Multivariable Calculus

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With references to *Calculus Early Transcendentals*, 7th Edition, by Stewart

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Introduction

Mathematical Truths

From Dr. Woodruff:

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
>> Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Teaching philosophy

From Dr. Schmitt:

Over time, I've come to view teaching and learning as a shared journey on which my students and I embark each semester. I am the subject matter expert responsible for providing information and guidance, setting expectations, and assessing how well students meet those expectations. My students are responsible for much hard work, including preparing in advance for class, participating in class activities, and doing out-of-class assignments, regardless of whether or not they are graded. There is only so much that can be conveyed in 50 minutes, and my own personal experience and educational research agree that students get far less out of a 50-minute lecture than their professors hope. Thus, I have chosen to take an approach that is more work both for you and for me but has been shown to produce better results. During class you will work on a carefully chosen series of problems designed to build the mathematical knowledge and experience you need to succeed. These problems will be done in a collaborative, small group setting where you can grapple with and truly understand the material. I'll be there to support, guide, and correct misconceptions. Sure, I could expect you to do this alone outside of class, but over time I've realized a few things about working in groups. As a student, I usually understood something better when I went over it with classmates, even if I was the one who thought I understood it completely and explained it to a peer. As a researcher, I am more productive and effective when I collaborate. Friends in industry report that teams are increasingly used to produce the best results. Furthermore, having me there to help in the early stages ensures that we're traveling together on this journey.

-Dr. Karl Schmitt
Modified with permission from Dr. Mitchel Keller at Washington & Lee University

Modification Notes

This work is based almost entirely off of Ben Woodruff's IBL textbook (see copyright information on previous pages). Some modifications have been made by Dr. Karl Schmitt at Valparaiso University to more closely match the teaching and content for Valparaiso's Calculus III course (Math 253). In particular several chapters/units have been split apart or reorganized.

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Unit 0

Review: Calculus I & II

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When you complete this unityou should be able to:

1. Summarize the ideas you learned in Calculus I and II Math 131 and 132, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).

2. Compute the differential dy of a function and use it to approximate the change in a function.

3. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely many solutions.

4. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

0.1 Review of First Semester Calculus

0.1.1 Graphing

We'll need to know how to graph by hand some basic functions. If you have not spent much time graphing functions by hand before this class, then you should spend some time graphing the following functions by hand. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

Exercise 0.1

Provide a rough sketch of the following functions, showing their basic shapes:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Then use a computer algebra system, such as Wolfram Alpha, to verify your work. I suggest Wolfram Alpha, because it is now built into Mathematica 8.0+. If you can learn to use Wolfram Alpha, you will be able to use Mathematica.

To Valpo Students: You are also welcome to use Maple. On the surface both Mathematica and Maple are very similar.

0.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. The following rules are crucial.

- Power rule $(x^n)' = nx^{n-1}$
- Sum and difference rule $(f \pm g) = f' \pm g'$
- Product $(fg)' = f'g + fg'$ and quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Chain rule (arguably the most important) $(f \circ g)' = f'(g(x)) \cdot g'(x)$

Exercise 0.2

Compute the derivative of $e^{\sec x} \cos(\tan(x) + \ln(x^2 + 4))$.
Show each step in your computation, making sure to show what rules you used.

Stewart: See sections 3.1-3.6 for more practice with derivatives.

Exercise 0.3

If $y(p) = \dfrac{e^{p^3} \cot(4p + 7)}{\tan^{-1}(p^4)}$ find dy/dp.
Again, show each step in your computation, making sure to show what rules you used.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Exercise 0.4

Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1-x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

Stewart: See section 3.5 for more examples involving inverse trig functions and implicit differentiation.

0.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Exercise 0.5

Compute $\int x\sqrt{x^2+4}dx$.

Stewart: For practice with u-substitution, see sections 5.5. For practice with integration by parts, see section 7.1

Exercise 0.6

Compute $\int x \sin 2x dx$.

Exercise 0.7

Compute $\int \arctan x dx$.

Exercise 0.8

Compute $\int x^2 e^{3x} dx$.

0.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 0.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Exercise 0.9

If $f(x) = x^2 \ln(3x + 2)$ and $g(t) = e^{2t} \tan(t^2)$ then compute df and dg.

Stewart: See 3.10:11-22

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Exercise 0.10

Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

Stewart: See 3.10:1-10 and 3.10:23-31.
The linearization of a function is just an equation of the tangent line where you solve for y.

1. Find the differential of y with respect to x.
2. Draw a graph of $f(x)$. Place a dot at the point $(3, 9)$ and label it on your graph. Sketch a tangent line to the graph at the point $(3, 9)$ on the same axes. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.
3. Using the two points $(3, 9)$ and (x, y), compute the slope of the line connecting these two points. Your answer should involve x and y. What is the rise (i.e, the change in y called dy)? What is the run (i.e, the change in x called dx)?
4. We know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. These two must be equal. Use this fact to give an equation of the tangent line to $f(x)$ at $x = 3$. (Hint: do NOT simplify to slope-intercept form)
5. How is the equation for the tangent line related to the differentials dy and dx?

Exercise 0.11

The manufacturer of a spherical storage tank needs to create a tank with a radius of 3 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 3 m, plus or minus some small amount dr. The actual radius will be $3 + dr$.

Stewart: See 3.10

1. Find the differential dV. This should be a formula with dV, r, and dr.
2. If the actual radius is 3.02
   a What is r?
   b What is dr?
3. What is dV equal to? What does dV tell you about the volume of the manufactured sphere?

Exercise 0.12
A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about 60°. If this angle of 60° is correct, then what is the height of the tree? If the ranger's angle measurement could be off by as much as 5°, then how much could his estimate of the height be off? Use differentials to give an answer.

