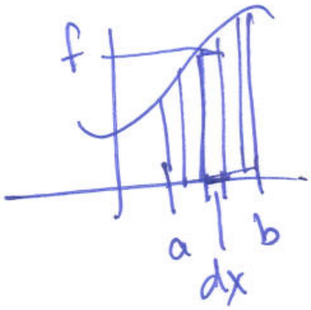
Line Integrals

$\int_a^b f(x)\,dx$



$dA = f\,dx$

$A = \int_{[a,b]} dA = \int_a^b f\,dx$

$f = 9 - x^2 - y^2$

$\vec{r} = \langle \cos t, 2\sin t \rangle.$

$f.$

$A = \int_C dA = \int_C f\,ds$

$$\frac{ds}{dt} = |r'|$$

$$ds = |r'|\,dt$$

$f = 9 - x^2 - y^2$

$\vec{r} = \langle \overset{x}{\cos t}, \overset{y}{2\sin t} \rangle$

$r' = \langle -\sin t, 2\cos t \rangle.$

$|r'| = \sqrt{\sin^2 t + 4\cos^2 t}$

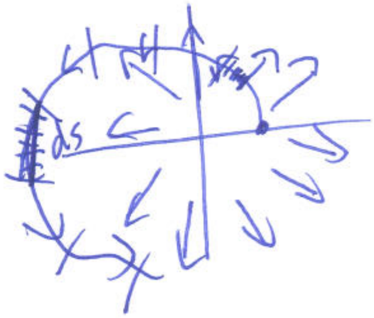
$$\int_C f\,ds$$

$$= \int_0^{2\pi} (9 - x^2 - y^2)\sqrt{\sin^2 t + 4\cos^2 t}\,dt$$

$$\int_0^{2\pi} (9 - (\cos t)^2 - (2\sin t)^2)\sqrt{\sin^2 t + 4\cos^2 t}\,dt$$

Work $\quad W = F \cdot d \implies W = \vec{F} \cdot \vec{d}$

$= |\vec{F}||\vec{d}| \cos\theta$



$\frac{d\vec{r}}{dt} = \vec{r}\,'$

$d\vec{r} = \vec{r}\,' dt$

$dW = \vec{F} \cdot (\vec{T} ds)$

$W = \int dW = \int_c \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \frac{\vec{r}\,'}{|\vec{r}\,'|} |\vec{r}\,'| dt$

$$\boxed{W = \int_a^b \vec{F} \cdot \vec{r}\,' dt}$$

$$= \int_c \vec{F} \cdot d\vec{r}$$

$\vec{F} = \langle M, N, P \rangle$.

$\vec{r} = \langle x, y, z \rangle$

$d\vec{r} = \langle dx, dy, dz \rangle$

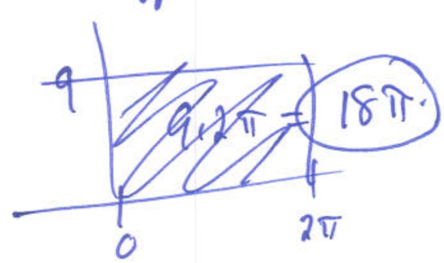
$$W = \int_c M dx + N dy + P dz$$

$\vec{F} = \langle \overset{M}{-y}, \overset{N}{x} \rangle \quad r(t) = \langle \overset{x}{3\cos t}, \overset{y}{3\sin t} \rangle \quad 0 \le t \le 2\pi$.

$r'(t) = \langle \overset{dx}{-3\sin t}, \overset{dy}{3\cos t} \rangle$.

$F(t) = \langle -3\sin t, 3\cos t \rangle$.

$W = \int_c F \cdot d\vec{r} = \int_0^{2\pi} 9\sin^2 t + 9\cos^2 t \cdot dt = \int_0^{2\pi} 9(1)\, dt$



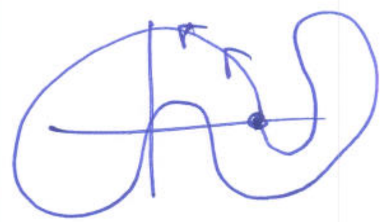
Flow + Circulation. (work).

$$\text{Flow} = \int_C \underset{\text{velocity}}{(\vec{F}\cdot\vec{T})}\, ds. \qquad \overset{m/s}{}\ \overset{m}{}\ \overset{A/s}{}$$

