MATH 240 EXERCISES

Contents

1.1-1.2: linear systems 1
1.3-1.4: matrix arithmetic 5
1.5-1.6: inverses, diagonal and symmetric matrices 7
2.1-2.3: the determinant 11
3.1: vector spaces 13
3.2: linear transformations 15
3.3: subspaces 15
3.4-3.5: linear independence and bases 17
3.6: dimension 19
4.1-4.2: linear transformations, bases, rank-nullity 21
4.3: coordinate vectors 24
4.4: matrix representations 25
4.5: change of basis 26
4.6: eigenvectors 29
4.7: diagonalization 30
5.1: inner products 33
5.2: orthogonal bases and projection 35

1.1-1.2: linear systems

(1) Recall that the set of solutions (x, y) to a single linear equation in 2 variables constitutes a line ℓ in $\mathbb{R}^2$. We denote this $\ell\colon ax + by = c$.
Similarly, the set of solutions (x, y, z) to a single linear equation in 3 variables $ax+by+cz = d$ constitutes a plane $\mathcal{P}$ in $\mathbb{R}^3$. We denote this $\mathcal{P}\colon ax + by + cz = d$.
  (a) Fix $m > 1$ and consider a system of m linear equations in the 2 unknowns x and y. What do solutions (x, y) to this *system* of linear equations correspond to geometrically?
  (b) Use your interpretation to give a *geometric* argument that a system of m equations in 2 unknowns will have either (i) 0 solutions, (ii) 1 solution, or (iii) infinitely many solutions.
  (c) Use your geometric interpretation to help produce explicit examples of systems in 2 variables satisfying these three different cases (i)-(iii).
  (d) Now repeat (a)-(b) for systems of linear equations in 3 variables x, y, z.

(2) We made the claim that each of our three row operations
  i scalar multiplication ($e_i \mapsto c \cdot e_i$ for $c \neq 0$),
  ii swap ($e_i \leftrightarrow e_j$),
  iii addition ($e_i \mapsto e_i + c \cdot e_j$ for some c)

do not change the set of solutions of a linear system.

To prove this claim, let L be a general linear system

$$\begin{array}{rccccccc} e_1: & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 \\ e_2: & a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & \vdots & \\ e_m: & a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & = & b_m \end{array}.$$

Now consider each type of row operation separately, write down the new system L' you get by applying this row operation, and prove that an n-tuple $s = (s_1, s_2, \dots, s_n)$ is a solution to the original system L $\boxed{\text{if and only if}}$ it is a solution to the new system L'.

(3) Consider the linear system

$$\begin{aligned} 7x_5 &= 2x_3 + 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1 \end{aligned}$$

Using augmented matrices, try and solve this system by reducing it to a simpler one using row operations. How many solutions (0, 1, or ∞) does the system have?

(4) Compute the set of solutions S to the following system of linear equations:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 1 \\ -2x_1 - 2x_2 + 2x_3 - 2x_4 &= -2 \\ x_1 + x_2 + x_3 + 2x_4 &= 3 \end{aligned}$$

Follow the Gaussian elimination algorithm *to the letter*, indicating what row operations are used. Make sure to indicate how the x_i are sorted into free and leading variables.

Your final answer should be described in set notation.

(5) Determine all values of a for which the system has (a) no solutions, (b) exactly one solution, or (c) infinitely-many solutions.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

(6) Use Gaussian elimination to find the general solution to the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= x_3 + x_4 + 3 \\ 3x_1 + 6x_2 &= 2x_3 - 4x_4 + 8 \\ -x_1 + 2x_3 &= 2x_2 - x_4 - 1 \end{aligned}$$

Follow the Gaussian elimination algorithm to the letter, indicating what row operations are used. Make sure to indicate how the x_i are sorted into free and leading variables.

(7) Show that a linear system with more unknowns than equations has has either 0 solutions or infinitely many solutions.

(8) True or false. If true you must prove it so; if false, you must give an explicit counterexample.
 (a) Every matrix has a unique row echelon form.

(b) Any homogeneous linear system with more unknowns than equations has infinitely many solutions.

(c) The only solution to a homogeneous system of n equations in n unknowns with n leading 1's is the trivial solution $s = (0, 0, \dots, 0)$.

(9) Use Gaussian elimination to solve the following system:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

(10) Solve by Gaussian elimination:

$$\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

(11) Solve the following system using Gauss-Jordan elimination: that is, first reduce the corresponding augmented matrix to reduced row echelon form.

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

(12) Solve the following system of equations by any method:

$$\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \end{aligned}$$

(13) Interpret each matrix below as an augmented matrix of a linear system. Asterisks represent an unspecified real number. For each matrix, determine whether the corresponding system is consistent or inconsistent.

If the system is consistent, decide further whether the solution is unique or not.

If there is not enough information answer 'inconclusive' and back up your claim by giving an explicit example where the system is consistent, and an explicit example where the system is inconsistent.

(a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$

(d) $\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

(14) Determine the values of a for which the system below has no solutions, many solutions, or one solution.

$$\begin{aligned} x + 2y + \qquad\quad z &= 2 \\ 2x - 2y + \qquad\quad 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

(15) What condition must a,b, and c satisfy for the system to be consistent?

$$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

(16) Solve for x, y, and z:

$$\begin{aligned} \tfrac{1}{x} + \tfrac{2}{y} - \tfrac{4}{z} &= 1 \\ \tfrac{2}{x} + \tfrac{3}{y} + \tfrac{8}{z} &= 0 \\ -\tfrac{1}{x} + \tfrac{9}{y} + \tfrac{10}{z} &= 5 \end{aligned}$$

(17) If A is a matrix with three rows and five columns, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(18) If B is a matrix with three rows and six columns, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(19) If C is a matrix with five rows and three columns, then what is the minimum possible number of rows of zeros in any row echelon form of C?

(20) True or False. If true, prove it; if false, give an explicit counterexample.
(a) If a matrix is in reduced row echelon form, then it is also in row echelon form.
(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.
(c) Every matrix has a unique row echelon form.
(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.
(e) All leading 1's in a matrix in row echelon form must occur in different columns.
(f) If every column of a matrix in row echelon form has a leading 1, then all entries that are not leading 1's are zero.
(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.
(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.
(i) If a linear system has more unknowns than equations, then it must have infinitely many

solutions.

1.3-1.4: MATRIX ARITHMETIC

(1) Use the row or column method to quickly compute the following product:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(2) Each of the 3×3 matrices B_i below performs a specific row operation when multiplying a $3 \times n$ matrix $A = \begin{bmatrix} -\mathbf{r}_1- \\ -\mathbf{r}_2- \\ -\mathbf{r}_3- \end{bmatrix}$ on the left; i.e., the matrix B_iA is the result of performing a certain row operation on the matrix A.

Use the row method of matrix multiplication to decide what row operation each B_i represents.

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(3) For each part below write down the most general 3×3 matrix $A = [a_{ij}]$ satisfying the given condition (use letter names a, b, c,etc. for entries).
(a) $a_{ij} = a_{ji}$ for all i, j.
(b) $a_{ij} = -a_{ji}$ for all i, j
(c) $a_{ij} = 0$ for $i \neq j$.

(4) Let A be a square $n \times n$ matrix. We define its **square** to be the matrix $A^2 := AA$.

Given a square $n \times n$ matrix B, we define a **square-root** of B to be a matrix A such that $A^2 = B$.
(a) Find *all* square-roots of $\underset{2\times 2}{\mathbf{0}}$.
(b) Find *all* square-roots of I_2.

Your answers for both parts should be a useful description of the entries of an arbitrary square-root matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, ideally some sort of parametrization.

(5) Suppose A is $n \times n$ and satisfies $A^2 + 3A - 5I = \mathbf{0}_n$. Prove A is invertible by providing an explicit inverse.

(6) Show that in general $(AB)^r \neq A^rB^r$. Why does the argument from real number algebra fail?

(7) Show that $(A + B)^2 = A^2 + 2AB + B^2$ if and only if $AB = BA$.

(8) Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Compute the following matrices, or else explain why the given expression is not well defined.
(a) $(2D^T - E)A$
(b) $(4B)C + 2B$
(c) $B^T(CC^T - A^TA)$

(9) Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}.$$

Compute the following using either the row or column method of matrix multiplication:
(a) the first column of AB;
(b) the second row of BB;
(c) the third column of AA.

(10) Find all values of k, if any, that satisfy the equation:

$$\begin{bmatrix} 2 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$$

(11) Let $\mathbf{0}_{2\times 2}$ denote the 2×2 matrix, each of whose entries is zero.
(a) Is there a 2×2 matrix A such that $A \neq \mathbf{0}$ and $AA = \mathbf{0}$?
(b) Is there a 2×2 matrix A such that $A \neq \mathbf{0}$ and $AA = A$?

(12) Answer true or false. If true, provide a proof; if false, provide an explicit counterexample.
(a) If B has a column of zeros, then so does AB if this product is defined.
(b) If B has a column of zeros, then so does BA if this product is defined.
(c) If A, B, C are *nonzero* $n \times n$ matrices and $AC = BC$, then $A = B$. (Remember: a matrix is nonzero if it is not the zero matrix.)
(d) If $AB + BA$ is well-defined, then A and B are square matrices of the same size.

(13) Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$.
Compute A^3, A^{-3} and $A^2 - 2A + I$.

(14) Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. Compute $p(A)$, where $p(x) = x^3 - 2x + 5$.

(15) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Find all choices of a, b, c, and d for which $AC = CA$.

(16) Consider the system

$$\begin{aligned} 2x_1 - 2x_2 &= 4 \\ x_1 + 4x_2 &= 4 \end{aligned}$$

(a) Find a matrix A and column vector $\mathbf{b}$ such that solutions (x_1, x_2) to the system correspond to solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the matrix equation $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}$.

(b) Now solve your matrix equation in (a) algebraically using inverses.

(17) Find at least nine solutions to $A^2 = I_3$

(18) Can a matrix with two identical rows or two identical columns have an inverse?