0.3 Solving Systems of equations

Exercise 0.13
Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x+y &= 3 \\ 2x-y &= 4 \end{cases}$
- $\begin{cases} -x+4y &= 8 \\ 3x-12y &= 2 \end{cases}$

Exercise 0.14

Find all solutions to the linear system $\begin{cases} x+y+z &= 3 \\ 2x-y &= 4 \end{cases}$. Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

This link will show you how to specify which variable is t when using Wolfram Alpha.

- If you let $x = t$, what are y and z. Write your solution in the form (x, y, z) where you replace x, y, and z with what they are in terms of t.
- If you let $y = t$, what are x and z (in terms of t).
- If you let $z = t$, what are x and y.

0.4 Optional: Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Exercise 0.15
Let $f(x) = e^x$. You should find that your work on each step can be reused to do the next step.

- Find a first degree polynomial $P_1(x) = a + bx$ so that $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. In other words, give me a line that passes through the same point and has the same slope as $f(x) = e^x$ does at $x = 0$. Set up a system of equations and then find the unknowns a and b. The next two are very similar.
- Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$.
- Find a third degree polynomial $P_3(x) = a+bx+cx^2+dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$.
- Now compute $e^{.1}$ with a calculator. Then compute $P_1(.1)$, $P_2(.1)$, and $P_3(.1)$. How accurate are the line, parabola, and cubic in approximating $e^{.1}$?

Exercise 0.16
Now let $f(x) = \sin x$. Find a 7th degree polynomial so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. Evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$? You may find it valuable to use the notation

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7.$$

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

The previous two problems involved finding polynomial approximations to the function at $x = 0$. The next problem shows how to move this to any other point, such as $x = 1$.

Exercise 0.17
Let $f(x) = e^x$.

- Find a second degree polynomial

$$T(x) = a + bx + cx^2$$

so that $T(1) = f(1)$, $T'(1) = f'(1)$, and $T''(1) = f''(1)$. In other words, give me a parabola that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

- Find a second degree polynomial written in the form

$$S(x) = a + b(x-1) + c(x-1)^2$$

so that $S(1) = f(1)$, $S'(1) = f'(1)$, and $S''(1) = f''(1)$. In other words, find a quadratic that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

Notice that we just replaced x with $x-1$. This centers, or shifts, the approximation to be at $x = 1$. The first part will be much simpler now when you let $x = 1$.

- Find a third degree polynomial written in the form

$$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$$

so that $P(1) = f(1)$, $P'(1) = f'(1)$, $P''(1) = f''(1)$, and $P'''(1) = f'''(1)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 1$.

Example 0.2. This example refers back to problem 0.11. We wanted a spherical tank of radius 3m, but due to manufacturing error the radius was slightly off. Let's now illustrate how we can use polynomials to give a first, second, and third order approximation of the volume if the radius is 3.02m instead of 3m.

We start with $V = \frac{4}{3}\pi r^3$ and then compute the derivatives

$$V' = 4\pi r^2, V'' = 8\pi r, \text{ and } V''' = 8\pi.$$

Because we are approximating the increase in volume from $r = 3$ to something new, we'll create our polynomial approximations centered at $r = 3$. We'll consider the polynomial

$$P(r) = a_0 + a_1(r-3) + a_2(r-3)^2 + a_3(r-3)^3,$$

whose derivatives are

$$P' = a_1 + 2a_2(r-3) + 3a_3(r-3)^2, P'' = 2a_2 + 6a_3(r-3), P''' = 6a_3.$$

So that the derivatives of the volume function match the derivatives of the polynomial (at $r = 3$), we need to satisfy the equations in the table below.

k	Value of V at the kth derivative	Value of P at the the kth derivative	Equation
0	$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi$	$P(3) = a_0$	$a_0 = 36\pi$
1	$V'(3) = 4\pi(3)^2 = 36\pi$	$P'(3) = a_1$	$a_1 = 36\pi$
2	$V''(3) = 8\pi(3) = 24\pi$	$P''(3) = 2a_2$	$2a_2 = 24\pi$
3	$V'''(3) = 8\pi$	$P'''(3) = 6a_3$	$6a_3 = 8\pi$

This tells us that the third order polynomial is

$$P(r) = a_0 + a_1(r-3) + a_2(r-3)^2 + a_3(r-3)^3 = 36\pi + 36\pi(r-3) + 12\pi(r-3)^2 + \frac{4}{3}\pi(r-3)^3.$$

We wanted to approximate the volume if $r = 3.2$, so our change in r is $dr = 3.2 - 3 = 0.2$. We can rewrite our polynomial as

$$P(r) = 36\pi + 36\pi(dr) + 12\pi(dr)^2 + \frac{4}{3}\pi(dr)^3.$$

We are now prepared to approximate the volume using a first, second, and third order approximation.

1. A first order approximation yields $P = 36\pi + 36\pi \cdot 0.02 = 36.72\pi$. The volume increased by 0.72π m^3.

2. A second order approximation yields
$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 = 36.7248\pi.$$

3. A third order approximation yields
$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 + \frac{4}{3}\pi(0.02)^3 = 36.724810\bar{6}\pi.$$

With each approximation, we add on a little more volume to get closer to the actual volume of a sphere with radius $r = 3.02$. The actual volume of a sphere involves a cubic function, so when we approximate the volume with a cubic, we should get an exact approximation (and $V(3.02) = \frac{4}{3}\pi(3.02)^3 = (36.724810\bar{6})\pi$.)

We'll end this section with a problem to practice the example above.

Exercise 0.18

Suppose you are constructing a cube whose side length should be $s = 2$ units. The manufacturing process is not exact, but instead creates a cube with side lengths $s = 2 + ds$ units. (You should assume that all sides are still the same, so any error on one side is replicated on all. We have to assume this for now, but before the semester ends we'll be able to do this with high dimensional calculus.)

