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TOPOLOGY

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Contents

1 Subspaces of Euclidean space

1.1 Basic definitions

Our starting place is the study of the *subspaces* of Euclidean space.

Notation 1.1.1. As usual, $\boldsymbol{R}$ will denote the ordered field of real numbers. For any $n \geq 0$, we write $\boldsymbol{R}^n$ for the vector space of n-tuples $x = (x_1, \dots, x_n)$ with $x_i \in \boldsymbol{R}$.

For our purposes, one of the most important features $\boldsymbol{R}^n$ has to offer is its *metric*; this is the map

$$d\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R} ,$$

called the *distance*, given by the formula

$$d(x, y) \coloneqq \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} .$$

This map enjoys four key properties:

1. (Nonnegativity.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) \geq 0$.
2. (Identity.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) = 0$ if and only if $x = y$.
3. (Symmetry.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) = d(y, x)$.
4. (Triangle inequality.) For every triple of points $x, y, z \in \boldsymbol{R}^n$, one has

$$d(x, y) + d(y, z) \geq d(x, z) .$$

If $r \geq 0$ and $x \in \boldsymbol{R}^n$, then we define the *ball of radius r centered at x*:

$$B^n(x, r) \coloneqq \{y \in \boldsymbol{R}^n : d(x, y) < r\} .$$

In topology, we are not interested in the metric d itself but in what it can tell us about the nearness of points and subsets:

Definition 1.1.2. Let $S \subseteq \boldsymbol{R}^n$ be a subset. A point $x \in \boldsymbol{R}^n$ will be said to be *close to S* if and only if, for every $\varepsilon > 0$, there exists an element $s \in S$ such that $d(x, s) < \varepsilon$.

We may express this notion with the aid of an auxiliary quantity. Define the *distance from x to S* as[1]

$$d(x, S) \coloneqq \inf\{d(x, s) : s \in S\} .$$

Then x is *close to S* if and only if $d(x, S) = 0$.

[1] If $S = \varnothing$, then this infimum doesn't exist as a real number. So we'll declare formally that $d(x, \varnothing) = +\infty$.

Let $X \subseteq \mathbf{R}^n$ be a subset such that $S \subseteq X$. We shall write $\tau_X(S) \subseteq X$ for the set of $x \in X$ that are close to S:

$$\tau_X(S) \coloneqq \{x \in X : (\forall \varepsilon > 0)(\exists s \in S)(d(x,s) < \varepsilon)\} = \{x \in X : d(x,S) = 0\}$$

This is called the *closure of S in X*. In other words,

$$\tau_X(S) = X \cap \bigcap_{\varepsilon>0} \bigcup_{s \in S} B^n(s,\varepsilon) .$$

We will refer to X along with the map[2] $\tau_X : \mathbf{P}(X) \to \mathbf{P}(X)$ as the *subspace* $X \subseteq \mathbf{R}^n$.

[2] The set $\mathbf{P}(X)$ is the *powerset* of X; it is the set of subsets of X.

Every time you see a new definition in mathematics, you should begin immediately to think about as many examples as you can. You should also have a couple of *counterexamples* at hand, preferably at least one for each condition in a definition. Then, when you encounter a theorem with a proof, you can take your example and 'run' the proof on your example, and you can run it on your counterexample to see where the conditions are really being used. This is one of the most important ways to understand how theorems really work.

In that spirit, let's dig into some examples.

Example 1.1.3.
- If $X = \varnothing$, then this definition becomes a little boring: the only subset of $\varnothing$ is $\varnothing$ itself, and $\tau_\varnothing(\varnothing) = \varnothing$.
- If $X = \mathbf{R}$ itself, then let's see what happens for various choices of S:
  - Let's consider various unit intervals:

$$\tau_{\mathbf{R}}(]0,1[) = \tau_{\mathbf{R}}([0,1]) = \tau_{\mathbf{R}}(]0,1]) = [0,1]$$

  - If $\mathbf{Q} \subset \mathbf{R}$ denotes the set of rational numbers, then $\tau_{\mathbf{R}}(\mathbf{Q}) = \mathbf{R}$.
  - At the same time, $\tau_{\mathbf{R}}(\mathbf{R} \smallsetminus \mathbf{Q}) = \mathbf{R}$.
  - Confirm that $\tau_{\mathbf{R}}(\varnothing) = \varnothing$.
- If $X \subseteq \mathbf{R}^n$ is a subspace, then we have

$$\tau_X(S) = \tau_{\mathbf{R}^n}(S) \cap X .$$

This is an important point: the closure of S is done *relative to the subspace X*. Let's see how the closure of $S = \,]0,1[$ depends on which subspace $X \subseteq \mathbf{R}$ in which we are closing:

  - If $X = \mathbf{R}$, then $\tau_{\mathbf{R}}(]0,1[) = [0,1]$.
  - If $X =]0,1[$, then $\tau_{]0,1[}(]0,1[) =]0,1[$.
  - If $X =]0,1]$, then $\tau_{]0,1]}(]0,1[) =]0,1]$.
  - If $X =]0,+\infty[$, then $\tau_{]0,+\infty[}(]0,1[) =]0,1]$.

Example 1.1.4. Let $X \subseteq \mathbf{R}$ be a subspace. Let $S \subseteq X$ be a nonempty subset that is bounded above. The completeness property of $\mathbf{R}$ then ensures that S has a supremum $\sup S$. If $\sup S \in X$, then $\sup S$ is close to S in X; that is,

$$\sup S \in \tau_X(S) .$$

Let's prove this. Let $\varepsilon > 0$. If $]x-\varepsilon, x+\varepsilon[\cap S \neq \varnothing$, then $x-\varepsilon$ is also an upper bound for S, contradicting the assumption that x is the *least* upper bound for S. Consequently, there exists an element $s \in S$ such that $d(x,s) < \varepsilon$.

The same argument shows that if $S \subseteq X$ is nonempty and bounded below, then if $\inf S \in X$, then

$$\inf S \in \tau_X(S) .$$

Here are the basic facts about the closure operator $\tau_X \colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$. As we shall see, these facts are in many ways even more important than the definition of τ_X itself.

Proposition 1.1.5. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace.*

1. *For every subset $S \subseteq X$, every point of S is close to S. That is, $S \subseteq \tau_X(S)$.*
2. *No points of X are close to $\varnothing$. That is, $\tau_X(\varnothing) = \varnothing$.*
3. *If $S_1, \dots, S_n \subseteq X$ are a finite collection of subsets, then a point $x \in X$ is close to $S_1 \cup \cdots \cup S_n$ if and only if, for some i, the point x is close to S_i. That is,*

$$\tau_X(S_1 \cup \cdots \cup S_n) = \tau_X(S_1) \cup \cdots \cup \tau_X(S_n) .$$

4. *If $S \subseteq X$, then if $x \in X$ is a point that is close to $\tau_X(S)$, then $\boldsymbol{x}$ is close to S itself. That is, $\tau_X(\tau_X(S)) = \tau_X(S)$.*

Proof. Let us prove these in order.

1. If $x \in S$, then for every $\varepsilon > 0$, we certainly have $d(x,x) = 0 < \varepsilon$.
2. For every $\varepsilon > 0$, the sentence[3] $(\exists s \in \varnothing)(d(x,s) < \varepsilon)$ is false.
3. If x is close to $S_1 \cup \cdots \cup S_n$, then for every natural number[4] $m \in \boldsymbol{N}^*$, we may select $s_m \in S_1 \cup \cdots \cup S_n$ such that $d(x, s_m) < 1/m$. By the pigeonhole principle, there exists an i such that the set $\{m \in \boldsymbol{N}^* : s_m \in S_i\}$ is infinite. Consequently, for any $\varepsilon > 0$, there exists a natural number $m \geq 1$ such that $s_m \in S_i$ and $1/m < \varepsilon$. In particular

$$d(x, s_m) < 1/m < \varepsilon .$$

Thus x is close to S_i.

Conversely, if x is close to some S_i, then for every $\varepsilon > 0$, there exists $s \in S_i \subseteq S_1 \cup \cdots \cup S_n$ such that $d(x,s) < \varepsilon$, so x is close to $S_1 \cup \cdots \cup S_n$.

4. If x is close to $\tau_X(S)$, then for any $\varepsilon > 0$, we may select $t \in \tau_X(S)$ such that $d(x,t) < \varepsilon/2$. Since t is close to S, we may select $s \in S$ such that $d(t,s) < \varepsilon/2$. Now by the triangle inequality,

$$d(x,s) \leq d(x,t) + d(t,s) < \varepsilon .$$

Thus x is close to S. □

[3] Sentences of the form $(\exists s \in \varnothing)(\phi(s))$ are *always* false, no matter what $\phi(s)$ says!

[4] In this text, $\boldsymbol{N}$ will denote the natural numbers $\{0, 1, \dots\}$, and $\boldsymbol{N}^*$ will denote the natural numbers $\{1, 2, \dots\}$.

Corollary 1.1.6. *If $X \subseteq \boldsymbol{R}^n$ is a subspace, then for any subsets $S, T \in \boldsymbol{P}(X)$, if $S \subseteq T$, then $\tau_X(S) \subseteq \tau_X(T)$.*

Proof. We may write $T = S \cup (T \smallsetminus S)$. Then by the third point of the proposition above,

$$\tau_X(T) = \tau_X(S) \cup \tau_X(T \smallsetminus S)\ ,$$

and so $\tau_X(S) \subseteq \tau_X(T)$. □

If $X \subseteq \boldsymbol{R}^n$ is a subspace, then there are two different kinds of subset that play a big role in topology:

Definition 1.1.7. A subset $Z \subseteq X$ is *closed in* X if and only if

$$\tau_X(Z) = Z\ .$$

In other words, Z is closed if and only if every point of X that is close to Z is contained in Z.

A subset $U \subseteq X$ is *open in* X if and only if $X \smallsetminus U$ is closed. In other words, U is open if and only if no point of U is close to the complement of U.

Example 1.1.8.
- For any subspace $X \subseteq \boldsymbol{R}^n$, the subset $X \subseteq X$ is both open and closed.[5]
- For any subspace $X \subseteq \boldsymbol{R}^n$, the subset $\varnothing \subseteq X$ is both open and closed.
- In $\boldsymbol{R}$ itself, the subset $[0,1] \subset \boldsymbol{R}$ is closed but not open.
- In $\boldsymbol{R}$, the subset $]0,1[\subset \boldsymbol{R}$ is open but not closed.
- In $\boldsymbol{R}$, the subsets $[0,1[$ and $]0,1]$ are neither open nor closed.
- In the subspace $]0,+\infty[\subseteq \boldsymbol{R}$, the subset $]0,1] \subseteq]0,+\infty[$ is closed but not open.
- In the subspace $[0,+\infty[\subseteq \boldsymbol{R}$, the subset $[0,1[\subseteq [0,+\infty[$ is open but not closed.

[5] This is sometimes a source of confusion – or at least irritation. One may expect that 'open' and 'closed' are opposites, but they are not: there are subsets that are neither, and there are subsets that are both. Instead, they are *complementary* notions.

Proposition 1.1.9. *For any subspace $X \subseteq \boldsymbol{R}^n$ and any subset $S \subseteq X$, the closure $\tau_X(S)$ is the smallest closed subset*[6] *of X that contains S.*

[6] That is, $\tau_X(S) \subseteq X$ is a closed subset that contains S, and for any closed subset $Z \subseteq X$ that contains S, one has $\tau_X(S) \subseteq Z$.

Proof. Let $Z \subseteq X$ be a closed subset that contains S. Then

$$\tau_X(S) \subseteq \tau_X(Z) = Z\ .$$

Hence $\tau_X(S)$ is contained in any closed subset of X that contains S. It remains to note that $\tau_X(S)$ is itself a closed subset of X that contains S. By the proposition above,

$$\tau_X(\tau_X(S)) = \tau_X(S)\ ,$$

so $\tau_X(S)$ is closed. □

Proposition 1.1.10. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. Then $U \subseteq X$ is open if and only if, for every $x \in U$, there exists an $\varepsilon > 0$ such that*

$$B^n(x,\varepsilon) \cap X \subseteq U\ .$$

Proof. Suppose that $U \subseteq X$ is open, and let $x \in U$ be a point. Then by definition, x is not close to $X \smallsetminus U$. Hence there exists $\varepsilon > 0$ such that for any point $y \in X \smallsetminus U$, one has $d(x, y) \geq \varepsilon$. That is, U contains $B^n(x, \varepsilon) \cap X$.

