MATHEMATICS AND STATISTICS DEPARTMENT

EXPLORING CALCULUS WITH MAPLE

FIRST EDITION

OKANAGAN COLLEGE

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Contents

Lab Description

Course Calendar Description

Math 112: Calculus I

An introductory course in differential calculus for science and engineering students, beginning with a review of basic algebra, equations and inequalities, analytic geometry, functions and graphs. Further topics include limits; continuity; rate of change; the derivative; differentiation of algebraic, trigonometric, exponential, logarithmic and inverse trigonometric functions; local and global extrema; Mean Value theorem; graph-sketching; related rates; linear approximation; L'Hopital's Rule; optimization; Newton's method.

Math 122: Calculus II

This course is a continuation of MATH 112. Topics include antiderivatives; the definite integral; Fundamental Theorem of Calculus; applications of integration including area, volume, average value; techniques of integration; numerical integration; improper integrals; introduction to differential equations; direction fields; Euler's method; separable differential equations and applications; infinite sequences and series; convergence; power series; Taylor series and Taylor polynomial approximation.

Purpose of Lab

Your lab section is designed to supplement the content that you are learning in lecture. You will be applying those lecture topics during the lab through Maple, which is a powerful computational software. It is the goal of the lab to help aide in your understanding of calculus as well as introduce you to the tools of mathematics that are available through computers.

Lab Expectations

Each week, you are expected to arrive few minutes prior to the start of lab in order to log into the lab computer. You should begin each lab by opening up the latest version of Maple as well as a web browser to your lab section's Moodle page. You are expected to have this lab manual with you during lab and it is a good idea to have a pen or pencil handy to take notes in your manual.

Some lab computers are faster than others. Give yourself a few extra minutes at the start of lab.

Always have your lab manual. Take lots of notes in it!

Your lab instructor will inform you which activity you will be working on during lab and you are expected to complete all of the exercises associated with that activity. Your lab instructor will be available for questions regarding those exercises, though you will find that many of your questions can be answered by looking through the tutorial sections at the back of this book.

Once you have completed the exercises, open up the associated **activity quiz** on Moodle and complete the questions there. The questions should be relatively quick to answer once you have completed the exercises from the activity. Often, you can simply copy and paste the output from Maple into the blanks on the Moodle quiz.

The questions on your Moodle quiz relate to the exercises in this book. Make sure to finish your exercises prior to the quiz!

During your final lab, you will be expected to complete an open-book **lab test**. The test will be based upon the activities that you have completed throughout the semester. You are permitted to have this manual during the test.

Evaluation

Your lab grade is determined as follows:

Activity Quizzes	50%
Lab Test	50%
Total	100%

Your lab grade is transferred to instructor of your lecture session at the end of the semester and accounts for 10% of your overall Math 112/122 grade.

Lab Test

During your lab test, you will be permitted the following materials:

- This lab manual
- Handwritten notes
- Maple help
- Previously completed activities

The following are not allowed:

- Talking
- Cell phones
- Other electronic devices
- Email
- Internet (other than Moodle for file submission)

You will be allowed to ask for clarification on questions, but will not be allowed to ask for assistance in completing a question or resolving an error in your work.

You will be expected to submit your work electronically and any duplicate submissions will receive a grade of zero. It is important to save your work frequently in case your computer encounters a serious error.

Don't forget to save your file in your **network folder**. This folder is accessible from any computer. Saving your files is explained in detail in the next chapter.

Introduction to Maple

The Maple Computer Algebra System

In lab, you will be using the latest version of Maple, which is a symbolic and numeric computing environment. Maple provides an interface for analyzing, exploring, visualizing, and solving mathematical problems. The interface also allows you to maintain an easy-to-follow document so that you can retrace your thought process. If you plan to pursue any branch of mathematics or field that relies on mathematics, having basic knowledge of a computer algebra system is a very useful tool.

How to Use This Manual

This book is divided into two main parts:

Part I consists of a activities for you to complete. These activities are divided up into two chapters: Math 112 and Math 122. Each activity focuses on one topic from that course and contains a list of exercises for you to complete with Maple. Some of these exercises may give very explicit instructions for typing a command into Maple, while others may require you to use your own intuition and understanding of the capabilities of Maple.

Part II consists of several chapters that provide examples of the usage of common commands in Maple. Many of these examples are designed to be minimalistic in order to show you their basic usage. Each of these examples was completed within Maple and should be reproducible on your own computer.

At the beginning of each activity, a list of the most relevant tutorials is given. It is expected that you read through those tutorials as you complete the exercises of that activity. Many activities build upon previously learned commands, so it is a good idea to thoroughly read through all of the relevant tutorials as you progress through lab.

Accessing Maple and Saving Your Work

Maple is installed on all lab computers as well as on computers in the library. However, personal copies of Maple are not available through Okanagan College at this time.

Opening Maple on School Computers

Figure 1: You may need to open the VMware Horizon Client first using this desktop icon.

On some Windows computers, the most recent version of Maple may be installed directly. In this case, you can simply hit the Start Menu key on your keyboard (in the lower left corner) and begin typing the word "Maple". If Maple is installed directly, then Windows Search will automatically find the version installed and display a link to the application. Alternatively, you should be able to find Maple by browsing the list of installed programs on the Start Menu.

On other computers, you may need to log into a virtual machine to access Maple remotely. On these machines, you will typically find a link to VMware Horizon Client on the desktop. You will need to open this client and log into the remote machine using the same information as you previously used to log into the computer. Once the remote machine loads, you should find Maple in the Start Menu as described above.

After opening Maple, you may be prompted to update the version. You can click Cancel on this dialog box.

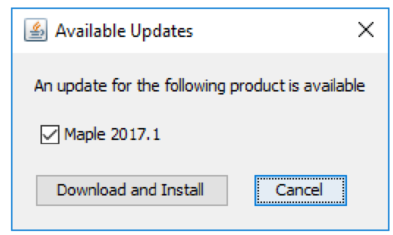


Figure 2: You can click Cancel if prompted to update Maple.

Organizing Files in your Network Folder

You are given your own personal folder on the network in which to save your files. This folder is accessible from any computer on campus, so you should save all of your work in this folder. You can find this folder by using the desktop shortcut or by opening up File Explorer and clicking on This PC.

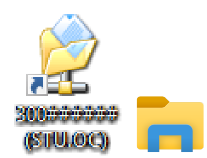


Figure 3: The shortcut for your network folder is on the desktop. You can also open File Explorer with the shortcut on the taskbar.

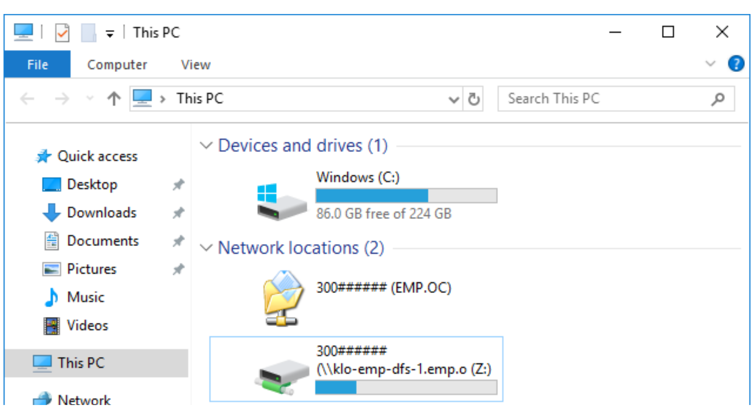


Figure 4: Your network folder should appear under This PC when using File Explorer.

It is best to create a new subfolder in your network folder for your lab. This folder could be called something like "Math 112 Labs" so that it is easy to find and organize all of your saved Maple files. When you wish to save your file in Maple, now you can navigate to this folder and save your file with the name of the activity.

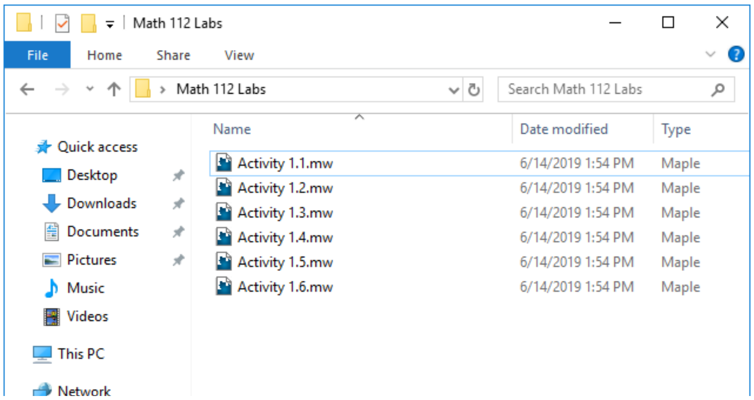


Figure 5: Make sure to create a folder for your lab inside your network folder. Save all of your files here using obvious names so that they are easy to find again later.

Accessing Your Files from Home

By default, files saved from Maple cannot be opened at home without having Maple installed on your computer. However, if you wish to access your files from home or from a mobile device, Maple can export your worksheet as a PDF file. You can find the Export As tool under the File menu. From there, you can change the file type to PDF and save your document.

Note that you can only view your output in the PDF file and cannot edit it.

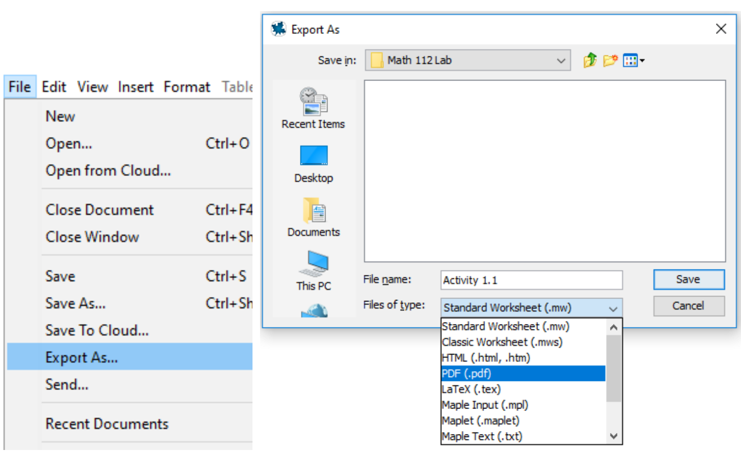


Figure 6: You can export your Maple worksheet as a PDF file type, which can be opened on most devices without Maple.

You can access your files from home using myFiles: `https://myfiles.okanagan.bc.ca/`

Part I

Activities

1 *Activities for Math 112*

The following list serves as a suggestion for which activities are considered core versus optional:

1.1 *Creating a Maple Worksheet*

Recommended Tutorials:

- Tutorial A, pg. 89
- Tutorial B, pg. 93

Introduction:

This activity will give you practice with:

- setting up a new worksheet in Maple for future activities.
- switching between paragraphs for text and execution groups for Maple input.
- evaluating expressions in exact and decimal form.

Exercises:

1. Open up Maple on your lab computer. If you are asked whether you would like to update Maple, you can select No. You should now have a blank worksheet for the following exercises.

2. On the first line of your worksheet you should be in Maple input, which is indicated by the > at the start of the line. Type in your first two commands, hitting Enter to run each command:

```
> restart;
```

```
> Digits := 15;
```

The output of these commands are not useful on their own. However, you have now told Maple to clear its memory and that you would like numerical answers to be displayed with up to 15 digit accuracy.

3. Create a new paragraph for text using T and type in the following information:

   (a) Your first and last name.

   (b) Your student number.

   (c) The name of your instructor.

   (d) The title of the activity.

Ctrl+Shift+J and Ctrl+Shift+K are also useful shortcuts for creating paragraphs after or before the current line.

4. Open up the palettes menu on the top left-hand side of Maple by clicking the small black triangle that points to the right. You may wish to use the "Expressions" palette for the next few exercises.

5. Create a new execution group by using the [> button and evaluate $5 + 2(7 - 4)$.

Ctrl+J and Ctrl+K are also useful shortcuts for creating execution groups after or before the current line.

6. Create a new execution group and evaluate $\left(\dfrac{5^2-9}{9-5}\right)^2$.

You can create a fraction by encasing the numerator, $5^2 - 9$, in a set of brackets before dividing. Alternatively, you can highlight $5^2 - 9$ and hit the `/` key to make a fraction.

7. Create a new execution group and evaluate $\sqrt{\dfrac{3+6^2}{4^2-3}}$.

To input a square root, use the `sqrt()` command.

8. Create a new execution group and evaluate $\sqrt{\dfrac{3.0+6.0^2}{4.0^2-3.0}}$.

Whenever possible, Maple will produce output in exact form. However, using decimals in a Maple input will produce output in decimal form (called a floating point decimal).

9. For each of the following, create a new execution group and evaluate the expression.

   (a) $\cos\left(\frac{\pi}{6}\right)$

   (b) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

   (c) $\exp(2)$

   (d) $\ln\left(5^{1/2}\right)$

The mathematical constant $\pi = 3.14\ldots$ must be typed into Maple as capitalized `Pi`. Do not type `pi`, as this is simply the Greek letter π.

The `exp(x)` function is the exponential function, e^x.

10. On each of the four Maple inputs you created in exercise 9, add a semicolon at the end of the line, followed by the command `evalf(%)`. For example:

```
> cos(Pi/6); evalf(%);
```

The semicolon `;` separates commands if you want to run multiple commands on the same Maple input.

11. If you have accidentally created any additional paragraphs or execution groups that you wish to delete, then delete them now by clicking on that line and pressing Ctrl+Del on the keyboard.

12. Organize your work by using a new section for each of exercises 5-10. Be sure to clearly label each question by creating a section title immediately to the right of the arrow at the top of the section.

Use the  buttons to create or remove sections. Alternatively, the shortcuts Ctrl+. and Ctrl+ can be used to enclose execution groups of paragraphs in a section or remove any section enclosing an input.

You can highlight several execution groups or paragraphs with the mouse before combining them into one section.

13. (Optional) Go back to the top of your worksheet and change the second line to:

```
> Digits := 50;
```

Now run each Maple input again by hitting Enter on each line or by pressing the  button. What changed?

You do not need to have your cursor at the end of the line to run the command. Hitting Enter with your cursor anywhere on the Maple input will run the command.

Pressing the  button executes every command in the entire worksheet.

1.2 The Basics

Recommended Tutorials:

- Tutorial C, pg. 97
- Tutorial D, pg. 101

Introduction:

In this activity, you will learn basic usage of some of the most common Maple commands:

- `expand()`
- `simplify()`
- `factor()`
- `plot()`

Exercises:

1. Expand the polynomial $(2x - y)^6$.

2. Factor the polynomial $16x^4 - 160x^3y + 600x^2y^2 - 1000xy^3 + 625y^4$.

When two or more variables appear next to each other, be sure to include a $*$ or space between them, so that Maple knows that they are multiplied together.

3. Simplify the expression $\dfrac{x^3 - 1}{x - 1}$.

4. Now we would like Maple perform all three commands together.

It is a good practice when using the % shortcut to run the commands simultaneously on the same Maple input.

   (a) Have Maple expand the rational expression $\dfrac{(x-y)^2 + (x+y)^2}{x^3 - y^3}$.

   (b) Add a semicolon to the end of the line, followed by `simplify(%)`.

   (c) Add another semicolon to the end of the line, followed by `factor(%)`.

   (d) Hit Enter to run all three commands together.

   You should see three outputs now: expanding, simplifying, and factoring.

You can add a line break between commands without running them with Shift+Enter.

5. (Optional) Consider polynomials of the form $x^p - 1$, where p is a prime number. Try factoring each of the following:

   (a) $x^2 - 1$

   (b) $x^3 - 1$

   (c) $x^5 - 1$

   (d) $x^7 - 1$

   (e) $x^{19} - 1$

   Can you notice a pattern and show that these polynomials follow a particular form when factored? To explain your solution, use the 

 button to create a new paragraph after the current line.

Ctrl+Shift+J can also be used to create a paragraph after the current line.

6. Plot the following two functions using separate `plot()` commands and note the difference in domain:

   (a) $x^{1/3}$

   (b) `surd`$(x, 3)$

The `surd(x,3)` function is equivalent to $\sqrt[3]{x}$. Similarly, `surd(x,5)` is equivalent to $\sqrt[5]{x}$, etc.

7. On a new Maple input, create a plot of the following list of functions

```
[ x^2, x^3, sqrt(x), surd(x,3), abs(x) ]
```

   and include the following options (separated by commas).

   - `x = -5..10` *(This specifies the x-axis)*
   - `y = -5..10` *(This specifies the y-axis)*
   - `colour = [red,blue,green,purple,orange]`

Square brackets in Maple are used to create a comma-separated list of items in the specified order.

1.3 The Assignment Operator and Creating Functions

Recommended Tutorials:

- Tutorial D, pg. 101
- Tutorial E, pg. 105

Introduction:

In this activity, you will be using the assignment operator `:=`, which allows you assign Maple output to a name of your choice. This is especially useful for assigning expressions and functions on one Maple input, before manipulating those expressions later on in your worksheet.

Exercises:

1. Assign the expression $\dfrac{\sin(x)+3}{\cos(x)+1}$ to `expr` and then use the `subs()` command to substitute $x = 3$ into `expr`. Evaluate this as a decimal with 15 digits.

2. Assign the expression $x^2 + 2^x$ to `expr2` and substitute $y =$ `expr2` into the expression $y^2 + 3y$.

3. Assign the expression $2x^2 - 4x + 7$ to `poly` and then substitute $x = 5 + h$ into `poly` and simplify.

4. Consider the function $f(x) = (1 - x^2)\mathrm{e}^{-x^2/2}$.

   (a) Assign the function to $f(x)$.

   (b) This function $f(x)$ is known as the *Mexican Hat Function*. Can you see why? Plot the graph of $f(x)$.

5. Maple, by default, does not know the function

   $$sinc(x) = \frac{\sin(\pi x)}{\pi x},$$

   which is important in engineering.

   (a) Assign this function to $sinc(x)$.

   (b) Evaluate $sinc(3)$, $sinc\left(\frac{1}{2}\right)$, and $sinc(0.25)$.

   (c) Plot the graph of $sinc(x)$.

Instead of using the `subs` command multiple times, it is often a better practice to define a function and use function notation instead.

The exponential function, e^x, in Maple is denoted as `exp(x)`.

When assigning the function to $f(x)$, use the $:=$ operator.

Often, this function is denoted as $\mathrm{sinc}_\pi(x)$ and $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$.

Be sure to include a multiplication symbol or space between π and x.

The mathematical constant $\pi = 3.14\ldots$ must be typed into Maple as `Pi`.

When assigning the function to $sinc(x)$, use the $:=$ operator.

1.4 *Transformations of Graphs*

Recommended Tutorials:

- Tutorial D, pg. 101
- Tutorial E, pg. 105

Introduction:

In this activity, you will plot multiple functions at once to investigate basic transformations of functions.

Exercises:

1. Plot the graphs of $\sin(x)$ and $\cos(x)$ on the same set of axes. Use *red* for $\sin(x)$ and *blue* for $\cos(x)$.

   If you do not specify the x-axis interval, Maple will default to increments of $\pi/8$ from -2π to 2π.

   (a) By what amount do you need to shift the graph of $\sin(x)$ *to the left* (negative x direction) so that it coincides with the graph of $\cos(x)$? Answer the question in sentence form by using the **T** button to create a new paragraph after the current line.

   You can also use Ctrl+Shift+J to insert a paragraph after the current line.

   (b) By what amount do you need to shift the graph of $\sin(x)$ *to the right* (positive x direction) so that it coincides with the graph of $\cos(x)$? Write your answer in sentence form using a new paragraph.

2. The graphs of e^x and $\ln(x)$ should appear to be reflected over the line $y = x$. Plot all three graphs on the same axes using `linestyle=[solid,solid,dash]`.

   The `exp(x)` function is the exponential function, e^x.

   When plotting functions of the form $y = f(x)$ using the `plot()` command, we omit the $y =$.

3. Assign the function $f(x) = \sqrt{-x^2 + 4x + 21}$.
   Plot each of the following functions on the same set of axes:

   $$y = f(x),\ y = f(2x),\ y = 3f(x),\ y = f(-x),\ y = -f(x).$$

   Make sure that the graph is displayed with constrained scaling $(1 : 1)$. Describe each transformation. Write your answer using a new paragraph.

   When assigning the function to $f(x)$, use the $:=$ operator.

   It is always a good practice to specify the colours of multiple graphs in order to tell them apart. A list of plot colours can be found by typing `?colours` on a new Maple input.

4. Plot each of the following graphs separately:

   The absolute value function $|\cdot|$ in Maple is denoted as `abs( )`.

   (a) $y = \sin(x)$

   (b) $y = \sin(|x|)$

   (c) $y = |\sin(x)|$

   Create a new paragraph to describe the transformations of $y = \sin(x)$ in parts (b) and (c).

1.5 Solving Equations in Maple

Recommended Tutorials:

- Tutorial F, pg. 113

Introduction:

In this activity, students will be asked to assign expressions and then evaluate and manipulate them.

Exercises:

1. Suppose we want to find the x-intercepts of the function

$$f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1.$$

   (a) Assign the function to $f(x)$.

   (b) Plot $f(x)$. Choose an interval for x that shows all x-intercepts. How many do you see?

   (c) Factor $f(x)$. Does $f(x)$ factor?

   (d) Solve $f(x) = 0$ using both the `solve()` and `fsolve()` commands.

When assigning the function to $f(x)$, use the $:=$ operator.

With `fsolve()`, you can also specify an interval for solutions if you wish to only find a particular solution. An example of this can be found in Tutorial F, pg. 115.

2. Solve the quadratic equation $x^2 + 4x + 6 = 0$ three ways:

   (a) Using the quadratic formula by hand.

   (b) Using `solve()`, followed by evaluating the result as a decimal.

   (c) Using `fsolve()`.

   Compare your results from (b) and (c).

In Maple, $\sqrt{-1}$ is denoted as I.

3. Consider the curves $y = x^2$ and $y = \frac{3}{x}$.

   (a) Find the point (x, y) where the two curves intersect. Try using both the `solve()` and `fsolve()` commands.

   (b) Plot the two curves on the same set of axes to verify that your intersection point from part (a) is correct.

An example that shows how to find the intersection point of two functions is given in Tutorial F, pg. 115.

When plotting equations of the form $y = f(x)$ using the `plot()` command, we never include the $y =$.

When multiple curves are plotted on the same set of axes, it is a good practice to specify the colour of each one, so that you can tell them apart.

1.6 *Limits*

Recommended Tutorials:

- Tutorial G, pg. 117

Introduction:

In this activity, students will use the `limit()` command to find various limits.

Exercises:

There is a shortcut for limits on the palettes toolbar under Expression.

$$\lim_{x \to a} f$$

1. Calculate the limit $\lim\limits_{x\to 1} \dfrac{\ln(x^4+1)-1}{x-2}$.

In Maple, the absolute value function $|\cdot|$ is denoted as `abs( )`.

When plotting functions that may contain discontinuities, make sure to use the optional parameter `discont = true`.

2. Plot the function
$$f(x) = \frac{|x|}{x}$$
over an appropriate range, and then use the `limit()` command to calculate the left-hand limit, the right-hand limit, and the two-sided limit as x tends to zero.

Use `sqrt()` to input a square root in Maple.

3. Consider the function $g(x) = \dfrac{x^2+x}{\sqrt{x^3+x^2}}$.
   (a) Use the `plot()` command between $x = -2$ and $x = 2$ to estimate the left-hand-limit, the right-hand limit, and the two-sided limit as x tends to zero.
   (b) Use the `limit()` command to verify your results from part (a).

You need to find the limit as x approaches zero, not as it approaches -2 or 2.

4. Estimate the value of
$$\lim_{x\to 0} \frac{\sin(x)}{\sin(\pi x)}$$
by using Maple's `plot()` command, and then check your answer with Maple's `limit()` command.

Do not forget to use `Pi` for π and include multiplication between π and x.

5. Calculate $\lim\limits_{x\to\infty} \sqrt{x+1} - x$.

To denote ∞ in Maple, you need to type the word `infinity`.

6. (Optional) Use the Limit Methods Tutor to explore the steps to calculate
$$\lim_{x\to -\infty} \sqrt{x^2+x+1} + x.$$

1.7 *Limits and Asymptotes*

Recommended Tutorials:

- Tutorial G, pg. 117

Introduction:

It is incorrect to assume that a vertical asymptote is always found whenever the denominator of a rational function is equal to zero. Instead, we say that $f(x)$ has a vertical asymptote at $x = a$ whenever

$$\lim_{x\to a^+} f(x) = \infty$$

or

$$\lim_{x\to a^-} f(x) = \infty.$$

In either case, the equation of the vertical asymptote is $x = a$.

Similarly, a horizontal asymptote of $f(x)$ is also defined in terms of limits. A function $f(x)$ has a horizontal asymptote $y = L$ if

$$\lim_{x\to\infty} f(x) = L$$

or

$$\lim_{x\to-\infty} f(x) = L.$$

While a function may have many vertical asymptotes, it cannot have more than two horizontal asymptotes.

In this case, the equation of the horizontal asymptote is $y = L$.

You will need to use both of these definitions while answering the following exercises.

Exercises:

1. Plot the function $f(x) = \dfrac{x-1}{x^2-x-2}$ and estimate its vertical asymptotes. Use Maple to prove that your guesses are, in fact, the vertical asymptotes by taking the one-sided limits at those values. Insert a new paragraph and state the equations of the vertical asymptotes.

When plotting functions that may contain vertical asymptotes, make sure to use `discont = true` as a plot option when required.

The **T** button or Ctrl+Shift+J can be used to create a paragraph after the current line.

2. Consider the function $g(x) = \dfrac{x+2}{x^2-x-6}$.

   (a) Use Maple to factor the denominator of $g(x)$.

   (b) By factoring the denominator of $g(x)$, you might suspect that the vertical asymptotes are $x = -2$ and $x = 3$. Plot $g(x)$ and show why this may or may not be the case.

If you use Maple's `denom()` command and type `denom(`$g(x)$`)`, you will get the denominator of $g(x)$.

3. Consider the function $h(t) = \dfrac{\sin(t)}{t}$.

   (a) Plot a graph of $h(t)$.

(b) Use Maple's `limit()` command to find the horizontal asymptote(s) of $h(t)$.

(c) Insert a new paragraph and state the equation(s) of the horizontal asymptote(s).

4. Consider the function $Q(x) = \dfrac{\sqrt{2x^2+1}}{3x+5}$.

The `sqrt()` command is used to input a square root.

(a) Plot a graph of $Q(x)$. Be sure to specify appropriate intervals for the x-axis and y-axis.

Does $Q(x)$ have any vertical asymptotes? If so, make sure to use `discont = true` as a plot option.

(b) Use Maple's `limit()` command to find the horizontal asymptote(s) of $Q(x)$.

(c) Insert a new paragraph and state the equation(s) of the horizontal asymptote(s).

Maple provides a useful `Asymptotes()` command for finding the asymptotes of a function. Try typing `?Asymptotes` to learn more.

1.8 *A Shrinking Circle Problem*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial G, pg. 117

Introduction:

Limits may seem trivial when you first learn them, but they are fundamental building blocks in calculus. They are used to explain terms like "infinitesimally small" and "infinitely large". It may be interesting to know that they can also be used in applied problems. This activity will explore a geometry problem and will solve it using limits, instead of an elementary geometric approach.

Figure 1.1 below shows two circles:

- C_1, centred at the point $(1, 0)$ with radius 1 and equation
$$(x-1)^2 + y^2 = 1.$$
- C_2, centred at the origin with radius r and equation
$$x^2 + y^2 = r^2.$$

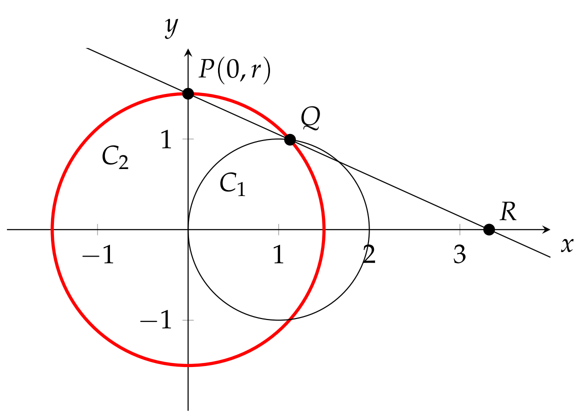


Figure 1.1: What happens to the point R as the radius of the thicker circle C_2 shrinks?

If we define P as the point $(0, r)$ at the top of the circle C_2, and Q as the upper point of intersection of the two circles, then we can construct the line PQ and see that it crosses the x-axis. Let R be the x-intercept of the line PQ.

Note that neither P nor Q is fixed as the radius of C_2 shrinks.

Now, begin to shrink the radius of circle C_2; that is, let $r \to 0^+$. What happens to the point R as C_2 shrinks?

Exercises:

You can find the point of intersection with a single `solve()` command, but you may need to include the optional parameter `explicit=true` to avoid the *RootOf()* output.

1. Assign names to the equations of both circles, such as `C1` and `C2`.

2. Find the point of intersection of C_1 and C_2 in quadrant I. This is the point Q. The coordinates of Q should be expressions of r.

3. We now have two points on the line, P and Q, so we can construct the equation of this line. For parts (a) and (b), let $P = (0, r)$ be (x_1, y_1) and let Q (the point you found in exercise 2) be (x_2, y_2).

   (a) Find the slope of the line PQ using the slope equation

   $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

   (b) Define the equation of the line PQ using the slope from part (a) and the point-slope equation of a line

   $$y - y_0 = m(x - x_0).$$

Remember that R is the x-intercept of the line PQ, so its y-coordinate is 0. The x-coordinate can be found using the `subs()` and `solve()` commands.

4. Using the equation of the line you just found, find the coordinates of R.

Use the `limit()` command as r approaches 0 from the right.

5. Now what happens as $r \to 0^+$?

1.9 *The Derivative of a Function*

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial H, pg. 125

Introduction:

In this activity, we will investigate the derivative of a function and use Maple's powerful computational skills to simplify the process of finding a derivative.

Exercises:

1. Consider the function $f(x) = \sqrt{9 - x}$.

   (a) Define this function in Maple.

   (b) Find the derivative, $f'(x)$, using the `limit()` command and the limit definition of the derivative.

   (c) Find the derivative, $f'(x)$, using the `diff()` command.

   (d) Find the derivative, $f'(x)$, using `'` notation.

To input a square root, use the `sqrt()` command.

When assigning the function to $f(x)$, use the := operator.

Don't forget!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. A toy rocket is fired straight upward, and its height (in meters) is given by: $h = t + 10 - \sqrt{2t^2 + 100}$, with $0 \leq t \leq 20$, where t is the time in seconds.

   (a) Plot a graph of height as a function of time.

   (b) Find a formula for velocity in terms of t.

   (c) At what time is the velocity 0 m/s?

   (d) What is the maximum height that the rocket attains?

If you decide to define a height function, be sure that the variable in your function name matches the variable in your function.

When the velocity of the rocket is 0 m/s, the rocket has reached its maximum height.

3. Molten lava can fill a chamber in the earth's crust before it builds up enough pressure to erupt. Let the pressure be modeled by $P(t) = 0.47t^2 e^{0.0035t}$, where t is the time in months.

   (a) What is the rate of change of pressure with respect to time?

   (b) Suppose that an eruption is highly likely to occur if pressure increases at a rate greater than 20. Is an eruption likely when $t = 30$ months? Use a new paragraph to state your answer.

Recall that `exp()` is used to denote the exponential function in Maple.

4. Consider the function $g(x) = \sin(2\pi^2 x)$.

   (a) Use Maple to find the first derivative, the second derivative, the third derivative, and the fourth derivative of $g(x)$. You should notice a pattern, so use a new paragraph to describe it.

   (b) Use your answer from part (a) to predict what the 77$^{\text{th}}$ derivative of $g(x)$ is, and then verify that your prediction is correct by computing this derivative using Maple.

   (c) (Optional) Use the information in Tutorial M on page 163 to write a loop that will output the first 100 derivatives of $g(x)$.

To type the mathematical constant $\pi = 3.14\ldots$, be sure to use `Pi`.

