Multivariable Calculus

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With references to *Calculus Early Transcendentals*, 7th Edition, by Stewart

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Introduction

Mathematical Truths

From Dr. Woodruff:

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
> > Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Teaching philosophy

From Dr. Schmitt:

Over time, I've come to view teaching and learning as a shared journey on which my students and I embark each semester. I am the subject matter expert responsible for providing information and guidance, setting expectations, and assessing how well students meet those expectations. My students are responsible for much hard work, including preparing in advance for class, participating in class activities, and doing out-of-class assignments, regardless of whether or not they are graded. There is only so much that can be conveyed in 50 minutes, and my own personal experience and educational research agree that students get far less out of a 50-minute lecture than their professors hope. Thus, I have chosen to take an approach that is more work both for you and for me but has been shown to produce better results. During class you will work on a carefully chosen series of problems designed to build the mathematical knowledge and experience you need to succeed. These problems will be done in a collaborative, small group setting where you can grapple with and truly understand the material. I'll be there to support, guide, and correct misconceptions. Sure, I could expect you to do this alone outside of class, but over time I've realized a few things about working in groups. As a student, I usually understood something better when I went over it with classmates, even if I was the one who thought I understood it completely and explained it to a peer. As a researcher, I am more productive and effective when I collaborate. Friends in industry report that teams are increasingly used to produce the best results. Furthermore, having me there to help in the early stages ensures that we're traveling together on this journey.

-Dr. Karl Schmitt
Modified with permission from Dr. Mitchel Keller at Washington & Lee University

Modification Notes

This work is based almost entirely off of Ben Woodruff's IBL textbook (see copyright information on previous pages). Some modifications have been made by Dr. Karl Schmitt at Valparaiso University to more closely match the teaching and content for Valparaiso's Calculus III course (Math 253). In particular several chapters/units have been split apart or reorganized.

Contents

Unit 0

Review: Calculus I & II

Contents

When you complete this unityou should be able to:

1. Summarize the ideas you learned in Calculus I and II Math 131 and 132, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).

2. Compute the differential dy of a function and use it to approximate the change in a function.

3. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely many solutions.

4. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

0.1 Review of First Semester Calculus

0.1.1 Graphing

We'll need to know how to graph by hand some basic functions. If you have not spent much time graphing functions by hand before this class, then you should spend some time graphing the following functions by hand. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

Exercise 0.1
Provide a rough sketch of the following functions, showing their basic shapes:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Then use a computer algebra system, such as Wolfram Alpha, to verify your work. I suggest Wolfram Alpha, because it is now built into Mathematica 8.0+. If you can learn to use Wolfram Alpha, you will be able to use Mathematica.

To Valpo Students: You are also welcome to use Maple. On the surface both Mathematica and Maple are very similar.

0.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. The following rules are crucial.

- Power rule $(x^n)' = nx^{n-1}$
- Sum and difference rule $(f \pm g) = f' \pm g'$
- Product $(fg)' = f'g + fg'$ and quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- Chain rule (arguably the most important) $(f \circ g)' = f'(g(x)) \cdot g'(x)$

Exercise 0.2
Compute the derivative of $e^{\sec x} \cos(\tan(x) + \ln(x^2 + 4))$.
Show each step in your computation, making sure to show what rules you used.

Stewart: See sections 3.1-3.6 for more practice with derivatives.

Exercise 0.3
If $y(p) = \dfrac{e^{p^3} \cot(4p + 7)}{\tan^{-1}(p^4)}$ find dy/dp.
Again, show each step in your computation, making sure to show what rules you used.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Exercise 0.4
Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1 - x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

Stewart: See section 3.5 for more examples involving inverse trig functions and implicit differentiation.

0.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Exercise 0.5

Compute $\int x\sqrt{x^2+4}dx$.

Stewart: For practice with u-substitution, see sections 5.5. For practice with integration by parts, see section 7.1

Exercise 0.6

Compute $\int x\sin 2x dx$.

Exercise 0.7

Compute $\int \arctan x dx$.

Exercise 0.8

Compute $\int x^2 e^{3x} dx$.

0.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 0.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Exercise 0.9

If $f(x) = x^2 \ln(3x + 2)$ and $g(t) = e^{2t} \tan(t^2)$ then compute df and dg.

Stewart: See 3.10:11-22

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Exercise 0.10

Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

Stewart: See 3.10:1-10 and 3.10:23-31.
The linearization of a function is just an equation of the tangent line where you solve for y.

1. Find the differential of y with respect to x.
2. Draw a graph of $f(x)$. Place a dot at the point $(3, 9)$ and label it on your graph. Sketch a tangent line to the graph at the point $(3, 9)$ on the same axes. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.
3. Using the two points $(3, 9)$ and (x, y), compute the slope of the line connecting these two points. Your answer should involve x and y. What is the rise (i.e, the change in y called dy)? What is the run (i.e, the change in x called dx)?
4. We know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. These two must be equal. Use this fact to give an equation of the tangent line to $f(x)$ at $x = 3$. (Hint: do NOT simplify to slope-intercept form)
5. How is the equation for the tangent line related to the differentials dy and dx?

Exercise 0.11

The manufacturer of a spherical storage tank needs to create a tank with a radius of 3 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 3 m, plus or minus some small amount dr. The actual radius will be $3 + dr$.

Stewart: See 3.10

1. Find the differential dV. This should be a formula with dV, r, and dr.
2. If the actual radius is 3.02
   a What is r?
   b What is dr?
3. What is dV equal to? What does dV tell you about the volume of the manufactured sphere?

Exercise 0.12

A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about 60°. If this angle of 60° is correct, then what is the height of the tree? If the ranger's angle measurement could be off by as much as 5°, then how much could his estimate of the height be off? Use differentials to give an answer.

0.3 Solving Systems of equations

Exercise 0.13

Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x + y &= 3 \\ 2x - y &= 4 \end{cases}$
- $\begin{cases} -x + 4y &= 8 \\ 3x - 12y &= 2 \end{cases}$

Exercise 0.14

Find all solutions to the linear system $\begin{cases} x + y + z &= 3 \\ 2x - y &= 4 \end{cases}$. Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

This link will show you how to specify which variable is t when using Wolfram Alpha.

- If you let $x = t$, what are y and z. Write your solution in the form (x, y, z) where you replace x, y, and z with what they are in terms of t.
- If you let $y = t$, what are x and z (in terms of t).
- If you let $z = t$, what are x and y.

