).$$

This allows a very useful interpretation: the left side is the slope of a graph, the right side is a function on $\mathbb{R}^2$, giving a value at every point in the plane. Together, we determine a slope at every point in the plane.

Definition 2.2.6. A determination of a slope at all point in a subset U of RR^2 is called a *direction field*.

If there are solutions to the DE, then they must be functions which fit these slopes. Therefore, the slopes give us a sense of what the functions look like. Let's go back to the simplest example.

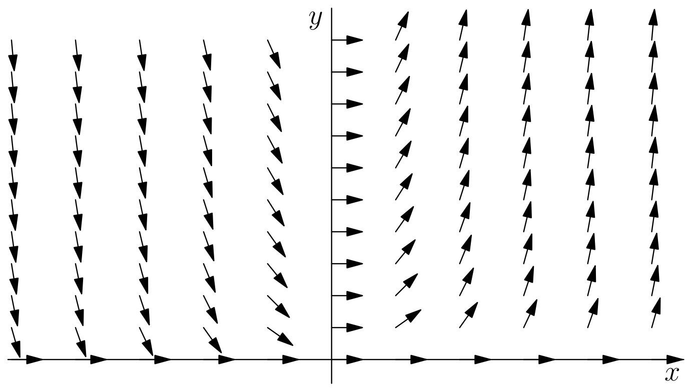

Figure 2.8: The Direction Field for $\frac{dy}{dx} = x\sqrt{y}$.

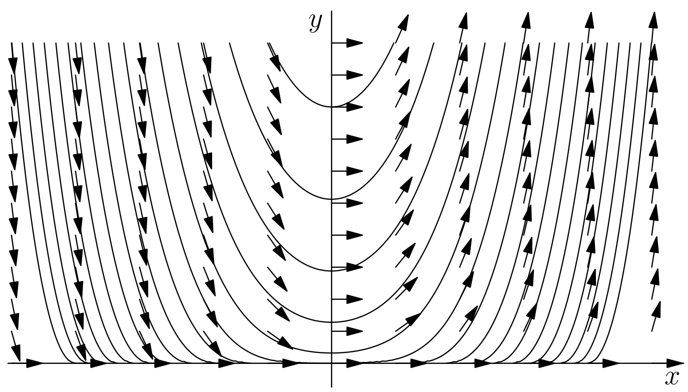

Figure 2.9: The Integral Curves for $\frac{dy}{dx} = x\sqrt{y}$.

Example 2.2.7.

$$\frac{dy}{dx} = y$$

The slope at a point (x, y) is y, so the slope at $(3, 4)$ is 4, the slope at $(-2, -3)$ is -3 and the slope at $(32, 0)$ is 0. Figure 2.6 shows the direction field.

The solutions exactly fit the direction field. Therefore, if we can draw and understand the direction field, we get a sense of the solutions. Notice that since the direction field fills $\mathbb{R}^2$ (or a portion of it), we expect an infinite family of graphs of functions to match all the slopes. Figure 2.6 shows the graphs of the infinite family of solutions.

Let's go back to the strange examples in the previous section and draw their direction fields.

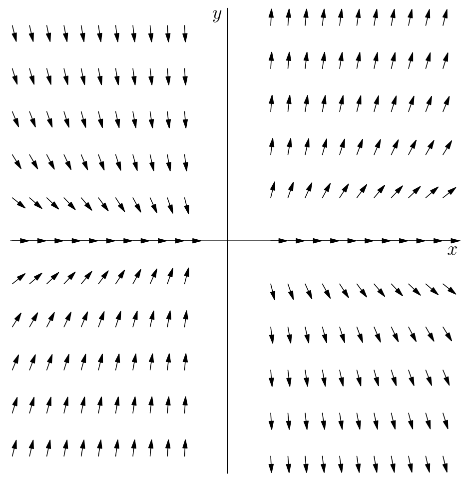

Figure 2.10: The Direction Field for $\frac{dy}{dx} = \frac{4y}{x}$.

Example 2.2.8.

$$\frac{dy}{dx} = x\sqrt{y}$$

Figure 2.7 shows the direction field and the infinite family of solutions. We see the entire family $y = (\frac{x^2}{4} + c)^2$, but we also see the singular solution $y = 0$.

Example 2.2.9.

$$\frac{dy}{dx} = \frac{4y}{x}$$

There family of solutions $y = cx^4$ which fits the direction field. The singular solutions are put together from one positive and one negative piece. Figure 2.8 shows the direction field and the infinite family of solutions.

Example 2.2.10.

$$\frac{dy}{dx} = xy$$

This is solved by $y = ce^{x^2}$, including $c = 0$ for $y = 0$ as a trivial solution. The direction field also shows the stability behaviour of the function: in this case, the functions grows very quickly away from the origin except for the stable and trivial $y = 0$ solution. Figure 2.9 shows the direction field and the infinite family of solutions.

Example 2.2.11.

$$\frac{dy}{dx} = -\frac{x}{y}$$

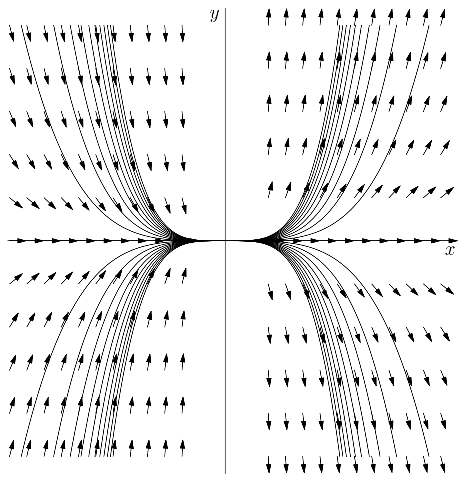

Figure 2.11: The Integral Curves for $\frac{dy}{dx} = \frac{4y}{x}$.

This is solved by $y = \pm\sqrt{c - x^2}$, which gives a series of circles. The direction field shows the bounded, relatively stable behaviour which is confirmed by the solutions. Notice that these solution have finite domains: we are not guaranteed solutions that apply to all real numbers. Figure 2.10 shows the direction field and the infinite family of solutions.

2.3 Stability and Linearization

2.3.1 Stability

Coming from applied mathematics, the language of stability is a very useful language for talking about DEs. We again think of a general autonomous DE as a population model.

$$\frac{dp}{dt} = f(p)$$

We defined steady states when we did phase-line analysis: these were roots of the function $f(p)$ and hence values of p where the rate of change is zero.

Definition 2.3.1. The steady states of an autonomous DE are classified by their *stability* A steady state P is *stable* if $p(t) \to P$ when the initial value is close to P. We can also call these steady states *attractors*. In the phase line diagram, the trajectories points toward such states. Similar, we can call a steady state *stable or attractive from above or below* if only one of the trajectories points towards the steady state. If both trajectories point away, the steady state is *unstable*.

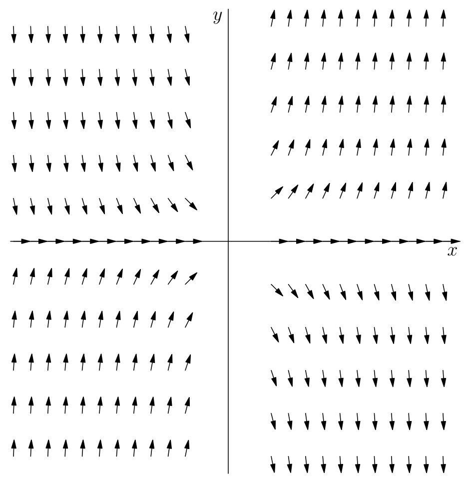

Figure 2.12: The Direction Field for $\frac{dy}{dx} = xy$.

2.3.2 Linearization

Once we have identified a steady state of a DE system, we often are only interested in the behaviour of slight perterbations from the steady state. This is the behaviour that stability capture: whether we approach or diverge from the steady state when we start a small distance away. If P is a steady state, the we can define a new function $q(t) = p(t) - P$. A change of variables results in

$$\frac{dq}{dt} = f(P + q).$$

If we expand f as a taylor series centered at p, this becomes

$$\frac{dq}{dt} = f(P + q) = f(P) + f'(P)(P + q - P) + \ldots = f'(P)q + \ldots.$$

Definition 2.3.2. The *linearization* of the DE at the steady state P is

$$\frac{dq}{dt} = f'(P)q.$$

The solution to the linearized DE is

$$q(t) = q_0 e^{f'(P)t}.$$

In particular, this linearized solution is either exponential growth or decay, depending on the sign of $f'(P)$. Therefore, the sign of $f'(P)$ determines the stability: positive and the solution is unstabe, negative and the solution is stable. This can also be seen in the phase line, since the sign of the derivative can indicate the trajectories on either side of the steady state. If $f'(P) = 0$, then the stability is determined by the higher order terms of the taylor series expansion.

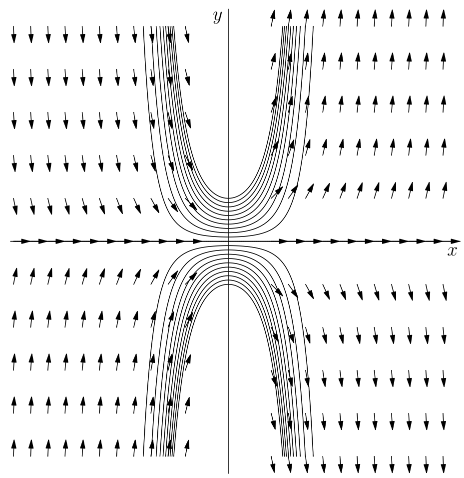

Figure 2.13: The Integral Curves for $\frac{dy}{dx} = xy$.

For now, there isn't much more we will do with this linearization. However, it is worth introducing here as a theme since it is so central to applied mathematics. Linear equations are almost always the first kind of DE that we try to use, typically since they have elegant and accessible solutions. Everything else gets simply referred to a 'non-linear'; in many ways, 'non-linear' is a synonmy for annoying and complication. However, linear models only go so far and often the non-linearity holds the key to understanding a model. Even so, we will often try to understand the linear part and then figure out how to add in the non-linearity in a reasonable fashion to add subtleties to various models.

2.4 Seperable Equations

Definition 2.4.1. A *seperable equation* is a DE which has the form

$$\frac{dy}{dt} = f(y)g(t).$$

The method of solving seperable equations treats the dt and dy terms as independent infinitesimals, a strange but historically reasonable treatment. If we allow for these independent infinitesimals, we can seperate (hence the name) the DE into two pieces, one involving each variable.

$$\frac{1}{f(y)}dy = g(t)dt$$

Then, again acting somewhat strangely by modern notational conventions, we integrate both sides with respect to their own variables.