The n^{th} derivative of the function can be computed by using the `diff()` command or `'` notation. See Tutorial H, pg. 125 for more information. You can also make use of the Calculus palette:

$$\frac{d}{dx} f \quad \frac{d^2}{dx^2} f \quad \frac{d^n}{dx^n} f$$

1.10 *Tangent Lines*

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial H, pg. 125

Introduction:

In this activity, students will find tangent lines to various functions, and then will display the tangent lines and function on the same graph.
To find the tangent line to a function $f(x)$ at $x = a$, two pieces of information are needed:

- The point, $(a, f(a))$.
- The slope of the tangent line, $f'(a)$.

Plugging these two pieces of information into the point-slope form of a line gives the following equation.

$$\begin{aligned} y - y_0 &= m_{tan} \cdot (x - x_0) \\ y - f(a) &= f'(a) \cdot (x - a) \\ y &= f'(a) \cdot (x - a) + f(a) \qquad (1.1) \end{aligned}$$

We will use equation (1.1) for the following exercises.

Exercises:

1. Consider the function function $f(x) = \sqrt{9 - x}$.
   (a) Define the equations of the tangent lines at $x = 0$ and $x = 5$.
   (b) Plot $f(x)$ and the two tangent lines on the same axes.

Be sure to choose different names for each tangent line. If you assign a new expression to an old name, the new expression will overwrite what was previously assigned.

Since you are plotting more than one tangent line on the same axes, it is a good idea to specify plot colours.

2. Consider the function $g(x) = xe^x$.
   (a) Define the equations of the tangent lines at $x = 1$ and $x = -1$.
   (b) Plot $g(x)$ and the two tangent lines on the same axes.

In Maple, `exp(x)` is used to denote the exponential function, e^x.

3. Consider the function $h(x) = x^3 - x^2 - 9x + 9$.
   (a) Find the x-values where the function has tangent lines with slope equal to 1.
   (b) Find the x-values where the function has horizontal tangent lines.

To solve part (b), you need to remember what the slope of a horizontal tangent line is.

1.11 *Building a Roller Coaster*

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial H, pg. 125

Introduction:

You are in charge of designing the first hill of a new roller coaster. For an initial design, you connect a straight stretch of track for the lift hill followed by two parabolas, as shown in Figure 1.2.

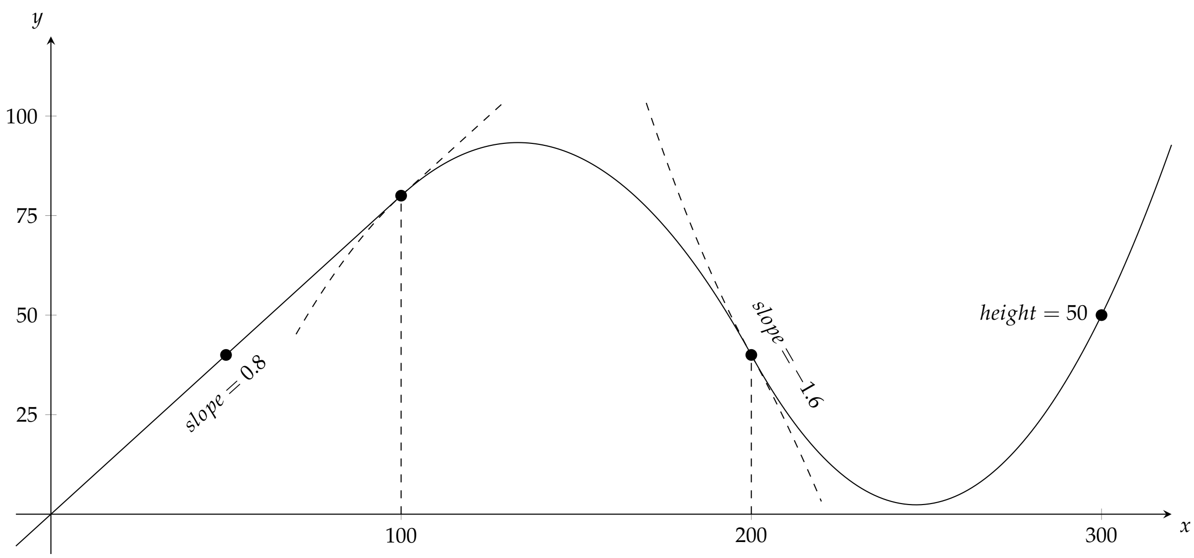


Figure 1.2: A simple design for the initial hill of a roller coaster.

The following criteria must be met to build the roller coaster:

- The lift hill will have a slope of 0.8.
- The straight section of the lift hill will cover a horizontal distance of 100 ft.
- The slope of the first descent will reach a maximum magnitude of 1.6 after another 100 ft.
- The next hill will reach a height of 50 ft after another 100 ft.
- The track must be smooth (i.e. there cannot be any sudden changes in the slope of the track).

Exercises:

The goal of the following exercises is to develop a piecewise function that satisfies all of the above criteria.

1. Let $L(x)$ be the function for the lift hill, which is a linear function passing through the origin.

   (a) Using the given criteria, assign the function $L(x)$ in Maple for the lift hill.

   The lift hill function can easily expressed in $y = mx + b$ form.

   (b) Determine the x- and y-coordinates at which the lift hill connects to the first parabola, $f(x)$.

2. Let $f(x)$ be the function for the first parabola, opening downward. We will define this function in a few steps.

   (a) Assign the function $f(x) = ax^2 + bx + c$ as a general quadratic function. We will solve for the coefficients using the above criteria over the next two steps.

   Be sure to include multiplication between variables.

   (b) We can solve for a, b, and c, by using three equations. Write an equation in function notation for each of the following conditions.

   Each of these equations can be written in the form of
   $$f(\alpha) = \beta$$
   or
   $$f'(\alpha) = \beta.$$

   - f must pass through the point from exercise 1(b).
   - The derivative of f at $x = 100$ is 0.8, so that the track is smooth (differentiable) where it connects to the lift hill.
   - The derivative of f at $x = 200$ is -1.6, according to the given criteria.

   (c) Use a `solve()` or `fsolve()` command and function notation to solve the system of equations for a, b, and c.

   (d) Reassign the function $f(x)$ using the values you have just calculated.

   (e) Determine the x- and y-coordinates at which the $f(x)$ connects to the next parabola (when $x = 200$).

3. Let $g(x)$ be the function for the second parabola, opening upward. We will define this function in a few steps.

   (a) Assign a function $g(x) = px^2 + qx + r$ as a general quadratic function. We will solve for the coefficients using the above criteria over the next two steps.

   Be sure to include multiplication between variables.

   (b) We can solve for p, q, and r, by using three equations. Write an equation in function notation for each of the following conditions.

   - g must pass through the point from exercise 2(e).

- The derivative of g at $x = 200$ is -1.6 so that the track is smooth (differentiable) where it connects to $f(x)$.
- The height of g at $x = 300$ is 50, according to the given criteria.

(c) Use a `solve()` or `fsolve()` command and function notation to solve the system of equations for p, q, and r.

(d) Reassign the function $g(x)$ using the values you have just calculated.

Each of these equations can be written in the form of

$$g(\alpha) = \beta$$

or

$$g'(\alpha) = \beta.$$

4. Use the `piecewise()` command to define a piecewise function called *coaster(x)* that consists of the three pieces:

$$coaster(x) = \begin{cases} L(x) & 0 \leq x < 100 \\ f(x) & 100 \leq x < 200 \\ g(x) & 200 \leq x \leq 300 \end{cases}$$

The `piecewise()` command can be found in Tutorial E, 105.

5. Plot the piecewise function, *coaster(x)*.

1.12 *Implicit Functions*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial I, pg. 135

Introduction:

In this activity, we will learn how to plot implicit functions, as well as compute their derivatives.

There are a few important things to remember about the `implicitplot()` command during this activity:

- The `plots` package needs to be loaded using the `with()` command.
- To produce a smooth plot, use either the `numpoints=30000` or `grid=[250,250]` parameter.
- The optional `scaling=constrained` parameter can be included to enforce 1 : 1 scaling.
- If you are plotting multiple graphs on the same set of axes, it is a good idea to specify plot colours.

Exercises:

1. Consider the circle centred at the origin with radius 5.

   (a) Define the equation for this circle and plot it. Ensure that the circle appears smooth.

   (b) Find the y-coordinates of the points when $x = 2$.

   (c) Compute the derivative, $\frac{dy}{dx}$, of the circle.

   (d) Find the slopes of the circle at both points when $x = 2$.

   (e) Define the equations of the tangent lines at both points when $x = 2$.

   (f) Plot the circle and the two tangent lines.

The equation of a circle that is centred at the point (a, b) and has radius r is

$$(x-a)^2 + (y-b)^2 = r^2.$$

The order of the parameters in the `implicitdiff()` command is important; you must use `implicitdiff( ,y,x)` to find dy/dx.

Be sure to assign different names for each tangent line and include $y =$ in the equations of the lines so that the `implicitplot()` command can plot them.

Do not forget to include multiplication between variables.

2. Consider the Folium of Descartes: $x^3 + y^3 = 6xy$.

   (a) Define this curve and plot it. Ensure the curve appears smooth.

   (b) Compute the derivative, $\frac{dy}{dx}$.

   (c) Find the equations of each of the tangent lines at the points where $x = 1$. Plot the Folium of Descartes and the tangent lines on the same graph.

   (d) Find any points on the Folium of Descartes where the tangent lines are horizontal.

In exercise (c), the `fsolve()` command works much better than the `solve()` command to find the y-coordinates of the points where $x = 1$.

To solve part (d), you need to remember what the slope of a horizontal tangent line is equal to.

In order to find points on a curve with a given slope m, consider that you must have $\frac{dy}{dx} = m$. This equation, plus the equation of the curve give a system of equations that Maple can easily solve for both x and y.

1.13 Orthogonal Curves

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial I, pg. 135

Introduction:

Orthogonal curves are curves that are perpendicular whenever they intersect. Perpendicular lines are an elementary example of this. We know that if m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 m_2 = -1$. Similarly, two curves are orthogonal if their derivatives multiply to -1 whenever they intersect.

There are a few important things to remember about the `implicitplot()` command during this activity:

- The `plots` package needs to be loaded using the `with()` command.
- To produce a smooth plot, use either the `numpoints=30000` or `grid=[250,250]` parameter.
- The optional `scaling=constrained` parameter can be included to enforce $1:1$ scaling.
- If you are plotting multiple graphs on the same set of axes, it is a good idea to specify plot colours.

Exercises:

1. Consider the curves $y^2 - x^2 = 3$ and $xy = 2$.

   Do not forget to include multiplication between x and y.

   (a) Plot both of these curves on the same set of axes. Make sure both curves appear smooth.

   (b) Find the x- and y-coordinates of the points of intersection.

   In order to find points of intersection, Maple can solve a system of equations in one `solve()` or `fsolve()` command for both x and y.

   (c) Compute the derivatives of both curves.

   (d) Are these curves orthogonal at the points of intersection? Confirm this using the fact that $m_1 m_2 = -1$ for perpendicular slopes.

2. Consider the two families of curves given by

$$y = cx^2$$

   and

$$x^2 + 2y^2 = k,$$

   where c and $k > 0$ are arbitrary constants.

   (a) Plot several curves of the first family using

$$c = -2, -1, 0, 1, 2$$

   and several curves of the second family using

$$k = 1, 4, 9$$

   on the same axes.

   You may wish to make all curves of the same family the same colour.

   (b) Compute the derivative of each family of curves and show that they are orthogonal, regardless of c and k.

1.14 Newton's Method

Recommended Tutorials:

- Tutorial H, pg. 125
- Tutorial M, pg. 163

Introduction:

Newton's method is an algorithm that can be used to approximate the root of a continuous function. In class, we learned that the linear approximation for $f(x)$ at $x = a$ is given by the function

$$L(x) = f'(a)(x - a) + f(a).$$

Now, suppose we are given a continuous function f with a root r and wish to approximate r. One way we could do this is to choose a value a close to r, determine the linear approximation $L(x)$ at $x = a$ and find the root of $L(x)$ by setting it to zero:

$$f'(a)(x - a) + f(a) = 0.$$

This gives the root of $L(x)$ as

$$x = a - \frac{f(a)}{f'(a)}.$$

Now, assuming that this value of x is closer to r than our initial value a, we could repeat this process again. Therefore, our formula for each new approximation can be simplified to

$$x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})}.$$

By repeated evaluation of this equation, we can continue to find better approximations of r.

There are some practical considerations you should be aware of when using Newton's method including:

1. how difficult it is to compute the derivative,
2. poor initial guesses, and
3. convergence to the wrong root.

Any of these will lead Newton's method to be less useful.

Newton's method will fail for a number of different reasons:

1. if the starting point leads to a cycle between two or more points,
2. if the iteration point is at a critical point, or
3. if the derivative is discontinuous.

Be careful of such situations.

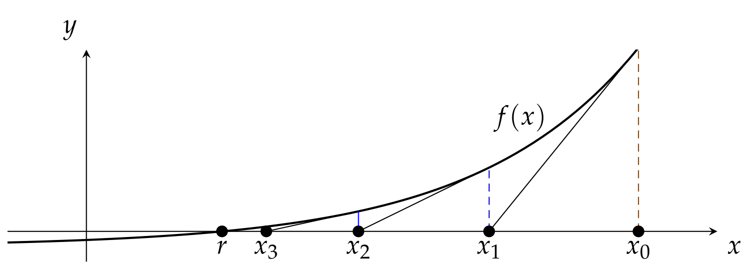


Figure 1.3: Using three iterations of Newton's Method to approximate the root r.

In the following exercises, you can make use of the `NewtonsMethod()` command, provided as part of the `Student[Calculus1]` package. A useful example can be found in Tutorial H.6 on page 133. However, it is also beneficial to try more advanced coding methods and implement a "while" loop if you are interested; information regarding loops is given in Tutorial M on page 163.

Newton's method can be used to approximate the critical points (max or min) of a function. Since a function f has a maximum or minimum at a point c for which $f'(c) = 0$, we can simply change our Newton's method to be

$$x_{new} = x_{old} - \frac{f'(x_{old})}{f''(x_{old})}.$$

To load the `StudentCalculus1` package, the `with()` command needs to be used. Make sure you capitalize the appropriate letters in the package name, or else it will not load the necessary commands.

Exercises:

1. Use Maple's `NewtonsMethod()` command to determine the value of the root of the function $f(x) = x^2 - 2$. Use an initial guess of 2. Iterate until you get 10 decimal places of accuracy.

2. Apply Newton's method to determine the value of the root of the function $f(x) = x^2 - 2$. Use an initial guess of -2. Iterate until you get 10 decimal places of accuracy.

3. We cannot use Newton's method to find the root of the function $f(x) = x^2 - 2$ if we use an initial guess of 0. Use a graph to help explain why, and discuss your answer by using a new paragraph.

4. Suppose you wish to find the value of $\sqrt{7}$ using Newton's method.

   (a) What function would we use if we wanted to apply Newton's method to determine the value of $\sqrt{7}$? State your answer using a new paragraph.

   (b) Evaluate $\sqrt{7}$ to 15 digits using `evalf()`. Apply Newton's method to the function from part (a) with an appropriate initial guess for x and verify that the values agree.

5. Newton's method converges with *quadratic convergence*. That roughly means that you will get twice as many correct digits for x_{new} as you did for x_{old}. Iterate Newton's method for

$$g(x) = x^2 - \sin(x) - 0.5$$

   with an initial guess of $x = 2$. Find the value of the zero of $g(x)$ to 16 decimal places.

6. (Optional) Write a "while" loop that allows you to solve exercises 1 and 2.

Be sure to load the `StudentCalculus1` package before using the `NewtonsMethod()` command. The package only needs to be loaded once in your document, not every time you use one of its commands.

Adding the optional `iterations` parameter to the `NewtonsMethod()` command allows you to choose how many iterations are performed. By default, `NewtonsMethod()` carries out 5 iterations.

The output of Maple's `NewtonsMethod()` command can be displayed in different ways, depending on whether the `output` parameter is set equal to `value`, `plot`, `animation`, or `sequence`. See Tutorial H.6 on pg. 133 for more information.

1.15 The Shape of a Can

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial H, pg. 125

Introduction:

Your goal in this question is to design the most economical shape of a perfectly cylindrical can. First, let us assume the can has to hold a

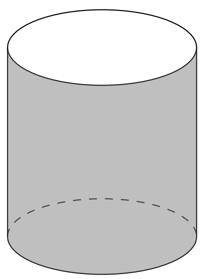


Figure 1.4: A simple can in the shape of a circular cylinder.

volume of 250 mL (250 cm^3). The cylindrical sides as well as the top and bottom circles are cut from sheets of aluminum. The material for the cylindrical sides of the cans is made from rectangles that are bent, so no material is wasted. However, some amount of material is always wasted when trying to cut circles from a sheet of metal. To make the most economical can, your goal is to reduce the *total* amount of material needed, including any wasted metal.

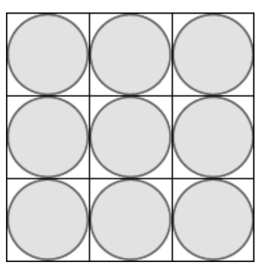


Figure 1.5: Cutting circles in a square pattern.

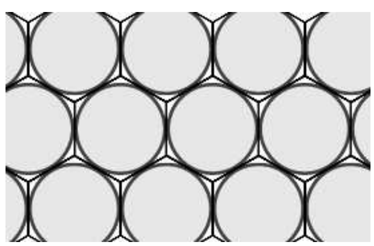


Figure 1.6: Cutting circles in a hexagonal pattern.

If the circles are cut according to Figure 1.5, then the total amount of material required for each can is a rectangle and two squares, which wastes a lot of material. If we plan to minimize the amount of material required, we will instead cut out circles according to Figure 1.6. The total area of material required for each can then becomes a rectangle and two hexagons.

Exercises:

We will set up an equation for the total surface area of sheet metal required and then find when it is a minimum.

1. The side of the can is in the shape of a rectangle that has width h and a length equal to the circumference of the lid. Give an equation for this area in terms of r and h.

Don't forget that the mathematical constant π is represent in Maple as `Pi`.

2. The two hexagons that are needed for the top and bottom of the can circumscribe a circle of radius r. Give an equation for the area of this hexagon in terms of r. Figure 1.7 may be helpful for splitting the hexagon up into six equilateral triangles.

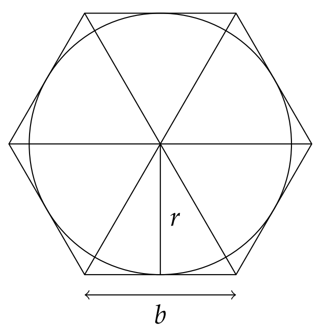


Figure 1.7: A hexagon circumscribing a circle.

Using trigonometric ratios of $\frac{\pi}{3} = 60°$ and $\frac{\pi}{6} = 30°$, it is possible to compute the length of b in terms of r.

The area of each of the six triangles is $\frac{1}{2}br$.

3. Give an equation for the total area A required (including waste) as a function of r and h. This total area should include two hexagons and the rectangle that is used for the side of the can.

The volume of a circular cylinder is

$$V = \pi r^2 h.$$

4. Given the required total volume $V = 250$, use a substitution for h to give the area from exercise 3 in terms of r only. Assign this function as $A(r)$.

5. Plot a graph of $A(r)$ over the interval $0 \leq r \leq 5$. Limit the vertical range to $-200 \leq A \leq 400$.

6. Use the second derivative test to find the value of r that minimizes the total area $A(r)$.

For the second derivative test, find Type I critical numbers where $A'(r) = 0$. Then show that $A''(r) > 0$ for all $r > 0$ to indicate that the area is an absolute minimum at this critical number.

7. Calculate the height of the can using this radius.

8. Show that the ratio of height to radius is $\dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi} \approx 2.21$.

It can be shown that the ratio of height to radius of the optimized can is $\frac{h}{r} = \frac{4\sqrt{3}}{\pi}$, regardless of volume.

1.16 *The Calculus of Rainbows*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial H, pg. 125

Introduction:

Rainbows are created when raindrops scatter sunlight. In this project, we use the ideas of Descartes and Newton to explain the shape, location, and colours of rainbows.

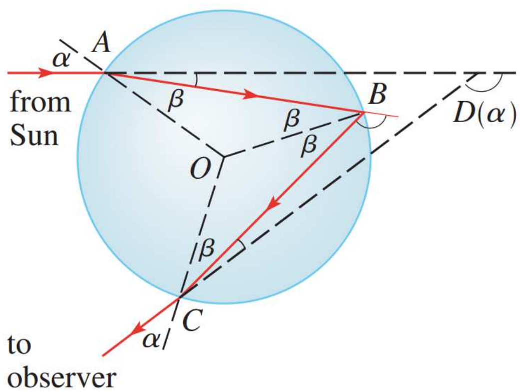


Figure 1.8: Formation of the primary rainbow.

Figure 1.8 shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that

$$\sin(\alpha) = k \sin(\beta), \tag{1.2}$$

where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water. At B, some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected.

Exercises:

1. When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. The angle of deviation $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. The formula for the angle of deviation is given by

$$D(\alpha) = \pi + 2\alpha - 4\beta. \tag{1.3}$$

The Greek letters alpha (α) and beta (β) can be found on the Greek palette, or typed out and autocompleted using ESC or Ctrl+Space.

(a) Solve Equation (1.2) for β and substitute this into Equation (1.3).

(b) Show that the minimum value of the deviation is $D(\alpha) \approx 138^\circ$ and occurs when $\alpha \approx 59.4^\circ$.

(c) Plot the Angle of Deviation function in Maple to verify your answer to (b).

You may use the closed interval method over the interval $\alpha \in \left[0, \frac{\pi}{2}\right]$. Note that Maple will give you answers in radians, so you will have to convert to degrees.

The significance of the minimum deviation is that when $\alpha \approx 59.4^\circ$, we have $D'(\alpha) \approx 0$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4^\circ$ become deviated by approximately the same amount. It is the concentration of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. Figure 1.9 shows that the angle of elevation from the observer up to the highest point on the rainbow is $180^\circ - 138^\circ = 42^\circ$. (This angle is called the rainbow angle.) The rainbow angle is found by subtracting the angle of deviation from 180° (i.e. $180^\circ - D(\alpha)$).

The rainbow angle is found by

$$180^\circ - D(\alpha).$$

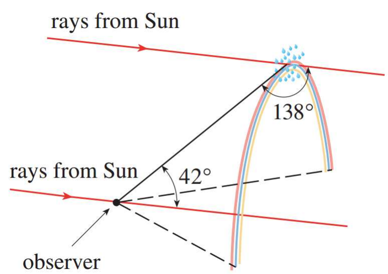


Figure 1.9: Angle of elevation from observer.

Exercise 1 explains the location of the primary rainbow, but how do we explain the colours? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet (ROYGBIV! Ring a bell?). As Newton discovered in his prism experiments of 1666, the index of refraction is different for each colour. (The effect is called dispersion.)

2. For red light, the refractive index is $k \approx 1.3318$, whereas for violet light it is $k \approx 1.3435$. By repeating the calculation of exercise 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colours.

In this example, you will repeat exercise 1 using different refractive indices of different colours of line

Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see Figure 1.10).

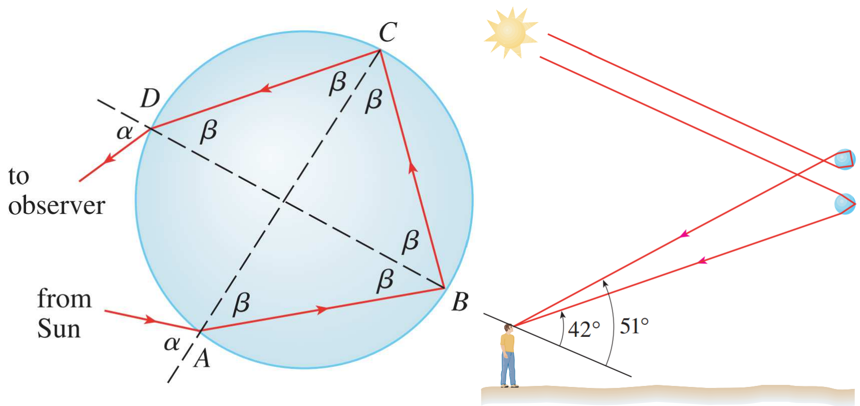


Figure 1.10: Angles of elevation from observer.

3. This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. In this case, the angle of deviation is given by the formula

$$D(\alpha) = 2\alpha - 6\beta + 2\pi, \tag{1.4}$$

and $D(\alpha)$ has a minimum value when

$$\cos(\alpha) = \sqrt{\frac{k^2 - 1}{8}}. \tag{1.5}$$

(a) Solve Equation (1.2) for β and substitute this into Equation (1.4).

(b) When $k \approx \frac{4}{3}$, find the angle of incidence, α, using Equation (1.5).

(c) Find the angle of deviation, $D(\alpha)$, using your answers from exercises 3(a) and 3(b).

(d) Verify that the rainbow angle for the secondary rainbow is approximately 51°, as shown in Figure 1.10.

It is still necessary to use Snell's Law (Equation (1.2)) to determine the angle of deviation:

$$\sin(\alpha) = k\sin(\beta)$$

The rainbow angle is found by

$$180^\circ - D(\alpha).$$

1.17 *Sweet 16*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial G, pg. 117
- Tutorial H, pg. 125

Introduction:

We all know that 16 exhibits the following interesting property:

$$16 = 2^4 = 4^2$$

Are there any other positive integers N for which there exist positive integers a and b ($a \neq b$) such that

$$N = a^b = b^a?$$

This activity will explore this question.

If we remove the restriction that a and b have to be integers, this problem becomes lot more interesting.

It can be shown that for any value $N > \mathrm{e}^{\mathrm{e}}$, there exist real numbers a and b that satisfy $N = a^b = b^a$ with $a \neq b$. To prove this, one needs to be very careful in evaluating the limit

$$\lim_{\substack{a\to 1\\ b\to\infty}} a^b.$$

It is not obvious that this limit goes to infinity. However, if it does, all $N > \mathrm{e}^{\mathrm{e}}$ will have the desired property. It may help to recall that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}.$$

So, just because the base tends to 1 and the exponent tends to infinity, the value of a^b is not necessarily infinity.

Exercises:

Let's assume that for some $N \neq 16$, there are positive integers a and b such that $N = a^b = b^a$.

1. If $a^b = b^a$, can we solve for a or b? Why or why not? Try to do this by hand and comment on the result in a new paragraph.
2. Rearrange the equation to separate the a's and b's, so that all a's are on one side and all b's are on the other side. Comment about what you notice about the left and right sides of the equation in a new paragraph.
3. Assign the function $f(x) = \ln(x)/x$. Graph $f(x)$ using Maple.
4. Determine the critical point(s) and the maximum and minimum values (if they exist).
5. Insert a new paragraph and comment on what happens to $f(x)$ as $x \to \infty$.
6. On what intervals is $f(x)$ increasing and decreasing? Answer on a new paragraph.
7. If $f(a) = f(b)$ and $a < b$, what does that mean in terms of the graph? Answer on a a new paragraph.
(*Hint: Rolle's Theorem*)
8. If $f(a) = f(b)$ and $a < b$, prove that $a = 2$ is the only positive integer such that $a^b = b^a$.

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

(a) f is continuous on the closed interval $[a, b]$.

(b) f is differentiable on the open interval (a, b).

(c) $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

2 Lab Test Review for Math 112

The following exercises are provided as examples of potential questions on the final lab test at the end of the semester.

2.1 The Basics

1. Evaluate $\sqrt[3]{\dfrac{5.0^2 - 3.0}{1.5 + 9.0^2}}$ to 15 digits.

2. Give the numerical value of e^2 to 15 decimal places.

3. Factor the given cubic to find its roots.

$$2x^3 - 7x^2 + 7x - 2$$

4. Find the point of intersection of the curves $y = \mathrm{e}^x$ and $y = \frac{1}{x}$. Give both the x- and y-coordinates to 15 digits.

2.2 Limits

5. Find the left- and right-hand limits of

$$\frac{x^2|x-2|}{x-2}$$

at $x = 2$. Does the two-sided limit at $x = 2$ exist? Explain.

6. Consider the function

$$f(x) = \frac{\sin(x)}{x^2+1} + \arctan(x).$$

Evaluate the following limits to 15 decimal places.

(a) $\lim_{x \to \infty} f(x)$

(b) $\lim_{x \to -\infty} f(x)$

2.3 *Vertical and Horizontal Asymptotes*

7. Sketch the plot of $y = \dfrac{\sqrt{4x^2+1}}{2x-1}$ and state the equations of vertical and horizontal asymptotes.

8. Find the horizontal asymptote of $\dfrac{5x^3-2x+1}{1-8x^3}$.

2.4 *The Derivative of a Function*

9. Consider the function

$$f(x) = (1-x^2)e^{-x^2/2}.$$

Find all critical numbers of f to 15 digits.

10. Consider the function

$$g(x) = \frac{x+2}{3\sqrt{x^2+5}}.$$

(a) Plot $g(x)$, $g'(x)$, and $g''(x)$ on the same axes.

(b) Find the maximum value of $g(x)$.

(c) Find the maximum value of $g'(x)$.

11. The height in metres of a ball thrown from the top of a building is given by the function

$$h(t) = -9.80t^2 + 5.00t + 40.$$

(a) Find the velocity of the ball at $t = 2$ seconds.

(b) At what time does the ball reach its maximum height?

(c) What is the maximum height of the ball?

2.5 *Tangent Lines*

12. Given the function

$$f(x) = e^x \cos x,$$

find the equation of the tangent line to $f(x)$ at $x = 1$ in the form $y = mx + b$.

13. Given the function

$$g(x) = x^{\ln x},$$

find the equation of the tangent line to $g(x)$ at $x = 5$ in the form $y = mx + b$.

2.6 *Implicit Functions*

14. Given the Folium of Descartes,

$$x^3 + y^3 = 6xy,$$

find the slopes at all points on the curve where $x = 2$.

15. Find all points on the curve

$$(x^2 + y^2)^2 = 16(y^2 - x^2)$$

where the slope is equal to 4.
Hint: Find one point and use symmetry for the others.

3 Activities for Math 122

The following list serves as a suggestion for which activities are considered core versus optional:

3.1 *Riemann Sums for Monotonic Functions*

Recommended Tutorials:

- Tutorial J, pg. 141

Introduction:

In this activity, students will use Maple's `ApproximateInt()` command to visualize and calculate Riemann sums for the area below the monotonic function

$$f(x) = x - 2\ln(x).$$

Exercises:

1. A monotonic function is one that is strictly increasing or strictly decreasing. Using 10 subintervals, plot the Riemann sum for

$$f(x) = x - 2\ln(x)$$

on the interval $[2, 10]$. Use both `left` and `right` endpoint methods.

Using left and right endpoint methods, it is easy to come up with an overestimate and underestimate for monotonic functions. In particular, to come up with an overestimate requires only one Riemann sum calculation.

For exercises 2 through 5, answer the question in a new paragraph on your Maple worksheet.

2. What do you notice about using the left-point method to estimate the area below a monotonically increasing function?

3. What do you notice about using the right-point method to estimate the area below a monotonically increasing function?

4. What do you suspect will happen if you use the left-point method to estimate the area below a monotonically decreasing function?

5. What do you suspect will happen if you use the right-point method to estimate the area below a monotonically decreasing function?

6. Using 10 boxes, estimate the Riemann sum for $f(x)$ on the interval $[2, 10]$. Use both left- and right-point methods.

7. Using n boxes, estimate the Riemann sum for $f(x)$ on the interval $[2, 10]$. Use both left- and right-point methods. What happens as $n \to \infty$?

8. Estimate the Riemann sum for $f(x)$ on the interval $[0.5, 3]$ using 10 boxes and left- and right-point methods. What do you notice about your answers? Does this contradict your findings about monotonic functions?

Any function that is not monotonic is not as easy to find an overestimate or underestimate for.

3.2 Riemann Sums and the Net Change Theorem

Recommended Tutorials:

- Tutorial J, pg. 141

Introduction:

The Net Change Theorem states that if a quantity $Q = F(t)$ is a differentiable function on the interval $[a, b]$, then

$$\begin{aligned}\int_a^b F'(t)\, dt &= F(b) - F(a) \\ &= \text{net change in } Q \text{ over } [a, b].\end{aligned}$$

In other words, the Net Change Theorem states that the definite integral of the rate of change of Q from a to b is given by the difference in the initial quantity and the final quantity.

We may also be interested in finding the total change of the quantity Q, given by the integral

$$\int_a^b |F'(t)| dt.$$

In this case, all area is positive.

