1. Let X be a finite set and let τ be a topology on X. In this exercise we will show that there is a minimal basis for the topology τ. That is, there is a basis $\mathcal{B}_{\text{min}}$ of τ such that if $\mathcal{B}$ is any other basis for τ, then $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$.

   (a) If $x \in X$, let U_x be the intersection of all open sets that contain x. Explain why U_x is an open set.

   (b) Let $\mathcal{B}_{\text{min}} = \{U_x \mid x \in X\}$. Show that $\mathcal{B}_{\text{min}}$ is a basis for τ.

   (c) Show that if $\mathcal{B}$ is a basis for τ, then $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$.

   (d) Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$. You may assume that τ is a topology on X. Find the unique minimal basis for τ.

   Solution.

   (a) Let $x \in X$. Since X is finite, there are only finitely many open sets that contain x. The fact that the intersection of a finite number of open sets is open implies that U_x is an open set.

   (b) If $x \in X$, then $x \in U_x$. Now let $B_1 = U_x$ and $B_2 = U_y$ for some x, y in X, and suppose $z \in B_1 \cap B_2$. Then $B_1 \cap B_2$ is an open set that contains z and so $U_z \subseteq B_1 \cap B_2$. So $\mathcal{B}_{\text{min}}$ is a basis for τ.

   (c) Let $\mathcal{B}$ be a basis for τ. To show that $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$, let U_x be an element of $\mathcal{B}_{\text{min}}$ we will show that $U_x \in \mathcal{B}$. The fact that $U_x \in \tau$ means that U_x is a union of basis elements from $\mathcal{B}$. That is, there is a set $U \in \mathcal{B}$ such that $x \in U \subseteq U_x$. But U_x is the intersection of all open sets that contain x, so it follows that $C = U_x$ and $U_x \in \mathcal{B}$.

   (d) The minimal basis for τ is the set $\{U_a, U_b, U_c, U_d\}$. Since $\{a\}$ is in τ, it follows that $U_a = \{a\}$. Also

$$
\begin{aligned}
U_b &= \{a, b\} \cap \{a, b, c\} \cap \{a, b, d\} \cap X = \{a, b\} \\
U_c &= \{a, c\} \cap \{a, b, c\} \cap \{a, c, d\} \cap X = \{a, c\} \\
U_d &= \{a, d\} \cap \{a, b, d\} \cap \{a, c, d\} \cap X = \{a, d\}.
\end{aligned}
$$

   So the minimal basis for τ is $\{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}\}$.

2. You may wonder why we can't define a basis for a topology on a set X to be any collection of subsets whose union is X. Consider the example of $X = \{a, b, c\}$ and $S = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$.

   (a) Determine the collection of all of the unions of elements of S.

   (b) Explain why the collection of unions of the elements of S, along with the empty set, is not a topology on X. What property of a basis is not satisfied?

3. Let a and b be integers with $b \neq 0$. Let $A_{a,b} = a\mathbb{Z} + b = \{a + kb \mid k \in \mathbb{Z}\}$.

   (a) Show that $\{A_{a,b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ is a basis for a topology τ on $\mathbb{Z}$. (Hint: If $B_1 = A_{a_1,b_1}$ and $B_2 = A_{a_2,b_2}$, and if $x \in B_1 \cap B_2$, what can we say about $A_{x,b_1 b_2}$?)

   (b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n + (-1)^n$.

      i. Prove that f is a bijection.

      ii. If O is an open set in $\mathbb{Z}$, is $f(O)$ an open set?

      iii. If U is an open set in $\mathbb{Z}$, is $f^{-1}(U)$ an open set? (Hint: What is f^{-1}?)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$, with the Euclidean metric on both the domain and the codomain. Is f continuous at $x = 0$? Prove your answer.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$ Is f continuous at $x = 0$? Prove your answer.

6. Let f and g be functions from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$.

   (a) Is it true that if $f + g$ is a continuous function, then f and g are continuous functions? Verify your answer.

(b) Is it true that if fg is a continuous function, then f and g are continuous functions? Verify your answer.

