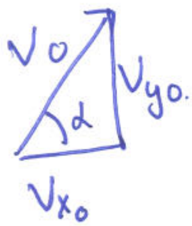
Projectile motion.

$\vec{r}(t)$ = position at time t

$\vec{v}(t) = \vec{r}\,'(t)$

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t).$

$\vec{r}(0) = \langle 0,0 \rangle.$ $\quad \vec{a} = \langle 0, -g \rangle \quad g = 32\,ft/s^2$ or $9.81\,m/s^2$



$$\vec{v} = \int \vec{a}\,dt = \int \langle 0, -g \rangle dt = \underline{\langle 0 + c_1, -gt + c_2 \rangle}$$

$$\vec{r} = \int \vec{v}\,dt = \int \langle c_1, -gt + c_2 \rangle = \langle \underline{c_1} t + c_3, -g\frac{t^2}{2} + \underline{c_2} t + c_4 \rangle$$

$$\vec{v}(0) = \langle V_{x_0}, V_{y_0} \rangle = \langle \underline{c_1}, -0 + c_2 \rangle$$

$V_{x_0} = V_0 \cos\alpha$

$V_{y_0} = V_0 \sin\alpha.$

$$\vec{r}(0) = \langle 0, 0 \rangle = \langle c_3, c_4 \rangle$$

$$\vec{v}(t) = \langle V_{x_0}, -gt + V_{y_0} \rangle.$$

$$\vec{r}(t) = \langle V_{x_0} t + 0, -g\frac{t^2}{2} + V_{y_0} t + 0 \rangle$$

Finding Max height, flight time and range.

$\vec{a}(t) = \langle 0, -g \rangle$

$\vec{v}(t) = \langle v_{x_0}, -gt + v_{y_0} \rangle$

$\vec{r}(t) = \langle v_{x_0} t, -g\frac{t^2}{2} + v_{y_0} t \rangle$

$-gt + v_{y_0} = 0 \qquad t = \frac{v_{y_0}}{g}$ time to max height.

max height $= -g\frac{\left(\frac{v_{y_0}}{g}\right)^2}{2} + v_{y_0}\left(\frac{v_{y_0}}{g}\right) = -\frac{v_{y_0}^2}{2g} + \frac{v_{y_0}^2}{g} = \frac{v_{y_0}^2}{g}\left(-\frac{1}{2}+1\right)$

$= \frac{1}{2}\frac{v_{y_0}^2}{g}$

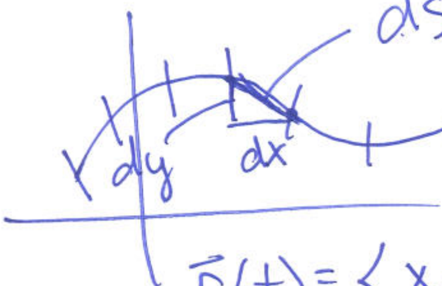
flight time $= 2\left(\frac{v_{y_0}}{g}\right)$

Range $= v_{x_0}\left(2\frac{v_{y_0}}{g}\right) = \frac{2 v_{x_0} v_{y_0}}{g} = \frac{2 v_0 \cos\alpha \, v_0 \sin\alpha}{g}$

$= \frac{\cancel{2} v_0^2 (2\cos\alpha \sin\alpha)}{g}$ $= \cancel{2}\sin 2\alpha$

$= \frac{v_0^2}{g}\sin\alpha$

Arc length



$ds = \sqrt{dx^2 + dy^2}$

$$S = \int_C ds = \int_C \sqrt{dx^2 + dy^2}\,\frac{dt}{dt}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$\vec{r}(t) = \langle x, y \rangle$

$a \le t \le b$

$\vec{v} = \vec{r}\,' = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

$\text{speed} = |r'| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

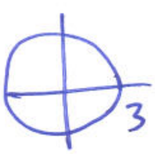
$$S = \int_a^b |r'|\,dt \;*$$

Think distance = speed · time

$\underline{Ex}$: $\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle \quad 0 \le t \le 2\pi$.