We will use Maple's `ApproximateInt()` command to help visualize the net change and total change of a function.

Exercises:

1. Let $f(x) = \dfrac{x}{x^2 + 4}$. Plot $f(x)$ on the interval $[-10, 10]$.
2. Plot and evaluate the Riemann sum for $f(x)$ on the interval $[-5, 10]$ with $n = 15$. Use both upper and lower boxes.
3. Evaluate the Riemann sum for $f(x)$ on the interval $[-5, 10]$ with n boxes. Use the limit command to find a numerical solution.
4. Compute $\int_{-5}^{10} f(x)\, dx$. Insert a new paragraph and describe how this compares to the Riemann sum value.
5. Plot and evaluate the Riemann sum for $|f(x)|$ on the interval $[-5, 10]$ with $n = 15$. Use both upper and lower boxes.
6. Evaluate the Riemann sum for $|f(x)|$ on the interval $[-5, 10]$ with n boxes. Use the limit command to find a numerical solution.
7. Compute $\int_{-5}^{10} |f(x)|\, dx$. Insert a new paragraph and describe how this compares to the Riemann sum value.

Recall that the `abs()` command is used for absolute values in Maple.

Think about how you would evaluate this integral if you could not integrate the absolute value function. In general, computing $\int_a^b |f(x)|\, dx$ is difficult.

8. What is the net area bounded by the function $f(x)$ and the x-axis on the interval $[-5, 10]$?

9. What is the total area bounded by the function $f(x)$ and the x-axis on the interval $[-5, 10]$?

The net area and the total area between a curve and the x-axis can be different quantities. It is important to know when they are different and when they are the same.

3.3 Other Integral Approximation Techniques

Recommended Tutorials:

- Tutorial J, pg. 141

Introduction:

The trapezoid rule, the midpoint rule, and Simpson's rule are all useful methods for approximating a definite integral of a function $f(x)$. However, each of these methods has some error in its approximation for a finite number of subintervals. It is possible to calculate an upper bound for this error, which relies on calculating the the maximum value of $|f''(x)|$ (or $|f^{(4)}(x)|$ for Simpson's rule) over the interval first.

In this activity, students will use Maple's `ApproximateInt()` command to help visualize these three approximation methods and then calculate the error associated with them.

Exercises:

Consider the definite integral $\int_0^1 e^x \sin(x)\, dx$.

1. Plot $e^x \sin(x)$ on the interval $[-1, 2]$.

Don't forget that the `exp()` function is used for e^x.

2. Consider the area under $e^x \sin(x)$ using the **trapezoid** rule with 4 subintervals over the interval $[0, 1]$.

   (a) Display this area using `output=plot`.

   (b) Display the sum for this area using `output=sum`.

   (c) Find the value of this area using `output=value`.

   (d) Calculate an upper bound on the error of this approximation.

   (e) Compare the approximated area to the true value of definite integral. What is the actual error of this approximation?

The upper bound for the error of the trapezoid rule over the interval $[a, b]$ is

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ and n is the number of subintervals.

3. Consider the area under $e^x \sin(x)$ using the **midpoint** rule with 4 subintervals over the interval $[0, 1]$.

   (a) Display this area using `output=plot`.

   (b) Display the sum for this area using `output=sum`.

   (c) Find the value of this area using `output=value`.

   (d) Calculate an upper bound on the error of this approximation.

   (e) Compare the approximated area to the true value of definite integral. What is the actual error of this approximation?

The upper bound for the error of the midpoint rule over the interval $[a, b]$ is

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ and n is the number of subintervals.

4. Consider the area under $e^x \sin(x)$ using **Simpson's** rule with 4 subintervals over the interval $[0, 1]$.

   (a) Display this area using `output=plot`.

   (b) Display the sum for this area using `output=sum`.

   (c) Find the value of this area using `output=value`.

   (d) Calculate an upper bound on the error of this approximation.

   (e) Compare the approximated area to the true value of definite integral. What is the actual error of this approximation?

The upper bound for error of Simpson's rule over the interval $[a, b]$ is

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $\left|f^{(4)}(x)\right| \leq K$ and n is **twice** the number of subintervals.

5. Find an example of a function (possibly in the form of a graph) with 4 partitions, in which the trapezoid rule will yield a better approximation than either Simpson's rule or the midpoint rule.

6. Find an example of a function (possibly in the form of a graph) with 4 partitions, in which the midpoint rule will yield a better approximation than either Simpson's rule or the trapezoid rule.

7. Find an example of a function (possibly in the form of a graph) with 4 partitions, in which Simpson's rule will yield a better approximation than either the trapezoid rule or the midpoint rule.

3.4 Describing the Shapes of Integral Functions

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial H, pg. 125
- Tutorial J, pg. 141

Introduction:

In this activity, we will examine two functions that are defined by integrals. We may view these integral functions as the accumulated area under a function of t over an interval from 0 to x. We can analyze the shape of these integral functions by looking for relative extrema inflection points.

Recall that a critical point of a function $f(x)$ occurs when $f'(x) = 0$ or when $f'(x)$ does not exist. An inflection point of $f(x)$ is a point at which the concavity of $f(x)$ changes direction.

Exercises:

1. Plot the graph of the sine integral function

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt.$$

Integral functions show up all the time in analysis and in differential equations. Determining critical points and inflection points is incredibly important in the analysis of these types of problems.

2. Determine where the sine integral function has a local maximum value.

3. Determine the maximum value of the sine integral function.

4. Plot the graph of the following function:

$$f(x) = \int_0^x \frac{1}{1+t+t^2}\,dt.$$

When faced with a non-obvious problem in calculus, it is often a good idea to use Maple to graph the function. This helps you get an idea of what the function looks like.

5. Determine the inflection points of the function.

6. Determine the intervals in which the function is concave up and where the function is concave down.

3.5 The Sine Integral Function

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial G, pg. 117
- Tutorial H, pg. 125
- Tutorial J, pg. 141

Introduction:

The sine integral function

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\, dt$$

is important in electrical engineering. Since the integrand is not defined at $t = 0$, we define a continuous function $Si(x)$ by defining $Si(0) = 1$ as seen belown.

$$Si(x) = \begin{cases} 1 & x = 0 \\ \displaystyle\int_0^x \frac{\sin(t)}{t}\, dt & \text{otherwise.} \end{cases}$$

By taking the limit as the integration bounds approach $\pm\infty$, one can show that $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi$.

It is an interesting exercise to integrate the sine integral function. Indeed, $\int Si(x)\, dx = \cos(x) + xSi(x) + c$.

Exercises:

1. Assign $Si(x)$ using the `piecewise()` command. Since `Si` is a *protected variable*, you may wish to define it as $SI(x)$ (with a capital I).
2. Use Maple to plot the graph of $Si(x)$.
3. Determine the values of x at which this function has local maximum values.
4. Find the coordinates of the first inflection point to the right of the origin.
5. Does this function have a horizontal asymptote? Explain in a new paragraph.

By the Fundamental Theorem of Calculus,

$$\frac{d}{dx} Si(x) = \frac{\sin(x)}{x} = \text{sinc}(x)$$

($x \neq 0$). This function is used in signal processing and the theory of Fourier transforms.

3.6 Applying the Fundamental Theorem of Calculus to a Limit Problem

Recommended Tutorials:

- Tutorial G, pg. 117
- Tutorial J, pg. 141

Introduction:

In this activity, students will be asked to demonstrate their knowledge of the Fundamental Theorem of Calculus in order to evaluate

$$\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^3}.$$

Exercises:

It is important to be able to use theorems from both Calculus I and II in order to solve problems. Maple is a great tool to help you find an answer to a problem; however, Maple graphs do not constitute a proof. Even when you know the answer, it is important to learn to verify your Maple result.

1. Plot the graph of $\frac{\int_0^x \sin(t^2)\,dt}{x^3}$. Graphically determine the value of
$$\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^3}.$$
2. Can we evaluate $\int_0^x \sin(t^2)\,dt$? If not, why? If so, what is it?
3. What rule do we need to use in order to evaluate the limit? State the theorem, including conditions under which we can apply the theorem.
4. Apply the theorem. What happens now?
5. What is the value of the limit?
6. Write your answer out, from start to finish, in paragraph form.

For questions that require an explanation, use a new paragraph in Maple.

Always be careful when you apply a theorem. Ensure that the hypotheses are fully satisfied before you apply the theorem.

Write out your answer in paragraph form. Use many different words to link sentences. This is how you begin to get better at completing proofs.

3.7 Average Value of a Function on a Shrinking Interval

Recommended Tutorials:

- Tutorial G, pg. 117
- Tutorial J, pg. 141

Introduction:

The average value of a function f on the interval $[a, b]$ is defined as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In this activity, we will investigate the function

$$f(t) = \sqrt{1+t^3}$$

over a shrinking interval with $a = 2$ fixed and b approaching a. Specifically, we will determine the value of the integral

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt.$$

Integral functions show up all the time in analysis and in differential equations. Determining critical points and inflection points is incredibly important in the analysis of these types of problems.

When faced with a non-obvious problem in calculus, it is often a good idea to use Maple to graph the function. This helps you visualize the function before you begin working with it.

Exercises:

1. Plot the graph of the function $\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$.

Note that you should use h as the independent variable here.

2. From the plot you displayed in the previous question, what do you expect $\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$ to be?
3. In a new paragraph on your worksheet, explain why is this difficult to show by hand.
4. Determine $\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$ by hand. State any theorems, and the hypothesis that must be true, that you used to solve this problem.

It is important to state the hypothesis of a theorem before you use it.

5. Write out your full solution in paragraph form in your Maple worksheet.

3.8 Find the Function with the Fundamental Theorem of Calculus

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial J, pg. 141

Introduction:

In this activity, students will be asked to find a function f and a number a that satisfy the equation

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}.$$

Exercises:

1. Find a function f and a number a that satisfy the equation

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}.$$

Hint: Use the Fundamental Theorem of Calculus to begin.

2. Write down a step by step procedure to solve this problem that you could give to someone so that they could follow your logic.

3.9 Shark Attack

Recommended Tutorials:

- Tutorial E, pg. 105
- Tutorial F, pg. 113
- Tutorial J, pg. 141

Introduction:

In this activity, you will determine whether you can swim to safety before a shark attacks. It will use piecewise functions and the Net Change Theorem to find the outcome.

You are surfing on the ocean and there is a shark 50 m away from you at rest ($v(0) = 0$).

The shark senses you and begins accelerating toward you at a rate of 5 m/s², up to a top speed of 13 m/s. You see the shark coming and begin swimming towards shore at a speed of 2 m/s. Assume there is no time needed for you to accelerate up to your top speed. If the shore is 20 m away, do you make it to shore before the shark attacks?

Exercises:

1. If the shark accelerates at 5 m/s², how long does it take for it to reach its top speed of 13 m/s? Note that the shark will no longer accelerate once it reaches its top speed.

The shark's velocity will contain an acceleration term for $0 \le t \le t_1$ (where t_1 is the time you find in part (a)) but will not have an acceleration term after t_1 seconds.

2. Write down a piecewise function for the velocity of the shark over time using $v(t) = v(0) + at$. Define it in Maple.

You can read how to define piecewise functions in Tutorial G.

3. Integrate your velocity equation to find the distance.
4. Define the velocity equation for you (the person swimming to shore) in Maple.
5. Find the distance equation for you using integration. Since you start 50 m ahead of the shark ($d(0) = 50$), add the 50 m between you and the shark so that you compensate for your head start.
6. Set the two equations equal and solve for the time using Maple.

Try using the `solve()` command first, then go to `fsolve()` if necessary.

7. Plug this time into your distance formula to see if you would make it to shore by that time.
8. Plot the two functions on the same graph to check your answer. You may want to define the colours for each plot so that you know which plot is which.

9. Instead of swimming, let's suppose you surf the nearest wave away from the shark, accelerating you at 2 m/s^2. Starting from rest, the wave accelerates you to a top speed of 4 m/s. Describe the outcome of this new scenario in your worksheet.

3.10 *Areas Between Curves*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial J, pg. 141

Introduction:

Consider two functions, $f(x)$ and $g(x)$. We can find the area between these two curves, over an interval $a \leq x \leq b$, by integrating the absolute value of the difference between these two functions. In other words,

$$\text{Area} = \int_a^b |f(x) - g(x)| \, dx.$$

If $f(x) \geq g(x)$ over that interval, then the absolute value can be dropped. If $g(x) \geq f(x)$ over that interval, then the absolute value can be dropped and the order of subtraction reversed.

However, if $f(x) \geq g(x)$ for some subintervals of $a \leq x \leq b$ and $g(x) \geq f(x)$ for other subintervals of $a \leq x \leq b$, then the intersection points of the two functions must be found and the integral must be split into multiple parts for each subinterval. The absolute value will factor in all these possibilities and we will always find the *total* area between $f(x)$ and $g(x)$.

In this activity, students will be asked to find the value of c for which the region bounded by the curves

$$y = x^2 - c^2 \quad \text{and} \quad y = c^2 - x^2,$$

has area equal to 576.

There is nothing fancy about 576 nor is there anything fancy about the curves. In fact, any curve that is entirely non-negative and crosses the x-axis in two different places (along with the negative of that curve) will work.

Exercises:

1. Choose a few different values of c and plot the two parabolas on the same graph for each value chosen. What do you notice? You may want to specify the colours of the two curves when you plot them to ensure you know which is which.

When first solving a problem like this, selecting a few different values for c and playing around with the problem, is a good idea. Here, it made sense to graph the curves to get a sense of what the problem was asking.

2. For the graphs in the preceding question, what are the coordinates (x, y) of the points of intersection of the curves?

3. Use Maple's `Int()` command to set up an integral that represents the area bounded by the two parabolas.

4. Evaluate the integral from exercise 3 using Maple and solve the problem.

3.11 *Volumes of Revolution*

Recommended Tutorials:

- Tutorial F, pg. 113
- Tutorial I, pg. 135
- Tutorial J, pg. 141

Introduction:

In this activity, students will use the Volume of Revolution Tutor to find the volume of a region rotated about a vertical or horizontal axis.

Exercises:

1. Define the two functions $f(x) = x^5 - x^3$ and $g(x) = \sin(x)$ in Maple.
2. Plot the graphs of $f(x)$ and $g(x)$ on the same set of axes to get an idea where the curves intersect.
3. Find the point of intersection of the curves $f(x)$ and $g(x)$ for $x > 0$. You can assign this to an expression and use the expression name in the tutor.

   You will probably need to use `fsolve()` for this.
4. Find the volume of revolution if the region between the curves for $x \geq 0$ is rotated about the line $x = \pi$.
5. Use the Volume of Revolution Tutor. Make sure the plot looks like what you are trying to calculate. Record the value in your Maple worksheet.

   You can invoke this tutor by either typing `VolumeOfRevolutionTutor()` available in the `Student[Calculus1]` package, or by selecting Tools -> Tutors -> Calculus - Single Variable -> Volume of Revolution in the top bar of Maple.
6. Find the volume of revolution if the region between the curves for $x \geq 0$ is rotated about the line $y = -4$.
7. Use the Volume of Revolution Tutor. Make sure the plot looks like what you are trying to calculate. Record the value in your Maple worksheet.
8. Suppose you want to find the volume of an egg which has an elliptical shape defined in the (x, y) plane by $\frac{x^2}{2} + y^2 = 1$. Plot the curve using the `implicitplot()` command.
9. Which part of the curve can you rotate about what line in order to get the volume of the egg?
10. Solve the equation for y. Find the endpoints of your interval for the volume of revolution.

11. Use the Volume of Revolution Tutor to find the volume. Make sure to copy and paste resulting volume into your Maple worksheet to save it for later.

3.12 *The Golden Gate Bridge Problem*

Recommended Tutorials:

- Tutorial J, pg. 141

Introduction:

Arc length is the distance between two points along a section of a curve. If this curve can be represented by a function $f(x)$, then we can calculate the length of this curve from $x = a$ to $x = b$ with the formula

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

You will have to use the `sqrt()` command to enter this into Maple.

In this activity, students will be asked to work with the above arc length equation and eventually determine the length of the cable that holds up the Golden Gate Bridge.

Exercises:

1. Plot the graph of $y = \cos(\sin(x))$.
2. Determine the arc length of this curve between the points $(0, 1)$ and $(\pi, 1)$.

Remember that you must type `Pi` in Maple for π.

3. Plot the graph of $y^2 = x^3$.

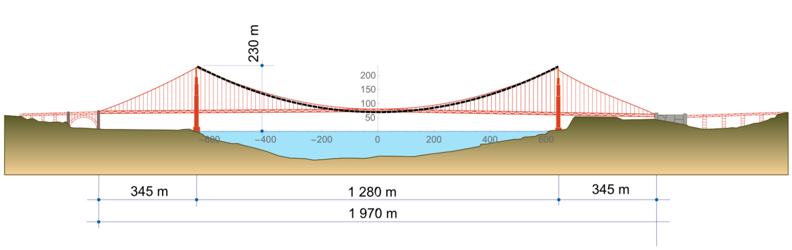
Use the `implicitplot()` command for this.

4. Determine the arc length of this curve between the points $(1, 1)$ and $(4, 8)$.
5. Plot the graph of $x = \frac{1}{3}\sqrt{y}(y - 3)$.
6. Determine the arc length of this curve for $1 \leq y \leq 9$.
7. The span of the Golden Gate Bridge is 1280 m long. The top of the tower is 230 m above the surface of the water. We will assume that a freely hanging cable between two towers takes the form of a **catenary**. The general form for a catenary passing through its lowest point k at $x = 0$ is

$$y = a\left(\cosh\left(\tfrac{x}{a}\right) - 1\right) + k = a\left(\frac{e^{x/a} + e^{-x/a}}{2} - 1\right) + k.$$

A catenary is a curve that an idealized hanging chain or cable assumes under its own weight when supported only at its ends. The Golden Gate Bridge cable is almost a catenary and almost a parabola, but not quite either (because of the weight of the cables, the suspender ropes and the roadway).

Functions like cosh() and sinh() are called hyperbolic functions. These functions are analogs of the ordinary trigonometric, or circular, functions; just as the points $(\cos(t), \sin(t))$ form a circle with a unit radius, the points $(\cosh(t), \sinh(t))$ form the right half of the equilateral hyperbola.



Interesting fact 1: The total length of each cable is actually 2,332 m.

Assuming that the Golden Gate Bridge cable takes the shape of a catenary with its lowest point at a height of 70 m, determine the length of one of the cables holding up the bridge deck if $a = 1304$ for this catenary.

Interesting fact 2: The main cables of the Golden Gate Bridge are nearly one meter in diameter (actually, 0.91 m) and the total length of galvanized steel wire used in both main cables is 129,000 km.

3.13 *Infinite Integrals*

Recommended Tutorials:

- Tutorial D, pg. 101
- Tutorial E, pg. 105
- Tutorial J, pg. 141

Introduction:

In this activity, students will be asked to evaluate a number of improper integrals.

Infinite integrals are used in a variety of applications, including finding solutions to differential equations by way of the Laplace transform.

Exercises:

1. (a) Define the function $f(x) = \dfrac{1}{\sqrt{2-x}}$ in Maple.

   (b) Plot $f(x)$.

   (c) Evaluate the integral $\int_{-\infty}^{-1} f(x)dx$.

Don't forget to use `sqrt()` when defining this function.

It is important to check that the function does not have any points of discontinuity before you integrate.

2. (a) Define the function $g(x) = x\mathrm{e}^{-x^2}$ in Maple.

   (b) Plot $g(x)$.

   (c) Evaluate the integral $\int_{-\infty}^{\infty} g(x)dx$.

Don't forget to use `exp()` when defining this function.

3. (a) Define the function $h(x) = \dfrac{\ln x}{x}$ in Maple.

   (b) Plot $h(x)$.

   (c) Evaluate the integral $\int_{1}^{\infty} h(x)dx$.

4. Let $J(t) = t$. Evaluate $F(s) = \int_{0}^{\infty} J(t)\mathrm{e}^{-st}dt$. This is called the Laplace transform for t.

3.14 Probability

Recommended Tutorials:

- Tutorial J, pg. 141

Introduction:

In this activity, we define probability density functions (pdf) for some continuous probability distributions. Let's define our probability function as $f(x)$. If $f(x)$ is a valid probability function, then

$$\int_{-\infty}^{\infty} f(x)dx = 1,$$

and $0 \leq f(x) \leq 1$.

If we want to compute the probability that a given observation is less than a

$$P(x < a) = \int_{-\infty}^{a} f(x)dx.$$

If we want to compute the probability that the given observation is more than a

$$P(x > a) = \int_{a}^{\infty} f(x)dx.$$

If we want to compute the probability that the given observation is between two values a and b then we calculate

$$P(a < x < b) = \int_{a}^{b} f(x)dx.$$

Define the exponential distribution as

$$\frac{1}{\lambda}\mathrm{e}^{-x/\lambda} \text{ for } x > 0,$$

where λ is the mean and standard deviation of the probability distribution.

Define the normal distribution as

$$\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/2\sigma^2},$$

where σ is the standard deviation and μ is the mean.

Exercises:

1. Suppose that the lifetime of a certain tire is exponentially distributed with mean $\lambda = 45,000$ miles.

   (a) Define the pdf using a piecewise function since it is only true for $x > 0$ (it is 0 otherwise).

Use the `piecewise()` command to define this function.

(b) Verify that the pdf you created is a valid pdf.

Hint: take the integral and plot the function to verify this.

(c) Find the probability that a given tire will last more than 40,000 miles.

(d) Find the probability that a given tire will last less than 50,000 miles.

(e) Find the probability that a given tire will last between 40,000 and 50,000 miles.

(f) Find the probability that a given tire will last exactly 40,000 miles.

2. Suppose that the height of a male is normally distributed with mean $\mu = 178$ cm and standard deviation $\sigma = 10$ cm.

(a) Define the normal pdf using these values of μ and σ.

The commands `exp()`, `sqrt()`, and `Pi` will all have to be used for this function in Maple.

(b) Verify that this is a valid pdf.

(c) You have a friend who is 7 ft tall (213 cm). Find the probability that a given individual is that height or taller.

(d) Find the probability that a given individual is 213 cm or smaller.

(e) What is the probability of selecting an individual with a height of exactly 213 cm?

3.15 Motion of a Mass Connected to a Spring

Recommended Tutorials:

- Tutorial K, pg. 155

Introduction:

According to Hooke's law and Newton's second law ($F = ma$), the differential equation for the motion of a mass (m) on the end of a spring is

$$m\frac{d^2x}{dt^2} = -kx,$$

where k is the spring constant. This equation assumes no damping (resistance). The displacement of the mass from equilibrium is denoted by x, and thus $\frac{dx}{dt}$ is the velocity, and $\frac{d^2x}{dt^2}$ is the acceleration.

For the exercises, we will assume that the mass is $m = 2$ kg and the spring constant is $k = 3$ kg/s^2. Additionally, we will assume the initial conditions $x(0) = 1$ m and $x'(0) = -1$ m/s.

Exercises:

1. Solve the following differential equation using the given initial conditions:
$$2\frac{d^2x}{dt^2} = -3x.$$

Since we are treating x as a function of time, be sure to use $x(t)$ and $x''(t)$ in Maple.

2. Plot the solution of the differential equation.

The `rhs()` command may be used to refer to only the right hand side of the differential equation solution. You can use this command to assign a name to the solution.

3. Insert a new paragraph and describe what you observe about the motion of a mass on the spring.

Now we are going to add damping (resistance) to the spring. We assume that the damping is opposite the direction of the motion and proportional to the velocity. Therefore we have the equation,

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt},$$

where c is the damping constant. We will try two damping constants, $c = 4$ (overdamping) and $c = 0.5$ (underdamping).

4. Solve the following differential equation using the given initial conditions:
$$2\frac{d^2x}{dt^2} = -3x - 4\frac{dx}{dt}.$$

5. Plot the solution of the differential equation.

6. Insert a new paragraph and describe what you observe about the motion of a mass on the spring with overdamping.

7. Solve the following differential equation using the given initial conditions:
$$2\frac{d^2x}{dt^2} = -3x - 0.5\frac{dx}{dt}.$$

8. Plot the solution of the differential equation.

9. Insert a new paragraph and describe what you observe about the motion of a mass on the spring with underdamping.

To solve these differential equations algebraically you assume that solutions are of the form $x(t) = e^{rt}$ and then plug it into the differential equation to get the "characteristic equation" to solve for r. If the roots are complex, you will have oscillations (sine and cosine functions) and if the roots are real, then you have strictly exponential solutions. Notice that the overdamped case has no oscillations whereas the underdamping and no damping cases have oscillations in their solutions.

We can also try to force the spring to oscillate at a given frequency. Let's suppose that we add a forcing term $3\sin(2t)$.

10. Solve the differential equation
$$2x''(t) = -3x(t) + 3\sin(2t)$$
with $x(0) = 1$ and $x'(0) = -1$.

11. Plot the differential equation solution.

12. Insert a new paragraph and describe what you observe about the motion of a mass on the spring with forcing.

13. Repeat exercises 10-12 with a damping constant of 0.5. Insert a new paragraph and describe what you observe about the motion of a mass on the spring with forcing and underdamping.

This underdamping term is identical to the $\frac{dx}{dt}$ term from exercises 7-9.

3.16 *Tank Mixing Problem*

Recommended Tutorials:

- Tutorial K, pg. 155

Introduction:

Suppose you are having a wedding and you need to have a 5 L tank of coffee that has a concentration of 60 g/L. The guests are drinking the coffee at a rate of $\frac{1\text{ L}}{5\text{ min}}$. You are refilling the tank at a rate of $\frac{0.75\text{ L}}{5\text{ min}}$ with coffee of a concentration of 50 g/L.

Exercises:

1. Find the rate of change of volume, $\frac{dV}{dt}$, using (rate in) − (rate out). Setup the appropriate differential equation.

Remember to use $V(t)$ and not just V when setting up the differential equation.

2. In your worksheet, describe the initial condition for the volume.
3. Solve the differential equation for volume using your initial condition.
4. On your worksheet, compute how long it will be until you run out of coffee.
5. Find the rate of change equation for the mass of coffee, $\frac{dm}{dt}$, using (rate in) − (rate out). Notice that the rate in/out can be found by multiplying two quantities; concentration×(rate of volume in/out). Set up the appropriate differential equation.
6. In your worksheet, describe the initial condition for the mass.
7. Solve the differential equation for mass using your initial condition.
8. Write down an equation for the concentration = mass/volume.
9. Use separate graphs to plot what happens to the mass, volume, and concentration over time.
10. Calculate the concentration of coffee in your last drop of coffee.

You may use the `limit()` command to determine this concentration.

3.17 Direction Fields

Recommended Tutorials:

- Tutorial K, pg. 155

Introduction:

In this activity, you will use direction fields to predict the population dynamics for a population of rabbits.

Suppose that you have a population of rabbits. They reproduce according to the equation.

$$\frac{dn}{dt} = an - bn,$$

where a is the birth rate and b is the death rate.

For all below calculations use two initial conditions $n(0) = 1$ and $n(0) = 50$.

Exercises:

1. Draw the direction field, including both of the initial conditions given. Try $a = 2, b = 1$ and $a = 1, b = 2$.

2. In a new paragraph in your worksheet, describe what you can conclude about the importance of the death to birth rate comparison.

You will need to access the `DEplot()` command through the `DETools` package.

Remember to use $n(t)$ and not just n in your differential equation.

Now suppose you have a rabbit population that is more realistic and limited by habitat or food. Then the equation grows logistically according to

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{k}\right),$$

where r is the intrinsic growth rate and k is the carrying capacity. Suppose that $r = 2$ and $k = 30$.

2. Draw the direction field, including both of the initial conditions given. In a new paragraph, describe what do you observe about the solutions.

Now we add an extra death rate (hunting) represented by the differential equation

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{k}\right) - bn,$$

where $b = 1$.

3. Draw the direction field, including both of the initial conditions given. In a new paragraph in your worksheet, describe what changed by adding the death rate to the differential equation.

Most mammal population growth is dependent upon other species in the region, via an interconnected food web. One simple predator-prey model is the the Lotka-Volterra model

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = cxy - dx,$$

where x is prey and y is predator. In this equation, the prey grow and are eaten by predators. Then the predators' growth depends on eating the prey and the predators have some death rate.

Make sure you insert multiplication between x and y here, otherwise Maple will think you want to use a variable called xy.

4. Go to Tools, Tutors, Differential Equations, DE Plots, select Lotka-Volterra Model, and then DEPlot. If you want, you can change the parameters or the initial conditions. Then click quit to plot it on your Maple worksheet.

5. In a new paragraph, explain what the prey and predator populations do on this direction field. Notice that the prey is on the x-axis and the predator is on the y-axis of the direction field.

3.18 Series Convergence and Divergence

Recommended Tutorials:

- Tutorial L, pg. 159

Introduction:

In this activity, students will set up series and evaluate whether they converge or diverge.

Exercises:

1. Evaluate whether each of the following series converge or diverge. If they converge, state the value to which they converge.

   (a) $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

   (b) $\sum_{n=0}^{\infty} \sin(n\pi)\arctan(n)$

   (c) $\sum_{n=1}^{\infty} \ln(n)$

   (d) $\sum_{n=0}^{\infty} \frac{n!}{n^2}$

Use the `Sum()` command to set up the summation symbolically, then take the limit.

For this sum, make sure you use "`Pi`" for π and place multiplication between the n and `Pi`.

2. The Integral Test for Convergence states that for a non-negative, monotonically decreasing function $f(n)$ and an integer N, the infinite series

$$\sum_{n=N}^{\infty} f(n)$$

   converges to a real number if and only if the improper integral

$$\int_N^{\infty} f(x)\,dx$$

   is finite. From this, we can also conclude that if the integral diverges, then the series diverges as well.

   Use the Integral Test to determine whether or not the series

$$\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$$

   converges or diverges. Make sure to check that all conditions of the Integral Test are satisfied. The easiest way to do this is to graph the integrand function. In a new paragraph, explain why we have to start the sum at $n = 3$.

3.19 Taylor Series

Recommended Tutorials:

- Tutorial L, pg. 159

Introduction:

Recall that the Taylor series of the function f centred at a is given by the equation

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \end{aligned}$$

For the special case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ,$$

known as the Maclaurin series.

In this activity, the student will compute and plot Taylor series approximations to understand how the radius of convergence works.

The radius of convergence for a Taylor series is the values of x for which a Taylor series will converge to the function $f(x)$.

Exercises:

1. Consider the function $f(x) = \cos(x)$.

This is called a Maclaurin series.

   (a) Find the Taylor series of $f(x)$ centred at $x = 0$.
   (b) Compute the polynomial accurate to orders 3, 6, 12, and 24.
   (c) Plot the graph of $f(x)$ and its Taylor series approximation. Repeat this for each order from Exercise (b).

You may want to choose different colours for each of these four curves to keep track of which is which if you plan to plot all of them on the same set of axes.

   (d) In a new paragraph, describe what happens to the graph of the Taylor series approximation as we increase its order (that is, have higher degree polynomials).
   (e) If we increase our order to infinity, we expect that the approximation will converge to the Radius of Convergence (ROC) on either side of 0. In a new paragraph in your worksheet, state what you expect the radius of convergence to be.

2. Consider the function $g(x) = \ln(1 + x)$.
   (a) Find the Taylor series of $g(x)$ centred at $x = 0$.
   (b) Compute the polynomial accurate to orders 3, 6, 12, and 24.
   (c) Plot the graph of $g(x)$ and its Taylor series approximation. Repeat this for each order from Exercise (b).

If you plan to plot all of the graphs on the same set of axes, you may want to choose different colours for each curve to keep track of which is which.

(d) In a new paragraph in your worksheet, state what you expect the radius of convergence to be.

3. It turns out that the full Taylor series for the functions from exercises 1 and 2 are

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

Find the ROC by hand for these two series. In a new paragraph, affirm whether or not it matches your predictions of the ROC from your Maple plots.

3.20 *Approximations of π*

Recommended Tutorials:

- Tutorial L, pg. 159

Introduction:

In this activity, students will use Taylor series in order to approximate π.

How quickly can we determine the value of π accurate to 15 digits using the formula

$$\arctan(x) = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-1)^n x^{2n+1}}{2n+1}?$$

Borwein, Bailey, and Plouffe published a formula for π that allows someone to find the n^{th} binary digit of π without needing to find any preceding digit. Their formula,

$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right],$$

was published in the paper "On the Rapid Computation of Various Polylogarithmic Constants" in which they compute the 10 billionth hexadecimal digit of π. Can you see how this formula lends itself well for this type of calculation?