Now assume that for every $x \in U$, there exists an $\varepsilon > 0$ such that

$$B^n(x, \varepsilon) \cap X \subseteq U\,.$$

We aim to show that $X \smallsetminus U$ is closed; hence let $x \in U$ be a point that is close to $X \smallsetminus U$. For some $\varepsilon > 0$, the intersection $B^n(x, \varepsilon) \cap X$ is contained in U. But at the same time, since x is close to $X \smallsetminus U$, the intersection $B^n(x, \varepsilon) \cap (X \smallsetminus U) \neq \varnothing$. This is a contradiction that shows that no point of U is close to $X \smallsetminus U$; thus U is open. □

Proposition 1.1.11. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. Then a subset $Z \subseteq X$ is closed if and only if $Z = Z' \cap X$ for some closed subset $Z' \subseteq \boldsymbol{R}^n$. Similarly, a subset $U \subseteq X$ is open if and only if $U = U' \cap X$ for some open subset $U' \subseteq \boldsymbol{R}^n$.*

Proof. Assume that $Z' \subseteq \boldsymbol{R}^n$ is a closed subset. Now consider the intersection $Z' \cap X$. The closure $\tau_{\boldsymbol{R}^n}(Z' \cap X)$ is contained in $\tau_{\boldsymbol{R}^n}(Z') = Z'$. Consequently,

$$\tau_X(Z' \cap X) = \tau_{\boldsymbol{R}^n}(Z' \cap X) \cap X \subseteq Z' \cap X\,,$$

so $Z' \cap X$ is closed.

Conversely, assume that $Z \subseteq X$ is a closed subset. Then $Z' \coloneqq \tau_{\boldsymbol{R}^n}(Z)$ is closed in $\boldsymbol{R}^n$, and

$$Z = \tau_X(Z) = \tau_{\boldsymbol{R}^n}(Z) \cap X = Z' \cap X.$$

The statements about open subsets of X now follow by taking complements.[7] Indeed, if $U' \subseteq \boldsymbol{R}^n$ is an open subset, then let Z' denote its complement. Since $Z' \cap X$ is closed by what we've already proved, its complement in X, which is $U' \cap X$ is open. Conversely, if $U \subseteq X$ is open, then let Z denote its complement in X. By what we've already shown, there is a closed subset $Z' \subseteq \boldsymbol{R}^n$ such that $Z = Z' \cap X$, so if U' denotes the open complement of Z', then $U = U' \cap X$. □

[7] For proofs of this kind, it is usually enough just to offer such a short sentence. In this case, we will unpack this sentence a little more.

Proposition 1.1.12. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. For any family $\{Z_a\}_{a \in A}$ of closed subsets of X, the intersection $\bigcap_{a \in A} Z_a$ is closed as well; if A is* finite, *then the union $\bigcup_{a \in A} Z_a$ is closed as well. Dually, for any family $\{U_a\}_{a \in A}$ of open subsets of X, the intersection $\bigcup_{a \in A} U_a$ is open as well; if A is* finite, *then the intersection $\bigcap_{a \in A} U_a$ is open as well.*

Proof. Let $W = \bigcap_{a \in A} Z_a$. We aim to prove that $\tau_X(W) \subseteq W$. Let $x \in X$ be a point that is close to W. For every $a \in A$, since $W \subseteq Z_a$, it follows that x is close to Z_a as well. Since Z_a is closed, $x \in Z_a$. This happens for every $a \in A$, so $x \in W$.

If A is finite, then we have shown that

$$\tau_X\left(\bigcup_{a \in A} Z_a\right) = \bigcup_{a \in A} \tau_X(Z_a) = \bigcup_{a \in A} Z_a\,,$$

so the union is closed.

The dual statement follows by forming complements. □

Example 1.1.13. Let's consider examples in $\boldsymbol{R}$ in which the finiteness hypotheses of the previous proposition are necessary.

- Infinite unions of closed subsets need not be closed:

$$\bigcup_{n\in\boldsymbol{N}^*}\left[-1+\frac{1}{n},1-\frac{1}{n}\right]=\left]-1,1\right[\ ,$$

  or open:

$$\bigcup_{n\in\boldsymbol{N}^*}\left[0,1-\frac{1}{n}\right]=\left[0,1\right[\ ..$$

- Infinite intersections of open subsets need not be open:

$$\bigcap_{n\in\boldsymbol{N}^*}\left]-1-\frac{1}{n},1+\frac{1}{n}\right[=[-1,1]\ ,$$

  or closed:

$$\bigcap_{n\in\boldsymbol{N}^*}\left]0,1+\frac{1}{n}\right[=\left]0,1\right]\ .$$

We conclude with some examples of subspaces of Euclidean space to which we will return regularly.

Example 1.1.14.
- For any $n\in\boldsymbol{N}$, the *n-sphere* S^n is the subspace

$$S^n\coloneqq\{x\in\boldsymbol{R}^{n+1}\ :\ \|x\|=1\}\ .$$

- The *open unit ball* is the subspace

$$B^n\coloneqq B^n(0,1)=\{x\in\boldsymbol{R}^n\ :\ \|x\|<1\}\ ,$$

  and its closure in $\boldsymbol{R}^n$ is the *closed unit ball*

$$\overline{B}^n\coloneqq\{x\in\boldsymbol{R}^n\ :\ \|x\|\leq 1\}\ .$$

- If we have a finite collection of subspaces $X_1\subseteq\boldsymbol{R}^{n_1},\dots,X_k\subseteq\boldsymbol{R}^{n_k}$ then we obtain the subspace

$$X_1\times\cdots\times X_k\coloneqq\{x=(x^1,\dots,x^k)\in\boldsymbol{R}^{n_1}\times\cdots\times\boldsymbol{R}^{n_k}\ :\ (\forall i)(x^i\in X_i)\}\ .$$

- For any $n\in\boldsymbol{N}$, the *n-torus* is the subspace

$$T^n\coloneqq S^1\times\cdots\times S^1\subseteq\boldsymbol{R}^{2n}\ .$$

  That is,

$$T^n\coloneqq\{(x_1,y_1,\dots,x_n,y_n)\in\boldsymbol{R}^{2n}\ :\ (\forall i)(x_i^2+y_i^2=1)\}\ .$$

1.2 *Continuity*

Another good name for this bit would be, 'A first introduction to what topology is really about'.

Definition 1.2.1. Let $X\subseteq\boldsymbol{R}^m$ and $Y\subseteq\boldsymbol{R}^n$ be subspaces. A map $f\colon X\to Y$ is *continuous* if and only if, for any subset $S\subseteq X$ and any point $x\in X$, if x is close to S, then $f(x)$ is close to $f(S)$. In other words, f is continuous if and only if, for any subset $S\subseteq X$, one has $f(\tau_X(S))\subseteq\tau_Y(f(S))$.

Proposition 1.2.2. *Let $X \subseteq \boldsymbol{R}^m$ and $Y \subseteq \boldsymbol{R}^n$ be subspaces. The following are equivalent for a map $f\colon X \to Y$.*

1. *The map f is continuous.*
2. *For any subset $T \subseteq Y$, one has $\tau_X(f^{-1}(T)) \subseteq f^{-1}(\tau_Y(T))$.*
3. *For any closed subset $Z \subseteq Y$, the inverse image $f^{-1}(Z) \subseteq X$ is closed.*
4. *For any open subset $U \subseteq Y$, the inverse image $f^{-1}(U) \subseteq X$ is open.*
5. *For any $x \in X$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0 \in X$ is a point such that $d(x_0, x) < \delta$, then*[8] *$d(f(x_0), f(x)) < \varepsilon$.*

[8] This is probably the characterization of continuity that is most familiar to you from analysis. In this text, we are happy to assume basic facts of analysis about the continuity of well-known functions on subspaces of Euclidean space.

Proof. Assume (1); we aim to prove (2). Let $T \subseteq Y$, and assume that x is close to $f^{-1}(T)$. Then by assumption $f(x)$ is close to $f(f^{-1}(T))$. Since $f(f^{-1}(T)) \subseteq T$, it follows that $f(x)$ is close to T, so $x \in f^{-1}(\tau_Y(T))$.

Assume (2); we aim to prove (3). Let $Z \subseteq Y$ be closed. Then

$$\tau_X(f^{-1}(Z)) \subseteq f^{-1}(\tau_Y(Z)) = f^{-1}(Z)\,,$$

so $f^{-1}(Z)$ is closed.

Assume (3); then (4) follows by taking complements.

Assume (4); we aim to prove (5). Let $x \in X$, and let $\varepsilon > 0$. The subset $f^{-1}(B^n(f(x), \varepsilon) \cap Y) \subseteq X$ is open. Consequently, there exists $\delta > 0$ such that $B^m(x, \delta) \cap X \subseteq f^{-1}(B^n(f(x), \varepsilon) \cap Y)$; that is, for any $x_0 \in X$ such that $d(x_0, x) < \delta$, one has $d(f(x_0), f(x)) < \varepsilon$, as desired.

Assume (5); we aim to prove (1). Let $S \subseteq X$, and let $x \in X$ be a point that is close to S. Let $\varepsilon > 0$. By assumption there is a $\delta > 0$ such that for any $x_0 \in X$ with $d(x_0, x) < \delta$, we have $d(f(x_0), f(x)) < \varepsilon$. Since x is close to S, we may select $s \in S$ such that $d(x, s) < \delta$. Thus $d(f(x), f(s)) < \varepsilon$. This proves that $f(x)$ is close to $f(S)$. □

To be the 'same', topologically speaking, is to be *homeomorphic*.

Definition 1.2.3. Let $X \subseteq \boldsymbol{R}^m$ and $Y \subseteq \boldsymbol{R}^n$ be subspaces. Then a *homeomorphism* $f\colon X \to Y$ is a continuous bijection whose inverse f^{-1} is continuous. In other words, a map $f\colon X \to Y$ is a homeomorphism if and only if f is a bijection such that for any subset $S \subseteq X$, one has $\tau_Y(f(S)) = f(\tau_X(S))$. Equivalently, $f\colon X \to Y$ is a homeomorphism if and only if it is a bijection such that both the image and the inverse image of closed (respectively, open) subsets remains closed (resp., open).

The two subspaces X and Y are said to be *homeomorphic* if and only if there is a homeomorphism $f\colon X \to Y$.

The following is a very important example to understand what *topology* is all about:

Example 1.2.4. The subspaces $\boldsymbol{R}$, $]0, +\infty[$, and $]0, 1[$ are all homeomorphic. To prove this, let's construct some homeomorphisms. The exponential map $\exp\colon \boldsymbol{R} \to]0, +\infty[$ is a continuous bijection[9] whose inverse is the logarithm $\log\colon]0, +\infty[\to \boldsymbol{R}$, which is also continuous. This shows that $\boldsymbol{R}$ and $]0, +\infty[$ are homeomorphic.

[9] In this course, we will accept these basic facts of analysis.

The map $f\colon \boldsymbol{R} \to \,]0,1[$ given by

$$f(x) = \frac{1}{1+\exp(-x)}$$

is a continuous bijection with inverse $g\colon \;]0,1[\, \to \boldsymbol{R}$ given by

$$g(x) = -\log\left(\frac{1-x}{x}\right),$$

which is also continuous. Thus $\boldsymbol{R}$ and $]0,1[$ are homeomorphic.

Please observe that what we are saying here is that an open interval and the real line are *topologically the same*. The fact that the real line is 'infinitely long' is of no importance to the topologist; the topology 'knows' about the relation of closeness, but nothing about distance.

You can think about topology as a certain 'filter' that you've applied to the information supplied by a geometric object. That filter removes a lot of what we ordinarily think of as important about these objects – distances, angles, and so on. There's still some information left over, however – a kind of 'proto-geometric' information.

We imagine our subspace as made up of some infinitely deformable material. As long as we don't tear it, or poke holes in it, we are free to stretch it, twist it, and move it around, and we haven't changed its 'topological nature'. Understanding that topological nature is the central topic of this course.

Example 1.2.5. Consider the following two subpsaces. First, we have $X \coloneqq \boldsymbol{R} \smallsetminus \{0\} \subset \boldsymbol{R}$. Next, we have

$$Y \coloneqq \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\} \subset \boldsymbol{R}^2 .$$

These spaces are homeomorphic. To see this, let us define a map $f\colon X \to Y$. We shall let

$$f(t) = \left(\frac{t}{|t|}, \log|t|\right).$$

This is a continuous bijection with inverse $g \coloneqq Y \to X$ given by

$$g(x,y) = x\exp(y),$$

which is also continuous.

By taking the real line and removing a point from it, we obtained two lines.[10]

[10] You may be wondering: is there some weird homeomorphism between $\boldsymbol{R}$ itself and Y? Maybe one line is somehow homeomorphic to two lines? It turns out that $\boldsymbol{R}$ and Y are not homeomorphic, and we'll see why in the next section.

Roughly speaking, the pursuit of *topology* is the study of subspaces of $\boldsymbol{R}^n$ and more general *topological spaces* (which we will define in the next section) *up to homeomorphism*. As topologists, we are constantly interested in the following question: *are these two spaces the same, topologically?* That is, *are these two spaces homeomorphic?*

Proving that two spaces *are* homeomorphic is, in principle, straightforward: you simply construct a homeomorphism between them.[11] But how does one prove that two spaces are *not* homeomorphic? That is, how does one *distinguish* two spaces?

[11] In practice, it often takes a certain amount of inspiration to cook up a homeomorphism.

For example, it seems intuitive that there isn't a homeomorphism between $\boldsymbol{R}$ and the subspace

$$Y = \{(x,y) : |x| = 1\}$$

from the example above. Somehow, a continuous map from $\boldsymbol{R}$ has to be 'drawn without picking up your pen', and Y has two lines that are separated from each other.

This is where *topological properties* become useful. these are properties that are *homeomorphism invariant*, so that if X has the property, and X is homeomorphic to Y, then Y will also have the property. Let's study the first of these for subspaces of Euclidean space: *connectedness*.