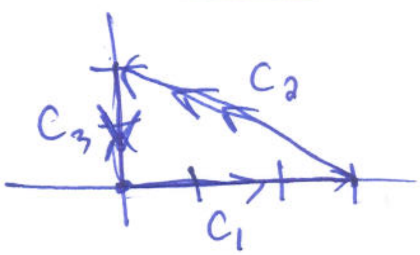
$$W = \int_C \vec{F}\cdot(\vec{T}\,ds)$$

Circulation = flow along a closed curve

$$\oint_C M\,dx + N\,dy$$



$$\vec{F} = \langle x+y,\ xy\rangle \qquad \oint_C M\,dx + N\,dy$$



$$= \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$r_1(t) = \langle 3,0\rangle t + \langle 0,0\rangle.$$
$$= \langle 3t,\ 0\rangle \qquad \underline{0 \le t \le 1}$$
$$r_1'(t) = \langle \overset{dx}{3}, \overset{dy}{0}\rangle.$$
$$F(t) = \langle 3t+0,\ 3t\cdot 0\rangle$$
$$= \langle \underset{M}{3t}, \underset{N}{0}\rangle$$

$$r_2(t) = \langle -3,2\rangle t + \langle 3,0\rangle$$
$$r_2^t = \langle \underset{x}{-3t+3}, \underset{y}{2t}\rangle$$
$$r_2' = \langle \overset{dx}{-3}, \overset{dy}{2}\rangle$$
$$F(t). = \langle \underset{M}{-3t+3+2t}, \underset{N}{(-3t+3)(2t)}\rangle$$

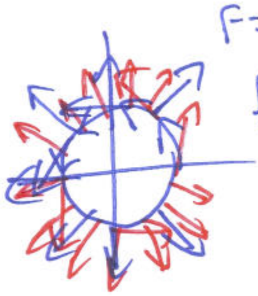
$$r_3(t) = \langle 0,-2\rangle t + \langle 0,2\rangle$$
$$r_3(t). = \langle \overset{x}{0}, \overset{y}{-2t+2}\rangle$$
$$r_3 = \langle \overset{dx}{0}, \overset{dy}{-2}\rangle$$
$$\vec{F} = \langle \underset{M}{-2t+2}, \underset{N}{0}\rangle$$

$$\int_0^1 (3t)(3) + 0\cdot 0\,dt + \int_0^1 (-t+3)(-3) + (-3t+3)(2t)(2)\,dt + \underbrace{\int_0^1 0+0\,dt}_{0}$$

Flux (flow across a curve).

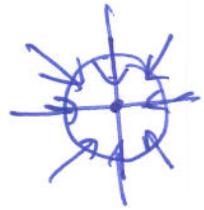
$F = \langle x, y \rangle$

flux > 0.



$F = \langle -x, -y \rangle$.

flux < 0



$W = \int_C \vec{F} \cdot \vec{T}\, ds$

$= \int_C M\,dx + N\,dy$

F $\quad T = \langle dx, dy \rangle$

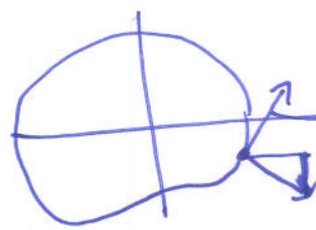
Outward normal vector.

$\text{Flux} = \int \vec{F} \cdot \vec{N}\, ds$

$\Phi = \boxed{= \oint_C M\,dy - N\,dx}$

F $\quad N = \langle dy, -dx \rangle$

$\langle dx, dy \rangle \cdot \langle dy, -dx \rangle = 0$



$\langle \overset{>0}{dx}, \overset{>0}{dy} \rangle$

$\langle \underbrace{dy}_{>0}, \underbrace{-dx}_{<0} \rangle = \vec{N} ds$

$F = \langle \overset{M}{x+y}, \overset{N}{y} \rangle$.

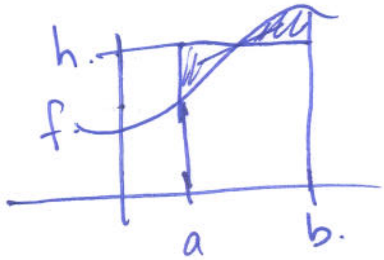
$\vec{r} = \langle \overset{x}{2\cos t}, \overset{y}{2\sin t} \rangle$.

