CLARK BARWICK

TOPOLOGY

UNIVERSITY OF EDINBURGH

CONTENTS

1 SUBSPACES OF EUCLIDEAN SPACE

1.1 Basic definitions

Our starting place is the study of the *subspaces* of Euclidean space.

Notation 1.1.1. As usual, $\boldsymbol{R}$ will denote the ordered field of real numbers. For any $n \geq 0$, we write $\boldsymbol{R}^n$ for the vector space of n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \boldsymbol{R}$.

For our purposes, one of the most important features $\boldsymbol{R}^n$ has to offer is its *metric*; this is the map

$$d\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}\,,$$

called the *distance*, given by the formula

$$d(x, y) \coloneqq \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\,.$$

This map enjoys four key properties:

1. (Nonnegativity.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) \geq 0$.
2. (Identity.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) = 0$ if and only if $x = y$.
3. (Symmetry.) For every pair of points $x, y \in \boldsymbol{R}^n$, one has $d(x, y) = d(y, x)$.
4. (Triangle inequality.) For every triple of points $x, y, z \in \boldsymbol{R}^n$, one has

$$d(x, y) + d(y, z) \geq d(x, z)\,.$$

If $r \geq 0$ and $x \in \boldsymbol{R}^n$, then we define the *ball of radius r centered at x*:

$$B^n(x, r) \coloneqq \{y \in \boldsymbol{R}^n : d(x, y) < r\}\,.$$

In topology, we are not interested in the metric d itself but in what it can tell us about the nearness of points and subsets:

Definition 1.1.2. Let $S \subseteq \boldsymbol{R}^n$ be a subset. A point $x \in \boldsymbol{R}^n$ will be said to be *close to* S if and only if, for every $\varepsilon > 0$, there exists an element $s \in S$ such that $d(x, s) < \varepsilon$.

We may express this notion with the aid of an auxiliary quantity. Define the *distance from x to S* as[1]

$$d(x, S) \coloneqq \inf\{d(x, s) : s \in S\}\,.$$

Then x is *close to* S if and only if $d(x, S) = 0$.

[1] If $S = \varnothing$, then this infimum doesn't exist as a real number. So we'll declare formally that $d(x, \varnothing) = +\infty$.

Let $X \subseteq \boldsymbol{R}^n$ be a subset such that $S \subseteq X$. We shall write $\tau_X(S) \subseteq X$ for the set of $x \in X$ that are close to S:

$$\tau_X(S) \coloneqq \{x \in X : (\forall \varepsilon > 0)(\exists s \in S)(d(x,s) < \varepsilon)\} = \{x \in X : d(x,S) = 0\}$$

This is called the *closure of S in X*. In other words,

$$\tau_X(S) = X \cap \bigcap_{\varepsilon>0} \bigcup_{s\in S} B^n(s,\varepsilon) .$$

We will refer to X along with the map[2] $\tau_X : \boldsymbol{P}(X) \to \boldsymbol{P}(X)$ as the *subspace $X \subseteq \boldsymbol{R}^n$*.

[2] The set $\boldsymbol{P}(X)$ is the *powerset* of X; it is the set of subsets of X.

Every time you see a new definition in mathematics, you should begin immediately to think about as many examples as you can. You should also have a couple of *counterexamples* at hand, preferably at least one for each condition in a definition. Then, when you encounter a theorem with a proof, you can take your example and 'run' the proof on your example, and you can run it on your counterexample to see where the conditions are really being used. This is one of the most important ways to understand how theorems really work.

In that spirit, let's dig into some examples.

Example 1.1.3.
- If $X = \varnothing$, then this definition becomes a little boring: the only subset of $\varnothing$ is $\varnothing$ itself, and $\tau_\varnothing(\varnothing) = \varnothing$.
- If $X = \boldsymbol{R}$ itself, then let's see what happens for various choices of S:
  - Let's consider various unit intervals:

    $$\tau_{\boldsymbol{R}}(]0,1[) = \tau_{\boldsymbol{R}}([0,1]) = \tau_{\boldsymbol{R}}(]0,1]) = [0,1]$$

  - If $\boldsymbol{Q} \subset \boldsymbol{R}$ denotes the set of rational numbers, then $\tau_{\boldsymbol{R}}(\boldsymbol{Q}) = \boldsymbol{R}$.
  - At the same time, $\tau_{\boldsymbol{R}}(\boldsymbol{R} \smallsetminus \boldsymbol{Q}) = \boldsymbol{R}$.
  - Confirm that $\tau_{\boldsymbol{R}}(\varnothing) = \varnothing$.
- If $X \subseteq \boldsymbol{R}^n$ is a subspace, then we have

  $$\tau_X(S) = \tau_{\boldsymbol{R}^n}(S) \cap X .$$

  This is an important point: the closure of S is done *relative to the subspace X*. Let's see how the closure of $S = \,]0,1[$ depends on which subspace $X \subseteq \boldsymbol{R}$ in which we are closing:
  - If $X = \boldsymbol{R}$, then $\tau_{\boldsymbol{R}}(]0,1[) = [0,1]$.
  - If $X = \,]0,1[$, then $\tau_{]0,1[}(]0,1[) = \,]0,1[$.
  - If $X = \,]0,1]$, then $\tau_{]0,1]}(]0,1[) = \,]0,1]$.
  - If $X = \,]0,+\infty[$, then $\tau_{]0,+\infty[}(]0,1[) = \,]0,1]$.

Example 1.1.4. Let $X \subseteq \boldsymbol{R}$ be a subspace. Let $S \subseteq X$ be a nonempty subset that is bounded above. The completeness property of $\boldsymbol{R}$ then ensures that S has a supremum $\sup S$. If $\sup S \in X$, then $\sup S$ is close to S in X; that is,

$$\sup S \in \tau_X(S) .$$

Let's prove this; let $x := \sup S$. Let $\varepsilon > 0$. If $]x-\varepsilon, x+\varepsilon[\, \cap S = \varnothing$, then $x-\varepsilon$ is also an upper bound for S, contradicting the assumption that x is the *least* upper bound for S. Consequently, there exists an element $s \in S$ such that $d(x,s) < \varepsilon$.

The same argument shows that if $S \subseteq X$ is nonempty and bounded below, then if $\inf S \in X$, then

$$\inf S \in \tau_X(S) .$$

HERE ARE the basic facts about the closure operator $\tau_X : \boldsymbol{P}(X) \to \boldsymbol{P}(X)$. As we shall see, these facts are in many ways even more important than the definition of τ_X itself.

Proposition 1.1.5. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace.*

1. *For every subset $S \subseteq X$, every point of S is close to S. That is, $S \subseteq \tau_X(S)$.*
2. *No points of X are close to $\varnothing$. That is, $\tau_X(\varnothing) = \varnothing$.*
3. *If $S_1, \dots, S_n \subseteq X$ are a finite collection of subsets, then a point $x \in X$ is close to $S_1 \cup \cdots \cup S_n$ if and only if, for some i, the point x is close to S_i. That is,*
$$\tau_X(S_1 \cup \cdots \cup S_n) = \tau_X(S_1) \cup \cdots \cup \tau_X(S_n) .$$
4. *If $S \subseteq X$, then if $x \in X$ is a point that is close to $\tau_X(S)$, then x is close to S itself. That is, $\tau_X(\tau_X(S)) = \tau_X(S)$.*

Proof. Let us prove these in order.

1. If $x \in S$, then for every $\varepsilon > 0$, we certainly have $d(x,x) = 0 < \varepsilon$.
2. For every $\varepsilon > 0$, the sentence[3] $(\exists s \in \varnothing)(d(x,s) < \varepsilon)$ is false.
3. If x is close to $S_1 \cup \cdots \cup S_n$, then for every natural number[4] $m \in \boldsymbol{N}^*$, we may select $s_m \in S_1 \cup \cdots \cup S_n$ such that $d(x, s_m) < 1/m$. By the pigeonhole principle, there exists an i such that the set $\{m \in \boldsymbol{N}^* : s_m \in S_i\}$ is infinite. Consequently, for any $\varepsilon > 0$, there exists a natural number $m \geq 1$ such that $s_m \in S_i$ and $1/m < \varepsilon$. In particular
$$d(x, s_m) < 1/m < \varepsilon .$$
Thus x is close to S_i.
Conversely, if x is close to some S_i, then for every $\varepsilon > 0$, there exists $s \in S_i \subseteq S_1 \cup \cdots \cup S_n$ such that $d(x,s) < \varepsilon$, so x is close to $S_1 \cup \cdots \cup S_n$.
4. If x is close to $\tau_X(S)$, then for any $\varepsilon > 0$, we may select $t \in \tau_X(S)$ such that $d(x,t) < \varepsilon/2$. Since t is close to S, we may select $s \in S$ such that $d(t,s) < \varepsilon/2$. Now by the triangle inequality,
$$d(x,s) \leq d(x,t) + d(t,s) < \varepsilon .$$
Thus x is close to S. □

[3] Sentences of the form $(\exists s \in \varnothing)(\phi(s))$ are *always* false, no matter what $\phi(s)$ says!

[4] In this text, $\boldsymbol{N}$ will denote the natural numbers $\{0, 1, \dots\}$, and $\boldsymbol{N}^*$ will denote the natural numbers $\{1, 2, \dots\}$.

Corollary 1.1.6. *If $X \subseteq \boldsymbol{R}^n$ is a subspace, then for any subsets $S, T \in \boldsymbol{P}(X)$, if $S \subseteq T$, then $\tau_X(S) \subseteq \tau_X(T)$.*

Proof. We may write $T = S \cup (T \smallsetminus S)$. Then by the third point of the proposition above,

$$\tau_X(T) = \tau_X(S) \cup \tau_X(T \smallsetminus S)\,,$$

and so $\tau_X(S) \subseteq \tau_X(T)$. ☐

If $X \subseteq \boldsymbol{R}^n$ is a subspace, then there are two different kinds of subset that play a big role in topology:

Definition 1.1.7. A subset $Z \subseteq X$ is *closed in* X if and only if

$$\tau_X(Z) = Z\,.$$

In other words, Z is closed if and only if every point of X that is close to Z is contained in Z.

A subset $U \subseteq X$ is *open in X* if and only if $X \smallsetminus U$ is closed. In other words, U is open if and only if no point of U is close to the complement of U.

Example 1.1.8.
- For any subspace $X \subseteq \boldsymbol{R}^n$, the subset $X \subseteq X$ is both open and closed.[5]
- For any subspace $X \subseteq \boldsymbol{R}^n$, the subset $\varnothing \subseteq X$ is both open and closed.
- In $\boldsymbol{R}$ itself, the subset $[0,1] \subset \boldsymbol{R}$ is closed but not open.
- In $\boldsymbol{R}$, the subset $]0,1[\subset \boldsymbol{R}$ is open but not closed.
- In $\boldsymbol{R}$, the subsets $[0,1[$ and $]0,1]$ are neither open nor closed.
- In the subspace $]0,+\infty[\subseteq \boldsymbol{R}$, the subset $]0,1] \subseteq]0,+\infty[$ is closed but not open.
- In the subspace $[0,+\infty[\subseteq \boldsymbol{R}$, the subset $[0,1[\subseteq [0,+\infty[$ is open but not closed.

[5] This is sometimes a source of confusion – or at least irritation. One may expect that 'open' and 'closed' are opposites, but they are not: there are subsets that are neither, and there are subsets that are both. Instead, they are *complementary* notions.

Proposition 1.1.9. *For any subspace $X \subseteq \boldsymbol{R}^n$ and any subset $S \subseteq X$, the closure $\tau_X(S)$ is the smallest closed subset*[6] *of X that contains S.*

[6] That is, $\tau_X(S) \subseteq X$ is a closed subset that contains S, and for any closed subset $Z \subseteq X$ that contains S, one has $\tau_X(S) \subseteq Z$.

Proof. Let $Z \subseteq X$ be a closed subset that contains S. Then

$$\tau_X(S) \subseteq \tau_X(Z) = Z\,.$$

Hence $\tau_X(S)$ is contained in any closed subset of X that contains S. It remains to note that $\tau_X(S)$ is itself a closed subset of X that contains S. By the proposition above,

$$\tau_X(\tau_X(S)) = \tau_X(S)\,,$$

so $\tau_X(S)$ is closed. ☐

Proposition 1.1.10. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. Then $U \subseteq X$ is open if and only if, for every $x \in U$, there exists an $\varepsilon > 0$ such that*

$$B^n(x,\varepsilon) \cap X \subseteq U\,.$$

Proof. Suppose that $U \subseteq X$ is open, and let $x \in U$ be a point. Then by definition, x is not close to $X \smallsetminus U$. Hence there exists $\varepsilon > 0$ such that for any point $y \in X \smallsetminus U$, one has $d(x, y) \geq \varepsilon$. That is, U contains $B^n(x, \varepsilon) \cap X$.

Now assume that for every $x \in U$, there exists an $\varepsilon > 0$ such that

$$B^n(x, \varepsilon) \cap X \subseteq U \ .$$

We aim to show that $X \smallsetminus U$ is closed; hence let $x \in U$ be a point that is close to $X \smallsetminus U$. For some $\varepsilon > 0$, the intersection $B^n(x, \varepsilon) \cap X$ is contained in U. But at the same time, since x is close to $X \smallsetminus U$, the intersection $B^n(x, \varepsilon) \cap (X \smallsetminus U) \neq \varnothing$. This is a contradiction that shows that no point of U is close to $X \smallsetminus U$; thus U is open. □

Proposition 1.1.11. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. Then a subset $Z \subseteq X$ is closed if and only if $Z = Z' \cap X$ for some closed subset $Z' \subseteq \boldsymbol{R}^n$. Similarly, a subset $U \subseteq X$ is open if and only if $U = U' \cap X$ for some open subset $U' \subseteq \boldsymbol{R}^n$.*

Proof. Assume that $Z' \subseteq \boldsymbol{R}^n$ is a closed subset. Now consider the intersection $Z' \cap X$. The closure $\tau_{\boldsymbol{R}^n}(Z' \cap X)$ is contained in $\tau_{\boldsymbol{R}^n}(Z') = Z'$. Consequently,

$$\tau_X(Z' \cap X) = \tau_{\boldsymbol{R}^n}(Z' \cap X) \cap X \subseteq Z' \cap X \ ,$$

so $Z' \cap X$ is closed.

Conversely, assume that $Z \subseteq X$ is a closed subset. Then $Z' \coloneqq \tau_{\boldsymbol{R}^n}(Z)$ is closed in $\boldsymbol{R}^n$, and

$$Z = \tau_X(Z) = \tau_{\boldsymbol{R}^n}(Z) \cap X = Z' \cap X.$$

The statements about open subsets of X now follow by taking complements.[7] Indeed, if $U' \subseteq \boldsymbol{R}^n$ is an open subset, then let Z' denote its complement. Since $Z' \cap X$ is closed by what we've already proved, its complement in X, which is $U' \cap X$ is open. Conversely, if $U \subseteq X$ is open, then let Z denote its complement in X. By what we've already shown, there is a closed subset $Z' \subseteq \boldsymbol{R}^n$ such that $Z = Z' \cap X$, so if U' denotes the open complement of Z', then $U = U' \cap X$. □

[7] For proofs of this kind, it is usually enough just to offer such a short sentence. In this case, we will unpack this sentence a little more.

Proposition 1.1.12. *Let $X \subseteq \boldsymbol{R}^n$ be a subspace. For any family $\{Z_a\}_{a \in A}$ of closed subsets of X, the intersection $\bigcap_{a \in A} Z_a$ is closed as well; if A is* finite, *then the union $\bigcup_{a \in A} Z_a$ is closed as well. Dually, for any family $\{U_a\}_{a \in A}$ of open subsets of X, the union $\bigcup_{a \in A} U_a$ is open as well; if A is* finite, *then the intersection $\bigcap_{a \in A} U_a$ is open as well.*

Proof. Let $W = \bigcap_{a \in A} Z_a$. We aim to prove that $\tau_X(W) \subseteq W$. Let $x \in X$ be a point that is close to W. For every $a \in A$, since $W \subseteq Z_a$, it follows that x is close to Z_a as well. Since Z_a is closed, $x \in Z_a$. This happens for every $a \in A$, so $x \in W$.

If A is finite, then we have shown that

$$\tau_X\left(\bigcup_{a \in A} Z_a\right) = \bigcup_{a \in A} \tau_X(Z_a) = \bigcup_{a \in A} Z_a \ ,$$

so the union is closed.

The dual statement follows by forming complements. □

Example 1.1.13. Let's consider examples in $\boldsymbol{R}$ in which the finiteness hypotheses of the previous proposition are necessary.

- Infinite unions of closed subsets need not be closed:
$$\bigcup_{n \in \boldsymbol{N}^*} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] = \left]-1,1\right[\ ,$$
or open:
$$\bigcup_{n \in \boldsymbol{N}^*} \left[0, 1-\frac{1}{n}\right] = \left[0,1\right[\ ..$$
- Infinite intersections of open subsets need not be open:
$$\bigcap_{n \in \boldsymbol{N}^*} \left]-1-\frac{1}{n}, 1+\frac{1}{n}\right[= [-1,1]\ ,$$
or closed:
$$\bigcap_{n \in \boldsymbol{N}^*} \left]0, 1+\frac{1}{n}\right[= \left]0,1\right]\ .$$

We conclude with some examples of subspaces of Euclidean space to which we will return regularly.

Example 1.1.14.
- For any $n \in \boldsymbol{N}$, the ***n-sphere*** S^n is the subspace
$$S^n \coloneqq \{x \in \boldsymbol{R}^{n+1} : \|x\| = 1\}\ .$$
- The ***open unit ball*** is the subspace
$$B^n \coloneqq B^n(0,1) = \{x \in \boldsymbol{R}^n : \|x\| < 1\}\ ,$$
and its closure in $\boldsymbol{R}^n$ is the ***closed unit ball***
$$\overline{B}^n \coloneqq \{x \in \boldsymbol{R}^n : \|x\| \leq 1\}\ .$$
- If we have a finite collection of subspaces $X_1 \subseteq \boldsymbol{R}^{n_1}, \dots, X_k \subseteq \boldsymbol{R}^{n_k}$ then we obtain the subspace
$$X_1 \times \cdots \times X_k \coloneqq \{x = (x^1, \dots, x^k) \in \boldsymbol{R}^{n_1} \times \cdots \times \boldsymbol{R}^{n_k} : (\forall i)(x^i \in X_i)\}\ .$$
- For any $n \in \boldsymbol{N}$, the ***n-torus*** is the subspace
$$T^n \coloneqq S^1 \times \cdots \times S^1 \subseteq \boldsymbol{R}^{2n}\ .$$
That is,
$$T^n \coloneqq \{(x_1, y_1, \dots, x_n, y_n) \in \boldsymbol{R}^{2n} : (\forall i)(x_i^2 + y_i^2 = 1)\}\ .$$

1.2 Continuity

Another good name for this bit would be, 'A first introduction to what topology is really about'.

Definition 1.2.1. Let $X \subseteq \boldsymbol{R}^m$ and $Y \subseteq \boldsymbol{R}^n$ be subspaces. A map $f: X \to Y$ is ***continuous*** if and only if, for any subset $S \subseteq X$ and any point $x \in X$, if x is close to S, then $f(x)$ is close to $f(S)$. In other words, f is continuous if and only if, for any subset $S \subseteq X$, one has $f(\tau_X(S)) \subseteq \tau_Y(f(S))$.