$C = 2\pi r = 2\pi \cdot 3$
$= 6\pi$

$r' = \langle -3\sin t, 3\cos t \rangle$

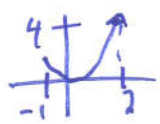


$|r'| = \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9(1)} = 3$

$S = \int_0^{2\pi} 3\,dt$

$= \underline{(2\pi)(3)}$

$r(t) = \langle t, t^2 \rangle \quad -1 \le t \le 2$



$r' = \langle 1, 2t \rangle$

$|r'| = \underline{\sqrt{1 + 4t^2}}$

$$S = \int_{-1}^{2} |r'|\,dt = \int_{-1}^{2} \sqrt{1 + 4t^2}\,dt$$

The Arclength Parameter (finding derivatives wrt s)

Arc length

$\vec{r}(t)$ $a \le t \le b$

$s = \int_a^b |r'(t)|\,dt$

$$S(t) = \int_a^t |r'(t)|\,dt = \int_a^t |r'(\tau)|\,d\tau$$

$$\frac{ds}{dt} = \frac{d}{dt}\int_a^t |r'(t)|\,dt = |r'(t)| = \frac{ds}{dt}$$

speed = $\frac{\text{distance}}{\text{time}}$.

$r(t) = \langle \cos t, \sin t, t \rangle$ $0 \le t$

$\frac{d\vec{r}}{dt} = r' = \langle -\sin t, \cos t, 1 \rangle$

$|r'| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1} = \sqrt{1+1} = \sqrt{2}$

$S(t) = \int_0^t \sqrt{2}\,dt$

$= \sqrt{2}\,t$

$S = \sqrt{2}\,t$

$\vec{r}(t(s)) = \langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \rangle$ $\qquad \frac{s}{\sqrt{2}} = t$

$\frac{d\vec{r}}{ds} = \langle -\sin\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}}, \cos\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\sin^2\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{2} + \cos^2\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$

$\frac{d\vec{r}}{ds}$ is a unit vector $\quad \frac{d\vec{r}}{ds} = \vec{T}$

The Unit Tangent Vector

$$\frac{d\vec{r}}{ds} \qquad \frac{d\,\vec{r}(t(s))}{ds} \qquad D(r(t(s)))$$

$$= Dr \cdot Dt$$

$$\frac{dr}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds} = \frac{dr/dt}{ds/dt}$$

$$\frac{d\vec{r}/dt}{ds/dt} = \frac{r'}{|r'|} = \vec{T}$$

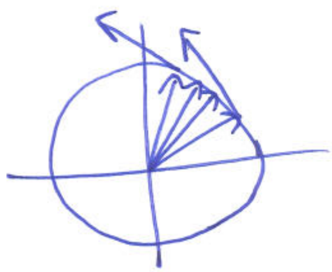
$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$r' = \langle -\sin t, \cos t, 1 \rangle$

$|r'| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1} = \sqrt{1+1} = \sqrt{2}.$

$$\vec{T} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

Vectors of constant length have a derivative which is orthogonal to itself.



If $|v(t)| = c$

then $\vec{v}\cdot\vec{v}\,' = 0$

Proof: $|\vec{v}| = c$

$|\vec{v}|^2 = c^2$ recall $\vec{v}\cdot\vec{v} = |\vec{v}|^2$

$\vec{v}\cdot\vec{v} = c^2$ diff both sides.

Product rule.

$\vec{v}\cdot\vec{v}\,' = \vec{v}\,'\cdot\vec{v}$

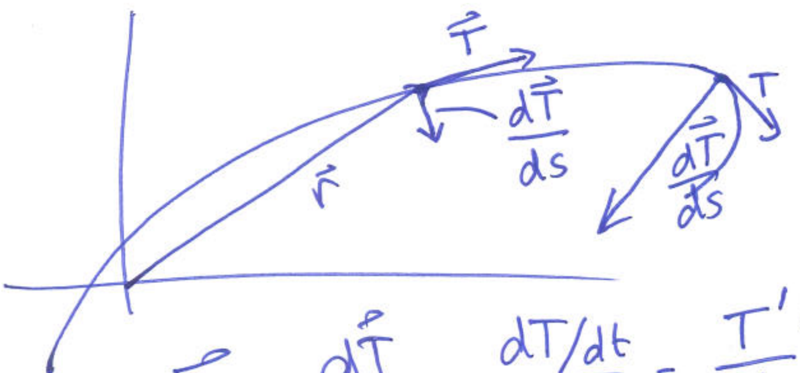
$$\vec{v}\cdot\vec{v}\,' + \underbrace{\vec{v}\,'\cdot\vec{v}}_{\vec{v}\cdot\vec{v}\,'} = 0$$