Suppose that the machine creates a cube with side length 2.3 units instead of 2 units. Note that the volume of the cube is $V = s^3$. Use a first, second, and third order approximation to estimate the increase in volume caused by the .3 increase in side length. Then compute the actual increase in volume $V(2.3) - V(2)$.

As a challenge, try to draw a 3D graph which illustrates the volume added on by each successive approximation. If you have it before we get here as a class, let me know and I'll let you share what you have discovered with the class.

Unit 1

Vectors

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In this unit you will learn how to...

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

1.1 Vectors and Lines

Topical Objectives:

- understand vectors as quantities having length and direction, independent of position
- express curves with parametric (vector) equations

Learning to work with vectors will be key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors. ‘

Exercise 1.1

To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

1. Draw a 3-D axis then plot the points $(1, 5)$ and $(2, 3)$
2. Sketch the triangle that connects these two points and the origin – $(0, 0)$
3. Find Δx and Δy
4. Now extend your picture into 3 and use it to show that the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
5. Use this formula to find the distance between the points $(1, 5, 2)$ and $(2, 3, 3)$

Exercise 1.2

Recall that a circle (or sphere in 3-D) is just a collection of points which are all equal distance away from a center point.

Stewart: See 12.1:11-18

1. If the center of a circle is the point $(1, 5)$ what is the equation for a circle which passes through the point $(2, 3)$?
2. Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$.
3. Now find an equation of the sphere passing though point Q whose center is at P.
   Hint: Each point on the surface of a sphere is r distance away from the center.

Definition 1.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

$$\vec{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The entries of the vector are called the x, y, and z (or i, j, k) components of the vector.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 1.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors.
Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane.
Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 1.3. Suppose $\vec{x} = \langle x_1, x_2, x_3\rangle$ and $\vec{y} = \langle y_1, y_2, y_3\rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Exercise 1.3

Consider the vectors $\vec{u} = \langle 1, 2\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$. Stewart: See 12.2:5-6

Exercise 1.4

Consider the vector $\vec{v} = \langle 3, -1\rangle$.

1. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$.

Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time.

2. At the following times, where is the donkey?
   (a) $t = 0$?
   (b) $t = 1$?
   (c) $t = 2$?
3. Draw an axis, then sketch the path followed by the donkey.
4. Where is the donkey at time $t = 0, 1, 2$? Put markers with labels on your graph to show the donkey's location.

In a previous problem (1.4) you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function is called a parametrization of the donkey's path. Because the output is a vector, we call the function a vector-valued function. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Let's look at another, similar problem.

Exercise 1.5

Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1,2)t + (3,4)$. (Remember this is the same as writing $(x,y) = (1,2)t + (3,4)$ or similarly $(x,y) = (1t+3, 2t+4)$.)

1. Where is the horse at time $t = 0, 1, 2$?
2. Draw an set of axis and put markers on your graph to show the horse's location.
3. Draw the path followed by the horse.

In the last two problems we described the donkey and horse's paths using vectors. The use of vectors actually lets us do a lot more like give a generalized direction or easily compute their speed at any point along the path. To do that we need to define two more mathematical ideas, the magnitude and unit-vector.

Definition 1.4. The magnitude, or length, or norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin.

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

In the special circumstance where a vector represents an object's velocity, the magnitude of the vector gives the objects *speed* or non-directional velocity.

Definition 1.5. A unit vector is a vector whose length is one unit. The notation for a unit vector is generally a "^" or **bold face**. Equivalently, the unit vector of a vector $\vec{v}$ is: $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$
The standard unit vectors are $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$, $\mathbf{k} = \langle 0,0,1 \rangle$.

Exercise 1.6: Magnitude and Unit Vector Practice

For each of the following vectors, compute the magnitude and unit vector in the same direction.

1. $-3\vec{\imath} + 7\vec{\jmath}$
2. $< -4, 2, 4 >$
3. $8\vec{\imath} - \vec{\jmath} + 4\vec{k}$

Exercise 1.7

Consider the two points $P = (1,2,3)$ and $Q = (2,-1,0)$. Stewart: See 12.2: 23-26, 41, 42

1. Write the vector $\vec{PQ}$ in component form $< a, b, c)$.
2. Find the length (magnitude) of vector $\vec{PQ}$.
3. Find a unit vector for $\vec{PQ}$.
4. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$.

Exercise 1.8

Let's return to problems 1.4 and 1.5. We now have the tools to determine the donkey's and horse's speed and direction.

1. The donkey's path was given by $(x, y) = \vec{v}t = (3t, -t)$. Find
   (a) The donkey's speed
   (b) A unit vector that gives the donkey's direction of travel.
2. The horse's path was given by $\vec{r}(t) = (1, 2)t + (3, 4)$.
   (a) State the part of $\vec{r}(t)$ which gives the direction of travel.
   (b) What is a unit vector in the same direction?
   (c) what is the speed of the horse?

Exercise 1.9

A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2\rangle$.

Stewart: Look at 12.2:34-40

Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 1.5, and base your work off of what you saw there. It's almost identical.]

Generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = \langle v_1, v_2, v_3\rangle$.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Exercise 1.10

Let $P = (3, 1)$ and $Q = (-1, 4)$.

Stewart: 12.5: 6-15

1. Write a vector equation $\vec{r}(t) = (x, y)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
2. Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
3. Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

1.2 The Dot Product

Topical Objectives:
- perform the dot-product of two vectors
- recognize when two vectors are orthogonal

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. We'll use the dot product to help us find find angles. First, we need to recall the law of cosines.

Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of* ***length*** *a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90^\circ$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Exercise 1.11

Sketch in $\mathbb{R}^2$ the vectors $\langle 1, 2\rangle$ and $\langle 3, 5\rangle$. Use the law of cosines to find the angle between the vectors. Stewart: See 12.3:15-20

Exercise 1.12

Sketch in $\mathbb{R}^3$ the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$. Use the law of cosines to find the angle between the vectors. Stewart: See 12.3:15-20

Definition 1.6: The Dot Product. If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, then we define the dot product of these two vectors to be

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process used in matrix multiplication.

Exercise 1.13

1. Use the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$.
2. Which was easier, 1.12 or this method? (You will derive this formula in a later problem)

Definition 1.7. We say that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Exercise 1.14

Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

Exercise 1.15

Mark each statement true or false. Use Definitions 1.3 - 1.7 to explain and justify or prove your claim. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Exercise 1.16

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ (which is often written $\vec{u}, \vec{v} \in \mathbb{R}^3$), then show that

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2.$$

Exercise 1.17

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

Stewart: See pages 801-802 if you are struggling

1. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos \theta$.
2. Use the above together with problem 1.16 to derive

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta.$$

Exercise 1.18

Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is $90°$. Then show that if the angle between them is $90°$, then the vectors are orthogonal. I.E. expand and compute both sides of the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$ with non-zero, orthogonal vectors.

Stewart: See page 804

The dot product provides a really easy way to find when two vectors meet at a right angle. The dot product is precisely zero when this happens.

1.3 Interlude:Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. Alternatively see texample.net for a visualization of the idea.

To Valpo Students: The electronic version has links that will open your browser and take you to the web.

Exercise 1.19

Compute the following matrix products.

To Valpo Students: For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

- $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}$

Exercise 1.20

Compute the product $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

1.3.1 Determinants

Associated with every square matrix is a number, called the determinant. Determinants are only defined for square matrices. Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Definition 1.8. The determinant of a 2×2 matrix is the number

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad-bc.$$

The determinant of a 3×3 matrix is the number

$$\begin{aligned}\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} &= a\det\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\det\begin{vmatrix}d & f\\ g & i\end{vmatrix} + c\det\begin{vmatrix}d & e\\ g & h\end{vmatrix}\\ &= a(ei-hf)-b(di-gf)+c(dh-ge).\end{aligned}$$

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Exercise 1.21

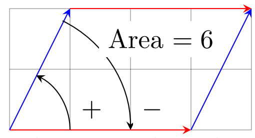
Compute $\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix}$ and $\begin{vmatrix}1 & 2 & 0\\ -1 & 3 & 4\\ 2 & -3 & 1\end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use Wolfram Alpha to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$ whose determinant is $3\cdot 2 - 0\cdot 1 = 6$. Draw the column vectors $\begin{bmatrix}3\\ 0\end{bmatrix}$ and $\begin{bmatrix}1\\ 2\end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0\le x\le 1$, $0\le y\le 2$, $0\le z\le 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign



$$\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 \text{ and } \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Exercise 1.22

1. Use determinants to find the area of the triangle with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.
2. What would you change if you wanted to find the area of the triangle with vertices $(-3,1)$, $(-2,5)$, and $(3,4)$? Find this area.

1.4 The Cross Product

Topical Objective:

- perform the cross-product of vectors

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vectors. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 1.9: The Cross Product. The cross product of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is a new vector $\vec{u} \times \vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$. The formula for the cross product is

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

This definition is not really a definition. It is actually a theorem. If you use the formula given as the definition, then you would need to prove the three facts. We have the tools to give a complete proof of (1) and (3), but we would need a course in linear algebra to prove (2). It shouldn't be too much of a surprise that the cross product is related to area, since it is defined in terms of determinants

Exercise 1.23

Let $\vec{u} = \langle 1, -2, 3 \rangle$ and $\vec{v} = \langle 2, 0, -1 \rangle$.

Stewart: See 12.4:1-7

1. Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. How are they related?
2. Compute $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$. Why did you get the answer you got?
3. Compute $\vec{u} \times (2\vec{u})$. Why did you get the answer you got?

Exercise 1.24

Consider the vectors $\vec{\imath} = (1, 0, 0)$, $2\vec{\jmath} = (0, 2, 0)$, and $3\vec{k} = (0, 0, 3)$.

1. Compute $\vec{\imath} \times 2\vec{\jmath}$ and $2\vec{\jmath} \times \vec{\imath}$.
2. Compute $\vec{\imath} \times 3\vec{k}$ and $3\vec{k} \times \vec{\imath}$.
3. Compute $2\vec{\jmath} \times 3\vec{k}$ and $3\vec{k} \times 2\vec{\jmath}$.

Give a geometric reason as to why some vectors above have a plus sign, and some have a minus sign.

Exercise 1.25

Let $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR [Hint: What shape does two triangles side-by-side make?]. Construct a 3D graph of this triangle.

Stewart: See 12.4: 29-32

1.5 The Cross Product and Planes

Topical Objectives

- use the normal vector to find the equation for a plane

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Exercise 1.26

Suppose the point $P = (1, 2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n} = \langle 3, 4\rangle$ is $90°$ (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q = (x, y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$ to obtain an equation of the line L.

Exercise 1.27

Let $P = (a, b, c)$ be a point on a plane in 3D. Let $\vec{n} = \langle A, B, C\rangle$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is $90°$). Let $Q = (x, y, z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is

Stewart: See pages 819-822

$$A(x - a) + B(y - b) + C(z - c) = 0.$$

Exercise 1.28

Find an equation of the plane containing the lines $\vec{r}_1(t) = (1,3,0)t + (1,0,2)$ and $\vec{r}_2(t) = (2,0,-1)t + (2,3,2)$.

Exercise 1.29

Consider the three points $P = (1,0,0), Q = (2,0,-1), R = (0,1,3)$. Find an equation of the plane which passes through these three points. [Hint: First find a normal vector to the plane.]