Proposition 1.2.2. *Let $X \subseteq \boldsymbol{R}^m$ and $Y \subseteq \boldsymbol{R}^n$ be subspaces. The following are equivalent for a map $f : X \to Y$.*

1. *The map f is continuous.*
2. *For any subset $T \subseteq Y$, one has $\tau_X(f^{-1}(T)) \subseteq f^{-1}(\tau_Y(T))$.*
3. *For any closed subset $Z \subseteq Y$, the inverse image $f^{-1}(Z) \subseteq X$ is closed.*
4. *For any open subset $U \subseteq Y$, the inverse image $f^{-1}(U) \subseteq X$ is open.*
5. *For any $x \in X$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0 \in X$ is a point such that $d(x_0, x) < \delta$, then[8] $d(f(x_0), f(x)) < \varepsilon$.*

[8] This is probably the characterization of continuity that is most familiar to you from analysis. In this text, we are happy to assume basic facts of analysis about the continuity of well-known functions on subspaces of Euclidean space.

Proof. Assume (1); we aim to prove (2). Let $T \subseteq Y$, and assume that x is close to $f^{-1}(T)$. Then by assumption $f(x)$ is close to $f(f^{-1}(T))$. Since $f(f^{-1}(T)) \subseteq T$, it follows that $f(x)$ is close to T, so $x \in f^{-1}(\tau_Y(T))$.

Assume (2); we aim to prove (3). Let $Z \subseteq Y$ be closed. Then

$$\tau_X(f^{-1}(Z)) \subseteq f^{-1}(\tau_Y(Z)) = f^{-1}(Z)\,,$$

so $f^{-1}(Z)$ is closed.

Assume (3); then (4) follows by taking complements.

Assume (4); we aim to prove (5). Let $x \in X$, and let $\varepsilon > 0$. The subset $f^{-1}(B^n(f(x), \varepsilon) \cap Y) \subseteq X$ is open. Consequently, there exists $\delta > 0$ such that $B^m(x, \delta) \cap X \subseteq f^{-1}(B^n(f(x), \varepsilon) \cap Y)$; that is, for any $x_0 \in X$ such that $d(x_0, x) < \delta$, one has $d(f(x_0), f(x)) < \varepsilon$, as desired.

Assume (5); we aim to prove (1). Let $S \subseteq X$, and let $x \in X$ be a point that is close to S. Let $\varepsilon > 0$. By assumption there is a $\delta > 0$ such that for any $x_0 \in X$ with $d(x_0, x) < \delta$, we have $d(f(x_0), f(x)) < \varepsilon$. Since x is close to S, we may select $s \in S$ such that $d(x, s) < \delta$. Thus $d(f(x), f(s)) < \varepsilon$. This proves that $f(x)$ is close to $f(S)$. □

To be the 'same', topologically speaking, is to be *homeomorphic*.

Definition 1.2.3. Let $X \subseteq \boldsymbol{R}^m$ and $Y \subseteq \boldsymbol{R}^n$ be subspaces. Then a ***homeomorphism*** $f : X \to Y$ is a continuous bijection whose inverse f^{-1} is continuous. In other words, a map $f : X \to Y$ is a homeomorphism if and only if f is a bijection such that for any subset $S \subseteq X$, one has $\tau_Y(f(S)) = f(\tau_X(S))$. Equivalently, $f : X \to Y$ is a homeomorphism if and only if it is a bijection such that both the image and the inverse image of closed (respectively, open) subsets remains closed (resp., open).

The two subspaces X and Y are said to be ***homeomorphic*** if and only if there is a homeomorphism $f : X \to Y$.

The following is a very important example to understand what ***topology*** is all about:

Example 1.2.4. The subspaces $\boldsymbol{R}$, $]0, +\infty[$, and $]0, 1[$ are all homeomorphic. To prove this, let's construct some homeomorphisms. The exponential map $\exp\colon \boldsymbol{R} \to \,]0, +\infty[$ is a continuous bijection[9] whose inverse is the logarithm $\log\colon \,]0, +\infty[\, \to \boldsymbol{R}$, which is also continuous. This shows that $\boldsymbol{R}$ and $]0, +\infty[$ are homeomorphic.

[9] In this course, we will accept these basic facts of analysis.

The map $f\colon \boldsymbol{R} \to \,]0,1[$ given by

$$f(x) = \frac{1}{1+\exp(-x)}$$

is a continuous bijection with inverse $g\colon\ ]0,1[\ \to \boldsymbol{R}$ given by

$$g(x) = -\log\left(\frac{1-x}{x}\right),$$

which is also continuous. Thus $\boldsymbol{R}$ and $]0,1[$ are homeomorphic.

Please observe that what we are saying here is that an open interval and the real line are *topologically the same*. The fact that the real line is 'infinitely long' is of no importance to the topologist; the topology 'knows' about the relation of closeness, but nothing about distance.

You can think about topology as a certain 'filter' that you've applied to the information supplied by a geometric object. That filter removes a lot of what we ordinarily think of as important about these objects – distances, angles, and so on. There's still some information left over, however – a kind of 'proto-geometric' information.

We imagine our subspace as made up of some infinitely deformable material. As long as we don't tear it, or poke holes in it, we are free to stretch it, twist it, and move it around, and we haven't changed its 'topological nature'. Understanding that topological nature is the central topic of this course.

Example 1.2.5. Consider the following two subpsaces. First, we have $X \coloneqq \boldsymbol{R} \smallsetminus \{0\} \subset \boldsymbol{R}$. Next, we have

$$Y \coloneqq \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\} \subset \boldsymbol{R}^2 .$$

These spaces are homeomorphic. To see this, let us define a map $f\colon X \to Y$. We shall let

$$f(t) = \left(\frac{t}{|t|}, \log|t|\right).$$

This is a continuous bijection with inverse $g \coloneqq Y \to X$ given by

$$g(x,y) = x\exp(y),$$

which is also continuous.

By taking the real line and removing a point from it, we obtained two lines.[10]

[10] You may be wondering: is there some weird homeomorphism between $\boldsymbol{R}$ itself and Y? Maybe one line is somehow homeomorphic to two lines? It turns out that $\boldsymbol{R}$ and Y are not homeomorphic, and we'll see why in the next section.

Roughly speaking, the pursuit of *topology* is the study of subspaces of $\boldsymbol{R}^n$ and more general *topological spaces* (which we will define in the next section) *up to homeomorphism*. As topologists, we are constantly interested in the following question: *are these two spaces the same, topologically?* That is, *are these two spaces homeomorphic?*

Proving that two spaces *are* homeomorphic is, in principle, straightforward: you simply construct a homeomorphism between them.[11] But how does one prove that two spaces are *not* homeomorphic? That is, how does one *distinguish* two spaces?

[11] In practice, it often takes a certain amount of inspiration to cook up a homeomorphism.

For example, it seems intuitive that there isn't a homeomorphism between $\boldsymbol{R}$ and the subspace

$$Y = \{(x,y) : |x| = 1\}$$

from the example above. Somehow, a continuous map from $\boldsymbol{R}$ has to be 'drawn without picking up your pen', and Y has two lines that are separated from each other.

This is where *topological properties* become useful. these are properties that are ***homeomorphism invariant***, so that if X has the property, and X is homeomorphic to Y, then Y will also have the property. Let's study the first of these for subspaces of Euclidean space: *connectedness*.

1.3 Connectedness

Definition 1.3.1. Let $X \subseteq \boldsymbol{R}^n$ be a subspace. A subset $S \subseteq X$ that is both open and closed (in X) is said to be ***clopen***.[12] We shall say that X is ***connected*** if there are *exactly two* clopen subsets $S \subseteq X$.

[12] I know, this is silly.

1.3.2. If $X \subseteq \boldsymbol{R}^n$ is a nonempty subspace, then there are always *at least two* clospen subsets of X: the empty set $\varnothing$, and X itself. Thus if X is nonempty, then it is connected if and only if the only nonempty clopen subset is X itself.

Example 1.3.3. The empty set $\varnothing$, however, is not connected: it has only one subset, itself, which is clopen.

Example 1.3.4. The space $\boldsymbol{R}$ itself is connected. Let's prove this. There are at least two clopen subsets: $\varnothing$ and $\boldsymbol{R}$ itself. Now we have to prove that there are no more. So let $S \subseteq \boldsymbol{R}$ be a clopen subset. If $S \neq \varnothing$, then there exists a point $x \in S$. Since S is open, there exists $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\subseteq S$. Now let us consider the set

$$E \coloneqq \{\varepsilon > 0 :]x-\varepsilon, x+\varepsilon[\subseteq S\} \subseteq \boldsymbol{R}\,.$$

The set E is unbounded if and only if $S = \boldsymbol{R}$, so assume that E in bounded. We aim to generate a contradiction. Since E is bounded, it admits a supremum ε_0. In particular, for every $\varepsilon < \varepsilon_0$, one has $]x-\varepsilon, x+\varepsilon[\subseteq S$. Note that $]x-\varepsilon_0, x+\varepsilon_0[= \bigcup_{\varepsilon<\varepsilon_0}]x-\varepsilon, x+\varepsilon[$, so it follows that $]x-\varepsilon_0, x+\varepsilon_0[\subseteq S$ as well. Thus $\varepsilon_0 \in E$.

Now consider the point $x+\varepsilon_0$. This is close to S, because the closure of S contains the closure of $]x-\varepsilon_0, x+\varepsilon_0[$, which is $[x-\varepsilon_0, x+\varepsilon_0]$. Since S is closed, it follows that $x+\varepsilon_0 \in S$. The same analysis shows that $x-\varepsilon_0 \in S$.

Again since S is open, we may choose a $\delta > 0$ such that $]x+\varepsilon_0-\delta, x+\varepsilon_0+\delta[\subset S$ *and* $]x-\varepsilon_0-\delta, x-\varepsilon_0+\delta[\subset S$. But now we have that $]x-\varepsilon_0-\delta, x+\varepsilon_0+\delta[\subset S$, so $\varepsilon_0+\delta \in E$, which contradicts the maximality of ε_0.

This contradiction shows that E is unbounded, and so the only nonempty clopen subset of $\boldsymbol{R}$ is $\boldsymbol{R}$ itself.

Example 1.3.5. The subspace

$$Y \coloneqq \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\}$$

we introduced above is not connected. Indeed, in addition to $\varnothing$ and Y itself, the subset

$$Y_+ \coloneqq \{(x,y) \in \boldsymbol{R}^2 : x = 1\}$$

is clopen. To see this, let us show that both Y_+ and its complement

$$Y_- := Y \smallsetminus Y_+ = \{(x,y) \in \boldsymbol{R}^2 \,:\, x = -1\}$$

are open. The key observation here is that if $u \in Y_+$ and $v \in Y_-$, then $d(u,v) \geq 2$. Consequently, if $u \in Y_+$, then $B^2(u,1) \cap Y \subset Y_+$, and, similarly, if $v \in Y_-$, then $B^2(v,1) \cap Y \subset Y_-$.

Proposition 1.3.6. *Let $\{X_a\}_{a \in A}$ be a nonempty family of connected subspaces $X_a \subseteq \boldsymbol{R}^n$. Assume that for any $a, b \in A$, one has $X_a \cap X_b \neq \varnothing$. Then the union $x := \bigcup_{a \in A} X_a$ is connected as well.*

Proof. Since $X_a \cap X_b$ is nonempty, the union X is nonempty too. Hence it suffices to show that if $V \subseteq X$ is a nonempty clopen, then $V = X$.

Note that for any $a \in A$, the intersection $V \cap X_a$ is clopen in X_a. For each $a \in A$, the subspace X_a is connected, so $V \cap X_a$ is either $\varnothing$ or X_a. In other words, for each $a \in A$, either V is disjoint from X_a or else $X_a \subseteq V$. Since $V \neq \varnothing$, there is at least one $a_0 \in A$ such that $X_{a_0} \subseteq V$.

But now for any other $a \in A$, the nonempty intersection $X_a \cap X_{a_0}$ is contained in $X_a \cap V$; thus $X_a \subseteq V$ as well. We thus conclude that $X \subseteq V$. $\square$

We are now in a position to classify all the connected subspaces of $\boldsymbol{R}$.

Example 1.3.7. Let $X \subseteq \boldsymbol{R}$ be a nonempty subset. We'll say that X is *an interval* if and only if, for every $a, b \in X$ and every $x \in \boldsymbol{R}$ such that $a \leq x \leq b$, we have $x \in X$. Assume that X is an interval. If X is bounded above, then there exists a supremum $b \in \boldsymbol{R}$. If X is bounded below, then there exists an infimum $a \in \boldsymbol{R}$. Now, depending upon whether a and b exist, there are three options for an interval X:

- a interval of finite length such as $[a,b]$, $]a,b]$, $[a,b[$, or $]a,b[$;
- a ray $]a,+\infty[$, $[a,+\infty[$, $]-\infty,b[$, or $]-\infty,b]$; or
- the line $\boldsymbol{R}$ itself.

Here now is our claim: a subspace $X \subseteq \boldsymbol{R}$ is connected if and only if it is an interval – hence if and only if it is of one of the three forms above. To prove this, we must prove two things:

- first, that any interval is connected, and
- second, that any subspace that is not an interval is not connected.

To prove the first statement, assume first that X is a closed interval $[a,b]$. Assume that $V \subseteq [a,b]$ is a clopen subset. Suppose that $x \in [a,b]$ is a point such that $x \notin V$; we aim to show that V is empty.

If there are points $s \in V$ such that $s < x$, let

$$s_0 := \sup\{s \in V \,:\, s < x\}\,.$$

Thus $s_0 \leq x$. Observe that s_0 is close[13] to V. Since V is closed, it follows that $s_0 \in V$. Since $x \notin V$, it follows that $s_0 < x$. Since V is open, there is a $\varepsilon > 0$ such that $]s_0 - \varepsilon, s_0 + \varepsilon[\cap [a,b]$ is contained in V. In particular, there exist

[13] **??**

points $s \in V$ such that $s_0 < s < x$, which contradicts the definition of s_0. Consequently, there are no points $s \in V$ such that $s < x$.

On the other hand, if there are points $s \in V$ such that $s > x$, then let

$$s_1 \coloneqq \inf\{s \in V : s > x\} .$$

We can replay the same argument as above to generate a contradiction. Since V is closed, $s_1 \in V$. Since V is open, there are points $s \in V$ such that $x < s < s_1$, which is a contradiction. It therefore follows that $V = \varnothing$, as desired.

So far, we have shown that a closed interval $[a, b]$ is always connected. Now for a general interval X, note that if $x \in X$, then we may write

$$X = \bigcup_{\substack{a,b \in X, \\ a \le x \le b}} [a, b] .$$

This is a union of connected subspaces of $\boldsymbol{R}$ such that any two intersect (because they all contain x). Hence X is connected as well, thanks to the previous proposition.

Now let us prove the converse. Assume that $X \subset \boldsymbol{R}$ is *not* an interval. We aim to prove that X is not connected; it will suffice to construct a nonempty proper clopen $V \subsetneq X$. For this, choose real numbers $a < x < b$ such that $a, b \in X$ but $x \notin X$. Now consider the subset

$$V = [x, +\infty[\cap X =]x, +\infty[\cap X .$$

The first expression proves that V is closed; the second proves that V is open. Since $b \in V$ but $a \notin V$, it follows that V is nonempty and proper.

Here's our first truly *topological* theorem. We will think of this as a generalization of the intermediate value theorem.

Theorem 1.3.8. *Let $f : X \to Y$ be a continuous surjection. If X is connected, so is Y.*

Proof. Assume that X is connected. In particular, X is nonempty, hence so is Y.

Let $V \subseteq Y$ be a nonempty clopen. Then $f^{-1}(V) \subseteq X$ is clopen since f is continuous, and it is nonempty since f is a surjection. Since X is connected, it follows that $f^{-1}(V) = X$ itself. This implies that the image $f(X)$ is contained in V. Since f is a surjection, it follows that $V = Y$. Thus Y has exactly two clopen subsets: $\varnothing$ and Y. □

Example 1.3.9. Let $X \subseteq \boldsymbol{R}^n$ be a connected subspace, and let $f : X \to \boldsymbol{R}$ be a continuous map. Then the image $f(X)$ is an interval. This is our friend, the intermediate value theorem.

Consider the following assertion: there are two *antipodal*[14] points on the Earth's equator that, at ground level, have exactly the same temperature.[15] In other words, there are two coordinates of the form

$$0°\text{S}, x°\text{E} \qquad \text{and} \qquad 0°\text{N}, (180 - x)°\text{W}$$

where at ground level the temperature is exactly the same.

[14] Two points are antipodal if and only if they are in exactly opposite directions from the center of the Earth.

[15] For the purposes of this discussion, we assume that temperature is a continuous function of position.

Here's the proof. The Earth's equator is homeomorphic to the circle S^1, which is a connected subspace of $\boldsymbol{R}^2$. Temperature at the equator is thus a continuous function $T\colon S^1 \to \boldsymbol{R}$. Now define a related continuous map $g\colon S^1 \to \boldsymbol{R}$ by the formula

$$g(x) = T(x) - T(-x)\,.$$

Now since x and $-x$ are antipodal points, our claim will be proved if we can show that there is a point $x \in S^1$ such that $g(x) = 0$. If $g(1,0) = 0$, then we are done. Otherwise, our formula ensures that $g(-1,0) = g(1,0)$, so $g(1,0)$ and $g(-1,0)$ have different signs.[16] But since the image $g(S^1) \subseteq \boldsymbol{R}$ is an interval, it follows that $0 \in g(S^1)$. Hence there exists $x \in S^1$ such that $g(x) = 0$.

[16] That is, one is positive and one is negative.

Example 1.3.10. Since $\boldsymbol{R} \smallsetminus \{0\}$ and the subspace

$$Y = \{(x,y) \in \boldsymbol{R}^2 : |x| = 1\}$$

are homeomorphic, it follows that $\boldsymbol{R} \smallsetminus \{0\}$ is not connected.

Since $\boldsymbol{R}$ is connected, it follows that it is not homeomorphic to $\boldsymbol{R} \smallsetminus \{0\}$.

Example 1.3.11. A good way to confirm that a nonempty subspace $X \subseteq \boldsymbol{R}^n$ is connected is to confirm that you can 'walk' from every point to every other point without leaving X. For any two points $x, y \in X$, a *path* from x to y in X is a continuous map

$$\gamma\colon [0,1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. If X has the property that for every pair of points from x to y, there exists a path from x to y in X, then X is connected.[17] Why? Well, choose $x \in X$. For every path γ from x to another point in X, consider the image $\gamma([0,1])$; this is connected, and our assumption on X ensures that X is the union of all these images, all of which contain x. Hence X is connected.

[17] The converse is, annoyingly, false. This is the difference between *connectedness* and what is called *path connectedness*.

Example 1.3.12. The line $\boldsymbol{R}$ is not homeomorphic to S^1. Indeed, suppose it were; choose a homeomorphism $f\colon \boldsymbol{R} \to S^1$. This will restrict to a homeomorphism $\boldsymbol{R} \smallsetminus \{0\} \to S^1 \smallsetminus \{f(0)\}$. However, we claim that $S^1 \smallsetminus \{f(0)\}$ is connected. In fact, we'll do better: we shall prove that $S^1 \smallsetminus \{f(0)\}$ is homeomorphic to $\boldsymbol{R}$.

To make our formulas simpler, let's use complex numbers.[18] We have:

$$S^1 = \{z \in \boldsymbol{C} : |z| = 1\}.$$

[18] The field of complex numbers $\boldsymbol{C}$ is really nothing more than $\boldsymbol{R}^2$, equipped with its complex multiplication rule $(a,b)\cdot(c,d) = (ac - bd, ad + bc)$.

Let $w = f(0)$. We may first construct a homeomorphism $S^1 \smallsetminus \{w\} \to S^1 \smallsetminus \{1\}$ by the rule $z \mapsto z/w$. Since the inverse map is $z \mapsto wz$, which is also continuous, we are done.

With this done, we now construct a homeomorphism $]0,1[\to S^1 \smallsetminus \{1\}$ by the rule $t \mapsto \exp(2\pi i t)$. This map is a continuous bijection, but what about its inverse r? To make it easy to deduce the continuity of r from well-known bits of analysis, we write it as:[19]

$$r(x,y) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y}{1-x}\right).$$

[19] Please note, however, that this formula will not work to define a continuous map from $S^1 \to \boldsymbol{R}$.