1.3 Connectedness

Definition 1.3.1. Let $X \subseteq \boldsymbol{R}^n$ be a subspace. A subset $S \subseteq X$ that is both open and closed (in X) is said to be *clopen*.[12] We shall say that X is *connected* if there are *exactly two* clopen subsets $S \subseteq X$.

[12] I know, this is silly.

1.3.2. If $X \subseteq \boldsymbol{R}^n$ is a nonempty subspace, then there are always *at least two* clospen subsets of X: the empty set $\varnothing$, and X itself. Thus if X is nonempty, then it is connected if and only if the only nonempty clopen subset is X itself.

Example 1.3.3. The empty set $\varnothing$, however, is not connected: it has only one subset, itself, which is clopen.

Example 1.3.4. The space $\boldsymbol{R}$ itself is connected. Let's prove this. There are at least two clopen subsets: $\varnothing$ and $\boldsymbol{R}$ itself. Now we have to prove that there are no more. So let $S \subseteq \boldsymbol{R}$ be a clopen subset. If $S \neq \varnothing$, then there exists a point $x \in S$. Since S is open, there exists $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\subseteq S$. Now let us consider the set

$$E \coloneqq \{\varepsilon > 0 :]x-\varepsilon, x+\varepsilon[\subseteq S\} \subseteq \boldsymbol{R}\,.$$

The set E is unbounded if and only if $S = \boldsymbol{R}$, so assume that E in bounded. We aim to generate a contradiction. Since E is bounded, it admits a supremum ε_0. In particular, for every $\varepsilon < \varepsilon_0$, one has $]x-\varepsilon, x+\varepsilon[\subseteq S$. Note that $]x-\varepsilon_0, x+\varepsilon_0[= \bigcup_{\varepsilon<\varepsilon_0}]x-\varepsilon, x+\varepsilon[$, so it follows that $]x-\varepsilon_0, x+\varepsilon_0[\subseteq S$ as well. Thus $\varepsilon_0 \in E$.

Now consider the point $x+\varepsilon_0$. This is close to S, because the closure of S contains the closure of $]x-\varepsilon_0, x+\varepsilon_0[$, which is $[x-\varepsilon_0, x+\varepsilon_0]$. Since S is closed, it follows that $x+\varepsilon_0 \in S$. The same analysis shows that $x-\varepsilon_0 \in S$.

Again since S is open, we may choose a $\delta > 0$ such that $]x+\varepsilon_0-\delta, x+\varepsilon_0+\delta[\subset S$ *and* $]x-\varepsilon_0-\delta, x-\varepsilon_0+\delta[\subset S$. But now we have that $]x-\varepsilon_0-\delta, x+\varepsilon_0+\delta[\subset S$, so $\varepsilon_0+\delta \in E$, which contradicts the maximality of ε_0.

This contradiction shows that E is unbounded, and so the only nonempty clopen subset of $\boldsymbol{R}$ is $\boldsymbol{R}$ itself.

Example 1.3.5. The subspace

$$Y \coloneqq \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\}$$

we introduced above is not connected. Indeed, in addition to $\varnothing$ and Y itself, the subset

$$Y_+ \coloneqq \{(x,y) \in \boldsymbol{R}^2 : x = 1\}$$

is clopen. To see this, let us show that both Y_+ and its complement

$$Y_- := Y \smallsetminus Y_+ = \{(x, y) \in \boldsymbol{R}^2 : x = -1\}$$

are open. The key observation here is that if $u \in Y_+$ and $v \in Y_-$, then $d(u, v) \geq 2$. Consequently, if $u \in Y_+$, then $B^2(u, 1) \cap Y \subset Y_+$, and, similarly, if $v \in Y_-$, then $B^2(v, 1) \cap Y \subset Y_-$.

Proposition 1.3.6. *Let $\{X_a\}_{a \in A}$ be a nonempty family of connected subspaces $X_a \subseteq \boldsymbol{R}^n$. Assume that for any $a, b \in A$, one has $X_a \cap X_b \neq \varnothing$. Then the union $x := \bigcup_{a \in A} X_a$ is connected as well.*

Proof. Since $X_a \cap X_b$ is nonempty, the union X is nonempty too. Hence it suffices to show that if $V \subseteq X$ is a nonempty clopen, then $V = X$.

Note that for any $a \in A$, the intersection $V \cap X_a$ is clopen in X_a. For each $a \in A$, the subspace X_a is connected, so $V \cap X_a$ is either $\varnothing$ or X_a. In other words, for each $a \in A$, either V is disjoint from X_a or else $X_a \subseteq V$. Since $V \neq \varnothing$, there is at least one $a_0 \in A$ such that $X_{a_0} \subseteq V$.

But now for any other $a \in A$, the nonempty intersection $X_a \cap X_{a_0}$ is contained in $X_a \cap V$; thus $X_a \subseteq V$ as well. We thus conclude that $X \subseteq V$. □

We are now in a position to classify all the connected subspaces of $\boldsymbol{R}$.

Example 1.3.7. Let $X \subseteq \boldsymbol{R}$ be a nonempty subset. We'll say that X is *an interval* if and only if, for every $a, b \in X$ and every $x \in \boldsymbol{R}$ such that $a \leq x \leq b$, we have $x \in X$. Assume that X is an interval. If X is bounded above, then there exists a supremum $b \in \boldsymbol{R}$. If X is bounded below, then there exists an infimum $a \in \boldsymbol{R}$. Now, depending upon whether a and b exist, there are three options for an interval X:

- a interval of finite length such as $[a, b]$, $]a, b]$, $[a, b[$, or $]a, b[$;
- a ray $]a, +\infty[$, $[a, +\infty[$, $]-\infty, b[$, or $]-\infty, b]$; or
- the line $\boldsymbol{R}$ itself.

Here now is our claim: a subspace $X \subseteq \boldsymbol{R}$ is connected if and only if it is an interval – hence if and only if it is of one of the three forms above. To prove this, we must prove two things:

- first, that any interval is connected, and
- second, that any subspace that is not an interval is not connected.

To prove the first statement, assume first that X is a closed interval $[a, b]$. Assume that $V \subseteq [a, b]$ is a clopen subset. Suppose that $x \in [a, b]$ is a point such that $x \notin V$; we aim to show that V is empty.

If there are points $s \in V$ such that $s < x$, let

$$s_0 := \sup\{s \in V : s < x\}.$$

Thus $s_0 \leq x$. Observe that s_0 is close[13] to V. Since V is closed, it follows that $s_0 \in V$. Since $x \notin V$, it follows that $s_0 < x$. Since V is open, there is a $\varepsilon > 0$ such that $]s_0 - \varepsilon, s_0 + \varepsilon[\cap [a, b]$ is contained in V. In particular, there exist

[13] ??

points $s \in V$ such that $s_0 < s < x$, which contradicts the definition of s_0. Consequently, there are no points $s \in V$ such that $s < x$.

On the other hand, if there are points $s \in V$ such that $s > x$, then let

$$s_1 \coloneqq \inf\{s \in V : s > x\} .$$

We can replay the same argument as above to generate a contradiction. Since V is closed, $s_1 \in V$. Since V is open, there are points $s \in V$ such that $x < s < s_1$, which is a contradiction. It therefore follows that $V = \varnothing$, as desired.

So far, we have shown that a closed interval $[a, b]$ is always connected. Now for a general interval X, note that if $x \in X$, then we may write

$$X = \bigcup_{\substack{a,b \in X, \\ a \le x \le b}} [a, b] .$$

This is a union of connected subspaces of $\boldsymbol{R}$ such that any two intersect (because they all contain x). Hence X is connected as well, thanks to the previous proposition.

Now let us prove the converse. Assume that $X \subset \boldsymbol{R}$ is *not* an interval. We aim to prove that X is not connected; it will suffice to construct a nonempty proper clopen $V \subseteq X$. For this, choose real numbers $a < x < b$ such that $a, b \in X$ but $x \notin X$. Now consider the subset

$$V = [x, +\infty[\cap X =]x, +\infty[\cap X .$$

The first expression proves that V is closed; the second proves that V is open. Since $b \in V$ but $a \notin V$, it follows that V is nonempty and proper.

Here's our first truly *topological* theorem. We will think of this as a generalization of the intermediate value theorem.

Theorem 1.3.8. *Let $f : X \to Y$ be a continuous surjection. If X is connected, so is Y.*

Proof. Assume that X is connected. In particular, X is nonempty, hence so is Y.

Let $V \subseteq Y$ be a nonempty clopen. Then $f^{-1}(V) \subseteq X$ is clopen since f is continuous, and it is nonempty since f is a surjection. Since X is connected, it follows that $f^{-1}(V) = X$ itself. This implies that the image $f(X)$ is contained in V. Since f is a surjection, it follows that $V = Y$. Thus Y has exactly two clopen subsets: $\varnothing$ and Y. □

Example 1.3.9. Let $X \subseteq \boldsymbol{R}^n$ be a connected subspace, and let $f : X \to \boldsymbol{R}$ be a continuous map. Then the image $f(X)$ is an interval. This is our friend, the intermediate value theorem.

Consider the following assertion: there are two *antipodal*[14] points on the Earth's equator that, at ground level, have exactly the same temperature.[15] In other words, there are two coordinates of the form

$$0°\text{S}, x°\text{E} \qquad \text{and} \qquad 0°\text{N}, (180 - x)°\text{W}$$

where at ground level the temperature is exactly the same.

[14] Two points are antipodal if and only if they are in exactly opposite directions from the center of the Earth.

[15] For the purposes of this discussion, we assume that temperature is a continuous function of position.

Here's the proof. The Earth's equator is homeomorphic to the circle S^1, which is a connected subspace of $\boldsymbol{R}^2$. Temperature at the equator is thus a continuous function $T\colon S^1 \to \boldsymbol{R}$. Now define a related continuous map $g\colon S^1 \to \boldsymbol{R}$ by the formula

$$g(x) = T(x) - T(-x)\,.$$

Now since x and $-x$ are antipodal points, our claim will be proved if we can show that there is a point $x \in S^1$ such that $g(x) = 0$. If $g(1,0) = 0$, then we are done. Otherwise, our formula ensures that $g(-1,0) = g(1,0)$, so $g(1,0)$ and $g(-1,0)$ have different signs.[16] But since the image $g(S^1) \subseteq \boldsymbol{R}$ is an interval, it follows that $0 \in g(S^1)$. Hence there exists $x \in S^1$ such that $g(x) = 0$.

[16] That is, one is positive and one is negative.

Example 1.3.10. Since $\boldsymbol{R} \smallsetminus \{0\}$ and the subspace

$$Y = \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\}$$

are homeomorphic, it follows that $\boldsymbol{R} \smallsetminus \{0\}$ is not connected.

Since $\boldsymbol{R}$ is connected, it follows that it is not homeomorphic to $\boldsymbol{R} \smallsetminus \{0\}$.

Example 1.3.11. A good way to confirm that a nonempty subspace $X \subseteq \boldsymbol{R}^n$ is connected is to confirm that you can 'walk' from every point to every other point without leaving X. For any two points $x, y \in X$, a *path* from x to y in X is a continuous map

$$\gamma\colon [0,1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. If X has the property that for every pair of points from x to y, there exists a path from x to y in X, then X is connected.[17] Why? Well, choose $x \in X$. For every path γ from x to another point in X, consider the image $\gamma([0,1])$; this is connected, and our assumption on X ensures that X is the union of all these images, all of which contain x. Hence X is connected.

[17] The converse is, annoyingly, false. This is the difference between *connectedness* and what is called *path connectedness*.

Example 1.3.12. The line $\boldsymbol{R}$ is not homeomorphic to S^1. Indeed, suppose it were; choose a homeomorphism $f\colon \boldsymbol{R} \to S^1$. This will restrict to a homeomorphism $\boldsymbol{R} \smallsetminus \{0\} \to S^1 \smallsetminus \{f(0)\}$. However, we claim that $S^1 \smallsetminus \{f(0)\}$ is connected. In fact, we'll do better: we shall prove that $S^1 \smallsetminus \{f(0)\}$ is homeomorphic to $\boldsymbol{R}$.

To make our formulas simpler, let's use complex numbers.[18] We have:

$$S^1 = \{z \in \boldsymbol{C} : |z| = 1\}.$$

[18] The field of complex numbers $\boldsymbol{C}$ is really nothing more than $\boldsymbol{R}^2$, equipped with its complex multiplication rule $(a,b)\cdot(c,d) = (ac - bd, ad + bc)$.

Let $w = f(0)$. We may first construct a homeomorphism $S^1 \smallsetminus \{w\} \to S^1 \smallsetminus \{1\}$ by the rule $z \mapsto z/w$. Since the inverse map is $z \mapsto wz$, which is also continuous, we are done.

With this done, we now construct a homeomorphism $]0,1[\to S^1 \smallsetminus \{1\}$ by the rule $t \mapsto \exp(2\pi i t)$. This map is a continuous bijection, but what about its inverse r? To make it easy to deduce the continuity of r from well-known bits of analysis, we write it as:[19]

$$r(x,y) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y}{1-x}\right).$$

[19] Please note, however, that this formula will not work to define a continuous map from $S^1 \to \boldsymbol{R}$.