(19) Assuming A, B, C and D are all invertible square matrixes, solve the following equation for D in terms of A, B and C.

$$ABC^T DBA^T C = AB^T.$$

(20) Prove: $A + B = B + A$, where A and B are any $m \times n$ matrices. (This is Theorem 1.18 (a).)

(21) Prove: $(AB)^T = B^T A^T$, where A is any $m \times n$ matrix and B is any $n \times r$ matrix.

(22) Answer true or false. If true, provide a proof; if false, give an explicit counterexample.

(a) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

(b) A square matrix containing a row or column of zeros cannot be invertible.

(c) If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible.

(d) If A is invertible, then A^T is invertible.

1.5-1.6: Inverses, Diagonal and Symmetric Matrices

(1) Let

$$B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix},$$

For each part below, find an explicit *elementary matrix* E satisfying the given matrix equation.

Justify your answer by first explaining what row operation E should perform.

(a) $EB = D$

(b) $ED = B$

(c) $EB = F$

(d) $EF = B$.

(2) For each matrix below use the inversion algorithm to find the inverse, if it exists.

(a) $A = \begin{bmatrix} 1/5 & 1/5 & -2/5 \\ 1/5 & 1/5 & 1/10 \\ 1/5 & -4/5 & 1/10 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1/5 & 1/5 & -2/5 \\ 2/5 & -3/5 & -3/10 \\ 1/5 & -4/5 & 1/10 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

(3) For each matrix below find the inverse using the inverse algorithm.
Assume $k_1, k_2, k_3, k_4, k \neq 0$.

(a) $A = \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

(4) Find all values of c, if any, making the matrix $A = \begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$ invertible.

(5) For each A below, express A and A^{-1} as products of elementary matrices.

(a) $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$

(6) Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

Produce B from A using row operations, the use this sequence of row operations to find C such that $CA = B$.

(7) Let A be $n \times n$. Show that the following are equivalent by proving a "cycle of implications".
  (i) A is invertible.
  (ii) For any column vector $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.
  (iii) For any column vector $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

  This completes the proof of the invertibility theorem in its current form.

(8) Let A be $n \times n$ and suppose A has two identical columns. Use an appropriate statement from the invertibility theorem to show A is not invertible.

(9) Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

(10) Determine conditions on b_i's to guarantee a consistent system.

$$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

(11) Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible n x n matrix.
  Prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(12) Answer true or false. If true, provide a proof; if false, give an explicit counterexample.
  (a) The product of two elementary matrices of the same size must be an elementary matrix.
  (b) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.
  (c) A is a singular $n \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
  (d) If A is invertible, then if the second row of A is replaced with the second row plus the first row, the resulting matrix is invertible.
  (e) If A is a square matrix, and if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x} = \mathbf{c}$ also must have a unique solution.
  (f) If A and B are row equivalent matrices, then the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.
  (g) Let A and B be n x n matrices. If A or B (or both) are not invertible, then neither is AB.

(13) Let $D = \begin{bmatrix} d_1 & 0 & \cdots & \\ 0 & d_2 & 0 & \cdots \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ be an $n \times n$ diagonal matrix.
  (a) Let A be $n \times r$. Use either the row/column method to describe the product DA as on operation on the rows/columns of A. (You choose which is appropriate, rows or columns.)
  (b) Let B be $m \times n$. Describe the product BD as on operation of the rows/columns of B. (You choose which is appropriate, rows or columns.)

(14) Let $A = [a_{ij}]$ be a triangular $n \times n$ matrix.
  Prove: A is invertible if and only if $a_{ii} \neq 0$ for all $1 \leq i \leq n$.
  Hint: use a convenient equivalent statement of invertibility from the invertibility theorem.

(15) Prove that if A and B are symmetric, then $cA + dB$ is symmetric for any $c, d \in \mathbb{R}$.

(16) Suppose A is symmetric and invertible. Prove A^{-1} is symmetric.

(17) Let A and B be symmetric. Prove that AB is symmetric if and only if $AB = BA$.

(18) For each matrix below (i) decide whether it is lower triangular, upper triangular, diagonal or neither, and (ii) decide whether it is invertible.

(a) $\begin{bmatrix} 4 & 0 \\ 1 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 0 & 0 \end{bmatrix}$

(19) Find the product by inspection:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(20) Find A^2, A^{-2}, and A^{-k} by inspection:

$$A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(21) Find the diagonal entries of AB by inspection.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 0 & 0 \\ -3 & 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$

(22) Find all values of x for which A is invertible.

$$\begin{bmatrix} x - 1/2 & 0 & 0 \\ x & x - 1/3 & 0 \\ x^2 & x^3 & x + 1/4 \end{bmatrix}$$

(23) Show that if A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix, the the following products are symmetric: $B^T B$, BB^T, $B^T AB$.

(24) Find all 3×3 diagonal matrices A such that

$$A^2 - 3A - 4I = 0$$

(25) Prove: If $A^T A = A$, then A is symmetric and $A = A^2$.

2.1-2.3: THE DETERMINANT

(1) Let A be $n \times n$. Let c be any constant. State and prove a formula relating $\det(cA)$ with $\det(A)$.
(Look at the $n = 2$ case to see what this should be.)

(2) Let A, B be $n \times n$, and suppose B is invertible. Prove the following:
 (a) $\det(B^{-1}) = \dfrac{1}{\det(B)}$.
 (b) $\det(B^{-1}AB) = \det(A)$.

(3) Let A be $n \times n$, and suppose two rows of A are identical. Show that $\det(A) = 0$.
Now prove the same result for a matrix with two identical columns.

(4) Suppose A is 3×3 and satisfies:

$$\underset{r_1 - r_2}{E} \; \underset{2r_2}{E} \; \underset{r_1 \leftrightarrow r_2}{E} A = \begin{bmatrix} 1 & 0 & 0 \\ 85 & 2 & 0 \\ -18 & 201 & 3 \end{bmatrix}.$$

Find $\det(A)$.

(5) State and prove a formula for $\det(A)$ where

$$A = \begin{bmatrix} a & b & b \dots & \\ b & a & b & \dots \\ \vdots & & \vdots & \\ b & b & \dots & a \end{bmatrix},$$

the matrix with a's along the diagonal and b's everywhere else.
Look at the $n = 2$ and $n = 3$ cases first. Your proof of the formula in the general case will involve row operations.

(6) Given $r_1, r_2, \dots, r_n \in \mathbb{R}$ the *Vandermonde* matrix is defined as

$$A_{r_1, r_2, \dots, r_n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} = [r_j^{i-1}]_{1 \le i, j \le n}.$$

Prove: $\det(A_{r_1, r_2, \dots, r_n}) = \prod_{1 \le i < j \le n} (r_j - r_i)$.

Hint. Do a proof by induction. For the induction step, first clear out the entries under the 1 of the first column, working from the bottom up. You will also use linearity in columns.

(7) Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

(a) Find M_{32} and C_{32}.
(b) Find M_{44} and C_{44}.

(8) Let

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}$$

(a) Compute $\det(A)$ by expanding along the second row.
(b) Compute $\det(A)$ by expanding along the third column.

(9) Compute

$$\det \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$$

(10) Compute

$$\det \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$$

(11) We know now how elementary row operations affect the determinant. State and prove analogous statements about how elementary *column operations* affect the determinant of a matrix.

(12) Let

$$A = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$$

Show that $\det(A) = 0$ without directly computing the determinant.

(13) Use determinants to decide whether the given matrix is invertible.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

(14) Use determinants to decide whether the given matrix is invertible.

$$A = \begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$$

(15) Find the values of k for which the matrix A is invertible.

$$\begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$$

(16) Use the adjoint formula to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(17) In each part, find the determinant given that A is a 4x4 matrix for which $\det(A) = -2$.
  (a) $\det(-A)$
  (b) $\det(A^{-1})$
  (c) $\det(2A^T)$
  (d) $\det(A^3)$

(18) Prove that a square matrix A is invertible if and only if $A^T A$ is invertible.

(19) True or false. If true, provide a proof; if false, give an explicit counterexample.
  (a) If A is a 4x4 matrix and B is obtained from A by interchanging the first two rows and then interchanging the last two rows, then $\det(A) = \det(B)$.
  (b) If A is a 3x3 matrix and B is obtained from A by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $\det(B) = 3\det(A)$.
  (c) If A is a 3x3 matrix and B is obtained from A by adding 5 times the first row to each of the second and third rows, then $\det(B) = 25\det(A)$.
  (d) If the sum of the second and fourth row vectors of a 6x6 matrix A is equal to the last row vector, then $\det(A) = 0$
  (e) If A is a 3x3 matrix, then $\det(2A) = 2\det(A)$.
  (f) If A and B are square matrices of the same size and A is invertible, then $\det(A^{-1}BA) = \det(B)$.
  (g) If A is a square matrix and the linear system $A\mathbf{x} = \mathbf{0}$ has multiple solutions for $\mathbf{x}$, then $\det(A) = 0$.
  (h) If E is an elementary matrix, then $E\mathbf{x} = \mathbf{0}$ has only the trivial solution.

3.1: VECTOR SPACES

(1) Let $V = \{A \in M_{nn} \colon A \text{ is invertible}\}$. Show that the usual matrix addition and scalar multiplication do NOT make V into a vector space.

(2) Let $V = \mathbb{R}^n$. Define vector addition on V to be the usual n-vector addition, but define scalar multiplication as

$$r(v_1, v_2, \ldots, v_n) = (r^2 v_1, r^2 v_2, \ldots, r^2 v_n).$$

Show that V is NOT a vector space with these choices of vector addition and scalar multiplication.

(3) Let $V = \{(x, y) \in \mathbb{R}^2 \colon x > 0,\ y < 0\}$: i.e., V is the set of pairs whose first component is positive, and whose second component is negative. The following operations satisfy the axioms of a vector space.

Vector addition: $(x_1,\ y_1) + (x_2,\ y_2) = (x_1 x_2,\ -y_1 y_2)$.
(In other words, vector addition takes the product of the first components, and takes the negative of the product of the second components.)

Scalar multiplication: $r(x,\ y) = (x^r,\ -|y|^r)$.
Below you are asked to verify some (but not all!) of the vector space axioms.