Stewart: See 12.5: 23-40

Exercise 1.30

Consider the points $P = (2,-1,0)$, $Q = (0,2,3)$, and $R = (-1,2,-4)$.

1. Give an equation $(x,y,z) = (?,?,?)$ of the line through P and Q.
2. Give an equation of the line through P and R.
3. Give an equation of the plane through P, Q, and R.

Exercise 1.31

Consider the two planes $x+2y+3z = 4$ and $2x - y + z = 0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line.

Stewart: See 12.5: 23-40, 45-47

1.6 Projections and Their Applications

Course Objective:

- See applications of multi-variable calculus

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of 200 Newtons, which we will write in vector notation as $\vec{F} = \langle 0, -200\rangle$. The ramp was placed so that the box needs to be moved right 6 m, and down 3 m, so we need to get from the origin $(0,0)$ to the point $(6,-3)$. This displacement can be written as $\vec{d} = \langle 6, -3\rangle$. The force F acts straight down, which means the ramp takes some of the force. Our goal is to find out how much of the 200N the ramp takes, and how much force must be applied to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

Exercise 1.32

Read the preceding paragraph.
We want to write $\vec{F}$ as the sum of two vectors: $\vec{F} = \vec{w} + \vec{n}$

- where $\vec{w}$ is parallel to $\vec{d}$
- and $\vec{n}$ is orthogonal to $\vec{d}$

Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c.

1. Rewrite $\vec{F}$ in terms of $\vec{d}$

2. Take the dot-product of both sides with $\vec{d}$
3. Since $\vec{n}$ is orthogonal to $\vec{d}$ we know that $\vec{n} \cdot \vec{d} = ?$
4. Substitute and solve for the unknown c

The solution to the previous problem gives us the definition of a projection.

Definition 1.10. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}.$$

Exercise 1.33

Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Compute the $\text{proj}_{\vec{v}}\vec{u}$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$ all on one set of axis. Then draw a line segment from the head of $\vec{u}$ to the head of the projection. Stewart: See 12.3: 39-44

Now let $\vec{w} = (-2, 0)$. Compute $\text{proj}_{\vec{v}}\vec{w}$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{w}$. Then draw a line segment from the head of $\vec{w}$ to the head of the projection.

One application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 1.32. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200\rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3\rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Exercise 1.34: Projection Application: Work

Find the amount of work done by the force $\vec{F} = \langle 0, -200\rangle$ through the displacement $\vec{d} = \langle 6, -3\rangle$. Find this by doing the following: Stewart: See 12.3:49-52

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F} \cdot \vec{d}$. You have just shown that $W = \vec{F} \cdot \vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

Exercise 1.35: Projection Application: Planes

Consider the points $P = (2, 4, 5)$, $Q = (1, 5, 7)$, and $R = (-1, 6, 8)$.

Stewart: See 12.5:69-72

1. What is the area of the triangle PQR.
2. Give a normal vector to the plane through these three points.
3. What is the distance from the point $A = (1, 2, 3)$ to the plane PQR. [Hint: Compute the projection of $\vec{PA}$ onto $\vec{n}$. How long is it?]

1.7 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 2

Review: Conic Sections

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Upon completing this unit you will be prepared to...

1. Describe, graph, give equations of, and find foci for conic sections (parabolas, ellipses, hyperbolas).

2.1 Conic Sections

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to object in 3D. These examples are conic sections. We call them conic sections because you can obtain each one by intersecting a cone and a plane (I'll show you in class how to do this). Here's a definition.

Definition 2.1. Consider two identical, infinitely tall, right circular cones placed vertex to vertex so that they share the same axis of symmetry. A conic section is the intersection of this three dimensional surface with any plane that does not pass through the vertex where the two cones meet.

These intersections are called circles (when the plane is perpendicular to the axis of symmetry), parabolas (when the plane is parallel to one side of one cone), hyperbolas (when the plane is parallel to the axis of symmetry), and ellipses (when the plane does not meet any of the three previous criteria).

The definition above provides a geometric description of how to obtain a conic section from cone. We'll not introduce an alternate definition based on distances between points and lines, or between points and points. Let's start with one you are familiar with.

Definition 2.2. Consider the point $P = (a, b)$ and a positive number r. A circle circle with center (a, b) and radius r is the set of all points $Q = (x, y)$ in the plane so that the segment PQ has length r.

Using the distance formula, this means that every circle can be written in the form $(x - a)^2 + (y - b)^2 = r^2$.

Exercise 2.1

The equation $4x^2 + 4y^2 + 6x - 8y - 1 = 0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x - a)^2 + (y - b)^2 = r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2 + y^2 + Dx + Ey + F = 0$.

2.1.1 Parabolas

Before proceeding to parabolas, we need to define the distance between a point and a line.

Definition 2.3. Let P be a point and L be a line. Define the distance between P and L (written $d(P, L)$) to be the length of the shortest line segment that has one end on L and the other end on P. Note: This segment will always be perpendicular to L.

Definition 2.4. Given a point P (called the focus) and a line L (called the directrix) which does not pass through P, we define a parabola as the set of all points Q in the plane so that the distance from P to Q equals the distance from Q to L. The vertex is the point on the parabola that is closest to the directrix.

Exercise 2.2

Consider the line $L : y = -p$, the point $P = (0, p)$, and another point $Q = (x, y)$. Use the distance formula to show that an equation of a parabola with directrix L and focus P is $x^2 = 4py$. Then use your work to explain why an equation of a parabola with directrix $x = -p$ and focus $(p, 0)$ is $y^2 = 4px$.