$$\frac{2\vec{v}\cdot\vec{v}\,'}{2} = \frac{0}{2}$$

$$\vec{v}\cdot\vec{v}\,' = 0$$

Curvature and the Normal vector $\vec{N}$

Since $|\vec{T}| = 1$ $\quad \dfrac{d\vec{T}}{ds} \cdot \vec{T} = 0$ $\quad \vec{T}$ + $\dfrac{d\vec{T}}{ds}$ are orthogonal.



$\dfrac{d\vec{T}}{ds}$ is long when the curve is rapidly changing direction

$$\vec{K} = \frac{d\vec{T}}{ds} = \frac{dT/dt}{ds/dt} = \frac{T'}{|r'|} \qquad |\vec{K}| = \text{curvature} = K$$

Curvature vector.

$$\vec{N} = \frac{\vec{K}}{|\vec{K}|} = \frac{T'}{|T'|}$$

$r(t) = \langle 3\cos t, 3\sin t \rangle$

$r'(t) = \langle -3\sin t, 3\cos t \rangle.$

$|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9} = 3$

$\vec{T} = \langle -\frac{\cancel{3}}{\cancel{3}}\sin t, \frac{\cancel{3}}{\cancel{3}}\cos t \rangle$

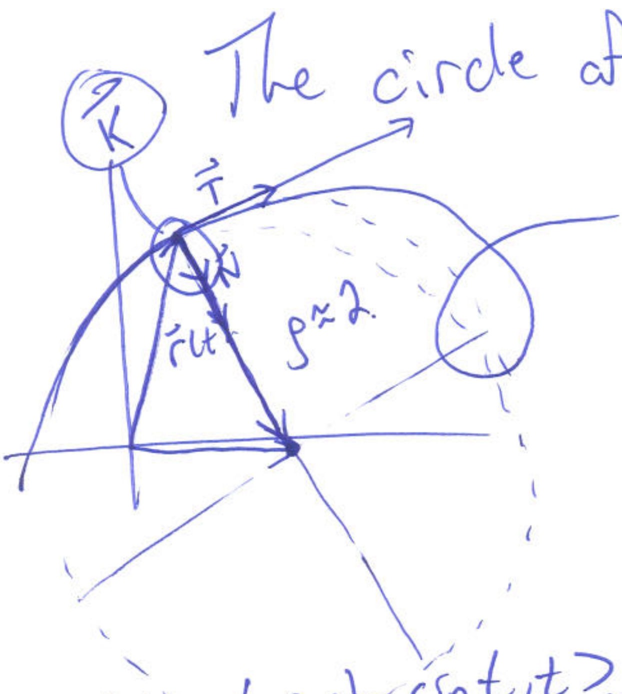
$\dfrac{d\vec{T}}{dt} = T' = \langle -\cos t, -\sin t \rangle$

$|\vec{T}| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$$\vec{K} = \frac{T'}{|V|} = \frac{\langle -\cos t, -\sin t \rangle}{3} \qquad |\vec{K}| = K = \frac{1}{3}$$

$$\vec{N} = \frac{T'}{|T'|} = \frac{\langle -\cos t, -\sin t \rangle}{1}$$

The circle of curvature



$\rho = 2 \quad K = \frac{1}{\rho} = \frac{1}{2}$

$\vec{K} = \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{T'}{|r'|}$

$\vec{r} + \overset{\text{mag.}}{\rho} \overset{\text{dir}}{\vec{N}} = $ center of curvature

$r(t) = \langle \cos t, \sin t, t \rangle$ @ $\pi/2$ $\quad r(\pi/2) = \langle 0, 1, \pi/2 \rangle$

$\rho = 2$

$\vec{N}(\pi/2) = \langle 0, -1, 0 \rangle$

$r' = \langle -\sin t, \cos t, 1 \rangle$

$|r'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

Center $= \langle 0, 1, \pi/2 \rangle + 2\langle 0, -1, 0 \rangle$

$= \langle 0, -1, \pi/2 \rangle$

$\vec{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$

$\vec{T}' = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$

$\vec{K} = \frac{T'}{|r'|} = \frac{\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle}{\sqrt{2}} = \frac{1}{2} \langle -\cos t, -\sin t, 0 \rangle$