0.4 Optional: Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Exercise 0.15

Let $f(x) = e^x$. You should find that your work on each step can be reused to do the next step.

- Find a first degree polynomial $P_1(x) = a + bx$ so that $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. In other words, give me a line that passes through the same point and has the same slope as $f(x) = e^x$ does at $x = 0$. Set up a system of equations and then find the unknowns a and b. The next two are very similar.

- Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$.

- Find a third degree polynomial $P_3(x) = a+bx+cx^2+dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$.

- Now compute $e^{.1}$ with a calculator. Then compute $P_1(.1)$, $P_2(.1)$, and $P_3(.1)$. How accurate are the line, parabola, and cubic in approximating $e^{.1}$?

Exercise 0.16

Now let $f(x) = \sin x$. Find a 7th degree polynomial so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. Evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$? You may find it valuable to use the notation

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7.$$

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

The previous two problems involved finding polynomial approximations to the function at $x = 0$. The next problem shows how to move this to any other point, such as $x = 1$.

Exercise 0.17

Let $f(x) = e^x$.

- Find a second degree polynomial

$$T(x) = a + bx + cx^2$$

so that $T(1) = f(1)$, $T'(1) = f'(1)$, and $T''(1) = f''(1)$. In other words, give me a parabola that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

- Find a second degree polynomial written in the form

  $$S(x) = a + b(x-1) + c(x-1)^2$$

  so that $S(1) = f(1)$, $S'(1) = f'(1)$, and $S''(1) = f''(1)$. In other words, find a quadratic that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

- Find a third degree polynomial written in the form

  $$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$$

  so that $P(1) = f(1)$, $P'(1) = f'(1)$, $P''(1) = f''(1)$, and $P'''(1) = f'''(1)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 1$.

Notice that we just replaced x with $x-1$. This centers, or shifts, the approximation to be at $x = 1$. The first part will be much simpler now when you let $x = 1$.

Example 0.2. This example refers back to problem 0.11. We wanted a spherical tank of radius 3m, but due to manufacturing error the radius was slightly off. Let's now illustrate how we can use polynomials to give a first, second, and third order approximation of the volume if the radius is 3.02m instead of 3m.

We start with $V = \frac{4}{3}\pi r^3$ and then compute the derivatives

$$V' = 4\pi r^2, V'' = 8\pi r, \text{ and } V''' = 8\pi.$$

Because we are approximating the increase in volume from $r = 3$ to something new, we'll create our polynomial approximations centered at $r = 3$. We'll consider the polynomial

$$P(r) = a_0 + a_1(r-3) + a_2(r-3)^2 + a_3(r-3)^3,$$

whose derivatives are

$$P' = a_1 + 2a_2(r-3) + 3a_3(r-3)^2, P'' = 2a_2 + 6a_3(r-3), P''' = 6a_3.$$

So that the derivatives of the volume function match the derivatives of the polynomial (at $r = 3$), we need to satisfy the equations in the table below.

k	Value of V at the kth derivative	Value of P at the the kth derivative	Equation
0	$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi$	$P(3) = a_0$	$a_0 = 36\pi$
1	$V'(3) = 4\pi(3)^2 = 36\pi$	$P'(3) = a_1$	$a_1 = 36\pi$
2	$V''(3) = 8\pi(3) = 24\pi$	$P''(3) = 2a_2$	$2a_2 = 24\pi$
3	$V'''(3) = 8\pi$	$P'''(3) = 6a_3$	$6a_3 = 8\pi$

This tells us that the third order polynomial is

$$P(r) = a_0 + a_1(r-3) + a_2(r-3)^2 + a_3(r-3)^3 = 36\pi + 36\pi(r-3) + 12\pi(r-3)^2 + \frac{4}{3}\pi(r-3)^3.$$

We wanted to approximate the volume if $r = 3.2$, so our change in r is $dr = 3.2 - 3 = 0.2$. We can rewrite our polynomial as

$$P(r) = 36\pi + 36\pi(dr) + 12\pi(dr)^2 + \frac{4}{3}\pi(dr)^3.$$

We are now prepared to approximate the volume using a first, second, and third order approximation.

1. A first order approximation yields $P = 36\pi + 36\pi \cdot 0.02 = 36.72\pi$. The volume increased by 0.72π m^3.

2. A second order approximation yields

$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 = 36.7248\pi.$$

3. A third order approximation yields

$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 + \frac{4}{3}\pi(0.02)^3 = 36.724810\bar{6}\pi.$$

With each approximation, we add on a little more volume to get closer to the actual volume of a sphere with radius $r = 3.02$. The actual volume of a sphere involves a cubic function, so when we approximate the volume with a cubic, we should get an exact approximation (and $V(3.02) = \frac{4}{3}\pi(3.02)^3 = (36.724810\bar{6})\pi$.)

We'll end this section with a problem to practice the example above.

Exercise 0.18

Suppose you are constructing a cube whose side length should be $s = 2$ units. The manufacturing process is not exact, but instead creates a cube with side lengths $s = 2 + ds$ units. (You should assume that all sides are still the same, so any error on one side is replicated on all. We have to assume this for now, but before the semester ends we'll be able to do this with high dimensional calculus.)

Suppose that the machine creates a cube with side length 2.3 units instead of 2 units. Note that the volume of the cube is $V = s^3$. Use a first, second, and third order approximation to estimate the increase in volume caused by the .3 increase in side length. Then compute the actual increase in volume $V(2.3) - V(2)$.

As a challenge, try to draw a 3D graph which illustrates the volume added on by each successive approximation. If you have it before we get here as a class, let me know and I'll let you share what you have discovered with the class.

Unit 1

Vectors

Contents

In this unit you will learn how to...

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

1.1 Vectors and Lines

Topical Objectives:

- understand vectors as quantities having length and direction, independent of position
- express curves with parametric (vector) equations

Learning to work with vectors will be key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors. ‘

Exercise 1.1

To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

1. Draw a 3-D axis then plot the points $(1, 5)$ and $(2, 3)$
2. Sketch the triangle that connects these two points and the origin – $(0, 0)$
3. Find Δx and Δy
4. Now extend your picture into 3 and use it to show that the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
5. Use this formula to find the distance between the points $(1, 5, 2)$ and $(2, 3, 3)$

Exercise 1.2

Recall that a circle (or sphere in 3-D) is just a collection of points which are all equal distance away from a center point. Stewart: See 12.1:11-18

1. If the center of a circle is the point $(1, 5)$ what is the equation for a circle which passes through the point $(2, 3)$?
2. Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$.
3. Now find an equation of the sphere passing though point Q whose center is at P.
   Hint: Each point on the surface of a sphere is r distance away from the center.