7. Let $f(x) = 2x^2 + 1$ map from $\mathbb{R}$ to $\mathbb{R}$, with both the domain and codomain having the Euclidean metric.

   (a) Let $\epsilon = \frac{1}{4}$. Find a value of δ such that $|x - 1| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. You might use the applet at to confirm your value of δ.

   (b) Prove that f is continuous at $x = 1$.

8. Let f and g be continuous functions from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$. In this exercise we will prove that fg is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let a be in $\mathbb{R}$, and follow the steps below to show that fg is continuous at $x = a$. Let ϵ be a positive number.

   (a) We will first want to express $f(x)g(x) - f(a)g(a)$ in a more useful way. Use the fact that $f(x) = f(a) + (f(x) - f(a))$ and $g(x) = g(a) + (g(x) - g(a))$ to show that

   $$f(x)g(x) - f(a)g(a) = f(a)(g(x) - g(a)) + g(a)(f(x) - f(a)) + (f(x) - f(a))(g(x) - g(a)).$$

   (b) Explain why there exist positive numbers δ_1, δ_2, δ_3, and δ_4 such that

   $$\begin{aligned}
   |f(x) - f(a)| &< \sqrt{\frac{\epsilon}{3}} \text{ when } |x - a| < \delta_1 \\
   |g(x) - g(a)| &< \sqrt{\frac{\epsilon}{3}} \text{ when } |x - a| < \delta_2 \\
   |f(x) - f(a)| &< \frac{\epsilon}{3(1 + |g(a)|)} \text{ when } |x - a| < \delta_3 \\
   |g(x) - g(a)| &< \frac{\epsilon}{3(1 + |f(a)|)} \text{ when } |x - a| < \delta_4.
   \end{aligned}$$

   (c) Use the results of (a) and (b) to show that fg is continuous at $x = a$. (Hint: $1 + |f(a)| > |f(a)|$.)

9. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Prove that $(g \circ f)$ is a continuous function from X to Z.

10. Let $X = \{a, b, c\}$, and let $\tau_1 = \{\emptyset, \{a\}, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Both τ_1 and τ_2 are topologies in X, but every element in τ_1 is also an element in τ_2. Then this happens we say that τ_1 is a weaker topology than τ_2. More formally,

    Definition 1. Let τ_1 and τ_2 be two topologies on a set X. If $\tau_1 \subseteq \tau_2$, then τ_1 is a **coarser** (or **weaker**) topology than τ_2. We also say that τ_2 is a **finer** (or **stronger**) topology than τ_1.

    (a) What is the weakest topology on any set?

    (b) What is the strongest topology on any set?

    (c) Let $X = \{a, b, c\}$. Are there any topologies γ on X such that γ is not the indiscrete topology but there are no weaker topologies on X other than the indiscrete topology? Explain.

    (d) Below is a list of 9 distinct topologies on $X = \{a, b, c\}$. Each topology lies in one or more sequences of topologies ordered by coarseness. For each topology τ, list the longest sequence(s) of topologies that start $\{\emptyset, X\} \subset \tau$, ordered by coarseness.

    1. $\{\emptyset, X\}$
    2. $\{\emptyset, \{a\}, X\}$
    3. $\{\emptyset, \{a, b\}, X\}$
    4. $\{\emptyset, \{a\}, \{a, b\}, X\}$
    5. $\{\emptyset, \{a\}, \{b, c\}, X\}$
    6. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
    7. $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$
    8. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$
    9. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$
    10. the discrete topology

11. Let (X, d) be a metric space, let A be a subset of X, and let $x \in X$. In this exercise we will prove that x is a limit point of A if and only if there is a sequence (a_n) in A that converges to x.

    (a) First show that if $x \in X$ is a limit point of A, then there is a sequence (a_n) in A that converges to x. (Hint: For $n \in \mathbb{Z}^+$, consider the neighborhood $B\left(x, \frac{1}{n}\right)$.)

    (b) Now prove that if there is a sequence (a_n) in X that converges to x, then x is a limit point of A. (Hint: What is true about the sequence $d(a_n, x)$?)