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$

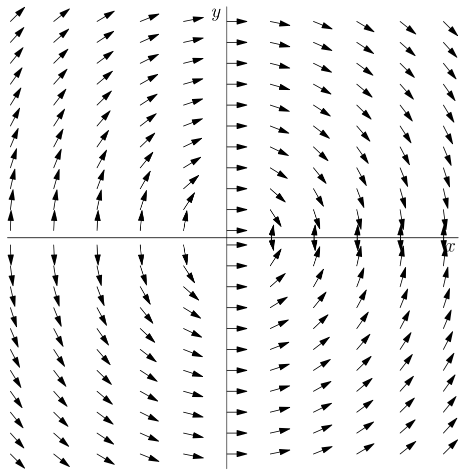

Figure 2.14: The Direction Field for $\frac{dy}{dx} = \frac{x}{y}$.

The solution is then left implicit unless we can reasonable solve for y, in which case we can write y as a conventional function of t.

If you are interested in the justification of this splitting procedure, we could think of the operation alternatively, writing $f(y(t))$ to remember the independent variable. If we bring the $f(y)$ to the left side, we get the expression

$$\frac{1}{f(y(t))}\frac{dy}{dt} = g(t).$$

Then we can integrate both sides with respect to t, which is a reasonable and justified operation.

$$\int \frac{1}{f(y(t))}\frac{dy}{dt}dt = \int g(t)dt$$

Finally, we change of variables from t to y in the left side integral.

$$\int \frac{1}{f(y)}dy = \int g(t)dt$$

In theory, we should get two constants of integration, one from each side. However, we can move the left side constant to the ride side and have the difference of two arbitrary constants, which is equivalent to one arbitrary constant. Therefore, we will only write on constant of integration for seperable equations.

In general, mathematicians have a practice of being somewhat careless with this constant. Since it doesn't need to be determined until we use an initial condition, we often forgoe various operations on the constant. For example, if we had $2(t + c)$, we would often simplify this to $2t + c$, since whether we figure out the constant from c or from $2c$ later, its value is still determined by the initial condition. It's useful to become accustomed to this carelessness constants of integration.

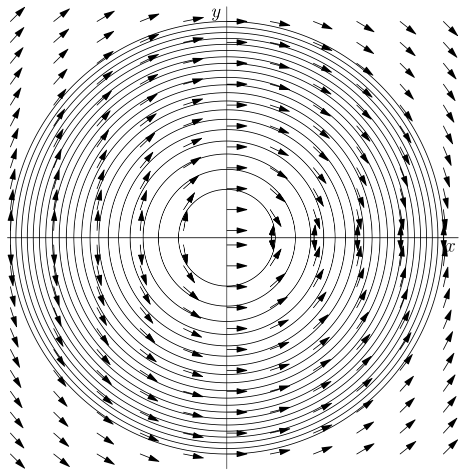

Figure 2.15: The Integral Curves for $\frac{dy}{dx} = \frac{x}{y}$.

Example 2.4.2.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin x}{y} \\ y dy &= \sin x dx \\ \frac{y^2}{2} &= -\cos x + c \\ y &= \pm\sqrt{c - 2\cos x} \end{aligned}$$

It is interesting to note where the constant of integration ends up. Since integration isn't the final step (we have to also solve for y), the constant moves around in the resulting algebra. In DE, we can't just add $+c$ at the very end of the process.

If we impose an initial condition of $y(0) = 1$, we can determined the value of the constant of integration.

$$\begin{aligned} 1 &= \sqrt{c - 2\cos(0)} = \sqrt{c - 2} \\ 1 &= c - 2 \\ c &= 3 \\ y &= \sqrt{3 - 2\cos x} \end{aligned}$$

Figure 2.11 shows the direction field and solution for this example. Notice the strange domain issues with this implicit plot. When $|c| \leq 2$, we have restricted domain solutions, represented by the closed curves. There are no solutions at all when $c < -2$. We have solutions with domain $\mathbb{R}$ only for $c \geq 2$. When $c = 2$, we get the strange crossed graph, which is not always differentiable. When $c = -2$. the "solution" is only defined at discrete points.

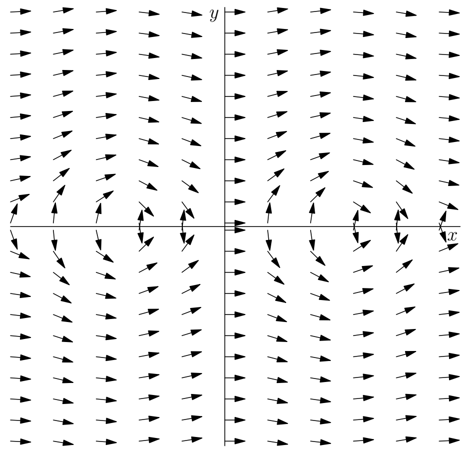

Figure 2.16: Direction Field for $\frac{dy}{dx} = \frac{\sin x}{y}$

Example 2.4.3. This is an autonomous example.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 - 4 \\
\int \frac{1}{y^2-4} dy &= \int 1 dx \\
\frac{-1}{2}\text{arctanh}\left(\frac{y}{2}\right) &= x + c \\
\text{arctanh}\left(\frac{y}{2}\right) &= -2x + c \\
\frac{y}{2} &= \tanh(-2x + c) \\
y &= 2\tanh(-2x + c)
\end{aligned}$$

Since this is an autonomous equation, we can look for steady (constant) singular solutions when the right side of the equation vanishes. Here, $y = 2$ and $y = -2$ are steady states. Moreover, $y = 2$ is stable and $y = -2$ is unstable. This can be seen in the direction field and implicit plot in Figure 2.12

It is also interesting to note that the output of tanh is only -1 to 1, so it is impossible to get $y \le -2$ or $y \ge 2$. We should wonder if there are solutions in this range at all. In the implicit plot, we could draw curves with these y values. These other curves are found by doing the integral differently, since both hyperbolic tangent and hyperbolic cotangent have the same anti-derivative. $y = 2\coth(-2x + c)$ is also a solution.

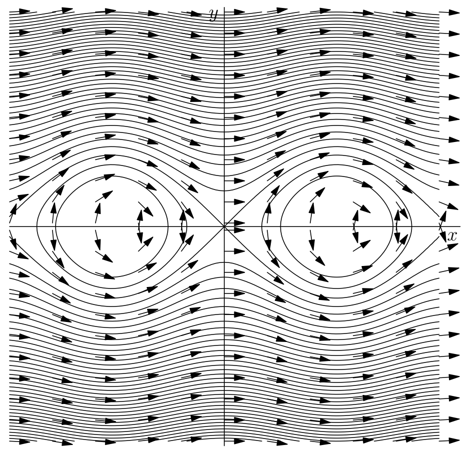

Figure 2.17: Direction Field with Solutions for $\frac{dy}{dx} = \frac{\sin x}{y}$

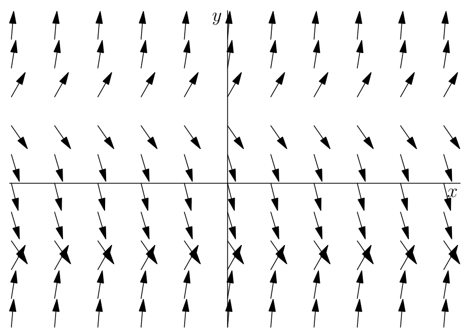

Figure 2.18: The Direction Field for $\frac{dy}{dx} = y^2 - 4$.

2.5 Existence and Uniqueness of Solutions

Before moving on with other techniques for solving first order equations, this is a nice place to take a pure-mathematical detour and talk about existence and uniqueness of solutions. We're going to deal with first order equations where we can isolate the derivative term, that is, equations of the form

$$\frac{dy}{dx} = F(x, y).$$

The details of existence and uniqueness theorems rely on the properties of F as a function of two variable. In this treatment, we think of y and x as two unrelated, independent variables, even though the DE itself implies that y is a function of x.

2.5.1 Existence

Existence of solutions to first order DES is established by the Peano Existence Theorem.

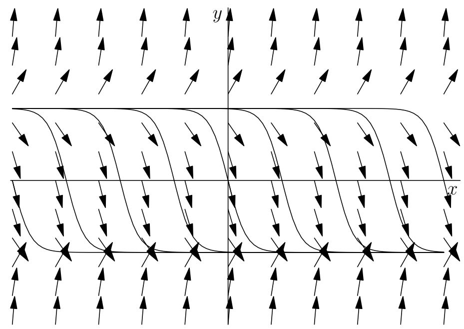

Figure 2.19: The Integral Curves for $\frac{dy}{dx} = y^2 - 4$.

Theorem 2.5.1. *In a equation of this form above, if F is continuous in both variables in an open set $U \subset \mathbb{R}^2$ and if $(x_0, y_0) \in U$, then there exists $\epsilon > 0$ such that the Initial Value Problem associated to the DE and the initial condition $y(x_0) = y_0$ has a solution with domain at least $[x_0 - \epsilon, x_0 + \epsilon]$.*

This is a very local result: the small positive ϵ only guarantees a tiny piece of a function as a solution. Existence (and later uniqueness) are only guaranteed very close to the initial value of the function. We do know that the function is differentiable in this small inteval, but we don't know anything else: outside the interval, anything could happen with the solution.

It's also useful to note, particularly for students with experience in other senior mathematics classes, that this theorem relies on topological considerations: we need an open subset U where F is continuous.

Example 2.5.2.

$$y' = y^{\frac{2}{3}} \qquad\qquad y(0) = 0$$

This satisfies the conditions of the theorem, so a solutions exists. However, there are two solutions: $y = 0$ and $y = x^3/27$. Peano's theorem doesn't imply the uniqueness of a solution.

Example 2.5.3.

$$y' = \sqrt{|y|} \qquad\qquad y(0) = 0$$

This F is also continuous near $(0, 0)$, so a solution exists by Peano's theorm. This IVP is solved by $y = 0$ and $y = x^2/4$. It's obvious that we need something more to ensure uniqueness.

Before we move on to the next result, we could ask for the proof of this theorem. Unforunately, that proof doesn't fall within the scope of this course. We would need to establish a number of new definitions and techniques from real analysis, as well as struggle through a bunch of tricky ϵ and δ arguments. It's interesting stuff, but very challenging and therefore left aside for this course.