Definition 1.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

$$\vec{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The entries of the vector are called the x, y, and z (or i, j, k) components of the vector.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 1.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors.
Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane.
Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 1.3. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Exercise 1.3

Consider the vectors $\vec{u} = \langle 1, 2 \rangle$ and $\vec{v} = \langle 3, 1 \rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$. Stewart: See 12.2:5-6

Exercise 1.4

Consider the vector $\vec{v} = \langle 3, -1 \rangle$.

1. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$.

Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time.

2. At the following times, where is the donkey?
   (a) $t = 0$?
   (b) $t = 1$?
   (c) $t = 2$?
3. Draw an axis, then sketch the path followed by the donkey.
4. Where is the donkey at time $t = 0, 1, 2$? Put markers with labels on your graph to show the donkey's location.

In a previous problem (1.4) you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function is called a parametrization of the donkey's path. Because the output is a vector, we call the function a vector-valued function. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Let's look at another, similar problem.

Exercise 1.5

Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1,2)t + (3,4)$. (Remember this is the same as writing $(x,y) = (1,2)t + (3,4)$ or similarly $(x,y) = (1t+3, 2t+4)$.)

1. Where is the horse at time $t = 0, 1, 2$?
2. Draw an set of axis and put markers on your graph to show the horse's location.
3. Draw the path followed by the horse.

In the last two problems we described the donkey and horse's paths using vectors. The use of vectors actually lets us do a lot more like give a generalized direction or easily compute their speed at any point along the path. To do that we need to define two more mathematical ideas, the magnitude and unit-vector.

Definition 1.4. The magnitude, or length, or norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin.

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

In the special circumstance where a vector represents an object's velocity, the magnitude of the vector gives the objects *speed* or non-directional velocity.

Definition 1.5. A unit vector is a vector whose length is one unit. The notation for a unit vector is generally a "^" or **bold face**. Equivalently, the unit vector of a vector $\vec{v}$ is: $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$
The standard unit vectors are $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$, $\mathbf{k} = \langle 0,0,1 \rangle$.

Exercise 1.6: Magnitude and Unit Vector Practice

For each of the following vectors, compute the magnitude and unit vector in the same direction.

1. $-3\vec{\imath} + 7\vec{\jmath}$
2. $< -4, 2, 4 >$
3. $8\vec{\imath} - \vec{\jmath} + 4\vec{k}$

Exercise 1.7

Consider the two points $P = (1,2,3)$ and $Q = (2,-1,0)$. Stewart: See 12.2: 23-26, 41, 42

1. Write the vector $\vec{PQ}$ in component form $< a, b, c)$.
2. Find the length (magnitude) of vector $\vec{PQ}$.
3. Find a unit vector for $\vec{PQ}$.
4. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$.

Exercise 1.8

Let's return to problems 1.4 and 1.5. We now have the tools to determine the donkey's and horse's speed and direction.

1. The donkey's path was given by $(x, y) = \vec{v}t = (3t, -t)$. Find
   (a) The donkey's speed
   (b) A unit vector that gives the donkey's direction of travel.
2. The horse's path was given by $\vec{r}(t) = (1, 2)t + (3, 4)$.
   (a) State the part of $\vec{r}(t)$ which gives the direction of travel.
   (b) What is a unit vector in the same direction?
   (c) what is the speed of the horse?

Exercise 1.9

A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2\rangle$.

Stewart: Look at 12.2:34-40

Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 1.5, and base your work off of what you saw there. It's almost identical.]
Generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = \langle v_1, v_2, v_3\rangle$.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Exercise 1.10

Let $P = (3, 1)$ and $Q = (-1, 4)$.

Stewart: 12.5: 6-15

1. Write a vector equation $\vec{r}(t) = (x, y)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
2. Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
3. Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

1.2 The Dot Product

Topical Objectives:
- perform the dot-product of two vectors
- recognize when two vectors are orthogonal

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. We'll use the dot product to help us find find angles. First, we need to recall the law of cosines.

Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of* ***length*** *a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90°$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Exercise 1.11

Sketch in $\mathbb{R}^2$ the vectors $\langle 1, 2\rangle$ and $\langle 3, 5\rangle$. Use the law of cosines to find the angle between the vectors. Stewart: See 12.3:15-20

Exercise 1.12

Sketch in $\mathbb{R}^3$ the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$. Use the law of cosines to find the angle between the vectors. Stewart: See 12.3:15-20

Definition 1.6: The Dot Product. If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, then we define the dot product of these two vectors to be

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process used in matrix multiplication.

Exercise 1.13

1. Use the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$.
2. Which was easier, 1.12 or this method? (You will derive this formula in a later problem)

Definition 1.7. We say that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Exercise 1.14

Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

Exercise 1.15

Mark each statement true or false. Use Definitions 1.3 - 1.7 to explain and justify or prove your claim. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Exercise 1.16
If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ (which is often written $\vec{u}, \vec{v} \in \mathbb{R}^3$), then show that

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2.$$

Exercise 1.17
Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

Stewart: See pages 801-802 if you are struggling

1. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.
2. Use the above together with problem 1.16 to derive

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

Exercise 1.18
Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is 90°. Then show that if the angle between them is 90°, then the vectors are orthogonal. I.E. expand and compute both sides of the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ with non-zero, orthogonal vectors.

Stewart: See page 804

The dot product provides a really easy way to find when two vectors meet at a right angle. The dot product is precisely zero when this happens.

1.3 Interlude:Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. Alternatively see texample.net for a visualization of the idea.

To Valpo Students: The electronic version has links that will open your browser and take you to the web.

To Valpo Students: For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

Exercise 1.19
Compute the following matrix products.