Exercises:

1. As $k \to \infty$, we know that $\arctan(1) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

2. As $k \to \infty$, we know that $\arctan(1/2) + \arctan(1/3) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

3. As $k \to \infty$, we know that $2\arctan(1/2) - \arctan(1/7) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

4. As $k \to \infty$, we know that $2\arctan(1/3) + \arctan(1/7) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

5. As $k \to \infty$, we know that $4\arctan(1/5) - \arctan(1/239) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

6. As $k \to \infty$, we know that $\arctan(1/2) + \arctan(1/5) + \arctan(1/8) = \pi/4$. How large does k need to be in order to approximate π accurate to 15 digits?

7. On a separate sheet of paper, use the power series for $\arctan(x)$ to prove that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

It may help to recall that $\arctan(1/\sqrt{3}) = \pi/6$.

Many other formulas exist that can be used to approximate π including $\frac{\pi}{4} = 5\arctan(1/7) + 2\arctan(3/79)$.

The only possible formulas with two terms using $\arctan(1/k)$ to approximate $\pi/4$ are the ones listed in the exercises in this activity (Borwein and Bailey 2003).

It is important to understand why some of these series converge quicker than others.

4 Lab Test Review for Math 122

The following exercises are provided as examples of potential questions on the final lab test at the end of the semester.

4.1 Riemann Sums

1. Assign the function $f(x) = x\left(1 - x^2 e^{-\frac{1}{6}x^2}\right)$ in Maple.

   (a) Evaluate the Riemann sum over the interval $[-4, 2]$ using the `method=lower` option with 8 partitions.

   (b) Evaluate the Riemann sum over the interval $[-4, 2]$ using the `method=upper` option with 8 partitions.

4.2 Integral Approximation Techniques

2. Assign the following function in Maple:

$$g(x) = \sqrt{x}\sin(x)$$

   (a) Give an approximate value of $\int_1^8 g(x)\,dx$ using the midpoint rule and 10 partitions.

   (b) Give an approximate value of $\int_1^8 g(x)\,dx$ using the trapezoid rule and 10 partitions.

   (c) Give an approximate value of $\int_1^8 g(x)\,dx$ using Simpson's rule and 10 partitions.

   (d) Give the exact value of the definite integral $\int_{-2}^4 g(x)\,dx$. Do not evaluate as a decimal.

4.3 Integral Functions and the Fundamental Theorem of Calculus

3. Consider the function

$$f(x) = \int_0^x 10e^{-0.5t}\sin(t)\,dt.$$

(a) Assign the function to $f(x)$ and plot it over the interval $[0, 10]$.

(b) What is the derivative, $f'(x)$?

(c) At what value in the interval $[0, 10]$ does $f(x)$ reach its maximum?

(d) What is the maximum value of $f(x)$ over the interval $[0, 10]$?

4.4 *Areas Between Curves*

4. Assign the following function in Maple:

$$h(x) = \frac{2x}{x^2 + 6}$$

(a) Find the **net** area bounded by $h(x)$ and the x–axis on the interval $[-2, 6]$.

(b) Find the **total** area bounded by $h(x)$ and the x–axis on the interval $[-2, 6]$.

5. Plot the region between the curves $f(x) = \tan^2(x)$ and $g(x) = \sqrt{x}$ and compute the area to 15 digits.

4.5 *Average Value of a Function*

6. Find the average value of $f(x) = 2\sin(x) - \sin(2x)$ on the interval $[0, \pi]$.

4.6 *Volumes of Revolution*

7. Find the volume of the egg-shaped solid obtained by revolving the region bounded by the implicit curve

$$4x^2 + y^2 = 12$$

about the x-axis.

8. Find the volume bounded by the curves $y^2 - x^2 = 1$ and $y = 2$ rotated about the x-axis.

4.7 *Arc Length*

9. Determine the arc length of the curve $f(x) = x\sqrt[3]{4 - x}$ over the interval $[0, 4]$ to 15 digits.

4.8 Infinite Integrals and Probability

10. At an annual triathlon, the finishing times for male athletes can be modeled by the probability density function

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},$$

where $\mu = 4313$ seconds (the average finish time) and $\sigma = 583$ seconds (the standard deviation of finish times). Define this function in Maple using the specified values of μ and σ.

(a) Plot the function over the interval $[0, 6500]$.

(b) What is the probability that a male athlete finishes the triathlon in under 3600 seconds (1 hour)?

(c) What is the probability that a male athlete finishes the triathlon in over 4200 seconds (1 hour 10 min)?

(d) What is the probability that a male athlete takes between 3600 and 5400 seconds to finish the triathlon (1 hour to 1.5 hours)?

4.9 Differential Equations

11. After 500 fish are introduced to a lake, the rate of growth for the population of fish is given by the differential equation

$$\frac{dN}{dt} = \frac{N\,(7000 - N)}{10000},$$

where $N = N(t)$ is the population of fish after t years. Define this differential equation in Maple.

(a) Use `dsolve()` to give the solution to the differential equation for $N(t)$ using the initial condition $N(0) = 500$.

(b) How many fish will there be after 6 years (to the nearest fish)?

(c) What does the population of fish approach after a long time? (Take the limit as t tends to infinity or use a plot).

4.10 Taylor Series

12. Find the Taylor series expansion of $f(x) = e^{2x}\tan(x)$ centred at $x = 0$ (Maclaurin series) and give the coefficient of the x^8 term.

Part II

Tutorials

A The Maple Environment

A.1 Execution Groups

Maple input is used for computations and must use recognizable commands. Maple output gives the result of the computation after hitting the Enter key. Together, Maple input and output are called an *execution group*.

- Place a new execution group after the current line with the [> button.
- Place a new execution group after the current line with Ctrl+J.
- Place a new execution group before the current line with Ctrl+K.

Maple input is preceded by the > character (sometimes called a "carrot"). Maple output is displayed in the centre of the following line.

```
> 2 + 2;
```

$$4$$

```
> 8 / 2;
```

$$4$$

If at any point you wish to correct a previous Maple input line, you may simply go back to that line and modify it. However, there will be no change to the output until you hit Enter. You do not need to be at the end of the line in order to run it.

You may wish to have more than one calculation or command on a single Maple input. For more information on this, see Tutorial B on page 93.

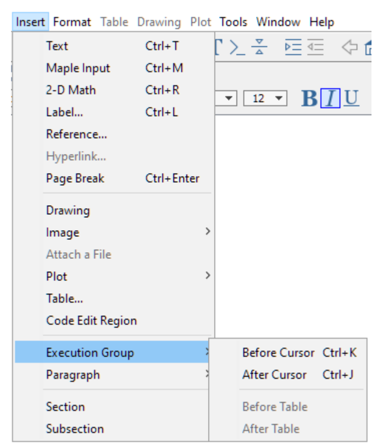


Figure A.1: Using the Insert menu to include a new Maple execution group.

A.2 Paragraphs

Paragraphs are used for text, rather than for computation. Hitting Enter simply creates a new line of text and does not try to display a result.

- Create new text after the current line with the T button.

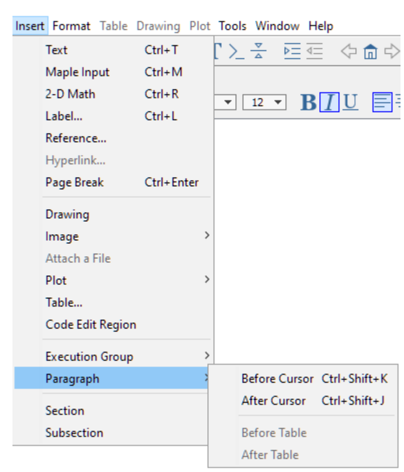


Figure A.2: Using the Insert menu to include a new text paragraph.

- Create new text after the current line with Ctrl+Shift+J.
- Create new text before the current line with Ctrl+Shift+K.

A.3 Deleting Inputs

Delete a section of Maple Input or Text using Ctrl+Delete. You will notice that hitting backspace will not delete a section by default.

It's a good habit to delete unnecessary Maple inputs.

A.4 Three Font Styles

There are two font styles that are commonly used in execution groups.

- "2D Math" (the default in modern versions)
  - In this mode, mathematical expressions are formatted nicely, as one would write them on paper.
  - Text will appear in *italic*, with the exception of common mathematical functions and constants.
  - An example of an expression in 2D Math:

$$\frac{x^5}{e^x + \sin(x)}.$$

  - Switch to 2D Math using Ctrl+R.

While 2D Math has the advantage of looking prettier, it often treats spaces as multiplication and it doesn't always treat brackets as multiplication when desired. Maple Input causes fewer issues with how Maple interprets what has been typed, but it often requires the use of many more parenthesis. This lab manual will make frequent use of Maple Input for clarity, but you will likely use 2D Math for most of your exercises.

- "Maple Input" (the default for old versions)
  - In this mode, mathematical expressions are displayed inline, with no additional formatting.
  - All text is shown in `monospaced font`.
  - The above expression in Maple input:

```
x^5/(exp(x)+sin(x))
```

  - Switch to Maple Input using Ctrl+M.

Finally, paragraphs use a third style of font.

- "Plain Text"
  - In this mode, mathematical expressions are displayed inline, with no additional formatting.
  - All text is shown in a standard font, such as this line.
  - Switch to Plain Text using Ctrl+T.

It can be useful to switch to 2D Math in paragraphs for an expression or equation, then switch back to text afterwards.

A.5 Using Sections

Sections are groups of one or more execution groups or paragraphs that are indented together. At the top of a section, you can create a section title. An arrow to the left of the title will allow you to expand or collapse that section. Subsections may be created inside other sections.

- Use the  buttons to create or remove sections.
- Use Ctrl+. to enclose input in a section.
- Use Ctrl+, to remove any section enclosing an input.

You may wish to highlight several execution groups or paragraphs with the mouse before combining them into one section.

A.6 Palettes Toolbar

You can open or close the palettes toolbar by clicking on the black arrows at the left side of the page.

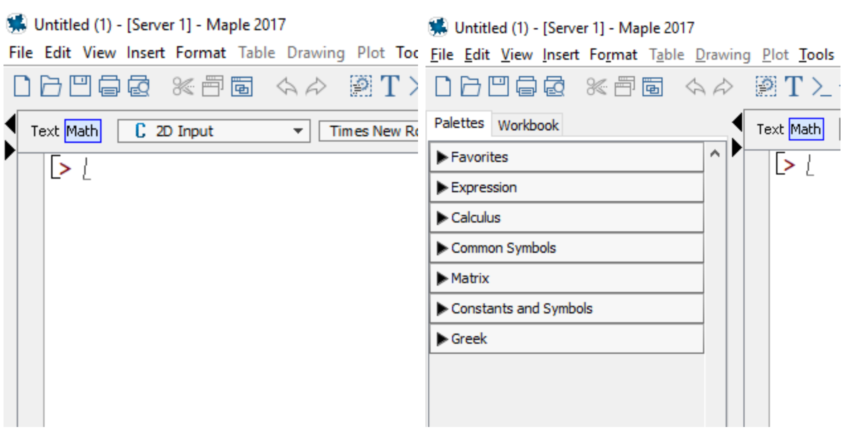


Figure A.3: Opening and closing the palettes toolbar.

With the palettes toolbar open, you can see several palettes that are available. These can be used to quickly access common operations and procedures by clicking on the appropriate buttons. Each palette can be expanded or closed. The Expression palette is especially useful for common functions.

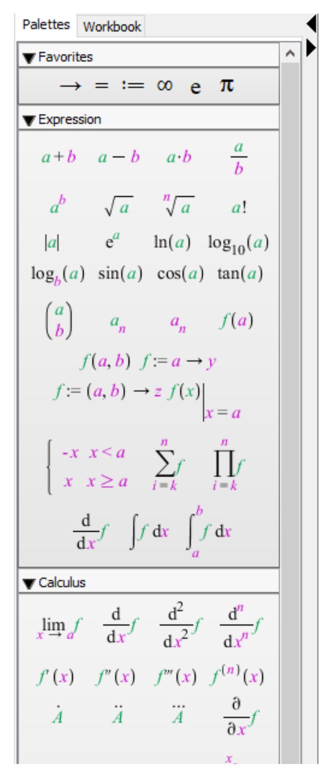


Figure A.4: The first few default palettes expanded.

B Basics of Maple Syntax

B.1 Algebraic Operations

At its elementary level, Maple can be used as a really large, powerful calculator. It can do any arithmetic operation including, but not limited to, addition (+), subtraction (-), multiplication (*), division (/), powers (^), and other more complicated operations such as radicals (roots), logarithms, and exponentials. Anytime Maple performs an arithmetic operation, it is creating an *expression*.

```
> a + b;
```

$$a+b$$

```
> a - b;
```

$$a-b$$

```
> a * b;
```

$$ab$$

In 2D Math mode (the default font), it is also possible to perform multiplication by including a space between two variables. However, do not use brackets for multiplication. Maple has no way of knowing whether $a(b)$ is meant as multiplication or as a function of b.

```
> a / b;
```

$$\frac{a}{b}$$

```
> a ^ b;
```

$$a^b$$

B.2 Maple Commands

If we wish to do anything more complicated than basic arithmetic in Maple, we likely need to use a special *command* to perform the desired task. Commands have two parts:

- the *command name*: this is usually one or more words with no spaces that describes what the command does.
- the *parameters* of the command: these are the objects that the command needs to be given so that it can complete its procedure.

The syntax of a command is as follows:

```
command( parameter1, parameter2, ...  )
```

Never include a space between the name of the command and the parentheses around the parameters. Maple will erroneously treat this space as multiplication

You should note that the parameters are always enclosed in parentheses `( )` and there is no space between the name of the command and the first parenthesis.

Some commands only need one parameter, while others need multiple parameters. In many cases, additional, optional parameters can be added to perform a more specialized procedure.

B.3 *Radical Functions*

For your first commands, you should know how to type a square root or other root into Maple. The `sqrt()` command can be used for square roots, while it is usually better to use `surd()` for higher roots.

```
> sqrt(a);
```

$$\sqrt{a}$$

```
> surd(a, 3);
```

$$\sqrt[3]{a}$$

```
> surd(a, 4);
```

$$\sqrt[4]{a}$$

The `sqrt()` command can also be found in the Expressions palette by clicking $\sqrt{a}$. However, the `surd()` command is better than using the $\sqrt[n]{a}$ button in most cases.

B.4 *Exponential Function*

Another very common command we'll be using is `exp()`. This is meant to be used as the exponential function. Unfortunately, simply typing in e^x using the 'e' key on the keyboard won't perform the correct operation. Instead, Maple treats 'e' as it does any other letter, such as 'x' or 'y'.

```
> exp(a);
```

$$e^a$$

The `exp()` command can also be found in the Expressions palette by clicking the e^a button. Similarly, you can get the numerical value e by clicking on its button in the Common Symbols palette.

B.5 *Other Common Functions*

There are a multitude of built-in functions that Maple possesses. All of these functions act like the functions that would exist in your calculator. They include, but are not limited to, trigonometric and inverse trigonometric functions, logarithms, exponentials, and radical functions.

Factorial notation ! works much like algebraic operations and does not need parentheses. However, a `factorial()` command is also available.

```
> !            > arcsec( )      > cot( )
> abs( )       > arcsin( )      > csc( )
> arccos( )    > arctan( )      > csch( )
> arccot( )    > ceil( )        > exp( )
> arccsc( )    > cos( )         > floor( )
```

```
> ln( )        > min( )       > sqrt( )
> log( )       > root( )      > surd( )
> log10( )     > sec( )       > tan( )
> max( )       > sin( )
```

Note that capitalization is important in Maple. Capitalizing a character that is not supposed to be capitalized will result in an error or useless output.

B.6 *Maple Help*

For more information on any of these or other commands (for example, the "sin" command), you can type a question mark followed by the command name:

```
> ?sin
```

A help menu will open, which describes the required parameters in the order they must be listed, as well as a few examples. These examples can be very useful for more complicated procedures. There may also be several options described for you to customize the procedure.

If you are unsure of the command name for a procedure that you think Maple should have built in, you can also search Maple help. For example,

```
> ?gcd
```

will give a description of commands for finding the greatest common divisor and lowest common multiple of two values.

B.7 *Autocompleting Commands*

For convenience, Maple can automatically fill in the command that you wish to type. To do this, type in the first few letters of the command and then hit ESC. For example, typing in `exp` and hitting ESC will provide the following list of commands. From here, you

Ctrl+Space is another shortcut for this feature, if you find it more convenient while typing.

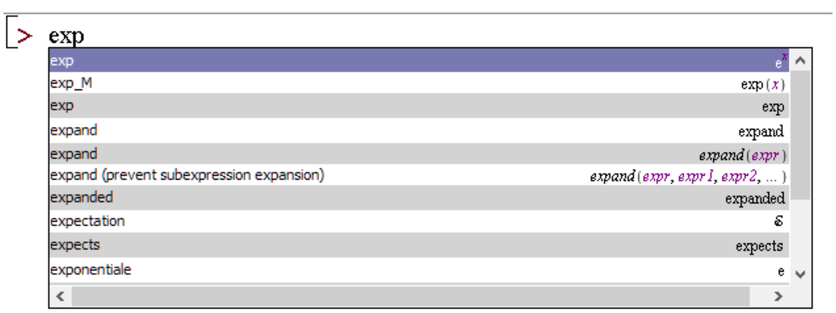


Figure B.1: Using autocomplete with ESC gives a menu of possible commands you are trying to use.

can select the `exp` command for the exponential function, or possibly the `expand` command, which we will look at later. If the command requires a parameter, it should already be highlighted so that you can type in its value. If a command has multiple parameters, you can hit the Tab key to highlight the next parameter.

B.8 Multiple Commands at Once

As you may have noticed, each time that we have used Maple input in this tutorial, there is a semicolon ; at the end of the line. This used to be required in older versions of Maple, though it is no longer mandatory in modern versions.

The reason why the semicolon continues to be important is because it tells Maple when one command ends and another begins. You can use this to run multiple commands within one Maple input:

```
> surd(x,3); exp(x);
```

$$\sqrt[3]{x}$$
$$e^x$$

Multiple commands can be run at the same time by placing a semicolon ; in between them.

If you use a full colon : instead of a semicolon, then the output of that command is hidden from the screen when it is executed.

B.9 The % Shortcut

Many of the exercises in the activities will involve executing multiple commands to obtain the answer. Often, this means running a command and then running a second command on the result of the first. Although copying and pasting the result from the first command takes less time than typing it out, it often causes many syntax problems.

Fortunately, Maple provides us with the % shortcut. Every time a command is run, its output is temporarily stored, much like a scientific calculator will remember what is currently on its screen. Using the % symbol within another command will use the result of the first command automatically.

The trouble with this shortcut comes from the fact that you can run Maple input anywhere on the page *in any order*! In the example below, you will only get the correct output if you run the second line *immediately* after running the first line.

```
> x^2 + 5;
```

$$x^2 + 5$$

```
> sqrt(%);
```

$$\sqrt{x^2 + 5}$$

To make better use of the % shotcut, it is the best practice to combine the two consecutive commands on the same Maple input:

```
> x^2 - 4; sqrt(%);
```

$$x^2 - 4$$
$$\sqrt{x^2 + 5}$$

The % shorcut in Maple works much like the *ANS* button on many scientific calculators. The % will only remember the output of whatever input was run most recently.

C Basic Operations

C.1 Expressing a Result as a Decimal

Maple tries to use exact, symbolic values whenever it can. If you need a decimal representation of a value or expression, you can use the `evalf()` command as seen below.

```
> sqrt(2);
```

$$\sqrt{2}$$

```
> evalf( sqrt(2) );
```

$$1.414213562$$

Maple will default to a decimal approximation anytime the input already uses decimals. For example, try `sqrt(2.0)` and see the result.

It is often useful to give the exact value as well as the decimal approximation in one execution group, using the % shorcut:

```
> sqrt(2); evalf(%);
```

$$\sqrt{2}$$

$$1.414213562$$

By default, Maple will express a decimal with 10-digit accuracy. This default can be changed by assigning a new value to `Digits`, or you can specify the number of digits anytime you use the `evalf()` command.

The first letter of `Digits` must be capitalized and there must be a colon included before the equals sign.

The assignment operator `:=` is explained in detail in Tutorial E on page 105.

```
> Digits := 15;
```

$$Digits := 15$$

```
> Pi;
```

$$\pi$$

```
> evalf(Pi);
```

$$3.14159265358979$$

```
> evalf(Pi,50);
```

$$3.1415926535897932384626433832795028841971693993751$$

C.2 Expanding Expressions

You can ask Maple to expand any expression with the `expand()` command, including products of polynomials and rational functions.

Maple will also expand various logarithmic and trigonometric expressions.

The multiplication operation * (or a space in 2D Math mode) is not required between numerical coefficients and variable. However, multiplication between multiple variables is required.

```
> expand( (2*x + 3*y)^6 );
```

$$64\,x^6 + 576\,x^5 y + 2160\,x^4 y^2 + 4320\,x^3 y^3 + 4860\,x^2 y^4 + 2916\,x y^5 + 729\,y^6$$

```
> expand( cos(2*x) );
```

$$2\,(\cos(x))^2 - 1$$

```
> expand( tan(a + b) );
```

$$\frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

C.3 *Factoring Expressions*

Maple can also factor expressions with the `factor()` command. It will factor polynomials (even those with multiple variables) and will even factor more complicated expressions (like those involving trig functions as seen below).

```
> factor( x^2 - 1 );
```

$$(x-1)(x+1)$$

In some cases, Maple may factor the expression and perform some additional basic simplification.

```
> factor( (x^2 + 2*x - 15)/(x^2 + 7*x + 10) );
```

$$\frac{x-3}{x+2}$$

```
> factor( sin(x)^2 - 3*sin(x) + 2 );
```

$$(\sin(x) - 1)(\sin(x) - 2)$$

If Maple cannot factor the expression given, it will output the original expression.

```
> factor( x^2 + 1);
```

$$x^2 + 1$$

Maple will factor expressions with multiple variables as well. Be sure to include multiplication (*) between variables.

```
> factor( a^3 + 3*a^2*b + 3*a*b^2 + b^3);
```

$$(a+b)^3$$

C.4 *Simplifying Expressions*

Likewise, you can also simplify any expression with `simplify()`. This includes basic simplifications such as collecting like terms as well as more complicated algebraic simplifications such as canceling and simplifying radicals, exponents, logarithms, trigonometric functions, etc.

```
> simplify( 3*sin(x)^2 + 3*cos(x)^2 );
```

$$3$$

The `simplify()` command can sometimes produce unexpected results. In some cases, the `factor` command may be more appropriate. In other cases, you may need to include some optional parameters in the command.

```
> simplify( 4*(tan(x)^2 + 1) );
```

$$4\,(\cos(x))^{-2}$$

```
> simplify( ln(3*x^3*y) );
```

$$\ln(3) + \ln\left(x^3 y\right)$$

```
> simplify( ln(3*x^3*y), assume=positive );
```

$$\ln(3) + 3\ \ln(x) + \ln(y)$$

Sometimes additional parameters need to be supplied in order for Maple to simplify the expression as you intend. Here, we add the assumption that all variables are positive so that $\ln(x)$ and $\ln(y)$ are defined.

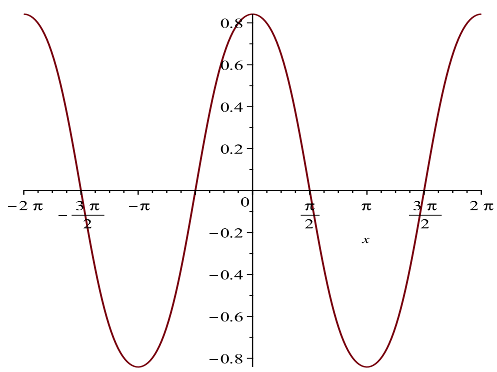
D Basics of Plotting

D.1 Plotting a Single Function

You can plot a function in Maple by using the `plot()` command. The only required parameter is the function you wish to plot. However, there are many additional parameters you can add to customise the way that the graph looks.

By default, Maple will plot trigonometric functions with an x-axis from -2π to 2π.

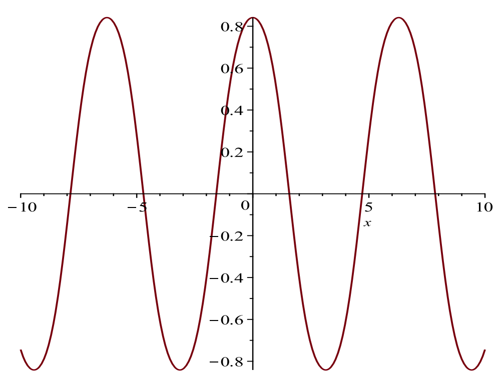
```
>  plot( sin(cos(x)) );
```



It is often useful to specify the interval for the x-axis. Choose left and right endpoint for the x-axis that are appropriate for your graph.

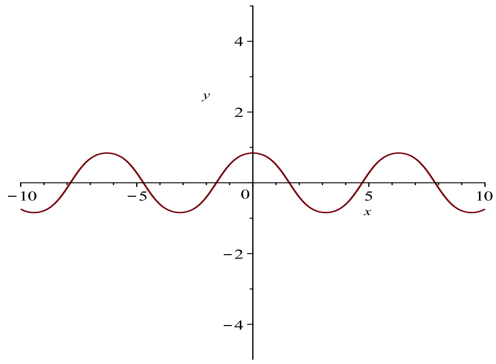
If you are plotting a function of t, then make sure to specify the interval as `t=a..b`.

```
>  plot( sin(cos(x)), x=-10..10);
```



You can also specify the interval for the y-axis for your graph.

```
>  plot( sin(cos(x)), x=-10..10, y=-5..5);
```



D.2 *Common Plot Options*

Table J.1 lists the most frequently used optional parameters.

Table D.1: A list of common optional parameters for the `plot()` command.

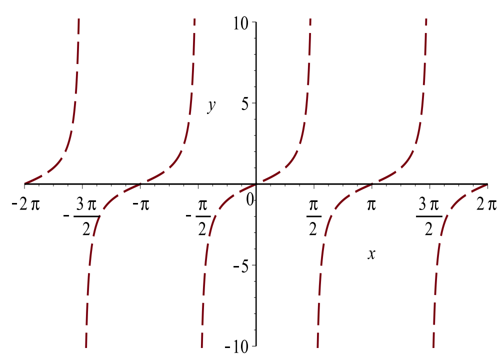
Parameter	Description
`x=a..b`	Plot over the interval $x \in [a, b]$.
`y=c..d`	Plot over the interval $y \in [c, d]$.
`colour=`*cname*	Specify the colour of the graph.
`discont=true`	Show discontinuities in a function.
`linestyle=`*lstyle*	Specify the style of the line (`solid`, `dash`, `dot`, etc.).
`gridlines=true`	Include gridlines.
`numpoints=`*n*	Increase the minimum number of points plotted for a smoother graph (Default 200).
`grid=[`m`,`n`]`	Set the number of initial points used to plot a graph (Default 26×26).
`scaling=constrained`	Force axes to use the same scale (so a circle should appear perfectly round).

A list of plot colours can be found by typing `?colours` on a new Maple input.

Choosing an extremely large value for `numpoints` may cause the output to take a long time to generate. A value of 30000 will likely be sufficient for complicated graphs (see Tutorial I).

To plot $\tan(x)$ properly, you should include the `discont=true` option, which properly indicates discontinuities in the function.

```
> plot(tan(x), x=-2*Pi..2*Pi, y=-10..10, linestyle=dash,
discont=true);
```



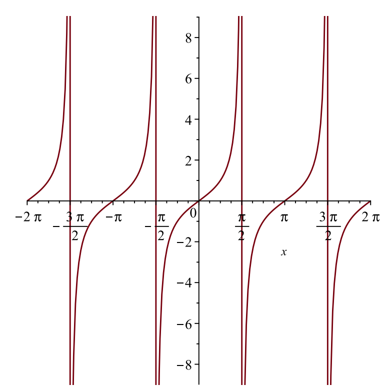


Figure D.1: Plotting $\tan(x)$ without the additional options.

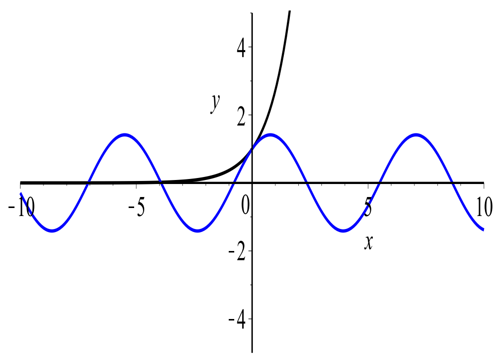
D.3 *Plotting Multiple Functions*

You can also plot multiple expressions and functions on the same graph using the `plot()` command. Instead of specifying a single function as your first parameter, create a list of functions using square brackets.

Similarly, instead of specifying a single colour, specify a list of colours using square brackets. The order of the functions matches the order of the colours.

If a list of colours is not specified for multiple graphs, it may be difficult to match the graphs to the corresponding function. The first three colours in Maple's default palette are dull shades of red, blue, and green.

```
> plot( [exp(x), sin(x) + cos(x)], x=-10..10, y=-5..5,
colour=[black, blue]);
```



If you need several complicated graphs to display at once, you may wish to explore the potential of the `display()` command, found in section E.8 on page 110.

E The Assignment Operator and Defining Functions

E.1 Assigning Expressions

Having to retype the same expression multiple times is tedious, but using copy and paste in Maple can sometimes produce unwanted effects. A better way to reuse an expression multiple times is to assign a name to it. You can assign any expression to a name of your choice (with some exceptions that Maple has protected) by using the := operator.

Protected names include common functions such as exp. For example,

```
> exp := 2*x;
```

would cause an error.

```
> poly := 3*x^4 - 2*x + 1;
```

$$poly := 3\,x^4 - 2\,x + 1$$

```
> poly;
```

$$3\,x^4 - 2\,x + 1$$

```
> poly^2;
```

$$\left(3\,x^4 - 2\,x + 1\right)^2$$

Never assign anything to single-letter names such as x or y. It is best to keep single letters as arbitrary variables.

```
> expr := ( 4^x - x^4 ) / exp(x + 1);
```

$$expr := \frac{4^x - x^4}{e^{x+1}}$$

```
> expr;
```

$$\frac{4^x - x^4}{e^{x+1}}$$

```
> sqrt(expr);
```

$$\sqrt{\frac{4^x - x^4}{e^{x+1}}}$$

It is important to assign expressions to names that make sense to you and are easy to remember. It is also recommended not to reuse a name in the same document. If you assign a new expression to an old name, the new expression will overwrite what was previously assigned.

E.2 Making a Substitution into an Expression

Let's suppose you have assigned an expression a name, and wish to replace one of its variables with a value or expression. As an example, we will assign an expression a name of `expr` and then substitute the numerical value for π, which is denoted as `Pi` in

Maple, into `expr`. The command used to substitute a value into an expression is `subs()`.

```
> expr := sin(x) - 1;
```

$$expr := \sin(x) - 1$$

```
> subs(x = Pi, expr);
```

$$\sin(\pi) - 1$$

Always be sure to use a capital P in `Pi` for the mathematical constant. You can also find π in the palettes toolbar.

You can make use of the % shortcut if you wish, but recall that it is best used on the same Maple input:

```
> x^2 + 3*x - 4; subs(x = 2, %);
```

$$x^2 + 3x - 4$$

$$6$$

You can also substitute one expression into another:

Note that using the `subs()` command does not permanently assign the substitution.

```
> expr2 := tan(2*x) + 1;
```

$$expr2 := \tan(2x) + 1$$

```
> subs(x = a+h, expr2);
```

$$\tan(2a + 2h) + 1$$

E.3 *Defining a Function*

Instead of defining an expression, it is sometimes more useful to define a function. A function specifies the variables that are included in the expression. This allows us to substitute values into our functions easily using function notation, such as $f(5)$.

If you have properly defined a function, you should see an arrow ($\rightarrow$) in your output. If you do not see an arrow within the output, then you have not defined the function properly.

```
> f(x) := sin(x) - exp(x);
```

$$f := x \mapsto \sin(x) - e^x$$

```
> g(t) := -4.9*t^2 + 5*t + 20
```

$$g := t \mapsto -4.9\,t^2 + 5\,t + 20$$

Make sure that variable in the function name matches the variable in the function.

