General topology

The Problems

Autumn 2020

Products

Problem 1

Consider the product topological space

$$\boldsymbol{R}^{\omega} \coloneqq \prod_{n \geq 1} \boldsymbol{R} \,.$$

The elements of $\boldsymbol{R}^{\omega}$ are sequences $(x_1, x_2, \dots)$ of real numbers. For each of the following subsets $S \subset \boldsymbol{R}^{\omega}$, compute the closure of S in $\boldsymbol{R}^{\omega}$.

- the set of sequences $(x_1, x_2, \dots)$ that are *eventually zero*: i.e., such that there exists an N such that for every $k > N$, one has $x_k = 0$;
- the set of sequences $(x_1, x_2, \dots)$ that converge to 0;
- the set of Cauchy sequences.

Notation. For any set S, let S^{δ} denote S with the discrete topology.

Problem 2

Let $S_1, S_2, \dots, S_N$ be a finite collection of sets. Show that

$$S_1^{\delta} \times S_2^{\delta} \times \cdots \times S_N^{\delta}$$

is discrete.

Problem 3

Prove that for any set S, the topological space

$$S^{\omega} \coloneqq \prod_{n \geq 1} S^{\delta}$$

is discrete if and only if S has ≤ 1 points.

Problem 4

Consider the product topological space

$$\{0,1\}^{\omega} \coloneqq \prod_{n \geq 1} \{0,1\}^{\delta} \,.$$

For any $x \in C$, let $a(x)$ be the sequence $(a(x)_1, a(x)_2, \dots)$ such that

$$a(x)_n \coloneqq \begin{cases} 0 & \text{if } x \in \left[\dfrac{3k}{3^n}, \dfrac{3k+1}{3^n}\right] \subset \bigcap_{i=1}^{n} C_i \text{ for some } k \,, \\[2ex] 1 & \text{if } x \in \left[\dfrac{3k+2}{3^n}, \dfrac{3k+3}{3^n}\right] \subset \bigcap_{i=1}^{n} C_i \text{ for some } k \,. \end{cases}$$

Show that $a \colon C \to \{0,1\}^{\omega}$ is a homeomorphism.

Problem 5

Define a map $c\colon C \to [0,1]$ by the formula

$$c(x) \coloneqq \sum_{n=1}^{\infty} \frac{a(x)_n}{2^n} \,.$$

Prove that c is a continuous surjection.

Problem 6

Construct a homeomorphism $s\colon C \to C \times C$.

Problem 7

Use c and s to construct a continuous surjection

$$h\colon C \to [0,1] \times [0,1] \,.$$

Extend h to a continuous surjection

$$[0,1] \to [0,1] \times [0,1] \,,$$

a *space-filling curve*.

Challenge problem

Problem 8

Show that if S and T are finite sets of cardinality at least 2, then S^ω is homeomorphic to T^ω.