Ask me about the reflective properties of parabolas in class, if I have not told you already. They are used in satellite dishes, long range telescopes, solar ovens, and more. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Exercise: Optional

Consider the parabola $x^2 = 4py$ with directrix $y = -p$ and focus $(0, p)$. Let $Q = (a, b)$ be some point on the parabola. Let T be the tangent line to L at point Q. Show that the angle between PQ and T is the same as the angle between the line $x = a$ and T. This shows that a vertical ray coming down towards the parabola will reflect of the wall of a parabola and head straight towards the vertex.

The next two problems will help you use the basic equations of a parabola, together with shifting and reflecting, to study all parabolas whose axis of symmetry is parallel to either the x or y axis.

Exercise 2.3

Once the directrix and focus are known, we can give an equation of a parabola. For each of the following, give an equation of the parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(0, 3)$ and the directrix is $y = -3$.
2. The focus is $(0, 3)$ and the directrix is $y = 1$.

Exercise 2.4

Give an equation of each parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(2, -5)$ and the directrix is $y = 3$.
2. The focus is $(1, 2)$ and the directrix is $x = 3$.

Exercise 2.5

Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = x^2$
2. $(y - 2)^2 = 4(x - 1)$

Exercise 2.6

Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = -8x^2 + 3$
2. $y = x^2 - 4x + 5$

2.1.2 Ellipses

Definition 2.5. Given two points F_1 and F_2 (called foci) and a fixed distance d, we define an ellipse as the set of all points Q in the plane so that the sum of the distances F_1Q and F_2Q equals the fixed distance d. The center of the ellipse is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the ellipse that intersect this line is called a vertex. The major axis is the segment between the two vertexes. The minor axis is the largest segment perpendicular to the major axis that fits inside the ellipse.

We can derive an equation of an ellipse in a manner very similar to how we obtained an equation of a parabola. The following problem will walk you through this.

Exercise: Optional

Consider the ellipse produced by the fixed distance d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the ellipse.

1. Show that $d = 2a$ by considering the distances from F_1 and F_2 to the point $Q = (a, 0)$.
2. Let $Q = (0, b)$ be a point on the ellipse. Show that $b^2 + c^2 = a^2$ by considering the distance between Q and each focus.
3. Let $Q = (x, y)$. By considering the distances between Q and the foci, show that an equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the fixed distance d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $(a, 0)$ be a point on the x axis that intersect the ellipse. Show that we still have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

but now we instead have $a^2 + c^2 = b^2$.

You'll want to use the results of the previous problem to complete the problems below. The key equation above is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The foci will be on the x-axis if $a > b$, and will be on the y-axis if $b > a$. The second part of the problem above shows that the distance from the center of the ellipse to the vertex is equal to the hypotenuse of a right triangle whose legs go from the center to a focus, and from the center to an end point of the minor axis.

The next three problems will help you use the basic equations of an ellipse, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Exercise 2.7

For each ellipse below, graph the ellipse and give the coordinates of the foci and vertexes.

1. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} + \dfrac{(y-2)^2}{9} = 1$

Exercise 2.8

For the ellipse $x^2 + 2x + 2y^2 - 8y = 9$, sketch a graph and give the coordinates of the foci and vertexes.

Exercise 2.9

Given an equation of each ellipse described below, and provide a rough sketch.

1. The foci are at $(2 \pm 3, 1)$ and vertices at $(2 \pm 5, 1)$.
2. The foci are at $(-1, 3 \pm 2)$ and vertices at $(-1, 3 \pm 5)$.

Exercise: Optional

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be some point on the ellipse. Let T be the tangent line to the ellipse at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that a ray from F_1 to Q will reflect off the wall of the ellipse at Q and head straight towards the other focus F_2.

2.1.3 Hyperbolas

Definition 2.6. Given two points F_1 and F_2 (called foci) and a fixed number d, we define a hyperbola as the set of all points Q in the plane so that the difference of the distances F_1Q and F_2Q equals the fixed number d or $-d$. The center of the hyperbola is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the hyperbola that intersect this line is called a vertex.

We can derive an equation of a hyperbola in a manner very similar to how we obtained an equation of an ellipse. The following problem will walk you through this.

Exercise: Optional

Consider the hyperbola produced by the fixed number d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the hyperbola.

1. Show that $d = 2a$ by considering the difference of the distances from F_1 and F_2 to the vertex $(a, 0)$.
2. Let $Q = (x, y)$ be a point on the hyperbola. By considering the difference of the distances between Q and the foci, show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$, or if we let $c^2 - a^2 = b^2$, then the equation is
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
3. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the number d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $a^2 = c^2 - b^2$. Show that an equation of the hyperbola is
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

You'll want to use the results of the previous problem to complete the problems below.

Exercise 2.10

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Construct a box centered at the origin with corners at $(a, \pm b)$ and $(-a, \pm b)$. Draw lines through the diagonals of this box. Rewrite the equation of the hyperbola by solving for y and then factoring to show that as x gets large, the hyperbola gets really close to the lines $y = \pm\frac{b}{a}x$. [Hint: rewrite so that you obtain $y = \pm\frac{b}{a}x\sqrt{\text{something}}$]. These two lines are often called oblique asymptotes.

Now apply what you have just done to sketch the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and give the location of the foci.

The next three problems will help you use the basic equations of a hyperbola, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Exercise 2.11

For each hyperbola below, graph the hyperbola (include the box and asymptotes) and give the coordinates of the foci and vertexes.

1. $16x^2 - 25y^2 = 400$ [Hint: divide by 400.]

2. $\dfrac{(x-1)^2}{5} - \dfrac{(y-2)^2}{9} = 1$

Exercise 2.12

For the hyperbola $x^2 + 2x - 2y^2 + 8y = 9$, sketch a graph (include the box and asymptotes) and give the coordinates of the foci and vertexes.

Exercise 2.13

Given an equation of each hyperbola described below, and provide a rough sketch.