- $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}$

Exercise 1.20

Compute the product $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

1.3.1 Determinants

Associated with every square matrix is a number, called the determinant. Determinants are only defined for square matrices. Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Definition 1.8. The determinant of a 2×2 matrix is the number

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

The determinant of a 3×3 matrix is the number

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge). \end{aligned}$$

Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Exercise 1.21

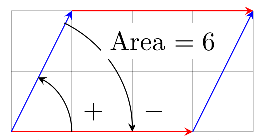
Compute $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -3 & 1 \end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use Wolfram Alpha to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ whose determinant is $3 \cdot 2 - 0 \cdot 1 = 6$. Draw the column vectors $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign



$$\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 \text{ and } \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Exercise 1.22

1. Use determinants to find the area of the triangle with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.
2. What would you change if you wanted to find the area of the triangle with vertices $(-3,1)$, $(-2,5)$, and $(3,4)$? Find this area.

1.4 The Cross Product

Topical Objective:

- perform the cross-product of vectors

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vectors. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 1.9: The Cross Product. The cross product of two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is a new vector $\vec{u} \times \vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$. The formula for the cross product is

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

This definition is not really a definition. It is actually a theorem. If you use the formula given as the definition, then you would need to prove the three facts. We have the tools to give a complete proof of (1) and (3), but we would need a course in linear algebra to prove (2). It shouldn't be too much of a surprise that the cross product is related to area, since it is defined in terms of determinants

Exercise 1.23

Let $\vec{u} = \langle 1, -2, 3\rangle$ and $\vec{v} = \langle 2, 0, -1\rangle$.

Stewart: See 12.4:1-7

1. Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. How are they related?
2. Compute $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$. Why did you get the answer you got?
3. Compute $\vec{u} \times (2\vec{u})$. Why did you get the answer you got?

Exercise 1.24

Consider the vectors $\vec{\imath} = (1,0,0)$, $2\vec{\jmath} = (0,2,0)$, and $3\vec{k} = (0,0,3)$.

1. Compute $\vec{\imath} \times 2\vec{\jmath}$ and $2\vec{\jmath} \times \vec{\imath}$.
2. Compute $\vec{\imath} \times 3\vec{k}$ and $3\vec{k} \times \vec{\imath}$.
3. Compute $2\vec{\jmath} \times 3\vec{k}$ and $3\vec{k} \times 2\vec{\jmath}$.

Give a geometric reason as to why some vectors above have a plus sign, and some have a minus sign.

Exercise 1.25

Let $P = (2,0,0)$, $Q = (0,3,0)$, and $R = (0,0,4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR [Hint: What shape does two triangles side-by-side make?]. Construct a 3D graph of this triangle.

Stewart: See 12.4: 29-32

1.5 The Cross Product and Planes

Topical Objectives

- use the normal vector to find the equation for a plane

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Exercise 1.26

Suppose the point $P = (1,2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n} = \langle 3,4 \rangle$ is 90° (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q = (x,y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$ to obtain an equation of the line L.

Exercise 1.27

Let $P = (a,b,c)$ be a point on a plane in 3D. Let $\vec{n} = \langle A,B,C \rangle$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is 90°). Let $Q = (x,y,z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is

$$A(x-a) + B(y-b) + C(z-c) = 0.$$

Stewart: See pages 819-822

Exercise 1.28

Find an equation of the plane containing the lines $\vec{r}_1(t) = (1,3,0)t + (1,0,2)$ and $\vec{r}_2(t) = (2,0,-1)t + (2,3,2)$.

Exercise 1.29

Consider the three points $P = (1,0,0), Q = (2,0,-1), R = (0,1,3)$. Find an equation of the plane which passes through these three points. [Hint: First find a normal vector to the plane.]

Stewart: See 12.5: 23-40

Exercise 1.30

Consider the points $P = (2,-1,0)$, $Q = (0,2,3)$, and $R = (-1,2,-4)$.

1. Give an equation $(x,y,z) = (?,?,?)$ of the line through P and Q.
2. Give an equation of the line through P and R.
3. Give an equation of the plane through P, Q, and R.

Exercise 1.31

Consider the two planes $x + 2y + 3z = 4$ and $2x - y + z = 0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line.

Stewart: See 12.5: 23-40, 45-47

1.6 Projections and Their Applications

Course Objective:

- See applications of multi-variable calculus

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of 200 Newtons, which we will write in vector notation as $\vec{F} = \langle 0, -200\rangle$. The ramp was placed so that the box needs to be moved right 6 m, and down 3 m, so we need to get from the origin $(0,0)$ to the point $(6,-3)$. This displacement can be written as $\vec{d} = \langle 6, -3\rangle$. The force F acts straight down, which means the ramp takes some of the force. Our goal is to find out how much of the 200N the ramp takes, and how much force must be applied to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

Exercise 1.32

Read the preceding paragraph.
We want to write $\vec{F}$ as the sum of two vectors: $\vec{F} = \vec{w} + \vec{n}$

- where $\vec{w}$ is parallel to $\vec{d}$
- and $\vec{n}$ is orthogonal to $\vec{d}$

Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c.

1. Rewrite $\vec{F}$ in terms of $\vec{d}$

2. Take the dot-product of both sides with $\vec{d}$
3. Since $\vec{n}$ is orthogonal to $\vec{d}$ we know that $\vec{n} \cdot \vec{d} = ?$
4. Substitute and solve for the unknown c

The solution to the previous problem gives us the definition of a projection.

Definition 1.10. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}.$$

Exercise 1.33

Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Compute the $\text{proj}_{\vec{v}}\vec{u}$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$ all on one set of axis. Then draw a line segment from the head of $\vec{u}$ to the head of the projection. Stewart: See 12.3: 39-44

Now let $\vec{w} = (-2, 0)$. Compute $\text{proj}_{\vec{v}}\vec{w}$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{w}$. Then draw a line segment from the head of $\vec{w}$ to the head of the projection.

One application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 1.32. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200\rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3\rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Exercise 1.34: Projection Application: Work

Find the amount of work done by the force $\vec{F} = \langle 0, -200\rangle$ through the displacement $\vec{d} = \langle 6, -3\rangle$. Find this by doing the following: Stewart: See 12.3:49-52

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F} \cdot \vec{d}$. You have just shown that $W = \vec{F} \cdot \vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

Exercise 1.35: Projection Application: Planes

Consider the points $P = (2, 4, 5)$, $Q = (1, 5, 7)$, and $R = (-1, 6, 8)$.

Stewart: See 12.5:69-72

1. What is the area of the triangle PQR.
2. Give a normal vector to the plane through these three points.
3. What is the distance from the point $A = (1, 2, 3)$ to the plane PQR. [Hint: Compute the projection of $\vec{PA}$ onto $\vec{n}$. How long is it?]