2.5.2 Lipschitz Continuity

In order to state the theorem about uniqueness, we need a new definition of continuity.

Definition 2.5.4. A function from an open set U in $\mathbb{R}$ to $\mathbb{R}$ is called *Lipschitz continuous* if $\exists K > 0$ such that $\forall x_1, x_2 \in U$ we have $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$.

This is a strange kind of continuity. The definition is stronger than normal: Lipschitz continuity implies normal continuity. Moreover, the definition is global over U, not just local like conventional continuity. Therefore, the domain matters: a function might be Lipschitz continuous on a small set, but not on a larger one.

As an interpretation, Lipschitz continuity is a control on the global growth over the designated set. For a Lipschitz continuous function, there is a linear function that bounds the function on the designated set. Unsurprisingly, this means that the defintion usually only works on bounded sets. As a visualization, we can think of the definition as comparing f to a function $g(x) = K|x|$. The graph of g gives a cone in $\mathbb{R}^2$ and the f must stay inside this cone to be Lipschitz continuous. In this way, the definition limits the local growth of f and its derivatives.

Example 2.5.5. Here are some examples to illustrate the idea of Lipschitz continuity.

- $f(x) = x$ is Lipschitz continuous for $K = 1$ on any interval, since it is bounded by itself, a linear function.
- $f(x) = x^2$ is Lipschitz continuous on any *bounded* interval. Specifically, on $(-7, 7)$ we can take $K = 14$. However, $f(x) = x^2$ is not Lipschitz continuous on all of $\mathbb{R}$, since no linear function bounds it.
- $f(x) = x^{\frac{2}{3}}$ is not Lipschitz continuous on $(-1, 1)$, since its slope gets arbitrarily steep near the origin. This means that very close to 0, it cannot be bounded by a linear function through $(0, 0)$.

The previous example was not Lipschitz continuous at 0, and it also failed to be differentiable at that point. We might wonder if differentiability is a sufficient condition. That would be convenient, since we know how to check for differentiability. However, consider a strange example.

Example 2.5.6.

$$f(X) = \begin{cases} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right) & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

This isn't Lipschitz continuous, but it is differentiable near zero, showing that differentiability isn't sufficient. However, there is some good news: this example is a strange aberation.

Proposition 2.5.7. *A function which is C^1 at a point a in its domain is also Lipschitz continuous at a.*

$f \in C^1$ is roughly equivalent to saying that $\frac{\partial f}{\partial y}$ must exist and be bounded. This last criterion is the one we will use in practice: we check if the derivatives exists and if it is bounded.

Consider the $f(x) = x^2$. The derivaitve is $2x$, which is bounded locally near 0. $f(x) = x^{\frac{2}{3}}$ has derivative $\frac{2}{3}x^{\frac{-1}{3}}$ which is unbounded near 0. The former function is Lipschitz continuous and the later is not.

2.5.3 Uniqueness

The theorem for Uniqueness is called the Picard-Lindelöf thereom. It is a small improvement and adjustment of the Peano existence theorem.

Theorem 2.5.8. *If F is Lipschitz continuous in y and continuous is x on an open set U in $\mathbb{R}^2$ and if $(x_0, y_0) \in U$ then the initial value problem*

$$\frac{dy}{dx} = F(x, y) \qquad\qquad y(x_0) = y_0$$

has a unique solution $y = f(x)$ defined on the domain $[x_0 - \epsilon, x_0 + \epsilon]$ for some $\epsilon > 0$.

All of the comments from the previous theorem apply here. The result is very local, relies on topology, and the proof is beyond the scope of the course.

Example 2.5.9.

$$\frac{dy}{dx} = x\sqrt{y}$$

We considered this differential equation earlier. It had multiple solutions for initial value $y(0) = 0$. The function $F(x, y) = x\sqrt{y}$ has y deriavtive $\frac{\partial F}{\partial y} = \frac{x}{2\sqrt{y}}$, which is unbounded near 0. Therefore, it is not Lipschitz continuous and doesn't satisfy the continious of the Picard-Lidenlöf theorem, meaning that we don't expect a unique solution.

Lastly, anticipating the next section, consider the general first-order linear DE where P and Q are continuous functions.

$$\frac{dy}{dx} = Q(x) - P(x)y$$

In this case, $F(x, y) = Q(x) - P(x)y$ and the y derivative is $-P(x)$, which is bounded assuming that F is continuous. We can then apply the Picard-Lindelöf theorem to conclude that linear equations have unique solutions whenever their coefficient functions P and Q are continuous.

2.6 Linear Equations and Integrating Factors

First order linear equation have the general form

$$a(t)\frac{dy}{dt} + b(t)y = f(t).$$

Here $a(t)$, $b(t)$ and $f(t)$ are various (usually continuous) functions. If we avoid the values where $a(t) = 0$, we can divide by $a(t)$ to isolate the derivative, giving the following more typical form. (Remember the denominators! We have to pay attention to the roots of $a(t)$ throughout the solution.)

$$\frac{dy}{dt} + P(t)y = Q(t)$$

2.6.1 Homogeneous Solutions

The linear DE is called homogeneous if $Q(t) = 0$. In the homogeneous case, the DE is relatively easy to solve as a seperable equation.

$$\begin{aligned} \frac{dy}{dt} &= -P(t)y \\ \frac{1}{y}dy &= -P(t)dt \\ \int \frac{1}{y}dy &= -\int P(t)dt \\ \ln|y| &= -\int P(t)dt + c \\ y &= ce^{-\int P(t)}dt \end{aligned}$$

I should make a couple of notes about this calculation.

- We are informal with the constant. When we take exponents of each side of the equation, we should have multiplication by e^c. However, since this is still an undetermined constant, we simply write c instead of e^c. Also, when we drop the absolute value bars from y, we should have a $\pm$ term. Again, since c can be either positive or negative, we don't worry about that $\pm$.
- We had a y in the denominator for this process, which means that we have to be careful at points where $y = 0$. We use limits to figure out behaviour when y gets close to zero.
- If P is constant, this is our most basic exponential equation with solution $y = ce^{-\alpha t}$.

2.6.2 Linear Operators and Superposition

Recall from Definition 2.1.4 the idea of linear operators. If we write $L = \frac{d}{dt} + P(t)$, we can write a second order DE as

$$Ly = Q(t).$$

This is a *linear* differential operator, so it behaves linearly, that is, it has the two linearity properties.

$$\begin{aligned} L(y_1 + y_2) &= Ly_1 + Ly_2 \\ L(cy) &= cL(y) \end{aligned}$$

Then consider both the homoegenous equation $Ly = 0$ and a non-homogenous equation $Ly = Q(t)$. If f is a solution to the non-homogenous equation, then $Lf = Q(t)$, and if g is a solution to the homogenous equation, then $Lg = 0$. By linearity, we can conclude that

$$L(f + \alpha g) = Lf + \alpha Lg = Q(t) + \alpha \cdot 0 = Q(t).$$

Definition 2.6.1. If we have a solution to $Ly = Q(t)$, we can get other solutions by adding multiples of the solution to the homogeneous equation $Ly = 0$. The solution to $Ly = Q(t)$ is called the *particular solution* and this process is called *superposition of solutions.*

For those who remember their linear algebra, we can think of the solutions of a linear equation as a linear subspace of the vector space of differentiable functions (on an appropriate domain). In this sense the solution set of a linear equation is an offset span: the non-homogenous solution is the offset and the basis of the span is any homogenous solution. Superposition gives us the family-structure of solutions to linear equations. Any multiple of a homogenous solutions can be added to the particular solution, so the family is the particular solution plus any multiple of a homogeneous solutions: $y_p + \alpha y_h$. This α is the parameter of the family.

2.6.3 Integrating Factors

We know how to solve homogenous linear equations, since they are seperable. Finding non-homogenous solutions is somewhat more difficult; since these are not seperable equations, we need a new technique.

Let $L = \frac{d}{dt} + Q(t)$ be as before, and let y_h be a homogeneous solutions (a solution to $Ly = 0$). The technique we will use is called *variation of parameters.* This technique says we should look for a particular solution of the type $y_p = g(t)y_h$. Assuming that our solution has this special form, we have to try to find this $g(t)$. In order to do that, we put $g(t)y_h$ into the equation.

$$\begin{aligned}
Ly_p(t) = Lg(t)y(t) &= Q(t) \\
g'(t)y(t) + g(t)y'(t) + g(t)P(t)y(t) &= Q(t) \\
g'(t)y(t) + g(t)\Big(y'(t) + P(t)y(t)\Big) = g'(t) + g(t)Ly &= Q(t) \\
g'(t)y(t) + g(t)0 &= Q(t) \\
g'(t) &= \frac{Q(t)}{y(t)} \\
g(t) &= \int \frac{Q(t)}{y(t)}
\end{aligned}$$

We can put this back into our special form and use the fact that the homogeneous solution is $y = e^{-\int P(t)dt}$.

$$\begin{aligned}
y_p = g(t)y &= \left(\int \frac{Q(t)}{e^{-\int P(t)dt}} dt\right) e^{-\int P(t)dt} \\
y_p &= e^{-\int P(t)dt} \int e^{\int P(t)dt} Q(t)dt
\end{aligned}$$

Definition 2.6.2. In solving a linear equation of the for $\frac{dy}{dt} + P(t)y = Q(t)$, the expression $e^{\int P(t)dt}$ is called an *integrating factor* and it is typically written $\mu(t)$.

We rearrange the expression using the integrating factor.

$$
\begin{aligned}
e^{\int P(t)y} y_p &= \int e^{\int P(t)dt} Q(t)dt \\
\frac{d}{dt}\left(e^{\int P(t)dt} y_p\right) &= e^{\int P(t)dt} Q(t) \\
\frac{d}{dt}(\mu(t)y_p(t)) &= \mu(t)Q(t) \\
y_p &= \frac{\int \mu(t)Q(t)dt + c}{\mu(t)}
\end{aligned}
$$

Multiplying by the integrating factor turns the $y_p'(t) + P(t)y_p(t)$ side of the equation into a product rule derivative, which we can just integrate to solve. The integrating factor turns a linear equation into something that can be directly solved by integration, hence the name. It is best to remember the process this way: the original DE becomes a product rule derivative problem by multiplying both sides of the original DE by the integrating factor and then isolating y_p.

This variation of parameters is the first instance of a very common approach to solving DEs. Often, working with no idea of what kind of function we are looking for is simply too open-ended and too difficult. Therefore, we make a reasonable guess regarding the form of the solution. Here, that guess was $y_p = g(t)y_h$ where $g(t)$ was some unknown function. Then we put out special form back into the DE and try to find specific information about the parameter or unknown involved.