A function's name does not have to be a single letter. In fact, it is often a good idea to have a function's name correspond to what the function represents.

```
> area(r) := Pi*r^2
```

$$area := r \mapsto \pi r^2$$

Figure E.1: You can find a shortcut for defining functions in the palettes toolbar. This button makes use of the arrow notation.

E.4 *Using Function Notation*

Once we have defined a function, we can use function notation much like we would in class. For example, instead of defining an expression involving x to a name such as `expr` and then using the `subs()` command to substitute $x = 0$ into `expr`, we can now use the function $f(x)$ as assigned in Section E.3 and type $f(0)$.

```
> f(0);
```

$$-1$$

It is easy to substitute an expression into a function using function notation.

```
> g(x) := 2*x^2 - 3*x
```

$$g := x \mapsto 2\,x^2 - 3\,x$$

```
> g(x+1)
```

$$2\,(x+1)^2 - 3\,x - 3$$

E.5 *Operations on Functions*

Once one or more functions are assigned, we can make expressions using function notation, or perform other operations on those functions.

```
> f(x) := 2*x^3;
```

$$f := x \mapsto 2\,x^3$$

```
> g(t) := t+1;
```

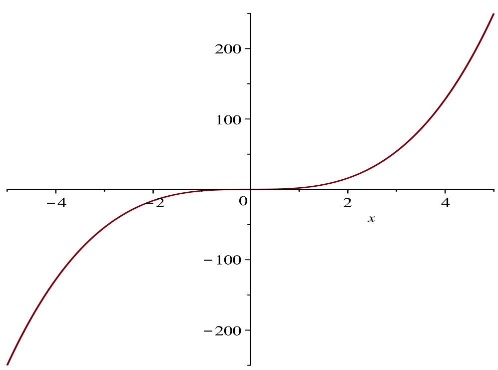
$$g := t \mapsto t+1$$

```
> f(g(t)); expand(%);
```

$$2\,(t+1)^3$$

$$2\,t^3 + 6\,t^2 + 6\,t + 2$$

```
> plot( f(x), x=-5..5 );
```



E.5.1 *Average Value of a Function over an Interval*

In this example, we will find the average rate of change of the function $f(x) = -2x^3 + 25x^2 + 15$ over the interval $[2,7]$. We begin by defining the function:

```
> f(x) := -2*x^3 + 25*x^2 + 15;
```

$$f := x \mapsto -2\,x^3 + 25\,x^2 + 15$$

Once the function is defined, we can calculate the average rate of change over an interval $[a, b]$ by using the formula

$$\frac{f(b) - f(a)}{b - a}.$$

In this case, we let $a = 2$ and $b = 7$:

This fraction will look much nicer on your screen if you use 2D Math font.

```
>  ( f(7) - f(2) )/( 7 - 2 );
```

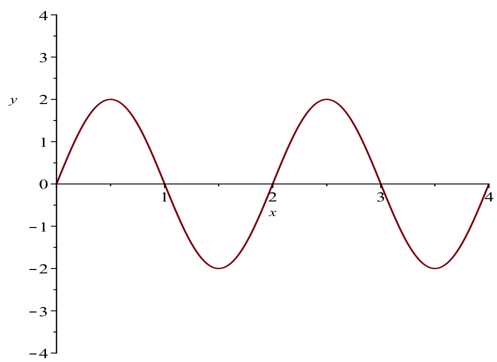
$$91$$

The average rate of change over the interval $[2, 7]$ is 91. The units of this rate would be given in (units of y)/(units of x).

E.5.2 *Plotting Transformations of Functions*

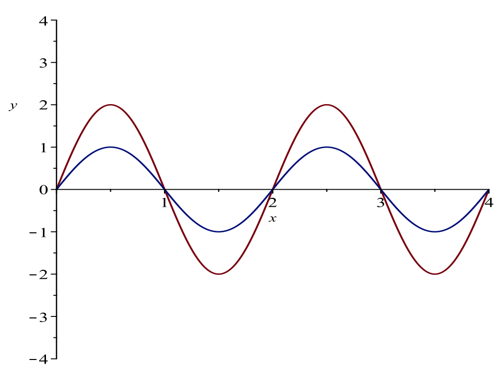
Suppose that we start with a sinusoidal function $g(x)$ with amplitude 2 and period 2.

```
>  g(x) := 2*sin( Pi*x );
>  plot( g(x), x=0..4, y=-4..4 );
```



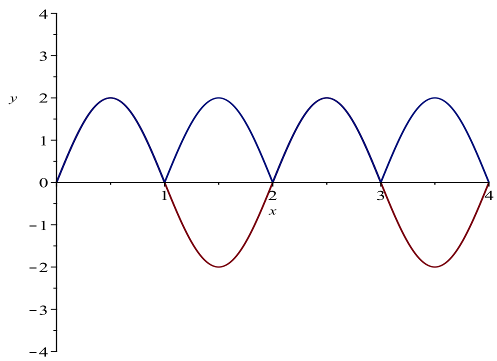
We can then plot the original function $g(x)$ and and the transformation $\frac{1}{2}g(x)$ on the same axes to see that the amplitude has been halved.

```
>  plot( [ g(x), 0.5*g(x) ], x=0..4, y=-4..4);
```



We can also see how an absolute value transformation of $g(x)$ compares to the original function. The result here is known as a fully rectified sine wave.

```
> plot( [ g(x), abs(g(x)) ], x=0..4, y=-4..4);
```



E.6 Assigning Piecewise Functions

A piecewise-defined function is a function defined by multiple sub-functions. Each sub-function applies to a certain interval of the domain, called a sub-domain. We can define piecewise functions in Maple by using the `piecewise()` command. The sub-domain of each sub-function is specified before its expression.

```
> P(x) := piecewise(x<=-1, x^2, x<=1, -x, 1<x, x-4);
```

$$P := x \mapsto \begin{cases} x^2 & x \le -1 \\ -x & x \le 1 \\ x-4 & 1 < x \end{cases}$$

```
> P(x);
```

$$\begin{cases} x^2 & x \le -1 \\ -x & x \le 1 \\ x-4 & 1 < x \end{cases}$$

E.7 Functions of More than one Variable

We can also define functions of multiple variables and use the notation the same way as one variable functions.

```
> g(x,y) := sin(x) - cos(y);
```

$$g := (x, y) \mapsto \sin(x) - \cos(y)$$

Evaluating at a point:

```
> g( 0, Pi);
```

$$1$$

An example of a function of more than one variable is the volume of a circular cylinder.

```
> cylindervol(r,h) := Pi*r^2*h;
```

$$cylindervol := (r,h) \mapsto \pi\, r^2 h$$

```
> cylindervol(3,5)
```

$$45\,\pi$$

E.8 Assigning Plots and the `display()` Command

You can use the assignment operator to assign just about any type of output to a variable name, including plots. This can be useful when different types of plots need to be displayed on the same graph. You can assign the output of several `plot()` commands into variables and then 'display' them all on the same set of axes.

To make use of the `display()` command, you need to include the `plots` package. Packages are loaded using the `with()` command, where the name of the package appears within the parentheses. The command

```
> with(plots);
```

$$[animate, animate3d, animatecurve \ldots]$$

will display all of the commands that are included in the package. If a full colon is added, then the package is loaded but the output is hidden.

```
> with(plots):
```

With the `plots` package loaded, plotting options can be defined for each plot individually and assigned to a name. Multiple plots can then be displayed together.

A package does not need to be loaded more than once in your document. However, you will need to reload the package if you have previously closed the document.

```
> p1(x) := (x+2)^2 + 2*(x+2) - 5;
> p2(x) := 3*(x-4)^3 - 2*(x-4)^2 - (x-4) + 1;
> p3(x) := x - 3;
```

$$p1 := x \mapsto (x+2)^2 + 2\,x - 1$$

$$p2 := x \mapsto 3\,(x-4)^3 - 2\,(x-4)^2 - x + 5$$

$$p3 := x \mapsto x - 3$$

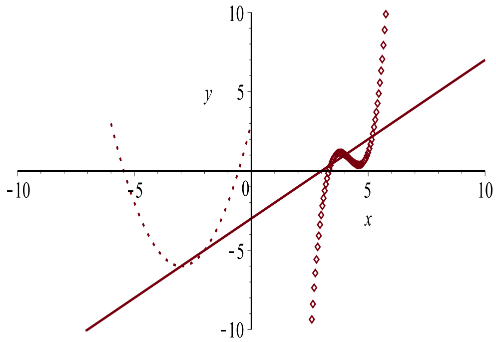
Full colons are used at the end of each line here to hide the output of the individual plot.

```
> A := plot( p1(x), x=-6..0, y=-6..3, linestyle=dot):

> B := plot( p2(x), x=-0..10, y=-10..10, style=point):

> C := plot( p3(x), x=-10..10, y=-10..10):

> display([A,B,C]);
```


10
5
y
-10
-5
0
5
10
x
-5
-10

F Equation Solvers

F.1 Assigning Equations

The assignment operator:= can be used to assign a name to nearly any type of output. Often, it is useful to assign an equation (involving a regular $=$ sign) a name. Some of the operations that we discussed in Tutorial C (such as simplifying, expanding, substituting, etc.) can then be applied to that equation.

```
> circle := x^2 + y^2 = 25;
```

$$circle := x^2 + y^2 = 25$$

Here we can see that the point $(3, 4)$ lies on the circle of radius 5, since $x = 3$ and $y = 4$ satisfy the equation.

```
> subs(x = 3, y = 4, circle);
```

$$25 = 25$$

```
> eqn := x^4 + 1 = 2*x^2;
```

$$eqn := x^4 + 1 = 2\,x^2$$

Here we can see that it is possible to add or subtract a value from both sides of an equation and factor the result.

```
> eqn - 2*x^2; factor(%);
```

$$x^4 - 2\,x^2 + 1 = 0$$

$$(x-1)^2\,(x+1)^2 = 0$$

F.2 Two Types of Solvers

We will make use of two different equations solvers in Maple:

- `solve()`
  - This solver attempts to solve an equation and then display the solutions in their exact form.
  - This solver will give both real and complex solutions.
  - Solutions to high-degree polynomials can be very large and may be displayed symbolically using "*RootOf*" placeholders.
- `fsolve()`
  - This solver uses numerical approximation methods and displays the solutions in decimal form.

Maple will give complex solutions using $I = \sqrt{-1}$. For most of the exercises in the activities, these solutions can be ignored.

- This solver will give real solutions with the number of digits assigned to `Digits`.
- Some solutions may not be found by the methods used by the solver.

It is a good idea to see the output of both solvers to decide which output is more useful. A good strategy is to use `solve()` and see if the output helpful. If it is not, then type an `f` at the start of that input and rerun the new command.

F.3 *Solving an Equation of One Variable*

The parameters of `solve()` and `fsolve()` are the same in most cases. You must include the equation to be solved and you can specify the variable that you wish to solve for.

```
> solve(x^2 = 5, x);
```

$$\sqrt{5}, -\sqrt{5}$$

```
> fsolve(x^2 = 5, x);
```

$$-2.236067977, 2.236067977$$

If there is only one variable in the equation, then it is not necessary to specify the desired variable.

It may not make sense to use *both* `solve()` and `fsolve()`. Choose the solver that produces the most useful output.

```
> solve(x^2 = 5);
```

$$\sqrt{5}, -\sqrt{5}$$

```
> fsolve(x^2 = 5);
```

$$-2.236067977, 2.236067977$$

If you provide `solve()` or `fsolve()` with an *expression* rather than an *equation*, then the solver will set that expression equal to 0 and solve the resulting equation.

Note that `solve(x^2 - 5)` is equivalent to typing `solve(x^2 - 5 = 0, x)`.

```
> solve(x^2 - 5, x);
```

$$\sqrt{5}, -\sqrt{5}$$

```
> fsolve(x^2 - 5, x);
```

$$-2.236067977, 2.236067977$$

Maple will give complex solutions using $I = \sqrt{-1}$ when using `solve()`. Typically, `fsolve()` will not display complex solutions.

```
> poly := 12*x^3-14*x^2+13*x-6;
```

$$poly := 12\,x^3 - 14\,x^2 + 13\,x - 6$$

```
> factor(poly = 0);
```

$$\left(4\,x^2 - 2\,x + 3\right)(3\,x - 2) = 0$$

```
> solve(poly = 0, x);
```

$$1/4 - I/4\sqrt{11}, 1/4 + I/4\sqrt{11}, 2/3$$

```
> fsolve(poly = 0, x);
```

$$0.6666666667$$

When trying to solve high-degree polynomials, solutions may be displayed symbolically using `solve()`, while `fsolve()` may display a more useful output.

```
> high := x^4 + 133*x + 200;
```

$$high := x^4 + 133\,x + 200$$

```
> solve(high);
```

$$RootOf\left(\_Z^4 + 133\,\_Z + 200, index = 1\right),$$
$$RootOf\left(\_Z^4 + 133\,\_Z + 200, index = 2\right),$$
$$RootOf\left(\_Z^4 + 133\,\_Z + 200, index = 3\right),$$
$$RootOf\left(\_Z^4 + 133\,\_Z + 200, index = 4\right)$$

This output is Maple's way of representing four solutions to the quartic symbolically. Switching to `fsolve()`, we see only two real solutions. The other two solutions are either complex or were not found using the methods used in `fsolve()`.

```
> fsolve(high);
```

$$-4.448682310,\ -1.546800745$$

When using the `fsolve()` command, you may also specify an interval in which to look for a solution.

In this example, solutions will only be found on the interval $[5, 10]$.

```
> fsolve(cos(x) = tan(x), x = 5..10);
```

$$6.949424740$$

F.3.1 *Finding the Intersection of Two Functions*

In this example, we will find the intersection point of $f(x) = x\ln(x)$ and $g(x) = \sin(x)$.

```
> f(x) := x*ln(x);
```

$$f := x \mapsto x\ln(x)$$

```
> g(x) := sin(x);
```

$$g := x \mapsto \sin(x)$$

Equating $f(x) = g(x)$, we can use either `solve()` or `fsolve()` to obtain a solution. Since this is an equation in x only, we will get the x-coordinate of the point.

```
> solve( f(x) = g(x) );
```

$$RootOf\left(\_Z\ \ln(\_Z) - \sin(\_Z)\right)$$

```
> fsolve( f(x) = g(x) );
```

$$1.752677281$$

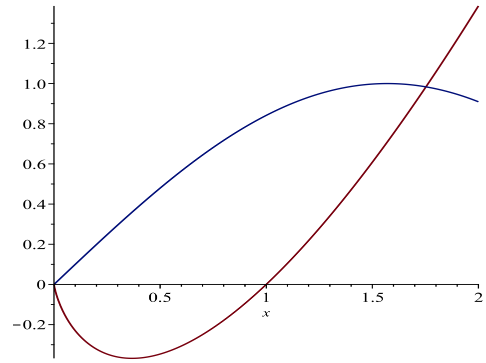
We can use this result and substitute into $f(x)$ or $g(x)$ to determine the corresponding y-coordinate.

```
> f(%);
```

$$0.9835052055$$

So, the point of intersection is $(1.752677281, 0.9835052055)$. This can be verified by plotting both functions.

```
> plot( [f(x), g(x)], x=0..2 );
```



Although it appears that there is another intersection point at $x = 0$, $f(0)$ is undefined.

F.4 Solving a System of Equations in Multiple Variables

We can also solve a system of equations by placing the various equations in a list (by using curly brackets) inside the `solve()` command.

```
> eq1 := x - y = 2;
```

$$eq1 := x - y = 2$$

```
> eq2 := y = x^2 - 4;
```

$$eq2 := y = x^2 - 4$$

```
> solve( {eq1, eq2}, {x, y} );
```

$$\{x = 2, y = 0\}, \{x = -1, y = -3\}$$

F.4.1 Finding the Intersection of Two Functions (Continued)

Using a system of equations, we can complete the example from F.3.1 in a single command.

```
> solve( {y = x*ln(x), y = sin(x)}, {x,y} );
```

$$\{x = RootOf\left(\_Z \ln\left(\_Z\right) - \sin\left(\_Z\right)\right),$$
$$y = \sin\left(RootOf\left(\_Z \ln\left(\_Z\right) - \sin\left(\_Z\right)\right)\right)\}$$

Once again, we find that `fsolve()` provides a more useful output.

```
> fsolve( {y = x*ln(x), y = sin(x)}, {x,y} );
```

$$\{x = 1.752677281, y = 0.9835052061\}$$

G Limits

G.1 Limits

We can use the `limit()` command to evaluate the limit of a function as x approaches a. The `limit()` command needs two parameters. The first parameter is the expression and the second parameter gives the value for a variable to approach.

```
> f(x) := x^2 + 2*x -4;
```

$$f := x \mapsto x^2 + 2\,x - 4$$

```
> limit( f(x), x=3);
```

$$11$$

```
> limit( ( f(x+h) - f(x) )/h, h=0);
```

$$2x + 2$$

It is important to note that $h = 0$ here means that h *approaches* 0, but we are not simply substituting $h = 0$ into the expression.

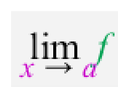


Figure G.1: You can find a shortcut for limits on the palettes toolbar.

G.2 One-Sided Limits

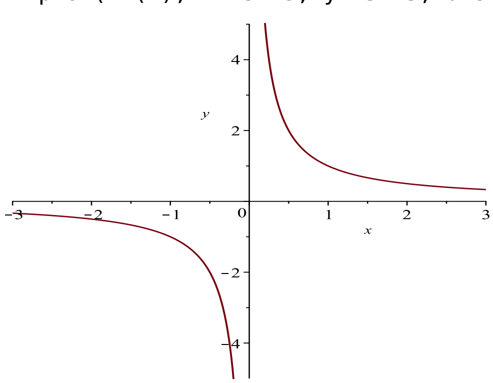
For one-sided limits, you will need to add an additional parameter to the `limit()` command, specifying which side (left or right) to approach the value from. In the case of a vertical asymptote, these limits will be equal to $\pm\infty$.

```
> L(x) := 1/x;
```

$$L := x \mapsto x^{-1}$$

```
> plot( L(x), x=-3..3, y=-5..5, discont=true );
```

It is best to include the `discont=true` option to see discontinuities in the function.



```
> limit( L(x), x=0, right);
```

$$\infty$$

```
> limit( L(x), x=0, left);
```

$$-\infty$$

G.2.1 *Vertical Asymptotes and One-Sided Limits*

In this example, we will examine a rational function and use limits to determine its vertical asymptotes.

```
> f(x) := (x^2-x-6)/(x^2-8*x+15);
```

$$f := x \mapsto \frac{x^2 - x - 6}{x^2 - 8\,x + 15}$$

We can factor the denominator to find the domain of $f(x)$ and predict where we might find vertical asymptotes.

There is a useful `denom()` command that Maple provides. You can type `denom(f(x))` to get the denominator of $f(x)$.

```
> factor(x^2-8*x+15);
```

$$(x - 3)\,(x - 5)$$

It looks like $x = 3$ and $x = 5$ are not in the domain of $f(x)$. We can find the limit of $f(x)$ as $x \to 3$.

```
> limit( f(x), x=3 );
```

$$-5/2$$

Since this limit exists but $f(3)$ does not, this is a *removable discontinuity* and not a vertical asymptote. Now we can find the limit of $f(x)$ as $x \to 5$.

```
> limit( f(x), x=5 );
```

$$\mathit{undefined}$$

Even though this limit does not exist, we cannot automatically conclude that $f(x)$ has a vertical asymptote at $x = 5$. We need to compute the one-sided limits to see if there is asymptotic behaviour.

Maple provides an `Asymptotes()` command that you can investigate using Maple help. Try typing `?Asymptotes` to learn more.

```
> limit( f(x), x=5, left);
> limit( f(x), x=5, right);
```

$$-\infty$$

$$\infty$$

Since these limits are given as $\pm\infty$, we know that $f(x)$ has a vertical asymptote at $x = 5$.

G.3 *Limits at Infinity*

To take the limit of a function as x becomes infinitely large, Maple recognizes `infinity` and `-infinity`. These can be used to find horizontal asymptotes. If the function does not have a horizontal asymptote, the limit may result in $\pm\infty$.

```
> g(x) := (3*x^2 + 5*x - 10) / (5*x^2 - 20*x + 1);
```

$$g := x \mapsto \frac{3\,x^2+5\,x-10}{5\,x^2-20\,x+1}$$

```
> limit( g(x), x=infinity);
```

$$3/5$$

An oscillating function such as $\sin(x)$ may not have a definable limit. Maple will attempt to determine a range for the minimum and maximum of the function.

```
> h(x) := sin(x);
```

$$h := x \mapsto \sin(x)$$

```
> limit( h(x), x=-infinity);
```

$$-1\ldots1$$

Since $\sin(x)$ oscillates between -1 and 1, Maple cannot determine a unique value for the limit as $x \to -\infty$.

G.3.1 *Horizontal Asymptotes and Limits at Infinity*

In this example, we will examine the function $f(t) = \frac{2000}{1+e^{-t+2}}$, which is known as a logistic function.

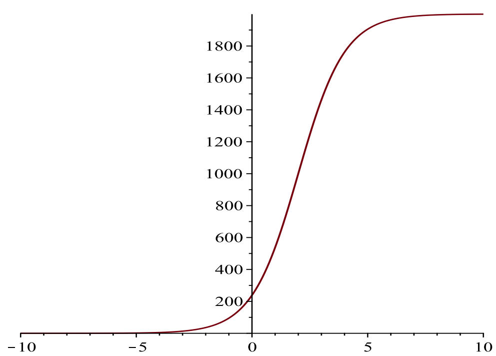
```
> logistic(t) := 2000/(1 + exp(-t+2))
```

$$logistic := t \mapsto \frac{2000}{1+e^{-t+2}}$$

Judging by the plot of the logistic function, it appears that the function may have horizontal asymptotes.

Logistic functions have many applications, such as population modeling.

```
> plot(logistic(t));
```



To find the right-hand asymptote, we take the limit as $t \to \infty$.

Here we are using `t=infinity` rather than x, since the variable of this function is t.

```
> limit( logistic(t), t=infinity );
```

$$2000$$

To find the left-hand asymptote, we take the limit as $t \to -\infty$.

```
> limit( logistic(t), t=-infinity );
```

$$0$$

G.4 *Limits and Piecewise Functions*

A piecewise function is a good opportunity to practice plotting discontinuities and investigating one- and two-sided limits.

```
> P(x) := piecewise(x<=-1, x^2, x<=1, -x, 1<x, x-4);
```

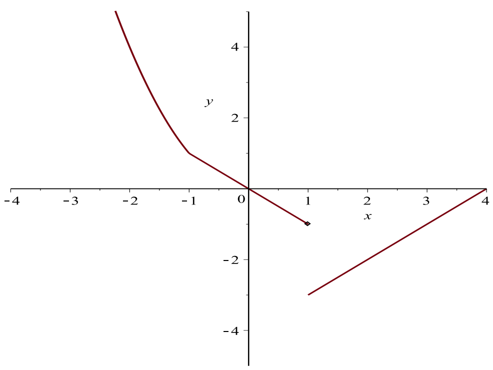
$$P := x \mapsto \begin{cases} x^2 & x \leq -1 \\ -x & x \leq 1 \\ x-4 & 1 < x \end{cases}$$

```
> P(x);
```

$$\begin{cases} x^2 & x \leq -1 \\ -x & x \leq 1 \\ x-4 & 1 < x \end{cases}$$

```
> plot( P(x), x=-4..4, y=-5..5, discont=true);
```

Unfortunately, even with the `discont=true` option, Maple does not include an open dot at $(1, -3)$.




```
> limit( P(x), x=1);
```

$$\mathit{undefined}$$

```
> limit( P(x), x=1, left);
```

$$-1$$

```
> limit( P(x), x=1, right);
```

$$-3$$

G.5 *The Limit Methods Tutor*

The Limit Methods tutor will walk you through each step needed to evaluate a limit, including all of the limit laws learned in class. The tutor will open in a new interactive window and will output all steps in your Maple worksheet once the window is closed.

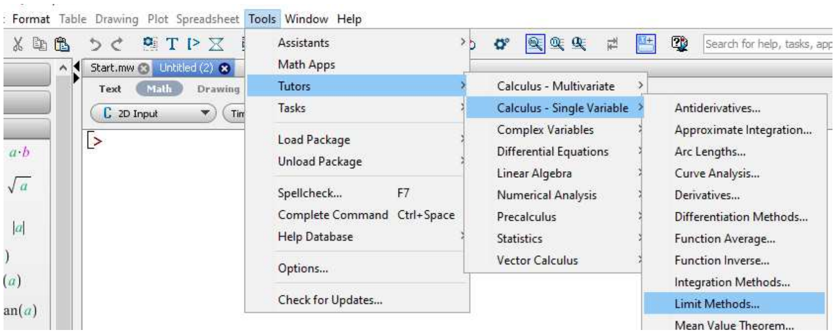


Figure G.2: Opening up the Limit Methods tutor using menus.

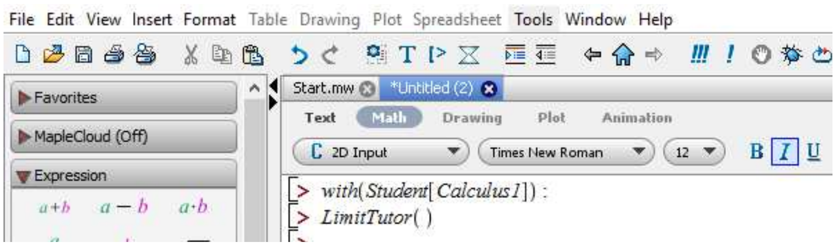


Figure G.3: Opening up the Limit Methods tutor using commands. The `Student[Calculus1]` package is required.

G.5.1 *Using Limit Laws for a One-sided Limit*

This example will illustrate all of the steps required to evaluate $\lim\limits_{x\to 2^+} \dfrac{x-2}{x^2-x-2}$.

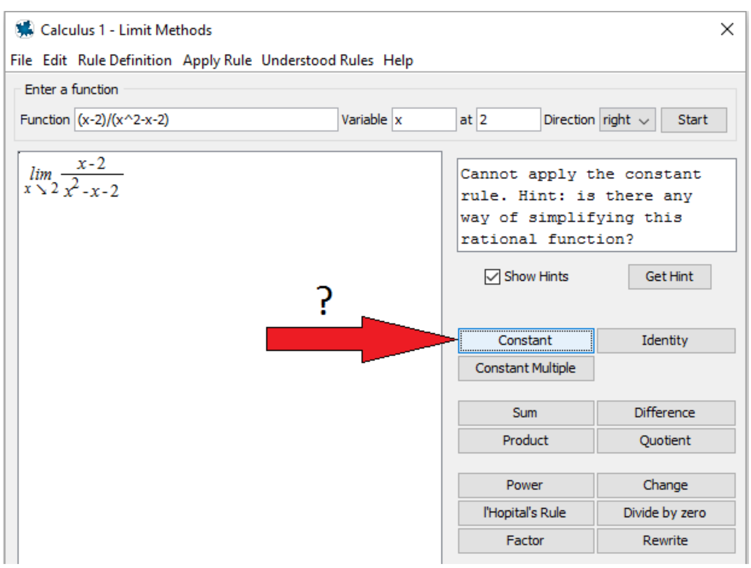


Figure G.4: Begin by typing out the function, the variable name, and the value that you want the variable to approach. The direction can be specified in the drop down menu to the right of the variable information. Hit Start. You can click on individual limit laws to see whether they apply to the given limit.

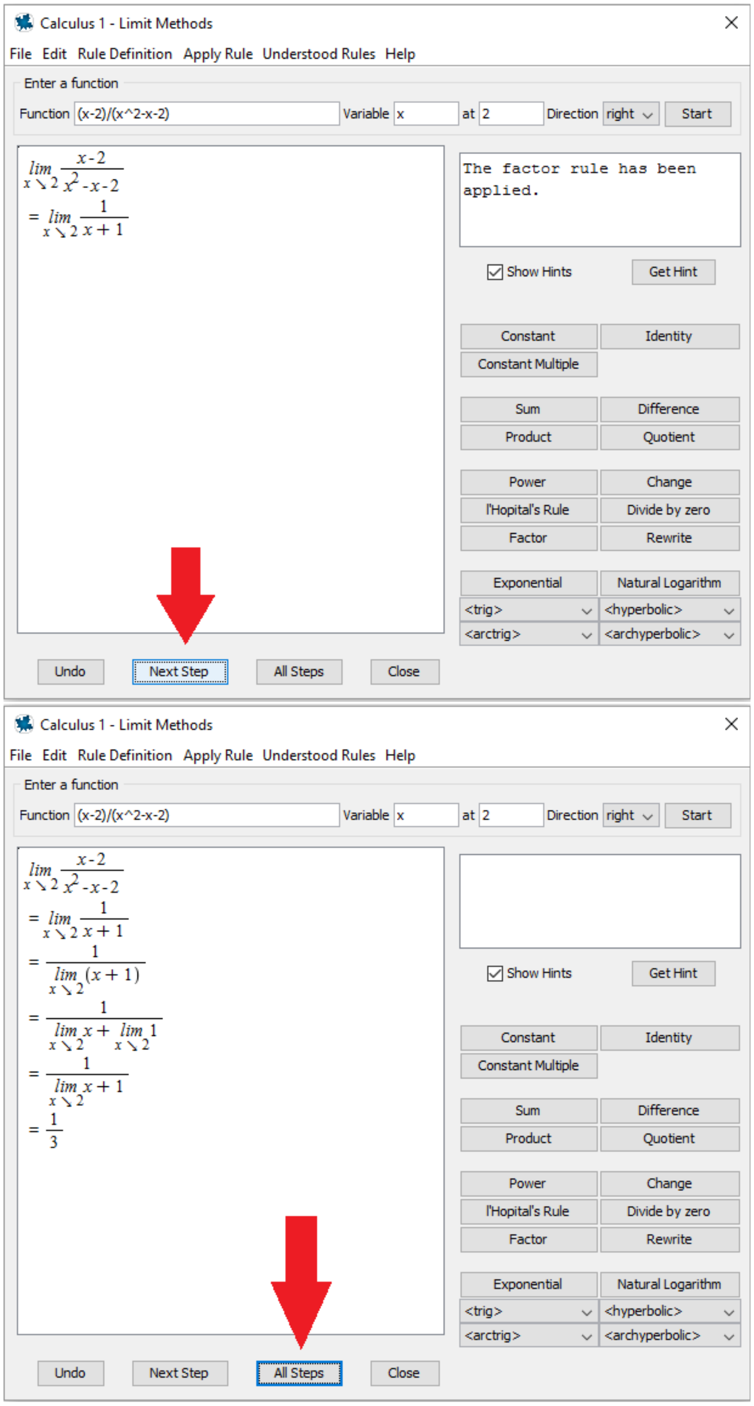


Figure G.5: You can press the Next Step or All Steps buttons to have Maple show you a step-by-step solution.

G.5.2 *Using Limit Laws for a Limit at Infinity*

In this example, we will see all of the limit laws used to evaluate
$$\lim_{x\to-\infty} \frac{2x-1}{\sqrt{x^2-x+3}}.$$

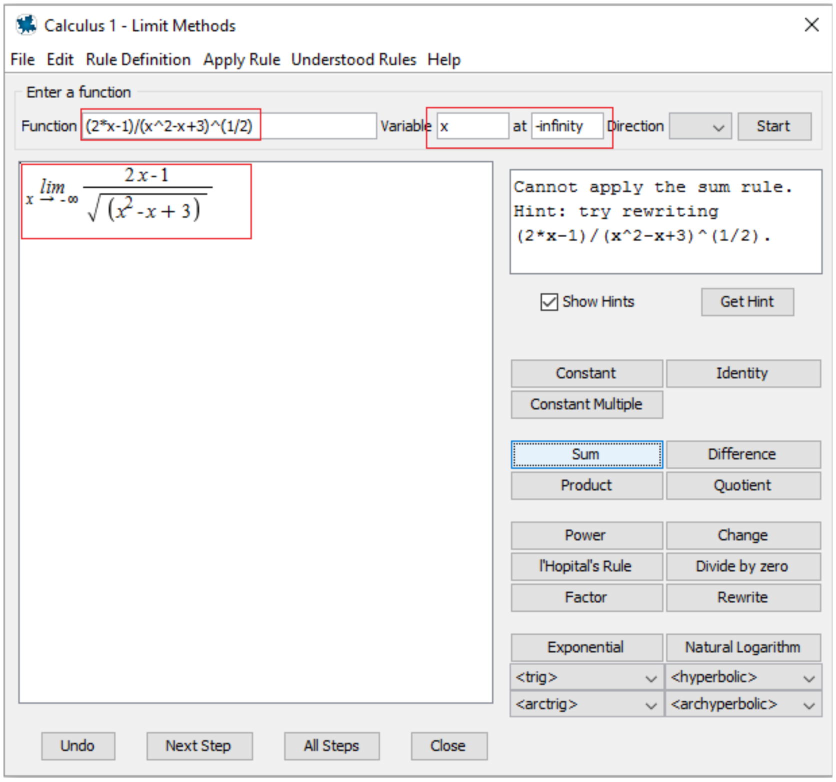


Figure G.6: For a limit as $x \to \infty$ or $x \to -\infty$, Maple recognizes the word `infinity`.