1. The vertexes are at $(2 \pm 3, 1)$ and foci at $(2 \pm 5, 1)$.
2. The vertexes are at $(-1, 3 \pm 2)$ and foci at $(-1, 3 \pm 5)$.

Exercise: Optional

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be a point on the hyperbola. Let T be the tangent line to the hyperbola at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that if you begin a ray from a point in the plane and head towards F_1 (where the wall of the hyperbola lies between the start point and F_1), then when the ray hits the wall at Q, it reflects off the wall and heads straight towards the other focus F_2.

Unit 3

Parametric Equations

Contents

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3.1.2 Arc Length 32

3.2 Wrap Up **33**

After completing this unit you will be able to...

1. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.

2. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.

3. Use integrals to find the lengths of parametric curves.

3.1 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as:

$$(x, y) = (1, m)t + (0, b)$$

This style of equation is called a vector equation. It is equivalent to writing the two equations

$$x = 1t + 0, y = mt + b,$$

which we call the parametric equations of the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.1. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve.

You can generalize this definition to 3D and beyond by just adding more variables.

Exercise 3.1

By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion.

1. $x = \cos t, y = \sin t$, for $0 \leq t \leq 2\pi$.
2. $x = \sin t, y = \cos t$, for $0 \leq t \leq 2\pi$.
3. $x = \cos t, y = \sin t, z = t$, for $0 \leq t \leq 4\pi$.

Exercise 3.2

For the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$:

1. Plot the path traced out by the curve.
2. Use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y).
3. What are the foci of the resulting object (it's a conic section)?

Exercise 3.3

Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the vector unit should help here.

Exercise 3.4

For the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$:

1. Plot the path traced out by the curve.

2. Give a Cartesian equation of the curve (eliminate the parameter t)

3. Now find the focus of the resulting curve.

[Hint: Set up a system of equations then use substitution.]

Exercise 3.5

Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve (a trig identity will help). Then find the foci of the resulting conic section. [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

3.1.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.2. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then the derivative is defined as:

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The subtraction above requires vector subtraction. The following problem will provide a simple way to take derivatives which we will use all semester long.

Exercise 3.6

Show that if $\vec{r}(t) = (f(t), g(t))$, then the derivative is just $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.

[The definition above says that $\frac{d\vec{r}}{dt} = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$. We were told $\vec{r}(t) = (f(t), g(t))$, so use this in the derivative definition. Then try to modify the equation to obtain $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.]

The previous problem shows you can take the derivative of a vector valued function by just differentiating each component separately. The next problem shows you that velocity and acceleration are still connected to the first and second derivatives.

Exercise 3.7

Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$.

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.

2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.

3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 3.3. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length (or magnitude) of the velocity vector.

Exercise 3.8

Consider the curve $\vec{r}(t) = (2t + 3, 4(2t - 1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.
2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?
3. Find a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Show that the slope of the line is
$$\frac{dy}{dx}\Big|_{x=5} = \frac{(dy/dt)\big|_{t=1}}{(dx/dt)\big|_{t=1}}.$$
[How can you turn the direction vector, which involves (dx/dt) and (dy/dt) into a slope (dy/dx)?]

3.1.2 Arc Length

This section covers:

- finding rates of change along space curves

If an object moves at a constant speed, then the distance travelled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

Exercise 3.9: Derivation of the arc length formula

Suppose an object moves along the path given by:

$$\vec{r}(t) = (x(t), y(t))$$

for

$$a \leq t \leq b$$

.

1. Show that the object's speed at any time t is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
2. If you move over a really small time interval, say of length dt, then the speed is almost constant. If you move at a constant speed of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a time length of dt, what's the distance ds you have traveled.

   [Hint: This should be really fast to write down]

3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$\text{Arc Length} = s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

Exercise 3.10

Use the equation from 3 to find the length of the curve $\vec{r}(t) = \left(t^3, \frac{3t^2}{2}\right)$ for $t \in [1, 3]$. The notation $t \in [1, 3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a u-substitution).

Exercise 3.11

For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

1. The parabola $\vec{p}(t) = (t, t^2)$ for $t \in [0, 3]$.
2. The ellipse $\vec{e}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{h}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

3.2 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 4

Polar and New Coordinate Systems

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4.1.1 Graphing and Intersections 37

4.1.2 Calculus with Polar Coordinates 37

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4.3 Wrap Up **40**

After completing this unit you will be able to...

1. Convert between rectangular and polar coordinates in 2D. Convert between rectangular and cylindrical or spherical in 3D.
2. Graph polar functions in the plane. Find intersections of polar equations, and illustrate that not every intersection can be obtained algebraically (you may have to graph the curves).
3. Find derivatives and tangent lines in polar coordinates.
4. Find area and arc length using polar equations.

4.1 Polar Coordinates

Up to now, we have most often given the location of a point (or coordiantes of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Exercise 4.1

Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$.

1. Find the distance r from P to the origin.

2. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.

   [Hint: Use a triangle and trigonometry]

Exercise 4.2

Suppose that a point $Q = (a, b)$ is 6 units from the origin, and the angle the ray $\vec{OQ}$ makes with the x-axis is $\pi/4$ radians. Find the Cartesian (rectangular) coordinates (a, b) of Q.

Definition 4.1. Let Q be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We define the polar coordinates of Q to be the ordered pair (r, θ) where r is the displacement from the origin to Q, and θ is an angle of rotation (counter-clockwise) from the x-axis to the ray $\vec{OP}$.

Exercise 4.3

The following points are given using their polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

$$(1, \pi), \left(3, \frac{5\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

The next problem provides general formulas for converting between the Cartesian (rectangular) and polar coordinate systems.

Exercise 4.4

Suppose that Q is a point in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ).