2.6.4 Examples

Example 2.6.3.

$$
\frac{dy}{dt} + \frac{y}{t} = 2e^t
$$

Since there is a t in the denominator, we must avoid $t = 0$ in the domain of solutions. We first look for the homogeneous solution.

$$
y_h(t) = ce^{-\int \frac{1}{t} dt} = ce^{-\ln|t|} = \frac{c}{t}
$$

Then the integrating factor is

$$
\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = t.
$$

We use the integrating factor to get a new DE.

$$\begin{aligned}
\mu(t)\frac{dy}{dt} + \mu(t)\frac{y}{t} &= \mu(t)2e^t \\
t\frac{dy}{dt} + t\frac{y}{t} &= t2e^t \\
\frac{d}{dt}(ty) &= 2te^t \\
ty &= \int (2te^t)dt \\
ty &= 2(te^t - e^t) + c \\
y &= 2\left(e^t - \frac{e^t}{t}\right) + \frac{c}{t}
\end{aligned}$$

Notice that we actually get the the homogeneous pieces here from the constant of integration, getting the whole linear family. Also notice that $t = 0$ is excluded from the domain.

Example 2.6.4.

$$(t^2 - 9)\frac{dy}{dt} + ty = 0$$

This is just a homogeneous DE. We note that $t \neq \pm 3$ in the domain. The solution to the homogeneous case is

$$y = e^{-\int P(t)dt} = e^{-\int \frac{t}{t^2-9}dt} = e^{-\frac{1}{2}\ln|t^2-9|+c} = \frac{c}{\sqrt{t^2-9}}.$$

We should be careful with the absolute value in $\ln|f(t)|$. The calculation can be done in two pieces, when $f(t) > 0$ and when $f(t) < 0$. In the previous calculation, we were a little careless with this detail.

Example 2.6.5.

$$\frac{dy}{dt} + y = f(t)$$

To add some complication, the non-homogeneous function here is going to be a step function.

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

This is still allowed: often the coefficients and functions involved in DEs are only piecewise continuous and/or piecewise differentiable. We can still work with them. The integrating factor is $\mu(t) = e^{\int 1dt} = e^t$. We have to work with two different intervals. First look at $[0, 1]$.

$$\frac{d}{dt}e^t y = e^t \implies e^t y = e^t + c_1 \implies y = 1 + c_1 e^{-t}$$

Alternatively, look at $(1, \infty)$.

$$\frac{d}{dt}e^t y = 0 \implies e^t y = t + c_2 \implies y = c_2 e^{-t}$$

We have two constants, but we want a continuous solution. (It will fail to be differentiable at $t = 1$, but that's alright. There is a sudden change in the situation, so that's expected.) Continuity at 1 means that $e - c_1 = c_2$. The final solution is a continuous piecewise function.

$$y = \begin{cases} 1 - ce^{-t} & t \in [0,1] \\ (e-c)e^{-t} & t \in (1,\infty) \end{cases}$$

Even with this piecewise function, we can still do initial value problems. If $y(0) = 0$, we find that $c = 1$ and we get a specific solution.

2.7 Substitutions

The third and final technique we will study for first order DEs is substitution. At some level, solving DEs is a more complicated and involved version of doing integrals, since we are trying to undo the results of differentiation. Substitution is the most common and important technique for solving integrals. It takes complicated integrals and changes their setup to make them more approachable. The technique is exactly the same here: we change the DE with some substitution operation to turn it into something we already know how to do. As with substitution for integrals, we can recognize some typical forms, but many others require creativity and ingenuity to solve. In this section, we'll introduce two forms which use substitution.

2.7.1 Homogeneous DEs

The first substitution is for a class of terribly named DEs called homogenous equations. Please note, this homogeneous has *nothing* to do with the previous definition for linear equations. The word here comes from a different use of the term in algebra. In any case, a homogenous DE has the form

$$\frac{dy}{dt} = f\left(\frac{y}{t}\right).$$

The substitution is the relatively obvious replacement $v = \frac{y}{t}$. The right side of the equation easily turns into $f(v)$, but the transformation of the left side is trickier.

$$\begin{aligned} y &= tv \\ f(v) = \frac{dy}{dt} &= v + t\frac{dv}{dt} \\ \frac{dv}{dt} &= \frac{f(v) - v}{t} \\ \int \frac{dv}{f(v) - v} &= \int \frac{dt}{t} \end{aligned}$$

In the last step, we've already started to solve the homogeneous equation as a seperable equation after the substitution is complete.

Example 2.7.1. This example is from Roberts and Marion. It is technically a seperable equation, but it is still useful to see how the substitution technique works.

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{t}{y} \\
v &= \frac{y}{t} \\
f(v) &= \frac{1}{v} \\
\frac{dv}{dt} &= \frac{\frac{1}{v} - v}{t} \\
\int \frac{dv}{\frac{1}{v} - v} &= \int \frac{dt}{t} \\
\int \frac{v}{1 - v^2} dv &= \ln|t| + c \\
\frac{-1}{2} \ln|1 - v^2| &= \ln|t| + C \\
\frac{1}{\sqrt{1 - v^2}} &= ct \\
\sqrt{1 - v^2} &= \frac{c}{t} \\
1 - v^2 &= \frac{c}{t^2} \\
v^2 &= 1 - \frac{c}{t^2} \\
v &= \pm\sqrt{1 - \frac{c}{t^2}} \\
\frac{y}{t} &= \pm\sqrt{1 - \frac{c}{t^2}} \\
y &= \pm t\sqrt{1 - \frac{c}{t^2}} = \pm\sqrt{t^2 - c}
\end{aligned}
$$

Notice that we reverse the substitution at the end, so that we end with the same variables as we started with.

Example 2.7.2.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{-y^2 - yt}{t^2} = \frac{-y^2}{t^2} - \frac{y}{t} \\
v &= \frac{y}{t} \\
f(v) &= -v^2 - v \\
\frac{dv}{dt} &= \frac{f(v) - v}{t} = \frac{-v^2 - v - v}{t} = \frac{-v^2 - 2v}{t} \\
\int \frac{dv}{-v^2 - 2v} &= \int \frac{dt}{t} = \ln|t| + c \\
\frac{1}{-v^2 - 2v} &= \frac{-1}{v(2+v)} = \frac{1}{2}\frac{1}{v+2} + \frac{-1}{2}\frac{1}{v} && \text{(Partial Fractions)} \\
\frac{1}{2}\int \frac{dv}{v+2} + \frac{-1}{2}\int \frac{dv}{v} &= \ln|t| + c \\
\frac{1}{2}\ln|v+2| + \frac{-1}{2}\ln|v| &= \ln|t| + c \\
\ln|v+2|^{\frac{1}{2}} + \ln|v|^{\frac{-1}{2}} &= \ln|t| + c \\
\sqrt{1 + \frac{2}{v}} &= ct \\
\frac{1+\sqrt{2}}{v} &= ct \\
1 + \frac{2}{v} &= ct^2 \\
\frac{2}{v} &= ct^2 - 1 \\
v &= \frac{2}{ct^2 - 1} \\
\frac{y}{t} &= \frac{2}{ct^2 - 1} \\
y &= \frac{2t}{ct^2 - 1}
\end{aligned}$$

2.7.2 Bernoulli Equations

The other substitution is for a class of DEs called Bernoulli equations.

These equations are almost linear, but they have an extra y^n term. The most general form is

$$\frac{dy}{dt} + P(t)y = Q(t)y^n.$$

The substitution is $v = y^{1-n}$. Be careful: this is y^{1-n}, not y^{n-1}. It's a very easy mistake to get this exponent wrong. We transform the DE by looking at the following derivations.

$$\begin{aligned}
\frac{dv}{dt} &= (1-n)y^{1-n-1}\frac{dy}{dt} = (1-n)y^{-n}\frac{dy}{dt} \\
&= (1-n)y^{-n}\left(Q(t)y^n - P(t)y\right) = (1-n)Q(t) - (1-n)P(t)y^{1-n} \\
&= (1-n)Q(t) - (1-n)P(t)v
\end{aligned}$$

$$\frac{dv}{dt} + (1-n)vP(t) = (1-n)Q(t)$$

This is now a linear equation in v. We can solve it as a linear equation in v and then use the reverse substitution to get back to y.

Example 2.7.3. This example is from Roberts and Marion.

$$\begin{aligned}
\frac{dy}{dt} - \frac{1}{2}\frac{y}{t} &= -e^t y^3 \\
n &= 3 \\
v &= y^{-2} \\
\frac{dv}{dt} - (-2)v\frac{1}{2t} &= -2(-e^t) = 2e^t \\
\frac{dv}{dt} + \frac{v}{t} &= 2e^t
\end{aligned}$$

This is a familiar linear equation which was solved in Example 2.6.3. We recall the solution from before.

$$\begin{aligned}
v &= 2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t} \\
y^{-2} &= 2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t} \\
y &= \left(2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t}\right)^{\frac{-1}{2}} \\
y &= \frac{\pm 1}{\sqrt{2e^t\left(1 + \frac{1}{t}\right) + \frac{c}{t}}}
\end{aligned}$$

Example 2.7.4.

$$\begin{aligned}
\frac{dy}{dt} &= y(ty^3 - 1) = ty^4 - y \\
\frac{dy}{dt} + y &= ty^4 \\
v &= y^{-3} \\
\frac{dv}{dt} - 3v &= -3t \\
P(t) &= -3 \\
Q(t) &= -3t \\
\mu(t) &= e^{\int P(t)dt} = e^{-3t} \\
\frac{d}{dt}e^{-3t}v &= -3te^{-3t} \\
e^{-3t}v &= \int -3te^{-3t}dt = -3\left(\frac{te^{-3t}}{-3} - \int \frac{e^{-3t}-3}{d}t\right) \\
e^{-3t}v &= te^{-3t} - \int e^{-3t}dt = te^{-3t} + \frac{e^{-3t}}{3} + c \\
v &= t + \frac{1}{3} + ce^{3t} = \frac{3t+1+ce^{3t}}{3} \\
y^{-3} &= \frac{3t+1+ce^{3t}}{3} \\
y^3 &= \frac{3}{3t+1+ce^{3t}} \\
y &= \sqrt[3]{\frac{3}{3t+1+ce^{3t}}}
\end{aligned}$$

2.8 Approximation Methods and Applications for First Order Equations

As this point, many DE courses might include a section on approximation methods or a section on applications, or both. We will be not spending any time on either of these.