$r' = \langle \overset{dx}{-2\sin t}, \overset{dy}{2\cos t} \rangle$.

$$\boxed{\int_0^{2\pi} (2\cos t + 2\sin t)(2\cos t) - 2\sin t(-2\sin t)\, dt}$$

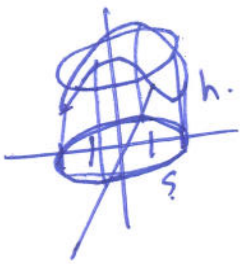
$F(t) = (\underset{M}{2\cos t + 2\sin t}, \underset{N}{2\sin t})$

Average Value.



$$h(b-a) = A = \int_a^b f\,dx$$

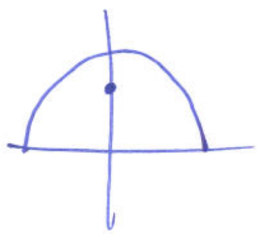
$$h = \frac{1}{b-a}\int_a^b f(x) = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\int_a^b f\,dx}{\int_a^b dx}$$



$$hs = A = \int_C f\,ds$$

$$h = \frac{1}{s}\int_C f\,ds = \frac{\int_C f\,ds}{\int_C ds}$$

Centroids



$$\bar{x} = \frac{\int x\,ds}{\int ds} \quad \bar{y} = \frac{\int y\,ds}{\int ds} \quad \bar{z} = \frac{\int_C z\,ds}{\int_C ds}$$

Average temperature

$$f(x,y,z) = x^2 + yz \qquad \vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 2\pi$$

$$r' = \langle -\sin t, \cos t, 1 \rangle$$

$$\bar{T} = \frac{\int_C T\,ds}{\int_C ds} = \frac{\int_0^{2\pi} \left((\cos^2 t) + (\sin t)(t)\right)\sqrt{2}\,dt}{\int_0^{2\pi} \sqrt{2}\,dt}$$

$$|r'| = \sqrt{1+1} = \sqrt{2}$$

Density, Mass, and Center of Mass

$$\text{Density} = \delta = \frac{\text{mass}}{\text{unit volume}} = \frac{\text{mass}}{\text{unit length}} = \frac{dm}{ds} = \delta.$$

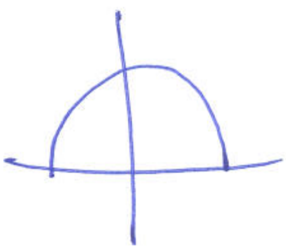
$$dm = \delta\, ds$$

$$\text{mass} = \int dm = \int_C \delta\, ds = \int \delta\, |r'|\, dt$$

If δ = constant then $\int_C \delta\, ds = \delta \int_C ds = \delta \cdot s = M$

Center of Mass

$$\bar{x} = \frac{\int x\, dm}{\int dm} \qquad \bar{y} = \frac{\int y\, dm}{\int dm} \qquad \bar{z} = \frac{\int z\, dm}{\int dm}.$$



$\delta = x^2 + y.$

$$r(t) = \langle \overset{x}{3\cos t}, \overset{y}{3\sin t} \rangle.$$

$$r' = \langle -3\sin t, 3\cos t \rangle$$

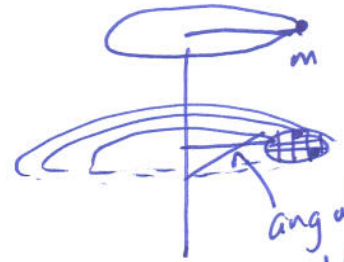
$$|r'| = \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9} = 3$$

$$dm = \delta \cdot |r'|\, dt = ((3\cos t)^2 + 3\sin t)(3)\, dt.$$

$$\bar{x} = \frac{\int_0^\pi 3\cos t\,((3\cos t)^2 + 3\sin t)(3)\, dt.}{\int_0^\pi ((3\cos t)^2 + 3\sin t)(3)\, dt}$$

$$\bar{y} = \frac{\int_0^\pi 3\sin t\,((3\cos t)^2 + 3\sin t)(3)\, dt}{\int_0^\pi ((3\cos t)^2 + 3\sin t)(3)\, dt}$$

(second) Moments of Inertia



angular velocity

$\omega = \frac{d\theta}{dt}$ is constant..

$KE = \frac{1}{2} m v^2$

$= \frac{1}{2} m (r\omega)^2$

$S = r\theta$

$KE = \frac{1}{2} m r^2 \omega^2 \qquad V = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$

