General topology

The Problems

Autumn 2020

Quotients

Notation. We're going to study an interesting way to construct some topological spaces. Choose an integer $n \geq 3$. We will start with the n-th roots of unity in $\boldsymbol{C}$:

$$\zeta_n^0, \zeta_n^1, \ldots, \zeta_n^{n-1} ,$$

where $\zeta_n \coloneqq \exp(2\pi i/n)$. Let $P \subset \boldsymbol{C}$ be the *convex hull* of these points:

$$P \coloneqq \{a_0\zeta_n^0 + \cdots + a_{n-1}\zeta_n^{n-1} \in \boldsymbol{C} : (\forall i)((a_i \in \boldsymbol{R}_{\geq 0}) \wedge (a_1 + \cdots + a_n = 1))\} .$$

This is a (filled) regular n-gon inscribed in the unit circle in $\boldsymbol{C}$. We will also consider the *interior* $\imath P$, which, recall, is the largest open subset of $\boldsymbol{C}$ that is contained in P.

A *gluing datum* (ϕ, p) consists of an *involution*[1] ϕ on $\{0, \ldots, n-1\}$ along with a map $a\colon \{0, \ldots, n-1\} \to \{+, -\}$ such that for every i, one has $a(i) = a(\phi(i))$, and if $i = \phi(i)$, then $a(i) = +$.

[1] that is, a map $\phi\colon \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ such that $\phi^2 = id$.

As a bit of notation, we'll write ϕ as a product of disjoint transpositions:[2]

[2] This is possible because ϕ is an involution!

$$(j_1\ j_2)(j_3\ j_4) \cdots (j_{k-1}\ j_k) ,$$

and we'll add the information of the map a by decorating each transposition $(j_{m-1}\ j_m)$ with a plus or minus, according to whether $a(i_m)$ is $+$ or $-$. Thus if $n = 5$, for example,

$$(0\ 3)^+(2\ 4)^-$$

represents the gluing datum in which $\phi = (0\ 3)(2\ 4)$ and $a(0) = a(1) = a(3) = +$, and $a(2) = a(4) = -$.

We'll consider the equivalence relation $\sim$ on P generated by the following requirements:

- For every j such that $a(j) = +$, and for every $t \in [0, 1]$, we require that

$$t\zeta_n^j + (1-t)\zeta_n^{j+1} \sim t\zeta_n^{\phi(j)} + (1-t)\zeta_n^{\phi(j)+1} .$$

- For every j such that $a(j) = -$, and for every $t \in [0, 1]$, we require that

$$t\zeta_n^j + (1-t)\zeta_n^{j+1} \sim (1-t)\zeta_n^{\phi(j)} + t\zeta_n^{\phi(j)+1} .$$

We can then form the quotient topological space[3]

$$\Sigma \coloneqq P/\sim ,$$

which depends on n, ϕ, and a.

[3] On the final page of this PDF, we have a drawing of the identifications made by $\sim$ when $n = 4$ and the gluing datum is $(0\ 2)^-(1\ 3)^+$.

Problem 1

Let $n = 3$. Describe all the different possibilities for gluing data, and describe all the different topological spaces Σ that result from these choices. You need not justify your answer.

Problem 2

Let $n = 4$. In the following examples, indicate whether the topological spaces Σ are homeomorphic to a topological space we have already seen.

- $(0\ 2)^-$
- $(0\ 2)^+$
- $(0\ 2)^-(1\ 3)^-$
- $(0\ 2)^-(1\ 3)^+$
- $(0\ 2)^+(1\ 3)^+$

Again, you need not justify your answer.

Problem 3

Let $n = 2m$, and consider the gluing datum given by

$$(0 \quad m)^-(1 \quad m+1)^- \cdots (m-1 \quad 2m-1)^- .$$

What does the resulting topological space Σ look like? Once again, you need not justify your answer.

Notation. Let $n \in \boldsymbol{N}^*$. Recall that $\boldsymbol{P}^n_{\boldsymbol{R}}$ is the set of 1-dimensional $\boldsymbol{R}$-linear subspaces of $\boldsymbol{R}^{n+1}$; we call these subspaces *(real) lines*. Define a map

$$q\colon \boldsymbol{R}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n_{\boldsymbol{R}}$$

that carries any $\boldsymbol{x} = (x_0, \ldots, x_n) \in \boldsymbol{R}^{n+1} \smallsetminus \{0\}$ to the line $[\boldsymbol{x}] = [x_0, \ldots, x_n] \in \boldsymbol{P}^n_{\boldsymbol{R}}$ spanned by $\boldsymbol{x}$. Endow $\boldsymbol{P}^n_{\boldsymbol{R}}$ with the finest topology such that q is continuous.

Equivalently, $\boldsymbol{P}^n_{\boldsymbol{R}}$ is the quotient space $\left(\boldsymbol{R}^{n+1} \smallsetminus \{0\}\right)/\!\sim$, where we declare $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if there exists a nonzero real number λ such that $\boldsymbol{x} = \lambda \boldsymbol{y}$.

Let $\boldsymbol{P}^n$ denote the set of 1-dimensional $\boldsymbol{C}$-linear subspaces of $\boldsymbol{C}^{n+1}$; we call these subspaces *complex lines*.[4] Define a map

$$q\colon \boldsymbol{C}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n$$

that carries any $\boldsymbol{w} = (w_0, \ldots, w_n) \in \boldsymbol{C}^{n+1} \smallsetminus \{0\}$ to the line $[\boldsymbol{w}] = [w_0, \ldots, w_n] \in \boldsymbol{P}^n$ spanned by $\boldsymbol{w}$. Endow $\boldsymbol{P}^n$ with the finest topology such that q is continuous.

Equivalently, $\boldsymbol{P}^n$ is the quotient space $\left(\boldsymbol{C}^{n+1} \smallsetminus \{0\}\right)/\!\sim$, where we declare $\boldsymbol{w} \sim \boldsymbol{z}$ if and only if there exists a nonzero complex number λ such that $\boldsymbol{w} = \lambda \boldsymbol{z}$.

[4] These subspaces have 1 complex dimension but two real dimensions.

Problem 4

Construct a homeomorphism between $\boldsymbol{P}^2_{\boldsymbol{R}}$ and the quotient space Σ formed as described above by selecting $n = 4$ and gluing data

$$(0\ 2)^+(1\ 3)^+ .$$

Problem 5

Prove that $\boldsymbol{P}^1$ and S^2 are homeomorphic.

Problem 6

Prove that $\boldsymbol{P}_{\boldsymbol{R}}^1$ and $\boldsymbol{P}_{\boldsymbol{R}}^2$ are not homeomorphic.

Problem 7

Find an open subset $U \subset \boldsymbol{P}^n$ such that U is homeomorphic to $\boldsymbol{C}^n$ and such that the complement $\boldsymbol{P}^n \smallsetminus U$ is homeomorphic to $\boldsymbol{P}^{n-1}$.

Problem 8

Identify S^{2n+1} with the subspace

$$\left\{\boldsymbol{w} \in \boldsymbol{C}^{n+1} : \|\boldsymbol{w}\| = 1\right\} \subset \boldsymbol{C}^{n+1} \smallsetminus \{0\} .$$

Write h for the composite map $S^{2n+1} \subset \boldsymbol{C}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n$. Prove that h is surjective, and identify the fibres $h^{-1}\{x\}$, up to homeomorphism.