An alternate method for inputting the function into the tutor involves the `sqrt()` command.

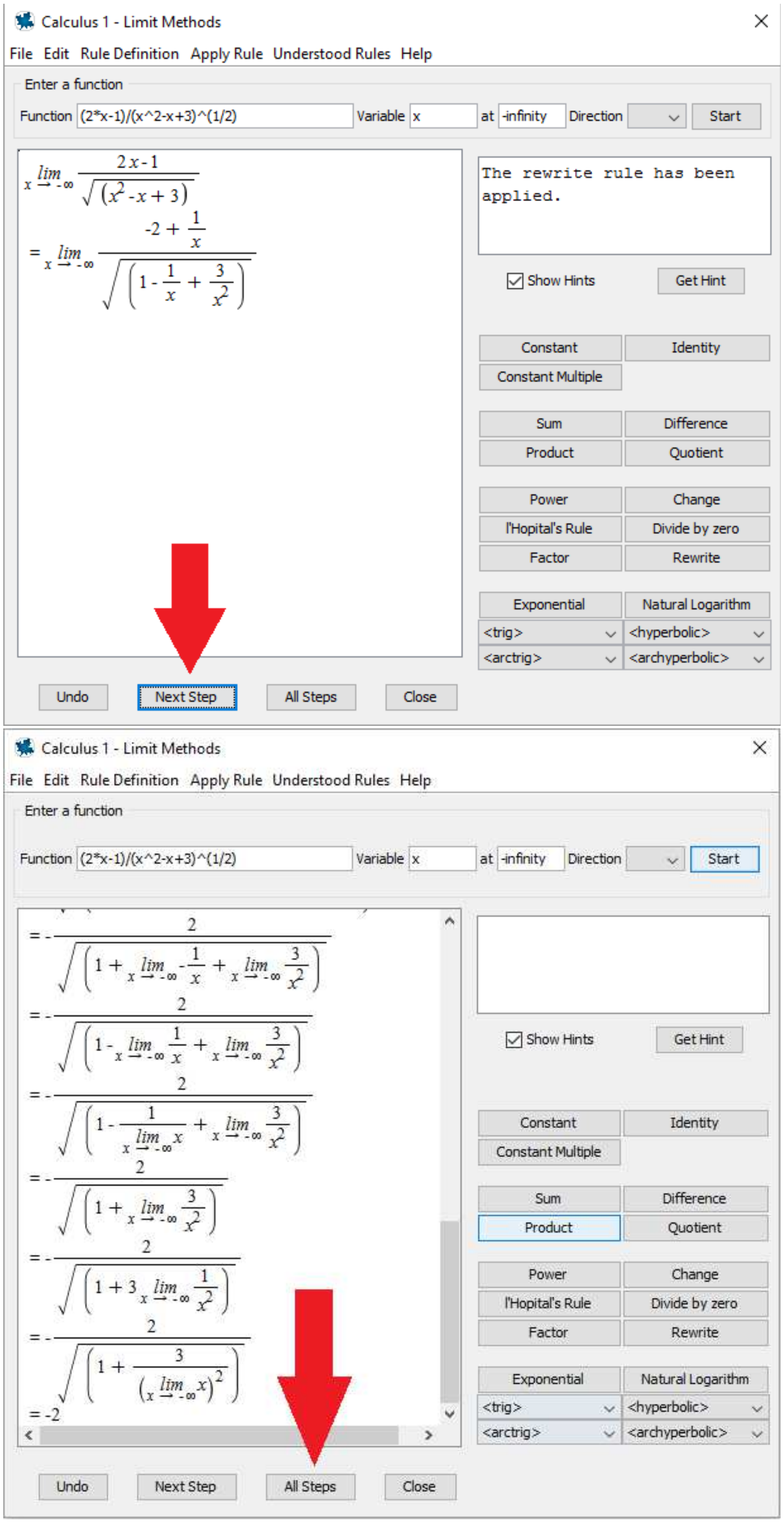


Figure G.7: You can click the Next Step or All Steps buttons to have Maple show you a step-by-step solution.

H The Derivative

H.1 The `diff()` Command

The `diff()` command is probably the most basic way of finding the derivative of an expression. The first parameter is the expression to be differentiated and the second parameter is the variable that the expression is to be differentiated with respect to.

```
> diff( arctan(t), t);
```

$$\frac{1}{t^2+1}$$

If you have assigned a function, then make sure to use proper function notation inside the `diff()` command.

```
> f(x) := sin(x);
```

$$f := x \mapsto \sin(x)$$

```
> deriv1 := diff( f(x), x);
```

$$deriv1 := \cos(x)$$

Make sure that you are taking the derivative with respect to the desired variable.

```
> slope := subs( x = Pi/2, deriv1); simplify(%)
```

$$slope := \cos(\pi/2)$$

$$0$$

H.1.1 Higher Derivatives using `diff()`

Higher derivatives can be evaluated by applying the `diff()` command multiple times, by specifying the variable repetitively, or using the $ notation, as shown below.

```
> diff(arctan(t), t); diff(%,t);
```

$$\frac{1}{t^2+1}$$

$$-\frac{2t}{(t^2+1)^2}$$

```
> diff( f(x), x, x);
```

$$-\sin(x)$$

```
> diff( f(x), x$2);
```

$$-\sin(x)$$

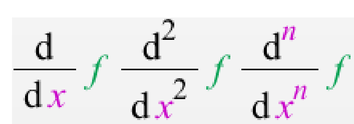


Figure H.1: You can also make use of quick shortcuts from the Calculus palette.

H.2 Using Function Notation

If you have properly defined a function $f(x)$, you may also make use of the familiar $f'(x)$ notation used in class.

```
> f(x) := sin(x) + x^2;
```

$$f := x \mapsto \sin(x) + x^2$$

```
> deriv1 := f'(x);
```

$$deriv1 := \cos(x) + 2x$$

This notation is especially useful for evaluating the derivative at a value, without using the `subs()` command separately.

```
> slope1 := f'(0);
```

$$deriv1 := 1$$

```
> slope2 := f'(Pi);
```

$$slope2 := -1 + 2\pi$$

While using m is a common choice for slope, it is a good idea to avoid overusing it in your Maple worksheet.

H.2.1 Higher Derivatives using Function Notation

Higher derivatives are also notated in much the same way as in class. Rather than using a string of tick marks, we use a raised exponent in brackets to specify the n^{th} derivative.

```
> deriv2 := f"(x);
```

$$deriv2 := -\sin(x) + 2$$

```
> deriv3 := f(3)(x);
```

$$deriv3 := -\cos(x)$$

H.3 Applications of the Derivative

In these examples, we will use the function notation method for derivatives, though the `diff()` command may also be used.

H.3.1 Finding the Equation of a Tangent Line

In this example, we will find the equation of the tangent line to the function $f(x) = 6\sqrt{x} - 2x$ at $x = 4$. We start by assigning the function a name.

```
> f(x) := 6*sqrt(x) - 2*x;
```

$$f := x \mapsto 6\sqrt{x} - 2x$$

```
> f'(x);
```

$$\frac{3}{\sqrt{x}} - 2$$

The y-coordinate and the slope can be found by substituting $x = 4$ into the function and its derivative, respectively.

```
> f(4);
```

$$4$$

```
> f'(4); simplify(%);
```

$$\frac{3}{4}\sqrt{4}-2$$

$$-\frac{1}{2}$$

Sometimes Maple output can be easily simplified, such as $\sqrt{4}$ here. Alternatively, an `evalf(%)` command would produce a decimal output.

The equation of the tangent line at $x = a$ is

$$L(x) = f'(a)(x-a) + f(a).$$

In this case, the tangent line is at $x = 4$.

The equation of a tangent line is a major topic of Math 112. Make sure you know this equation well!

```
> line := f'(4)*(x-4) + f(4); expand(line);
```

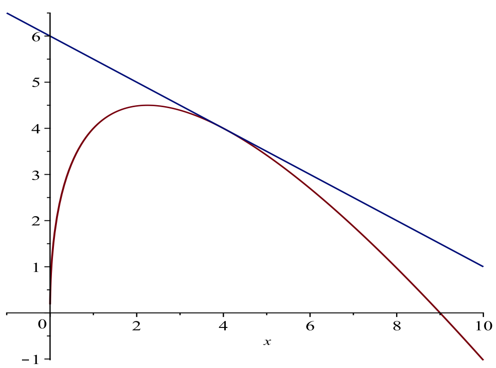
$$line := \left(\frac{3}{4}\sqrt{4}-2\right)(x-4)+4$$

$$-\frac{1}{2}x+6$$

After expanding the equation, we have the equation of a line in standard $y = mx + b$ form.

Notice that the line is only defined as an expression and is not in function notation. We can now plot the function and the line.

```
> plot( [f(x),line], x=-1..10);
```



It is a good idea to specify plot colours, especially if plotting more than one tangent line on the same axes.

H.3.2 *The Closed Interval Method for Min/Max Problems*

In this example, we will find the absolute minimum and maximum values of

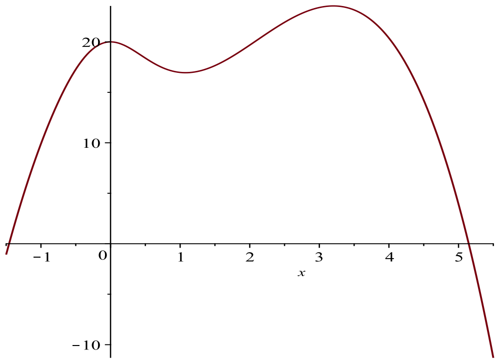
$$f(x) = \frac{-x^4+5x^3+20}{\sqrt{x^2+1}}$$

on the interval $[-1, 5]$. It is best to define the function and plot it first to get an idea of where the critical numbers are located.

```
> f(x) := (20 + 5*x^3 - x^4)/sqrt(x^2 + 1);
```

$$f := x \mapsto \frac{-x^4+5\,x^3+20}{\sqrt{x^2+1}}$$

```
> plot(f(x), x=-1.5..5.5);
```



By factoring the derivative, we can see that $x = 0$ is a critical number. We can use the `fsolve()` command over smaller intervals to find the remaining critical numbers.

```
> factor(f'(x));
```

$$-\frac{x\left(3x^4 - 10x^3 + 4x^2 - 15x + 20\right)}{\left(x^2+1\right)^{3/2}}$$

```
> CN1:=0;
```

$$CN1 := 0$$

```
> CN2:=fsolve(f'(x)=0, x=1..2);
```

$$CN2 := 1.078091128$$

```
> CN3:=fsolve(f'(x)=0, x=3..4);
```

$$CN3 := 3.201521345$$

If a closed interval was not specified, we could use the second derivative test to find local minima and maxima.

```
> f"(CN1); f"(CN2); f"(CN3);
```

$$-20$$
$$8.886020129$$
$$-8.221546325$$

Since $f(x)$ is concave down at `CN1` and `CN3`, we have found two local maxima. Since $f(x)$ is concave up at `CN2`, we have found one local minimum.

To apply the closed interval method, we must evaluate the function at all critical numbers in the interval as well as the two endpoints.

```
> f(CN1); f(CN2); f(CN3);
```

$$20$$
$$16.94310958$$
$$23.55848432$$

```
> f(-1):  evalf(%);
```

$$9.899494934$$

```
> f(5):  evalf(%);
```

$$3.922322703$$

The full colon : here hides the output of inputting the endpoint into the function. Only the output of the `evalf(%)` command is displayed.

Comparing these values, we see that $x = 3.201521345$ gives an absolute maximum of 23.55848432 and that $x = 5$ gives an absolute minimum of 3.922322703 on the interval $[-1, 5]$.

H.4 *The Derivatives Tutor*

The Derivatives tutor is a useful way for visualising the graphs of the first (and second) derivatives of a given function.

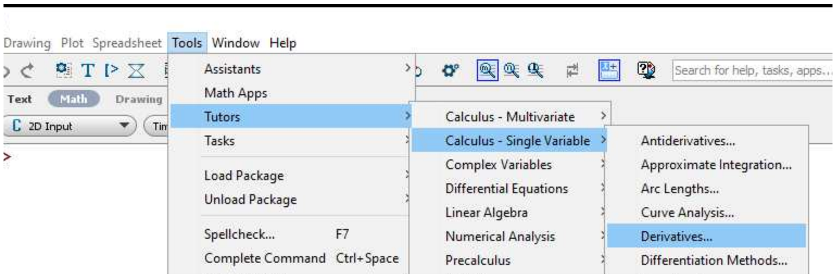


Figure H.2: Opening up the Derivatives tutor using menus.

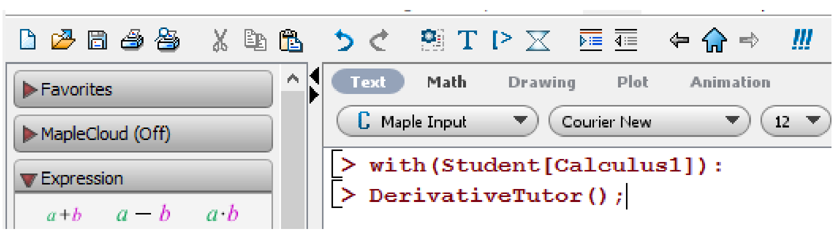


Figure H.3: Opening up the Derivatives tutor using commands. The `Student[Calculus1]` package is required.

H.4.1 *Visualising the Derivative of* $(x-1)^2 \sin(x)$

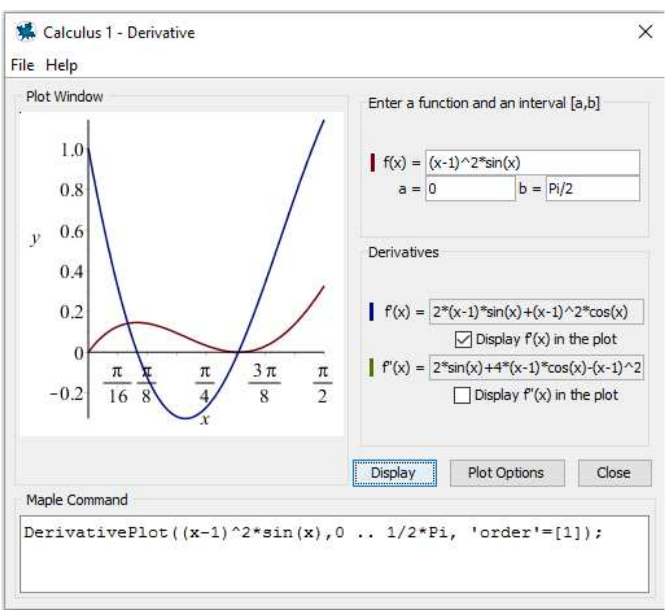


Figure H.4: The Derivatives tutor displays the given function $f(x)$ as well as its derivative(s) $f'(x)$ (and optionally $f''(x)$).

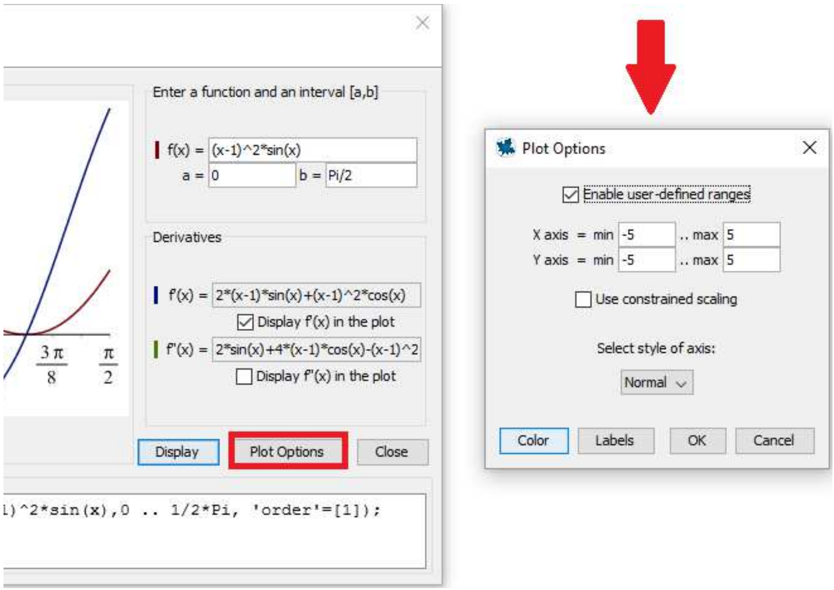


Figure H.5: You can optionally specify the axes ranges as well as 1 : 1 scaling.

H.5 *The Differentiation Methods Tutor*

The Differentiation Methods tutor is useful for showing each individual differentiation rule as a separate step when finding the derivative of a given function.

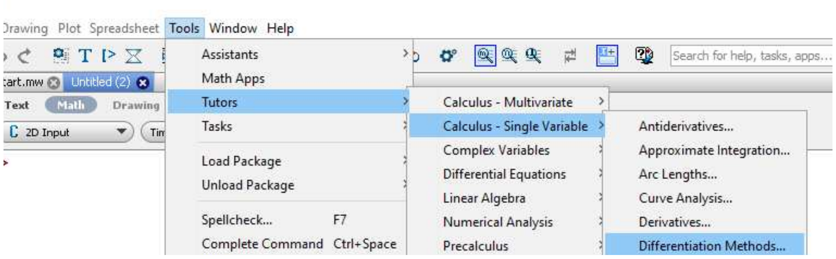


Figure H.6: Opening up the Differentiation Methods tutor using menus.

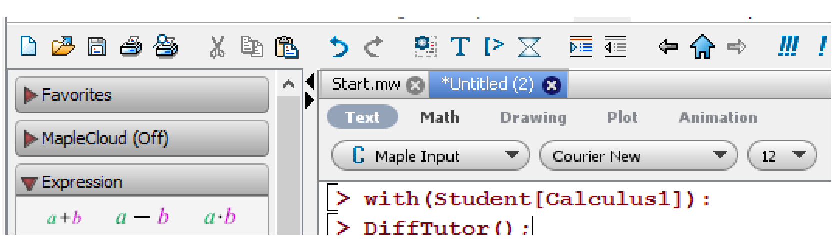


Figure H.7: Opening up the Differentiation Methods tutor using commands. The `Student[Calculus1]` package is required.

H.5.1 Differentiating $x^2 \cos(x)$ using Product and Power Rules

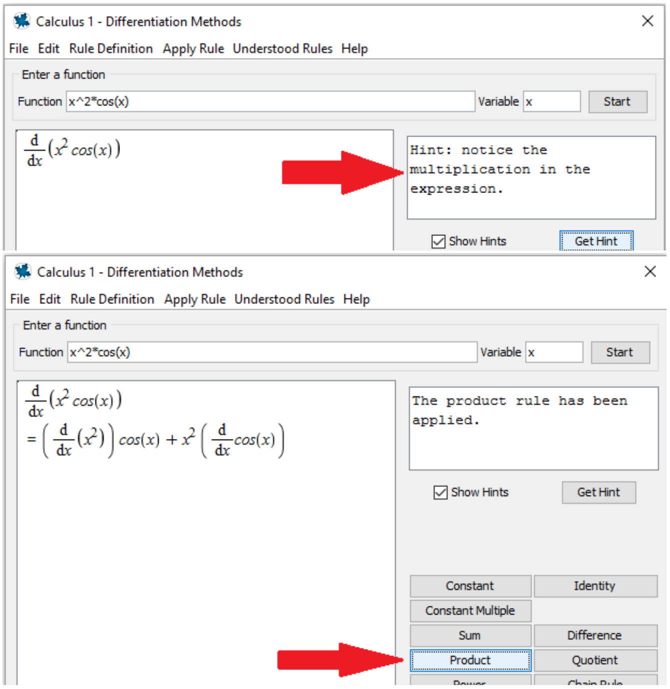


Figure H.8: Using the Get Hint button gives you suggestions as to what differentiation rule to use.

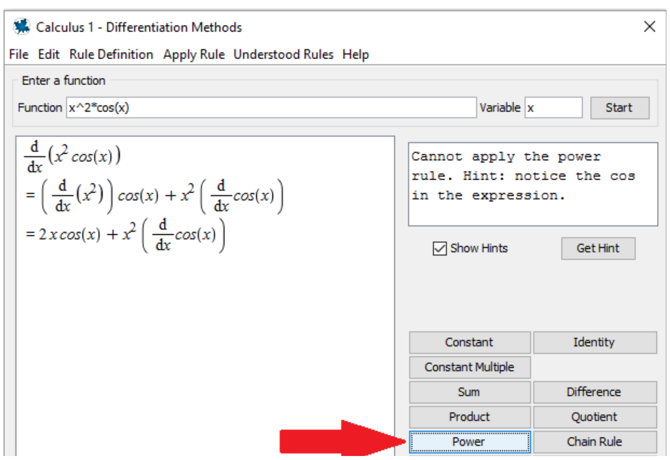


Figure H.9: If a rule cannot be applied, then the tutor will provide additional hints.

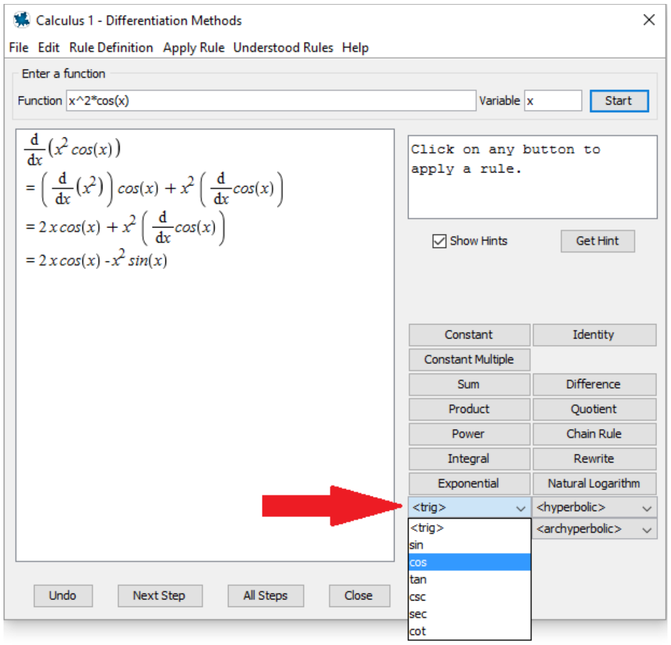


Figure H.10: Using the product rule, the power rule, and the derivative of $\cos(x)$ to differentiate $x^2 \cos(x)$.

H.5.2 Differentiating $\frac{\sin(x)}{x\tan(x)}$ using Rewrite and Quotient Rule

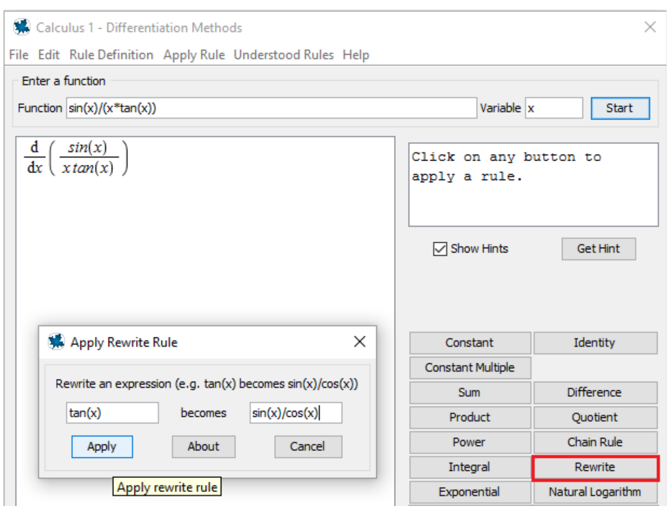


Figure H.11: Using the Rewrite button allows you to replace an expression with another equivalent expression.

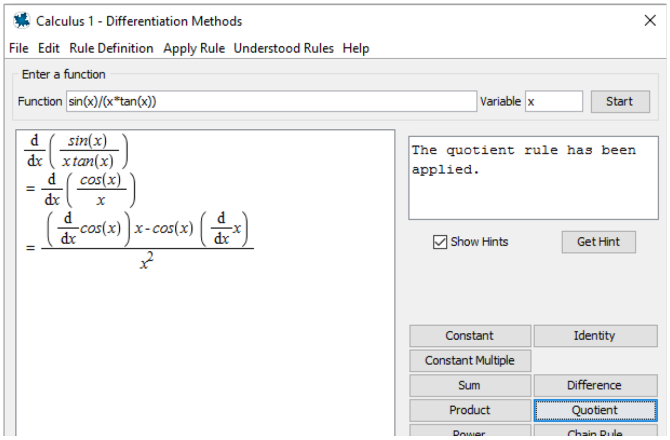


Figure H.12: Applying the quotient rule to differentiate $\frac{\sin(x)}{x\tan(x)}$.

H.6 *Newton's Method*

Suppose we start with the function $f(x) = e^x - 2$. We know that the root of this function is $x = \ln 2$. If we wish to evaluate $\ln 2$ as a decimal, we can simply use `fsolve()`.

```
> f(x) := exp(x) - 2;
```

$$f := x \mapsto e^x - 2$$

```
> fsolve( f(x) = 0 );
```

$$0.6931471806$$

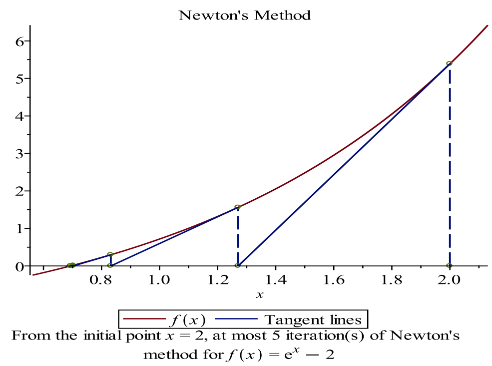
We can also find this root by applying Newton's method with an initial guess of $x = 2$. We need to load the `Student[Calculus1]` package before we use the `NewtonsMethod()` command.

Optional parameters may be included to change how the result is displayed and how many iterations of the method are performed.

Parameter	Description
`output = value`	Outputs the numerical result of Newton's method.
`output = plot`	Outputs a plot showing the tangent line approximation approach to finding the root.
`output = animation`	Much like the `plot` output, only with each iteration as a separate frame.
`output = sequence`	Outputs the original guess and the result of each iteration of Newton's method.
`iterations =` n	Specifies the number of iterations to perform in Newton's method.

Table H.1: A list of optional parameters for the `NewtonsMethod()` command.

```
> with(Student[Calculus1]):
> NewtonsMethod( f(x), x=2, output=plot );
```



If we wish to simply evaluate the root, the `output=plot` option may be omitted. For more accuracy, the algorithm can be run with additional iterations.

```
> NewtonsMethod( f(x), x=2 );
```

$$0.6931471814$$

The default number of iterations for `NewtonsMethod()` with the parameter `ouput=sequence` is 5.

```
> NewtonsMethod( f(x), x=2, output=sequence);
```

$$2, 1.270670566, 0.8319573035, 0.7023505839, 0.6931894021, 0.6931471814$$

```
> NewtonsMethod( f(x), x=2, iterations=10 );
```

$$0.6931471804$$

I Implicit Functions

I.1 Implicit Functions

An implicit function cannot be defined as a normal function. Instead, the curve is defined as an *equation* of multiple variables. It is easiest to assign a name to the entire equation, including the $=$ sign.

```
> E := y^2 = x^3 - 2*x + 1;
```

$$E := y^2 = x^3 - 2\,x + 1$$

To find points on the curve, we can substitute a value for x and solve for y.

```
> subs(x=2, E);
```

$$y^2 = 5$$

```
> solve(%, y);
```

$$\sqrt{5}, -\sqrt{5}$$

Many implicit functions cannot be expressed as a single function $y = f(x)$. However, it may be possible to be split up implicit functions into explicit functions by solving for y.

```
> L := x^2 + (y - root[3](x^2))^2 = 1;
```

Here, the command `root[3]()` is equivalent to $\sqrt[3]{(\,)}$.

$$L := x^2 + \left(y - \sqrt[3]{x^2}\right)^2 = 1$$

```
> solve(L, y);
```

$$\sqrt[3]{x^2} + \sqrt{-x^2 + 1}, \sqrt[3]{x^2} - \sqrt{-x^2 + 1}$$

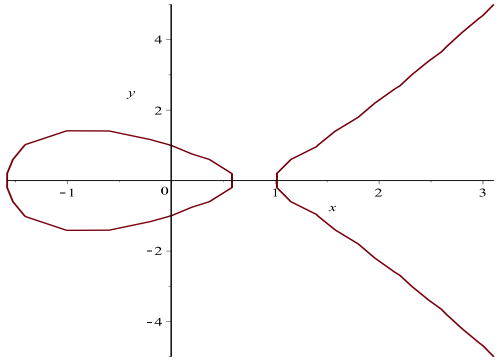
I.2 Plotting Implicit Functions

The `implicitplot()` command can be used to plot implicit functions. It requires use of the `plots` package. Unlike the normal `plot()` command, each curve that is being plotted must be in the form of an *equation* of two variables, including the $=$ sign. Additionally, you must specify an interval for *both* variables.

```
> with(plots):
```

The `plots` package can be loaded using the `with()` command. This only needs to be loaded once per Maple worksheet and needs to be run each time you open a new or previously closed document.

```
> implicitplot(E, x=-5..5, y=-5..5);
```

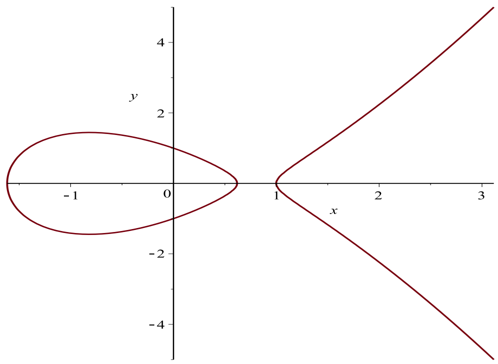


Often, Maple's plot will not appear smooth. We can force Maple to produce a smoother plot by including more points in the graph.

An alternate option for smoother plots is the `grid=[x,y]` parameter:

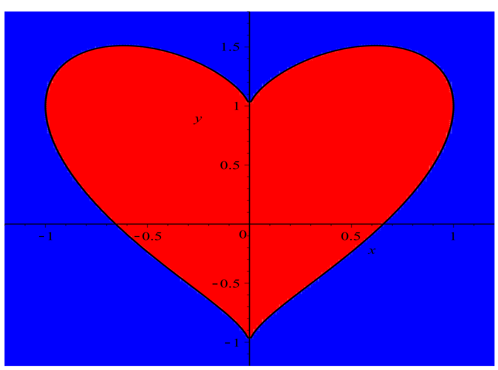
```
> implicitplot(E, x=-5..5,
y=-5..5, grid=[200,200])
```

```
> implicitplot(E, x=-5..5, y=-5..5, numpoints=25000);
```



Be careful not to choose too large of a value for `numpoints`, or else the output may take a very long time to produce.

```
> implicitplot(L, x=-1.2..1.2, y=-1.2..1.8, numpoints =
30000, coloring=["red","blue"], filledregions=true);
```



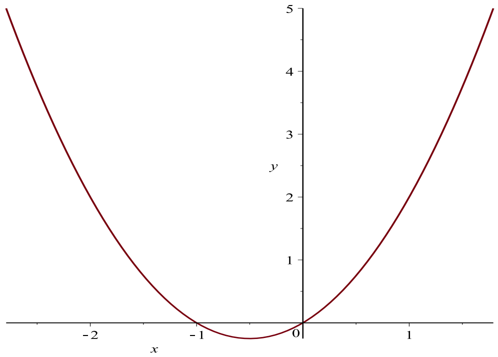
I.3 *Implicit Differentiation*

We need to use the `implicitdiff()` command to find the derivative of an implicit function. It is easiest to first assign a name to the equation.

```
> E := y = x^2 + x;
```

$$E := y = x^2 + x$$

```
> with(plots):
> implicitplot(E, x=-5..5, y=-5..5,numpoints=30000);
```




```
> dydx := implicitdiff(E, y, x);
```

$$dydx := 2\,x + 1$$

The order in which you list the variables matters; the first variable is treated as the dependent variable and the second variable is treated as the independent variable.