(a) Explicitly identify the element of V that acts as the additive identity $\mathbf{0}$ with respect to the above vector operations.
Show directly that your choice of $\mathbf{0}$ satisfies $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} = (x, y) \in V$.
Caution: at the very least, the $\mathbf{0}$ you provide should be an actual element of the given V. For example, note that $(0, 0)$ is not even an element of V !!

(b) Given $\mathbf{v} = (x, y) \in V$, give a formula in terms of x and y for its additive inverse $-\mathbf{v}$ with respect to the above vector operations.
Show directly that your formula for $-\mathbf{v}$ satisfies $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Again, you must always use the definition of the vector space operations given above.

(c) Prove that Axiom (h) holds: i.e., that $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

(4) Let $V = \{f(x) = 1 + ax \colon a \in \mathbb{R}\}$ be the set of all linear polynomials $f(x)$ with constant coefficient 1: i.e., $f(0) = 1$. Define:

$$\text{Vector addition: } (1 + ax) + (1 + bx) := (1 + ax)(1 + bx) - abx^2$$
$$\text{Scalar multiplication: } r \cdot (1 + ax) := 1 + rax$$

Prove V along with these operations forms a vector space. Make explicit what the additive identity and inverses are.

(5) Let $V = \mathbb{R}^2$. The following operations do **NOT** satisfy all of the vector space axioms.
Vector addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$. (The usual vector addition on $\mathbb{R}^2$.)
Scalar multiplciation: $r(x, y) = (rx, 0)$.

(a) Verify that these operations satisfy Axiom (f): i.e., that $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$ for all $r, s \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$.

(b) Find an axiom that these operations do **NOT** satisfy, and provide an explicit counterexample showing that the axiom fails.

(6) Prove the vector space properties theorem. (Some of these parts have been proved in the lecture notes or elsewhere. No matter, recreate those proofs or produce new ones!)

(7) Prove that a vector space V is either trivial (i.e., $V = \{\mathbf{0}\}$) or infinite. Hint: if $\mathbf{v} \neq \mathbf{0}$, show that $r\mathbf{v} \neq s\mathbf{v}$ for two distinct real numbers $r \neq s$.

(8) Let V be a vector space. Show that for all $\mathbf{v} \in V$ there is a unique additive inverse for $\mathbf{v}$. That is, show if $\mathbf{v} + \mathbf{w} = \mathbf{0}$, then $\mathbf{w} = -\mathbf{v}$.

(9) Let V be a vector space. Prove, using only the axioms of a vector space: if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.

3.2: LINEAR TRANSFORMATIONS

(1) For each of the following functions T, decide whether T is linear. If yes, provide a proof; if no, given an explicit example that violates one of the axioms.
  (a) $T\colon M_{nn} \to M_{nn}$, $T(A) = A^2$
  (b) $T\colon M_{nn} \to \mathbb{R}$, $T(A) = \operatorname{tr} A$
  (c) $T\colon M_{nn} \to \mathbb{R}$, $T(A) = \det A$.
  (d) $T\colon F((-\infty,\infty)) \to F((-\infty,\infty))$, $T(f) = 1 + f$
  (e) $T\colon F((-\infty,\infty)) \to F((-\infty,\infty))$, $T(f(x)) = f(x+3)$
  (f) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (xy, yz)$.

(2) Define $T\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f) = g$, where $g(x) = f(x) + f(-x)$.
Show that T is a linear transformation.

(3) *Reflections in* $\mathbb{R}^2$. Given a fixed angle α, $0 \leq \alpha < \pi$, let ℓ_α be the line through the origin that makes an angle of α with the positive x-axis.
We define $T_\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ as $T_\alpha(P) =$ the reflection P' of P through ℓ_α
In more detail, given point $P \in \mathbb{R}^2$, let ℓ' be the line passing through P that is perpendicular to ℓ_α, and let Q be the intersection of ℓ and ℓ_α. If P already lies on ℓ_α, then $T_\alpha(P) = P$; otherwise $T_\alpha(P)$ is the *other* point P' lying on ℓ, whose distance to Q is equal to the distance between P and Q.
  (a) Give a geometric argument (i.e., draw a picture) that strongly suggests T_α is linear.
  (b) Now prove T_α is linear by finding a 2×2 matrix A such that $T_\alpha = T_A$.

(4) Let $V = \mathbb{R}^\infty = \{(a_1, a_2, \ldots,)\colon a_i \in \mathbb{R}\}$, the space of all infinite sequences. Define the "shift left function", T_ℓ, and "shift right function", T_r, as follows:

$$T_\ell\colon \mathbb{R}^\infty \to \mathbb{R}^\infty \qquad\qquad T_r\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$$

$$s = (a_1, a_2, a_3, \ldots) \longmapsto T_\ell(s) = (a_2, a_3, \ldots) \qquad s = (a_1, a_2, a_3, \ldots) \longmapsto T_r(s) = (0, a_1, a_2, \ldots)$$

Prove that T_ℓ and T_r are linear transformations.

(5) (Conjugation). Fix an invertible matrix $Q \in M_{nn}$. Define $T\colon M_{nn} \to M_{nn}$ as $T(A) = QAQ^{-1}$. Show that T is a linear transformation. (This operation is called **conjugation by** Q.)

3.3: SUBSPACES

(1) Given any square matrix $A = [a_{ij}] \in M_{nn}$, we define the **trace** of A as $\operatorname{tr} A = \sum_{i=1}^n a_{ii}$. In other words, the trace $\operatorname{tr} A$ is the sum of the diagonal entries of A.
Let $V = M_{22}$, and define $W = \{A \in M_{22}\colon \operatorname{tr}(A) = 0\}$.
  (a) Prove that W is a subspace.
  (b) Find a spanning set for the subspace

$$W = \{A \in M_{22}\colon \operatorname{tr}(A) = 0\}.$$

(2) Let $V = M_{22}$, and define $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Compute $W = \operatorname{span}(\{A_1, A_2, A_3\})$, identifying it as a certain *familiar* set of matrices.

(3) Let $V = \mathbb{R}^4$ and define $W = \{(x_1, x_2, x_3, x_4)\colon x_1 - x_4 = x_2 + x_3 = x_1 + x_2 + x_3 = 0\}$.
  (a) Show W is a subspace by identifying it as $\operatorname{null} T$ for an explicit linear transformation.
  (b) Describe W in as simple a manner as possible.

(4) Determine which of the following subsets W are subspaces of M_{nn}.
  (a) $W = \{\text{diagonal } n \times n \text{ matrices}\}$
  (b) $W = \{A \in M_{nn} \colon \det A = 0\}$
  (c) $W = \{\text{upper triangular } n \times n \text{ matrices}\}$
  (d) $W = \{A \in M_{nn} \colon \operatorname{tr} A = 0\}$. (See Exercise 1 above for the definition of $\operatorname{tr} A$.)
  (e) $W = \{\text{symmetric } n \times n \text{ matrices}\} = \{A \in M_{nn} \colon A^T = A\}$.
  (f) $W = \{A \in M_{nn} \colon A^T = -A\}$. (These are called **skew-symmetric matrices**.)
  (g) The set W of all n by n matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  (h) The set W of all n by n matrices A such that $AB = BA$, where B is a fixed $n \times n$ matrix.

(5) Determine which of the following are subspaces of P_3.
  (a) The set W of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$
  (b) The set W of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
  (c) All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, a_3 are rational numbers.
  (d) All polynomials of the form $a_0 + a_1x$, where a_0, a_1 are real numbers.

(6) Which of the following are subspaces of $F(-\infty, \infty)$?
  (a) The set W of all functions f in $F(-\infty, \infty)$ for which $f(0) = 0$.
  (b) All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.
  (c) The set W of all functions f in $F(-\infty, \infty)$ for which $f(-x) = x$.
  (d) All polynomials of degree 2.

(7) Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$
  (a) $(-9, -7, -15)$
  (b) $(6, 11, 6)$
  (c) $(0, 0, 0)$

(8) Which are the following are linear combinations of:
$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$
  (a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$
  (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
  (c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

(9) Express $p(x) = 7 + 8x + 9x^2$ as a linear combination of
$$p_1 = 2 + x + 4x^2, p_2 = 1 - x + 3x^2, p_3 = 3 + 2x + 5x^2$$

(10) Determine whether the following polynomials span P_2.
$$p_1 = 1 - x + 2x^2,\ p_2 = 3 + x,\ p_3 = 5 - x + 4x^2,\ p_4 = -2 - 2x + 2x^2$$

(11) Let $f(x) = \cos^2 x$ and $g(x) = \sin^2 x$. Determine whether the following functions lie in the space spanned by f and g. If yes, provide an explicit linear combination; if no, show no linear combination exists. (You may want to review your trig identities.)
  (a) $q(x) = \cos 2x$
  (b) $q(x) = 3 + x^2$
  (c) $q(x) = 1$
  (d) $q(x) = \sin x$

(e) $q(x) = 0$

(12) Show that the solution vectors of a consistent nonhomogeneous system of m linear equations in n unknowns do not form a subspace of R^n.

(13) Define the function $T\colon P_2 \to \mathbb{R}^2$ by
$T(p(x)) = \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix}$.
(a) Prove T is a linear transformation.
(b) Compute null(T) and range(T) and describe these in as simple a manner as possible.

(14) Define $T\colon P_2 \to P_3$ by $T(p) = q$, where $q(x) = \displaystyle\int_0^x p(t)\,dt$.
  (a) Prove that T is a linear transformation.
  (b) Compute null(T) and range(T) and describe these spaces in as simple a manner as possible.

(15) Define $S\colon M_{nn} \to M_{nn}$ by $S(A) = A^T - A$.
  (a) Show S is linear.
  (b) Compute null(S) and range(S) and identify both of these spaces as certain special named families of matrices. Make sure to prove your claims.

(16) Define $T\colon P_3 \to M_{22}$ by $T(p) = \begin{bmatrix} p(1) & p(-1) \\ p(2) & p(-2) \end{bmatrix}$.
  (a) Show T is linear.
  (b) Compute null(T) and range(T).
  Hint: a nonzero polynomial of degree 3 can have at most 3 roots!