The applications are useful for motivation. The approximation methods are interesting due to the fact, observed previously, that many DEs are terribly difficult to solve. The techniques we included above for first order equations cover only a small portion of all the possible equations and even then, we have to rely on integrals for seperable and linear equations. Integrals themselves are difficult and often only possible up to approximation. Therefore, a huge part of the mathematics of differential equations is the study of approximation techniques. Getting a sense of how these approximations are set up is an important insight into the field.

Chapter 3

Second Order Linear Differential Equations

A second order (ordinary) differential equation involves the second derivative of a function of one variable. In full generality, this could be any ridiculously complicated expression involving the independent variable, the function, its first derivative, and its second derivative.

Example 3.0.1.

$$\left(\frac{d^2 f}{dt^2}\right)^2 + \frac{\frac{df}{dt} - t^2 f^2}{\frac{df}{dt} - \frac{d^2 f}{dt^2}} = \frac{t^2 - \sin t}{f}$$

We have basiclaly no idea and no hope for approaching such complicated general cases. In this chapter, even more than with first order equations, we will restrict to very particular cases where there are useful techniques of solutions. The general restriction will be to linear equations, but even inside that restriction, we will make further assumptions about the structure.

As with first order equations, we solve initial value problems. However, since we have a second derivative involved, we expect (at least implicitly) to have to integrate twice. This results in two constants of integration. Therefore, we will require two initial values. Typically, for an equation about a function $f(t)$, there will be an initial value for the function $f(t_0) = a$ and an initial value for the derivative $f'(t_1) = b$. In most cases, a specific solution can only be found if both initial values are specified.

3.1 Linear DEs with Constant Coefficients

Definition 3.1.1. A *second order linear differential equation* is an equation of the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t).$$

As with the first order case, this is called homogeneous if $f(t) = 0$.

Definition 3.1.2. A *second order linear differential operator* is an operator of the form.

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

In operator notation, a second order linear differential equation is $Ly = f(t)$. The principles of linearity and superposition defined for first order equation in Section 2.6 still hold for higher-order linear equations.

Even though we look to linear equations for solvable second order DEs, linear equations are still very tricky, particularly if the coefficient functions are complicated. We need to restrict even further, to the simplest case where the coefficient functions are constant.

Definition 3.1.3. A *second order constant coefficient linear operator* has the form

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c.$$

The corresponding homogeneous equation is $Ly = 0$ and the corresponding non-homogeneous equation is $Ly = f(t)$. To avoid writing a lengthy name over and over again, I'll write SOCCLDE for a second order constant coefficient linear differential equation.

Second order constant coefficient linear differential equations, even with their very restricted form, are quite important for applied mathematics. They give models to understand both harmonic motion and alternating current electric curcuits. The harmonic motion interpretation is our starting point.

3.1.1 Harmonic Motion

Hooke's Law for a position function $x(t)$ in a harmonic system is the equation $F = -kx(t)$. We interpret this equation in terms of Newton's law of motion.

$$\begin{aligned} F &= -kx(t) \\ ma(t) + kx(t) &= 0 \\ m\frac{d^2x}{dt^2} + kx(t) &= 0 \end{aligned}$$

This gives us a homogeneous SOCCLDE with $a = m$, $b = 0$ and $c = k$. This is the simplest possible spring and there are two solutions.

$$x_1(t) = \sin\left(\sqrt{\frac{k}{m}}x\right) \qquad\qquad x_2(t) = \cos\left(\sqrt{\frac{k}{m}}x\right)$$

Since we can solve the homogeneous case, superposition applies and we can get a general solution.

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Note there are two parameters and the solution is a *linear combination* of the two independent solutions. That's expected for a second order equation. We expect two linearly independent solutiolns (neither is a constant multiple of the other) and the full family of homogeneous solutions is all linear combinations of the two solutions.

As an aside: the sums of sin and cos can be expressed as a single sinusoidal wave. We are helped by a relatively obscure but useful identity (if $a \geq 0$).

$$a\sin t + b\cos t = \sqrt{a^2+b^2}\sin(t+\phi) \qquad \text{where} \qquad \phi = \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

This identity lets us see linear combinations of sine and cosine as a single wave. In particular, the amplitude of a linear combination is the pythagorian combination $\sqrt{a^2+b^2}$ of the original amplitudes and frequency is altered, though the wave is shifted.

Hooke's Law is a nice starting point for understanding harmonic motion. However, these SOCCLDEs have no b term. We can wonder: what should b be? This b is a coefficient of the first derivative, the velocity of the system. Velocity causes friction in the system, and the greater the velocity the greater the friction. Hooke's law was an idealization which ignored friction; by adding in a non-zero b coefficient, we account for friction in the harmonic system. Again, we set up Hooke's law as a force quation using Newton's laws of motion.

$$F = -kx(t) - b\frac{dx}{dt}$$

$$ma(t) + b\frac{dx}{dt} + kx(t) = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0$$

Harmonic systems with friction are called *damped harmonic systems*. What do we expect out of such systems? The undamped system is a sine or cosine wave, which just oscillates forever. We expect that the friction should cause the oscillations to eventually slow down. Therefore, we expect decay in the amplitude of the sine way.

Now consider the non-homogeneous system. If we have $L = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$ where m is mass, b is the coefficient of friction and k is the spring constant, then what is the interpretation of $f(t)$ in $Lx(t) = f(t)$? The term $Lx(t)$ represents the force, since we derived it using Newton's law. Therefore, this must be an equation of forces and $f(t)$ must be some external force on the system. The homogenous equation gives the behaviour of the system on its own, and the non-homogenous equation add the complication of external forces.

3.1.2 Alternating Current Circuits

The second major interpretations of SOCCLDEs is as alternating current circuits. The DE that will result is exactly the same, but the interpretation of each of the coefficients is quite different. Instead of a position function $x(t)$, we will have a charge function $q(t)$ and the movement of that charge will constitute current. For our purposes, we will have four components to a circuit: resistors, capacitors, inductors and an external electro-motive force. Let's quickly given an account.

- Resistors allow for energy leaving the system and they represent the resistance to energy flow. The resistance is written R and measured in ohms. They act like friction in the mechanical system in that they want to slow down the flow of current. They will result in a decrease in current over time if there is no external forcing. Resistors represent the devices powered by the circuit, whatever those devices are.
- Capacitors are storage devices for electrical energy in electric fields. They have a measurement c called capacitance, which has units of farads (coloumbs per volt). They stabilize alternating current flow; as such, they can be see as controlling the natural way in which current flows. They (up to a reciprocal) align with the spring constant, which controlled the natural behaviour of the harmonic system (before friction and external forces).
- Inductors are storage devices for electrical energy in a magnetic field. They have a measurement L called inductance, which has units of henrys. Inductors block alternating current; as such, they represent the difficulty of moving charge through the system. In the harmonic system, the difficult of moving the object was its mass. Inductance takes the place of the mass term.
- Electromotive forces are external forces to the system, from batteries or generators. They are written $E(t)$ and have units of volts. Like the forces that add movement to a harmonic system, these electromotive forces add charge to a circuit.
- To give a complete account of the units, charge is written q and measured in coloumbs. The movement of charge is current, represented by the derivative $\frac{dq}{dt}$.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E(t)$$

3.1.3 Solving SOCCLDEs

Now that we have the two classic interpretations in place, we need to move on to actually solving SOCCLDEs. We'll use the operator

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c.$$

In the homogeneous constant-coefficient first order case, $\frac{dy}{dt} = at$, the solution was exponential $y = e^{at}$. We look for the same kind of solution here by assuming that $Ly = 0$ has solution $y = e^{rt}$. We put this potential solution in the DE.

$$\begin{aligned}
\frac{d}{dt}e^{rt} &= re^{rt} \\
\frac{d^2}{dt^2}e^{rt} &= r^2 e^{rt} \\
Le^{rt} &= ar^2 e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c)
\end{aligned}$$

If $Ly = 0$, then this equation is only satisfied if $ar^2 + br + c = 0$.

Definition 3.1.4. This equation is called the *characteristic equation* of the SOCCLDE.

The characteristic equation has two roots (with a possible repeated root).

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall the term $b^2 - 4ac$ is called the discriminant. In terms of the discriminat, there are three cases. We are making use here of the fundamental theorem of arithmetic, which says that polynoials always have roots over $\mathbb{C}$. See Section 1.5 for reference.

(a) $b^2 - 4ac > 0 \implies$ two real roots.
(b) $b^2 - 4ac = 0 \implies$ one real repeated root.
(c) $b^2 - 4ac < 0 \implies$ two complex roots.

3.1.4 Case 1: Two Real Roots

In this case, the solutions are normal exponential functions $e^{r_1 t}$ and $e^{r_1 t}$. Since the roots are distinct, these are linearly independent solutions (neither exponential function is a multiple of the other). We can simply check that they satisfy the DE.

$$Le^{rt} = e^{rt}(ar^2 + br + c) = 0$$

Since we expect two linearly indepedent solutions to a linear equation, the are no other solutions. The general solution is a superposition (linear combination) of the two solutions.

$$y = Ae^{r_1 t} + Be^{r_2 t}$$

The behaviour is this case exponential. If a, b and c are positive (as they are in both the major interpretations), then the roots are both negative, which means that both solutions are exponential decay. If we are modelling harmonic motion, the harmonic system decays to equilibrium with no oscillations. The discriminant condition is $b^2 > 4ac$ and recall that b was the coefficient of friction. The case of two real roots only happens if b is large enough. Exponential decay is the behaviour that results from a surplus of friction. There is too much friction even to have oscillations: we only have exponential decay to equilibrium. We called these situations *overamped* harmonic systems.

Example 3.1.5. We start with a classic example.

$$y'' - y = 0$$

The characteristic equation is is $r^2 - 1 = 0$ which has roots $r = \pm 1$. The solutions are $y = e^t$ and $y = e^{-t}$. The general solution is $y = Ae^t + Be^{-t}$. If we have initial conditions of $y(0) = 1$ and $y'(0) = 0$ then we get a system of equations for A and B: $A + B = 1$ and $A - B = 0$, which is solved by $A = B = \frac{1}{2}$. The final solution is $\frac{1}{2}e^t + \frac{1}{2}e^{-t}$. We should notice that this is just $\cosh t$, and then realize that we should have predicted this solution. The DE asks: what function returns to itself after two derivatives? Only the hyperbolics have this behaviour.