To Valpo Students:

1.7 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 2

Review: Conic Sections

Contents

Upon completing this unit you will be prepared to...

1. Describe, graph, give equations of, and find foci for conic sections (parabolas, ellipses, hyperbolas).

2.1 Conic Sections

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to object in 3D. These examples are conic sections. We call them conic sections because you can obtain each one by intersecting a cone and a plane (I'll show you in class how to do this). Here's a definition.

Definition 2.1. Consider two identical, infinitely tall, right circular cones placed vertex to vertex so that they share the same axis of symmetry. A conic section is the intersection of this three dimensional surface with any plane that does not pass through the vertex where the two cones meet.

These intersections are called circles (when the plane is perpendicular to the axis of symmetry), parabolas (when the plane is parallel to one side of one cone), hyperbolas (when the plane is parallel to the axis of symmetry), and ellipses (when the plane does not meet any of the three previous criteria).

The definition above provides a geometric description of how to obtain a conic section from cone. We'll not introduce an alternate definition based on distances between points and lines, or between points and points. Let's start with one you are familiar with.

Definition 2.2. Consider the point $P = (a, b)$ and a positive number r. A circle circle with center (a, b) and radius r is the set of all points $Q = (x, y)$ in the plane so that the segment PQ has length r.

Using the distance formula, this means that every circle can be written in the form $(x - a)^2 + (y - b)^2 = r^2$.

Exercise 2.1

The equation $4x^2 + 4y^2 + 6x - 8y - 1 = 0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x - a)^2 + (y - b)^2 = r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2 + y^2 + Dx + Ey + F = 0$.

2.1.1 Parabolas

Before proceeding to parabolas, we need to define the distance between a point and a line.

Definition 2.3. Let P be a point and L be a line. Define the distance between P and L (written $d(P, L)$) to be the length of the shortest line segment that has one end on L and the other end on P. Note: This segment will always be perpendicular to L.

Definition 2.4. Given a point P (called the focus) and a line L (called the directrix) which does not pass through P, we define a parabola as the set of all points Q in the plane so that the distance from P to Q equals the distance from Q to L. The vertex is the point on the parabola that is closest to the directrix.

Exercise 2.2

Consider the line $L : y = -p$, the point $P = (0, p)$, and another point $Q = (x, y)$. Use the distance formula to show that an equation of a parabola with directrix L and focus P is $x^2 = 4py$. Then use your work to explain why an equation of a parabola with directrix $x = -p$ and focus $(p, 0)$ is $y^2 = 4px$.

Ask me about the reflective properties of parabolas in class, if I have not told you already. They are used in satellite dishes, long range telescopes, solar ovens, and more. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Exercise: Optional

Consider the parabola $x^2 = 4py$ with directrix $y = -p$ and focus $(0, p)$. Let $Q = (a, b)$ be some point on the parabola. Let T be the tangent line to L at point Q. Show that the angle between PQ and T is the same as the angle between the line $x = a$ and T. This shows that a vertical ray coming down towards the parabola will reflect of the wall of a parabola and head straight towards the vertex.

The next two problems will help you use the basic equations of a parabola, together with shifting and reflecting, to study all parabolas whose axis of symmetry is parallel to either the x or y axis.

Exercise 2.3

Once the directrix and focus are known, we can give an equation of a parabola. For each of the following, give an equation of the parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(0, 3)$ and the directrix is $y = -3$.
2. The focus is $(0, 3)$ and the directrix is $y = 1$.

Exercise 2.4
Give an equation of each parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(2, -5)$ and the directrix is $y = 3$.
2. The focus is $(1, 2)$ and the directrix is $x = 3$.

Exercise 2.5
Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = x^2$
2. $(y - 2)^2 = 4(x - 1)$

Exercise 2.6
Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = -8x^2 + 3$
2. $y = x^2 - 4x + 5$

2.1.2 Ellipses

Definition 2.5. Given two points F_1 and F_2 (called foci) and a fixed distance d, we define an ellipse as the set of all points Q in the plane so that the sum of the distances F_1Q and F_2Q equals the fixed distance d. The center of the ellipse is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the ellipse that intersect this line is called a vertex. The major axis is the segment between the two vertexes. The minor axis is the largest segment perpendicular to the major axis that fits inside the ellipse.

We can derive an equation of an ellipse in a manner very similar to how we obtained an equation of a parabola. The following problem will walk you through this.

Exercise: Optional
Consider the ellipse produced by the fixed distance d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the ellipse.

1. Show that $d = 2a$ by considering the distances from F_1 and F_2 to the point $Q = (a, 0)$.
2. Let $Q = (0, b)$ be a point on the ellipse. Show that $b^2 + c^2 = a^2$ by considering the distance between Q and each focus.
3. Let $Q = (x, y)$. By considering the distances between Q and the foci, show that an equation of the ellipse is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the fixed distance d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $(a, 0)$ be a point on the x axis that intersect the ellipse. Show that we still have
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
but now we instead have $a^2 + c^2 = b^2$.

You'll want to use the results of the previous problem to complete the problems below. The key equation above is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The foci will be on the x-axis if $a > b$, and will be on the y-axis if $b > a$. The second part of the problem above shows that the distance from the center of the ellipse to the vertex is equal to the hypotenuse of a right triangle whose legs go from the center to a focus, and from the center to an end point of the minor axis.

The next three problems will help you use the basic equations of an ellipse, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Exercise 2.7

For each ellipse below, graph the ellipse and give the coordinates of the foci and vertexes.

1. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} + \dfrac{(y-2)^2}{9} = 1$

Exercise 2.8

For the ellipse $x^2 + 2x + 2y^2 - 8y = 9$, sketch a graph and give the coordinates of the foci and vertexes.

Exercise 2.9

Given an equation of each ellipse described below, and provide a rough sketch.

1. The foci are at $(2 \pm 3, 1)$ and vertices at $(2 \pm 5, 1)$.
2. The foci are at $(-1, 3 \pm 2)$ and vertices at $(-1, 3 \pm 5)$.

Exercise: Optional

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be some point on the ellipse. Let T be the tangent line to the ellipse at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that a ray from F_1 to Q will reflect off the wall of the ellipse at Q and head straight towards the other focus F_2.

2.1.3 Hyperbolas

Definition 2.6. Given two points F_1 and F_2 (called foci) and a fixed number d, we define a hyperbola as the set of all points Q in the plane so that the difference of the distances F_1Q and F_2Q equals the fixed number d or $-d$. The center of the hyperbola is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the hyperbola that intersect this line is called a vertex.