It now follows that $S^1 \smallsetminus \{1\}$ is homeomorphic to an open interval and thus to $\boldsymbol{R}$ itself.

2 Abstract topological spaces

2.1 *Topologies*

There are three observations we can make about our study of subspaces of Euclidean space:

1. When we think about the topological nature of a subspace $X \subseteq \boldsymbol{R}^n$, the ambient Euclidean space in which we find them isn't all that important; homeomorphic subspaces can be found in wildly different Euclidean spaces. The only thing we *do* want to remember from the ambient $\boldsymbol{R}^n$ is the definition of the closure operator τ_X.

2. For the vast majority of the work we've done so far, we didn't need the precise definition of τ_X in terms of the metric on $\boldsymbol{R}^n$. Rather, what we really used repeatedly were the properties given in Proposition 1.1.5. Almost everything we did was just a direct consequence of those!

3. There are examples of sets with a reasonable notion of closeness that don't arrive in your lap embedded in a Euclidean space. We want to *do topology* with these examples too!

Example 2.1.1. Let's illustrate this last point with an interesting example. Let $\boldsymbol{P}^1_{\boldsymbol{R}}$ be the set of 1-dimensional linear subspaces of $\boldsymbol{R}^2$. In other words, an element of $\boldsymbol{P}^1_{\boldsymbol{R}}$ is a line in $\boldsymbol{R}^2$ through the origin. If $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$, let $\mu(L)$ denote the slope of L; that is, if L is spanned by a nonzero vector $(x, y) \in \boldsymbol{R}^2$, then

$$\mu(L) = \begin{cases} y/x & \text{if } x \neq 0; \\ \infty & \text{if } x = 0\,. \end{cases}$$

We will say that a line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ is close to a subset $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ if and only if one of the following conditions holds:

- if $\mu(L) \neq \infty$, then for every $\varepsilon > 0$, there exists a line $L' \in S$, such that

  $$|\mu(L) - \mu(L')| < \varepsilon\,,$$

  or

- if $\mu(L) = \infty$, then for every real number N, there exists a line $L' \in S$ such that

  $$|\mu(L')| > N\,.$$

Equivalently, if we think of the circle $S^1 \subset \boldsymbol{R}^2$, then every line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ intersects S^1 in exactly two points[1] $i(L)$ and $j(L)$, where $i(L) = -j(L)$. We may now say that $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ is close to $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ if and only if, for every $\varepsilon > 0$, there exists a line $L' \in S$ such that either $\|i(L) - i(L')\| < \varepsilon$ or $\|i(L) - j(L)\| < \varepsilon$.

[1] For our purposes, it won't matter which of the two intersection points we call $i(L)$ and which we call $j(L)$.

To deal with examples like this one, we're going to pull a trick that is often quite powerful in mathematics: we're going to turn Proposition 1.1.5 into a *definition*. Thus a *topology* on a set X is a systematic way of asking whether an element of X – which we will call *point* – is 'close' to a subset. In order to constitute a topology on the set X, this notion of closeness must satsify three conditions:

- For every subset $S \subseteq X$ and any point $x \in X$, if $x \in S$, then x is close to S. More briefly, an element of a subset is close to that subset.
- For every finite collection $\{S_1, \dots, S_n\}$ of subsets of X, if a point $x \in X$ is close to the union $S_1 \cup \cdots \cup S_n$, then there exists $i \in \{1, \dots, n\}$ such that x is close to S_i. More briefly, if a point is close to a finite union of subsets, then it is close to one of those subsets. In particular, when $n = 0$, this condition says that no point is close to the empty set $\varnothing$.
- Suppose S and T are two subsets of X such that every point $s \in S$ is close to T; then if a point x is close to S, it is also close to T.

To formalise this idea, we introduce, for each subset $S \subseteq X$, the set $\tau(S)$ of all the elements of X that are close to S, and we list its properties.

Definition 2.1.2. A *topology* on a set X is a map

$$\tau\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

such that the following conditions hold.

1. For any $S \in \boldsymbol{P}(X)$, one has $S \subseteq \tau(S)$. In particular, $\tau(X) = X$.
2. For any finite subset $\Sigma \subseteq \boldsymbol{P}(X)$, one has
$$\tau\left(\bigcup_{S\in\Sigma} S\right) = \bigcup_{S\in\Sigma} \tau(S)\ .$$
In particular,[2] $\tau(\varnothing) = \varnothing$. Also, observe that τ is *inclusion-preserving*; that is, if $S \subseteq T$, then $\tau(S) \subseteq \tau(T)$.
3. For any $S \in \boldsymbol{P}(X)$, one has $\tau(\tau(S)) = \tau(S)$.

If $x \in X$ and $S \in \boldsymbol{P}(X)$, then we say that x *is close to* S if and only if $x \in \tau(S)$.

A pair (X, τ) consisting of a set X and a topology τ on X is called a *topological space*. We call the elements of a topological space *points*.

The set $\tau(S)$ is called the *closure* of S. We say that S is *closed* if $S = \tau(S)$, and we say that S is *open* if its complement is closed.[3]

[2] An empty union $\bigcup_{S\in\varnothing} Y_S$ is empty.

[3] In a little while, we'll discuss how one can specify a topology by listing the open sets, which is what you will frequently see in other books.

Notation 2.1.3. We will often write just X for the topological space, without introducing an extra symbol for the topology. When we have to name the topology, we shall often just write τ_X.

Example 2.1.4. We have already seen the first example. Any subset $X \subseteq \boldsymbol{R}^n$ inherits a topology τ_X that carries a subset $S \subseteq X$ to

$$\tau_X(S) = \{x \in X : (\forall\varepsilon > 0)(\exists s \in S)(d(x,s) < \varepsilon\}\ .$$

When $X = \boldsymbol{R}^n$, this is often called the *standard topology*, and for $X \subseteq \boldsymbol{R}^n$, this is called the *subspace topology*.

Example 2.1.5. The closure operation on $\boldsymbol{P}^1_{\boldsymbol{R}}$ (Example 2.1.1) is a topology.

1. Let $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$. If $L \in S$, then the conditions are automatic for $L \in \tau(S)$.

2. Let $S_1, \dots, S_n \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ be a collection of subsets of $\boldsymbol{P}^1_{\boldsymbol{R}}$, and let $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$. If L is close to $S_1 \cup \cdots \cup S_n$, then a Pigeonhole argument like the one given in Proposition 1.1.5 ensures that L is close to $S_1 \cup \cdots \cup S_n$.[4] If L is close to some S_i, then it is certainly close to $S_1 \cup \cdots \cup S_n$.

3. Let $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$, and let $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ be a line that is close to $\tau(S)$. We aim to show that L is close to S; so let $\varepsilon > 0$. There exists a line $L' \in \tau(S)$ such that either $\|i(L) - i(L')\| < \varepsilon$ or $\|i(L) - j(L')\| < \varepsilon/2$. Accordingly, there exists a line $L'' \in S$ such that either $\|i(L') - i(L'')\| < \varepsilon/2$ or $\|i(L') - j(L'')\| < \varepsilon/2$. Analyzing the four options and applying the triangle inequality, we find that either $\|i(L) - i(L'')\| < \varepsilon$ or $\|i(L) - j(L'')\| < \varepsilon$.

[4] Fill in the details here! Often, in writing, mathematicians say something like, 'by the same argument …,' or 'by the standard argument …,' or even, 'it is obvious that….' This can be a little annoying (and I don't recommend calling very many things 'obvious'), but it is a good opportunity for you to take a moment to sort through the details yourself.

Example 2.1.6. The empty set $\varnothing$ admits a unique topology. The powerset $\boldsymbol{P}(\varnothing)$ is a singleton $\{\varnothing\}$, and the identity is the only map $\{\varnothing\} \to \{\varnothing\}$. This map is a topology. The only subset available – $\varnothing$ itself – is both open and closed.

Example 2.1.7. More generally, on any set X, the identity map

$$\delta \colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

is always a topology. This is called the *discrete topology* on a set X.

In the discrete topology, a point $x \in X$ is close to a subset $S \subseteq X$ if and only if $x \in S$. Every subset of X is both open and closed in the discrete topology.

Example 2.1.8. Let us study topologies on the singleton $\{x\}$. The power set $\boldsymbol{P}(\{x\})$ is a two-element set $\{\varnothing, \{x\}\}$. There are four maps from this set to itself, but most of these are not topologies. In order for a map τ to be a topology, the first condition demands that $\tau(\{x\}) = \{x\}$, and the second condition demands that $\tau(\varnothing) = \varnothing$. Thus the only topology on the singleton $\{x\}$ is the discrete topology.

Example 2.1.9. Let $X \coloneqq \{x, y\}$. In order to specify a topology τ on X, it is really only necessary to specify the subsets $\tau(\{x\})$ and $\tau(\{y\})$. The first condition states that these subsets must contain the sets $\{x\}$ and $\{y\}$, respectively, and the second and third conditions become automatic. That gives us four topologies on $\{x, y\}$:

- the discrete topology, in which $\tau(\{x\}) = \{x\}$ and $\tau(\{y\}) = \{y\}$;
- the topology in which $\tau(\{x\}) = \{x\}$, and $\tau(\{y\}) = X$;
- the topology in which $\tau(\{y\}) = X$, and $\tau(\{x\}) = \{x\}$; and
- the topology in which $\tau(\{x\}) = \tau(\{y\}) = X$.

Let us describe the relation of closeness for each of these topologies. The only questions that aren't answered by the axioms of a topology is: 'is x close to $\{y\}$?' and 'is y close to $\{x\}$?' Let's answer these in each case:

- in the discrete topology, x is not close to $\{y\}$, and y is not close to $\{x\}$;
- in the second of these topologies, x is close to $\{y\}$, but y is not close to $\{x\}$;
- in the third topology, y is close to $\{x\}$, but x is not close to $\{y\}$; and
- finally, in the last of these topologies, x is close to $\{y\}$ and y is close to $\{x\}$.

The asymmetry in the second and third topologies is strange from our natural idea of what 'close' ought to mean. Nevertheless, these topologies are actually interesting examples; even rather bizarre ideas of closeness can be captured in topology!

Finally, for each of these examples, let us determine which sets are open and closed. In each case, the only questions that aren't answered by the axioms are: 'is $\{x\}$ open? closed?' and 'is $\{y\}$ open? closed?'

- in the discrete topology, both $\{x\}$ and $\{y\}$ are both open and closed;
- in the second topology, $\{x\}$ is closed and not open, and $\{y\}$ is open and not closed.
- in the third topology, the situation is reversed: $\{x\}$ is open and not closed, and $\{y\}$ is closed and not open;
- in the last of these topologies, neither $\{x\}$ nor $\{y\}$ is open or closed.

The discrete topology makes sense for any set. Similarly, the last of the topologies in Example 2.1.9 above is sensible for any set.

Example 2.1.10. For any set X, we define a map

$$\iota\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

by the formula

$$\iota(S) \coloneqq \begin{cases} \varnothing & \text{if } S = \varnothing\,; \\ X & \text{if } S \neq \varnothing\,. \end{cases}$$

This is a topology called the *indiscrete topology*. Thus in the indiscrete topology, any point is close to any nonempty subset. The only subsets of X that are open or closed in the indiscrete topology are $\varnothing$ and X itself.

The other two topologies of Example 2.1.9 are more interesting:

Example 2.1.11. Let P be a *preorder*. That is, P is a set equipped with a relation $\leq$ such that:

- for every $a \in P$, one has $a \leq a$; and
- for every $a, b, c \in P$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Then we can define a topology as follows. For any subset $S \subseteq P$, we define

$$\tau_{\leq}(S) \coloneqq \{a \in P : (\exists s \in S)(a \leq s)\}\,.$$

In other words, $a \in P$ is close to S if and only if some element of S exceeds it.

Let us quickly confirm that this is a topology:

1. Let $S \subseteq P$ be a subset. Since $s \leq s$ for every $s \in S$, it follows that $S \subseteq \tau_{\leq}(S)$.

2. Let $S_1, \dots, S_n \subseteq P$ be a finite collection of subsets of P. Then

$$\begin{aligned}\tau_{\leq}(S_1 \cup \dots \cup S_n) &= \{a \in P : (\exists s \in S_1 \cup \dots \cup S_n)(a \leq s)\} \\ &= \{a \in P : (\exists i)(\exists s \in S_i)(a \leq s)\} \\ &= \tau_{\leq}(S)_1 \cup \dots \cup \tau_{\leq}(S)_n .\end{aligned}$$

3. Let $S \subseteq P$ be a subset. Then

$$\begin{aligned}\tau_{\leq}(\tau_{\leq}(S)) &= \{a \in P : (\exists t \in \overline{S})(a \leq t)\} \\ &= \{a \in P : (\exists s \in S)(\exists t \in X)(a \leq t \leq s)\} \\ &= \{a \in P : (\exists s \in S)(a \leq s)\} \\ &= \tau_{\leq}(S) .\end{aligned}$$

This topology is called the *Alexandroff topology*.

The *closed subsets* are the subsets $Z \subseteq P$ with the property that if $a \in P$, $z \in Z$, and $a \leq z$, then $a \in Z$ as well. The *open subsets* are the subsets $U \subseteq P$ with the property that if $a \in P$, $u \in U$, and $a \geq u$, then $a \in U$ as well.