We could have made different choices for the initial linearly independent solutions by taking $y_1 = \cosh t$ and $y_2 = \sinh t$. The linearly independent solutions are not unique. For those who know linear algebra, these functions form a basis for the space of solutions and it is well known that a linear space has many different bases.

3.1.5 Case 2: Repeated Real Roots

If the characteristic equation factors as $(x - r)^2$, then r is a repeated real root (with value $\frac{-b}{2a}$). This gives us only one exponential solution: e^{rt}. This is a problem, since we expect two linearly independent solutions.

The solution to this problem is a bit strange, but turns out to be a common trick for linear equations: multiply by the independent variable. The second solution is te^{rt}.

$$\begin{aligned}
y &= te^{rt} \\
y' &= rte^{rt} + e^{rt} \\
y'' &= r^2te^{rt} + 2re^{rt} \\
ay'' + ay' + yc &= a(r^2te^{rt} + 2re^{rt}) + b(rte^{rt} + e^{rt}) + cte^{rt} \\
&= e^{rt}(ar^2t + 2ar + brt + b + ct) \\
&= e^{rt}(t(ar^2 + br + c) + 2ar + b) \\
&= e^{rt}(t \cdot 0 + 2a\frac{-b}{2a} + b) = 0 \qquad \text{Using } r = \frac{-b}{2a}
\end{aligned}$$

So we have two linearly indepedent solutions: $y_1 = e^{rt}$ and $y_2 = te^{rt}$. We already understand the first, since it is similar to the case of two distinct real roots. For harmonic systems, $r < 0$, the first solution is exponential decay and it corresponds to sufficient friction. The second solution is also exponential decay, since asymptotically the e^{-rt} dominates over t. However, it allows for one oscillation before decay.

The situation of repeated roots happens when $b^2 = 4ac$, so that the $\pm$ disappears from the solutions to the quadratic. For harmonic systems, this only happens if the friction is exactly as this special value $b = \sqrt{4ac}$. This is the exact friction that allows for this one oscillation before exponential decay. We call these systems *critically damped*. This is the tipping point for friction: there is exactly enough friction to result in exponential decay.

Example 3.1.6.

$$y'' - 2y' + y = 0$$

The characteristic equation is $r^2 = 2r + 1 = (r-1)^2$. The solutions are $y = e^t$ and $y = te^t$, so the general solution is $y = Ae^t + Bte^t$. If $y(0) = 1$ and $y'(0) = 0$ then we substitute into the equations to get $A = 1$ and $B = -1$ for a solution of $e^t - te^t$.

3.1.6 Case 3: Complex Roots

We have complex roots when $b^2 - 4ac < 0$. In that situation, we can factor $\imath$ out of the square root to get a real root.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \imath \frac{\sqrt{4ac - b^2}}{2a}$$

This gives a pair of complex numbers $x \pm y\imath$ with the real part $x = \frac{-b}{2a}$ and imaginary part $y = \frac{\sqrt{4ac-b^2}}{2a}$. They are a conjugate pair (each is the conjugate of the other). This is expected behaviour: the complex roots to a real quadratic always come as a conjugate pair. To write this more succinctly, we define two new constants.

$$\alpha = \frac{-b}{2a} \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

Then the complex roots can be written as $\alpha \pm \imath\beta$. The solutions to the DE are

$$e^{(\alpha \pm \imath\beta)t} = e^{\alpha t} e^{\imath\beta t}.$$

What do these complex functions means? The e^α term is fine, it is just a real exponential. The complex exponential is understood through Euler's formula (Proposition 1.5.5 in Section 1.5).

$$e^{\alpha t} e^{\pm \imath \beta t} = e^{\alpha t}(\cos(\beta t) \pm \imath \sin(\beta t))$$

These solutions are problematic because they involve complex numbers. We are trying to solve a real system with real coefficient and we want real solutions. To find them, we need to take clever linear combinations (over $\mathbb{C}$!) of the two solutions.

$$\frac{1}{2} e^{\alpha t}(\cos \beta t + \imath \sin \beta t) + \frac{1}{2} e^{\alpha t}(\cos \beta t - \imath \sin \beta t) = e^{\alpha t} \cos \beta t$$
$$\frac{1}{2\imath} e^{\alpha t}(\cos \beta t + \imath \sin \beta t) - \frac{1}{2\imath} e^{\alpha t}(\cos \beta t - \imath \sin \beta t) = e^{\alpha t} \sin \beta$$

On the basis of these linear combinations, we can take the following as our linearly independent real solutions.

$$y_1 = e^{\alpha t} \sin \beta t \qquad y_2 = e^{\alpha t} \cos \beta t$$

The general, real-valued solutions are linear combinations of the two linearly independent solutions.

$$y = Ae^{\alpha t} \sin \beta t + Be^{\alpha t} \cos \beta t$$

Example 3.1.7.

$$y'' + y = 0$$

The characteristic equation is $r^2 + 1$ which has roots $\pm\imath$. Therefore $\alpha = 0$ and $\beta = 1$, so the solutions are $\cos t$ and $\sin t$, with the general solution of $A\cos t + B\sin t$. If $y(0) = 1$ and $y'(0) = 0$, substitution into the equation gives $A = 1$ and $B = 0$ for $y = \cos t$ as the unique solution.

Example 3.1.8.

$$\begin{aligned}
y'' - 2y' + 5y &= 0 \\
r^2 + 2r + 5 &= 0 && \text{(Characteristic Equation)} \\
r &= \frac{2}{2} \pm \frac{\sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm \imath 2 \\
\alpha &= 1 \qquad \beta = 2 \\
y &= Ae^t \cos 2t + Be^t \sin 2t \\
y(0) &= 4 && \text{(Initial Conditions)} \\
y'(0) &= 6 \\
A &= 4 \qquad B = 1 \\
y &= 4e^t \cos 2t + e^t \sin 2t
\end{aligned}$$

Example 3.1.9.

$$\begin{aligned}
y'' + 3y' + 4y &= 0 \\
r^2 + 3r + 4 &= 0 && \text{(Characteristic Equation)} \\
r &= \frac{-3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{-3}{2} \pm \frac{\sqrt{-7}}{2} = \frac{-3}{2} \pm \imath\sqrt{7} \\
\alpha &= \frac{-3}{2} \qquad \beta = \sqrt{7} \\
y &= Ae^{-\frac{3t}{2}} \cos \sqrt{7}t + Be^{-\frac{3t}{2}} \sin \sqrt{7}t \\
y(0) &= 2 && \text{(Initial Conditions)} \\
y'(0) &= 2 \\
A &= 2 \qquad B = \frac{5}{\sqrt{7}} \\
y &= 2e^{-\frac{3t}{2}} \cos \sqrt{7}t + \frac{5}{\sqrt{7}} e^{-\frac{3t}{2}} \sin \sqrt{7}t
\end{aligned}$$

The previous example could be a harmonic system, since its coefficients are all positive. The results are sinusoidal functions with exponentially decaying amplitude. This fits our expectations for harmonic systems. We expect sinusoidal behaviour, but with decreasing amplitude. The complex roots happen when $b^2 < 4ac$, meaning that friction is small enough to allow for sinusoidal behaviour. We call such harmonic systems *underdamped*.

3.2 The Method of Undetermined Coefficients

Undetermined coefficients is the first of two methods we use to solve non-homogeneous SOCCLDEs. The second method, variation of parameters, is more general, but undetermined coefficients is easier and faster for particular types of non-homogemeous terms. In the spirit of applications to harmonic motion, I will often refer to the non-homogenous part of the SOCCLDE as a forcing term.

Recall, before we start, that if $Ly = f(t)$ is a non-homogeneous SOCCLDE, we know the structure of the general family of solutions. We expect two linearly independent solutions, y_1 and y_2, for the homogenous equaiton $Ly = 0$. We look for only one particular solution y_p of $Ly = f(t)$. The general solution has the form (where α and β are real parameter)

$$y = y_p + \alpha y_1 + \beta y_2.$$

Undetermined coefficents and variation of parameters try to find this particular solution y_p, and then we use the homogenous solutions to write the complete family.

The idea of undetermined coefficients is relatively simple: we try to guess a particular solution which is the same type of function as the forcing. If the forcing is polynomial, we look for a polynomial solution; if the forcing is exponential, we look for an exponential solution; and if the foring is sinusoidal, we look for a sinusoidal solution. Undertermined coefficients is going to work well for these three types of forcing terms or forcing terms given by products of functions of these types. However, it doesn't apply to other types of functions, where we can't reasonably expect the solution to look like the forcing term.

There is one possible pitfall to the process: sometimes the forcing terms is similar to the homogeneous solutions. In this case, the same type of function will only solve the homogeneous equation, not the non-homogeneous case. Our solution here is reminiscent of the case of repeated roots: we multiply by the independent variable t until we get something new. We'll see how this works out in examples.

Before we get into examples, here is a useful chart. As I said, we want to guess a similar type of functions to the forcing. What is similar? This chart gives the guesses. The constants C_i and D_j need to be determined: these constants are the undetermined coefficients which give the process its name.

$f(t)$	y_p
ke^{at}	Ce^{at}
kt^n	$C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0$
$k\cos(at)$ or $k\sin(at)$	$C\cos(at) + D\sin(at)$
$kt^n e^{at}$	$e^{at}\left(C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0\right)$
$kt^n\cos(at)$ or $kt^n\sin(at)$	$(C_n t^n + \cdots + C_0)\cos(at) + (D_n t^n + \cdots + D_0)\sin(at)$
$ke^{at}\cos(bt)$ or $ke^{at}\sin(bt)$	$e^{at}\left(C\cos(at) + D\sin(at)\right)$
$kt^n e^{at}\cos(bt)$ or $kt^n e^{at}\sin(bt)$	$(C_n t^n + \cdots + C_0)\, e^{at}\cos(bt) + (D_n t^n + \cdots + D_0)\, e^{bt}\sin(bt)$

With this chart, the process is simple. We find the right guess, put it into the DE, and try to work out the unknown coefficients. Let's see this in examples.