It now follows that $S^1 \smallsetminus \{1\}$ is homeomorphic to an open interval and thus to $\boldsymbol{R}$ itself.

2 ABSTRACT TOPOLOGICAL SPACES

2.1 Topologies

There are three observations we can make about our study of subspaces of Euclidean space:

1. When we think about the topological nature of a subspace $X \subseteq \boldsymbol{R}^n$, the ambient Euclidean space in which we find them isn't all that important; homeomorphic subspaces can be found in wildly different Euclidean spaces. The only thing we *do* want to remember from the ambient $\boldsymbol{R}^n$ is the definition of the closure operator τ_X.

2. For the vast majority of the work we've done so far, we didn't need the precise definition of τ_X in terms of the metric on $\boldsymbol{R}^n$. Rather, what we really used repeatedly were the properties given in Proposition 1.1.5. Almost everything we did was just a direct consequence of those!

3. There are examples of sets with a reasonable notion of closeness that don't arrive in your lap embedded in a Euclidean space. We want to *do topology* with these examples too!

Example 2.1.1. Let's illustrate this last point with an interesting example. Let $\boldsymbol{P}^1_{\boldsymbol{R}}$ be the set of 1-dimensional linear subspaces of $\boldsymbol{R}^2$. In other words, an element of $\boldsymbol{P}^1_{\boldsymbol{R}}$ is a line in $\boldsymbol{R}^2$ through the origin. If $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$, let $\mu(L)$ denote the slope of L; that is, if L is spanned by a nonzero vector $(x, y) \in \boldsymbol{R}^2$, then

$$\mu(L) = \begin{cases} y/x & \text{if } x \neq 0; \\ \infty & \text{if } x = 0\,. \end{cases}$$

We will say that a line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ is close to a subset $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ if and only if one of the following conditions holds:

- if $\mu(L) \neq \infty$, then for every $\varepsilon > 0$, there exists a line $L' \in S$, such that

  $$|\mu(L) - \mu(L')| < \varepsilon\,,$$

  or

- if $\mu(L) = \infty$, then for every real number N, there exists a line $L' \in S$ such that

  $$|\mu(L')| > N\,.$$

Equivalently, if we think of the circle $S^1 \subset \boldsymbol{R}^2$, then every line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ intersects S^1 in exactly two points[1] $i(L)$ and $j(L)$, where $i(L) = -j(L)$. We may now say that $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ is close to $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ if and only if, for every $\varepsilon > 0$, there exists a line $L' \in S$ such that either $\|i(L) - i(L')\| < \varepsilon$ or $\|i(L) - j(L)\| < \varepsilon$.

[1] For our purposes, it won't matter which of the two intersection points we call $i(L)$ and which we call $j(L)$.

To DEAL with examples like this one, we're going to pull a trick that is often quite powerful in mathematics: we're going to turn Proposition 1.1.5 into a *definition*. Thus a ***topology*** on a set X is a systematic way of asking whether an element of X – which we will call *point* – is 'close' to a subset. In order to constitute a topology on the set X, this notion of closeness must satsify three conditions:

- For every subset $S \subseteq X$ and any point $x \in X$, if $x \in S$, then x is close to S. More briefly, an element of a subset is close to that subset.
- For every finite collection $\{S_1, \ldots, S_n\}$ of subsets of X, if a point $x \in X$ is close to the union $S_1 \cup \cdots \cup S_n$, then there exists $i \in \{1, \ldots, n\}$ such that x is close to S_i. More briefly, if a point is close to a finite union of subsets, then it is close to one of those subsets. In particular, when $n = 0$, this condition says that no point is close to the empty set $\varnothing$.
- Suppose S and T are two subsets of X such that every point $s \in S$ is close to T; then if a point x is close to S, it is also close to T.

To formalise this idea, we introduce, for each subset $S \subseteq X$, the set $\tau(S)$ of all the elements of X that are close to S, and we list its properties.

Definition 2.1.2. A ***topology*** on a set X is a map

$$\tau\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

such that the following conditions hold.

1. For any $S \in \boldsymbol{P}(X)$, one has $S \subseteq \tau(S)$. In particular, $\tau(X) = X$.
2. For any finite subset $\Sigma \subseteq \boldsymbol{P}(X)$, one has

$$\tau\left(\bigcup_{S\in\Sigma} S\right) = \bigcup_{S\in\Sigma} \tau(S)\,.$$

   In particular,[2] $\tau(\varnothing) = \varnothing$. Also, observe that τ is ***inclusion-preserving***; that is, if $S \subseteq T$, then $\tau(S) \subseteq \tau(T)$.

3. For any $S \in \boldsymbol{P}(X)$, one has $\tau(\tau(S)) = \tau(S)$.

If $x \in X$ and $S \in \boldsymbol{P}(X)$, then we say that x ***is close to*** S if and only if $x \in \tau(S)$.

A pair (X, τ) consisting of a set X and a topology τ on X is called a ***topological space***. We call the elements of a topological space ***points***.

The set $\tau(S)$ is called the ***closure*** of S. We say that S is ***closed*** if $S = \tau(S)$, and we say that S is ***open*** if its complement is closed.[3]

Notation 2.1.3. We will often write just X for the topological space, without introducing an extra symbol for the topology. When we have to name the topology, we shall often just write τ_X.

Example 2.1.4. We have already seen the first example. Any subset $X \subseteq \boldsymbol{R}^n$ inherits a topology τ_X that carries a subset $S \subseteq X$ to

$$\tau_X(S) = \{x \in X : (\forall \varepsilon > 0)(\exists s \in S)(d(x, s) < \varepsilon\}\,.$$

When $X = \boldsymbol{R}^n$, this is often called the ***standard topology***, and for $X \subseteq \boldsymbol{R}^n$, this is called the ***subspace topology***.

[2] An empty union $\bigcup_{S\in\varnothing} Y_S$ is empty.

[3] In a little while, we'll discuss how one can specify a topology by listing the open sets, which is what you will frequently see in other books.

Example 2.1.5. The closure operation on $\boldsymbol{P}^1_{\boldsymbol{R}}$ (Example 2.1.1) is a topology.

1. Let $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$. If $L \in S$, then the conditions are automatic for $L \in \tau(S)$.

2. Let $S_1, \dots, S_n \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$ be a collection of subsets of $\boldsymbol{P}^1_{\boldsymbol{R}}$, and let $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$. If L is close to $S_1 \cup \cdots \cup S_n$, then a Pigeonhole argument like the one given in Proposition 1.1.5 ensures that L is close to $S_1 \cup \cdots \cup S_n$.[4] If L is close to some S_i, then it is certainly close to $S_1 \cup \cdots \cup S_n$.

3. Let $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$, and let $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ be a line that is close to $\tau(S)$. We aim to show that L is close to S; so let $\varepsilon > 0$. There exists a line $L' \in \tau(S)$ such that either $\|i(L) - i(L')\| < \varepsilon$ or $\|i(L) - j(L')\| < \varepsilon/2$. Accordingly, there exists a line $L'' \in S$ such that either $\|i(L') - i(L'')\| < \varepsilon/2$ or $\|i(L') - j(L'')\| < \varepsilon/2$. Analyzing the four options and applying the triangle inequality, we find that either $\|i(L) - i(L'')\| < \varepsilon$ or $\|i(L) - j(L'')\| < \varepsilon$.

[4] Fill in the details here! Often, in writing, mathematicians say something like, 'by the same argument …,' or 'by the standard argument …,' or even, 'it is obvious that….' This can be a little annoying (and I don't recommend calling very many things 'obvious'), but it is a good opportunity for you to take a moment to sort through the details yourself.

Example 2.1.6. The empty set $\varnothing$ admits a unique topology. The powerset $\boldsymbol{P}(\varnothing)$ is a singleton $\{\varnothing\}$, and the identity is the only map $\{\varnothing\} \to \{\varnothing\}$. This map is a topology. The only subset available – $\varnothing$ itself – is both open and closed.

Example 2.1.7. More generally, on any set X, the identity map

$$\delta\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

is always a topology. This is called the ***discrete topology*** on a set X.

In the discrete topology, a point $x \in X$ is close to a subset $S \subseteq X$ if and only if $x \in S$. Every subset of X is both open and closed in the discrete topology.

Example 2.1.8. Let us study topologies on the singleton $\{x\}$. The power set $\boldsymbol{P}(\{x\})$ is a two-element set $\{\varnothing, \{x\}\}$. There are four maps from this set to itself, but most of these are not topologies. In order for a map τ to be a topology, the first condition demands that $\tau(\{x\}) = \{x\}$, and the second condition demands that $\tau(\varnothing) = \varnothing$. Thus the only topology on the singleton $\{x\}$ is the discrete topology.

Example 2.1.9. Let $X \coloneqq \{x, y\}$. In order to specify a topology τ on X, it is really only necessary to specify the subsets $\tau(\{x\})$ and $\tau(\{y\})$. The first condition states that these subsets must contain the sets $\{x\}$ and $\{y\}$, respectively, and the second and third conditions become automatic. That gives us four topologies on $\{x, y\}$:

- the discrete topology, in which $\tau(\{x\}) = \{x\}$ and $\tau(\{y\}) = \{y\}$;
- the topology in which $\tau(\{x\}) = \{x\}$, and $\tau(\{y\}) = X$;
- the topology in which $\tau(\{y\}) = X$, and $\tau(\{x\}) = \{x\}$; and
- the topology in which $\tau(\{x\}) = \tau(\{y\}) = X$.

Let us describe the relation of closeness for each of these topologies. The only questions that aren't answered by the axioms of a topology is: 'is x close to $\{y\}$?' and 'is y close to $\{x\}$?' Let's answer these in each case:

- in the discrete topology, x is not close to $\{y\}$, and y is not close to $\{x\}$;
- in the second of these topologies, x is close to $\{y\}$, but y is not close to $\{x\}$;
- in the third topology, y is close to $\{x\}$, but x is not close to $\{y\}$; and
- finally, in the last of these topologies, x is close to $\{y\}$ and y is close to $\{x\}$.

The asymmetry in the second and third topologies is strange from our natural idea of what 'close' ought to mean. Nevertheless, these topologies are actually interesting examples; even rather bizarre ideas of closeness can be captured in topology!

Finally, for each of these examples, let us determine which sets are open and closed. In each case, the only questions that aren't answered by the axioms are: 'is $\{x\}$ open? closed?' and 'is $\{y\}$ open? closed?'

- in the discrete topology, both $\{x\}$ and $\{y\}$ are both open and closed;
- in the second topology, $\{x\}$ is closed and not open, and $\{y\}$ is open and not closed.
- in the third topology, the situation is reversed: $\{x\}$ is open and not closed, and $\{y\}$ is closed and not open;
- in the last of these topologies, neither $\{x\}$ nor $\{y\}$ is open or closed.

The discrete topology makes sense for any set. Similarly, the last of the topologies in Example 2.1.9 above is sensible for any set.

Example 2.1.10. For any set X, we define a map

$$\iota\colon \boldsymbol{P}(X) \to \boldsymbol{P}(X)$$

by the formula

$$\iota(S) \coloneqq \begin{cases} \varnothing & \text{if } S = \varnothing\,; \\ X & \text{if } S \neq \varnothing\,. \end{cases}$$

This is a topology called the ***indiscrete topology***. Thus in the indiscrete topology, any point is close to any nonempty subset. The only subsets of X that are open or closed in the indiscrete topology are $\varnothing$ and X itself.

The other two topologies of Example 2.1.9 are more interesting:

Example 2.1.11. Let P be a *preorder*. That is, P is a set equipped with a relation $\leq$ such that:

- for every $a \in P$, one has $a \leq a$; and
- for every $a, b, c \in P$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Then we can define a topology as follows. For any subset $S \subseteq P$, we define

$$\tau_{\leq}(S) \coloneqq \{a \in P : (\exists s \in S)(a \leq s)\}\,.$$

In other words, $a \in P$ is close to S if and only if some element of S exceeds it.

Let us quickly confirm that this is a topology:

1. Let $S \subseteq P$ be a subset. Since $s \leq s$ for every $s \in S$, it follows that $S \subseteq \tau_{\leq}(S)$.

2. Let $S_1, \ldots, S_n \subseteq P$ be a finite collection of subsets of P. Then

$$\begin{aligned} \tau_{\leq}(S_1 \cup \cdots \cup S_n) &= \{a \in P : (\exists s \in S_1 \cup \cdots \cup S_n)(a \leq s)\} \\ &= \{a \in P : (\exists i)(\exists s \in S_i)(a \leq s)\} \\ &= \tau_{\leq}(S)_1 \cup \cdots \cup \tau_{\leq}(S)_n . \end{aligned}$$

3. Let $S \subseteq P$ be a subset. Then

$$\begin{aligned} \tau_{\leq}(\tau_{\leq}(S)) &= \{a \in P : (\exists t \in \overline{S})(a \leq t)\} \\ &= \{a \in P : (\exists s \in S)(\exists t \in X)(a \leq t \leq s)\} \\ &= \{a \in P : (\exists s \in S)(a \leq s)\} \\ &= \tau_{\leq}(S) . \end{aligned}$$

This topology is called the *Alexandroff topology*.

The *closed subsets* are the subsets $Z \subseteq P$ with the property that if $a \in P$, $z \in Z$, and $a \leq z$, then $a \in Z$ as well. The *open subsets* are the subsets $U \subseteq P$ with the property that if $a \in P$, $u \in U$, and $a \geq u$, then $a \in U$ as well.

Example 2.1.12. Form the set $\boldsymbol{R} \sqcup \{\infty\}$, where ∞ is just some symbol such that $\infty \notin \boldsymbol{R}$. We already have a topology on $\boldsymbol{R}$; let's doctor it a mite to make a topology on $\boldsymbol{R} \sqcup \{\infty\}$. To do this, we have to understand how our notion of closeness has changed. So we have a point $x \in \boldsymbol{R} \sqcup \{\infty\}$ and a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$, and we want to see if x is close to S. There are two cases:

- If $x \in \boldsymbol{R}$, then we declare that x is close to S if and only if x is close to $S \cap \boldsymbol{R}$ (for the standard topology on $\boldsymbol{R}$).

- We declare that ∞ is close to S if and only if, for every $N \in \boldsymbol{R}_{\geq 0}$, there is a point of S that does not lie[5] in $[-N, N]$.

[5] Note that this includes two possibilities: the point of S may *be* ∞, or else the point may be a real element $s \in S$ such that $|s| > N$.

Let's see that this is a topology. In effect, you check everything by checking the two cases ($x \in \boldsymbol{R}$ or $x \notin \boldsymbol{R}$) separately. In the first case, you use the fact that $\boldsymbol{R}$ has a topology already, and in the second, you give a little argument.

- By definition if x lies in a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$, then x is close to S.

- Let $\{S_1, \ldots, S_n\}$ be a finite collection of subsets of $\boldsymbol{R} \sqcup \{\infty\}$. If $x \in \boldsymbol{R}$, then since one has

$$(S_1 \cup \cdots \cup S_n) \cap \boldsymbol{R} = (S_1 \cap \boldsymbol{R}) \cup \cdots \cup (S_n \cap \boldsymbol{R}),$$

it follows that x is close to $S_1 \cup \cdots \cup S_n$ if and only if x is close to $(S_1 \cap \boldsymbol{R}) \cup \cdots \cup (S_n \cap \boldsymbol{R})$ (for the standard topology on $\boldsymbol{R}$), if and only if x is close to $S_i \cap \boldsymbol{R}$ for some i (for the topology on $\boldsymbol{R}$), if and only if x is close to S_i. Now ∞ is close to $S_1 \cup \cdots \cup S_n$ if and only if, for any $N \in \boldsymbol{R}$, there is a point of $S_1 \cup \cdots \cup S_n$ that does not lie in $[-N, N]$. This happens if and only if there is an i such that for any $N \in \boldsymbol{R}$, there is[6] a point of S_i that does not lie in $[-N, N]$.

[6] This is the *pigeonhole principle*. In effect, if X is an infinite set, Y is a finite set, and $f : X \to Y$ is a map, there is an element $y \in Y$ such that $f^{-1}\{y\}$ is infinite.

- Finally, suppose that every point of a subset $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$ is close to a subset $T \subseteq \boldsymbol{R} \sqcup \{\infty\}$. Now if x is close to S, there are two options. First, if

$x \in \boldsymbol{R}$, then x is close to $S \cap \boldsymbol{R}$ and hence to $T \cap \boldsymbol{R}$. Otherwise, if $x = \infty$, then if $\infty \in T$ or $\infty \in S$, then we're done. If not, then for every $N \in \boldsymbol{R}$, there is a point $s_N \in S \subseteq \boldsymbol{R}$ such that $|s_N| > N$; this point s_N is close to T, so if we set $\epsilon_N \coloneqq |s_N| - N$, then there exists a point $t_N \in T$ such that $|s_N - t_N| < \epsilon_N$. Thus $|t_N| > N$, and so ∞ is close to T.

This isn't the only way to enlarge $\boldsymbol{R}$.

Example 2.1.13. Define a topology on $\boldsymbol{R} \sqcup \{-\infty, +\infty\}$ in which, for any point $x \in \boldsymbol{R} \sqcup \{-\infty, +\infty\}$ and any subset $S \subseteq \boldsymbol{R} \sqcup \{-\infty, +\infty\}$,

- If $x \in \boldsymbol{R}$, then x is close to S if and only if x is close to $S \cap \boldsymbol{R}$.
- We declare that $-\infty$ is close to S if and only if either $-\infty \in S$ or for any $N \in \boldsymbol{R}_{\geq 0}$, there is a point $s \in S \cap \boldsymbol{R}$ such that $s < -N$.
- We declare that $+\infty$ is close to S if and only if either $+\infty \in S$ or for any $N \in \boldsymbol{R}_{\geq 0}$, there is a point $s \in S \cap \boldsymbol{R}$ such that $s > N$.

This is a topology.[7]

[7] You should convince yourself of this!

There is another, more standard, way to specify topologies. It's a little less intuitive, but it becomes technically convenient when we want to do things like *generate* topologies.

Definition 2.1.14. Fix a set X. Let's define two different kinds of information on X.

- A *system of open sets* on X is a subset $\mathscr{O} \subseteq \boldsymbol{P}(X)$ that is stable under unions[8] and finite intersections[9]. That is, if $\Sigma \subseteq \mathscr{O}$, then the union
$$\bigcup_{U \in \Sigma} U$$
also lies in $\mathscr{O}$, and if Σ is finite, then the intersection
$$\bigcap_{U \in \Sigma} U$$
also lies in $\mathscr{O}$.
- Dually, a *system of closed sets* on X is a subset $\mathscr{C} \subseteq \boldsymbol{P}(X)$ that is stable under intersections and finite unions. That is, if $\Sigma \subseteq \mathscr{C}$, then the intersection
$$\bigcap_{Z \in \Sigma} Z$$
also lies in $\mathscr{C}$, and if Σ is finite, then the union
$$\bigcup_{Z \in \Sigma} Z$$
also lies in $\mathscr{C}$.

[8] possibly empty
[9] possibly empty

Specifying a system of open sets is the same as specifying a system of closed sets. More precisely, the formation of the complement in X defines a bijection between the set of systems of open sets and the set of systems of closed sets.