We can derive an equation of a hyperbola in a manner very similar to how we obtained an equation of an ellipse. The following problem will walk you through this.

Exercise: Optional

Consider the hyperbola produced by the fixed number d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the hyperbola.

1. Show that $d = 2a$ by considering the difference of the distances from F_1 and F_2 to the vertex $(a, 0)$.

2. Let $Q = (x, y)$ be a point on the hyperbola. By considering the difference of the distances between Q and the foci, show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$, or if we let $c^2 - a^2 = b^2$, then the equation is
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

3. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the number d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $a^2 = c^2 - b^2$. Show that an equation of the hyperbola is
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

You'll want to use the results of the previous problem to complete the problems below.

Exercise 2.10

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Construct a box centered at the origin with corners at $(a, \pm b)$ and $(-a, \pm b)$. Draw lines through the diagonals of this box. Rewrite the equation of the hyperbola by solving for y and then factoring to show that as x gets large, the hyperbola gets really close to the lines $y = \pm\frac{b}{a}x$. [Hint: rewrite so that you obtain $y = \pm\frac{b}{a}x\sqrt{\text{something}}$]. These two lines are often called oblique asymptotes.

Now apply what you have just done to sketch the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and give the location of the foci.

The next three problems will help you use the basic equations of a hyperbola, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Exercise 2.11

For each hyperbola below, graph the hyperbola (include the box and asymptotes) and give the coordinates of the foci and vertexes.

1. $16x^2 - 25y^2 = 400$ [Hint: divide by 400.]

2. $\dfrac{(x-1)^2}{5} - \dfrac{(y-2)^2}{9} = 1$

Exercise 2.12

For the hyperbola $x^2 + 2x - 2y^2 + 8y = 9$, sketch a graph (include the box and asymptotes) and give the coordinates of the foci and vertexes.

Exercise 2.13

Given an equation of each hyperbola described below, and provide a rough sketch.

1. The vertexes are at $(2 \pm 3, 1)$ and foci at $(2 \pm 5, 1)$.

2. The vertexes are at $(-1, 3 \pm 2)$ and foci at $(-1, 3 \pm 5)$.

Exercise: Optional

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be a point on the hyperbola. Let T be the tangent line to the hyperbola at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that if you begin a ray from a point in the plane and head towards F_1 (where the wall of the hyperbola lies between the start point and F_1), then when the ray hits the wall at Q, it reflects off the wall and heads straight towards the other focus F_2.

Unit 3

Parametric Equations

Contents

After completing this unit you will be able to...

1. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.

2. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.

3. Use integrals to find the lengths of parametric curves.

3.1 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as:

$$(x, y) = (1, m)t + (0, b)$$

This style of equation is called a vector equation. It is equivalent to writing the two equations

$$x = 1t + 0, y = mt + b,$$

which we call the parametric equations of the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.1. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve.

You can generalize this definition to 3D and beyond by just adding more variables.

Exercise 3.1

By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion.

1. $x = \cos t, y = \sin t$, for $0 \leq t \leq 2\pi$.
2. $x = \sin t, y = \cos t$, for $0 \leq t \leq 2\pi$.
3. $x = \cos t, y = \sin t, z = t$, for $0 \leq t \leq 4\pi$.

Exercise 3.2

For the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$:

1. Plot the path traced out by the curve.
2. Use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y).
3. What are the foci of the resulting object (it's a conic section)?

Exercise 3.3

Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the vector unit should help here.

Exercise 3.4

For the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$:

1. Plot the path traced out by the curve.

2. Give a Cartesian equation of the curve (eliminate the parameter t)

3. Now find the focus of the resulting curve.

[Hint: Set up a system of equations then use substitution.]

Exercise 3.5

Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve (a trig identity will help). Then find the foci of the resulting conic section. [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

3.1.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.2. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then the derivative is defined as:

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The subtraction above requires vector subtraction. The following problem will provide a simple way to take derivatives which we will use all semester long.

Exercise 3.6

Show that if $\vec{r}(t) = (f(t), g(t))$, then the derivative is just $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.

[The definition above says that $\frac{d\vec{r}}{dt} = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$. We were told $\vec{r}(t) = (f(t), g(t))$, so use this in the derivative definition. Then try to modify the equation to obtain $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.]

The previous problem shows you can take the derivative of a vector valued function by just differentiating each component separately. The next problem shows you that velocity and acceleration are still connected to the first and second derivatives.

Exercise 3.7

Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$.

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.

2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.

3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 3.3. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length (or magnitude) of the velocity vector.

Exercise 3.8

Consider the curve $\vec{r}(t) = (2t + 3, 4(2t - 1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.

2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?

3. Find a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.

4. Show that the slope of the line is

$$\frac{dy}{dx}\Big|_{x=5} = \frac{(dy/dt)\big|_{t=1}}{(dx/dt)\big|_{t=1}}.$$

   [How can you turn the direction vector, which involves (dx/dt) and (dy/dt) into a slope (dy/dx)?]

3.1.2 Arc Length

This section covers:

- finding rates of change along space curves

If an object moves at a constant speed, then the distance travelled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

Exercise 3.9: Derivation of the arc length formula

Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$.

1. Show that the object's speed at any time t is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

2. If you move over a really small time interval, say of length dt, then the speed is almost constant. If you move at a constant speed of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a time length of dt, what's the distance ds you have traveled.

$$Hint: Thisshouldbereallyfasttowritedown$$

3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$\text{Arc Length} = s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

Exercise 3.10

Use the equation from 3 to find the length of the curve $\vec{r}(t) = \left(t^3, \frac{3t^2}{2}\right)$ for $t \in [1, 3]$. The notation $t \in [1, 3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a u-substitution).

Exercise 3.11

For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

1. The parabola $\vec{p}(t) = (t, t^2)$ for $t \in [0, 3]$.
2. The ellipse $\vec{e}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{h}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

To Valpo Students:

3.2 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 4

Polar and New Coordinate Systems

Contents

After completing this unit you will be able to...

1. Convert between rectangular and polar coordinates in 2D. Convert between rectangular and cylindrical or spherical in 3D.
2. Graph polar functions in the plane. Find intersections of polar equations, and illustrate that not every intersection can be obtained algebraically (you may have to graph the curves).
3. Find derivatives and tangent lines in polar coordinates.
4. Find area and arc length using polar equations.

