General topology

The Problems

Autumn 2020

Subspaces of Euclidean space

Problem 1

Prove or disprove: every countable subspace $X \subset \boldsymbol{R}$ is discrete.[1]

[1] A subspace $X \subseteq \boldsymbol{R}$ is *discrete* if and only if every subset $S \subseteq X$ is both open and closed.

Problem 2

Let $X \subseteq \boldsymbol{R}^n$ be a subspace. For every pair of subsets $S, T \in \boldsymbol{P}(X)$, write

$$d(S,T) = \inf\{d(s,t) : (s \in S) \wedge (t \in T)\} .$$

Suppose that $S, T \in \boldsymbol{P}(X)$ are subsets with the property that $d(S,T) = 0$. Prove or provide counterexamples for each of the following claims:

- $S \cap T \neq \varnothing$;
- $S \cap \tau(T) \neq \varnothing$;
- $(S \cap \tau(T)) \cup (\tau(S) \cap T) \neq \varnothing$;
- $\tau(S) \cap \tau(T) \neq \varnothing$.

Problem 3

Define a map $e\colon\ ]0,1] \to S^1$ as follows:

$$e(\theta) \coloneqq (\cos(2\pi\theta), \sin(2\pi\theta)) .$$

Show that e is a continuous bijection, but not a homeomorphism.

Problem 4

Prove or disprove: S^1 is homeomorphic to the subspace

$$\{z \in \boldsymbol{C} : |z-1||z+1| = 1\} \subset \boldsymbol{C} .$$

Problem 5

Which pairs of the following five subspaces of $\boldsymbol{R}$ are homeomorphic?

- $\boldsymbol{R}$,
- $[0,1]$,
- $[0,+\infty[$,
- $]-\infty,0]$, and
- $]0,1[$.

Problem 6

Prove that for any $n \in \boldsymbol{N}$, the subspace

$$B = \{\boldsymbol{x} \in \boldsymbol{R}^n : \|\boldsymbol{x}\| < 1\} \subset \boldsymbol{R}^n$$

is homeomorphic to $\boldsymbol{R}^n$.

Problem 7

For every integer $n \in \boldsymbol{N}^*$, let $X_n \subseteq \boldsymbol{C}$ be the subset given by

$$X_n := \{z \in \boldsymbol{C} : z^n = |z|^n\} .$$

Prove or disprove: if $m \neq n$, then X_m is not homeomorphic to X_n.

Notation. Let $k \leq n$ be natural numbers. We will think of elements of $\boldsymbol{R}^{nk}$ as $(n \times k)$-matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} .$$

Using matrix multiplication, we may define a subspace

$$V(k, n) = \{A \in \boldsymbol{R}^{nk} : A^t A = I_k\} \subseteq \boldsymbol{R}^{nk} .$$

This is called the *Stiefel manifold*. If the columns of A are written as $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$, then

$$V(k, n) = \{(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \in \boldsymbol{R}^{nk} : (\forall i, j)(\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \delta_{ij})\} ,$$

where δ is the *Kronecker delta*:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j , \\ 0 & \text{if } i \neq j . \end{cases}$$

In particular, $V(1, n) \subseteq \boldsymbol{R}^n$ is the set of $\boldsymbol{v} \in \boldsymbol{R}^n$ such that $\|\boldsymbol{v}\| = 1$. In other words, $V(1, n) = S^{n-1}$.

Problem 8

Prove that the map $p\colon V(k, n) \to V(1, n) = S^{n-1}$ given by the assignment $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \mapsto \boldsymbol{v}_1$ is continuous. For any $\boldsymbol{v} \in V(1, n)$, consider the inverse image $p^{-1}\{\boldsymbol{v}\} \subseteq V(k, n) \subseteq \boldsymbol{R}^{nk}$. Show that $p^{-1}\{\boldsymbol{v}\}$ is homeomorphic to $V(k-1, n-1)$.

Notation. Define a subspace $C \subset [0, 1]$ as follows. For every natural number n, set

$$C_n := \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right) ,$$

and define

$$C := \bigcap_{n \geq 1} C_n .$$

The topological space C is called the *Cantor space*.

Problem 9

Prove that C is closed in $[0, 1]$. What is the interior of C as a subspace of $[0, 1]$? That is, what is the largest open subset $U \subseteq [0, 1]$ such that $U \subseteq C$?

(Don't worry; we'll have a lot more questions about the Cantor space!)