Example 2.1.12. Form the set $\boldsymbol{R} \sqcup \{\infty\}$, where ∞ is just some symbol such that $\infty \notin \boldsymbol{R}$. We already have a topology on $\boldsymbol{R}$; let's doctor it a mite to make a topology on $\boldsymbol{R} \sqcup \{\infty\}$. To do this, we have to understand how our notion of closeness has changed. So we have a point $x \in \boldsymbol{R} \sqcup \{\infty\}$ and a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$, and we want to see if x is close to S. There are two cases:

- If $x \in \boldsymbol{R}$, then we declare that x is close to S if and only if x is close to $S \cap \boldsymbol{R}$ (for the standard topology on $\boldsymbol{R}$).

- We declare that ∞ is close to S if and only if, for every $N \in \boldsymbol{R}_{\geq 0}$, there is a point of S that does not lie[5] in $[-N, N]$.

[5] Note that this includes two possibilities: the point of S may *be* ∞, or else the point may be a real element $s \in S$ such that $|s| > N$.

Let's see that this is a topology. In effect, you check everything by checking the two cases ($x \in \boldsymbol{R}$ or $x \notin \boldsymbol{R}$) separately. In the first case, you use the fact that $\boldsymbol{R}$ has a topology already, and in the second, you give a little argument.

- By definition if x lies in a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$, then x is close to S.

- Let $\{S_1, \dots, S_n\}$ be a finite collection of subsets of $\boldsymbol{R} \sqcup \{\infty\}$. If $x \in \boldsymbol{R}$, then since one has

$$(S_1 \cup \dots \cup S_n) \cap \boldsymbol{R} = (S_1 \cap \boldsymbol{R}) \cup \dots \cup (S_n \cap \boldsymbol{R}),$$

  it follows that x is close to $S_1 \cup \dots \cup S_n$ if and only if x is close to $(S_1 \cap \boldsymbol{R}) \cup \dots \cup (S_n \cap \boldsymbol{R})$ (for the standard topology on $\boldsymbol{R}$), if and only if x is close to $S_i \cap \boldsymbol{R}$ for some i (for the topology on $\boldsymbol{R}$), if and only if x is close to S_i. Now ∞ is close to $S_1 \cup \dots \cup S_n$ if and only if, for any $N \in \boldsymbol{R}$, there is a point of $S_1 \cup \dots \cup S_n$ that does not lie in $[-N, N]$. This happens if and only if there is an i such that for any $N \in \boldsymbol{R}$, there is[6] a point of S_i that does not lie in $[-N, N]$.

[6] This is the *pigeonhole principle*. In effect, if X is an infinite set, Y is a finite set, and $f : X \to Y$ is a map, there is an element $y \in Y$ such that $f^{-1}\{y\}$ is infinite.

- Finally, suppose that every point of a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$ is close to a subset $T \subseteq \boldsymbol{R} \sqcup \{\infty\}$. Now if x is close to S, there are two options. First, if

$x \in \boldsymbol{R}$, then x is close to $S \cap \boldsymbol{R}$ and hence to $T \cap \boldsymbol{R}$. Otherwise, if $x = \infty$, then if $\infty \in T$ or $\infty \in S$, then we're done. If not, then for every $N \in \boldsymbol{R}$, there is a point $s_N \in S \subseteq \boldsymbol{R}$ such that $|s_N| > N$; this point s_N is close to T, so if we set $\epsilon_N \coloneqq |s_N| - N$, then there exists a point $t_N \in T$ such that $|s_N - t_N| < \epsilon_N$. Thus $|t_N| > N$, and so ∞ is close to T.

This isn't the only way to enlarge $\boldsymbol{R}$.

Example 2.1.13. Define a topology on $\boldsymbol{R} \sqcup \{-\infty, +\infty\}$ in which, for any point $x \in \boldsymbol{R} \sqcup \{-\infty, +\infty\}$ and any subset $S \subseteq \boldsymbol{R} \sqcup \{-\infty, +\infty\}$,

- If $x \in \boldsymbol{R}$, then x is close to S if and only if x is close to $S \cap \boldsymbol{R}$.
- We declare that $-\infty$ is close to S if and only if either $-\infty \in S$ or for any $N \in \boldsymbol{R}_{\geq 0}$, there is a point $s \in S \cap \boldsymbol{R}$ such that $s < -N$.
- We declare that $+\infty$ is close to S if and only if either $+\infty \in S$ or for any $N \in \boldsymbol{R}_{\geq 0}$, there is a point $s \in S \cap \boldsymbol{R}$ such that $s > N$.

This is a topology.[7]

[7] You should convince yourself of this!

There is another, more standard, way to specify topologies. It's a little less intuitive, but it becomes technically convenient when we want to do things like *generate* topologies.

Definition 2.1.14. Fix a set X. Let's define two different kinds of information on X.

- A *system of open sets* on X is a subset $\mathscr{O} \subseteq \boldsymbol{P}(X)$ that is stable under unions[8] and finite intersections[9]. That is, if $\Sigma \subseteq \mathscr{O}$, then the union
$$\bigcup_{U \in \Sigma} U$$
also lies in $\mathscr{O}$, and if Σ is finite, then the intersection
$$\bigcap_{U \in \Sigma} U$$
also lies in $\mathscr{O}$.
- Dually, a *system of closed sets* on X is a subset $\mathscr{C} \subseteq \boldsymbol{P}(X)$ that is stable under intersections and finite unions. That is, if $\Sigma \subseteq \mathscr{C}$, then the intersection
$$\bigcap_{Z \in \Sigma} Z$$
also lies in $\mathscr{C}$, and if Σ is finite, then the union
$$\bigcup_{Z \in \Sigma} Z$$
also lies in $\mathscr{C}$.

[8] possibly empty
[9] possibly empty

Specifying a system of open sets is the same as specifying a system of closed sets. More precisely, the formation of the complement in X defines a bijection between the set of systems of open sets and the set of systems of closed sets.

2.1.15. Specifying a topology is the same as specifying a system of closed sets.

For any a topology τ on X, one may define $\mathscr{C}_\tau$ as the set of closed subsets of the topology:

$$\mathscr{C}_\tau := \{Z \in \boldsymbol{P}(X) : \tau(Z) = Z\} .$$

To see that $\mathscr{C}_\tau$ is stable under intersection and finite union, let $\Sigma \subseteq \mathscr{C}_\tau$ be a subset. Let W be the intersection $\bigcap_{Z\in\Sigma} Z$. For every $Z \in \Sigma$, since τ is inclusion-preserving, it follows that $\tau(W) \subseteq \tau(Z) = Z$. Consequently, $\tau(W)$ is contained in the intersection W, so W is closed. If Σ is finite, then since τ preserves finite unions, we have

$$\tau\left(\bigcup_{Z\in\Sigma} Z\right) = \bigcup_{Z\in\Sigma} \tau(Z) = \bigcup_{Z\in\Sigma} Z .$$

This proves that $\mathscr{C}_\tau$ is a system of closed sets.

In the opposite direction, for any system of closed subsets $\mathscr{C}$ on X, define a map $\tau_{\mathscr{C}} : \boldsymbol{P}(X) \to \boldsymbol{P}(X)$ that carries a subset $S \subseteq X$ to the smallest element of $\mathscr{C}$ that contains S: [10]

$$\tau_{\mathscr{C}}(S) := \bigcap_{\substack{Z\in\mathscr{C},\\ S\subseteq Z}} Z .$$

Let us see that $\tau_{\mathscr{C}}$ is a topology.

1. By definition, for every subset $S \subseteq X$, we have $S \subseteq \tau(S)$.

2. By definition, $\tau_{\mathscr{C}}$ is an inclusion-preserving operation. Thus if $S_1, \dots, S_n \subseteq X$ is a finite collection of subsets of X, then it follows that

   $$\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n) \supseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) .$$

   On the other hand, since $\mathscr{C}$ is stable under finite unions, it follows that $\tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) \in \mathscr{C}$. Since in addition, $S_1 \cup \cdots \cup S_n \subseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n)$, it follows[11] that

   $$\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n) \subseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) ,$$

   whence $\tau_{\mathscr{C}}$ preserves finite unions.

3. Finally, $\tau_{\mathscr{C}}(\tau_{\mathscr{C}}(S)$ is the smallest element of $\mathscr{C}$ that contains $\tau_{\mathscr{C}}(S)$. But since $\tau_{\mathscr{C}}(S)$ itself likes in $\mathscr{C}$, it follows that

   $$\tau_{\mathscr{C}}(\tau_{\mathscr{C}}(S)) \subseteq \tau_{\mathscr{C}}(S) .$$

Thus one may specify a topology on X by specifying the closed sets and checking stability under intersections and finite unions, or specifying the open sets and checking stability under unions and finite intersections.[12]

Example 2.1.16. Here's an extremely important example in modern mathematics. This one will be following us around throughout this text. Let $X \subseteq \boldsymbol{R}^m$ be any subspace, and let $n \in \boldsymbol{N}$. We are about to define a topological space $C_n(X)$ called the *configuration space of n points in X*. As a set, $C_n(X)$ is the set of subsets $\{x_1, \dots, x_n\} \subseteq X$ of n distinct points:

$$C_n(X) := \{S \subseteq \boldsymbol{P}(X) : \#S = n\} .$$

[10] Since $\mathscr{C}$ is stable under intersections, this formula indeed defines an element of $\mathscr{C}$. So $\tau_{\mathscr{C}}(S)$ is the smallest element of $\mathscr{C}$ that contains S.

[11] The point here is that $\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n)$ is the *smallest* closed subset that contains $S_1 \cup \cdots \cup S_n$.

[12] Most textbooks define topologies as a system of open sets. There's nothing wrong with this definition, but we've delayed this description, because it seems difficult to motivate fully.

To endow it with a topology, we need to define an auxiliary topological space.

Recall that we defined the subspace

$$X^n \coloneqq X \times \cdots \times X = \{x = (x_1, \ldots, x_n) \in \boldsymbol{R}^{mn} : (\forall i)(x_i \in X)\} \subseteq \boldsymbol{R}^{mn} .$$

We now pass to a further subspace:

$$E_n(X) \coloneqq \{(x_1, \ldots, x_n) \in X^n : (\forall i, j)((i \neq j) \Rightarrow (x_i \neq x_j))\} \subseteq X^m .$$

Now there is a map $q\colon E_m(X) \to C_m(X)$ defined by

$$q(x_1, \ldots, x_n) \coloneqq \{x_1, \ldots, x_n\} .$$

We now declare that $U \subseteq C_n(X)$ is open if and only if $q^{-1}(U) \subseteq E_n(X)$ is open. This defines a system of open sets (and hence a topology), since the formation of the inverse image preserves unions and intersections.

Example 2.1.17. Any set X can be given the *cofinite* topology, in which a subset $Z \subseteq X$ is declared to be closed if and only if either Z is finite or $Z = X$. Thus in the cofinite topology,

$$\tau(S) = \begin{cases} S & \text{if } S \text{ is finite;} \\ X & \text{if } S \text{ is infinite.} \end{cases}$$

Definition 2.1.18. Let X be a topological space, and let $x \in X$. Then an *open neighborhood* of x is an open subset $U \subseteq X$ such that $x \in U$. A *neighborhood* of x is a subset of X that contains an open neighborhood of x.

2.1.19. If X is a topological space, then $\tau(S)$ is the smallest closed subset that contains S. Equivalently, a point $y \in X$ lies in the closure $\tau(S)$ if and only if, for any open neighborhood U of y, the intersection $U \cap \tau(S)$ is nonempty.

Why are these equivalent? Well, if $y \notin \tau(S)$, then $X \smallsetminus \tau(S)$ is an open neighborhood of y that does not intersect S. Conversely, if U is an open neighborhood of x that does not intersect $\tau(S)$, then $X \smallsetminus U$ is a closed subset that contains S and therefore $\tau(S)$.

2.2 *Continuity*

The good news now is that continuity works almost exactly as it did in the example of subspaces of Euclidean space (Section 1.1). The only difference is that we no langer have access to the ε-δ characterization of continuity.

Definition 2.2.1. Let X and Y be topological spaces, and let $f\colon X \to Y$ be a map. Then f is *continuous* if and only if, for every subset $S \subseteq X$ any every point $x \in X$ that is close to S, the point $f(x)$ is close to $f(S)$.

Proposition 2.2.2. *Let X and Y be topological spaces. The following are equivalent for a map $f\colon X \to Y$.*

1. *The map f is continuous.*
2. *For any subset $T \subseteq Y$, one has $\tau_X(f^{-1}(T)) \subseteq f^{-1}(\tau_Y(T))$.*

3. *For any closed subset $Z \subseteq Y$, the inverse image $f^{-1}(Z) \subseteq X$ is closed.*

4. *For any open subset $U \subseteq Y$, the inverse image $f^{-1}(U) \subseteq X$ is open.*

Proof. Everything is exactly as in the proof of Proposition 1.2.2, except that since we do not have condition (5), we shall have to prove directly that (4) implies (1).