(17) Define the function $T\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f) = f'$.
  (a) Show T is a linear transformation.
  (b) Compute null(T) and range T.

(18) Define $T\colon F(-\infty,\infty) \to F(-\infty,\infty)$ by $T(f) = g$, where $g(x) = f(x) + f(-x)$.
Show that T is linear, and identify null(T) and range(T) as certain special named types of functions. Make sure to prove your claims.

3.4-3.5: Linear independence and bases

(1) Let $V = \mathbb{R}^n$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of r vectors. Prove (using some Gaussian elimination theory) that if $r > n$, then S is linearly dependent.
In other words: in $\mathbb{R}^n$ you can have *at most* n linearly independent vectors.

(2) Let $V = M_{22}$, and define $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
  (a) Compute $W = \text{span}(\{A_1, A_2, A_3\})$, identifying it as a certain *familiar* set of matrices.
  (b) Decide whether $S = \{A_1, A_2, A_3\}$ is independent.

(3) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -2 \end{bmatrix}$. Find a basis for $W = \text{null}(A)$ and compute its dimension. You should first use GE to solve the defining matrix equation of null(A).

(4) Let $V = M_{nn}$. For each of the following subspaces $W \subset M_{nn}$, give a basis B of W and compute $\dim(W)$.
Justify your answer.
  (a) W is the set of upper triangular $n \times n$ matrices.
  (b) W is the set of symmetric $n \times n$ matrices.
  (c) W is the set of skew-symmetric $n \times n$ matrices ($A^T = -A$).

(5) Determine if the vectors are linearly independent or not in $\mathbb{R}^3$
  (a) $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$
  (b) $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$

(6) Determine if the vectors are linearly independent or not in $\mathbb{R}^4$
  (a) $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (4, 2, 6, 4)$
  (b) $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

(7) Determine whether the vectors are linearly independent or are linearly dependent in P_2.
  (a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$
  (b) $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$

(8) Determine whether the matrices are linearly independent or dependent.
  (a) In M_{22}

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}; \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

  (b) In M_{23}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(9) Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}; \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

(10) Show that the following polynomials form a basis for P_3.

$$p_1 = 1 + x, p_2 = 1 - x, p_3 = 1 - x^2, p_4 = 1 - x^3$$

(11) Show that the following matrices form a basis for M_{22}.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix};\ A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix};\ A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};\ A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(12) Show that the following vectors do not form a basis for P_2

$$p_1 = 1 - 3x + 2x^2, p_2 = 1 + x + 4x^2, p_3 = 1 - 7x$$

(13) Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$.
  (a) Show that $S = \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is not a basis for V.
  (b) Find a basis for V.

(14) Let $V = \mathbb{R}^n$, considered as column vectors. Prove: vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$ if and only if the matrix $A = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$ is invertible.

(15) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$ (considered as column vectors), and let Q be an invertible $n \times n$ matrix.
Show that the vectors

$$\begin{aligned}\mathbf{w}_1 &= Q\mathbf{v}_1\\ \mathbf{w}_2 &= Q\mathbf{v}_2\\ &\vdots\\ \mathbf{w}_n &= Q\mathbf{v}_n\end{aligned}$$

form a basis for $\mathbb{R}^n$.

(16) True or false. If true, provide a proof; if false, give an explicit counterexample.
  (a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.
  (b) Every linearly independent subset of a vector space V is a basis for V.
  (c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
  (d) Every basis of P_4 contains at least one polynomial of degree 3 or less.
  (e) There is a basis for M_{22} consisting of invertible matrices.

3.6: DIMENSION

(1) Prove the following statement from the "street smarts theorem".
Let V be a vector space with basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then any collection of r vectors, with $r > n$, is linearly dependent.
Hint: suppose $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ with $r > n$. Begin by setting up the vector equation $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r = \mathbf{0}_V$; then write each $\mathbf{w}_j$ in this equation as a linear combination of the $\mathbf{v}_i$ (possible since $\mathbf{v}_i$ span); then collect like terms and use the fact that the $\mathbf{v}_i$ are linearly independent.

(2) For each of the following choices of a vector space V and subset $S \subseteq V$ decide whether S is a basis for V.
Keep an eye out for potential shortcuts using the dimension theorem compendium.
  (a) $V = P_3$, $S = \{1 - x^3, x - x^3, x^2 - x^3\}$.
  (b) $V = \mathbb{R}^3$, $S = \{(1, 2, 3), (2, 5, 6), (1, -1, 3)\}$.

(3) In multivariable calculus, a plane in $\mathbb{R}^3$ is defined as the set of solutions to an equation of the form $ax + by + cz = d$, where at least one of a, b, c is nonzero.
In particular, a plane passing through the origin $(0, 0, 0)$ is the set of solutions to an equation of the form $ax + by + cz = 0$, where at least one of a, b, c is nonzero.
  (a) Show that a plane passing through the origin is a subspace of $\mathbb{R}^3$ of dimension 2; conversely, show that a subspace of $\mathbb{R}^3$ of dimension 2 is a plane passing through the origin.
  In other words, show that the set of all planes passing through the origin is precisely the set of all 2-dimensional subspaces of $\mathbb{R}^3$.
  (b) Let $\mathcal{P}\colon ax + by + cz + d$ be any plane in $\mathbb{R}^3$, and let $P = (x_0, y_0, z_0)$ be a point of $\mathcal{P}$. Show that there is a 2-dimensional subspace $W \subseteq \mathbb{R}^3$ such that $\mathcal{P} = P + W := \{(x_0, y_0, z_0) + \mathbf{w} \colon \mathbf{w} \in W\}$.

The set $P+W$ is called the **translate of W by P**. This result shows that all planes in $\mathbb{R}^3$ are translates of 2-dimensional subspaces.

(4) Let $V = M_{33}$, $W = \{A \in M_{33} \colon A^T = -A\}$ (i.e., the subspace of skew-symmetric matrices), and
$$W' = \operatorname{span}\left\{ A_1 = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$
Show that $W' = W$ as follows:
(a) Show that $W' \subseteq W$ (easy);
(b) Compute the dimensions of W' and W and use the dimension theorem (compendium).

(5) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.
$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

(6) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.
$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

(7) Let
$$S = \{\mathbf{v}_1 = (1,1,1,1), \mathbf{v}_2 = (2,2,2,0)\} \subseteq \mathbb{R}^4.$$
Enlarge S into a basis of $\mathbb{R}^4$.

(8) Compute bases and dimensions for the following subspaces of $\mathbb{R}^4$.
(a) $W = \{(a,b,c,d) \in \mathbb{R}^4 \colon d = a+b, c = a-b\}$
(b) $W = \{(a,b,c,d) \in \mathbb{R}^4 \colon a+b = c+d\}$

(9) Let $V = M_{nn}$, W_1 the set of diagonal matrices, W_2 the set of symmetric matrices, W_3 the set of upper triangular matrices. Compute the dimensions of all these spaces. You may use your results from Exercise 4 in Section 3.3-3.4.

(10) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

(11) Let $V = M_{23}$ and define
$$W = \{A \in M_{23} \colon \text{ rows and columns of } A \text{ sum to } 0\}.$$
Find a basis for W and compute its dimension.

(12) Let $C^\infty(-\infty, \infty)$ and consider the subspace
$$W = \operatorname{span}(\{\cos(2x), \sin(2x), \sin(x)\cos(x), \cos^2(x), \sin^2(x)\}).$$
Compute the dimension of W by providing an explicit basis.

(13) Let $A \in M_{nn}$. Show that there is a nonzero polynomial $p(x) = a_{n^2}x^{n^2} + a_{n^2-1}x^{n^2-1} + \cdots + a_1x + a_0$ such that

$$p(A) = a_{n^2}A^{n^2} + a_{n^2-1}A^{n^2-1} + \cdots + a_1A + a_0I_n = \underset{n\times n}{\mathbf{0}}.$$

Hint: consider the set $S = \{I_n, A, A^2, \ldots, A^{n^2}\} \subseteq M_{nn}$ and use dimension theory.

(14) Prove that $V = C^\infty(\mathbb{R})$ is infinite dimensional.

Hint: use a proof by contradiction. More specifically, assume $\dim V = n$ for some n, and try and derive a contradiction to one of our dimension theorem compendium statements.

4.1-4.2: LINEAR TRANSFORMATIONS, BASES, RANK-NULLITY

(1) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is linear, and satisfies $T(1,1) = (1,1,1)$, $T(1,-1) = (1,2,1)$.

Find the matrix A such that $T = T_A$.

Hint: first write $(1,0)$ and $(0,1)$ as linear combinations of $(1,1)$ and $(1,-1)$.

(2) Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear, that $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and that $T(\mathbf{v}_i) = 2\mathbf{v}_i$ for all $1 \leq i \leq n$.

Compute the $n \times n$ matrix A such that $T = T_A$. Justify your answer.

(3) Use the rank-nullity theorem to compute the rank of the linear transformation T described.
- (a) $T\colon \mathbb{R}^7 \to M_{32}$ has nullity 2.
- (b) $T\colon P_3 \to \mathbb{R}$ has nullity 1.
- (c) The null space of $T\colon P_5 \to P_5$ is P_4.
- (d) $T\colon P_n \to M_{mn}$ has nullity 3.

(4) Give an explicit example of a matrix A such that $\mathrm{col}(A) \neq \mathrm{col}(U)$.

(5) Prove that $\mathrm{col}(A) = \{\mathbf{b} \in \mathbb{R}^m \colon A\mathbf{x} = \mathbf{b} \text{ has a solution}\}$.

(6) Find bases and dimensions for all fundamental spaces of

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(7) For each matrix below use the rank-nullity theorem to help determine the fundamental spaces "by inspection"–i.e., without having to do Gaussian elimination.