2.1.15. Specifying a topology is the same as specifying a system of closed sets.

For any a topology τ on X, one may define $\mathscr{C}_\tau$ as the set of closed subsets of the topology:

$$\mathscr{C}_\tau \coloneqq \{Z \in \boldsymbol{P}(X) : \tau(Z) = Z\} .$$

To see that $\mathscr{C}_\tau$ is stable under intersection and finite union, let $\Sigma \subseteq \mathscr{C}_\tau$ be a subset. Let W be the intersection $\bigcap_{Z\in\Sigma} Z$. For every $Z \in \Sigma$, since τ is inclusion-preserving, it follows that $\tau(W) \subseteq \tau(Z) = Z$. Consequently, $\tau(W)$ is contained in the intersection W, so W is closed. If Σ is finite, then since τ preserves finite unions, we have

$$\tau\left(\bigcup_{Z\in\Sigma} Z\right) = \bigcup_{Z\in\Sigma} \tau(Z) = \bigcup_{Z\in\Sigma} Z .$$

This proves that $\mathscr{C}_\tau$ is a system of closed sets.

In the opposite direction, for any system of closed subsets $\mathscr{C}$ on X, define a map $\tau_{\mathscr{C}} : \boldsymbol{P}(X) \to \boldsymbol{P}(X)$ that carries a subset $S \subseteq X$ to the smallest element of $\mathscr{C}$ that contains S: [10]

$$\tau_{\mathscr{C}}(S) \coloneqq \bigcap_{\substack{Z\in\mathscr{C},\\ S\subseteq Z}} Z .$$

[10] Since $\mathscr{C}$ is stable under intersections, this formula indeed defines an element of $\mathscr{C}$. So $\tau_{\mathscr{C}}(S)$ is the smallest element of $\mathscr{C}$ that contains S.

Let us see that $\tau_{\mathscr{C}}$ is a topology.

1. By definition, for every subset $S \subseteq X$, we have $S \subseteq \tau(S)$.

2. By definition, $\tau_{\mathscr{C}}$ is an inclusion-preserving operation. Thus if $S_1, \dots, S_n \subseteq X$ is a finite collection of subsets of X, then it follows that

$$\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n) \supseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) .$$

On the other hand, since $\mathscr{C}$ is stable under finite unions, it follows that $\tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) \in \mathscr{C}$. Since in addition, $S_1 \cup \cdots \cup S_n \subseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n)$, it follows[11] that

$$\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n) \subseteq \tau_{\mathscr{C}}(S_1) \cup \cdots \cup \tau_{\mathscr{C}}(S_n) ,$$

whence $\tau_{\mathscr{C}}$ preserves finite unions.

[11] The point here is that $\tau_{\mathscr{C}}(S_1 \cup \cdots \cup S_n)$ is the *smallest* closed subset that contains $S_1 \cup \cdots \cup S_n$.

3. Finally, $\tau_{\mathscr{C}}(\tau_{\mathscr{C}}(S)$ is the smallest element of $\mathscr{C}$ that contains $\tau_{\mathscr{C}}(S)$. But since $\tau_{\mathscr{C}}(S)$ itself likes in $\mathscr{C}$, it follows that

$$\tau_{\mathscr{C}}(\tau_{\mathscr{C}}(S)) \subseteq \tau_{\mathscr{C}}(S) .$$

Thus one may specify a topology on X by specifying the closed sets and checking stability under intersections and finite unions, or specifying the open sets and checking stability under unions and finite intersections.[12]

[12] Most textbooks define topologies as a system of open sets. There's nothing wrong with this definition, but we've delayed this description, because it seems difficult to motivate fully.

Example 2.1.16. Here's an extremely important example in modern mathematics. This one will be following us around throughout this text. Let $X \subseteq \boldsymbol{R}^m$ be any subspace, and let $n \in \boldsymbol{N}$. We are about to define a topological space $C_n(X)$ called the *configuration space of n points in X*. As a set, $C_n(X)$ is the set of subsets $\{x_1, \dots, x_n\} \subseteq X$ of n distinct points:

$$C_n(X) \coloneqq \{S \subseteq \boldsymbol{P}(X) : \#S = n\} .$$

To endow it with a topology, we need to define an auxiliary topological space.

Recall that we defined the subspace

$$X^n \coloneqq X \times \cdots \times X = \{x = (x_1, \dots, x_n) \in \boldsymbol{R}^{mn} : (\forall i)(x_i \in X)\} \subseteq \boldsymbol{R}^{mn} .$$

We now pass to a further subspace:

$$E_n(X) \coloneqq \{(x_1, \dots, x_n) \in X^n : (\forall i, j)((i \neq j) \Rightarrow (x_i \neq x_j))\} \subseteq X^m .$$

Now there is a map $q \colon E_m(X) \to C_m(X)$ defined by

$$q(x_1, \dots, x_n) \coloneqq \{x_1, \dots, x_n\} .$$

We now declare that $U \subseteq C_n(X)$ is open if and only if $q^{-1}(U) \subseteq E_n(X)$ is open. This defines a system of open sets (and hence a topology), since the formation of the inverse image preserves unions and intersections.

Example 2.1.17. Any set X can be given the ***cofinite*** topology, in which a subset $Z \subseteq X$ is declared to be closed if and only if either Z is finite or $Z = X$. Thus in the cofinite topology,

$$\tau(S) = \begin{cases} S & \text{if } S \text{ is finite;} \\ X & \text{if } S \text{ is infinite.} \end{cases}$$

Definition 2.1.18. Let X be a topological space, and let $x \in X$. Then an ***open neighborhood*** of x is an open subset $U \subseteq X$ such that $x \in U$. A ***neighborhood*** of x is a subset of X that contains an open neighborhood of x.

2.1.19. If X is a topological space, then $\tau(S)$ is the smallest closed subset that contains S. Equivalently, a point $y \in X$ lies in the closure $\tau(S)$ if and only if, for any open neighborhood U of y, the intersection $U \cap \tau(S)$ is nonempty.

Why are these equivalent? Well, if $y \notin \tau(S)$, then $X \smallsetminus \tau(S)$ is an open neighborhood of y that does not intersect S. Conversely, if U is an open neighborhood of x that does not intersect $\tau(S)$, then $X \smallsetminus U$ is a closed subset that contains S and therefore $\tau(S)$.

2.2 Continuity

The good news now is that continuity works almost exactly as it did in the example of subspaces of Euclidean space (Section 1.1). The only difference is that we no langer have access to the ε-δ characterization of continuity.

Definition 2.2.1. Let X and Y be topological spaces, and let $f \colon X \to Y$ be a map. Then f is ***continuous*** if and only if, for every subset $S \subseteq X$ any every point $x \in X$ that is close to S, the point $f(x)$ is close to $f(S)$.

Proposition 2.2.2. *Let X and Y be topological spaces. The following are equivalent for a map $f \colon X \to Y$.*

1. *The map f is continuous.*
2. *For any subset $T \subseteq Y$, one has $\tau_X(f^{-1}(T)) \subseteq f^{-1}(\tau_Y(T))$.*

3. *For any closed subset $Z \subseteq Y$, the inverse image $f^{-1}(Z) \subseteq X$ is closed.*

4. *For any open subset $U \subseteq Y$, the inverse image $f^{-1}(U) \subseteq X$ is open.*

Proof. Everything is exactly as in the proof of Proposition 1.2.2, except that since we do not have condition (5), we shall have to prove directly that (4) implies (1).

So assume (4); we aim to prove (1). Let $S \subseteq X$, and let $x \in \tau_X(S)$. Observe that the complement $V \coloneqq Y \smallsetminus \tau_Y(f(S))$ is open; hence so is the inverse image $U \coloneqq f^{-1}(V) \subseteq X$. Note also that S is disjoint from U. Thus the complement $X \smallsetminus U$ is a closed subset that contains S. Consequently, $\tau_X(S) \subseteq U$. So since $x \in \tau_X(S)$, it follows that $f(x) \notin V$. In other words, $f(x) \in \tau_Y(f(S))$, as desired. □

Example 2.2.3. Let X, Y, and Z be topological spaces. Then the identity map $\mathrm{id}\colon X \to X$ is continuous. Also, if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then the composition $g \circ f\colon X \to Z$ is continuous as well.

Example 2.2.4. Let X be a topological space, and let S be a set. Any map $f\colon S \to X$ is continuous if S is endowed with the *discrete* topology. Dually, any map $g\colon X \to S$ is continuous if S is endowed with the emphchaotic topology.

Definition 2.2.5. Suppose τ_1 and τ_2 are two topologies on the same set X. Then we say that τ_1 is ***finer*** than τ_2 – and that τ_2 is ***coarser*** than τ_1 – if the identity map on X is continuous as a map

$$(X, \tau_1)(X, \tau_2)\,.$$

Hence if τ_1 is finer than τ_2, then the closure of a set relative to τ_1 is contained in its closure relative to τ_2. In particular, the topology τ_1 has more closed sets than the topology τ_2. Forming complements, we deduce also that the topology τ_1 has more open sets than the topology τ_2.

There are two extremes: the finest topology on a set is the discrete topology, and the coarsest is the chaotic topology.

Example 2.2.6. Recall the topological space $\boldsymbol{P}^1_{\boldsymbol{R}}$ of Example 2.1.1. Define a map $L\colon S^1 \to \boldsymbol{P}^1_{\boldsymbol{R}}$ as follows: for every point $x \in S^1$, let $L(x)$ be the 1-dimensional subspace spanned by the vector x. That is, $L(x)$ is the unique line in $\boldsymbol{R}^2$ that passes through the origin and x. It follows from the definition of the topology on $\boldsymbol{P}^1_{\boldsymbol{R}}$ that L is continuous.

To unpack this a little, let $S \subseteq S^1$ be a subset, and let $x \in \tau(S)$. We want to see that $L(x)$ is close to the image $L(S)$. Let $\varepsilon > 0$. There exists $s \in S$ such that $\|x - s\| < \varepsilon$. Now $L(x)$ intersects S^1 at the points x and $-x$, and $L(s)$ intersects S^1 in the points s and $-s$. Thus the line $L(s)$ has the property that, in the notation of Example 2.1.1, $\|i(L(x)) - i(L(s))\| < \varepsilon$ or $\|i(L(x)) - j(L(s))\| < \varepsilon$.

Please note that while L is surjective, it is not injective, since for every $x \in S^1$, one has $L(x) = L(-x)$.

Definition 2.2.7. Let X and Y be topological spaces, and let $x \in X$. Then a map $f\colon X \to Y$ is *continuous at x* if and only if, for any subset $S \subseteq X$, if x is close to S then $f(x)$ is close to $f(S)$.

A map $f\colon X \to Y$ is continuous if and only if it is continuous at every point $x \in X$.

Example 2.2.8. Consider the map $s\colon \boldsymbol{R} \to \boldsymbol{R}$ given by the formula

$$s(x) \coloneqq \begin{cases} x/|x| & \text{if } x \neq 0\,; \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then s is continuous at every $x \in \boldsymbol{R} \smallsetminus \{0\}$. If s were continuous at 0, then it would be continuous. But even though the set $\{1\} \subset \boldsymbol{R}$ is closed, its inverse image

$$s^{-1}\{1\} = \,]0, +\infty[\, \subset \boldsymbol{R}$$

is not. Hence s is not continuous at 0, as we expect!

Example 2.2.9. Consider the function $f(x) = 1/x$. This is a continuous map

$$f\colon \boldsymbol{R} - \{0\} \to \boldsymbol{R} - \{0\}\,,$$

relative to the subspace topology on each side. Of course we have removed the point $0 \in \boldsymbol{R}$, because in primary school we were told that $1/0$ is 'undefined.' But let's try to define it anyhow.

We note that, as x approaches 0 from the right, $1/x$ increases without bound; as x approaches 0 from the left, $1/x$ decreases without bound. If we wanted to add a point that would play the role of $1/0$, then this leads us to the following idea: consider the topological space $\boldsymbol{R} \sqcup \{\infty\}$ constructed in **??**. Now we may extend our map $f\colon \boldsymbol{R} - \{0\} \to \boldsymbol{R} - \{0\}$ to a map $F\colon \boldsymbol{R} \sqcup \{\infty\} \to \boldsymbol{R} \sqcup \{\infty\}$ by

$$F(x) \coloneqq \begin{cases} 1/x & \text{if } x \in \boldsymbol{R} \smallsetminus \{0\}\,; \\ \infty & \text{if } x = 0\,; \\ 0 & \text{if } x = \infty\,. \end{cases}$$

With the topology we've given $\boldsymbol{R} \sqcup \{\infty\}$, this is continuous! The only thing left for us to check is continuity at 0 and ∞.

To do this, let $S \subseteq \boldsymbol{R} \sqcup \{\infty\}$ be a subset such that 0 is close to S. Let $N > 0$. There exists an element $s \in S$ such that $|s| < 1/N$. Thus $|F(s)| > N$. It follows that F is continuous at 0.

The proof of contiuity at ∞ is similar.[13]

[13] Please fill in the details!

Proposition 2.2.10. *Let X and Y be topological spaces, and let $x \in X$. The following are equivalent for a map $f\colon X \to Y$.*

1. *The map f is continuous at x.*
2. *For any open neighborhood V of $f(x)$, the inverse image $f^{-1}(V)$ is an open neighborhood of x.*

Proof. Assume (1); we aim to prove (2). Let V be an open neighborhood of $f(x)$. Let $S \coloneqq f^{-1}(Y \smallsetminus V)$. If $x \in \tau_X(S)$, then

$$f(x) \in \tau_Y(f(S)) \subseteq \tau(Y \smallsetminus V) = Y \smallsetminus V\,,$$

since $Y \smallsetminus V$ is closed. This is a contradiction, which shows that $x \notin \tau_X(S)$. Now let $U \coloneqq X \smallsetminus \tau_X(S)$. This is now an open neighborhood of x, and $f(U) \subseteq V$.

Conversely, assume (2); we aim to prove (1). Let $S \subseteq X$ be a subset, and let $x \in \tau_X(S)$. Let $V := Y \smallsetminus \tau_Y(f(S))$. If $f(x) \in V$, then by assumption there exists an open neighborhood U of x such that $f(U) \subseteq V$. Since S is disjoint from $f^{-1}(V)$, it follows that the closure $\tau_X(S)$ is disjoint from U as well. This is a contradiction that implies that $f(x) \notin V$, so that $f(x) \in \tau(f(S))$. $\square$

THE DEFINITION of *homeomorphism* is also just the same as for subspaces of Euclidean space:

Definition 2.2.11. Let X and Y be topological spaces. A ***homeomorphism*** $f: X \to Y$ is a continuous bijection whose inverse f^{-1} is continuous. The two topological spaces X and Y are ***homeomorphic*** if and only if there exists a homeomorphism $X \to Y$. In this case, we may write $X \cong Y$.

Example 2.2.12. Let's show that the following three topological spaces are homeomorphic:

- The circle $S^1 \subseteq \boldsymbol{R}^2$.
- The topological space $\boldsymbol{P}^1_{\boldsymbol{R}}$ from Example 2.1.1.
- The topological spaces $\boldsymbol{R} \sqcup \{\infty\}$ from Example 2.1.12.

The map $L: S^1 \to \boldsymbol{P}^1_{\boldsymbol{R}}$ defined above is not a homeomorphism, but that doesn't mean we can't find another homeomorphism between these two topological spaces. So let's try to go the other way. Let's define a map $h: \boldsymbol{P}^1_{\boldsymbol{R}} \to S^1$. For every line $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$, there is a point $z \in S^1$ such that $L = L(z)$. If we think of $S^1 \subset \boldsymbol{C}$, we can define $h(L)$ as z^2 for any $z \in S^1$ such that $L = L(z)$. In other words, if L is the line spanned by a nonzero vector $(x, y) \in \boldsymbol{R}^2$, then

$$h(L) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right) .$$

This is well-defined, since this formula gives the same value if you replace (x, y) with $(\alpha x, \alpha y)$ for $\alpha \in \boldsymbol{R} \smallsetminus \{0\}$.

Let us prove that h is continuous. We may be tempted to use the fact that the formula above defines a continuous function $\boldsymbol{R}^2 \smallsetminus \{0\} \to S^1$, but we have to be a little careful, because $\boldsymbol{P}^1_{\boldsymbol{R}}$ wasn't defined as a subspace of a Euclidean space. Assume that $S \subseteq \boldsymbol{P}^1_{\boldsymbol{R}}$, and assume that $L \in \tau(S)$. Let $\varepsilon > 0$; there exists a line $L' \in S$ such that either $\|i(L) - i(L')\| < \varepsilon/2$ or $\|i(L) - j(L')\| < \varepsilon/2$. Furthermore, we know that no two points in S^1 are separated by a distance of more than 2; therefore, both $\|i(L) - i(L')\| \le 2$ or $\|i(L) - j(L')\| \le 2$. We may therefore multiply these inequalities to obtain:

$$\|i(L) - i(L')\| \|i(L) - j(L')\| < \varepsilon .$$

Since $i(L') = -j(L')$, we therefore obtain

$$\|h(L) - h(L')\| = \|i(L)^2 - j(L')^2\| < \varepsilon .$$

It therefore follows that $h(L)$ is close to $h(S)$. So h is continuous.

Now let us show that h is a bijection. If $w \in S^1$, then the inverse image $h^{-1}\{w\}$ is the set of lines $L \in \boldsymbol{P}^1_{\boldsymbol{R}}$ such that either $i(L)$ or $j(L)$ is a square root of w. In fact, since $i(L) = -j(L)$, it follows that $i(L)$ is a square root of w if and only if $j(L)$ is a square root of w. Hence $h^{-1}\{w\}$ consists of exactly one line: the line passing through either square root of w and the origin.

Now let's demonstrate that h is a homeomorphism. For this, we note that the inverse h^{-1} carries $w \in S^1$ to the line passing through the square roots of w and the origin. To prove that this is continuous, assume that $T \subseteq S^1$, and assume that $w \in S^1$ is close to T. Let $\varepsilon > 0$; then there exists $w' \in T$ such that $\|w - w'\| < \varepsilon^2$. Now if $z, -z \in S^1$ are the two square roots of w, and if $z', -z' \in S^1$ are the two square roots of w', then either $|z - z'| < \varepsilon$ or $|z + z'| < \varepsilon$, since otherwise, we would have

$$|w - w'| = |z - z'||z + z'| \geq \varepsilon^2 .$$

Consequently, it follows that $h^{-1}(w)$ is close to $h^{-1}(T)$. This now completes the proof that $\boldsymbol{P}^1_{\boldsymbol{R}}$ and S^1 are homeomorphic.

Now we prove that $\boldsymbol{R} \sqcup \{\infty\}$ and S^1 are homeomorphic. Indeed, define maps

$$f : S^1 \to \boldsymbol{R} \sqcup \{\infty\} \qquad \text{and} \qquad g : \boldsymbol{R} \sqcup \{\infty\} \to S^1$$

by the formulas

$$f(x, y) \coloneqq \begin{cases} x/(1-y) & \text{if } y \neq 1 ; \\ \infty & \text{if } y = 1 . \end{cases}$$

and

$$g(t) \coloneqq \begin{cases} 1/(t^2+1)(2t, t^2-1) & \text{if } t \neq \infty ; \\ (0, 1) & \text{if } t = \infty . \end{cases}$$

A direct check confirms that f and g are inverses, but the interesting thing to prove is that they are each continuous.