So assume (4); we aim to prove (1). Let $S \subseteq X$, and let $x \in \tau_X(S)$. Observe that the complement $V \coloneqq Y \smallsetminus \tau_Y(f(S))$ is open; hence so is the inverse image $U \coloneqq f^{-1}(V) \subseteq X$. Note also that S is disjoint from U. Thus the complement $X \smallsetminus U$ is a closed subset that contains S. Consequently, $\tau_X(S) \subseteq U$. So since $x \in \tau_X(S)$, it follows that $f(x) \notin V$. In other words, $f(x) \in \tau_Y(f(S))$, as desired. □

Example 2.2.3. Let X, Y, and Z be topological spaces. Then the identity map $\mathrm{id}\colon X \to X$ is continuous. Also, if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then the composition $g \circ f\colon X \to Z$ is continuous as well.

Example 2.2.4. Let X be a topological space, and let S be a set. Any map $f\colon S \to X$ is continuous if S is endowed with the *discrete* topology. Dually, any map $g\colon X \to S$ is continuous if S is endowed with the emphchaotic topology.

Example 2.2.5. Recall the topological space $\boldsymbol{P}^1_{\boldsymbol{R}}$ of Example 2.1.1. Define a map $L\colon S^1 \to \boldsymbol{P}^1_{\boldsymbol{R}}$ as follows: for every point $x \in S^1$, let $L(x)$ be the 1-dimensional subspace spanned by the vector x. That is, $L(x)$ is the unique line in $\boldsymbol{R}^2$ that passes through the origin and x. It follows from the definition of the topology on $\boldsymbol{P}^1_{\boldsymbol{R}}$ that L is continuous.

To unpack this a little, let $S \subseteq S^1$ be a subset, and let $x \in \tau(S)$. We want to see that $L(x)$ is close to the image $L(S)$. Let $\varepsilon > 0$. There exists $s \in S$ such that $\|x - s\| < \varepsilon$. Now $L(x)$ intersects S^1 at the points x and $-x$, and $L(s)$ intersects S^1 in the points s and $-s$. Thus the line $L(s)$ has the property that, in the notation of Example 2.1.1, $\|i(L(x)) - i(L(s))\| < \varepsilon$ or $\|i(L(x)) - j(L(s))\| < \varepsilon$.

Please note that while L is surjective, it is not injective, since for every $x \in S^1$, one has $L(x) = L(-x)$.

Definition 2.2.6. Let X and Y be topological spaces, and let $x \in X$. Then a map $f\colon X \to Y$ is *continuous at* x if and only if, for any subset $S \subseteq X$, if x is close to S then $f(x)$ is close to $f(S)$.

A map $f\colon X \to Y$ is continuous if and only if it is continuous at every point $x \in X$.

Example 2.2.7. Consider the map $s\colon \boldsymbol{R} \to \boldsymbol{R}$ given by the formula

$$s(x) \coloneqq \begin{cases} x/|x| & \text{if } x \neq 0\,; \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then s is continuous at every $x \in \boldsymbol{R} \smallsetminus \{0\}$. If s were continuous at 0, then it would be continuous. But even though the set $\{1\} \subset \boldsymbol{R}$ is closed, its inverse image

$$s^{-1}\{1\} = \,]0, +\infty[\, \subset \boldsymbol{R}$$

is not. Hence s is not continuous at 0, as we expect!

Proposition 2.2.8. *Let X and Y be topological spaces, and let $x \in X$. The following are equivalent for a map $f: X \to Y$.*

1. *The map f is continuous at x.*

2. *For any open neighborhood V of $f(x)$, the inverse image $f^{-1}(V)$ is an open neighborhood of x.*

Proof. Assume (1); we aim to prove (2). Let V be an open neighborhood of $f(x)$. Let $S \coloneqq f^{-1}(Y \smallsetminus V)$. If $x \in \tau_X(S)$, then

$$f(x) \in \tau_Y(f(S)) \subseteq \tau(Y \smallsetminus V) = Y \smallsetminus V\,,$$

since $Y \smallsetminus V$ is closed. This is a contradiction, which shows that $x \notin \tau_X(S)$. Now let $U \coloneqq X \smallsetminus \tau_X(S)$. This is now an open neighborhood of x, and $f(U) \subseteq V$.

Conversely, assume (2); we aim to prove (1). Let $S \subseteq X$ be a subset, and let $x \in \tau_X(S)$. Let $V \coloneqq Y \smallsetminus \tau_Y(f(S))$. If $f(x) \in V$, then by assumption there exists an open neighborhood U of x such that $f(U) \subseteq V$. Since S is disjoint from $f^{-1}(V)$, it follows that the closure $\tau_X(S)$ is disjoint from U as well. This is a contradiction that implies that $f(x) \notin V$, so that $f(x) \in \tau(f(S))$. □

THE DEFINITION of *homeomorphism* is also just the same as for subspaces of Euclidean space:

Definition 2.2.9. Let X and Y be topological spaces. A *homeomorphism* $f: X \to Y$ is a continuous bijection whose inverse f^{-1} is continuous. The two topological spaces X and Y are *homeomorphic* if and only if there exists a homeomorphism $X \to Y$. In this case, we may write $X \cong Y$.

Example 2.2.10. Let's show that the following three topological spaces are homeomorphic:

- The circle $S^1 \subseteq \boldsymbol{R}^2$.
- The topological space $\boldsymbol{P}^1_{\boldsymbol{R}}$ from Example 2.1.1.
- The topological spaces $\boldsymbol{R} \sqcup \{\infty\}$ from Example 2.1.12.

The map $L: S^1 \to \boldsymbol{P}^1_{\boldsymbol{R}}$ defined above is not a homeomorphism, but that doesn't mean we can't find another homeomorphism between these two topological spaces. So let's try to go the other way. Let's define a map $h: \boldsymbol{P}^1_{\boldsymbol{R}} \to S^1$. For every line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$, there is a point $z \in S^1$ such that $L = L(z)$. If we think of $S^1 \subset \boldsymbol{C}$, we can define $h(L)$ as z^2 for any $z \in S^1$ such that $L = L(z)$. In other words, if L is the line spanned by a nonzero vector $(x, y) \in \boldsymbol{R}^2$, then

$$h(L) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2}\right).$$

This is well-defined, since this formula gives the same value if you replace (x, y) with $(\alpha x, \alpha y)$ for $\alpha \in \boldsymbol{R} \smallsetminus \{0\}$.

Let us prove that h is continuous. We may be tempted to use the fact that the formula above defines a continuous function $\boldsymbol{R}^2 \smallsetminus \{0\} \to S^1$, but we have to be a little careful, because $\boldsymbol{P}^1_{\boldsymbol{R}}$ wasn't defined as a subspace of a Euclidean space. Assume that $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$, and assume that $L \in \tau(S)$. Let $\varepsilon > 0$; there exists a line $L' \in S$ such that either $\|i(L) - i(L')\| < \varepsilon/2$ or $\|i(L) - j(L')\| < \varepsilon/2$. Furthermore, we know that no two points in S^1 are separated by a distance of more than 2; therefore, both $\|i(L) - i(L')\| \leq 2$ or $\|i(L) - j(L')\| \leq 2$. We may therefore multiply these inequalities to obtain:

$$\|i(L) - i(L')\|\|i(L) - j(L')\| < \varepsilon\,.$$

Since $i(L') = -j(L')$, we therefore obtain

$$\|h(L) - h(L')\| = \|i(L)^2 - j(L')^2\| < \varepsilon\,.$$

It therefore follows that $h(L)$ is close to $h(S)$. So h is continuous.

Now let us show that h is a bijection. If $w \in S^1$, then the inverse image $h^{-1}\{w\}$ is the set of lines $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ such that either $i(L)$ or $j(L)$ is a square root of w. In fact, since $i(L) = -j(L)$, it follows that $i(L)$ is a square root of w if and only if $j(L)$ is a square root of w. Hence $h^{-1}\{w\}$ consists of exactly one line: the line passing through either square root of w and the origin.

Now let's demonstrate that h is a homeomorphism. For this, we note that the inverse h^{-1} carries $w \in S^1$ to the line passing through the square roots of w and the origin. To prove that this is continuous, assume that $T \subseteq S^1$, and assume that $w \in S^1$ is close to T. Let $\varepsilon > 0$; then there exists $w' \in T$ such that $\|w - w'\| < \varepsilon^2$. Now if $z, -z \in S^1$ are the two square roots of w, and if $z', -z' \in S^1$ are the two square roots of w', then either $|z - z'| < \varepsilon$ or $|z + z'| < \varepsilon$, since otherwise, we would have

$$|w - w'| = |z - z'||z + z'| \geq \varepsilon^2\,.$$

Consequently, it follows that $h^{-1}(w)$ is close to $h^{-1}(T)$. This now completes the proof that $\boldsymbol{P}^1_{\boldsymbol{R}}$ and S^1 are homeomorphic.

Now we prove that $\boldsymbol{R} \sqcup \{\infty\}$ and S^1 are homeomorphic. Indeed, define maps

$$f\colon S^1 \to \boldsymbol{R} \sqcup \{\infty\} \qquad \text{and} \qquad g\colon \boldsymbol{R} \sqcup \{\infty\} \to S^1$$

by the formulas

$$f(x, y) \coloneqq \begin{cases} x/(1-y) & \text{if } y \neq 1\,; \\ \infty & \text{if } y = 1\,. \end{cases}$$

and

$$g(t) \coloneqq \begin{cases} 1/(t^2+1)(2t, t^2-1) & \text{if } t \neq \infty\,; \\ (0, 1) & \text{if } t = \infty\,. \end{cases}$$

A direct check confirms that f and g are inverses, but the interesting thing to prove is that they are each continuous.

For f, continuity away from the point $(0, 1)$ follows from the elementary analytic facts we are happy to assume here. But continuity at $(0, 1)$ is more interesting! So assume that $S \subseteq S^1$ is a subset to which $(0, 1)$ is close. We have to show that ∞ is close to $f(S)$. So let $N > 0$ be a real number; we aim to find an element $(x, y) \in S$ such that $|x/(1-y)| > N$ or, equivalently, that $x^2/(1-y)^2 = (1+y)/(1-y) > N^2$, Choose $\varepsilon > 0$ so that $\varepsilon \leq 2/N^2$; since $(0, 1)$ is close to S, there exists a point $(x, y) \in S$ such that $x^2 - (1-y)^2 < \varepsilon$

and $y \geq 0$. Since $x^2 + y^2 = 1$, this implies that $1 - y < \varepsilon/2$, and since $1 + y \geq 1$, we obtain

$$\frac{1+y}{1-y} \geq \frac{1}{1-y} > \frac{2}{\varepsilon} > N^2\,.$$

Once again, elementary analysis facts imply that g is continuous at every point of $\boldsymbol{R}$. We must show that g is continuous at ∞ as well. For this, assume that $T \subseteq \boldsymbol{R} \sqcup \{\infty\}$ is a subset such that ∞ is close to T. Let $\varepsilon > 0$; we aim to show that there exists an element $t \in T$ such that $\|g(t) - (0,1)\| < \varepsilon$. If $\infty \in T$, then this is immediate, so it suffices to consider the case in which $T \subseteq \boldsymbol{R}$ is an unbounded subset. There exists $t \in T$ such that $|t| > 2/\varepsilon$, and so

$$\left\|1/(t^2+1)(2t, t^2-1) - (0,1)\right\|^2 = \frac{4}{t^2+1} < \varepsilon^2\,,$$

so that $\|g(t) - (0,1)\| < \varepsilon$, just as we'd hoped.

The upshot here is that we have

$$\boldsymbol{P}^1_{\boldsymbol{R}} \cong S^1 \cong \boldsymbol{R} \sqcup \{\infty\}\,.$$

2.3 *Connectedness*

2.4 *Specifying topologies efficiently*

2.5 *Quotient spaces*

2.6 *Products*

2.7 *Function spaces*

2.8 *Coproducts*

2.9 *Exotica*

3 Compactness

3.1 Quasicompact topological spaces

3.2 Other forms of compactness

3.3 Compact hausdorff topological spaces

3.4 Ultrafilters

3.5 Stone topological spaces

3.6 Spectral topological spaces

3.7 Local compactness

3.8 Function spaces, redux

3.9 Paracompactness

4 Separation

5 Countability & metrisation

5.1 First countability

5.2 Second countability

5.3 Urysohn metrisation

5.4 Nagata–Smirnov metrisation

6 Generating topological spaces

6.1 Generation by a family

6.2 Finitely generated topological spaces

6.3 Numerically generated topological spaces

6.4 Compactly generated topological spaces

7 Sheaves

8 Manifolds

A Elements of set theory

Mathematicians treat the concepts of *set* and *element* as *undefined primitives*. Rules (in the form of axioms and axiom schemata) are provided for the manipulation of these objects. These extend the rules of first-order predicate calculus. In most foundational schemes (including the one presented below), absolutely every mathematical object – every number, every polynomial, every element of every set – is a set. So when we write $x \in X$, both x and X are sets.

The student who wants to go very deep into the subject of set theory should consult the astonishing text of Jech[**Jech:2003tt**]. The student who would prefer to work up to Jech's text should begin with Halmos's text[**MR0453532**].[1] This course won't require any set theory beyond Halmos's book, but because general topology and set theory interact in various nontrivial ways, it would be intellectually dishonest not to give at least a quick overview of some the basic elements of set theory.