$V = r\omega$

$$dKE = \frac{1}{2} dm\,(rad)^2 \omega^2$$

$$KE = \int dKE = \int \frac{1}{2}\omega^2 (rad)^2\, dm$$

$$= \frac{1}{2}\omega^2 \boxed{\int (rad)^2\, dm}$$

moment of Inertia

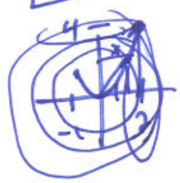
$$I = \int_C (rad)^2\, dm$$

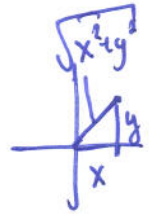
$\delta(x,y) = x^2 + 3y$

$|r'| = \sqrt{1+4t^2}$

$r(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle \quad -1 \le t \le 2$

$r' = \langle 1, 2t \rangle$





$dm = \delta \cdot ds = \delta \cdot |r'|\,dt$

rad = y

$$I_x = \int_C (y)^2\, dm = \int_C y^2 (x^2 + 3y)\sqrt{1+4t^2}\,dt$$

$$= \int_{-1}^{2} (t^2)^2 ((t)^2 + 3(t^2))\sqrt{1+4t^2}\,dt$$

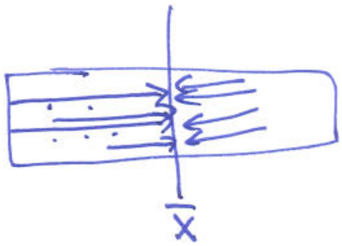
rad x

$$I_y = \int_C (x)^2\, dm = \int_{-1}^{2} (\overset{x}{t})^2 ((t)^2 + 3(t^2))\sqrt{1+4t^2}\,dt$$

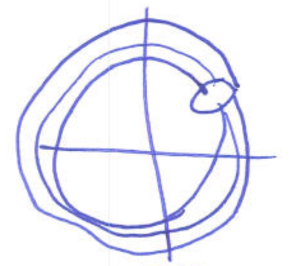
$rad = \sqrt{x^2+y^2}$

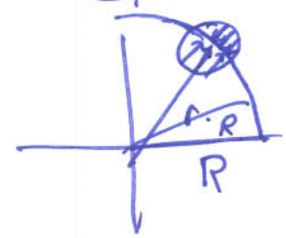
$$I_0 = \int_C \left(\sqrt{x^2+y^2}\right)^2 dm = \int_{-1}^{2} \left((t)^2 + (t^2)^2\right)\left((t)^2 + 3(t^2)\right)\sqrt{1+4t^2}\,dt$$

Center of Mass and Radii of gyration
(rotational center of mass)



$$\int x\,dm = \int \bar{x}\,dm$$
$$\int x\,dm = \bar{x}\int dm$$
$$\frac{\int x\,dm}{\int dm} = \bar{x}$$





$$I = \int (\text{rad})^2\,dm = \int R^2\,dm$$

constant

$\delta = 3$

$r(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 2\pi$

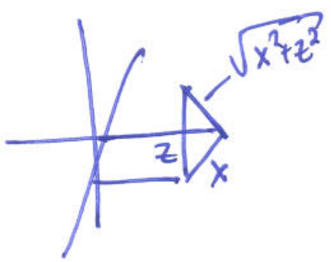
$r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|r'| = \sqrt{1+1} = \sqrt{2}$

$dm = \delta\,ds = \delta \cdot |r'|\,dt = 3\sqrt{2}\,dt$

$$I = R^2\left(\int dm\right) = R^2 m$$
$$\sqrt{I/m} = R$$

$$I_y = \int_C \left(\sqrt{x^2+z^2}\right)^2 dm$$
$$= \int_0^{2\pi} \left((\cos t)^2 + (t)^2\right) 3\sqrt{2}\,dt$$



$$R_y = \sqrt{I_y / m} = \sqrt{\frac{\int_0^{2\pi} (\cos^2 t + t^2)\, 3\sqrt{2}\,dt}{\int_0^{2\pi} 3\sqrt{2}\,dt}}$$

$m = \int dm$

Gradients and Potentials of a vector field

$$f = x^2 + 4xy + 3y^2 \quad Df = \begin{array}{c} \quad x \qquad\quad y \\ \left[\, 2x+4y \;\vdots\; 4x+6y \,\right] \end{array}$$

$$\nabla f = \langle 2x+4y,\ 4x+6y \rangle$$ — The gradient.

$$\vec{F} = \langle 2x+4y,\ 4x+6y \rangle$$

$$\int f_x\,dx \qquad \int f_y\,dy$$

$$x^2 + \underline{4xy} + C(y) \;\Big|\; \cancel{4xy} + \frac{6y^2}{2} + C(x)$$

$$f = x^2 + 4xy + 3y^2$$ — a potential for $\vec{F}$

a potential is a function f s.t. $\nabla f = \vec{F}$

$$\vec{F} = \langle \underset{f_x}{2x+3y+4z},\ \underset{f_y}{3x+4y+5z},\ \underset{f_z}{4x+5y+6z} \rangle$$

$$\underline{x^2} + \underline{3xy} + \underline{4xz} \;\Big|\; \cancel{3xy} + \frac{\underline{4y^2}}{2} + 5yz \;\Big|\; \cancel{4xz} + \cancel{5yz} + 3z^2$$

$$f = x^2 + 3xy + 4xz + 2y^2 + 5yz + 3z^2$$

$$\nabla f = \langle 2x+3y+4z,\ 3x+4y+5z,\ 4x+5y+6z \rangle$$

Component Test (Test for a conservative vector field)

$F = \langle -y, x \rangle$ $f_{xy} = -1$ $f_{yx} = 1$ $\boxed{-1 \neq 1}$

$\int -y\,dx$ $\int x\,dy$

$-yx \mid xy$

problem?