For f, continuity away from the point $(0, 1)$ follows from the elementary analytic facts we are happy to assume here. But continuity at $(0, 1)$ is more interesting! So assume that $S \subseteq S^1$ is a subset to which $(0, 1)$ is close. We have to show that ∞ is close to $f(S)$. So let $N > 0$ be a real number; we aim to find an element $(x, y) \in S$ such that $|x/(1-y)| > N$ or, equivalently, that $x^2/(1-y)^2 = (1+y)/(1-y) > N^2$, Choose $\varepsilon > 0$ so that $\varepsilon \leq 2/N^2$; since $(0, 1)$ is close to S, there exists a point $(x, y) \in S$ such that $x^2 - (1-y)^2 < \varepsilon$ and $y \geq 0$. Since $x^2 + y^2 = 1$, this implies that $1 - y < \varepsilon/2$, and since $1 + y \geq 1$, we obtain

$$\frac{1+y}{1-y} \geq \frac{1}{1-y} > \frac{2}{\varepsilon} > N^2 .$$

Once again, elementary analysis facts imply that g is continuous at every point of $\boldsymbol{R}$. We must show that g is continuous at ∞ as well. For this, assume that $T \subseteq \boldsymbol{R} \sqcup \{\infty\}$ is a subset such that ∞ is close to T. Let $\varepsilon > 0$; we aim to show that there exists an element $t \in T$ such that $\|g(t) - (0, 1)\| < \varepsilon$. If $\infty \in T$, then this is immediate, so it suffices to consider the case in which $T \subseteq \boldsymbol{R}$ is an unbounded subset. There exists $t \in T$ such that $|t| > 2/\varepsilon$, and so

$$\left\|1/(t^2+1)(2t, t^2-1) - (0, 1)\right\|^2 = \frac{4}{t^2+1} < \varepsilon^2 ,$$

so that $\|g(t) - (0, 1)\| < \varepsilon$, just as we'd hoped.

The upshot here is that we have

$$\boldsymbol{P}^1_{\boldsymbol{R}} \cong S^1 \cong \boldsymbol{R} \sqcup \{\infty\} .$$

2.3 Connectedness

2.4 Specifying topologies efficiently

A common way of specifying a topology is to speak of the *coarsest topology such that blah* or the *finest topology such that blah*. Specifying a topology this way has its pluses and minuses: on one hand, these properties often make it easy to verify features of these topologies, but on the other hand, one has to do the work to actually confirm that a finest or coarsest topology with the given property exists. Let's do some of that work now.

Example 2.4.1. Let X be a topological space, and let $Y \subseteq X$ is a subspace, then Y has the coarsest topology such that the inclusion map $Y \rightarrowtail X$ is continuous.

2.4.2. Here's a handy fact of set theory that we'll be using a lot in this section. Let X be a set, and let $A \subseteq \boldsymbol{P}(X)$ be a collection of subsets of X. Then the following are equivalent for a subset $U \subseteq X$:

- The set U can be expressed as the union of elements of A.
- For every point $u \in U$, there exists an element $V \in A$ such that $u \in V \subseteq U$.

Proposition 2.4.3. *Suppose X a set, and suppose $B \subseteq \boldsymbol{P}(X)$. Then there is a unique coarsest topology τ_B on X such that every element of B is open.*

Proof. Let's define $O \subseteq \boldsymbol{P}(X)$ as the collection of all those subsets $U \subseteq X$ such that for any point $x \in U$, there exist finitely many elements $V_1, \dots, V_n \in B$ such that $x \in V_1 \cap \cdots \cap V_n \subseteq U$. It is easy to see that O is a system of open sets for a topology on X. We claim that this is the desired topology τ_B. Indeed, every element of B is an element of O, and at the same time, every element of O is a union of finite intersections of elements of B, and hence must be open in any topology in which the elements of B are open. □

Definition 2.4.4. In the situation of the previous exercise, one says that B ***generates*** the topology on X or that B is a ***subbase***[14] for the topology on X.

[14] Some authors require additionally that *B cover X*; that is, that $\cup B = X$. Not us, though.

Proposition 2.4.5. *Let X be a set, and let $F \coloneqq \{f_\alpha \colon X \to Y_\alpha\}_{\alpha \in A}$ be a family of maps. Suppose that each target Y_α is equipped with a topology τ_α. Then there exists a unique coarsest topology on X such that each f_α is continuous. This is called the* **initial topology** *on X with respect to F.*

Proof. The desired topology on X is the one generated by the set

$$\{f_\alpha^{-1}(U) : (\alpha \in A) \wedge (U \subseteq Y_\alpha \text{ is open})\}. \qquad \square$$

Proposition 2.4.6. *Let X be a set, and let $F \coloneqq \{f_\alpha \colon X \to Y_\alpha\}_{\alpha \in A}$ be an indexed family of maps. Suppose that each target Y_α is equipped with a topology τ_α, and equip X with the initial topology with respect to F. For any topological space X', a map $g \colon X' \to X$ is continuous if and only if, for each $\alpha \in A$, the composite $f_\alpha \circ g$ is continuous.*

Proof. If g is continuous, then since the composition of continuous maps is continuous, it follows that each $f_\alpha \circ g$ is continuous too.

Conversely, let us assume that each $f_\alpha \circ g$ is continuous. Now let $U \subseteq X$ be an open set. Since X has the initial topology with respect to F, it follows that U is a union of finite intersections of sets of the form $f_\alpha^{-1}(V)$ for $\alpha \in A$ and $V \subseteq Y_\alpha$ open. Consequently, $g^{-1}(U)$ is a union of finite intersections of sets of the form $g^{-1}(f_\alpha^{-1}(V))$ for $\alpha \in A$ and $V \subseteq Y_\alpha$. Since $f_\alpha \circ g$ is continuous, it follows that $g^{-1}(U)$ is open. ☐

2.4.7. Dually, but simpler, suppose X a set, and suppose $F \coloneqq \{f_\alpha \colon Y_\alpha \to X\}_{\alpha \in A}$ an indexed family of maps. Suppose that each source Y_α is endowed with a topology τ_α.

- There exists a unique finest topology on X relative to which every map of F is continuous. This is called the ***final topology*** with respect to F. A subset $U \subseteq X$ is open with respect to the final topology if and only if, for every $\alpha \in A$, the inverse image $f_\alpha^{-1}(U) \subseteq Y_\alpha$ is open.
- A map $g \colon X \to X'$ is continuous if and only if, for each $\alpha \in A$, the composite $g \circ f_\alpha$ is continuous.

Example 2.4.8. Each of the following sets form a subbase for the standard topology on $\boldsymbol{R}$;

- the set of all open subsets of $\boldsymbol{R}$;
- the set $\{]a, b[\; : a, b \in \boldsymbol{R}\}$ of open intervals; and
- the set $\{]a, +\infty[\; : a \in \boldsymbol{R}\} \cup \{]-\infty, b[\; : b \in \boldsymbol{R}\}$ of open rays.

Example 2.4.9. Let $X \subseteq \boldsymbol{R}^n$ be a subspace of Euclidean space. The set $\{B(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$ is a subbase for the induced topology on X.

If B is a subbase for a topology, then any open set in that topology is a union of finite intersections of subbase elements. One might wish for a better subbase with enough sets therein to ensure that every open set can be written as a union of elements of that subbase.

Definition 2.4.10. Let X be a topological space. Let $x \in X$ be a point. Then a ***local base*** at $x \in X$ is a set N_x of open neighbourhoods of x with the following property: For any open neighbourhood W of x, there is an open neighbourhood $V \in N_x$ with $V \subseteq W$.

A ***base*** for X is a collection B of open sets such that for any point $x \in X$, the set

$$B_x \coloneqq \{U \in B : x \in U\}$$

is a local base at $x \in X$.

Proposition 2.4.11. *Let X be a topological space, let $x \in X$ be a point, and let N_x be a local base at x. Then x is close to a subset $S \subseteq X$ if and only if, for any $V \in N_x$, the intersection $S \cap V \neq \varnothing$.*

Proposition 2.4.12. *Let X be a topological space, and let B be a base for the topology. Then the following are equivalent for a subset $U \subseteq X$.*

- *The set U is open in X.*
- *For any point $x \in U$, there exists an element $V \in B_x$ such that $V \subset U$.*
- *The set U can be written as the union of elements of B.*

Proposition 2.4.13. *Let X be a topological space. Then a set B of open sets is a base for the topology if and only if the following conditions are satisfied.*

- *The elements of B cover X.*
- *For any $U, V \in B$ and any element $x \in U \cap V$, there is an element $W \in B_x$ such that $W \subseteq U \cap V$.*

Example 2.4.14. Let $X \subseteq \boldsymbol{R}^n$ be a subspace of Euclidean space. Then the set

$$\{B(x, \varepsilon) : (x \in X) \wedge (\epsilon > 0)\}$$

is a base for the induced topology on X.

Example 2.4.15. Let X be a set with two topologies τ_1 and τ_2; let B_1 be a base for (X, τ_1), and let B_2 be a base for (X, τ_2). Show that τ_1 is finer than τ_2 if and only if, for every point $x \in X$ and every element $U \in (B_2)_x$, there exists an element $V \in (B_1)_x$ such that $V \subseteq U$.

2.5 Quotient spaces

2.5.1. Let X be a set. Recall that an *equivalence relation* $\sim$ on X is a relation that satisfies the following conditions:

- For every $x \in X$, one has $x \sim x$.
- For every $x, y \in X$, one has $x \sim y$ if and only if $y \sim x$.
- For every $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

If $x \in X$, one can form the *equivalence class* $[x] \coloneqq \{y \in X : y \sim x\}$. Now the set $X/\sim : \{[x] : x \in X\}$ of equivalence classes relative to $\sim$ is called the *quotient* of X relative to $\sim$. The assignment $x \mapsto [x]$ is a map $q : X \to X/\sim$, which is called the *quotient map*; The map q is a surjection, and it enjoys the following pleasant property: for every set Y and every map $f : X \to Y$ with the property that for every $x, y \in X$, if $x \sim y$, then $f(x) = f(y)$, there exists a unique map $f' : X/\sim \to Y$ such that $f' \circ q = f$.

Every surjection is secretly a quotient map: if $p : X \to T$ is a surjection, then we may define an equivalence relation $\sim$ on X in which $x \sim y$ if and only if $p(x) = p(y)$. We may call this the equivalence relation *induced by p*. The unique map $p' : X/\sim \to T$ through which p factors is then a bijection.

Finally, if R is *any* relation on X, then it *generates* an equivalence relation. That is, recall that $\sim$ is *really* a subset[15] of $X \times X$. If we look at the collection of all subsets $Q \in \boldsymbol{P}(X \times X)$ that are equivalence relations that contain R, then the intersection of all these Q is itself an equivalence relation that contains R. This intersection is the *equivalence relation generated by R*.

[15] consisting of those pairs (x, y) such that xRy

Definition 2.5.2. Let X be a topological space, let Y be a set, and let $p\colon X \to Y$ be a surjection. The final topology with respect to p is the ***quotient topology*** on Y; this is the finest topology on Y such that the map p is continuous. We call p the ***quotient map***. We also say that Y is the *quotient space* of X under the equivalence relation induced by p.

The ability to form quotients of topological spaces is really the most powerful motivation for defining abstract topological spaces in the first place.

Example 2.5.3. Consider the real line $\boldsymbol{R}$, and consider the equivalence relation $\sim$ in which $x \sim y$ if and only if $y - x$ is an integer. Now consider the map $t \mapsto \exp(2\pi t)$; this is a continuous map $e\colon \boldsymbol{R} \to S^1$. By the discussion above, there exists a unique map $f\colon \boldsymbol{R}/\sim \longrightarrow S^1$ such that if $q\colon \boldsymbol{R} \to \boldsymbol{R}/\sim$ is the quotient map, then $f \circ q = e$. This map is continuous with respect to the quotient topology on $\boldsymbol{R}/\sim$. In fact, f is a homeomorphism.

Similarly, consider the closed interval $[0, 1]$, and consider the equivalence relation $\sim$ generated by the requirement that $0 \sim 1$. Then the inclusion map $[0, 1] \hookrightarrow \boldsymbol{R}$ descends to a continuous map $[0, 1]/\sim \longrightarrow \boldsymbol{R}/\sim$, and this map is a homeomorphism as well. This justifies our intuition that the circle is obtained from the interval by identifying the endpoints.

Example 2.5.4. Let $X \subseteq \boldsymbol{R}^n$ be a subspace, and let $k \in \boldsymbol{N}$ be a natural number. Then we have the subspace $X^k \subseteq \boldsymbol{R}^{nk}$. Furthermore, we can define the subsspace

$$Y \coloneqq \{(x_1, \dots, x_k) \in X^k : (\forall i \neq j)(x_i \neq x_j)\} \subseteq X^k .$$

Now conisder the equivalence relation $\sim$ on Y in which $(x_1, \dots, x_k) \sim (y_1, \dots, y_k)$ if and only if there exists a permutation[16] $\sigma \in \Sigma_k$ such that for each i, one has $y_i = x_{\sigma(i)}$. Now we can think of the points of the quotient space $C_k(X) \coloneqq Y/\sim$ as *unordered* collections of k points on X. This space $C_k(X)$ is called the ***configuration space*** of k marked points on X.

[16] We write Σ_k for the set of permutations of the set $\{1, \dots, k\}$.

The space $C_0(X)$ has exactly one point.

The space $C_1(X)$ is X itself.

The space $C_2(X)$ is much more interesting. For example, $C_2(\boldsymbol{R}^2) \cong \boldsymbol{R}^3 \times S^1$. The space $C_2(S^1)$ is the *Möbius band*.

As k goes higher, these spaces become even more involved.

The spaces $C_k(X)$ are extremely important in physics, since we may think of them as the space of possible states of k noninteracting particles on X.

Example 2.5.5. Let V be a finite dimensional real vector space. Since V is a finite dimensional, it is isomorphic to $\boldsymbol{R}^n$. Let us transport the topology from $\boldsymbol{R}^n$ to V along such an isomorphism.[17] Thus V is homeomorphic to $\boldsymbol{R}^n$.

[17] That is, if $\phi\colon V \to \boldsymbol{R}^n$ is a linear isomorphism, then $U \subseteq V$ is open if and only if $\phi(V) \subseteq \boldsymbol{R}^n$ is open. As an exercise, you can show that the topology is the same, irrespective of which isomorphism ϕ you select. (Hint: linear isomorphisms from $\boldsymbol{R}^n$ to itself are always homeomorphisms!)

For every $k \in \boldsymbol{N}^*$, the set V^k of k-tuples $(v_1, \dots, v_k)$ of vectors in V is also a vector space, and thus also a topological space that is homeomorphic to $\boldsymbol{R}^{nk}$. Now we let $L_{\boldsymbol{R}}(k, V) \subseteq V^k$ be the set of k-tuples $(v_1, \dots, v_k)$ of vectors in V that are linearly independent, so that they span a k-dimensional subspace of V.

Now we define an equivalence relation $\sim$ on $L_{\boldsymbol{R}}(k, V)$ by the rule that $(v_1, \dots, v_k) \sim (w_1, \dots, w_k)$ if and only if their spans are equal (as subspaces of V). The quotient space $G_{\boldsymbol{R}}(k, V) \coloneqq L_{\boldsymbol{R}}(k, V)/\sim$ is called the

Grassmannian of k-dimensional subspaces of V. The points of $G_{\boldsymbol{R}}(k,V)$ are k-dimensional real linear subspaces of V. One often writes $G_{\boldsymbol{R}}(k,n)$ as a shorthand for $G_{\boldsymbol{R}}(k,\boldsymbol{R}^n)$.

In the particular case in which $k = 1$, the space $G_{\boldsymbol{R}}(1,n)$ is the space of lines through the origin of $\boldsymbol{R}^n$; this is also called the (real) *projective space* $\boldsymbol{P}^{n-1}_{\boldsymbol{R}}$.

We can do this same thing with $\boldsymbol{C}$ in place[18] of $\boldsymbol{R}$.

In particular, $G(k,n)$ is the space of *complex* lines through the origin in $\boldsymbol{C}^n$; this is called the (complex) *projective space* $\boldsymbol{P}^{n-1}$.

[18] That is, we can look at a finite dimensional complex vector space V with the topology transported from $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$; we can consider the subspace $L(k,V)$ of $\boldsymbol{C}$-linearly independent k-tuples of V^k; and we can contemplate the quotient $G(k,V)$ of k-dimensional complex subspaces of V.

2.5.6. Let X and Y be topological spaces, and let $p\colon X \to Y$ be a continuous surjection. Then p need not be a quotient map. For example, the map $e\colon\ ]0,1] \to S^1$ given by $t \mapsto \exp(2\pi t)$ is not a quotient map: even though $e^{-1}(e(]1/2,1/2])) =]1/2,1/2]$ is open, the subset $e(]1/2,1/2]) \subseteq S^1$ is not. If p is an *open map* – that is, if it carries open sets to open sets – then it is a quotient map.

2.5.7. Let $\{X_a\}_{a\in A}$ be an indexed family of sets. Then the *coproduct of sets* is by definition the union

$$\coprod_{a\in A} X_a \coloneqq \bigcup_{a\in A} X_a \times \{a\}\,.$$

It is engineered precisely so that a map out of the coproduct is determined by each of its components. More precisely, the coproduct is equipped with inclusion maps

$$\iota_b\colon X_b \to \coprod_{a\in A} X_a\,,$$

one for every $b \in A$, given by $\iota_b(x) = (x,b)$. For any set Y, and for any collection $\{f_a\colon X_a \to Y\}_{a\in A}$ of maps, there exists a unique map

$$f\colon \coprod_{a\in A} X_a \to Y$$

such that for each $a \in A$, one has $f \circ \iota_a = f_a$. Thus composition with the inclusions induces a bijection

$$\prod_{a\in A} - \circ \iota_a\colon \mathrm{Map}\left(\coprod_{a\in A} X_a, Y\right) \to \prod_{a\in A} \mathrm{Map}(X_a,Y)\,.$$

Definition 2.5.8. Let X be a topological space, and let $A \subseteq X$ a subspace. Consider the map $q\colon X \to (X - A) \sqcup \{X - A\}$ defined by

$$q(x) \coloneqq \begin{cases} x & \text{if } x \notin A; \\ X - A & \text{if } x \in A. \end{cases}$$

We define X/A as the set $(X - A) \sqcup \{X - A\}$ equipped with the final topology with respect to q.

Example 2.5.9. Consider the subspace

$$D^n \coloneqq \{x \in \boldsymbol{R}^n : \|x\| \leq 1\} \subset \boldsymbol{R}^n$$

and the subspace

$$S^{n-1} \coloneqq \{x \in D^n : \|x\| = 1\} \subset D^n\,.$$

Then the space D^n/S^{n-1} is homeomorphic to $(\boldsymbol{R}^n)^+$, which is in turn homeomorphic to S^n.

Definition 2.5.10. Let $\{(X_a, \tau_a)\}_{a\in A}$ be an indexed family of topological spaces. Then the *coproduct topology* $\coprod_{a\in A} \tau_a$ is the final topology with respect to the set $\{\iota_a\}_{\alpha\in A}$.

2.5.11. Let $S = \coprod_{a\in A} S_a \subseteq \coprod_{a\in A} X_a$ (so that $S_a \subseteq X_a$). If τ is the coproduct topology, then

$$\tau(S) = \coprod_{a\in A} \tau_a(S_a)\,.$$

Example 2.5.12. If X is topological space, then X_+ denotes the coproduct $X \sqcup \{X\}$ with the coproduct topology.

Example 2.5.13. Let $A = (A, \delta)$ be a discrete topological space. Suppose $f\colon X \to A$ a continuous map. Then we have a bijection $G\colon \coprod_{a\in A} f^{-1}\{a\} \to X$. Let us show that this is a homeomorphism, where the right hand side is given the coproduct topology (and the fibers $f^{-1}\{a\} \subseteq X$ are given the subspace topology). By construction, the map G is continuous. It remains to show that G^{-1} is continuous; let $U \subseteq \coprod_{a\in A} f^{-1}\{a\}$ be an open subset. Then $U \cap f^{-1}\{a\}$ is open in $f^{-1}\{a\}$, and so since U is the union of these intersections, it suffices to prove that any open subset of $f^{-1}\{a\}$ is open in X. Indeed, this is true, since $f^{-1}\{a\}$ is itself open.

Conversely, if $\{X_a\}_{a\in A}$ is an indexed family of topological spaces, then define the map

$$g\colon \coprod_{a\in A} X_a \to A$$

such that $g(x) = a$ if and only if $x \in X_a$. The map g is continuous.