$N = \langle -\cos t, -\sin t, 0 \rangle$

$|\vec{K}| = K = \frac{1}{2}\sqrt{\cos^2 t + \sin^2 t + 0} = \frac{1}{2} \quad \rho = 2$

$\vec{N} = \frac{\vec{K}}{|K|}$

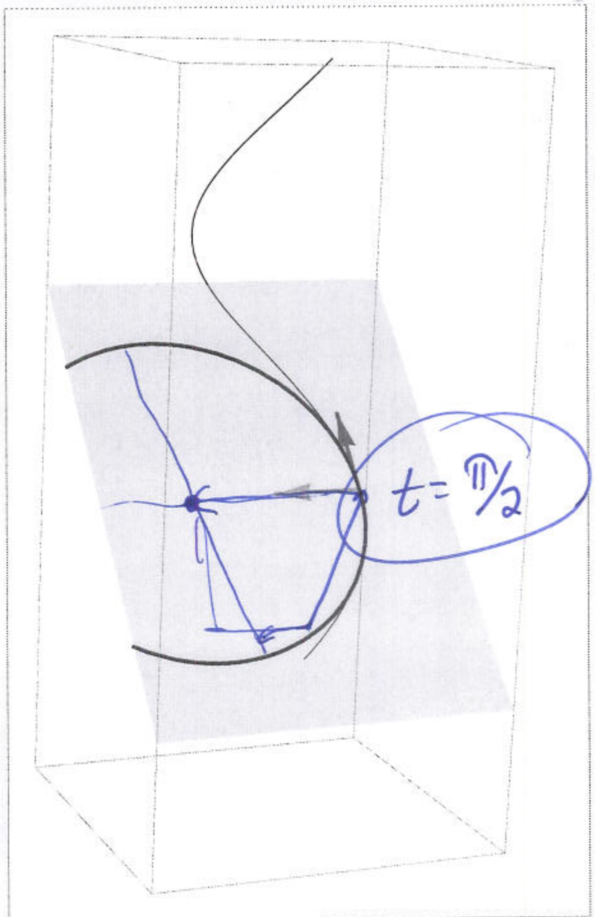
Out[64]=

cc

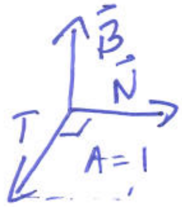
xbounds {x, −2, 2}

ybounds {y, −2, 2}

zbounds {z, −2, 2 π}



The Binormal vector and Torsion.



$\vec B = \vec T \times \vec N$ $\qquad$ $\vec N \times \vec T = -\vec B$

Since $|\vec B| = 1$ $\frac{d\vec B}{ds}$ is $\perp$ to $\vec B$

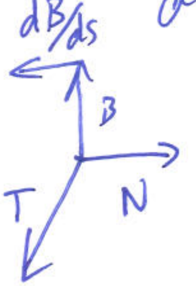
dB/ds is $\parallel$ or $\downarrow\uparrow$ to $\vec N$

$$\frac{d}{ds}\vec B = \frac{d}{ds}(\vec T \times \vec N) = \vec T \times \frac{d\vec N}{ds} + \underbrace{\overbrace{\frac{d\vec T}{ds}}^{\vec K} \times \vec N}_{\vec 0}$$

$$\frac{d\vec B}{ds} = \vec T \times \frac{d\vec N}{ds}$$

$\frac{d\vec B}{ds}$ & $\vec T$ are $\perp$

$\tau = \pm\left|\frac{dB}{ds}\right|$



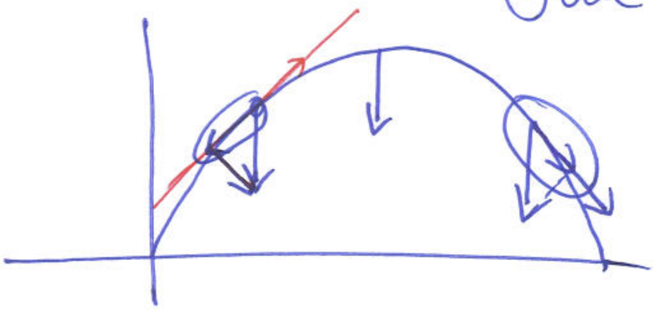
$\tau > 0$

$\tau < 0$

Tangential and Normal components of acceleration

Goal $\vec{a} = a_T \vec{T} + a_N \vec{N}$

find.