The various fundamental spaces will live either in $\mathbb{R}^2$ or $\mathbb{R}^3$. For each matrix give two sketches showing (a) $\mathrm{null}(A)$ and $\mathrm{row}(A)$ in one sketch, and (b) $\mathrm{col}(A)$ in another.
- (a) $A = \begin{bmatrix} 3 & 2 & 1 \\ -6 & -4 & -2 \end{bmatrix}$
- (b) $A = \begin{bmatrix} 2 & 1 \\ 8 & 4 \\ 6 & 3 \end{bmatrix}$

(8) Let A be $m \times n$ with $n < m$. Use fundamental spaces to show that there is a $\mathbf{b} \in \mathbb{R}^m$ such that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(9) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Use the rank-nullity theorem to find bases for all the fundamental spaces of A "by inspection".

(10) Find bases for all the fundamental spaces of the following matrices.

(a)

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

(11) Find bases for the row space and for the column space of the given matrix.

(a)

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(12) Let $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$.

(a) Use the usual procedure to find bases for $\mathrm{col}(A)$ and $\mathrm{row}(A)$.

(b) Now compute a basis for $\mathrm{row}(A)$ consisting of a *subset of the rows* of A. Hint: look at A^T.

(13) Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

$$\mathbf{v}_1 = (1, 0, 1, 1), \mathbf{v}_2 = (-3, 3, 7, 1), \mathbf{v}_3 = (-1, 3, 9, 3), \mathbf{v}_4 = (-5, 3, 5, -1)$$

(14) Find the rank and nullity of each matrix by reducing it to row echelon form.

(a)

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

(15) Matrix R is the reduced row echelon form of the matrix A.

$$A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Compute the rank and nullity of A.
(b) Confirm that the rank an nullity satisfy the rank-nullity theorem.

(16) Give two explicit matrices A and B of the same size satisfying $\text{rank}(A) = \text{rank}(B)$, but $\text{rank}(A^2) \neq \text{rank}(B^2)$.

(17) Prove: an $n \times n$ matrix A satisfies $\text{col}(A) \subseteq \text{null}(A)$ if and only if $A^2 = \mathbf{0}$.
Produce an explicit nonzero example of such a matrix.

(18) If A is an m by n matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

(19) Complete the following statements. Justify your answer.
(a) If A is a 3 by 5 matrix, then the rank of A is at most ...
(b) If A is a 3 by 5 matrix, then the nullity of A is at most ...

(20) Let A be a 5 by 7 matrix with rank 4.
(a) What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?
(b) Is $A\mathbf{x} = \mathbf{b}$ consistent for all vectors $\mathbf{b}$ in $\mathbb{R}^5$?

(21) Prove: If a matrix A is not a square, then either the row vectors or the column vectors of A are linearly dependent.

(22) True or false. If true, provide a proof; if false, give an explicit counterexample.
(a) Either the row vectors or the column vectors of a square matrix are linearly independent.
(b) A matrix with linearly independent row vectors and linearly independent column vectors is square.
(c) The nullity of a nonzero m by n matrix is at most m.
(d) Adding one additional column to a matrix increases its rank by one.
(e) The nullity of a square matrix with linearly dependent rows is at least one.
(f) If A is a square and $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, the the nullity of A is zero.
(g) If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space.
(h) $\text{rank}(A) = \text{rank}(A^T)$.
(i) If $\text{nullity}(A^T) = \text{nullity}(A)$, then A is square.

(23) Suppose $\underset{m \times n}{A}$ reduces to the row echelon matrix $\underset{m \times n}{U}$.
(a) Prove: the columns $\mathbf{u}_{i_1}, \mathbf{u}_{i_2}, \ldots, \mathbf{u}_{i_r}$ of U form a basis for $\text{col}\, U$ if and only if the corresponding columns $\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_r}$ of A form a basis for $\text{col}\, A$.

(b) Prove: the columns of U with leading 1's form a basis for $\operatorname{col} U$.
(c) Prove: $\dim \operatorname{null} U = \#($ free variables $)$ in linear system $U\mathbf{x} = \mathbf{0}$.
(d) Suppose $x_{j_1} = t_{j_1}, x_{j_2} = t_{j_2}, \dots, x_{j_s} = t_{j_s}$ are the free variables of the system $U\mathbf{x} = \mathbf{0}$. Let $\mathbf{v}_{j_k}$ be the element of $\operatorname{null} U$ obtained by setting $t_{j_k} = 1$ and $t_{j_\ell} = 0$ for $\ell \neq k$ in our parametric description of solutions to $U\mathbf{x} = \mathbf{0}$.
Prove: $\{\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_s}\}$ is a basis for $\operatorname{null} U$.

4.3: COORDINATE VECTORS

(1) Find the coordinate vector $[\mathbf{v}]_B$ relative to the basis $B = \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for $\mathbb{R}^3$.
  (a) $\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (3, 3, 3)$
  (b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

(2) Find the coordinate vector $[p]_B$ relative to the basis $B = \{p_1, p_2, p_3\}$ for P_2.
  (a) $p = 4 - 3x + x^2$, $p_1 = 1$, $p_2 = x$, $p_3 = x^2$
  (b) $p = 2 - x + x^2$, $p_1 + x = 1$, $p_2 = 1 + x^2$, $p_3 = x + x^2$

(3) The set $B = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

  (a) Compute $[A]_B$, where

$$A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}.$$

  (b) Now compute $[A]_B$ for the general matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(4) Suppose $\dim(V) = n$, and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Define

$$\begin{aligned} T\colon V &\to \mathbb{R}^n \\ \mathbf{v} &\mapsto [\mathbf{v}]_B. \end{aligned}$$

Prove all statements of the coordinate vector map theorem, as listed below.
  (a) T is a linear transformation.
  (b) $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ if and only if $\mathbf{v}_1 = \mathbf{v}_2$: i.e., T is *one-to-one*.
  In particular, $T(\mathbf{v}) = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}_V$.
  (c) $\operatorname{range} T = \mathbb{R}^n$: i.e. T is *onto*.
  (d) A set $S = \{w_1, w_2, \dots, w_r\} \subseteq V$ is linear independent if and only if $T(S) = \{T(\mathbf{w}_1), T(\mathbf{w}_2), \dots, T(\mathbf{w}_r)\} \subseteq \mathbb{R}^n$ is linearly independent.
  (e) A set $S = \{w_1, w_2, \dots, w_r\} \subseteq V$ spans V if and only if $T(S) = \{T(\mathbf{w}_1), T(\mathbf{w}_2), \dots, T(\mathbf{w}_r)\}$ spans $\mathbb{R}^n$.

(5) Let $S = \{p_1 = 1+x+2x^2, p_2 = 1-x, p_3 = 1+x^2, p_4 = 1+x-x^2\}$ and let $W = \operatorname{span}(S) \subset P$. (Recall P is the space of all polynomials.)
  (a) Use "street smarts" to decide whether S is linearly independent.
  (b) Use coordinate vectors and an appropriate fundamental space algorithm to choose a basis of $W = \operatorname{span}(S)$ *from among the elements of S*.
  (c) Give a satisfying description of W.

(6) Let $V = P_2$, and let $S = \{x^2 + 2x + 1, 3x^2 + 6x\}$. Extend S to a basis of P_2 by first translating the problem to $\mathbb{R}^3$ using coordinate vectors and applying the relevant algorithm there.

(7) Let $V = P_3$ and consider the polynomials

$$p_1 = x^3 + 1, p_2 = 2x^3 + x + 1, p_3 = 3x^3 + 2x + 1, p_4 = 2x^3 + x^2 + x + 1,$$

Let $S = \{p_1, p_2, p_3, p_4\}$ and define $W = \text{span}(S)$.

Find a subset of S that is a basis of W and compute $\dim(W)$. Use coordinate vectors!

(8) Let

$$S = \left\{ A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \ A_3 = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix}, \ A_4 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

and define $W = \text{span}(S) \subseteq M_{22}$.

(a) Compute a basis B of W from among the elements of S.
You should first "translate" the problem into $\mathbb{R}^4$ using coordinate vectors and apply an appropriate fundamental space algorithm.

(b) Show that in fact W is equal to the subspace $W' = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.
Make your life easier by using a dimension argument, but make sure all your claims are fully justified.

4.4: MATRIX REPRESENTATIONS

(1) Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ to be reflection through the plane $\mathcal{P}\colon x + y + z = 0$. In other words, T takes an input (x, y, z) and sends it to its reflection (x', y', z') on the other side of $\mathcal{P}$. You may take it on faith that T is a linear transformation.

(a) Draw a picture of $\mathcal{P}$, an arbitrary point (x, y, z), and its reflection (x', y', z'). Add some right triangles to your picture to indicate the relation between (x, y, z) and (x', y', z').

(b) Pick a *nonstandard* basis B' of $\mathbb{R}^3$, for which the behavior of T is easy to determine: i.e., pick a basis that reflects the geometry of this picture. Compute $[T]_{B'}$.

(2) Consider the differential equation

$$f''(x) + f'(x) = x^2 + x + 1 \qquad (*)$$

The theory of ODE's tells us that there are infinitely many solutions $f(x)$ to $(*)$. We will use linear algebra to find infinitely many polynomial solutions.

(a) First define $T\colon P_3 \to P_2$ as $T(f) = f''(x) + f'(x)$. Explain why $T(f)$ indeed lies in P_2 and show that T is linear.

(b) Compute $A = [T]_B^{B'}$ where B and B' are the standard bases of P_3 and P_2, respectively.

(c) Determine $\text{null}(A)$ and $\text{col}(A)$.

(d) Find all solutions in P_3 to the differential equation $(*)$. (First solve the relevant matrix equation involving A, then "lift up" to P_3.)

Note: though we have found all *polynomial solutions* to the differential equation $(*)$, we have not found *all* solutions. Observe that $g(x) = e^{-x}$ is a solution to the corresponding homogeneous equation $f'' + f' = 0$, It follows from basic ODE theory that we can add ce^{-x} to any of our solutions in (d) to get a new solution. Our analysis didn't catch these solutions as we restricted our attention to P_3.

(3) Define $T\colon M_{22} \to M_{22}$ by $T(A) = A^T - A$.

(a) Compute $A = [T]_B$ where B is the standard basis for M_{22}.

(b) Compute bases for $\text{null}(A)$ and $\text{col}(A)$.