No potential exists.

$F = \langle M, N \rangle$

$M = f_x$, $N = f_y$

$(f_x)_y = f_{yx}$

$M_y = N_x$ — If equal a potential exists

$F = \langle M, N, P \rangle$.

$f_x \quad f_y \quad f_z$.

$f_{xy} \quad f_{yx} \quad f_{zx}$

$f_{xz} \quad f_{yz} \quad f_{zy}$

$\frac{x^2}{2} + 2xy + 3xz \,/\, \cancel{2xy} + \frac{3y^2}{2} + 4yz \,/\, \cancel{3xz} + \cancel{4yz} + \frac{5z^2}{2}$

$\vec{F} = \langle x + 2y + 3z,\ 2x + 3y + 4z,\ 3x + 4y + 5z \rangle$.

$f_x \quad f_y \quad f_z$

$f_{xy} = 2$ ✓ $f_{yx} = 2$ ✓ $f_{zx} = 3$

$f_{xz} = 3$ $f_{yz} = 4$ ✓ $f_{zy} = 4$

$$f = \frac{x^2}{2} + 2xy + 3xz + \frac{3y^2}{2} + 4yz + \frac{5z^2}{2}$$

$F = \langle x + 2y,\ 3x + 4y \rangle$

$f_x \quad f_y$

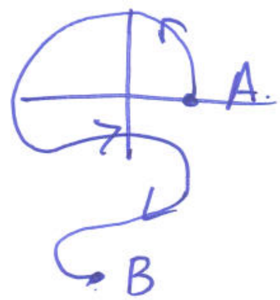
$f_{xy} = 2 \quad f_{yx} = 3$

$2 \neq 3$

So No potential.

Fundamental Theorem of Line Integrals
(The easy way to find work).

$$W = \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A). \quad \text{If } \vec{F} = \nabla f$$



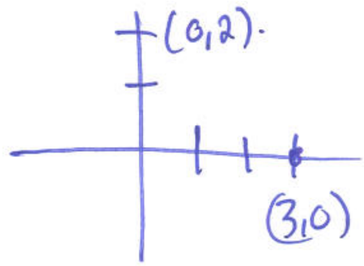
$F = \langle x + 2y, 2x - 4y \rangle.$

$f_x \qquad f_y$

$f_{xy} = 2 \qquad f_{yx} = 2$

$$\frac{x^2}{2} + 2xy \Big| \cancel{2xy} - \frac{4y^2}{2}$$

$$f = \frac{x^2}{2} + 2xy - \frac{4y^2}{2}$$



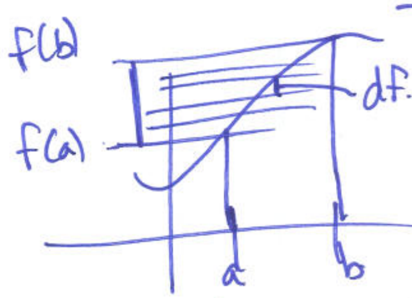
$$\int_{(3,0)}^{(0,2)} (x+2y)\,dx + (2x-4y)\,dy = \left. \frac{x^2}{2} + 2xy - \frac{4y^2}{2} \right|_{(3,0)}^{(0,2)}$$

$$= \left(\cancel{4}^{\,0} + 0 - \cancel{2}^{\,8}\right) - ($$

$$-8 \quad - \left(\frac{9}{2} + 0 + 0\right)$$

$$= -8 - 9/2$$

$$\int_a^b f'\,dx = f(b) - f(a)$$



$$f(b) - f(a) = \int df = \int \frac{df}{dx}\,dx$$

$$= \int f'\,dx$$

$$\int \frac{df}{dt}\,dt = \int_a^b \frac{d(f \circ r)}{dt}\,dt = \int_a^b Df\, Dr\,dt = \int \nabla f \cdot r'\,dt = \int \vec{F} \cdot d\vec{r}$$

$= \underline{\text{work}}$