To find dy/dx, you must use `implicitdiff(E,y,x)` and to find dx/dy, you must use `implicitdiff(E,x,y)`.

```
> dxdy := implicitdiff(E, x, y);
```

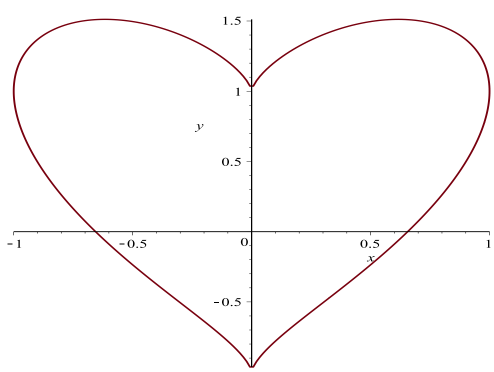
$$dxdy := \frac{1}{2\,x + 1}$$

When trying to find the slope of a tangent line at a point on an implicit curve, it helps to plot the curve first.

```
> L := x^2 + (y - root[3](x^2))^2 = 1;
```

$$L := x^2 + \left(y - \sqrt[3]{x^2}\right)^2 = 1$$

```
> implicitplot(L, x=-1.2..1.2, y=-1.2..1.8, numpoints =
30000);
```



To find a point on the curve at a specific x value, you must first substitute the value and then solve for the y-coordinates.

```
> subs(x=0.5, L); yCoords := fsolve(%, y);
```

$$0.25 + (y - 0.6299605249)^2 = 1$$

$$yCoords := 1.495985929, -0.2360648789$$

You should notice that a list of two y-coordinates is assigned to a single name here. Alternatively, you could assign the individual values to unique names.

Then you can find the slopes of the tangent lines by computing the derivative with `implicitdiff()` and substituting the x and y values for each point.

```
> dydx := implicitdiff(L, y, x);
```

$$dydx := -1/3\,\frac{x\left(3\,\left(x^2\right)^{2/3} + 2\,\sqrt[3]{x^2} - 2\,y\right)}{\left(x^2\right)^{2/3} y - x^2}$$

```
> subs(x=0.5, y=yCoords[1], dydx);
```

$$0.2625970976$$

```
> subs(x=0.5, y=yCoords[2], dydx);
```

$$1.417297636$$

`yCoords` is the name that both values are assigned to. To refer to an individual value, we use the index of the desired value in square brackets after the name `yCoords`.

I.4 Applications of Implicit Differentiation

In these examples, we will make use of the `implicitdiff()` and `implicitplot()` commands.

I.4.1 Finding the Equation of a Tangent Line

In this example, we will find the equations of the tangent lines to the circle of radius 5, centred at the origin, where $x = 3$.

```
> circle := x^2 + y^2 = 25;
```

$$circle := x^2 + y^2 = 25$$

We'll need to find the y-coordinates by substituting $x = 3$ and solving for y.

```
> subs(x=3, circle); solve(%, y);
```

$$y^2 + 9 = 25$$

$$4, -4$$

The derivative $\frac{dy}{dx}$ can be found using `implicitdiff()`. Then, by substituting the two points, we can find the slopes of both tangent lines.

```
> dydx := implicitdiff(circle, y, x);
```

$$dydx := -\frac{x}{y}$$

```
> m1 := subs(x=3, y=4, dydx);
```

$$m1 := -3/4$$

Make sure that you choose different names when assigning values. We will need both slopes later.

```
> m2 := subs(x=3, y=-4, dydx);
```

$$m2 := 3/4$$

Thinking ahead, in order to plot both lines using `implicitplot()`, they will need to be defined as *equations*.

```
> line1 := y = m1*(x-3) + 4;
```

$$line1 := y = -3/4\,x + \frac{25}{4}$$

Both `line1` and `line2` are defined with the inclusion of y =, making them equations that `implicitplot()` can plot.

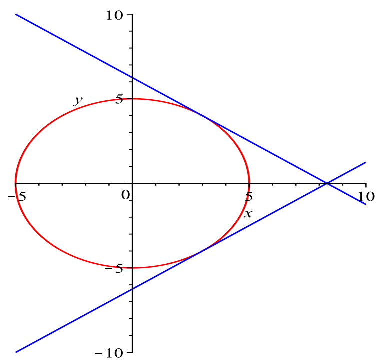
```
> line2 := y = m2*(x-3) + (-4);
```

$$line2 := y = 3/4\,x - \frac{25}{4}$$

We can now plot the circle and the two lines together.

```
> with(plots):
```

```
> implicitplot( [circle, line1, line2], x=-10..10,
y=-10..10, colour=[red,blue,blue], scaling=constrained,
numpoints=30000);
```



Using the `scaling=constrained` parameter will produce a proper circle in your worksheet.

I.4.2 *Orthogonal Curves*

In this example, we will show that the curves $y^2 - x^2 = 3$ and $xy = 2$ are always perpendicular (orthogonal) at their intersection points.

```
> curve1 := y^2 - x^2 = 3;
```

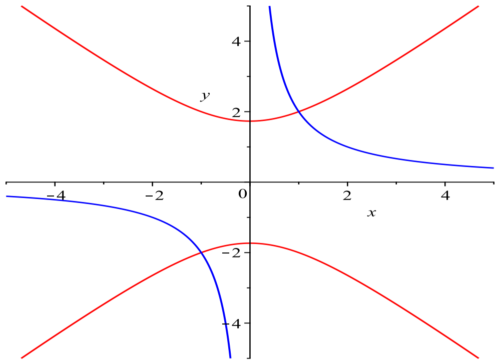
$$curve1 := -x^2 + y^2 = 3$$

```
> curve2 := x*y = 2;
```

$$curve2 := xy = 2$$

```
> with(plots):
```

```
> implicitplot( [curve1, curve2], x=-5..5, y=-5..5,
colour=[red, blue], numpoints=30000, scaling=constrained);
```



Using the `scaling=constrained` parameter will preserve right angles in your worksheet.

From the graphs of these two curves, it appears that their intersections are perpendicular. This can be proven by showing that the derivative of one curve is equal to the negative reciprocal of the other, or that they multiply to equal -1.

```
> dydx1 := implicitdiff(curve1, y, x);
```

$$dydx1 := \frac{x}{y}$$

```
> dydx2 := implicitdiff(curve2, y, x);
```

$$dydx2 := -\frac{y}{x}$$

```
> dydx1*dydx2;
```

$$-1$$

J Definite and Indefinite Integrals

In this tutorial, we use numerical approximation to calculate definite integrals (namely, the right-sum, left-sum, midpoint, Simpson's, and trapezoidal methods). We then calculate these definite integrals using the Fundamental Theorem of Calculus. We will also learn how to find indefinite integrals as functions.

J.1 Numerical Approximations Using ApproximateInt

To use the `ApproximateInt()` command, we must load the `Student[Calculus1]` package.

The function and interval must be specified. Other optional parameters may be included to change how the result is displayed and how the approximation is computed.

Parameter	Description
`method =` *method*	Select the method for appoximation (`left`, `right`, `lower`, `upper`, `midpoint`, `trapezoid`, `simpson`).
`output =` *output*	Change how the output is displayed (`plot`, `value`, `sum`).
`partition =` n	Change the number of subintervals to use for approximation.

Table J.1: A list of optional parameters for the `ApproximateInt()` command.

J.1.1 Approximating the Area under $x\sin(x)$

We can define a function and approximate the area under the curve over a specified interval. In this example, we will use the `upper`, `lower`, and `midpoint` methods for approximating rectangles.

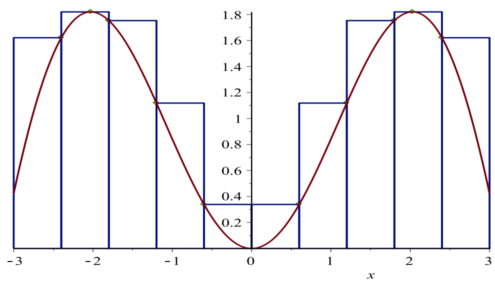
```
> with(Student[Calculus1]):
```

```
> f(x) := x*sin(x);
```

$$f := x \mapsto x\sin(x)$$

```
> ApproximateInt( f(x), x=-3..3, method=upper, output=plot,
partition=10);
```

The `method=upper` parameter always uses the **highest** rectangle in an interval (combination of left-sum and right-sum). This will force an overestimate of the actual area underneath the curve.



An upper Riemann sum approximation of $\int_{-3}^{3} f(x)\, \mathrm{d}x$, where $f(x) = x\sin(x)$ and the partition is uniform. The approximate value of the integral is 7.981170598. Number of subintervals used: 10.

```
> ApproximateInt( f(x), x=-3..3, method=upper, output=value,
partition=10);
```

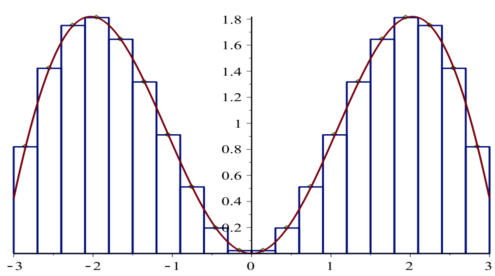
$$7.981170598$$

The `method=lower` parameter always uses the **lowest** rectangle in an interval (combination of left-sum and right-sum). This will force an underestimate of the actual area underneath the curve.

```
> ApproximateInt( f(x), x=-3..3, method=lower, output=value,
partition=10);
```

$$4.202044853$$

```
> ApproximateInt( f(x), x=-3..3, method=midpoint, output=plot,
partition=20);
```



A midpoint Riemann sum approximation of $\int_{-3}^{3} f(x)\, \mathrm{d}x$, where $f(x) = x\sin(x)$ and the partition is uniform. The approximate value of the integral is 6.243461668. Number of subintervals used: 20.

J.1.2 *Approximating the Area under* $10\mathrm{e}^{-x}$

We can use different methods for the rectangles, such as left-point and right-point. Let's define a different function, $g(x)$ and calculate some more approximations:

These are two common methods that we learn when first calculating Riemann sums.

```
> g(x) := 10*exp(-x);
```

$$g := x \mapsto 10\,\mathrm{e}^{-x}$$

```
> ApproximateInt(g(x), x = 0..4, method=left, output=value,
partition=8);
```

$$5 + 5\,\mathrm{e}^{-1/2} + 5\,\mathrm{e}^{-1} + 5\,\mathrm{e}^{-3/2} + 5\,\mathrm{e}^{-2} + 5\,\mathrm{e}^{-5/2} + 5\,\mathrm{e}^{-3} + 5\,\mathrm{e}^{-7/2}$$

```
> ApproximateInt(g(x), x = 0..4, method=right, output=value,
partition=8);
```

$$5\,\mathrm{e}^{-1/2} + 5\,\mathrm{e}^{-1} + 5\,\mathrm{e}^{-3/2} + 5\,\mathrm{e}^{-2} + 5\,\mathrm{e}^{-5/2} + 5\,\mathrm{e}^{-3} + 5\,\mathrm{e}^{-7/2} + 5\,\mathrm{e}^{-4}$$

J.1.3 *Numerical Approximation for a Given Integral*

We can also set up an integral to use numerical approximation on.

Use of the `Int()` command will be fully explained in the next section.

```
> int1 := Int( x^2, x=0..2);
```

$$int1 := \int_0^2 x^2 dx$$

We can evaluate this integral with 4 subintervals using three different methods.

```
> ApproximateInt( int1, method=midpoint, output=value,
partition=4);
```

$$\frac{21}{8}$$

```
> ApproximateInt( int1, method=trapezoid, output=value,
partition=4);
```

The `method=trapezoid` parameter uses trapezoid shapes instead of rectangles to approximate the area. This usually gives a more accurate estimate of the area.

$$\frac{11}{4}$$

```
> ApproximateInt( int1, method=simpson, output=value,
partition=4);
```

$$\frac{8}{3}$$

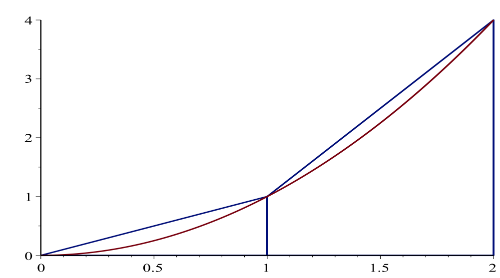
When using Simpson's rule, an additional sample point is used per subinterval. So, if n subintervals are used, then the number of sample points is $2n$.

Simpson's rule uses twice as many sample points as the other two methods used here. If we wish to have 4 points used for Simpson's rule, we use `partition=2`.

To visualize the approximation using any of the above methods, we use the option `output=plot`.

```
> ApproximateInt( int1, method=trapezoid, output=plot,
partition=2);
```

Notice that trapezoidal shapes are used instead of rectangles.



An approximation of $\int_0^2 f(x)\, dx$ using trapezoid rule, where $f(x) = x^2$ and the partition is uniform. The approximate value of the integral is 3.000000000. Number of subintervals used: 2.

Maple can also give the Riemann sum for any of the above approximations.

```
> ApproximateInt(int1, method=trapezoid, output=sum,
partition=4);
```

$$\frac{1}{4} \sum_{i=0}^{3} \frac{1}{4} i^2 + \left(\frac{1}{2} i + \frac{1}{2}\right)^2$$

The `output=sum` shows you what the expanded summation looks like **before** it is evaluated.

J.2 Definite Integrals

We now turn our attention to using the Fundamental Theorem of Calculus to calculate definite integrals to (in most cases) obtain the area underneath a curve.

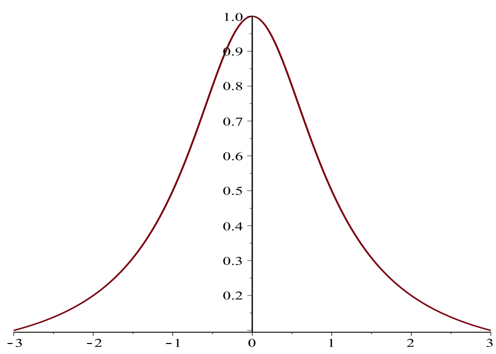
The `int()` command allows us to compute integrals (both definite and indefinite) directly. The `Int()` command allows us to symbolically view the integral, or assign it to a variable to use later for other computations or calculations. Capitalization is important in Maple.

J.2.1 The Definite Integral $\int_{-3}^{3} \frac{1}{x^2+1}\,dx$

```
> f(x) := 1/(1+x^2);
```

$$f := x \mapsto \frac{1}{x^2+1}$$

```
> plot( f(x), x=-3..3);
```



We use the `Int()` command to display the integral, and the `int()` command to evaluate the integral.

```
> Int( f(x), x=-3..3);
```

$$\int_{-3}^{3} \frac{1}{x^2+1}\,dx$$

Note the capital "I" in this command. This prevents Maple from automatically evaluating the integral.

```
> int( f(x), x=-3..3); evalf(%);
```

$$2\arctan(3)$$

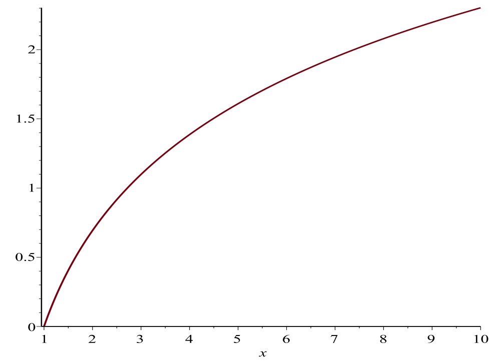
$$2.498091544$$

J.2.2 The Definite Integral $\int_{1}^{10} \ln(x)\,dx$

```
> g(x) := ln(x);
```

$$g := x \mapsto \ln(x)$$

```
> plot( g(x), x=1..10);
```




```
> int( g(x), x=1..10); evalf(%);
```

$$-9 + 10\ln(2) + 10\ln(5)$$
$$14.02585093$$

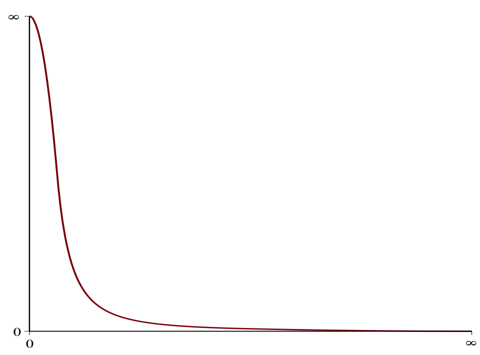
J.2.3 *Improper Integrals*

The `int()` and `Int()` commands can also be used to compute improper integrals.

```
> h(x) := 1/x^2;
```

$$h := x \mapsto x^{-2}$$

```
> plot( h(x), x=0..infinity);
```




```
> Int( h(x), x=1..infinity);
```

$$\int_1^\infty x^{-2} dx$$

```
> int( h(x), x=1..infinity);
```

$$1$$

J.3 *Indefinite Integrals and Antiderivatives*

If we do not have limits of integration we still can use the `Int()` and `int()` commands to show and evaluate the integrals respectively.

Recall that `Int()` displays the integral and `int()` evaluates the integral.

```
> Int(sin(x),x);
```

```
> int(sin(x),x);
```

$$\int \sin(x)\, dx$$

$$-\cos(x)$$

```
> p(x) := 1/sqrt(1 + x^2);
```

$$p := x \mapsto \frac{1}{\sqrt{x^2+1}}$$

```
> Int(p(x), x);
```

$$\int \left(\sqrt{x^2+1}\right)^{-1} dx$$

```
> int(p(x), x);
```

$$\operatorname{arcsinh}(x)$$

Maple does not include the addition of the constant of integration $+C$ when evaluating indefinite integrals.

J.4 The Integration Methods Tutor

The Integration Methods tutor can be used to evaluate an integral step-by-step. You can either manually attempt different techniques on the integral, or let Maple decide which techniques to use.

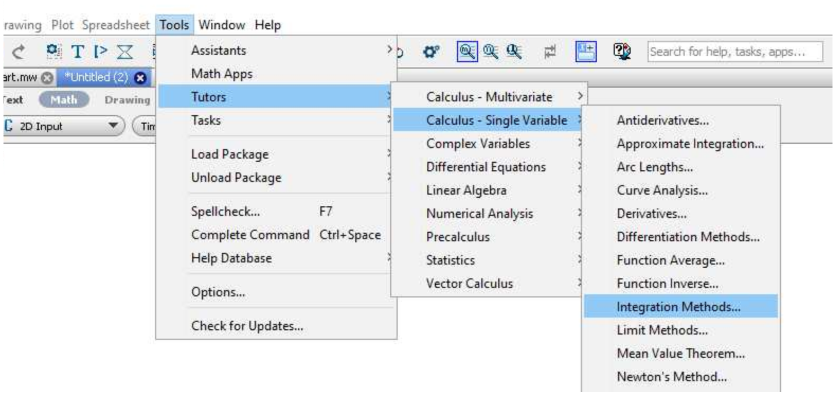


Figure J.1: Opening up the Integration Methods tutor using menus.

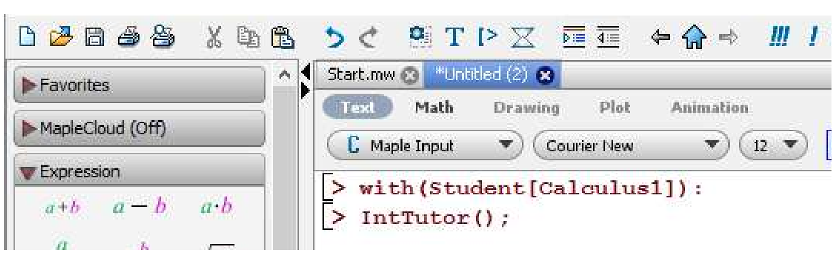


Figure J.2: Opening up the Integration Methods tutor using commands. The `Student[Calculus1]` package is required.

J.4.1 *Evaluating $\int x^2 e^x$ Using Integration by Parts*

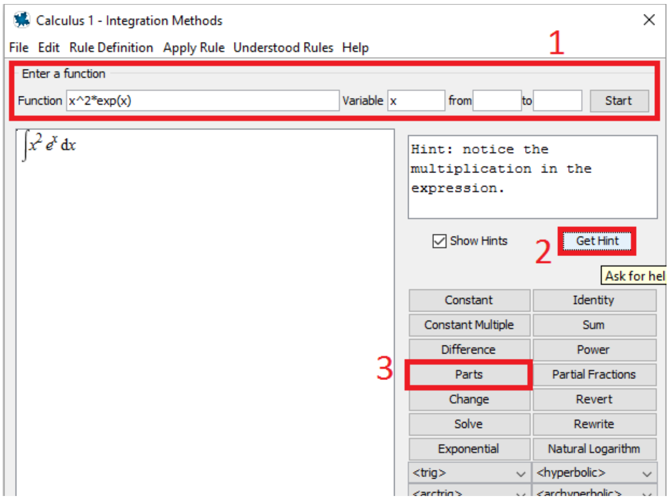


Figure J.3: If the Get Hit button is clicked, Maple will suggest an integration technique.

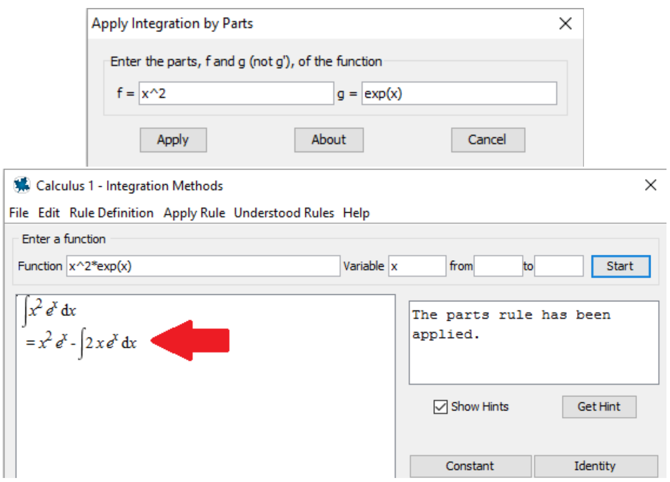


Figure J.4: Integration by parts can be applied to integrals of the form $\int u \, dv$, where $u = f(x)$ and $dv = g'(x)dx$.

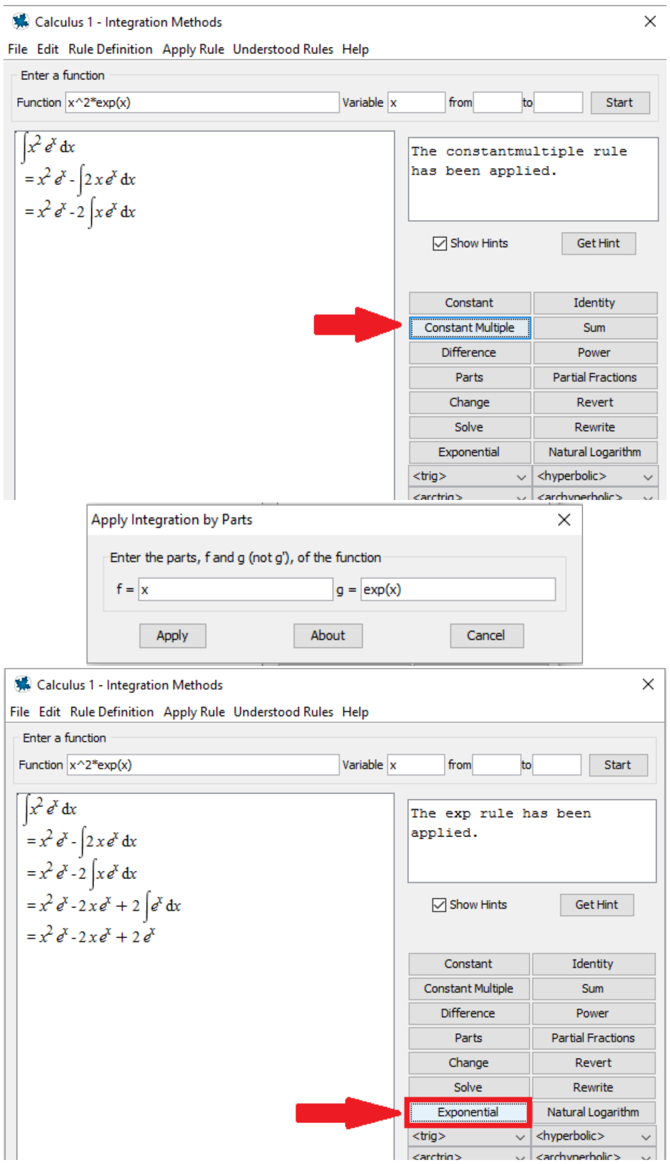


Figure J.5: Using the constant multiple rule, integration by parts a second time, and the exponential rule to evaluate the integral.

J.4.2 Evaluating $\int_{-\pi/2}^{\pi/4} x \sin(x^2+1)$ Using Substitution

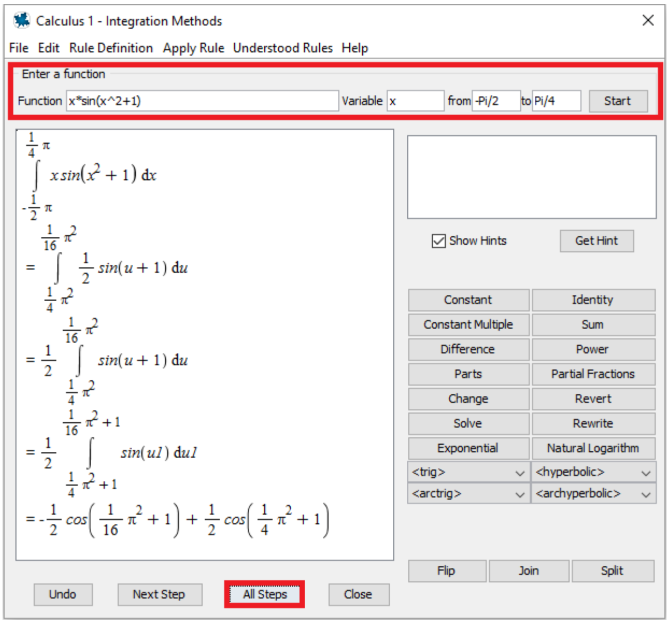


Figure J.6: If you use the All Steps button, then Maple will evaluate the integral, showing all steps.

J.5 *Volume of Revolution*

The Volume of Revolution tutor is used to evaluate the volume of a solid obtained by rotating a region about a specified horizontal or vertical axis.

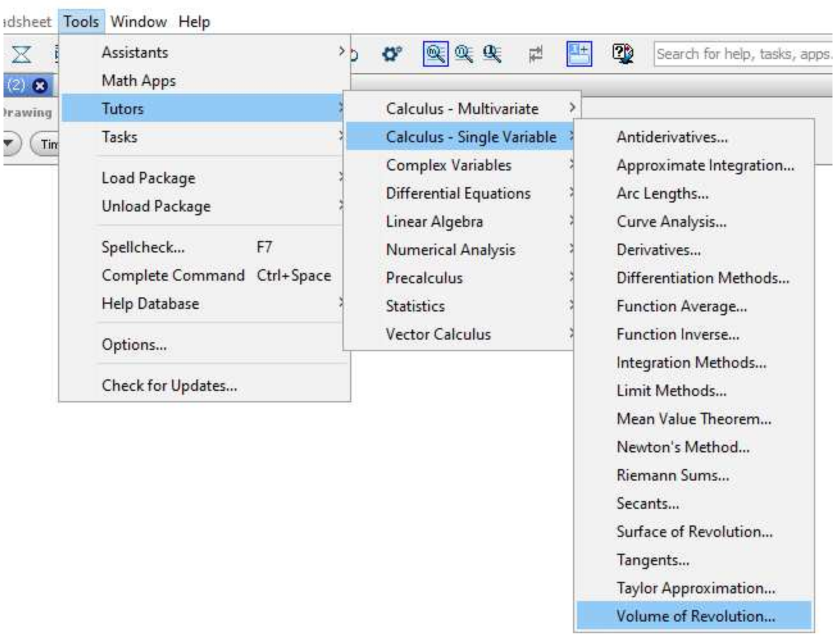


Figure J.7: Opening up the Volume of Revolution tutor using menus.

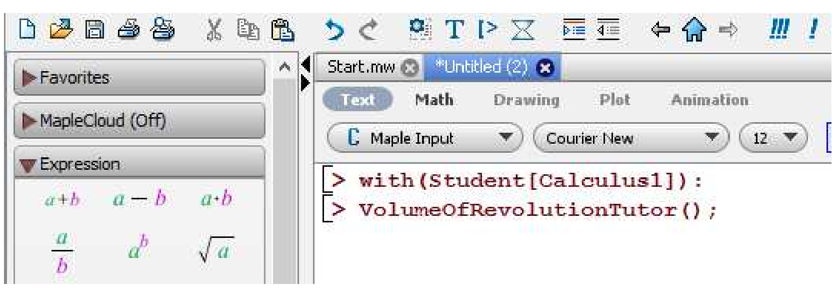


Figure J.8: Opening up the Volume of Revolution tutor using commands. The `Student[Calculus1]` package is required.

J.5.1 *Volume Obtained by Rotating the Region Bounded by the Parabolas $x^2 - 4$ and $-x^2 + 4$ about $y = 2$*

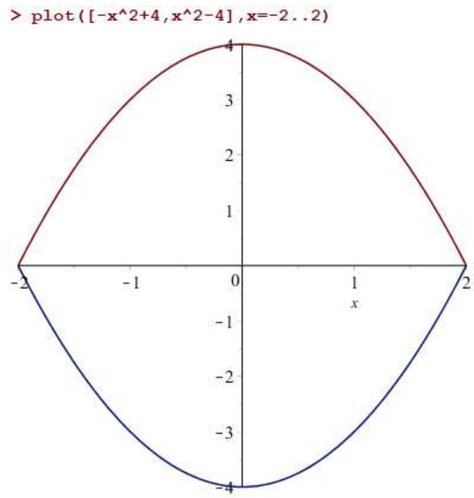


Figure J.9: Plotting the 2D region first is a good practice before revolving it around the given axis.

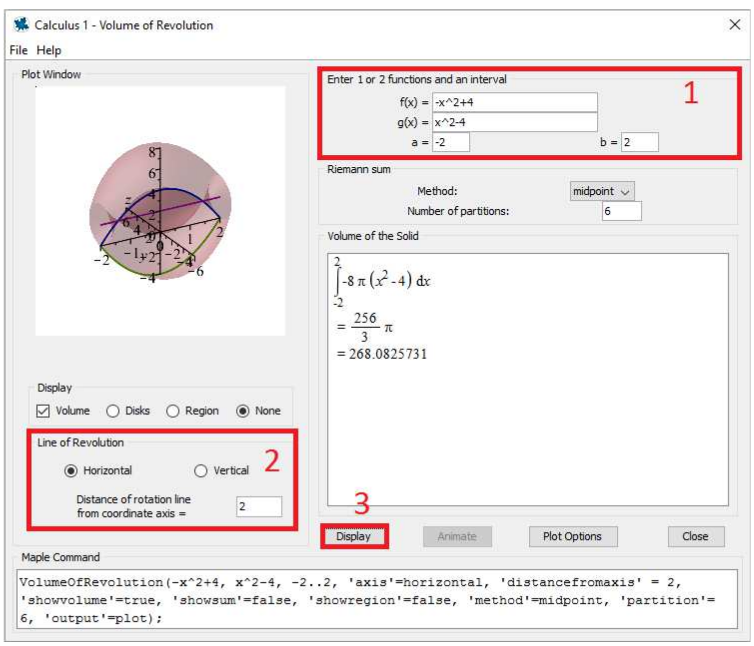


Figure J.10: Setting the Volume of Revolution tutor to revolve the region about a horizontal axis.