(c) Use your result in (b) to give bases for null(T) and range(T).
(d) Identify null(T) and range(T) as familiar subspaces of matrices.

(4) Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p) = x \cdot p(x-3)$.
(a) Find $[T]_B^{B'}$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.
(b) Use $[T]_B^{B'}$ to compute bases for null T and range T.

(5) Let $T : P_2 \to M_{22}$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) & p(1) \\ p(-1) & p(0) \end{bmatrix}$$

let B be the standard basis for M_{22}, let $B' = \{1, x, x^2\}$ and $B'' = \{1, 1+x, 1+x^2\}$ be bases for P_2.
(a) Find $[T]_{B'}^B$ and $[T]_{B''}^B$.
(b) Use either one of the matrix representations from (a) to compute bases for null T and range T.

(6) Let $T : M_{22} \to R^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ d \end{bmatrix}$$

and let B be the standard basis for M_{22}, B' be the standard basis for R^2 and let

$$B'' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

be another basis for R^2. Find $[T]_B^{B'}$ and $[T]_B^{B''}$.

4.5: CHANGE OF BASIS

(1) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Compute $\underset{B \to B'}{P}$ for each of the following bases B', and explain what happens to coordinate vectors as we change from B to B'.
(a) $B' = \{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$.
(b) $B' = \{3\mathbf{v}_1, 3\mathbf{v}_2, \ldots, 3\mathbf{v}_n\}$.

(2) Let $V = P_1$ with basis $B' = \{2x+1, 3x+2\}$.
(a) Compute $[p(x)]_{B'}$, where $p(x) = ax + b$, $a, b \in \mathbb{R}$. Your answer will be in terms of a and b.
(b) Now do the same computation using a change of basis matrix involving the standard basis B.

(3) Let V be a finite-dimensional vector space, and let B, B', B'' be three different bases of V. Let $P = \underset{B \to B'}{P}$ and $Q = \underset{B \to B''}{P}$.
(a) Express $\underset{B' \to B''}{P}$ in terms of P, Q, and/or their inverses.
Verify that your matrix satisfies the defining property of the change of basis matrix.
(b) Utilize the formula in (a) to compute $\underset{B' \to B''}{P}$, where $B' = \{2x+1, 3x+2\}$ and $B'' = \{x-1, x+1\}$: two nonstandard bases of P_1.

(4) Let B, B', and B'' be three different bases for a finite dimensional vector space V. Show that

$$\underset{B \to B''}{P} = \underset{B' \to B''}{P} \cdot \underset{B \to B'}{P}$$

using only the defining property and uniqueness of change of basis matrices.

(5) True or false. If true, provide a proof; if false, give an explicit counterexample.
  (a) If B is a basis for an n-dimensional space V, then $\underset{B\to B}{P} = I_n$.
  (b) If $\underset{B\to B'}{P}$ is diagonal, then each basis element of B is a scalar multiple of some basis element of B'.
  (c) If each basis element of B' is a scalar multiple of *some* basis element of B, then $\underset{B\to B'}{P}$ is diagonal.
  (d) The coordinate vector of a vector $\mathbf{x}$ in $\mathbb{R}^n$ relative to the standard basis for $\mathbb{R}^n$ is $\mathbf{x}$.

(6) Consider the bases $B = \{p_1 = 6 + 3x, p_2 = 10 + 2x\}$ and $B' = \{q_1 = 2, q_2 = 3 + 2x\}$ for P_1.
  (a) Compute the transition matrix from B' to B.
  (b) Compute the transition matrix from B to B'.
  (c) Compute the coordinate vector $[p]_B$, where $p = -4 + x$ and use the change of basis formula theorem to compute $[p]_{B'}$.
  (d) Now compute $[p]_{B'}$ directly and verify that your answer agrees with part (c).

(7) Let B be the standard basis for $\mathbb{R}^3$, and let $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 5, 0)$, and $\mathbf{v}_3 = (3, 3, 8)$.
  (a) Compute $\underset{B'\to B}{P}$.
  (b) Now compute $\underset{B\to B'}{P}$.
  (c) Confirm these two matrices are inverses of one another.
  (d) Let $\mathbf{w} = (5, -3, 1)$. Compute $[\mathbf{w}]_{B'}$ directly, then compute $[\mathbf{w}]_{B'}$ using $[\mathbf{w}]_B$ and the relevant change of basis matrix.
  (e) Let $\mathbf{w} = (3, -5, 0)$. Compute $[\mathbf{w}]_B$ by inspection, then compute $[\mathbf{w}]_{B'}$ using $[\mathbf{w}]_B$ and the relevant transition matrix.

(8) The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

is the change of basis matrix from what basis B to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ in $\mathbb{R}^3$?

(9) Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ as

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, -x_2, x_1 + 7x_3)$$

Let B be the standard basis of $\mathbb{R}^3$, and let B' be the nonstandard basis

$$B' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$

First compute $[T]_B$, then compute $[T]_{B'}$ using the change of basis formula for transformations.

(10) Define $T\colon P_1 \to P_1$ as

$$T(a_0 + a_1x) = a_0 + a_1(x + 1) = (a_0 + a_1) + a_1x.$$

The following are two different bases for P_1: $B = \{6 + 3x, 10 + 2x\}$; $B' = \{2, 3 + 2x\}$. First compute $[T]_B$, then use the change of basis formula for transformations to compute $[T]_{B'}$.

(11) Let $T\colon V \to W$ be a linear transformation between two finite-dimensional vector spaces.
Suppose B_1, B_2 are two different bases of V, and B_1', B_2' are two different bases of W.
State and prove a matrix equation relating $[T]_{B_2}^{B_2'}$ in terms of $[T]_{B_1}^{B_1'}$, $\underset{B_1 \to B_2}{P}$, $\underset{B_1' \to B_2'}{P}$, and/or the inverses of these change of basis matrices.

To prove your equality, you should show that the LHS and RHS matrices both satisfy the defining property of $[T]_{B_2}^{B_2'}$.

(12) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and that for any vector $\mathbf{v}$ of the form $\mathbf{v} = c(1,1) + d(-1,1)$, we have

$$T(\mathbf{v}) = T\left(c(1,1) + d(-1,1)\right) = (c+d)(1,1) + d(-1,1).$$

Note: this is an example of a *shearing transformation* along the vector $(1,1)$.

(a) Let $B' = \{(1,1), (-1,1)\}$. Compute $[T]_{B'}$.
(b) Use the change of basis formula to compute $A = [T]_B$, where B is the standard basis.
(c) Compute $T(1,2)$.

(13) For fixed direction vector $\mathbf{d} = (a,b) \neq \mathbf{0}$, let ℓ be the line in $\mathbb{R}^2$ passing through the origin that it defines.
Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be reflection through ℓ, as defined in Exercise 3 of Section 3.2. Recall that T is a linear transformation.

(a) Construct a basis B' of $\mathbb{R}^2$ which pays attention to the geometry involved in the definition of T, and compute $[T]_{B'}$.
(If you picked your basis appropriately, $[T]_{B'}$ should be diagonal.)
(b) Now use the change of basis formula to compute the A such that $T = T_A$.

(14) *Reflection through the plane* $\mathcal{P}\colon x+y+z=0$. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ to be reflection through the plane $\mathcal{P}\colon x+y+z=0$. See the next exercise for a precise definition of this operation.

(a) Construct an explicit basis of the form $B' = \{\mathbf{v}_1, \mathbf{v}_2, (1,1,1)\}$ where $\mathbf{v}_1, \mathbf{v}_2$ are elements of $\mathcal{P}$.
(b) Compute $[T]_{B'}$.
(c) Find the A such that $T = T_A$ using your result in (b) and a change of basis formula.
(d) Compute $T(1,2,3)$.

(15) *Reflection through a general plane in* $\mathbb{R}^3$. Fix $\mathbf{n} = (a,b,c) \neq \mathbf{0}$. Let W be the plane passing through the origin with normal vector $\mathbf{n} = (a,b,c)$: i.e., $W : ax+by+cz = 0$. Reflection through W is the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows:

> given $P = (x,y,z)$ let ℓ be the line passing through P and perpendicular to W; let Q be the intersection of this line with W; then the reflection $T(P)$ is the unique point $P' = (x',y',z')$ on ℓ such that the distance from P to P' is twice the distance from P to Q.

Intuitively, $T(P) = P'$ is the point "on the other side of the plane" from P and an equal distance away.
You may take for granted that T is a linear transformation. Also, observe that if P lies in W, then the definition implies $T(P) = P$: P is its own reflection.
Compute the matrix A such that $T = T_A$ as follows:

(a) First compute $A' = [T]_{B'}$, where $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ satisfying $\mathbf{v}_1, \mathbf{v}_2 \in W$, $\mathbf{v}_3 = (a,b,c)$.
(b) Now use the change of basis formula to compute $A = [T]_B$, where B is the standard basis.

(1) Compute the characteristic polynomial, eigenvalues, and eigenspace bases for each of the following matrices.
   (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
   (d) $A = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

(2) Compute the characteristic polynomial, eigenvalues, and eigenspace bases for

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

(3) Compute the characteristic polynomial, eigenvalues, and eigenspace bases for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(4) Compute the characteristic polynomial, eigenvalues, and eigenspace bases for

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

(5) Find the characteristic polynomial by inspection.

$$\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

(6) Let $A = \begin{bmatrix} -5 & 0 & 3 \\ -6 & 1 & 3 \\ -6 & 0 & 4 \end{bmatrix}$. Compute $p(t)$ and find bases for all eigenspaces of A.

(7) The matrix $B = \begin{bmatrix} -2 & -3 & 3 \\ -3 & -3 & 4 \\ -3 & -4 & 5 \end{bmatrix}$ has the same characteristic polynomial of A in Exercise 6. Find bases for all eigenspaces of B.

(8) Use induction to prove that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for all positive integers k. Make sure to invoke the definition of eigenvalue in your proof.

(9) Prove: A is invertible if and only if $\lambda = 0$ is *not* an eigenvalue of A.

(10) Let $B = \{\cos(x), \sin(x), \cosh(x), \sinh(x)\}$, let $V = \operatorname{span}(B)$, and define $T\colon V \to V$ by $T(f) = f'$.