Example 3.2.1. This example is from Roberts and Marion.

$$y'' - 3y' + 2y = t \qquad y(0) = \frac{3}{4} \qquad y'(0) = \frac{3}{2}$$

$$\begin{aligned}
L &= \frac{d^2}{dt^2} + 3\frac{d}{dt} + 2 \\
&\text{Solve the Homogeneous Case:} \\
Ly = 0 &\implies r^2 - 3r + 2 = 0 \\
(r-2)(r-1) = 0 &\implies r = 1, 2 \\
y_h &= Ae^{2t} + Be^t \\
&\text{Our Guess for Undetermined Coefficients:} \\
y_p &= Ct + D \\
y_p' &= C \\
y_p'' &= 0 \\
&\text{Solve for the Coefficients:} \\
Ly_p &= 0 - 3C + 2(Ct + D) = t \\
2C = 1 &\implies C = \frac{1}{2} \\
-3C + 2D &= 0 \\
-3\frac{1}{2} + 2D = 0 &\implies D = \frac{3}{4} \\
y_p &= \frac{t}{2} + \frac{3}{4} \\
y &= \frac{t}{2} + \frac{3}{4} + Ae^{2t} + Be^t \\
&\text{Use the Initial Values:} \\
y(0) &= \frac{3}{4} \qquad y'(0) = \frac{3}{2} \\
y' &= 2Ae^{2t} + Be^t + \frac{1}{2} \\
y(0) &= A + B + \frac{3}{4} = \frac{3}{4} \implies A = -B \\
y'(0) &= 2A + B + \frac{1}{2} = \frac{3}{2} \\
2A + B &= 1 \\
2A - B = 1 &\implies A = 1 \implies B = -1 \\
y &= \frac{t}{2} + \frac{3}{4} + e^{2t} - e^t
\end{aligned}$$

Example 3.2.2. This example lacks initial values. I've skipped over some of the algebra for the sake of brevity.

$$\begin{aligned}
y'' + 8y' + 3y &= e^{-2t}\cos 2t \\
L &= \frac{d^2}{dt^2} + 8\frac{d}{dt} + 3 \\
&\text{Solve the Homogeneous Case:} \\
Ly = 0 &\implies r^2 + 8r + 3 = 0 \\
r &= -4 \pm \sqrt{13} \\
y_h &= Ae^{(-4+\sqrt{13})t} + Be^{(-4-\sqrt{13})t} \\
&\text{Our Guess for Undetermined Coefficients:} \\
y_p &= e^{-2t}(C\cos 2t + D\sin 2t) \\
y' &= e^{-2t}((-2C+2D)\cos 2t + (-2D-2C)\sin 2t) \\
y'' &= e^{-2t}((-8D)\cos 2t + (8C)\sin 2t) \\
&\text{Solve for the Coefficients:} \\
Ly_p &= e^{-2t}((8D-13C)\cos 2t + (-8C-13D)\sin 2t) = e^{-2t}\cos 2t \\
8D - 13C &= 1 \\
-8C - 13D &= 0 \\
D &= \frac{-8C}{13} \\
C &= \frac{-13}{233} \implies D = \frac{8}{233} \\
y_p &= e^{-2t}\left(\frac{-13}{233}\cos 2t + \frac{8}{233}\sin 2t\right) \\
y &= Ae^{(-4+\sqrt{13})t} + Be^{(-4-\sqrt{13})t} + e^{-2t}\left(\frac{-13}{233}\cos 2t + \frac{8}{233}\sin 2t\right)
\end{aligned}$$

It's useful to go back to our harmonic system interpretation to understand these solutions. There are no oscillations in the homogeneous case here: sufficient friction gives exponential decay. However, there is oscillating forcing, though the forcing is also undering going exponential decay. This forcing is enough to add sinusoidal behaviour to the full solutions, but the decaying forcing terms means that the sinusoidal term will also decay over time. The amplitude of this combination of waves is given by the pythagorean combination: $\frac{\sqrt{13^2+8^2}}{233} = \frac{1}{\sqrt{233}}$.

Example 3.2.3. In this example, the forcing term is similar to one of the homogeneous solutions, so you will notice that we have multiplied by t in our guess for the particular solution. Again, I've skipped more of the algebra in this example, particularly omitting the long derivatives and calculation of Ly_p.

$$\begin{aligned}
y'' + 2y' + 2y &= f(t) = 4e^{-t}\sin t \\
&\text{Solve the Homogeneous Case:} \\
r^2 + 2r + 2 = 0 &\implies r = -1 \pm \imath \\
y_h &= e^{-t}(A\sin t + B\cos t) \\
&\text{Our Guess for Undetermined Coefficients:} \\
y_p &= te^{-t}(C\sin t + D\cos t) \\
y' &= e^{-t}(C\sin t + D\cos t) + te^{-t}((-C - D)\sin t + (-D + C)\cos t) \\
y'' &= e^{-t}((-2C - 2D)\sin t + (-2D + 2C)\cos t) + te^{-t}((2D)\sin t + (-2C)\cos t) \\
&\text{Solve for the Coefficients:} \\
Ly_p &= e^{-t}((-2D)\sin t + (2C)\cos t) = 4e^{-t}\sin t \\
-2D = 4 &\implies D = -2 \\
2C = 0 &\implies C = 0 \\
y_p &= -2te^{-t}\cos t \\
y &= -2te^{-t}\cos t + e^{-t}(A\sin t + B\cos t)
\end{aligned}$$

In this result, we still have exponential decay, since the exponential is asymptotically dominant in the te^{-t} term. However, the trajectory and behaviour of the decay differs, particular for small t, from the homogeneous solutions.

3.3 Resonance

The discussion in the previous section about the similarity between forcing terms and the homogeneous solutions leads us into the subject of resonance in harmonic sequences. The question of resonance is this: is there a particular frequency for an external force on a harmonic system which produces the strongest effect?

This is an important question in a number of situations. In music and acoustics, we may want to design explicitly for resonance. In the safety of structures, we would like to ensure that resonant behaviour is impossible.

Let's start with the SOCCLDE describing an underdamped harmonic system. (Underdamped is necessary to allow for the possibility of resonance, as we will see in the calculations). Recall that for harmonic systems, the coefficients can be identified as mass m, spring constant k, coefficient of friction b and forcing $f(t)$.

$$my'' + by' + ky = f(t)$$

Look at the characterstic equation $mr^2 + br + k = 0$. It has solutions

$$r = \frac{-b}{2m} \pm \frac{\sqrt{b^2 - 4km}}{2m}.$$

We can define a new constant to keep track of the behaviour. This constant is called the damping constant.

$$\zeta := \frac{b}{2\sqrt{km}}$$

The damping constant gives a nice measure of the friction. If $\zeta < 1$ then the system is underdamped and we have sinusoidal behaviour. If $\zeta = 1$ the situation is critically damped and if $\zeta > 1$, the situation is overdamped; in both cases, we just have exponential decay. The frictionless case is $\zeta = 0$. Let's go back to the frictionless case for a moment. The solutions when $\zeta = 0$ are

$$y = A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t.$$

We're going to define another useful constant.

$$\omega := \sqrt{\frac{k}{m}}$$

This constant is called the natural frequency. It represents the frequency of sinusoidal oscillation in a perfect system without friction. (Note that all of the frequencies in this section are not true frequencies: they are off by a factor of $\frac{1}{2\pi}$. However, we'll ignore this fact and keep referring to them as frequencies.)

Finally, we're going to define one more constant.

$$\lambda := \frac{b}{2m}$$

If we look back to the underdamped case, the exponential decay term can be written $e^{-\lambda t}$. Therefore, λ is called the decay coefficient. With these new constants, the roots of the characteristic equation become $r = -\lambda + \pm \imath \sqrt{\omega^2 - \lambda^2}$ and we can rewrite the homogeneous differential equation as

$$y'' + 2\lambda y' + \omega^2 y = 0.$$

Now let's return to the idea of a sinusoidal forcing term $f(t) = F \sin \gamma t$ with some frequency γ. We want to look at four situation to understand the effect of this force and the possibility of resonance.

3.3.1 Situation 1: No Friction, No Forcing

This is the trivial base case, where the system just oscillates forever with frequency ω. This frequency is called the natural frequency because it describes the dynamics of this trivla base case, even if the ideal frictionless situation isn't particularly natural.

3.3.2 Situation 2: No Friction, Forcing

The differential equation currently has the form

$$y'' + \omega^2 y = F \sin \gamma t.$$

This is something that we can solve with undetermined coefficients. If we assume that $\gamma \neq \omega$, then the forcing term is unlike the homogeneous solutions. We use undetermined coefficients.

$$\begin{aligned}
y_p &= C \sin \gamma t + D \cos \gamma C \\
y_p' &= \gamma C \cos \gamma t - \gamma D \sin \gamma C \\
y_p'' &= -\gamma^2 C \sin \gamma t - \gamma^2 D \cos \gamma C \\
L y_p &= (-\gamma^2 C + \omega^2 C) \sin \gamma t + (-\gamma^2 D + \omega^2 D) \cos \gamma t = F \sin \gamma t \\
-\gamma^2 C + \omega^2 C = F &\implies C = \frac{F}{\omega^2 - \gamma^2} \\
-\gamma^2 D + \omega^2 D = 0 &\implies D = 0 \\
y_p &= \frac{F}{\omega^2 - \gamma^2} \sin \gamma t \\
y &= A \sin \omega t + B \cos \omega t + \frac{F}{\omega^2 - \gamma^2} \sin \gamma t
\end{aligned}$$

We see that the particular solution is a sine wave with amplitutde $\frac{F}{\omega^2 - \gamma^2}$. The closer the forcing frequency is to the natural frequency, the greater the amplitude of the resulting oscillation. If we impose the initial conditions $y(0) = 0$ and $y'(0) = 0$, then the system is initially at rest and the only energy in the system comes from the external forcing. In this case, we easily get $B = 0$ from the first initial condition and we calculate for the second.

$$\begin{aligned}
y' &= \omega A \cos \omega t + \frac{\gamma F}{\omega^2 - \gamma^2} \cos \gamma t \\
y'(0) &= A\omega + \frac{\gamma F}{\omega^2 - \gamma^2} = 0 \implies A = \frac{-\gamma F}{\omega(\omega^2 - \gamma^2)} \\
y &= \frac{-\gamma F}{\omega(\omega^2 - \gamma^2)} \sin \omega t + \frac{F}{\omega^2 - \gamma^2} \sin \gamma t = \frac{-\gamma F \sin \omega t + \omega F \sin \gamma t}{\omega(\omega^2 - \gamma^2)}
\end{aligned}$$

Now, to answer the question of resonance, let's take the limit of this solution as $\gamma \to \omega$. The denominator is undefined, but the numerator also evaluates to 0, so we can apply L'Hôpital's rule to calculate the limit.

$$\begin{aligned}
\lim_{\gamma \to \omega} \frac{-\gamma F \sin \omega t + \omega F \sin \gamma t}{\omega(\omega^2 - \gamma^2)} &= \lim_{\gamma \to \omega} \frac{-F \sin \omega t + \omega t F \cos \gamma t}{-2\gamma\omega} \\
&= \frac{-F \sin \omega t + \omega t F \cos \omega t}{-2\omega^2} \\
&= \frac{F}{2\omega^2} \sin \omega t - \frac{F t \cos \omega t}{2\omega}
\end{aligned}$$

The result has a $t\cos\omega t$ term, which is a sinusoidal wave with linear growth in amplitude. The amplidute of the system will grow linearly without bound. This is the ideal (frictionless) understanding of resonance, where the oscillations of the system continue to grow.