4.1 Polar Coordinates

Up to now, we have most often given the location of a point (or coordiantes of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Exercise 4.1
Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$.

1. Find the distance r from P to the origin.
2. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.
[Hint: Use a triangle and trigonometry]

Exercise 4.2

Suppose that a point $Q = (a, b)$ is 6 units from the origin, and the angle the ray $\vec{OQ}$ makes with the x-axis is $\pi/4$ radians. Find the Cartesian (rectangular) coordinates (a, b) of Q.

Definition 4.1. Let Q be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We define the polar coordinates of Q to be the ordered pair (r, θ) where r is the displacement from the origin to Q, and θ is an angle of rotation (counter-clockwise) from the x-axis to the ray $\vec{OP}$.

Exercise 4.3

The following points are given using their polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

$$(1, \pi), \left(3, \frac{5\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

The next problem provides general formulas for converting between the Cartesian (rectangular) and polar coordinate systems.

Exercise 4.4

Suppose that Q is a point in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ).

1. Write formulas for x and y in terms of r and θ.
2. Write a formula to find the distance r from Q to the origin (in terms of x and y)
3. Write a formula to find the angle θ between the x-axis and a line connecting Q to the origin.
   [Hint: A picture of a triangle will help here.]

In problem 4.4, you should have obtained the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can write this in vector notation as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$. This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could write $\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name of the function because it is a transformation between coordinate systems. To emphasize that the domain and range are both two dimensional systems, we could also write $T : \mathbb{R}^2 \to \mathbb{R}^2$. In the next chapter, we'll spend more time with this notation.

The following problem will show you how to graph a coordinate transformation. When you're done, you should essentially have polar graph paper.

Exercise 4.5

Consider the coordinate transformation

For this problem, you are just drawing many parametric curves.

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

1. Let $r = 3$ and then graph $\vec{T}(3,\theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1,\theta), \vec{T}(2,\theta), \vec{T}(4,\theta)$ for $\theta \in [0, 2\pi]$
4. To the same axes as above, add the graphs of $\vec{T}(r, 0), \vec{T}(r, \pi/2), \vec{T}(r, 3\pi/4), \vec{T}(r, \pi)$ for $r \in [0, 5]$.

Exercise 4.6

In the plane,

1. Graph the curve $y = \sin x$ for $x \in [0, 2\pi]$ (make an x, y table)
2. Graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (an r, θ table).

The graphs should look very different. If one looks like a circle, you're on the right track.

Exercise 4.7

Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates by substituting in the formulas from 4.4, and then solve for r so that the equation is in the form $r = f(\theta)$.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$
3. $x^2 = y$

Exercise 4.8

Each of the following equations is written in the polar coordinate system. Convert each to an equation in the Cartesian coordinates.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

4.1.1 Graphing and Intersections

To construct a graph of a polar curve, just create an r, θ table. Choose values for θ that will make it easy to compute any trig functions involved. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases.

Exercise 4.9
Graph the polar curve $r = 2\sin 3\theta$.

Exercise 4.10
Graph the polar curve $r = 3\cos 2\theta$.

Be sure you actually plot out the next two problems, otherwise you'll probably miss a few points of intersection.

Exercise 4.11
Find the points of intersection of $r = 3 - 3\cos\theta$ and $r = 3\cos\theta$.

Exercise 4.12
Find the points of intersection of $r = 2\cos 2\theta$ and $r = \sqrt{3}$.

4.1.2 Calculus with Polar Coordinates

Recall that for parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ can be thought of as the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. So you can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Exercise 4.13
Consider the polar curve $r = 1 + 2\cos\theta$. (It wouldn't hurt to provide a quick sketch of the curve.)

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

You can find arc length for parametric curves using the formula:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

If we replace t with θ, this becomes a formula for arc length in polar coordinates. However, the formula can be simplified.

Exercise 4.14

Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is a continuous function, and that f' is continuous. Show that the arc length formula can be simplified to

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

[Hint: the product rule and Pythagorean identity will help.]

Exercise 4.15

Set up **(do not evaluate)** an integral formula to compute the length of

1. the rose $r = 2\cos 3\theta$, and
2. the rose $r = 3\sin 2\theta$.

4.2 Other Coordinate Systems

Sometimes a problem can't be solved until the correct coordinate system is chosen. You have previously done problems which showed you how to graph the coordinate transformation given by polar coordinates. The following problem shows you how to graph in a different coordinate system.

Exercise 4.16

Consider the coordinate transformation $T(a, \omega) = (a\cos\omega, a^2\sin\omega)$.

1. Let $a = 3$ and then graph the curve $\vec{T}(3, \omega) = (3\cos\omega, 9\sin\omega)$ for $\omega \in [0, 2\pi]$. [Hint: You can start by making a ω, x, y table.]

   See Sage. Click on the link to see how to check your answer in Sage.

2. Let $\omega = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(a, \frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a^2\frac{\sqrt{2}}{2}\right)$ for $a \in [0, 4]$.

   See Sage. Notice that you can add the two plots together to superimpose them on each other.

3. To the same axes as above, add the graphs of:

   Use Sage to check your answer.

   (a) $\vec{T}(1, \omega), \vec{T}(2, \omega), \vec{T}(4, \omega)$ for $\omega \in [0, 2\pi]$

   (b) $\vec{T}(a, 0), \vec{T}(a, \pi/2), \vec{T}(a, -\pi/6)$ for $a \in [0, 4]$.

[Hint: when you're done, you should have a bunch of parabolas and ellipses.]

In 3 dimensions, the most common coordinate systems are cylindrical and spherical. The equations for these coordinate systems are in the table below.

To Valpo Students:

Cylindrical Coordinates	Spherical Coordinates(Math)	Spherical (Physics/Engineering)
$x = r\cos\theta$	$x = \rho\sin\theta\cos\phi$	$x = r\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\theta\sin\phi$	$y = r\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\theta$	$z = r\cos\phi$

Exercise 4.17

Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. Construct a graph in 3D of this information, and use it to develop the equations for cylindrical coordinates given above.

Stewart: See pages 827-833

Exercise 4.18

Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ, and is called the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and is called the inclination angle, polar angle, or zenith angle. Construct a graph in 3D of this information, and use it to develop the equations for spherical coordinates given above.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

There is some disagreement between different fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given—the angle from the xy-plane (lines of lattitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating using them.