You may take as a fact that the set B is linearly independent, hence a basis.

Recall: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$.

(a) Compute $A = [T]_B$. What does A tell you about the invertibility of T?
(b) Find all eigenvalues of T.
(c) For each eigenvalue λ of T, find a basis for W_λ.
(d) Use your work in (c) to find a new basis B' of V that contains two eigenvectors of T.
(e) Compute $[T]_{B'}$.

(11) Suppose A is a 2×2 matrix satisfying $\operatorname{tr} A = 2$ and $\det A = -15$. Find the eigenvalues of A.

(12) Let $T\colon V \to V$ be a linear transformation, and suppose $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Prove: $[T]_B$ is a diagonal matrix if and only if $\mathbf{v}_j$ is an eigenvector of T for all $1 \leq i \leq n$. *Moral.* We can represent T by a diagonal matrix if and only if T has "enough" linearly independent eigenvectors to build a basis.

4.7: DIAGONALIZATION

(1) Find the eigenvalues of A. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$. Is A diagonalizable? Justify.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

(2) For each matrix A below, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find the matrix P that diagonalizes A, and find $P^{-1}AP$.

(a)

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

(3) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Derive necessary and sufficient conditions for A to be diagonalizable expressed in terms of equalities and inequalities involving a, b, c, d.

(4) Prove all statements of the properties of similarity theorem, stated below.

Properties of similarity theorem. Suppose A is similar to B: i.e., there is an invertible matrix P such that $B = P^{-1}AP$. Then:
(a) B is similar to A. (*Similarity is symmetric.*)
(b) A and B have the same trace and determinant.
(c) A and B have the same rank nullity and rank.
(d) A and B have the same characteristic polynomial.
(e) A and B have the same eigenvalues.
(f) Given any $\lambda \in \mathbb{R}$, let W_λ be the corresponding eigenspace for A, and W'_λ the corresponding eigenspace for B. Then $\dim W_\lambda = \dim W'_\lambda$.

(5) Similar matrices have the same rank. Show the converse is false by showing the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have the same rank, but are not similar.

(6) Similar matrices have the same eigenvalues. Show the converse is false by showing the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues, but are not similar.

(7) Prove that if $B = P^{-1}AP$, and $\mathbf{v}$ is an eigenvector of B corresponding to the eigenvalue λ, the $P\mathbf{v}$ is the eigenvector of A corresponding to λ.

(8) Let A be an n by n matrix, and let $q(A)$ be the matrix

$$q(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n$$

Prove that if $B = P^{-1}AP$, then $q(B) = P^{-1}q(A)P$.

(9) Let A be a 3×3 matrix with eigenvalues $1, -1, 0$.
(a) Prove that A is diagonalizable. What is D in this case?
(b) Show that $A^n = A$ for any **odd** $n \geq 0$: e.g., $A^{13} = A$, $A^{77} = A$, etc.

(10) The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined recursively as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-2} + F_{n-1} \qquad \text{for } n \geq 2 \end{aligned}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Show that
$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$
for all $n \geq 0$.

(b) Diagonalize A. Write its two eigenvalues as λ_1 and λ_2.

(c) Derive a closed formula for A^n.

(d) Derive a closed formula for F_n.

(11) Let $A \in M_{nn}$ be defined as the matrix of all 1's: i.e., $a_{ij} = 1$ for all $1 \leq i, j \leq n$.
Show that A has exactly 2 eigenvalues and that A is diagonalizable.
Hint: do NOT compute the characteristic polynomial of A. Instead, begin by first considering $\text{null}(A)$.

(12) The *Cayley-Hamilton theorem* states that any $A \in M_{nn}$ satisfies its own characteristic polynomial $p(t)$: i.e., $p(A) = \mathbf{0}_{n\times n}$.
Prove the Cayley-Hamilton theorem for *diagonalizable* matrices.

(13) Information about three matrices is given below. Find all eigenvalues of each matrix and decide whether it is diagonalizable (you are given enough information in each case).

(a) $A_1 \in M_{33}$, $p(t) = \det(tI - A_1) = t^3 - t^2$, $\text{null}(A_1)$ =the plane perpendicular to $(1, 0, 1)$.

(b) $A_2 \in M_{33}$, $p(t) = \det(tI - A_2) = t^3 + t^2 - t$.

(c) $A_3 \in M_{22}$, $\text{tr}\, A = 4$, $\det A = 3$.

(14) A matrix $A \in M_{nn}$ is called **nilpotent** if $A^k = 0$ for some $k \geq 1$.
In what follows, assume A is nilpotent.

(a) Prove that 0 is the only *possible* eigenvalue of A.

(b) Prove that 0 is indeed an eigenvalue of A.
Thus 0 is the only eigenvalue of A.

(15) A matrix $A \in M_{nn}$ is called **idempotent** if $A^2 = A$.
In what follows, assume A is idempotent.

(a) Prove that the only *possible* eigenvalues of A are 0 and 1.

(b) Prove that either 0 is an eigenvalue of A or else $A = I_n$.

(c) Give an example of an idempotent matrix for each of the following three possibilities: only 0 is an eigenvalue; only 1 is an eigenvalue; 0 and 1 are both eigenvalues.

(16) Let $A \in M_{nn}$ and suppose the characteristic polynomial of A factors as
$$p(t) = (t - \lambda)^m g(t),$$
where the $\lambda \in \mathbb{R}$ and λ is not a root of $g(t)$.
Recall that we define the algebraic multiplicity of λ to be m; and we define the geometric multiplicity of λ to be $\dim W_\lambda$.
In this exercise we will show $\dim W_\lambda \leq m$. To do so, set $\dim W_\lambda = r$.
Show that A is similar to a matrix B whose characteristic polynomial is divisible by $(t-\lambda)^r$.
Conclude that $(t - \lambda)^r$ divides $p(t)$, and hence that $r \leq m$.

(17) Each of the matrices below has characteristic polynomial $p(x) = (x - 1)^2(x + 2)$.
For each decide whether it is diagonalizable. If yes, provide explicit P and D witnessing this fact.

(a) $A = \begin{bmatrix} -5 & 0 & 3 \\ -6 & 1 & 3 \\ -6 & 0 & 4 \end{bmatrix}$.

(b) $B = \begin{bmatrix} -2 & -3 & 3 \\ -3 & -3 & 4 \\ -3 & -4 & 5 \end{bmatrix}$.

(18) Let $A = \begin{bmatrix} -5 & 0 & 3 \\ -6 & 1 & 3 \\ -6 & 0 & 4 \end{bmatrix}$. Use your work above to do the following to find a matrix C such that $C^3 = A$.

(19) Suppose A is 3×3, has determinant 0, and has among its eigenvalues $\lambda = 2$ with eigenspace W_2 a plane in $\mathbb{R}^3$.
   (a) Find all eigenvalues of A.
   (b) Find $\operatorname{tr}(A)$.
   (c) Decide whether A is diagonalizable.

(20) Give an example of a 3×3 matrix A satisfying the following conditions:
   (i) $\lambda = 2$ is an eigenvalue and $(1, 0, 1), (1, 1, 1)$ are 2-eigenvectors.
   (ii) $\lambda = -1$ is an eigenvalue and $(1, 0, -1)$ is a -1-eigenvector.

(21) True or false. If true, provide a proof; if false, give an explicit counterexample.
   (a) If $A \in M_{nn}$ has less than n distinct eigenvalues, then A is not diagonalizable.
   (b) If $A \in M_{nn}$ has fewer than n linearly independent eigenvectors, then A is not diagonalizable.
   (c) If $A \in M_{nn}$ is diagonalizable, then there is a *unique* matrix P such that $P^{-1}AP$ is diagonal.
   (d) If $A \in M_{nn}$ is diagonalizable, then A^{-1} is diagonalizable.
   (e) If $A \in M_{nn}$ is diagonalizable, then A^T is diagonalizable. s

5.1: INNER PRODUCTS

In what follows V is always a vector space along with an inner product $\langle\ ,\rangle$. Notions of norm and orthogonality always refer to this fixed choice of inner product.

(1) Show that $\frac{\mathbf{v}}{\|\mathbf{v}\|} := \frac{1}{\|\mathbf{v}\|} \cdot \mathbf{v}$ is a unit vector for any $\mathbf{v} \in V$.

(2) Let $(V, \langle\ ,\rangle)$ be an inner product space. Prove: $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$ for all $c \in \mathbb{R}$.
(Here $|c|$ is the absolute value of c.)
Your proof must be valid for a general inner product space; i.e., you can only invoke the definition of $\|\_\|$ in terms of $\langle\ ,\rangle$, the axioms of an inner product, and properties of real number arithmetic.

(3) Compute the angle between $\mathbf{v} = (1, 1, 1, 1)$ and $\mathbf{w} = (1, -1, 1, 1)$. Do NOT use a calculator.

(4) Let $V = C([0, 1])$ with inner product $\langle f, g\rangle = \displaystyle\int_0^1 f(x)g(x)\,dx$. Let $f(x) = 1$ and $g(x) = x$. Compute the angle θ between f and g with respect to this inner product.

Your answer cannot be expressed in terms of inverse trig functions: i.e., the angle is a familiar one that you can solve for by hand.

(5) Prove that if $\mathbf{w}$ is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then it is orthogonal to any linear combination of the $\mathbf{v}_i$. In other words prove the implication:

$$(\mathbf{w} \perp \mathbf{v}_i \text{ for all } i) \Rightarrow (\mathbf{w} \perp (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \text{ for any } c_i)$$

(6) For each of the following operations on $\mathbb{R}^2$, determine whether it defines an inner product on $\mathbb{R}^2$. If it fails to be an inner product, identify which of the three inner product axioms (if any) it does satisfy, and provide explicit counterexamples for any axiom that fails.
 (a) $\langle (x_1, x_2),\ (y_1, y_2)\rangle = x_1y_2 + x_2y_1$.
 (b) $\langle (x_1, x_2),\ (y_1, y_2)\rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$.
 (c) $\langle (x_1, x_2),\ (y_1, y_2)\rangle = x_1^2y_1^2 + x_2^2y_2^2$.