[1] Pleasant though Halmos's text is, it should be noted its attitude toward set theory is at times unfairly dismissive of the subject.

A.1 Sets and elements

The first thing you have to know about sets is that a set X is equal to a set Y if and only if X and Y have the same elements. That is, $X = Y$ if and only if for every A, one has $A \in X$ if and only if $A \in Y$.

The easiest set in the world is the *empty set* $\varnothing$. It has no elements. The sentence $x \in \varnothing$ is *always* false, no matter what x is. That means that any universally quantified sentence over the empty set – *i.e.,* $(\forall x \in \varnothing)(\phi(x))$ – is *true*, and any existentially quantified sentence over the empty set – *i.e.,* $(\exists x \in \varnothing)(\phi(x))$ – is *false*.[2]

[2] Be sure you see why this is true!

The next easiest sets in the world are *singletons*. A singleton is a set with exactly one element, $\{X\}$. Don't forget that that X has to be a set. The axioms of set theory let you take any set X and build the singleton $\{X\}$. More generally, if we have a pair of sets X, Y, we're permitted to form the set $\{X, Y\}$.

Example A.1.1. The sets $\varnothing$ and $\{\varnothing\}$ are unequal.

A.2 Bounded comprehension

We also want to be able to carve out pieces of our sets defined by suitable formulas. So if X is a set, and $\phi(x)$ is some formula of set theory (any sentence of predicate calculus along with $\in$ in which the only free variable is x), then the axioms of set theory allow us to form a set[3]

$$A = \{x \in X : \phi(x)\} .$$

[3] This bit of notation is sometimes called *set-builder* notation. The axioms say, in effect, that this notation actually means something, as long as the X that appears on the left side of the colon is known to be a set.

Thus A is the set whose elements are all and only those elements $x \in X$ such that $\phi(x)$ obtains.

Example A.2.1. It's important that sets defined by formulas are carved out of existing sets. This is called *bounded comprehension*. With an *unbounded* comprehension axiom, we would be able to build the following:

$$R \coloneqq \{X : \neg(X \in X)\} .$$

You may have seen this: this R creates some challenges, since $R \in R$ if and only if $R \notin R$. This is the example that Bertrand Russell cooked up, just to ruin Gottlob Frege's day.

A.3 Unions

For any set X, we also permit ourselves to form the *union* $\cup X$. This is the set whose elements are the elements of the elements of X; that is, $A \in \cup X$ if and only if there exists an element $S \in X$ such that $A \in S$. If $X = \{A, B\}$, then we write[4] $A \cup B$ for $\cup X$.

[4] This $\cup X$ notation may be unfamiliar to you. You might be happier with something like $\bigcup_{S \in X} S$, which means the same thing. The $\cup X$ notation is standard in set theory, however.

Example A.3.1. We can start building the *finite von Neumann ordinals* according to the following recipe: first, $0 \coloneqq \varnothing$. Then, for every von Neumann ordinal n, one can create its *successor von Neumann ordinal*

$$n + 1 \coloneqq n \cup \{n\} .$$

So the first few von Neumann ordinals look like this:

$$\begin{aligned}
0 &\coloneqq \varnothing; \\
1 &\coloneqq \{0\} = \{\varnothing\}; \\
2 &\coloneqq \{0, 1\} = \{\varnothing, \{\varnothing\}\}; \\
3 &\coloneqq \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}; \\
4 &\coloneqq \{0, 1, 2, 3\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}; \\
&\textit{etc.}
\end{aligned}$$

If n is a finite von Neumann ordinal, then for every $\alpha, \beta \in n$, exactly one of the following is the case:

$$\alpha \in \beta ; \quad \alpha = \beta ; \;\; or \quad \alpha \ni \beta .$$

The axioms then let you build the first *infinite von Neumann ordinal* ω, which is the set of all the finite von Neumann ordinals. Once you have that, you can build the successor to ω:

$$\omega + 1 \coloneqq \omega \cup \{\omega\} .$$

Wait, isn't that like $\infty + 1$*? Isn't that* ∞ *again?* That sort of thing is true if you're talking about *cardinals*. Here, we're talking about *ordinals*, and the distinction is important. We'll get into this more when we talk about number systems in the next section. But we absolutely can construct

$$\omega + 1, \omega + 2, \ldots, 2\omega = \omega + \omega ,$$

where $2\omega = \omega + \omega$ is the set

$$\{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}\ .$$

Likewise, you can build $3\omega, 4\omega, \dots$, and then even $\omega^2, \omega^3, \dots$, and even ω^ω. The key point is that each ordinal is the set of all the ordinals smaller than it. We'll investigate this more deeply in the next section.

A KEY CONSTRUCTION for building 'big' sets is the *power set* construction. To explain, a *subset* of a set X is a set S such that for any $s \in S$, one has $s \in X$ as well; in this case, we write $S \subseteq X$. The axioms of set theory permit us to form the *power set* $\boldsymbol{P}(X)$, which is the set of all subsets of X, so that $S \in \boldsymbol{P}(X)$ if and only if $S \subseteq X$.

Example A.3.2. The set $\boldsymbol{P}(\varnothing)$ is $\{\varnothing\}$. The set $\boldsymbol{P}(\boldsymbol{P}(\varnothing))$ is $\{\varnothing, \{\varnothing\}\}$.

A.4 Ordered pairs

We can also create *ordered pairs*. For sets X and Y, we write

$$\langle X, Y\rangle \coloneqq \{\{X\}, \{X, Y\}\}\ .$$

Now for any two sets X and Y, we define the *product* as the set of all ordered pairs:

$$X \times Y \coloneqq \left\{S \in \boldsymbol{P}(\boldsymbol{P}(X \cup Y)) : (\exists x \in X)(\exists y \in Y)(\langle x, y\rangle = S)\right\}\ .$$

We can also define an *ordered triple* by the rule

$$\langle X, Y, Z\rangle \coloneqq \langle\langle X, Y\rangle, Z\rangle.$$

We can keep going with this to build ordered quadruples and ordered quintuples, *etc.*, but once we have the concept of map up and running, we'll find more efficient and intuitive ways to talk about these things.

Exercise A.4.1. How many elements does the ordered pair $\langle x, x\rangle$ have? How about the ordered triple $\langle x, x, x\rangle$?

A.5 Maps

The real point of defining things as we have is so we can talk about *maps* as soon as possible. Maps – AKA *set maps*, *mappings*, *functions* – are an important notion in set theory. The idea is that a map $f : S \to T$ is a way of taking each element $s \in S$ and associating one and only one element $f(s) \in Y$ thereto. Importantly, though, a map $f : S \to T$ 'knows' its *source* S, its *target* T, and the method of associating elements of T with elements of S. Thus a map $f : S \to T$ is defined as an ordered triple $\langle S, T, \Gamma(f)\rangle$ in which $\Gamma(f) \subseteq S \times T$ is a subset with the property that for every element $s \in S$, there is a unique[5] element $f(x) \in Y$ such that $\langle x, f(x)\rangle \in \Gamma(f)$.

In practice, the way we describe maps is pretty relaxed. We typically identify the source S and the target T, and then we provide a *rule* for 'sending' elements of S to the associated elements of T. The idea is that for every

[5] The phrase 'there is a unique x such that $\phi(x)$' – sometimes written $(\exists! x)\ \phi(x)$ – is a helpful shorthand for the longer sentence

$$(\exists x)(\phi(x) \wedge ((\forall y)(\phi(y) \implies (x = y))))\ .$$

You knew that, of course, but our point here is that it's a formula of predicate calculus, and so the phrase 'such-and-so is a map' is a formula of predicate calculus as well.

$s \in S$, we have to specify a unique $f(s) \in T$ 'attached' to s in the sense that $\langle s, f(s)\rangle \in \Gamma(f)$.

Sometimes, it's handy to use the following notation: we may define a map $f : S \to T$ by the *assignment*

$$s \mapsto [\text{some formula involving } s] .$$

Example A.5.1. For instance, we can specify a map $f : \omega \to \{0, 1\}$ by saying that $f(0) = 0$ and for any successor $n + 1$, we let $f(n + 1)$ be the unique element of $\{0, 1\}$ such that $f(n + 1) \neq f(n)$. Since we can check that $f(n + 1)$ is indeed unique with this property, we are assured that f really is a map.

Example A.5.2. Suppose $S \subseteq X$. There is a map $i : S \to X$ such that $i(s) = s$, called the *inclusion map*. When $S = X$, this is called the *identity map* id.

If we specify two sets S and T, then we can write $\mathrm{Map}(S, T) \subseteq \boldsymbol{P}(S \times T)$ consisting of those subsets $\Gamma \subseteq S \times T$ such that $\langle S, T, \Gamma\rangle$ is a map. We can write this as:

$$\mathrm{Map}(S, T) \coloneqq \{\Gamma \subseteq \boldsymbol{P}(S \times T) : (\forall s \in S)(\exists! t \in T)(\langle s, t\rangle \in \Gamma)\} .$$

this is our set of maps from S to T.

Exercise A.5.3. Suppose X a set (possibly empty). The set $\mathrm{Map}(\varnothing, X)$ always consists of exactly one element: the inclusion map $\varnothing \to X$. The point here is that $\varnothing \times X = \varnothing$, so $\boldsymbol{P}(\varnothing \times X) = \{\varnothing\}$. The unique subset $\varnothing \subseteq \varnothing$ is a map $\varnothing \to X$: indeed, the condition is:

$$(\forall s \in \varnothing)(\exists! t \in X)(\langle s, t\rangle \in \varnothing) ,$$

which is always true. The set $\mathrm{Map}(X, \varnothing)$ is often, but not always, empty. Once again, we are looking at the unique subset $\varnothing \subseteq X \times \varnothing = \varnothing$, and the condition is:

$$(\forall s \in X)(\exists t \in \varnothing)(\langle s, t\rangle \in \varnothing) ,$$

which is true if $X = \varnothing$; otherwise it is false.

Maps can be *composed*. If $f : S \to T$ is a map, and $g : T \to U$ is another, then you can 'do f first, and then do g'. That is, one can form a new map $g \circ f : S \to U$ such that for any $s \in S$, one has

$$(g \circ f)(s) \coloneqq g(f(s)) .$$

More formally, f and g are triples $\langle S, T, \Gamma(f)\rangle$ and $\langle T, U, \Gamma(g)\rangle$ (respectively), and $g \circ f$ is the triple $\langle S, U, \Gamma(g \circ f)\rangle$, where

$$\Gamma(g \circ f) \coloneqq \{(s, u) \in S \times U : (\exists t \in T)((s, t) \in \Gamma(f) \wedge (t, u) \in \Gamma(g))\} .$$

It is easy enough to see that $g \circ \mathrm{id} = g$ and $\mathrm{id} \circ f = f$, and moreover composition is associative, so that $(h \circ g) \circ f = h \circ (g \circ f)$.

A.6 Bijections

One important kind of map is the *bijection*. When mathematicians are presented with a set, for many purposes, they won't be too very worried about what the elements are, but what structures they have.

The idea is that a bijection of sets is meant to be a 'mere labelling' of the elements of a set S by the elements of a set T. That labelling is meant to be a perfect match of information: you should never use the same label twice, and all the labels should be used. So a *bijection* of sets is a map $f: S \to T$ such that for any $t \in T$, there exists a unique $s \in S$ such that $f(s) = t$.

Example A.6.1. Let S and T be two sets. There is a bijection $\sigma: S \times T \to T \times S$, which is given by $\sigma(\langle s, t\rangle) = \langle t, s\rangle$.

If $f: S \to T$ is a bijection, then there exists a map $g: T \to S$ such that $g \circ f = \mathrm{id}$, and $f \circ g = \mathrm{id}$. To prove this, let us construct g: the function f gives us a subset $\Gamma(f) \subseteq S \times T$ such that for any $s \in S$, there exists a unique $t \in T$ such that $\langle s, t\rangle \in \Gamma(f)$. So now let's define $g = \langle T, S, \Gamma(g)\rangle$, where $\Gamma(g) = \{\langle t, s\rangle \in T \times S : \langle s, t\rangle \in \Gamma(f)\}$. Of course $\Gamma(g)$ makes perfect sense as a subset, but we aren't done: we have to show it is a map from S to T. For this we can use the fact that, since f is a bigjection, for every $t \in T$, there exists a unique $s \in S$ such that $\langle s, t\rangle \in \Gamma(f)$. In other words, for every $t \in T$, there exists a unique $s \in S$ such that $\langle t, s\rangle \in \Gamma(g)$. Thus if $t \in T$, then $g(t) \in S$ is the unique element such that $f(g(t) = t$. Thus $f \circ g = \mathrm{id}$. To see that $g \circ f = \mathrm{id}$, let $s \in S$; then $g(f(s)) \in S$ is the unique element such that $f(g(f(s))) = f(s)$. But since this element of S is *unique* with this property, it follows that $g(f(s)) = s$.

The converse is also correct: if $f: S \to T$ is a map such that there exists a function $g: T \to S$ such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$, then f is a bijection. Indeed, let $t \in T$ be an element; we aim to prove that there exists a unique element $s \in S$ such that $t = f(s)$. The function g provides us with exactly such an element: $g(t) \in T$ is an element, and $t = f(g(t))$. Now suppose that $s' \in S$ is an element such that $t = f(s')$; we see that $g(t) = g(f(s')) = s'$, so we have the uniqueness we sought!