This is a justification, if you want, of why multiplying by t gives the particular solutions when you have a forcing term similar to the homogeneous solutions. When $\gamma = \omega$, the forcing is already a homogeneous solution, so expect a solution multiplied by t. That's precisely what we have here. It is also what we would have found if we with $\omega = \gamma$ and used undetermined coefficients, instead of using the limit process.

3.3.3 Section 3: Friction, No Forcing

The homogeneous SOCCLDE with our choice of constants is

$$y'' + 2\lambda y' + \omega^2 y = 0.$$

The characteristic equation is $r^2 + 2\lambda r + \omega^2$ with roots $-\lambda \pm \imath\sqrt{\omega^2 - \lambda^2}$ If we let $\omega_d = \sqrt{\omega^2 - \lambda^2}$, then the homogeneous solutions are

$$y = e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t).$$

The new constant ω_d is called the damped frequency. It is not the same as the natural frequency, since friction changes the desire frequency of th system. We can observe from the previous definition that.

$$\lambda^2 = \frac{\zeta^2}{\omega^2} \qquad \text{or equivalently} \qquad \lambda^2\omega^2 = \zeta^2.$$

Then we can also observe that

$$\omega_d = \omega\sqrt{1 - \zeta^2}.$$

With friction involved and no forcing, we have sinusoidally decaying oscillations where the frequency is this ω_d. As an interesting aside, as ζ approaches 1 and we approach the critical damped situation with simple exponential decay, this damped frequency approaches zero. This explains the transition from sinusoidal to exponential behaviour: as we approach the critically damped situation, the wavelength of the sine wave (which is a reciprocal of frequency) grows to ∞. The wave stretches out further and further until there isn't any wave left at all, just an exponential decay.

3.3.4 Situation 4: Friction and Forcing

The differential equation is now

$$y'' + 2\lambda y' + \omega^2 y = F\sin\gamma t.$$

The homogeneous solutions are known from the third situation.

$$y_h = e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t)$$

As in the second situation, our forcing term is $F\sin\gamma t$ and has a particular frequency γ. Since the forcing lacks the exponential term, it is not the same as the homogeneous solutions. We can use undetermined coefficients without any alteration. (What does this already imply?) Here is the solution using undetermined coefficients, with some matrix algebra in the middle to solve the system of equations.

$$\begin{aligned}
&\text{Our Guess for Undetermined Coefficients:}\\
y_p &= C\sin\gamma t + D\cos\gamma t\\
y_p' &= C\gamma\cos\gamma t - D\gamma\sin\gamma t\\
y_p'' &= -C\gamma^2\cos\gamma t - D\gamma^2\sin\gamma t\\
&\text{Solve for the Coefficients:}\\
Ly_p &= (-C\gamma^2 - 2\lambda D\gamma + C\omega^2)\sin\gamma t + (-D\gamma^2 + 2\lambda C\gamma + D\omega^2)\cos\gamma t\\
-C\gamma^2 - 2\lambda D\gamma + C\omega^2 &= F\\
-D\gamma^2 + 2\lambda C\gamma + D\omega^2 &= 0\\
\begin{bmatrix}\omega^2-\gamma^2 & -2\lambda\gamma\\ 2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} &= \begin{bmatrix}F\\0\end{bmatrix}\\
M &= \begin{bmatrix}\omega^2-\gamma^2 & -2\lambda\gamma\\ 2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\\
\det M &= (\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2\\
M^{-1} &= \frac{1}{\det M}\begin{bmatrix}\omega^2-\gamma^2 & 2\lambda\gamma\\ -2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\\
\begin{bmatrix}C\\D\end{bmatrix} &= \frac{1}{\det M}\begin{bmatrix}\omega^2-\gamma^2 & 2\lambda\gamma\\ -2\lambda\gamma & \omega^2-\gamma^2\end{bmatrix}\begin{bmatrix}F\\0\end{bmatrix}\\
C &= \frac{(\omega^2-\gamma^2)F}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
D &= \frac{-2\gamma\lambda F}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
y_p &= \frac{(\omega^2-\gamma^2)F\sin\gamma t - 2\gamma\lambda F\cos\gamma t}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\\
y &= e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{(\omega^2-\gamma^2)F\sin\gamma t - 2\gamma\lambda F\cos\gamma t}{(\omega^2-\gamma^2) + 4\lambda^2\gamma^2}
\end{aligned}$$

As time passes, the homogeneous solutions fall out and only the term with the forcing frequency γ remains. What is its amplitude? The term is a linear combination of a sine and cosine wave, so we can use the identity mentioned previously that told us the amplitude of the combined wave was a pythagorean combination of the two amplitudes.

$$a = F\frac{\sqrt{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2}}{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2} = \frac{F}{\sqrt{(\omega^2-\gamma^2)^2 + 4\gamma^2\lambda^2}}$$

The question of resonance now becomes a question of amplitude. If F is fixed, what is the maximum amplitude we can achieve with a fixed force? Note this amplitude is always finite – with friction, there is no infinite growth of amplitude. (As predicted, since we didn't need to multiply by t when we setup the undetermined coefficients.) However, the amplitude can be quite large. This is a optimization problem, so we differentiate the expression for amplitude and find its critical points.

$$\begin{aligned}
a(\gamma) &= \frac{F}{\sqrt{(\omega^2-\gamma^2)^2+4\gamma^2\lambda^2}} \\
a'(\gamma) &= \frac{-F(2(\omega^2-\gamma^2)(-2\gamma)+8\gamma\lambda^2)}{2((\omega^2-\gamma^2)^2+4\gamma^2\lambda^2)^{\frac{3}{2}}} = 0 \\
4\gamma(\omega^2-\gamma^2) &= 8\gamma\lambda^2 \\
\omega^2-\gamma^2 &= 2\lambda^2 \\
\gamma^2 &= \omega^2-2\lambda^2 \\
\gamma &= \sqrt{\omega^2-2\lambda^2}
\end{aligned}$$

This is almost in the form we want. Recall that $\lambda^2 = \omega^2\zeta^2$ which allows the form

$$\gamma = \sqrt{\omega^2-2\zeta^2\omega^2} = \omega\sqrt{1-2\zeta^2}.$$

This is the critical point, and it is indeed a maximum for the amplitude. It is the resonant frequency. However, are we certain that it exists? We need $1-2\zeta^2$ to be positive to define this square root. This implies, certainly, that $\zeta < 1$, which we assumed when we decided to work with the underdamped case (that assumption is now justified). However, the inequality is stricter.

$$\zeta^2 \leq \frac{1}{2} \implies \zeta \leq \frac{1}{\sqrt{2}}$$

The bound for the damping coefficient is smaller than simply for the underdamped case. The conclusion we reach is that a resonant frequency only exists is the friction is minimal enough, measured by this $\zeta < \frac{1}{\sqrt{2}}$. In particular, if we are concerned about safety and want to avoid the situation of resonant frequency, we know how much friction we need to build into the system.

3.4 Variation of Parameters

Undetermined coefficients is an effective and efficient (relatively speaking!) way to solve for a particular non-homogeneous solution of a SOLCCDE. It's restriction is its scope: it only works for specific forcing terms. Variation of parameters is a much more general method of finding particular solutions. It's weakness is the reliance on difficult integrals – often we'll have to leave the solutions in a form which involves unfinished integrals. In the examples, we'll often stick to forcing terms which could have used undetermined coefficients, just to make the integrals reasonable.

3.4.1 Wronskians

Definition 3.4.1. Let $f_1, f_2, \dots f_n \in C^{n-1}(\mathbb{R})$. The *Wronskian* of this set of functions is defined to be the determinant of a matrix involving the f_i and their derivatives.

$$W(f_1, f_2, \dots, f_n) := \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ f_1'' & \dots & f_n'' \\ \dots & \dots & \dots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

The Wronskian of a set of functions has a number of different uses. Its first use is checking for linear independence. The set of functions linearly independent if and only if the Wronskian is never zero. Most of the time, this is easy to satisfy. In addition, if we have linearly independent functions, we know the Wronskian is never zero and we can divide by if with impunity.

For variation of parameters, we only need to the Wronskain of two functions. The definition here is relatively simple.

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$$

3.4.2 The Technique of Variations of Parameters

Let L be the second order operator

$$L = \frac{d^2}{dt^2} + P\frac{d}{dt} + Q.$$

Let y_1 and y_2 be solutions to the homogeneous equation $Ly = 0$. The general solution to the homogeneous equation is $Ay_1 + By_2$. As with the first-order case, variation of parameters replaces the constants A and B with functions. That is, we look for a solution which has the form (where $u_1(t)$ and $u_2(t)$ are unknown functions)

$$y_p = u_1 y_1 + u_2 y_2.$$

Then we put this y_p into the differential equation. A long and tedious calculation ensues.

$$\begin{aligned} Ly_p &= u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' + P\left(u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right) \\ &\qquad + Q\left(u_1 y_1 + u_2 y_2\right) = f \\ &= u_1\left(y_1'' + Py_1' + Qy_1\right) + u_2\left(y_2'' + Py_2' + Qy_2\right) + \left(y_1 u_1'' + u_1' y_1'\right) + \left(y_2 u_2'' + u_2' y_2'\right) \\ &\qquad + P\left(y_1 u_1' + y_2 u_2'\right) + y_1' u_1' + y_2' u_2' = f \\ &= u_1(Ly_1) + u_2(Ly_2) + \frac{d}{dt}\left(y_1 u_1'\right) + \frac{d}{dt}\left(y_2 u_2'\right) + P\left(y_1 u_1' + y_2 u_2'\right) + \left(y_1' u_1' + y_2' u_2'\right) = f \\ &= 0 + 0 + \frac{d}{dt}\left(y_1 u_1' + y_2 u_2'\right) + P\left(y_1 u_1' + y_2 u_2'\right) + \left(y_1' u_1' + y_2' u_2'\right) = f \end{aligned}$$