(7) Equip P_2 with the *evaluation inner product* $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Find all polynomials $p(x)$ orthogonal to $q(x) = x$ with respect to this inner product. Note: this is an infinite set of polynomials. Describe it by giving a parameter description of its elements.

(8) Equip $V = C([-\pi, \pi])$ with the *integral inner product* $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$.
 (a) Show that $f(x) = \cos(x)$ and $g(x) = \sin(x)$ are orthogonal with respect to this inner product.
 (b) Compute $\|\cos(x)\|$ with respect to this inner product.
 (c) Show that if $f(x)$ is an odd function (i.e., $f(x) = -f(-x)$ for all x) and $g(x)$ is an even function ($g(-x) = g(x)$ for all x), then f and g are orthogonal with respect to this inner product.
 Hint: use the area interpretation of the integral, as well as graphical properties of even/odd functions.

(9) Let $\mathbf{v}, \mathbf{w} \in V$, and let θ be the angle between them. Prove the following equivalence:

$$\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\| \text{ if and only if } \theta = 0.$$

Your proof should be a *chain of equivalences* with each step justified.

(10) Take $\mathbf{v}, \mathbf{w} \in V$. Suppose $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$. Find the maximum and minimum possible values of $\|\mathbf{v} - \mathbf{w}\|$, and give explicit examples where those values occur.

(11) Prove the *Pythagorean theorem for general inner product spaces*: if $\mathbf{v} \perp \mathbf{w}$, then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

(12) Prove each inequality below using a judicious choice of inner product in conjunction with the Cauchy-Schwarz inequality (and possibly a judicious choice of one of the vectors in the Cauchy-Schwarz inequality).
 (a) For all $f, g \in C([a, b])$

$$\left(\int_a^b f(x)g(x)\, dx\right)^2 \leq \int_a^b f^2(x)\, dx \int_a^b g^2(x)\, dx.$$

 (b) For all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$(x_1 + x_2 + \cdots + x_n) \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\sqrt{n}.$$

(c) For all $a, b, \theta \in \mathbb{R}$

$$(a\cos(\theta) + b\sin(\theta))^2 \leq a^2 + b^2.$$

(13) Let $(V, \langle\ ,\rangle)$ be an inner product space. Recall in this context that we define $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$, and $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$.

An **isometry** of V is a function $f\colon V \to V$ that preserves distance: i.e.,

$$d(f(\mathbf{v}), f(\mathbf{w})) = d(\mathbf{v}, \mathbf{w}) \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

In this exercise we will show that any isometry that maps $\mathbf{0}$ to $\mathbf{0}$ is a linear transformation. (This is a handy fact to know. For example, reflections through lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ clearly preserve distances and map $\mathbf{0}$ to itself; it follows immediately that these operations are linear.)

In what follows, assume that f is an isometry of V satisfying $f(\mathbf{0}) = \mathbf{0}$.

(a) Prove that $\|f(\mathbf{v})\| = \|\mathbf{v}\|$: i.e., f preserves norms.

(b) Prove $\langle f(\mathbf{v}), f(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$: i.e., f preserves inner products.
Hint: first prove that $\langle \mathbf{v}, \mathbf{w}\rangle = \frac{1}{2}(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2)$.

(c) To prove f is linear it is enough to show $f(\mathbf{v} + c\mathbf{w}) = f(\mathbf{v}) + cf(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$, $c \in \mathbb{R}$.
To do so, use the above parts to show that

$$\|f(\mathbf{v} + c\mathbf{w}) - (f(\mathbf{v}) + cf(\mathbf{w}))\| = 0.$$

Hint: regroup this difference in a suitable manner so that you can use parts (a)-(b). You may also want to use the identity

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2.$$

5.2: ORTHOGONAL BASES AND PROJECTION

All questions related to orthogonality, norm, distance, etc. in the following exercises are always understood to be with regard to the given inner product.

(1) Let $(V, \langle,\rangle)$ be an inner product space, and let $W \subseteq V$ be a subspace. Show that $W^\perp$ is a subspace.

(2) Let $(V, \langle,\rangle)$ be an inner product space, and let $W \subseteq V$ be a subspace. Show that $W \cap W^\perp = \{\mathbf{0}\}$.

(3) Consider the vector space $V = C([0, 1])$ with standard *integral inner product.*
Apply Gram-Schmidt to the basis $B = \{1, 2^x, 3^x\}$ of $W = \operatorname{span}(B)$ to obtain an orthogonal basis of W.

(4) Consider the vector space $V = P_2$ with the *evaluation inner product*

$$\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

Apply Gram-Schmidt to the standard basis of P_2 to obtain an orthogonal basis of P_2.

(5) Let $V = M_{22}$ with inner product $\langle A, B\rangle = \operatorname{tr}(A^T B)$, and let $W \subseteq V$ be the subspace of matrices whose trace is 0.

(a) Compute an orthogonal basis for W.

(b) Compute $\operatorname{proj}_W(A)$, where A is the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

(6) Let $V = M_{nn}$ with inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$.
Let W_1 be the subspace of all symmetric matrices ($A^T = A$), and let W_2 be the subspace of all skew-symmetric matrices ($A^T = -A$).
Recall (from a previous exercise) that $\dim W_1 = \frac{n(n+1)}{2}$ and $\dim W_2 = \frac{n(n-1)}{2}$.
   (a) Show that $W_2 \subseteq (W_1)^\perp$: i.e., skew-symmetric matrices are orthogonal to symmetric matrices!
   (b) Use the result of Exercise 12 and a dimension argument to show that in fact $W_2 = W_1^\perp$.
   (c) Use parts (a)-(b) and the orthogonal projection theorem to show that any matrix $A \in M_{nn}$ can be written uniquely in the form $A = A_1 + A_2$, where A_1 is a symmetric matrix and A_2 is a skew-symmetric matrix.

(7) Let $V = M_{22}$ with inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$, let W_1 be the subspace of symmetric matrices, and let W_2 be the subspace of skew-symmetric matrices.
   (a) Verify that $B_1 = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is an orthogonal basis of W_1.
   Verify that $B_2 = \left\{ A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is an orthogonal basis of W_2.
   (b) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.
      (i) Use the previous exercise and the orthogonal projection formula to write $A = A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric.
      (ii) Compute $d(A, W_1)$, the distance from A to the subspace W_1.

(8) Let $V = C([0,1])$ with the standard *integral inner product*, and let $f(x) = x$. Find the function of the form $g(x) = a + b\cos(2\pi x) + c\sin(2\pi x)$ that best approximates $f(x)$ (with respect to this inner product).

(9) Compute an orthogonal basis of W, where W is the plane $x + 2y - z = 0$. Then extend this orthgonal basis to an orthogonal basis of $\mathbb{R}^3$.

(10) Let A be an $n \times n$ matrix.
   Prove: $A^T = A^{-1}$ if and only if the columns of A are an orthonormal basis of $\mathbb{R}^n$.
   Matrices satisfying $A^T = A^{-1}$ are called **orthogonal matrices**.

(11) Let V an inner product space, and let $W \subseteq V$ be a finite-dimensional subspace.
Recall that $\mathrm{proj}_W(\mathbf{v})$ is defined as the unique $\mathbf{w} \in W$ satisfying $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$, where $\mathbf{w}^\perp \in W^\perp$.
Use this definition (including the uniqueness claim) to prove the following properties:
   (a) $\mathrm{proj}_W(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\mathrm{proj}_W(\mathbf{v}_1) + c_2\mathrm{proj}_W(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and all $c_1, c_2 \in \mathbb{R}$;
   (b) $\mathbf{v} \in W \iff \mathrm{proj}_W(\mathbf{v}) = \mathbf{v}$;
   (c) $\mathbf{v} \in W^\perp \iff \mathrm{proj}_W(\mathbf{v}) = \mathbf{0}$.

(12) Let $(V, \langle\ ,\ \rangle)$ be an inner product space of dimension n, and suppose $W \subseteq V$ is a subspace of dimension r. Prove: $\dim W^\perp = n - r$.
Hint: begin by picking an *orthogonal* basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ of W and extend to an *orthogonal* basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{u}_1, \dots, \mathbf{u}_{n-r}\}$ of all of V. Show the $\mathbf{u}_i$ form a basis for $W^\perp$.

(13) Let $(V, \langle\ ,\ \rangle)$ be an inner product space of dimension n. Let $T\colon V \to V$ be defined as $T(\mathbf{v}) = \mathrm{proj}_W(\mathbf{v})$: i.e., T is orthogonal projection onto W.
Let B_1 be a basis for W and B_2 be a basis for $W^\perp$.

(a) Using Exercises 11 and 12, as well as the orthogonal projection theorem, show that $B = B_1 \cup B_2$ is a basis for V consisting of eigenvectors of T.

(b) Compute the corresponding diagonal matrix $D = [T]_B$. Make sure to indicate how many times a given eigenvalue is repeated.

Note: this shows that any orthogonal projection of a finite-dimensional inner product space is diagonalizable.

(14) We consider the problem of fitting a collection of data points (x, y) with a quadratic curve of the form $y = f(x) = ax^2 + bx + c$.

Thus we are *given* some collection of points (x, y), and we *seek* parameters a, b, c for which the graph of $f(x) = ax^2 + bx + c$ " best fits" the points in some way.

(a) Show, using linear algebra, that if we are given any three points $(x, y) = (r_1, s_1), (r_2, s_2), (r_3, s_3)$, where the x-coordinates r_i are all distinct, then there is a *unique* choice of a, b, c such that the corresponding quadratic function agrees *precisely* with the data.

In other words, given just about any three points in the plane, there is a unique quadratic curve connecting them.

(b) Now suppose we are given the four data points

$$P_1 = (0, 2), P_2 = (1, 0), P_3 = (2, 2), P_4 = (3, 6).$$

(i) Use the least-squares method described in the lecture notes to come up with a quadratic function $y = f(x)$ that "best fits" the data.

(ii) Graph the function f you found, along with the points P_i. (You may want to use technology.)

Use your graph to explain precisely in what sense f "best fits" the data.