In this case, we say that g is the *inverse* of f, and we sometimes write f^{-1} for g.

Example A.6.2. Let S and T be two sets. Assume that $f: S \to T$ is a bijection between them. Now let U be another set. We can define a map $F: \mathrm{Map}(T, U) \to \mathrm{Map}(S, U)$ by the assignment $\alpha \mapsto \alpha \circ f$. Let's see that this is a bijection. If $g = f^{-1}: T \to S$ is the inverse to f, then we can define a map $G: \mathrm{Map}(S, U) \to \mathrm{Map}(T, U)$ by the assignment $\beta \mapsto \beta \circ g$. Now $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

A.7 *Products and sets of maps*

Here's a basic property that relates the product of sets and the set of maps. It's a bit of a tongue-twister, but it's worth it to unpack. Let S, T, and U be three sets. Then define a map

$$\phi: \mathrm{Map}(S \times T, U) \to \mathrm{Map}(S, \mathrm{Map}(T, U))$$

that carries an element $h \in \mathrm{Map}(S \times T, U)$ – that is, a map $h: S \times T \to U$ – to the element $\phi(h) \in \mathrm{Map}(S, \mathrm{Map}(T, U))$ – that is, the map $\phi(h): S \to \mathrm{Map}(T, U)$ – that carries an element $s \in S$ to the element $\phi(h)(s) \in Map(T, U)$ – that is, the map $\phi(h)(s): T \to U$ – that carries an element

$t \in T$ to the element

$$\phi(h)(s)(t) = h(\langle s, t\rangle) \in U\ .$$

Did you catch that? Let's say it differently: we're starting with a map $h\colon S \times T \to U$. We want to *get* a map $\phi(h)\colon S \to \mathrm{Map}(T, U)$. To describe *that*, we start with an element $s \in S$, and we want to *get* a map $\phi(h)(s)\colon T \to U$. To define that, we start with an element $t \in T$, and we want to *get* an element of U; that element is $h(\langle s, t\rangle)$. Some times the map $\phi(h)(s)\colon T \to U$ is written $h(\langle s, -\rangle)$, where the second position is treated as a blank where we can fill in $t \in T$.

Let's go the other way, and define a map

$$\psi\colon \mathrm{Map}(S, \mathrm{Map}(T, U)) \to \mathrm{Map}(S \times T, U)\ .$$

For this, we're starting with a map $k\colon S \to \mathrm{Map}(T, U)$, and we want to define a map $\psi(k)\colon S \times T \to U$. This is defined by

$$\psi(k)(\langle s, t\rangle) \coloneqq k(s)(t)\ .$$

Now if you inspect thes formulas carefully, you'll see that in fact $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. In other words, ϕ is a bijection, and ψ is its inverse. In still other words, the sets $\mathrm{Map}(S \times T, U)$ and $\mathrm{Map}(S, \mathrm{Map}(T, U))$ are the 'same', up to relabelling.

A.8 Injections and surjections

The condition to be a bijection is really the conjunction of two conditions: first, that we never use the same label twice, and second, that every label be used. Let's give names to these conditions. An *injection* is a map $f\colon S \to T$ such that for any $t \in T$, there is *at most one*[6] $s \in S$ such that $f(s) = t$. A *surjection* is a map $f\colon S \to T$ such that for any $t \in T$, there is (*at least one*) $s \in S$ such that $f(s) = t$. Of course, a bijection is a map that is both an injection and a surjection.

[6] The phrase 'there is at most one x such that $\phi(x)$' is a helpful shorthand for the longer sentence

$$(\forall x)(\forall y)((\phi(x) \wedge \phi(y)) \implies (x = y))\ .$$

This is a pretty annoying turn of language, one has to admit: we're actually using the phrase 'there is' in a sentence in which the only quantifier is universal.

Here's an important axiom that we will use in a nontrivial way a couple of times in this class. It's called the *Axiom of Choice*. It says that if you have a surjection $f\colon S \to T$, then there exists a map $s\colon T \to S$ such that $f \circ s = \mathrm{id}$. Note that we are *not* saying that $s \circ f = \mathrm{id}$ as well; that would imply that f is a bijection, which isn't always true.

The map s is not usually an inverse to f, and we do not usually use the notation f^{-1} for s. Rather it is what we call a *section* of f. Thus the Axiom of Choice says that every surjection has a section.

A.9 The algebra of subsets

The algebra of subsets of a set permits one to perform unions, intersections, and complements, and these satisfy certain rules.

So let X be a set, and let $U\colon A \to \boldsymbol{P}(X)$ be a map. Then there are two new subsets of X that can be constructed: the *indexed union*

$$\bigcup_{a \in A} U(a) \coloneqq \{x \in X : (\exists a \in A)\ x \in U(a)\}$$

and the *indexed intersection*

$$\bigcap_{a\in A} U(a) \coloneqq \{x \in X : (\forall a \in A)\ x \in U(a)\}.$$

Example A.9.1. There is only one map $I\colon\ \varnothing \boldsymbol{P}(X)$. The indexed union

$$\bigcup_{a\in\varnothing} I(a) = \varnothing\,.$$

The indexed intersection

$$\bigcap_{a\in\varnothing} I(a) = X\,.$$

Let's see what happens when you repeat these operations or mix them. Let A and B be sets, and let $U\colon\ A \times B \to \boldsymbol{P}(X)$. We're going to exploit some bijections now: we know that maps $A\times B \to \boldsymbol{P}(X)$ are in bijection with maps $A \to \mathrm{Map}(B, \boldsymbol{P}(X))$; we also know that $A \times B$ is in bijection with $B \times A$, and therefore that maps $A\times B \to \boldsymbol{P}(X)$ are in bijection with maps $B\times A \to \boldsymbol{P}(X)$, which are in turn in bijection with maps $B \to \mathrm{Map}(A, \boldsymbol{P}(X))$. Here are the formulas:

$$\bigcup_{a\in A}\bigcup b \in BU(a,b) = \bigcup_{b\in B}\bigcup a \in AU(a,b)\,;$$

$$\bigcap_{a\in A}\bigcap b \in BU(a,b) = \bigcap_{b\in B}\bigcap a \in AU(a,b)\,;$$

$$\bigcap_{a\in A}\bigcup_{b\in B} U(a,b) = \bigcup_{f\in\mathrm{Map}(A,B)}\bigcap_{a\in A} U(a,f(a))\,;$$

$$\bigcup_{a\in A}\bigcap_{b\in B} U(a,b) = \bigcap_{f\in\mathrm{Map}(A,B)}\bigcup_{a\in A} U(a,f(a))\,.$$

There is also the *complement* of any $A \in \boldsymbol{P}(X)$

$$\complement A = X \smallsetminus A \coloneqq \{x \in X : x \notin A)\}\,.$$

The *de Morgan laws* state that the formation of the complement exchanges union and intersection: for any map $U\colon\ A \to \boldsymbol{P}(X)$,

$$\complement\left(\bigcup_{a\in A} U(a)\right) = \bigcap_{a\in A} \complement U(a)\,;$$

$$\complement\left(\bigcap_{a\in A} U(a)\right) = \bigcup_{a\in A} \complement U(a)\,.$$

A.10 Inverse and direct image

A map $f\colon\ X \to Y$ induces a map

$$f^*\colon\ \boldsymbol{P}(Y) \to \boldsymbol{P}(X)$$

called the *inverse image* and a map

$$f_*\colon\ \boldsymbol{P}(X) \to \boldsymbol{P}(Y)$$

called the *direct image*. The inverse image of a subset $B \subseteq Y$ is the set

$$f^*(B) \coloneqq \{x \in X : f(x) \in B\}\ ,$$

and the direct image of a subset $A \subseteq X$ is the set

$$f_*(A) \coloneqq \{y \in Y : (\exists x \in A)(y = f(x))\} .$$

These operations are related in the following manner: one has $f_*(A) \subseteq B$ if and only if $A \subseteq f^*(B)$. In particular, we have

$$A \subseteq f^*(f_*(A)) \qquad \text{and} \qquad f_*(f^*(B)) \subseteq B .$$

In general, both of these containments are strict. However, if f is an injection, then $A = f^*(f_*(A))$, and if f is a surjection, then $f_*(f^*(B)) = B$.

In many respects, the inverse image is more natural than the direct image. For example, the inverse image preserves unions, intersections, and complements, so that one has:

$$f^*\left(\bigcup_{a\in A} U(a)\right) = \bigcup_{a\in A} f^*(U(a)) ;$$
$$f^*\left(\bigcap_{a\in A} U(a)\right) = \bigcap_{a\in A} f^*(U(a)) ;$$
$$f^*(\complement U) = \complement(f^*(U)) .$$

For the direct image, one only has

$$f_*\left(\bigcup_{a\in A} U(a)\right) = \bigcup_{a\in A} f_*(U(a)) ;$$
$$f_*\left(\bigcap_{a\in A} U(a)\right) \subseteq \bigcap_{a\in A} f_*(U(a)) .$$

There is no containment between $\complement f_*(A)$ and $f_*(\complement A)$ in general.

In the particular case where $B = \{y\}$, the inverse image $f^*\{y\}$ is called the *fiber* of f over y. This gives us a handy way to think about maps $f\colon X \to Y$: in effect, they organize X into a disjoint union of fibers. That is, if $y \neq y'$, then $f^*\{y\} \cap f^*\{y'\} = \varnothing$, and

$$X = \bigcup_{y\in Y} f^*\{y\} .$$

The map f is injective if and only if each fiber $f^*\{y\}$ has *at most one element*, and f is surjective if and only if each fiber $f^*\{y\}$ has *at least one element*.

Warning A.10.1. Now for the annoying news. The notations f^* and f_* are not the usual notations. The more typical notation for the inverse image of B is $f^{-1}(B)$. The more typical notation for the direct image of A is $f(A)$. This notation makes it look as though these operations are inverse – *in general they are not!!*

A.11 Products of sets

An *indexed set* is really just another name for a map[7] $U\colon A \to \Xi$; we typically abuse notation and write U_a instead of $U(a)$, and we write $(U_a)_{a\in A}$ for the map U. The union of the indexed set $(U_a)_{a\in A}$ will mean the union of the set $\{U_a : a \in A\}$:

$$\bigcup_{a\in A} U_a \coloneqq \bigcup\{U_a : a \in A\}.$$

[7] Remember that every element of every set is itself a set; thus the elements of Ξ are sets as well!

It may seem a bit silly to belabour this point, but the purpose will, we hope, become clear.

For any indexed set $(U_a)_{a\in A}$, the *product* is the set

$$\prod_{a\in A} U_a \coloneqq \left\{ x \in \mathrm{Map}\left(A, \bigcup_{a\in A} U_a \right) : (\forall a \in A)\ x(a) \in U_a \right\} .$$

An element of $\prod_{a\in A} U_a$ is thus a map $x\colon A \to \bigcup_{a\in A} U_a$ such that for every $a \in A$, one has $x(a) \in U_a$. One usually writes x_a instead of $x(a)$, and one often writes $x = (x_a)_{a\in A}$.

Example A.11.1. When $A = \{1, 2\}$, an indexed set consists of two sets U_1 and U_2. The product $\prod_{a\in\{1,2\}} U_a$ thus consists of pairs (x_1, x_2) with $x_1 \in U_1$ and $x_2 \in U_2$. The assignment $(x_1, x_2) \mapsto \langle x_1, x_2\rangle$ is a bijection $\prod_{a\in\{1,2\}} U_a \to U_1 \times U_2$. Most mathematicians are happy to pretend as if there is no difference between these sets; indeed, there is no *interesting* difference!

More generally, we think of the product $\prod_{a\in A} U_a$ as the set of ordered 'A-tuples'.

For every $b \in A$, there is an attached map

$$\pi_b\colon \prod_{a\in A} U_a \to U_b$$

given by the assignment $x \mapsto x_b$, called the *projection* onto the b-th factor.

For every set S, every indexed set $\{U_a\}_{a\in A}$, and every indexed set $\{f_a\colon S \to U_a\}$ of maps, there exists a unique map

$$f\colon S \to \prod_{a\in A} U_a$$

such that $\pi_a \circ f = f_a$. Indeed, the map f is given by the assignment $s \mapsto (f_a(x))_{a\in A}$.

A.12 Coproducts of sets

For any indexed set $\{U_a\}_{a\in A}$, the *coproduct* or *disjoint union* is the set

$$\coprod_{a\in A} U_a \coloneqq \bigcup_{a\in A} U_a \times \{a\}.$$

For every $b \in A$, there is an attached map

$$\iota_b\colon U_b \to \coprod_{a\in A} U_a$$

given by $\iota_b(x) = (x, b)$, called the *inclusion* onto the b-th summand.

The coproduct is really *dual* to the product. Here's how: for every set S, every indexed set $\{U_a\}_{a\in A}$, and every indexed set $\{f_a\colon U_a \to S\}$ of maps, there exists a unique map

$$f\colon \coprod_{a\in A} U_a \to S$$

such that $f \circ \iota_a = f_a$. Indeed, the map f is given by the assignment $(x, a) \mapsto f_a(x)$.