J.5.2 Volume Obtained by Rotating the Region Bounded by the Parabolas $x^2 - 4$ and $-x^2 + 4$ about $x = -3$

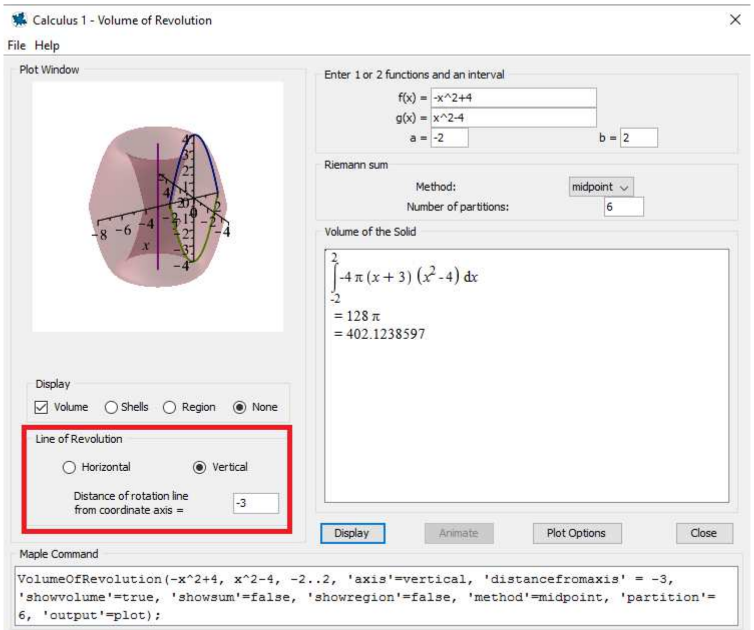


Figure J.11: Setting the Volume of Revolution tutor to revolve the region about a vertical axis.

J.6 Arc Length

We can use the `ArcLength()` command that is available in the `Student[Calculus1]` package to find the arc length of a function over a specified interval.

J.6.1 Arc Length of a Parabola

```
> with(Student[Calculus1]):
> g(x) := x^2;
```

$$g := x \mapsto x^2$$

```
> arcg := ArcLength( g(x), x=0..Pi);
```

$$arcg := 1/2\,\pi\,\sqrt{4\,\pi^2+1} - 1/4\,\ln\left(-2\,\pi + \sqrt{4\,\pi^2+1}\right)$$

```
> evalf(arcg);
```

$$10.62814707$$

J.6.2 Arc Length of a Sinusoid

```
> f(x) := sin(x);
```

$$f := x \mapsto \sin(x)$$

```
> arcf := ArcLength( f(x), x=0..Pi);
```

$$arcf := 2\,\sqrt{2}EllipticE\left(1/2\,\sqrt{2}\right)$$

`EllipticE()` is a special function in Maple that you don't have to know. We at least can evaluate it, however.

```
> evalf(arcf);
```

$$3.820197788$$

We can add the parameter `output = integral` to the `ArcLength()` command to display the integral for calculating the arc length.

```
> ArcLength( f(x), x=0..Pi, output=integral);
```

$$\int_0^{\pi} \sqrt{(\cos(x))^2 + 1}\, dx$$

K Differential Equations

We want to find solutions (and potentially graph said solutions) of **differential equations** – equations that contain functions and derivatives of these functions. For sake of ease, we will look at **first-order differential equations**: differential equations that only contain the **first** derivative (no higher-order).

K.1 Finding the General Solution to a Differential Equation

As an example, let's consider the differential equation

$$y' = x^2 y$$

which is a first-order differential equation, where we assume y is a function of x. The goal is to find the function $y(x)$ that satisfies this equation.

When we define differential equations in Maple, we must ensure that we write y in function notation; this means we write it as $y(x)$ and not as y.

```
> de1 := y'(x)= x^2*y(x);
```

$$de1 := \frac{d}{dx}y\left(x\right) = x^2 y\left(x\right)$$

We use the `dsolve()` command to solve a differential equation.

A package does not need to be imported to use the `dsolve()` command.

```
> dsolve( de1, y(x));
```

$$y\left(x\right) = \_C1\,\mathrm{e}^{1/3\,x^3}$$

Note that the result has _C1 as the coefficient. This is an arbitrary constant that is part of the solution of the differential equation. By default, Maple names the constants $\_C1, \_C2, \_C3, \ldots$ etc.

We can clean this up a bit by substituting a 'nicer' constant for _C1.

```
> desoln := subs(_C1=A, _C1*exp((1/3)*x^3));
```

$$desoln := A\,\mathrm{e}^{1/3\,x^3}$$

This solution with the arbitrary constant A, is known as the **general solution** to the differential equation.

K.2 *Finding the Particular Solution given Initial Conditions*

The constant A from the above example is arbitrary, meaning that the function $y(x) = Ae^{1/3\,x^3}$ is always a solution for the original differential equation, no matter the value of A.

If we are given an **initial condition** of the form $y(x_0) = y_0$ for constants x_0, y_0, then we can find a unique solution for the differential equation. This will give us a new value for A every time we change the initial condition.

For example, suppose that the function $y(x)$ goes through the point $(0, 5)$. In this case, we have the initial condition $y(0) = 5$.

To include this initial condition when solving the differential equation in Maple, we add it into the `dsolve()` command as follows:

```
> dsolve( [de1, y(0) = 5], y(x));
```

$$y(x) = 5\,e^{1/3\,x^3}$$

The particular solution to $y' = x^2y$ with the initial condition $y(0) = 5$ is $y(x) = 5\,e^{1/3\,x^3}$.

K.3 *Direction Fields*

A direction field is a way to plot an entire family of solutions on a graph. Each arrow on this direction field represents the slope of the tangent line at every point in the graph's range. To access this command, we need to import the `DETools` package.

```
> with(DETools):
```

The `DETools` package must be loaded to use the `DEplot()` command.

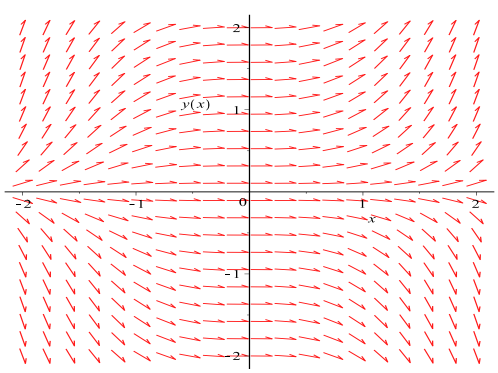
```
> de2 := y'(x) = x^2 * sin(y(x));
```

$$de2 := \frac{d}{dx}y(x) = x^2 \sin(y(x))$$

This differential equation may be written as $y' = x^2 \sin(y)$.

We can use `DEplot()` to plot the direction field for the differential equation. The required parameters for the `DEplot()` command are the differential equation, the function for which it is to be solved for, and the ranges of the two variables.

```
> DEplot( de2, y(x), x=-2..2, y=-2..2);
```



The above direction field allows us to track a "solution curve" for any particular initial condition. To track this curve, we start at a point (x_0, y_0) and follow the directions of the arrows (back and forth). This will give us the **one** unique curve that satisfies the initial condition $y(x_0) = y_0$.

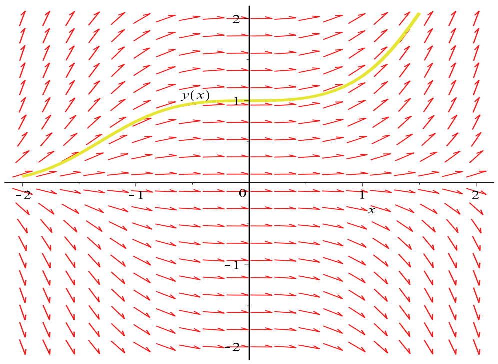
Maple will plot a particular solution if the initial condition in square brackets is added an as additional parameter.

Square brackets must be around the initial condition of a particular solution when plotting a solution curve.

K.3.1 *Plotting Solution Curves on a Direction Field*

If we want to force the `DEplot()` command to show a solution for one particular initial condition, say $y(0) = 1$, we add the parameter as follows:

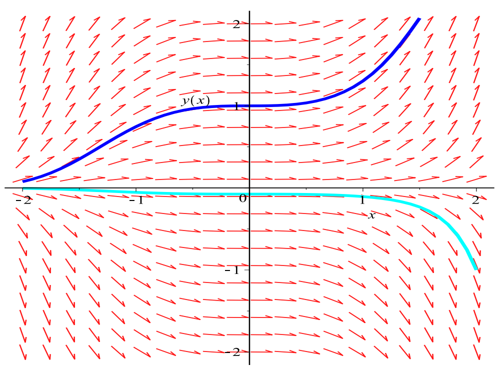
```
> DEplot( de2, y(x), x = -2 .. 2, y = -2 .. 2, [y(0) =
1]);
```



Notice that this curve goes through the point $(0, 1)$ and follows the direction of the arrows from the direction field.

We now plot the solutions that correspond to the initial conditions $y(0) = 1$ and $y(2) = -1$:

```
> DEplot( de2, y(x), x = -2 .. 2, y = -2 .. 2, [y(0) =
1, y(2) = -1], linecolour = [blue, cyan]);
```



The `linecolour = `*colour* parameter is used to adjust the colour of the solution curves, rather than the direction field.

L Sequences and Series

Suppose we want to generate a list of integers that are 2 more than a multiple of 3. For example, 32 is one of these numbers because $32 = 3(10) + 2$. In other words, we want to create a list of numbers of the form $3k + 2$.

Or perhaps we want a nice way to express and generate all the odd numbers $1, 3, 5, 7, 9, \ldots$ (numbers of the form $2k + 1$).

These lists are called *sequences*. In both of these examples, k is known as an *index*.

L.1 Sequences using $ Notation

The shortest way to list the terms in a sequence is using the $ symbol.

```
> 3*k^2+2 $ k=0..3;
```

$$2, 5, 14, 29$$

The notation for these sequences is $\left(3k^2+2\right)_{k=0}^{3}$ and $\left(2k^2-k\right)_{k=1}^{5}$.

Now let's generate five integers of the form $2k^2 - k$:

```
> 2*k^2-k $ k=1..5;
```

$$1, 6, 15, 28, 45$$

L.2 Sequences using the `seq()` Command

We can also list the terms in a sequence by using the `seq()` command. This command essentially performs the same operation as the $ symbol from the previous section.

```
> seq( 2*k^2-k, k=1..5);
```

$$1, 6, 15, 28, 45$$

The sequence $\left(2k^2-k\right)_{k=1}^{5}$

L.3 Defining and Evaluating Series

We use the capitalized `Sum()` command to display the summation notation for a series symbolically. This also makes it possible to assign a name to the sum and use it in other commands later.

Much like with sequences, we need a generating function involving an index, such as k. We then specify the range of indices for the terms in that sequence that are to be summed.

```
> s1 := Sum( 2*k^2-k, k=1..5);
```

$$s1 := \sum_{k=1}^{5} 2\,k^2 - k$$

This is a called a *finite* sum, since only finitely many terms are summed.

We use the lowercase `sum()` command to calculate the value of the sum. We may also use the `value()` command on the result of a capitalized `Sum()` command.

```
> value(s1);
```

$$95$$

```
> sum( 2*k^2-k, k=1..5);
```

$$95$$

To determine the value of the sum of infinitely many terms, we can use `infinite` as the a bound on the index. The value of an infinite sum may be a value (in the case of a convergent sum) or $\pm\infty$ (in the case of a divergent sum).

```
> s2 := Sum( (2/3)^k, k=0..infinity);
```

$$s2 := \sum_{k=0}^{\infty} (2/3)^k$$

The infinite sum $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$ is said to be convergent.

```
> value(s2);
```

$$3$$

```
> s3 := Sum( sin(k)+1, k=1..infinity);
```

$$s3 := \sum_{k=1}^{\infty} \sin(k) + 1$$

The infinite sum $\sum_{k=1}^{\infty} (\sin(k) + 1)$ is said to be divergent.

```
> value(s3);
```

$$\infty$$

L.4 Taylor and Maclaurin Series

One use of series is to find the Taylor series expansion of a function. Recall that the Taylor series of a function $f(x)$, centred at $x = a$, is the sum

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{n!}(x-a)^k.$$

In the case that a Taylor series is centred at $x = 0$, this is known as a Maclaurin Series.

L.4.1 Maclaurin Series Expansion of e^x

```
> taylor(exp(x), x = 0);
```

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + O\left(x^6\right)$$

Maple has given 7 terms as an output. The $O(x^6)$ term in this expression means "plus a bunch more terms with power 6 and higher".

We can specify the order (related to number of terms) of the Taylor series by adding a number as the final argument to the command.

```
> taylor(exp(x), x = 0, 10);
```

$$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5 +\frac{1}{720}x^6+\frac{1}{5040}x^7+\frac{1}{40320}x^8+\frac{1}{362880}x^9+O\left(x^{10}\right)$$

In this example, Maple has displayed all the terms with powers less than 10.

L.4.2 *Taylor Series Expansion of* $\sin(x)$*, Centred at* $x = 4$

```
> taylor(sin(x), x = 5, 4);
```

$$\sin(5)+\cos(5)(x-5)-\frac{1}{2}\sin(5)(x-5)^2 -\frac{1}{6}\cos(5)(x-5)^3+O\left((x-5)^4\right)$$

L.4.3 *Comparing the Graphs of* $\sin(x)$ *and its Maclaurin Series*

We can see how closely the Maclaurin (or Taylor) series expansion resembles the original function by plotting them on the same axes. To plot the Taylor series we need to get rid of the final term $O(\ldots)$. We use the `convert()` command to do this.

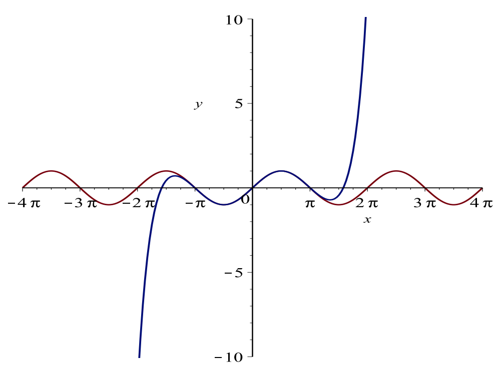
```
> taylor(sin(x), x = 0, 10); poly := convert(%, polynom);
```

This `convert()` command lets Maple know that we wish for the result to be a truncated polynomial.

$$x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\frac{1}{362880}x^9+O\left(x^{11}\right)$$

$$poly := x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\frac{1}{362880}x^9$$

```
> plot( [sin(x), poly], x = -4*Pi..4*Pi, y = -10..10);
```



M Conditional Statements and Loops

M.1 `if/else` Statements

'If' and 'else' statements allow us to test various conditions. The result then changes based on whether the given condition is **true** or **false**.

For example, let's say that someone makes the statement

"If it's sunny tomorrow, then I will go for a bike ride."

If it is, in fact, sunny tomorrow, then that person **will** go for a bike ride. In other words, since the "sunny tomorrow" condition becomes TRUE, then the following implication "go for a bike ride" will happen.

As the statement stands now, we have no idea what this person will do if it is **not** sunny. Perhaps they will still go for a bike ride, perhaps not. We can define this action with an 'else' statement.

In other words, we can define what will happen if it is **not sunny**. For example:

"If it's sunny tomorrow, then I will go for a bike ride. Otherwise, I will read a book at home."

From this example, we know that the following are true.

- If it's sunny tomorrow, then I will go for a bike ride.
- If it's not sunny tomorrow, then I read a book at home.

M.1.1 A Conditional Statement based on Numerical Value

We will outline this concept with a silly example. We define two variables $a = 2$ and $b = 5$. If $a < b$ is **true** (which it is because obviously 2 is less than 5), we print the word "Mario". Otherwise (if it's **false**), we print the word "Luigi". Therefore, we expect that "Mario" will be printed.

```
> a := 2: b := 5:
> if a < b then
    print("Mario")
  else
    print("Luigi")
  end if
```

You can use Shift+Enter to create multiple lines within the same Maple input.

"Mario"

M.1.2 *A Conditional Statement to Check Even/Odd*

Now we consider a more useful example. We define a function $f(x)$ and check whether substituting $x = 2$ into this function outputs an **even** number or an **odd** number.

```
> f(x) := x^2 + 3*x - 4:
> if type(f(2), even) then
    print(f(2), "even")
  else
    print(f(2), "odd")
  end if
```

The `type()` command is used to check if the expression has the specified property.

6, "even"

M.1.3 *A Conditional Statement to Check if a Limit Exists*

We define a function $f(x)$ and then use an 'if' statement to verify whether or not the limit of $f(x)$ as x approaches 0 is *numeric*. In other words, we are checking to see whether or not this limit exists.

```
> f(x) := 1/x:
> L := limit(f(x), x=0):
> if type(L, numeric) then
    print("Limit is defined")
  else
    print("Limit is undefined")
  end if
```

Does $\lim_{x \to 0} \frac{1}{x}$ exist? Does the limit give a number L as a result, or something that is **not** a number?

"Limit is undefined"

M.2 *`if/elif/else` Statements*

The `elif` (else if) command allows us to add more than one condition to our statement. For example, if we want to test whether a particular number is (i) positive, (ii) negative, or (iii) zero, and wish to have different outputs based on these **three** possibilities, we can do so with a combination of `if`, `else`, and `elif`.

M.2.1 *Using Conditional Statements to Interpret the First Derivative*

In this example, we will use one of these statements to illustrate the first derivative test in calculus. We would like to answer the following question: for a particular value $x = a$, is the function $g(x)$ (i) increasing (positive derivative), (ii) decreasing (negative derivative), or (iii) neither increasing nor decreasing (zero derivative)?

```
> g(x) := x^4 - 4*x^3 + 3*x^2:
> a := 0:
> if subs(x = a, diff(g(x), x) ) > 0 then
    print("increasing")
  elif subs(x = a, diff(g(x), x) ) < 0 then
    print("decreasing")
  else print("neither")
  end if
```

"neither"

This implies that $g'(0)$ is equal to 0.

M.3 `for` *Loops*

'For' loops allow us to carry out a computation repeatedly for an entire range of values. We can also combine these loops with conditional statements like 'if' and 'else'. To use a 'for' loop, we are required to type

```
for i from a to b do
   ...
   ...
end do
```

where i is a "dummy variable", referred to as an index. On the first iteration of the loop, the index begins at a. At the end of each iteration, the index is incremented by one. In the last iteration, the index will be equal to b.

M.3.1 *Outputting the First n Derivatives*

We will use a basic 'for' loop to output the first 10 derivatives of the function $f(x) = \sin(2x)$. To do this, we will use the `diff()` command. The 'for' loop will output the k^{th} derivative, starting from $k = 1$ and ending at $k = 10$.

Recall that `diff(f(x), x$k)` is the k^{th} derivative of f with respect to x. Use of the `diff()` command is explained in Tutorial H, pg. 125.

```
> f(x) := sin(2*x);
> for k from 1 to 10 do
     diff(f(x), x$k)
  end do
```

$$f := x \mapsto \sin(2\,x)$$

$$2\cos(2x)$$
$$-4\sin(2x)$$
$$-8\cos(2x)$$
$$16\sin(2x)$$
$$32\cos(2x)$$
$$-64\sin(2x)$$
$$-128\cos(2x)$$
$$256\sin(2x)$$
$$512\cos(2x)$$
$$-1024\sin(2x)$$

M.3.2 *Outputting the Primes up to 50*

Let's say we want to find all of the prime integers from 1 to 50. If we enter

```
> for i from 1 to 50 do
    print(i);
  end do
```

You can use Shift+Enter to create multiple lines within the same Maple prompt

into Maple, it will output all the integers from 1 to 50, rather than only the prime integers.. We include an 'if' statement that makes use of the `isprime()` command to only output primes.

```
> for i from 1 to 50 do
    if isprime(i) then
      print(i);
    end if
  end do
```

Since we do not want Maple to do anything for a composite integer (an integer that is not prime), we can omit the 'else' component of this 'if' statement.

$$2$$
$$3$$
$$5$$
$$7$$
$$11$$
$$13$$
$$17$$
$$19$$
$$\vdots$$

M.4 `for/while` *Loops*

Adding a 'while' is a short way of adding an inherent 'if' for every value of the 'for' loop. In the next example, we add '`while i^2 <`

`100'` to check that the square of the index i is less than 100 every time it increases in value. The moment that this condition is no longer met, the loop terminates.

Let's say we want to add the first few squares together: $1^2 + 2^2 + 3^2 + \cdots$ until i^2 becomes greater than or equal to 100. Instead of adding an 'if' statement every time the loop increases, we can do the same thing with a 'while':

We assign an initial total of 0 so that we can add a value to it for each iteration of the loop. The line

```
total := total + 2;
```

adds 2 to the current value of `total` before reassigning the new value.

```
> total := 0:
> for i from 1 while i^2 < 100 do
    total := total + i^2
  end do
```

$$\begin{aligned} total &:= 1 \\ total &:= 5 \\ total &:= 14 \\ total &:= 30 \\ total &:= 55 \\ total &:= 91 \\ total &:= 140 \\ total &:= 204 \\ total &:= 285 \end{aligned}$$

This loop has calculated the value $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 7^2 + 8^2 + 9^2 = 285$.

M.5 `for` Loops with Conditionals

We now combine all of the various conditional statements and loops together into one example. Let's assume we have a function $g(x)$ and want to test whether this function is (i) increasing ($g'(x) > 0$), (ii) decreasing ($g'(x) < 0$), or (iii) neither (critical point ($g'(x) = 0$)).

The loop we have constructed behaves according to the following steps:

Note that we can use any letter for the loop index, but the most common choices are i, j, k, and l.

1. Begin with the value $j = -2$.

2. If $g'(j) > 0$, then g is increasing at $x = j$.

3. If $g'(j) < 0$, then g is decreasing at $x = j$.

4. If $g'(j) = 0$, then g is neither increasing nor decreasing at $x = j$.

5. Update j to the next value in the list and repeat steps 2. through 4.

```
> g(x) := x^4 - 4*x^3 + 3*x^2:
> for j in [-2, 0, 4] do
    if subs(x = j, diff(g(x), x) ) > 0 then
      print(j, "increasing")
    elif subs(x = j, diff(g(x), x)) < 0 then
      print(j, "decreasing")
    else print(j, "neither")
    end if
  end do
```

$$-2, \text{“decreasing”}$$
$$0, \text{“neither”}$$
$$4, \text{“increasing”}$$

From the output, we an see that $g'(-2) < 0$, $g'(0) = 0$, and $g'(4) > 0$.

List of Common Errors

In this section, we will review some of the most common errors encountered when using Maple. Many of these errors are caused when using 2D Math, which makes complicated expressions look pretty, but can cause other issues. For a description of font styles, see Tutorial A, page 89.

Missing Brackets

In Maple, we will use several types of brackets such as parentheses, square brackets, and curly braces. Maple refers to these as delimiters. If these delimiters are not found in pairs, then Maple will be unable to understand the syntax of your command.

In this example, there is an open square bracket, but no square closed bracket.

$> plot([f(x), g(x), x = -5..5);$

```
Error, unable to match delimiters
```

Spelling Errors

Maple cannot correct for poor spelling. If a command is misspelled, then it will treat the command as a variable name.

Maple doesn't know to solve this equation if you don't spell the `solve()` command properly!

$> slove(x^2 = 4);$

$$slove\left(x^2 = 4\right)$$

The `%` Shortcut Not Working as Intended

The `%` shortcut will only use the output command that was last run.

In this example, the expected output of the second command is the decimal value of $\cos(\pi)$, which is equal to -1. However, it appears that the second command was run after the third, producing the decimal approximation of the third command.

It is usually a good practice to use the `%` operator on the same line as the previous command. For example:

```
> subs(x=Pi,cos(x)); evalf(%);
```

$> subs(x = \pi, \cos(x));$

$$\cos(\pi)$$

$> evalf(\%);$

$$0.6735930582$$

$> fsolve(x^3 + 4x = 3);$

$$0.6735930582$$

Missing Multiplication between Variables

Whenever two variables (such as x and y) are multiplied together, you must explicitly include the multiplication sign between them. If no * or space is included between the variables, it may be interpreted as an entirely different variable name.

$> implicitplot(x^2 + y^2 = 6xy, x = -5..5, y = -5..5);$

```
Error, (in plots/implicitplot) invalid input:  the following
extra unknowns were found in the input expression: xy
```

In this example, no multiplication was included between x and y. Maple mistakenly thinks that xy is the name of a third variable.

Spaces and Parentheses: Multiplication versus Functions

When typing in 2D Math (the default font), spaces and parentheses may be interpreted by Maple in unintended ways.

When using commands, make sure that there is no space between the command name and its parentheses. This will be treated as multiplication.

$> plot\ \left(x^2 + 5\right);$

$$plot\ \left(x^2 + 5\right)$$

In this example, Maple thinks that the word "plot" should be multiplied by the expression in parentheses. This is why the name of the command appears in the output.

Similarly, if there is no space or $*$ between a variable name and the parentheses, the notation may be mistaken as function notation.

$expand\ (2\,x(x^3 - 1));$

$$2\,x\left(x^3 - 1\right)$$

Clearly, Maple should be able to expand this basic expression. However, it misinterprets the user's input as being function notation. Maple reads this expression as "x of $x^3 - 1$".

Assigning Expressions to x or Other Common Variable Names

It's never a good idea to assign an expression to x, y, t, or other common variable names. If you want to unassign everything at once, you can do this with the `restart` command on a separate line.

$> x := 3; factor(x^2 - 4x - 12);$

$$x := 3$$
$$-15$$

$> restart; factor(x^2 - 4x - 12);$

$$(x + 2)\,(x - 6)$$

In this example, the value 3 is assigned to x, so the expression $x^2 - 4x - 12$ is equal to the value -15. This is remedied after using the `restart` command.

You will need to load any required packages again after running the `restart` command.

Equals Signs versus Assignment Operator

The equals sign = must be used for an equation and the assignment operator `:=` is used to store a value or expression for later use.

$> subs(x := 5, \cos\,(x) - 1);$

```
Error, invalid argument sequence
```

The equals sign is used for the `subs()` command, since we are not assigning the value for the remainder of the document.

Assigning a Function and Not Using Function Notation

If you have assigned a function (such as $f(x)$) in Maple, make sure to use function notation in other commands, rather than using only the name of the function.

In this example, the `factor()` command should be `factor(f(x))`.

$> f(x) := x^2 - 4; factor(f)$

$$f := x \mapsto x^2 - 4$$
$$f$$

Case-Sensitive Commands

Commands are case-sensitive. Make sure to use the correct capitalization.

The correct command here is `solve()`, which needs a lowercase 's'.

$> Solve(3\,x + 5 = 2)$

$$Solve\,(3\,x + 5 = 2)$$

There are examples of commands where the capitalized version behaves differently from the non-capitalized version.

In this example, the `Int()` command and `int()` command behave differently. Using `Int()` followed by `value(%)` will display the integral and then evaluate it.

$> Int(2x, x = 10..13);$

$$\int_{10}^{13} 2\,x\,\mathrm{d}x$$

$> int(2x, x = 10..13);$

$$69$$

List of Common Commands

Keyboard Shortcuts

Ctrl+J, Ctrl+K	New execution group (after or before current line)
Ctrl+Shift+J, Ctrl+Shift+K	New paragraph (after or before current line)
Ctrl+T	Change to inline text mode
Ctrl+M	Change to Maple input mode
Ctrl+R	Change to inline math mode
Ctrl+.	Indent section
Ctrl+,	Unindent section
Ctrl+Delete	Delete section

Common Expressions

exp(x)	The exponential function
sqrt(x)	The square root function
abs(x)	The absolute value function
root[n](x)	The nth root
surd(x,n)	The fractional exponent, $x^{1/n}$ (works better in the case where $x < 0$)

Manipulating Expressions

name := expression	Assigns an expression to a variable
subs(x=a, expression)	Evaluate an expression at $x = a$
evalf(expression)	Evaluate the given expression as a (floating point) decimal
factor(expression)	Factor the given expression
simplify(expression)	Simplify the given expression
expand(expression)	Expand the given expression
collect(expression,var)	Collect terms of the expression by the specified variable

Solving Equations

solve(equation,var)	Solves the given equation for the specified variable
fsolve(equation,var)	Solves the given equation for the specified variable (as a decimal)
fsolve(equation,var=a..b)	Solves the given equation for the specified variable (as a decimal) on the interval $[a, b]$
solve({eqn1,eqn2},{var1,var2})	Solves a system of equations for all specified variables
fsolve({eqn1,eqn2},{var1,var2})	Solves a system of equations for all specified variables (as a decimal)

	Defining Functions
name(var) := expression	Assigns a function of the specified variable
name := unapply(expression,var)	Convert an expression to a function of the specified variable
name(var) := piecewise(condition,expr, condition,expr,...)	Create a piecewise function of the specified variable
	Plotting Functions
plot(f(x),x=a..b)	Plot the given function, $f(x)$, over the interval $[a,b]$
plot([f(x),g(x)],x=a..b)	Plot two functions, $f(x)$ and $g(x)$, over the interval $[a,b]$
	Plot Options
y=c..d	Only plot the range $c \leq y \leq d$
discont=true	Includes discontinuities in a plot
colour=blue	Specify the colour for a graph (black, blue, red, etc.)
linestyle=dotted	Specify the style of the line (dash, dot, etc.)
	Limits
limit(expression,var=a)	Find the limit of the expression as var approaches a
limit(expression,var=a,right)	Find the limit of the expression as var approaches a from the right
limit(expression,var=a,left)	Find the limit of the expression as var approaches a from the left
limit(expression,var=infinity)	Find the limit of the expression as var approaches infinity
	Derivatives
diff(expression,var)	The derivative of the given expression with respect to variable
diff(expression,var,var) diff(expression,var$2)	The second derivative of the given expression with respect to variable
f'(var)	The derivative of the function f with respect to variable
f^(n)(var)	The nth derivative of the function f with respect to variable

Implicit Functions (requires "plots" package for plotting)

implicitplot(equation,x=a..b,y=c..d, numpoints=30000)	Plot the implicit function given by the equation over the region $a \leq x \leq b$, $c \leq y \leq d$ and ensure a smooth curve
implicitplot(equation,x=a..b,y=c..d, grid=[250,250])	Plot the implicit function given by the equation over the region $a \leq x \leq b$, $c \leq y \leq d$ and ensure a smooth curve
implicitdiff(equation,y,x)	The derivative of the implicit function, given as dy/dx

Riemann Sums and Numerical Integration (requires "Student[Calculus1]" package)

ApproximateInt(f(x),x=a..b)	Approximate the definite integral of $f(x)$ from $x = a$ to $x = b$

ApproximateInt Options

method=midpoint	Chose the method of approximation (left/right/lower/upper/ midpoint/trapezoid/simpson)
output=sum/value/plot/animation	Choose to output summation notation, exact value, graph (with value), or animate
partition=n	Use n equally spaced subintervals for approximation

Sequences and Series

expression $ var=a..b	Display the sequence of the expression from var= a up to var= b
seq(expression,var=a..b)	Display the sequence of the expression from var= a up to var= b
Sum(expression,var=a..b)	Display the sum for the expression from var= a up to var= b)
sum(expression,var=a..b)	Give the value of the sum of the expression from var= a up to var= b
taylor(f(x),x=a,n)	Give the Taylor series expansion of $f(x)$ about $x = a$, including terms up to power $n - 1$

Integrals

Int(f(x),x)	The indefinite integral of the function $f(x)$ with respect to x (inert form)
int(f(x),x)	The indefinite integral of the function $f(x)$ with respect to x, evaluated
Int(f(x),x=a..b)	The definite integral of $f(x)$ from $x = a$ to $x = b$ (inert form)
int(f(x),x=a..b)	The definite integral of $f(x)$ from $x = a$ to $x = b$, evaluated

	Differential Equations
dsolve(DE, y(x))	Solves the given differential equation for $y(x)$
dsolve([DE, ICs], y(x))	Solves the given differential equation with initial conditions for $y(x)$
	Direction Fields (Requires "DEtools" package)
DEplot(DE,y(x),x=a..b,y=a..b)	Plot the direction field for the differential equation dy/dx
DEplot(DE,y(x),x=a..b,y=a..b,[ICs])	Plot the direction field for the differential equation dy/dx with initial conditions
	DEplot Options
arrows=line	Use lines for the direction field, rather than arrows
colour=black	Change arrow colour
linecolour=blue	Change solution curve colour