In other words, a decomposition of a toploogical space into a coproduct of topological spaces is equivalent to a continuous map into a discrete topological space.

Much of the story of topology is the story of "gluing topological spaces together" to get new ones. We are now ready to say exactly what this means.

Example 2.5.14. Let U, V, and X be three topological spaces, and let $f\colon X \to U$ and $g\colon X \to V$ be two continuous maps. Consider the equivalence relation $\sim$ on $U \sqcup V$ generated by declaring that: for every element $x \in X$, $f(x) \sim g(x)$. Write $U \cup^X V$ for the quotient $(U \sqcup V)/\sim$ with the quotient topology.

Consider the subset

$$D^n \coloneqq \{x \in \boldsymbol{R}^n : \|x\| \leq 1\} \subset \boldsymbol{R}^n$$

and the subset

$$S^{n-1} \coloneqq \{x \in D^n : \|x\| = 1\} \subset D^n.$$

Here's a standard way to build a new space from an old one. Suppose X a topological space, and suppose $f\colon S^{n-1} \to X$ a continuous map. Then we can form the *cell attachment for attaching map f*

$$X \cup^{S^{n-1}} D^n.$$

A *cell complex* is a topological space Y that is built via cell attachments from discrete spaces; that is, a cell complex is a topological space Y along with a sequence of subspaces

$$Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y$$

such that:

- $Y = \bigcup_{n\geq 0} Y_n$;
- Y_0 is discrete;
- for any $n \geq 1$, there exist an indexed family $\{f_\alpha : S^{n-1} \to Y_{n-1}\}_{\alpha\in A_n}$ of attaching maps such that

$$Y_n = Y_{n-1} \cup^{\coprod_{\alpha\in A_n} S^{n-1}} \coprod_{\alpha\in A_n} D^n.$$

2.6 Products

2.6.1. The definition of a *product of sets* is engineered precisely so that a map into the product is determined by each of its components. More precisely, for any indexed family of sets $\{X_a\}_{a\in A}$, the product

$$\prod_{a\in A} X_a$$

is a set equipped with projection maps

$$\pi_b : \prod_{a\in A} X_a \to X_b \,,$$

one for every $b \in A$, which together *determine* the maps into $\prod_{a\in A} X_a$. That is, composition with the projections induce a bijection

$$\prod_{a\in A} \pi_a \circ - : \operatorname{Map}\left(Y, \prod_{a\in A} X_a\right) \to \prod_{a\in A} \operatorname{Map}(Y, X_a) \,.$$

If the sets X_a are all endowed with topologies, then we would like to topologise the product $\prod_{a\in A} X_a$ so that the same thing will happen with *continuous* maps. That is, we would like to ensure that a map $f : Y \to \prod_{a\in A} X_a$ is continuous if and only if, for every $a \in A$, the composite map $\pi_a \circ f : Y \to X_a$ is continuous. But we know just how to arrange that:

Definition 2.6.2. Let $\{(X_a, \tau_a)\}_{a\in A}$ be a family of topological spaces. Then the *product topology* $\prod_{a\in A} \tau_a$ on $\prod_{a\in A} X_a$ is the initial topology with respect to the set $\{\pi_a\}_{\alpha\in A}$.

2.6.3. The definition of the *product topology* is engineered precisely so that a continuous map into the product is determined by each of its continuous components. More precisely, for any indexed family of topological spaces $\{X_a\}_{a\in A}$, the product

$$\prod_{a\in A} X_a$$

is a set equipped with continuous projection maps

$$\pi_b : \prod_{a\in A} X_a \to X_b \,,$$

one for every $b \in A$, which together *determine* the maps into $\prod_{a\in A} X_a$. That is, composition with the projections induce a bijection

$$\prod_{a\in A} \pi_a \circ - : \operatorname{Map}\left(Y, \prod_{a\in A} X_a\right) \to \prod_{a\in A} \operatorname{Map}(Y, X_a) \,,$$

Example 2.6.4. In particular, for any set A and any topological space X, we may contemplate the product topology on the set $\mathrm{Map}(A, X)$. We write in particular

$$X^n := \mathrm{Map}(\{1, \dots, n\}, X) \quad \text{and} \quad X^\omega := \mathrm{Map}(\boldsymbol{N}, X).$$

Proposition 2.6.5. *For any $n \geq 0$, the product topology on $\boldsymbol{R}^n$ coincides with the standard topology (from the Euclidean metric).*

Proof. The product topology on $\boldsymbol{R}^n$ is generated by sets of the form $\boldsymbol{R} \times \cdots \times]a, b[\times \boldsymbol{R} \times \cdots \times \boldsymbol{R}$. Consequently, the sets of the form $C(x, \varepsilon) := \prod_{i=1}^n]x_i - \varepsilon, x_i + \varepsilon[$ are a base for the product topology. Now for any $x \in \boldsymbol{R}^n$, and for any $\varepsilon > 0$, the exists a $\delta > 0$ such that

$$B(x, \delta) \subseteq C(x, \varepsilon)\,,$$

and

$$C(x, \delta) \subseteq B(x, \varepsilon)\,.$$

Since the subsets $C(x, \varepsilon)$ generate the product topology, and the subsets $B(x, \varepsilon)$ generate the standard topology, the two topologies coincide. ☐

Warning 2.6.6. Consider the product topological space

$$\boldsymbol{R}^\omega = \mathrm{Map}(\boldsymbol{N}, \boldsymbol{R}) = \prod_{n \in \boldsymbol{N}} \boldsymbol{R}\,.$$

This topology has a base

$$\left\{ \prod_{i=0}^{N}]a_i, b_i[\times \prod_{i=N+1}^{\infty} \boldsymbol{R} \right\}\,.$$

Consequently, if U is an open subset of $\boldsymbol{R}^\omega$, then for any point $x \in U$, there exists $N \in \boldsymbol{N}$ such that any point of the form

$$(x_0, x_1, \dots, x_N, y_{N+1}, y_{N+2}, \dots)$$

also lies in U. Conequently, the subset

$$]0, 1[^\omega = \mathrm{Map}(\boldsymbol{N},]0, 1[) \subset \boldsymbol{R}^\omega$$

is not open!

Example 2.6.7. If $S_1, \dots, S_n$ is a finite collection of discrete topological spaces, then the product

$$\prod_{i=1}^{n} S_i = S_1 \times \cdots \times S_n$$

is also discrete. Indeed, for any i and any $x_i \in S_i$, the set $S_1 \times \cdots \times S_{i-1} \times \{x_i\} \times S_{i+1} \times \cdots \times S_n$ is open, and since finite intersections of opens are open, it follows that any singleton $\{(x_1, \dots, x_n)\}$ is open as well.

Warning 2.6.8. If $\{S_i\}_{i \in \boldsymbol{N}}$ is a countable family of discrete finite sets of cardinality at least 2, the product $S := \prod_{i \in \boldsymbol{N}} S_i$ is not discrete: if $(x_0, x_1, \dots) \in S$ is a point, then the open sets are unions of sets of the form

$$\{x_0\} \times \cdots \times \{x_N\} \times \prod_{i=N+1}^{\infty} S_i\,,$$

so that no singleton is open. In fact, it is always homeomorphic to our old pal the Cantor space C – irrespective of which finite sets S_i are chosen![19]

[19] This is not obvious, or even easy to prove. I personally found this fact totally implausible the first time I read it.

There is a *relative* form of the product as well.

Definition 2.6.9. Let U, V, and X be three topological spaces, and let $f: U \to X$ and $g: V \to X$ two continuous maps. Then the ***fiber product***[20] $U \times_X V$ is the subspace

$$\{(u, v) \in U \times V \,:\, f(u) = g(v)\}$$

of $U \times V$. In this case, the square

$$\begin{tikzcd} U \times_X V \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & V \arrow[d, "g"] \\ U \arrow[r, "f"'] & X \end{tikzcd}$$

is sometimes called a ***pullback square***.

[20] Please notice that this notation is a little sloppy. The fiber product $U \times_X V$ depends on the maps f and g. If one needs to make the choice of maps explicit, one may write

$$U \underset{f,X,g}{\times} V\,,$$

which is a bit busy, but very clear!

Example 2.6.10. As a special case of this construction, if V is a subspace of X and if g is the inclusion map $V \hookrightarrow X$, then the fiber product $U \times_X V$ is homeomorphic to the subspace $f^{-1}(V) \subseteq X$. In particular, if $V = \{x\}$ for some $x \in X$, then the fiber product $U \times_X V$ is the fiber $f^{-1}\{x\}$.

Example 2.6.11. Let $f: X \to Y$ be a continuous map of topological spaces. Then the ***graph*** of f is the fiber product

$$\Gamma(f) \coloneqq X \underset{f,Y,\mathrm{id}}{\times} Y \subseteq X \times Y\,.$$

The projection map $\mathrm{pr}_1 : \Gamma(f) \to X$ is a homeomorphism.

2.7 Esoterica

The universe of topological spaces is very large, and it includes some ***beasts***. These beasts exhibit some very strange behavior, at least from the standpoint of the topological spaces we construct from Euclidean spaces and finite sets. These are sometimes called ***pathological topological spaces*** or just ***pathologies***. They are useful as counterexamples to certain implications in the world of topology, but their importance should not be overestimated.

3 COMPACTNESS

3.1 Compact topological spaces

There are a lot of different characterisations of compactness out there. We'll give them all,[1] and we'll prove their equivalence.[2]

[1] or, at least, all the ones I know about!

[2] This is an important principle in mathematics: ***never have only one definition of any notion***. Instead, when possible, have multiple equivalent characterisations of the same idea. Then use whichever one is best adapted for whichever problem you happen to encounter.

Definition 3.1.1. Suppose X and Y two topological spaces. Then a map $f\colon X \to Y$ is ***closed*** if and only if, for any subset $S \subseteq X$ one has $f(\overline{S}) \supseteq \overline{f(S)}$. In other words, if y is close to $f(S)$, then $y = f(x)$ for some x that is close to S.

Example 3.1.2. Any homeomorphism is closed.

Example 3.1.3. Suppose X a topological space, and suppose $A \subseteq X$ a subspace thereof. Then the inclusion map $A \rightarrowtail X$ is closed if and only if A is closed in X.

Example 3.1.4. The map $f\colon \boldsymbol{R} \to \boldsymbol{R}$ given by the formula $f(x) = |x|$ is closed and, of course, continuous.

Example 3.1.5. The map $s\colon \boldsymbol{R} \to \boldsymbol{R}$ given by the formula

$$s(x) := \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ +1 & \text{if } x > 0 \end{cases}$$

is closed but not continuous.

Proposition 3.1.6. *Suppose X and Y two topological spaces. Then a map $f\colon X \to Y$ is closed if and only if, for any closed subset $Z \subseteq X$, the direct image $f(Z)$ is closed.*

Proof. Suppose f closed, and suppose $Z \subseteq X$ a closed subset; then

$$f(Z) = f(\overline{Z}) \supseteq \overline{f(Z)},$$

but the other inclusion $f(Z) \subseteq \overline{f(Z)}$ is automatic. Hence $f(Z) = \overline{f(Z)}$.

Conversely, suppose that f satisfies the condition that for any closed subset $Z \subseteq X$, the direct image $f(Z)$ is closed, and suppose $S \subseteq X$ any subset. Then $f(\overline{S})$ is certainly closed, and of course it contains $f(S)$; hence it contains $\overline{f(S)}$. □

Lemma 3.1.7. *If $f\colon X \to Y$ is a closed, continuous surjection, then f is a quotient map.*

Proof. Let $Z \subseteq Y$ be a subset such that $f^{-1}(Z) \subseteq X$ is closed. Then $Z = f(f^{-1}(Z)) \subseteq Y$ is closed, since f is closed. □

Example 3.1.8. The projection $\mathrm{pr}_1 : \boldsymbol{R}^2 \to \boldsymbol{R}$ (given by $\mathrm{pr}_1(x, y) = x$), which is certainly continuous, is not closed.[3] To see this, consider the closed subset

$$Z \coloneqq \{(x, y) \in \boldsymbol{R}^2 : y = 2^x\};$$

then $\mathrm{pr}_1(Z) =]0, +\infty[$, which is not closed in $\boldsymbol{R}$.

[3] It is, however, an *open map*: the direct image of any open subset is open.

Definition 3.1.9. A *cover* of a set X is a subset $\boldsymbol{U} \subseteq \boldsymbol{P}(X)$ such that $\cup\boldsymbol{U} = X$. An *open cover* of a topological space X is a cover whose members are all open in X.

Proposition 3.1.10. *Let X and Y be two topological spaces, and let $\{W_1, \dots, W_n\}$ be a finite cover of X. Then a map $f : X \to Y$ is closed if each restriction $f|_{W_j} : W_j \to Y$ is closed.*

Proof. Assume that each restriction $f|_{W_j}$ is closed, and let $Z \subseteq X$ a closed subset. Then $Z \cap W_j$ is closed in W_j, and so the subset

$$f(Z \cap W_j) = f|_{W_j}(Z \cap W_j) \subseteq Y$$

is closed, whence so is the subset $f(Z) = \bigcup_{j=1}^{n} f(Z \cap W_j)$. □

Proposition 3.1.11. *Let X and Y be two topological spaces, and let $f : X \to Y$ be a map. Then the following are equivalent.*

- *The map f is closed.*
- *For any $y \in Y$ and any open set $U \supseteq X_y$, there is an open neighbourhood V of y such that $f^{-1}(V) \subseteq U$.*

Proof. Assume that f is closed, and assume that $y \in Y$ and $U \supseteq X_y$ open. Then set $V \coloneqq Y \smallsetminus f(X \smallsetminus U)$; clearly $y \in V$, and since f is closed, V is open. Now an element of $f^{-1}(V)$ does not lie in $f^{-1}(f(X \smallsetminus U))$, whence we deduce that it does not lie in $X \smallsetminus U$, whence it lies U, as desired.

Conversely, suppose the second condition is satisfied, and suppose $Z \subseteq X$ a closed subset. By the second condition, for any point $y \notin f(Z)$, there is an open neighbourhood V of y such that $f^{-1}(V) \subseteq X \smallsetminus Z$, whence we deduce that $Y \smallsetminus f(Z)$ is open. □

Proposition 3.1.12. *The following are equivalent for a topological space X.*

- *Any open cover $\boldsymbol{U}$ of X contains a finite* subcover *– i.e., a finite subset $\boldsymbol{U}_0 \subseteq \boldsymbol{U}$ that covers X as well.*
- *Let T be a topological space, and let $t \in T$ be a point. Then for any open subset $U \subseteq T \times X$ that contains $\{t\} \times X$, there is an open neighbourhood V of t such that $V \times X \subseteq U$.*
- *For any topological space T, the projection map $\mathrm{pr}_1 : T \times X \to T$ is closed.*
- *Let $\boldsymbol{Z} \subseteq \boldsymbol{P}(X)$ be a collection of closed subsets of X such that $\cap\boldsymbol{Z} = \varnothing$; then there exists a finite subset $\boldsymbol{Z}_0 \subseteq \boldsymbol{Z}$ such that the intersection $\cap\boldsymbol{Z}_0 = \varnothing$.*

Proof. Let us assume the first condition and prove the second. So let T be a topological space and $t \in T$, and let U an open set of $T \times X$ containing $\{t\} \times X$. Now let $\boldsymbol{U}$ be the collection of those open subsets $W \subseteq X$ such that for some open neighbourhood E of t, one has $E \times W \subseteq U$. The collection $\boldsymbol{U}$ covers[4] X. The first condition now ensures that there is a finite subcover $\boldsymbol{U}_0 \subseteq \boldsymbol{U}$. For each $W \in \boldsymbol{U}_0$, select an open neighbourhood E_W of t such that $E_W \times W \subseteq U$. Now set

$$V \coloneqq \bigcap_{W \in \boldsymbol{U}_0} E_W;$$

so that V is an open neighbourhood of t, and $V \times W \subseteq U$ for any $W \in \boldsymbol{U}_0$. Hence

$$V \times X = \bigcup_{W \in \boldsymbol{U}_0} V \times W \subseteq U.$$

This verifies the second condition.

The equivalence of the second and third conditions is the content of the previous proposition.

Let us assume the third condition and prove the fourth. So let $\boldsymbol{Z} \subseteq \boldsymbol{P}(X)$ be a collection of closed subsets of X, and assume that for any finite subset $\boldsymbol{Z}_0 \subseteq \boldsymbol{Z}$, the intersection $\cap \boldsymbol{Z}_0$ is nonempty. Now let T be the set $X \sqcup \{\infty\}$, which we topologise with the coarsest topology such that[5]

- *any* subset $S \subseteq X$ is open in T, and
- for any $Z \in \boldsymbol{Z}$, the set $Z \sqcup \{\infty\}$ is open in T.

Any open neighbourhood of ∞ contains a finite intersection of elements of $\boldsymbol{Z}$. Since all such intersections are nonempty, any open neighbourhood of ∞ contains at least one point of X. Hence ∞ is not open or, equivalently, $X \subseteq T$ is dense. Now let

$$\Delta \coloneqq \{(x, x) : x \in X\} \subseteq T \times X,$$

and consider the closed set $\mathrm{pr}_1(\overline{\Delta}) \subseteq T$. Clearly $X = \mathrm{pr}_1(\Delta) \subseteq \mathrm{pr}_1(\overline{\Delta})$, so $\mathrm{pr}_1(\overline{\Delta}) = T$. In other words, $\overline{\Delta}$ contains a point (∞, x) for some $x \in X$. Thus for every $Z \in \boldsymbol{Z}$ and every open neighbourhood U of x, one has $((Z \sqcup \{\infty\}) \times U) \cap \Delta \neq \varnothing$, and so[6] $Z \cap U \neq \varnothing$. Thus x is close to every $Z \in \boldsymbol{Z}$, whence x lies in every $Z \in \boldsymbol{Z}$, whence $x \in \cap \boldsymbol{Z}$.

Finally, let's assume the fourth condition and prove the first. Let $\boldsymbol{U} \subseteq \boldsymbol{P}(X)$ be an open cover of X. Now let $\boldsymbol{Z}$ be the set of complements of elements[7] of $\boldsymbol{U}$. Clearly $\cap \boldsymbol{Z} = X - \cup \boldsymbol{U} = \varnothing$, so by the fourth condition, there exists a finite subset $\boldsymbol{Z}_0 \subseteq \boldsymbol{Z}$ such that $\cap \boldsymbol{Z}_0 = \varnothing$. Now let $\boldsymbol{U}_0$ be the set of complements of elements[8] of $\boldsymbol{Z}_0$. Now $\cup \boldsymbol{U}_0 = X \smallsetminus \cap \boldsymbol{Z}_0 = X$, whence $\boldsymbol{U}_0 \subseteq \boldsymbol{U}$ is a finite subcover. □

Definition 3.1.13. A topological space X is *compact*[9] if and only if it satisfies any of the equivalent conditions of 3.1.12. In this case, X is called a ***compactum***.

3.1.14. Note that compactness is a *topologically invariant* notion; that is, if X and Y are homeomorphic topological spaces, then X is compact if and only if Y is compact.[10]

[4] Since U is open in the product topology, for every $x \in X$, the point (t, x) lies in an open neighbourhood of the form $E \times W$ such that $E \times W \subseteq U$

[5] Please note that this is weird! The topology we're putting on T really has nothing to do with the topology we have on X; it's really all about $\boldsymbol{Z}$. We're using the class $\boldsymbol{Z}$ of *closed* subsets of X to describe some *open* sets of T.

[6] since a point (u, u) of that intersection has the property that $u \in Z \cap U$

[7] Hence $Z \in \boldsymbol{Z}$ if and only if $X \smallsetminus Z \in \boldsymbol{U}$.

[8] Hence $U \in \boldsymbol{U}_0$ if and only if $X \smallsetminus U \in \boldsymbol{Z}_0$.