1. Write formulas for x and y in terms of r and θ.

2. Write a formula to find the distance r from Q to the origin (in terms of x and y)

3. Write a formula to find the angle θ between the x-axis and a line connecting Q to the origin.
   [Hint: A picture of a triangle will help here.]

In problem 4.4, you should have obtained the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can write this in vector notation as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$. This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could write $\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name of the function because it is a transformation between coordinate systems. To emphasize that the domain and range are both two dimensional systems, we could also write $T : \mathbb{R}^2 \to \mathbb{R}^2$. In the next chapter, we'll spend more time with this notation.

The following problem will show you how to graph a coordinate transformation. When you're done, you should essentially have polar graph paper.

Exercise 4.5

Consider the coordinate transformation

For this problem, you are just drawing many parametric curves.

$$\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta).$$

1. Let $r = 3$ and then graph $\vec{T}(3, \theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1, \theta), \vec{T}(2, \theta), \vec{T}(4, \theta)$ for $\theta \in [0, 2\pi]$
4. To the same axes as above, add the graphs of $\vec{T}(r, 0), \vec{T}(r, \pi/2), \vec{T}(r, 3\pi/4), \vec{T}(r, \pi)$ for $r \in [0, 5]$.

Exercise 4.6

In the plane,

1. Graph the curve $y = \sin x$ for $x \in [0, 2\pi]$ (make an x, y table)
2. Graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (an r, θ table).

The graphs should look very different. If one looks like a circle, you're on the right track.

Exercise 4.7

Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates by substituting in the formulas from 4.4, and then solve for r so that the equation is in the form $r = f(\theta)$.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$

3. $x^2 = y$

Exercise 4.8

Each of the following equations is written in the polar coordinate system. Convert each to an equation in the Cartesian coordinates.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

4.1.1 Graphing and Intersections

To construct a graph of a polar curve, just create an r, θ table. Choose values for θ that will make it easy to compute any trig functions involved. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases.

Exercise 4.9

Graph the polar curve $r = 2\sin 3\theta$.

Exercise 4.10

Graph the polar curve $r = 3\cos 2\theta$.

Be sure you actually plot out the next two problems, otherwise you'll probably miss a few points of intersection.

Exercise 4.11

Find the points of intersection of $r = 3 - 3\cos\theta$ and $r = 3\cos\theta$.

Exercise 4.12

Find the points of intersection of $r = 2\cos 2\theta$ and $r = \sqrt{3}$.

4.1.2 Calculus with Polar Coordinates

Recall that for parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ can be thought of as the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. So you can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Exercise 4.13

Consider the polar curve $r = 1 + 2\cos\theta$. (It wouldn't hurt to provide a quick sketch of the curve.)

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

You can find arc length for parametric curves using the formula:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

If we replace t with θ, this becomes a formula for arc length in polar coordinates. However, the formula can be simplified.

Exercise 4.14
Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is a continuous function, and that f' is continuous. Show that the arc length formula can be simplified to

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

[Hint: the product rule and Pythagorean identity will help.]

Exercise 4.15
Set up **(do not evaluate)** an integral formula to compute the length of

1. the rose $r = 2\cos 3\theta$, and
2. the rose $r = 3\sin 2\theta$.

4.2 Other Coordinate Systems

Sometimes a problem can't be solved until the correct coordinate system is chosen. You have previously done problems which showed you how to graph the coordinate transformation given by polar coordinates. The following problem shows you how to graph in a different coordinate system.

Exercise 4.16
Consider the coordinate transformation $T(a, \omega) = (a\cos\omega, a^2\sin\omega)$.

1. Let $a = 3$ and then graph the curve $\vec{T}(3, \omega) = (3\cos\omega, 9\sin\omega)$ for $\omega \in [0, 2\pi]$. [Hint: You can start by making a ω, x, y table.]

   See Sage. Click on the link to see how to check your answer in Sage.

2. Let $\omega = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(a, \frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a^2\frac{\sqrt{2}}{2}\right)$ for $a \in [0, 4]$.

   See Sage. Notice that you can add the two plots together to superimpose them on each other.

3. To the same axes as above, add the graphs of:

   Use Sage to check your answer.

   (a) $\vec{T}(1, \omega), \vec{T}(2, \omega), \vec{T}(4, \omega)$ for $\omega \in [0, 2\pi]$
   (b) $\vec{T}(a, 0), \vec{T}(a, \pi/2), \vec{T}(a, -\pi/6)$ for $a \in [0, 4]$.

[Hint: when you're done, you should have a bunch of parabolas and ellipses.]

In 3 dimensions, the most common coordinate systems are cylindrical and spherical. The equations for these coordinate systems are in the table below.

To Valpo Students:

Cylindrical Coordinates	Spherical Coordinates(Math)	Spherical (Physics/Engineering)
$x = r\cos\theta$	$x = \rho\sin\theta\cos\phi$	$x = r\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\theta\sin\phi$	$y = r\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\theta$	$z = r\cos\phi$

Exercise 4.17

Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. Construct a graph in 3D of this information, and use it to develop the equations for cylindrical coordinates given above.

Stewart: See pages 827-833

Exercise 4.18

Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ, and is called the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and is called the inclination angle, polar angle, or zenith angle. Construct a graph in 3D of this information, and use it to develop the equations for spherical coordinates given above.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

There is some disagreement between different fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given—the angle from the xy-plane (lines of lattitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating using them.

To Valpo Students:

Exercise 4.19

If you have never worked in Maple, you will want to spend some time familiarizing yourself with the program before our Lab day (keep reading if you are!). The link here takes you to a set of modules which will review several of the vector concepts we learned last week, but also introduce you to Maple. The Vector exploration contains a very nice visual on vector projections, which you should look at even if you are comfortable in Maple.

The Connected Curriculum Project:
Introduction to Modules

Maple Tutorial

Vectors Exploration

4.3 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.