$$a_T \vec{T} = \text{proj}_T \vec{a} = \frac{\vec{a}\cdot\vec{T}}{|\vec{T}|}\cdot\frac{\vec{T}}{|\vec{T}|}$$

$$a_N = \vec{a}\cdot\vec{N}.$$

$$\vec{v} = \overset{\text{mag}}{|v|}\cdot\overset{\text{dir}}{\vec{T}}$$

$$\vec{a} = \frac{d}{dt}\vec{v}$$

$$= \frac{d}{dt}\left(|v|\vec{T}\right) = \left(\frac{d}{dt}|\vec{v}|\right)\vec{T} + |\vec{v}|\cdot\frac{d\vec{T}}{dt}$$

product rule.

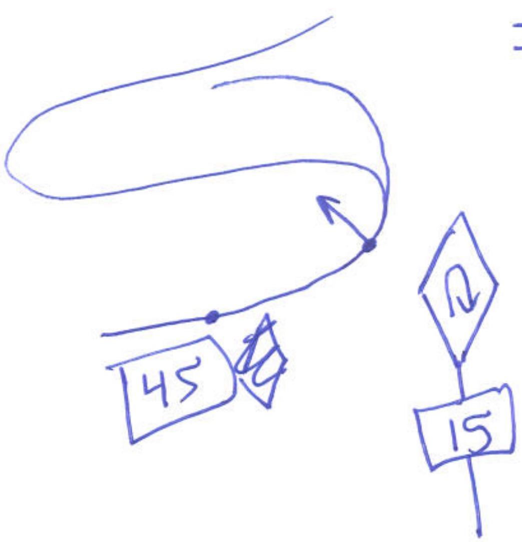
$$\vec{K} = \frac{d\vec{T}}{ds} = \frac{dT/dt}{ds/dt}$$

$$\vec{K} = \frac{T'}{|v|} \qquad \vec{T}' = \vec{K}|\vec{v}|$$

$$= \underbrace{\frac{d}{dt}|v|}_{a_T}\vec{T} + |\vec{v}||\vec{v}|\vec{K}$$

$$+ \underbrace{|v|^2 K}_{a_N}\vec{N}$$

$$\vec{K} = K\cdot\vec{N}$$



$$a_N = |v|^2 K.$$

$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ by reducing by $\frac{1}{3}$, I $\frac{1}{9}$th the a_N.

A final Example: $\vec{T}, \vec{N}, \vec{B}, \vec{K}, K, \rho$, center, τ, a_T, a_N

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 4 \rangle \qquad r'' = \langle -3\cos t, -3\sin t, 0 \rangle.$

$|v| = |r'| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$

$\vec{T} = \frac{r'}{|r'|} = \langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5} \rangle$

$T' = \langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0 \rangle \qquad K = |K| = \frac{3}{25}$

$\vec{K} = \frac{T'}{|v|} = \langle -\frac{3\cos t}{25}, -\frac{3}{25}\sin t, 0 \rangle \qquad \rho = \frac{25}{3}$

$\vec{N} = \frac{\vec{K}}{K} = \langle -\cos t, -\sin t, 0 \rangle \qquad |\vec{N}| = 1$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t, & \frac{4}{5} \\ -\cos t & -\sin t, & 0 \end{vmatrix} = \langle \frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t \rangle$$

$$= \langle \frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5} \rangle$$

$$\frac{d\vec{B}}{ds} = \frac{dB/dt}{|v|} = \frac{\langle \frac{4}{5}\cos t, +\frac{4}{5}\sin t, 0 \rangle}{5}$$

$$= \frac{4}{25}\langle \cos t, \sin t, 0 \rangle \qquad |dB/ds| = \frac{4}{25} \qquad \tau = +\frac{4}{25}$$

$a_T = \vec{a} \cdot \vec{T} = \cancel{\frac{9}{5}\cos^2 t + \frac{9}{5}\sin^2 t} = 0$