[9] Bourbaki use the term *quasicompact* for the notion we have introduced here, and reserve the term *compact* for topological spaces that are also *Hausdorff*. The Bourbaki convention is typical in French mathematical literature and in algebraic geometry literature.

[10] Please observe that this cannot be said for notions like openness or closedness; those depend upon the *ambient topological space*.

Example 3.1.15. Any finite topological space is compact, and the indiscrete topology on any set is compact.

Example 3.1.16. The discrete compacta are all finite.

Proposition 3.1.17. *Any closed interval $[a,b]$ is compact.*

Proof. Suppose $\boldsymbol{U}$ an open covering of $[a,b]$; then let

$$c = \sup\{x \in [a,b] \,:\, [a,x] \text{ is contained in the union of finitely many elements of } \boldsymbol{U}\}\ .$$

Now suppose, to generate a contradiction, that $c < b$; then let $U \in \boldsymbol{U}$ be an element of the open cover containing c. Then for some $\varepsilon > 0$, one has $]c-\varepsilon, c+\varepsilon[\ \subset U$. By definition of c, one has $[a, c-\varepsilon/2] \subseteq \cup\boldsymbol{U}_0$ for a finite subset $\boldsymbol{U}_0 \subseteq \boldsymbol{U}$; hence the union of the elements of $\boldsymbol{U}_0 \cup \{U\} \subseteq \boldsymbol{U}$ contains $[a, c+\varepsilon/2]$, yielding the desired contradiction. Thus $c = b$.

Now choose an element $V \in \boldsymbol{U}$ than contains b; for some $\varepsilon > 0$, one has $]b-\varepsilon, b] \subset U$. Now since $c = b$, there exists a finite subset $\boldsymbol{V}_0 \subseteq \boldsymbol{U}$ such that $[a, b-\varepsilon/2[\ \subseteq \cup\boldsymbol{V}_0$, and so $\boldsymbol{V}_0 \cup \{V\} \subseteq \boldsymbol{U}$ is a finite cover of $[a,b]$. □

Example 3.1.18. No Euclidean space $\boldsymbol{R}^n$ is compact;[11] indeed, the open balls $B^n(x,\varepsilon)$ form an open cover, but there is no finite subcover. To see this, consider any finite cover of $\boldsymbol{R}^n$ by balls $B^n(x_i, \varepsilon_i)$ for $i = 1, 2, \dots, m$, and suppose x a point of one of the balls. Then one may choose a real number r so that $B^n(x,r) \supset \bigcup_{i=1}^m B^n(x_i, \varepsilon_i)$.

[11] Please note that this example, combined with the previous one, implies that $\boldsymbol{R}$ is not homeomorphic to any closed interval. There are other ways to prove this, but this one really cuts through the treacle!

Example 3.1.19. Any set with the cofinite topology is compact.

3.1.20. If (X,τ) is a compactum, and if τ' is a topology on X that is coarser than τ, then (X,τ') is also a compactum.

Lemma 3.1.21. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous surjection. If X is compact, so is Y.*

Proof. Let T be a topological space. If $Z \subseteq T \times Y$ is closed, then $f^{-1}(Z) \subseteq X$ is closed, so[12]

$$\mathrm{pr}_1(Z) = \mathrm{pr}_1(f^{-1}(Z))$$

is closed as well. □

[12] The displayed equation is an *abuse of notation*: on the left side is the direct image under the projection $\mathrm{pr}_1 : T \times Y \to T$, and on the right is the direct image under the projection $\mathrm{pr}_1 : T \times X \to T$.

Lemma 3.1.22. *A closed subspace of a compactum is compact.*

Proof. Let X be a compactum, and suppose $Z \subseteq X$ closed. For any topological space T, the projection $\mathrm{pr}_1 : T \times X \to T$ is closed, and the inclusion $T \times Z \hookrightarrow T \times X$ is closed, so the composite $T \times Z \to T$, which is the projection, is closed. □

Proposition 3.1.23. [13] *Let $\{X_i\}_{i\in\{1,2,\dots,n\}}$ be a family of compacta. Then the product $\prod_{i=1}^n X_i$ is compact.*

[13] This is the easy case of Tychonoff's Theorem, to which we shall return.

Proof. Suppose T a topological space. Then the projections $T \times X_1 \to T$, $T \times X_1 \times X_2 \to T \times X_1$, …, and $T \times X_1 \times X_2 \times \cdots \times X_n \to T \times X_1 \times \cdots \times X_{n-1}$ are all closed, and so their composite

$$T \times X_1 \times X_2 \times \cdots \times X_n \to T$$

is closed as well. □

Theorem 3.1.24. *The following are equivalent for a subspace $A \subset \boldsymbol{R}^n$.*

- *A is compact.*
- *[Bolzano–Weierstraß] Every sequence of points in A has a convergent subsequence.*
- *[Heine–Borel] A is closed and bounded.*

Proof. Let us show that the first of these conditions implies the second. Suppose A compact, and suppose $(x_i)_{i \geq 0}$ a sequence of points in A. If there were no convergent subsequence of $(x_i)_{i \geq 0}$, then the set $\{x_i\}_{i \geq 0}$ would be a closed subset of A, hence compact, and there would be a sequence ε_i of positive real numbers such that the balls $B^n(x_i, \varepsilon_i)$ would be disjoint. But then $\{B^n(x_i, \varepsilon_i)\}_{i \geq 0}$ would be an open cover of $\{x_i\}_{i \geq 0}$ with no finite subcover.

Let us show that the second property implies the third. Suppose A has the property that every sequence has a convergent subsequence. Then A must be bounded, since otherwise there exists a sequence $(x_i)_{i \geq 0}$ of points in A such that $\|x_i\| \to \infty$. It must also be closed, since if $x \in \overline{A} - A$, one can construct a sequence of points of A converging to x.

Finally, let us show that the third condition implies the first. Suppose A is closed and bounded. Since A is bounded, it is contained in a box $[-a, a]^n$ for some $a \geq 0$. Since every closed subspace of a compact space is compact, it is enough to show that $[-a, a]^n$ is compact. Since a finite product of compacta is compact, it is enough to show that any closed interval is compact, and we've already shown this. □

Corollary 3.1.25. *Suppose X a compactum, and suppose $f : X \to \boldsymbol{R}$ a continuous function. Then f attains both a maximum and a minimum value.*

3.2 Compact Hausdorff topological spaces

Definition 3.2.1. A topological space X is said to be *Hausdorff* if for any two points $x, y \in X$, there exist open neighbourhoods U of x and V of y such that $U \cap V = \varnothing$.

Example 3.2.2. Any discrete topological space is Hausdorff.

Example 3.2.3. Any subspace of Euclidean space is Hausdorff: for any two points $x, y \in X$, if $r := d(x, y)/2$, then the balls $B(x, r)$ and $B(y, r)$ are disjoint.

Counterexample 3.2.4. The indiscrete topology on a set X is not Hausdorff unless $\#X \leq 1$.

Example 3.2.5. Any subspace of a Hausdorff space is Hausdorff.

Example 3.2.6. If (X, τ) is Hausdorff, then for any topology τ' on X that is finer than τ, the topological space (X, τ') is Hausdorff as well.

This points in the opposite direction from compactness. Whereas Hausdorffitude is stable under passage to a finer topology, compactness is stable under passage to a coarser topology.

Lemma 3.2.7. *Let X be a Hausdorff space, and let $K \subseteq X$ a compact subspace. Then K is closed in X.*

Proof. Suppose $y \in X \smallsetminus K$. We aim to prove that there is an open neighbourhood V of y that does not intersect K. For each point $x \in K$, select open neighbourhoods U_x of x and V_x of y such that $U_x \cap V_x = \varnothing$. Now $\{U_x \cap K : x \in X\}$ is an open cover of K, so it contains a finite subcover $\{U_{x_1} \cap K, \dots, U_{x_n} \cap K\}$. Now $V \coloneqq V_{x_1} \cap \cdots \cap V_{x_n}$ is an open neighbourhood of y that does not intersect K. □

Proposition 3.2.8. *Suppose X a set with two topologies τ and τ' such that τ is coarser than (or equal to) τ'. Assume that τ is Hausdorff and that τ' is compact. Then $\tau = \tau'$.*

Proof. Suppose $K \subseteq X$ a τ'-closed subset, hence compact with the subspace topology. Since the identity is continuous $(X, \tau') \to (X, \tau)$ and the continuous image of a compactum is compact, it follows that K is compact as a subspace of (X, τ). The previous lemma now implies that K is τ-closed. □

Proposition 3.2.9. *Suppose X a Hausdorff space, and suppose $K, L \subseteq X$ two disjoint compact subspaces. Then there exist open sets $U, V \subseteq X$ such that $K \subseteq U$, $L \subseteq V$, and $U \cap V = \varnothing$.*

Proof. For every pair of points $x \in K$ and $y \in L$, select open neighbourhoods $U_{x,y}$ of x and $V_{x,y}$ of y such that $U_{x,y} \cap V_{x,y} = \varnothing$. For any $y \in L$, we obtain an open cover $\{U_{x,y} \cap K : x \in K\}$ of K; it contains a finite subcover $\{U_{x_1,y} \cap K, \dots, U_{x_m,y} \cap K\}$. We form disjoint open sets

$$U_y \coloneqq \bigcup_{i=1}^{m} U_{x_i,y} \text{ and } V_y \coloneqq \bigcap_{i=1}^{m} V_{x_i,y}$$

such that $U_y \supseteq K$ and $y \in V_y$. Thus we obtain an open cover $\{V_y \cap L : y \in L\}$ of L; it contains a finite subcover $\{V_{y_1} \cap L, \dots, V_{y_n} \cap L\}$. We form disjoint open sets

$$U \coloneqq \bigcap_{j=1}^{n} U_{y_j} \text{ and } V \coloneqq \bigcup_{j=1}^{n} V_{y_j}$$

such that $U \supseteq K$ and $V \supseteq L$. □

Proposition 3.2.10. *A topological space X is Hausdorff if and only if, for any point $x \in X$, the intersection I_x of all* closed neighbourhoods *– i.e., all closed subsets of X that contain an open neighbourhood of x – of x is the singleton $\{x\}$.*

Proof. Assume that X is Hausdorff, and let $x \in X$. Surely $x \in I_x$, and we claim that any point $y \in X \smallsetminus \{x\}$ is not in I. Indeed, for any such point, one may find disjoint open neighbourhoods U of x and V of y, whence $X \smallsetminus V$ is a closed neighbourhood of x not containing y. Thus $I = \{x\}$.

Conversely, suppose that, for any point $x \in X$, one has $I_x = \{x\}$. Suppose x and y two distinct points of X. Then since $I_x = \{x\}$, there exists a closed neighbourhood W of x not containing y, and now the interior ιW and the complement $X \smallsetminus W$ are disjoint open neighbourhoods of x and y, respectively. □

Proposition 3.2.11. *A topological space X is Hausdorff if and only if the diagonal*

$$\Delta_X \coloneqq \{(x, x) \in X \times X\}$$

is a closed subspace of $X \times X$.

Proof. The key point is that if $U, V \subseteq X$ are subsets, then $U \cap V = \varnothing$ if and only if $(U \times V) \cap \Delta_X = \varnothing$.

Assume that X is Hausdorff. Let $(x, y) \in (X \times X) \smallsetminus \Delta_X$. Choose an open neighborhood U of x and V of y such that $U \cap V = \varnothing$. Thus $U \times V$ is an open neighborhood of (x, y) in $(X \times X) \smallsetminus \Delta_X$.

Conversely, assume that Δ_X is closed in $X \times X$. Now let $x, y \in X$ be points such that $x \neq y$. There exist open neighborhoods U of x and V of y such that $(U \times V) \subseteq (X \times X) \smallsetminus \Delta_X$. It follows that $U \cap V = \varnothing$. □

Corollary 3.2.12. *If X and Y are topological spaces, and Y is Hausdorff, then for any continuous map $f : X \to Y$, the* graph

$$\Gamma_f \coloneqq \{(x, y) \in X \times Y : f(x) = y\}$$

is closed in $X \times Y$.

Proof. This follows from the fact that $\Gamma_f = (f \times \mathrm{id})^{-1}(\Delta_Y)$. □

3.3 Ultrafilters

3.4 Stone topological spaces

3.5 Spectral topological spaces

3.6 Local compactness

3.7 Function spaces, redux

3.8 Paracompactness

4 SEPARATION

5 COUNTABILITY & METRISATION

6 GENERATING TOPOLOGICAL SPACES

7 SHEAVES

8 MANIFOLDS

A ELEMENTS OF SET THEORY

Mathematicians treat the concepts of *set* and *element* as *undefined primitives*. Rules (in the form of axioms and axiom schemata) are provided for the manipulation of these objects. These extend the rules of first-order predicate calculus. In most foundational schemes (including the one presented below), absolutely every mathematical object – every number, every polynomial, every element of every set – is a set. So when we write $x \in X$, both x and X are sets.

The student who wants to go very deep into the subject of set theory should consult the astonishing text of Jech[**Jech:2003tt**]. The student who would prefer to work up to Jech's text should begin with Halmos's text[**MR0453532**].[1] This course won't require any set theory beyond Halmos's book, but because general topology and set theory interact in various nontrivial ways, it would be intellectually dishonest not to give at least a quick overview of some the basic elements of set theory.

[1] Pleasant though Halmos's text is, it should be noted its attitude toward set theory is at times unfairly dismissive of the subject.

A.1 Sets and elements

The first thing you have to know about sets is that a set X is equal to a set Y if and only if X and Y have the same elements. That is, $X = Y$ if and only if for every A, one has $A \in X$ if and only if $A \in Y$.

The easiest set in the world is the *empty set* $\varnothing$. It has no elements. The sentence $x \in \varnothing$ is *always* false, no matter what x is. That means that any universally quantified sentence over the empty set – *i.e.*, $(\forall x \in \varnothing)(\phi(x))$ – is *true*, and any existentially quantified sentence over the empty set – *i.e.*, $(\exists x \in \varnothing)(\phi(x))$ – is *false*.[2]

[2] Be sure you see why this is true!

The next easiest sets in the world are *singletons*. A singleton is a set with exactly one element, $\{X\}$. Don't forget that that X has to be a set. The axioms of set theory let you take any set X and build the singleton $\{X\}$. More generally, if we have a pair of sets X, Y, we're permitted to form the set $\{X, Y\}$.

Example A.1.1. The sets $\varnothing$ and $\{\varnothing\}$ are unequal.

A.2 Bounded comprehension

We also want to be able to carve out pieces of our sets defined by suitable formulas. So if X is a set, and $\phi(x)$ is some formula of set theory (any sentence of predicate calculus along with $\in$ in which the only free variable is x), then the axioms of set theory allow us to form a set[3]

$$A = \{x \in X : \phi(x)\} \ .$$

[3] This bit of notation is sometimes called *set-builder* notation. The axioms say, in effect, that this notation actually means something, as long as the X that appears on the left side of the colon is known to be a set.

Thus A is the set whose elements are all and only those elements $x \in X$ such that $\phi(x)$ obtains.

Example A.2.1. It's important that sets defined by formulas are carved out of existing sets. This is called ***bounded comprehension***. With an *unbounded* comprehension axiom, we would be able to build the following:

$$R \coloneqq \{X : \neg(X \in X)\} .$$

You may have seen this: this R creates some challenges, since $R \in R$ if and only if $R \notin R$. This is the example that Bertrand Russell cooked up, just to ruin Gottlob Frege's day.

A.3 Unions

For any set X, we also permit ourselves to form the ***union*** $\cup X$. This is the set whose elements are the elements of the elements of X; that is, $A \in \cup X$ if and only if there exists an element $S \in X$ such that $A \in S$. If $X = \{A, B\}$, then we write[4] $A \cup B$ for $\cup X$.

[4] This $\cup X$ notation may be unfamiliar to you. You might be happier with something like $\bigcup_{S \in X} S$, which means the same thing. The $\cup X$ notation is standard in set theory, however.

Example A.3.1. We can start building the ***finite von Neumann ordinals*** according to the following recipe: first, $0 \coloneqq \varnothing$. Then, for every von Neumann ordinal n, one can create its ***successor von Neumann ordinal***

$$n + 1 \coloneqq n \cup \{n\} .$$

So the first few von Neumann ordinals look like this:

$$\begin{aligned}
0 &\coloneqq \varnothing; \\
1 &\coloneqq \{0\} = \{\varnothing\}; \\
2 &\coloneqq \{0, 1\} = \{\varnothing, \{\varnothing\}\}; \\
3 &\coloneqq \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}; \\
4 &\coloneqq \{0, 1, 2, 3\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}; \\
&\textit{etc.}
\end{aligned}$$

If n is a finite von Neumann ordinal, then for every $\alpha, \beta \in n$, exactly one of the following is the case:

$$\alpha \in \beta ; \qquad \alpha = \beta ; \ \ or \quad \alpha \ni \beta .$$

The axioms then let you build the first ***infinite von Neumann ordinal*** ω, which is the set of all the finite von Neumann ordinals. Once you have that, you can build the successor to ω:

$$\omega + 1 \coloneqq \omega \cup \{\omega\} .$$

Wait, isn't that like $\infty + 1$*? Isn't that* ∞ *again?* That sort of thing is true if you're talking about ***cardinals***. Here, we're talking about ***ordinals***, and the distinction is important. We'll get into this more when we talk about number systems in the next section. But we absolutely can construct

$$\omega + 1, \omega + 2, \dots , 2\omega = \omega + \omega ,$$

where $2\omega = \omega + \omega$ is the set

$$\{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\} .$$

Likewise, you can build $3\omega, 4\omega, \dots$, and then even $\omega^2, \omega^3, \dots$, and even ω^ω. The key point is that each ordinal is the set of all the ordinals smaller than it. We'll investigate this more deeply in the next section.

A KEY CONSTRUCTION for building 'big' sets is the *power set* construction. To explain, a ***subset*** of a set X is a set S such that for any $s \in S$, one has $s \in X$ as well; in this case, we write $S \subseteq X$. The axioms of set theory permit us to form the ***power set*** $\boldsymbol{P}(X)$, which is the set of all subsets of X, so that $S \in \boldsymbol{P}(X)$ if and only if $S \subseteq X$.

Example A.3.2. The set $\boldsymbol{P}(\varnothing)$ is $\{\varnothing\}$. The set $\boldsymbol{P}(\boldsymbol{P}(\varnothing))$ is $\{\varnothing, \{\varnothing\}\}$.

A.4 Ordered pairs

We can also create ***ordered pairs***. For sets X and Y, we write

$$\langle X, Y\rangle \coloneqq \{\{X\}, \{X, Y\}\} .$$

Now for any two sets X and Y, we define the ***product*** as the set of all ordered pairs:

$$X \times Y \coloneqq \left\{S \in \boldsymbol{P}(\boldsymbol{P}(X \cup Y)) : (\exists x \in X)(\exists y \in Y)(\langle x, y\rangle = S)\right\} .$$

We can also define an ***ordered triple*** by the rule

$$\langle X, Y, Z\rangle \coloneqq \langle\langle X, Y\rangle, Z\rangle.$$

We can keep going with this to build ordered quadruples and ordered quintuples, *etc.*, but once we have the concept of map up and running, we'll find more efficient and intuitive ways to talk about these things.

Exercise A.4.1. How many elements does the ordered pair $\langle x, x\rangle$ have? How about the ordered triple $\langle x, x, x\rangle$?

A.5 Maps

The real point of defining things as we have is so we can talk about ***maps*** as soon as possible. Maps – AKA ***set maps***, ***mappings***, ***functions*** – are an important notion in set theory. The idea is that a map $f : S \to T$ is a way of taking each element $s \in S$ and associating one and only one element $f(s) \in Y$ thereto. Importantly, though, a map $f : S \to T$ 'knows' its ***source*** S, its ***target*** T, and the method of associating elements of T with elements of S. Thus a map $f : S \to T$ is defined as an ordered triple $\langle S, T, \Gamma(f)\rangle$ in which $\Gamma(f) \subseteq S \times T$ is a subset with the property that for every element $s \in S$, there is a unique[5] element $f(x) \in Y$ such that $\langle x, f(x)\rangle \in \Gamma(f)$.

In practice, the way we describe maps is pretty relaxed. We typically identify the source S and the target T, and then we provide a *rule* for 'sending' elements of S to the associated elements of T. The idea is that for every

[5] The phrase 'there is a unique x such that $\phi(x)$' – sometimes written $(\exists! x)\ \phi(x)$ – is a helpful shorthand for the longer sentence

$$(\exists x)(\phi(x) \wedge ((\forall y)(\phi(y) \implies (x = y)))) .$$

You knew that, of course, but our point here is that it's a formula of predicate calculus, and so the phrase 'such-and-so is a map' is a formula of predicate calculus as well.

$s \in S$, we have to specify a unique $f(s) \in T$ 'attached' to s in the sense that $\langle s, f(s)\rangle \in \Gamma(f)$.

Sometimes, it's handy to use the following notation: we may define a map $f: S \to T$ by the *assignment*

$$s \mapsto [\text{some formula involving } s]\,.$$

Example A.5.1. For instance, we can specify a map $f: \omega \to \{0,1\}$ by saying that $f(0) = 0$ and for any successor $n+1$, we let $f(n+1)$ be the unique element of $\{0,1\}$ such that $f(n+1) \neq f(n)$. Since we can check that $f(n+1)$ is indeed unique with this property, we are assured that f really is a map.

Example A.5.2. Suppose $S \subseteq X$. There is a map $i: S \to X$ such that $i(s) = s$, called the ***inclusion map***. When $S = X$, this is called the ***identity map*** id.

If we specify two sets S and T, then we can write $\mathrm{Map}(S,T) \subseteq \boldsymbol{P}(S \times T)$ consisting of those subsets $\Gamma \subseteq S \times T$ such that $\langle S, T, \Gamma\rangle$ is a map. We can write this as:

$$\mathrm{Map}(S,T) \coloneqq \{\Gamma \subseteq \boldsymbol{P}(S \times T) : (\forall s \in S)(\exists! t \in T)(\langle s,t\rangle \in \Gamma)\}\,.$$

this is our set of maps from S to T.

Exercise A.5.3. Suppose X a set (possibly empty). The set $\mathrm{Map}(\varnothing, X)$ always consists of exactly one element: the inclusion map $\varnothing \to X$. The point here is that $\varnothing \times X = \varnothing$, so $\boldsymbol{P}(\varnothing \times X) = \{\varnothing\}$. The unique subset $\varnothing \subseteq \varnothing$ is a map $\varnothing \to X$: indeed, the condition is:

$$(\forall s \in \varnothing)(\exists! t \in X)(\langle s,t\rangle \in \varnothing)\,,$$

which is always true. The set $\mathrm{Map}(X, \varnothing)$ is often, but not always, empty. Once again, we are looking at the unique subset $\varnothing \subseteq X \times \varnothing = \varnothing$, and the condition is:

$$(\forall s \in X)(\exists t \in \varnothing)(\langle s,t\rangle \in \varnothing)\,,$$

which is true if $X = \varnothing$; otherwise it is false.

Maps can be ***composed***. If $f: S \to T$ is a map, and $g: T \to U$ is another, then you can 'do f first, and then do g'. That is, one can form a new map $g \circ f: S \to U$ such that for any $s \in S$, one has

$$(g \circ f)(s) \coloneqq g(f(s))\,.$$

More formally, f and g are triples $\langle S, T, \Gamma(f)\rangle$ and $\langle T, U, \Gamma(g)\rangle$ (respectively), and $g \circ f$ is the triple $\langle S, U, \Gamma(g \circ f)\rangle$, where

$$\Gamma(g \circ f) \coloneqq \{(s,u) \in S \times U : (\exists t \in T)((s,t) \in \Gamma(f) \wedge (t,u) \in \Gamma(g))\}\,.$$

It is easy enough to see that $g \circ \mathrm{id} = g$ and $\mathrm{id} \circ f = f$, and moreover composition is associative, so that $(h \circ g) \circ f = h \circ (g \circ f)$.

A.6 Bijections

One important kind of map is the ***bijection***. When mathematicians are presented with a set, for many purposes, they won't be too very worried about what the elements are, but what structures they have.

The idea is that a bijection of sets is meant to be a 'mere labelling' of the elements of a set S by the elements of a set T. That labelling is meant to be a perfect match of information: you should never use the same label twice, and all the labels should be used. So a ***bijection*** of sets is a map $f: S \to T$ such that for any $t \in T$, there exists a unique $s \in S$ such that $f(s) = t$.

Example A.6.1. Let S and T be two sets. There is a bijection $\sigma: S \times T \to T \times S$, which is given by $\sigma(\langle s, t\rangle) = \langle t, s\rangle$.

If $f: S \to T$ is a bijection, then there exists a map $g: T \to S$ such that $g \circ f = \mathrm{id}$, and $f \circ g = \mathrm{id}$. To prove this, let us construct g: the function f gives us a subset $\Gamma(f) \subseteq S \times T$ such that for any $s \in S$, there exists a unique $t \in T$ such that $\langle s, t\rangle \in \Gamma(f)$. So now let's define $g = \langle T, S, \Gamma(g)\rangle$, where $\Gamma(g) = \{\langle t, s\rangle \in T \times S : \langle s, t\rangle \in \Gamma(f)\}$. Of course $\Gamma(g)$ makes perfect sense as a subset, but we aren't done: we have to show it is a map from S to T. For this we can use the fact that, since f is a bigjection, for every $t \in T$, there exists a unique $s \in S$ such that $\langle s, t\rangle \in \Gamma(f)$. In other words, for every $t \in T$, there exists a unique $s \in S$ such that $\langle t, s\rangle \in \Gamma(g)$. Thus if $t \in T$, then $g(t) \in S$ is the unique element such that $f(g(t) = t$. Thus $f \circ g = \mathrm{id}$. To see that $g \circ f = \mathrm{id}$, let $s \in S$; then $g(f(s)) \in S$ is the unique element such that $f(g(f(s))) = f(s)$. But since this element of S is *unique* with this property, it follows that $g(f(s)) = s$.

The converse is also correct: if $f: S \to T$ is a map such that there exists a function $g: T \to S$ such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$, then f is a bijection. Indeed, let $t \in T$ be an element; we aim to prove that there exists a unique element $s \in S$ such that $t = f(s)$. The function g provides us with exactly such an element: $g(t) \in T$ is an element, and $t = f(g(t))$. Now suppose that $s' \in S$ is an element such that $t = f(s')$; we see that $g(t) = g(f(s')) = s'$, so we have the uniqueness we sought!

In this case, we say that g is the *inverse* of f, and we sometimes write f^{-1} for g.

Example A.6.2. Let S and T be two sets. Assume that $f: S \to T$ is a bijection between them. Now let U be another set. We can define a map $F: \mathrm{Map}(T, U) \to \mathrm{Map}(S, U)$ by the assignment $\alpha \mapsto \alpha \circ f$. Let's see that this is a bijection. If $g = f^{-1}: T \to S$ is the inverse to f, then we can define a map $G: \mathrm{Map}(S, U) \to \mathrm{Map}(T, U)$ by the assignment $\beta \mapsto \beta \circ g$. Now $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

A.7 Products and sets of maps

Here's a basic property that relates the product of sets and the set of maps. It's a bit of a tongue-twister, but it's worth it to unpack. Let S, T, and U be three sets. Then define a map

$$\phi: \mathrm{Map}(S \times T, U) \to \mathrm{Map}(S, \mathrm{Map}(T, U))$$

that carries an element $h \in \mathrm{Map}(S \times T, U)$ – that is, a map $h: S \times T \to U$ – to the element $\phi(h) \in \mathrm{Map}(S, \mathrm{Map}(T, U))$ – that is, the map $\phi(h): S \to \mathrm{Map}(T, U)$ – that carries an element $s \in S$ to the element $\phi(h)(s) \in Map(T, U)$ – that is, the map $\phi(h)(s): T \to U$ – that carries an element

$t \in T$ to the element

$$\phi(h)(s)(t) = h(\langle s, t\rangle) \in U\,.$$

Did you catch that? Let's say it differently: we're starting with a map $h\colon S \times T \to U$. We want to *get* a map $\phi(h)\colon S \to \mathrm{Map}(T, U)$. To describe *that*, we start with an element $s \in S$, and we want to *get* a map $\phi(h)(s)\colon T \to U$. To define that, we start with an element $t \in T$, and we want to *get* an element of U; that element is $h(\langle s, t\rangle)$. Some times the map $\phi(h)(s)\colon T \to U$ is written $h(\langle s, -\rangle)$, where the second position is treated as a blank where we can fill in $t \in T$.

Let's go the other way, and define a map

$$\psi\colon \mathrm{Map}(S, \mathrm{Map}(T, U)) \to \mathrm{Map}(S \times T, U)\,.$$

For this, we're starting with a map $k\colon S \to \mathrm{Map}(T, U)$, and we want to define a map $\psi(k)\colon S \times T \to U$. This is defined by

$$\psi(k)(\langle s, t\rangle) \coloneqq k(s)(t)\,.$$

Now if you inspect thes formulas carefully, you'll see that in fact $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. In other words, ϕ is a bijection, and ψ is its inverse. In still other words, the sets $\mathrm{Map}(S \times T, U)$ and $\mathrm{Map}(S, \mathrm{Map}(T, U))$ are the 'same', up to relabelling.

A.8 Injections and surjections

The condition to be a bijection is really the conjunction of two conditions: first, that we never use the same label twice, and second, that every label be used. Let's give names to these conditions. An ***injection*** is a map $f\colon S \to T$ such that for any $t \in T$, there is *at most one*[6] $s \in S$ such that $f(s) = t$. A ***surjection*** is a map $f\colon S \to T$ such that for any $t \in T$, there is (*at least one*) $s \in S$ such that $f(s) = t$. Of course, a bijection is a map that is both an injection and a surjection.

[6] The phrase 'there is at most one x such that $\phi(x)$' is a helpful shorthand for the longer sentence

$$(\forall x)(\forall y)((\phi(x) \wedge \phi(y)) \implies (x = y))\,.$$

This is a pretty annoying turn of language, one has to admit: we're actually using the phrase 'there is' in a sentence in which the only quantifier is universal.

Here's an important axiom that we will use in a nontrivial way a couple of times in this class. It's called the *Axiom of Choice*. It says that if you have a surjection $f\colon S \to T$, then there exists a map $s\colon T \to S$ such that $f \circ s = \mathrm{id}$. Note that we are *not* saying that $s \circ f = \mathrm{id}$ as well; that would imply that f is a bijection, which isn't always true.

The map s is not usually an inverse to f, and we do not usually use the notation f^{-1} for s. Rather it is what we call a ***section*** of f. Thus the Axiom of Choice says that every surjection has a section.

A.9 The algebra of subsets

The algebra of subsets of a set permits one to perform unions, intersections, and complements, and these satisfy certain rules.

So let X be a set, and let $U\colon A \to \boldsymbol{P}(X)$ be a map. Then there are two new subsets of X that can be constructed: the ***indexed union***

$$\bigcup_{a \in A} U(a) \coloneqq \{x \in X : (\exists a \in A)\ x \in U(a)\}$$

and the *indexed intersection*

$$\bigcap_{a\in A} U(a) \coloneqq \{x \in X : (\forall a \in A)\ x \in U(a)\}.$$

Example A.9.1. There is only one map $I\colon\ \varnothing\boldsymbol{P}(X)$. The indexed union

$$\bigcup_{a\in\varnothing} I(a) = \varnothing\,.$$

The indexed intersection

$$\bigcap_{a\in\varnothing} I(a) = X\,.$$

Let's see what happens when you repeat these operations or mix them. Let A and B be sets, and let $U\colon\ A\times B \to \boldsymbol{P}(X)$. We're going to exploit some bijections now: we know that maps $A\times B \to \boldsymbol{P}(X)$ are in bijection with maps $A \to \mathrm{Map}(B, \boldsymbol{P}(X))$; we also know that $A \times B$ is in bijection with $B \times A$, and therefore that maps $A\times B \to \boldsymbol{P}(X)$ are in bijection with maps $B\times A \to \boldsymbol{P}(X)$, which are in turn in bijection with maps $B \to \mathrm{Map}(A, \boldsymbol{P}(X))$. Here are the formulas:

$$\bigcup_{a\in A} \bigcup b \in BU(a,b) = \bigcup_{b\in B} \bigcup a \in AU(a,b)\,;$$

$$\bigcap_{a\in A} \bigcap b \in BU(a,b) = \bigcap_{b\in B} \bigcap a \in AU(a,b)\,;$$

$$\bigcap_{a\in A} \bigcup_{b\in B} U(a,b) = \bigcup_{f\in\mathrm{Map}(A,B)} \bigcap_{a\in A} U(a, f(a))\,;$$

$$\bigcup_{a\in A} \bigcap_{b\in B} U(a,b) = \bigcap_{f\in\mathrm{Map}(A,B)} \bigcup_{a\in A} U(a, f(a))\,.$$

There is also the *complement* of any $A \in \boldsymbol{P}(X)$

$$\complement A = X \smallsetminus A \coloneqq \{x \in X : x \notin A)\}\,.$$

The *de Morgan laws* state that the formation of the complement exchanges union and intersection: for any map $U\colon\ A \to \boldsymbol{P}(X)$,

$$\complement\left(\bigcup_{a\in A} U(a)\right) = \bigcap_{a\in A} \complement U(a)\,;$$

$$\complement\left(\bigcap_{a\in A} U(a)\right) = \bigcup_{a\in A} \complement U(a)\,.$$

A.10 Inverse and direct image

A map $f\colon\ X \to Y$ induces a map

$$f^*\colon\ \boldsymbol{P}(Y) \to \boldsymbol{P}(X)$$

called the *inverse image* and a map

$$f_*\colon\ \boldsymbol{P}(X) \to \boldsymbol{P}(Y)$$

called the *direct image*. The inverse image of a subset $B \subseteq Y$ is the set

$$f^*(B) \coloneqq \{x \in X : f(x) \in B\}\ ,$$

and the direct image of a subset $A \subseteq X$ is the set

$$f_*(A) := \{y \in Y : (\exists x \in A)(y = f(x))\} .$$

These operations are related in the following manner: one has $f_*(A) \subseteq B$ if and only if $A \subseteq f^*(B)$. In particular, we have

$$A \subseteq f^*(f_*(A)) \qquad \text{and} \qquad f_*(f^*(B)) \subseteq B .$$

In general, both of these containments are strict. However, if f is an injection, then $A = f^*(f_*(A))$, and if f is a surjection, then $f_*(f^*(B)) = B$.

In many respects, the inverse image is more natural than the direct image. For example, the inverse image preserves unions, intersections, and complements, so that one has:

$$f^*\left(\bigcup_{a\in A} U(a)\right) = \bigcup_{a\in A} f^*(U(a)) ;$$
$$f^*\left(\bigcap_{a\in A} U(a)\right) = \bigcap_{a\in A} f^*(U(a)) ;$$
$$f^*(\complement U) = \complement(f^*(U)) .$$

For the direct image, one only has

$$f_*\left(\bigcup_{a\in A} U(a)\right) = \bigcup_{a\in A} f_*(U(a)) ;$$
$$f_*\left(\bigcap_{a\in A} U(a)\right) \subseteq \bigcap_{a\in A} f_*(U(a)) .$$

There is no containment between $\complement f_*(A)$ and $f_*(\complement A)$ in general.

In the particular case where $B = \{y\}$, the inverse image $f^*\{y\}$ is called the ***fiber*** of f over y. This gives us a handy way to think about maps $f\colon X \to Y$: in effect, they organize X into a disjoint union of fibers. That is, if $y \neq y'$, then $f^*\{y\} \cap f^*\{y'\} = \varnothing$, and

$$X = \bigcup_{y\in Y} f^*\{y\} .$$

The map f is injective if and only if each fiber $f^*\{y\}$ has *at most one element*, and f is surjective if and only if each fiber $f^*\{y\}$ has *at least one element*.

Warning A.10.1. Now for the annoying news. The notations f^* and f_* are not the usual notations. The more typical notation for the inverse image of B is $f^{-1}(B)$. The more typical notation for the direct image of A is $f(A)$. This notation makes it look as though these operations are inverse – *in general they are not!!*

A.11 Products of sets

An ***indexed set*** is really just another name for a map[7] $U\colon A \to \Xi$; we typically abuse notation and write U_a instead of $U(a)$, and we write $(U_a)_{a\in A}$ for the map U. The union of the indexed set $(U_a)_{a\in A}$ will mean the union of the set $\{U_a : a \in A\}$:

$$\bigcup_{a\in A} U_a := \bigcup \{U_a : a \in A\}.$$

[7] Remember that every element of every set is itself a set; thus the elements of Ξ are sets as well!

It may seem a bit silly to belabour this point, but the purpose will, we hope, become clear.

For any indexed set $(U_a)_{a\in A}$, the ***product*** is the set

$$\prod_{a\in A} U_a \coloneqq \left\{ x \in \mathrm{Map}\left(A, \bigcup_{a\in A} U_a \right) : (\forall a \in A)\ x(a) \in U_a \right\}.$$

An element of $\prod_{a\in A} U_a$ is thus a map $x\colon A \to \bigcup_{a\in A} U_a$ such that for every $a \in A$, one has $x(a) \in U_a$. One usually writes x_a instead of $x(a)$, and one often writes $x = (x_a)_{a\in A}$.

Example A.11.1. When $A = \{1,2\}$, an indexed set consists of two sets U_1 and U_2. The product $\prod_{a\in\{1,2\}} U_a$ thus consists of pairs (x_1, x_2) with $x_1 \in U_1$ and $x_2 \in U_2$. The assignment $(x_1, x_2) \mapsto \langle x_1, x_2\rangle$ is a bijection $\prod_{a\in\{1,2\}} U_a \to U_1 \times U_2$. Most mathematicians are happy to pretend as if there is no difference between these sets; indeed, there is no *interesting* difference!

More generally, we think of the product $\prod_{a\in A} U_a$ as the set of ordered 'A-tuples'.

For every $b \in A$, there is an attached map

$$\pi_b\colon \prod_{a\in A} U_a \to U_b$$

given by the assignment $x \mapsto x_b$, called the ***projection*** onto the b-th factor.

For every set S, every indexed set $\{U_a\}_{a\in A}$, and every indexed set $\{f_a\colon S \to U_a\}$ of maps, there exists a unique map

$$f\colon S \to \prod_{a\in A} U_a$$

such that $\pi_a \circ f = f_a$. Indeed, the map f is given by the assignment $s \mapsto (f_a(x))_{a\in A}$.

A.12 Coproducts of sets

For any indexed set $\{U_a\}_{a\in A}$, the ***coproduct*** or ***disjoint union*** is the set

$$\coprod_{a\in A} U_a \coloneqq \bigcup_{a\in A} U_a \times \{a\}.$$

For every $b \in A$, there is an attached map

$$\iota_b\colon U_b \to \coprod_{a\in A} U_a$$

given by $\iota_b(x) = (x, b)$, called the ***inclusion*** onto the b-th summand.

The coproduct is really *dual* to the product. Here's how: for every set S, every indexed set $\{U_a\}_{a\in A}$, and every indexed set $\{f_a\colon U_a \to S\}$ of maps, there exists a unique map

$$f\colon \coprod_{a\in A} U_a \to S$$

such that $f \circ \iota_a = f_a$. Indeed, the map f is given by the assignment $(x, a) \mapsto f_a(x)$.