To Valpo Students:

Exercise 4.19

If you have never worked in Maple, you will want to spend some time familiarizing yourself with the program before our Lab day (keep reading if you are!). The link here takes you to a set of modules which will review several of the vector concepts we learned last week, but also introduce you to Maple. The Vector exploration contains a very nice visual on vector projections, which you should look at even if you are comfortable in Maple.

The Connected Curriculum Project:
Introduction to Modules

Maple Tutorial

Vectors Exploration

To Valpo Students:

4.3 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 5

Functions

Contents

When you have finished this unit you should be able to...

1. Describe uses for, and construct equations or graphs of, space curves and parametric surfaces.

2. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.

3. Describe uses for, construct graphs of, and find the domain or range of functions of several variables.

   (a) For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots.

   (b) For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.

4. Describe uses for, and construct graphs of, vector fields and transformations.

5. Explain how to obtain a function for a vector field, or a parametrization for a curve or surface if you are given a description of the vector field, curve, or surface (instead of a function or parametrization),

5.1 Function Terminology

In this section you will learn how to:

- identify the domain and range of a multivariable or vector-valued function

A function is a set of instructions involving two sets (called the domain and codomain). A function assigns to each element of the domain D exactly one element in the codomain R. We'll often refer to the codomain R as the target space. We'll write

$$f: D \to R$$

when we want to remind ourselves of the domain and target space. In this class, we will study what happens when the domain and target space are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular, we will study functions of the form

$$f: \mathbb{R}^n \to \mathbb{R}^m,$$

when m and n are 3 or less. The value of n is the dimension of the input vector (or number of inputs). The number m is the dimension of the output vector (or number of outputs). Our goal is to understand uses for each type of function, and be able to construct graphs to represent the function.

We will focus most of our time this semester on two- and three-dimensional problems. However, many problems in the real world require a higher number of dimensions. When you hear the word "dimension", it does not always represent a physical dimension, such as length, width, or height. If a quantity depends on 30 different measurements, then the problem involves 30 dimensions. As a quick illustration, the formula for the distance between two points depends on 6 numbers, so distance is really a 6-dimensional problem. As another example, if a piece of equipment has a color, temperature, age, and cost, we can think of that piece of equipment being represented by a point in four-dimensional space (where the coordinate axes represent color, temperature, age, and cost).

Exercise 5.1

A pebble falls from a 64 ft tall building. Its height (in ft) above the ground t seconds after it drops is given by the function $y = f(t) = 64 - 16t^2$.

See Sage or Wolfram Alpha. Follow the links to Sage or Wolfram Alpha in all the problems below to see how to get the computer to graph the function.

1. What is n (the number of inputs)?
2. What is m (the number of outputs)?
3. Construct a graph of this function.
4. How many dimensions do you need to graph this function?

The next several sections will explore different sorts of functions with a variety of n and m combinations. We will start with these same questions as we work to understand the functions. Later on in the semester, you can use these questions to help understand what sort of function you are studying.

5.2 Parametric Curves:

In this section you will learn how to:

- express curves and surfaces with parametric equations

- compute rates of change along space curves

Exercise 5.2

A horse runs around an elliptical track. Its position at time t is given by the function $\vec{r}(t) = (2\cos t, 3\sin t)$. We could alternatively write this as $x = 2\cos t, y = 3\sin t$.

See Sage or Wolfram Alpha.
Stewart: For review on parameterizing see 10.1 and 10.2. Find more practice in 13.4:3-6

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function.
3. Next to a few points on your graph, include the time t at which the horse is at this point on the graph. Include an arrow for the horse's direction.
4. How many dimensions do you need to graph this function?
5. If we wanted to plot time (t) on its own axis, how many dimensions would we need?

Notice in the problem above that we placed a vector symbol above the function name, as in $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$. When the target space (codomain) is 2-dimensional or larger, we place a vector above the function name to remind us that the output is more than just a number.

In the next problem, we keep the input as just a single number t, but the output is now a vector in $\mathbb{R}^3$.

Exercise 5.3

A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}_2(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha.
Stewart: More practice in 13.4:7-8, 15-18

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function by picking several values of t and plotting the points resulting from $(2\cos t, 2\sin t, t)$.
3. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
4. How many dimensions do you need to graph this function?

Exercise 5.4

On a separate piece of paper (you'll be expanding this later) create a table with columns for:

- problem number
- function
- n
- m
- number of dimensions required to graph

Go back over the previous problems in this unit and fill in the table.

5.3 Parametric Surfaces:

We now increase the number of inputs from 1 to 2. This will allow us to graph many space curves at the same time.

Exercise 5.5

The jet from problem 5.3 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from one jet spreads outwards to mix with the neighboring jet, so that it looks like the jets are leaving a rather wide sheet of smoke behind them as they fly.

See Sage or Wolfram Alpha.

The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream is $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $0 \le t \le 4\pi$ and $2 \le a \le 4$.

1. What are n (inputs) and m (outputs) when we write the function $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Start by graphing the position of the three jets. For $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ plot the position of each jet:
   (a) $\vec{r}_2(2,t) = (2\cos t, 2\sin t, t)$
   (b) $\vec{r}_3(3,t) = (3\cos t, 3\sin t, t)$
   (c) $\vec{r}_4(4,t) = (4\cos t, 4\sin t, t)$
3. We said in the initial problem statement that the smoke spreads and merges. So we really want to plot $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $2 \le a \le 4$ at each t value.
   (a) Describe how would this modify your graph from the previous part?
   (b) Let $t = 0$ and graph the curve $r(a,0) = (a,0,0)$ for $a \in [2,4]$.
   (c) Repeat this for $t = \pi/2, \pi, 3\pi/2$
4. Describe the resulting surface.

Contexual Aside: The function above is called a parametric surface. Parametric surfaces are formed by joining together many parametric space curves. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Exercise 5.6

Use the same set up as problem 5.3, namely

$$\vec{r}(t) = (2\cos t, 2\sin t, t).$$

See Section 0.10 and Definition 3.3.
Stewart: The text has more practice in 13.2:3-8

You'll need a graph of this function to complete this problem.

1. Find the first and second derivative of $\vec{r}(t)$.
   Note that $\vec{v}(t) = \vec{r}'(t)$ and $\vec{a}(t) = \vec{r}''(t)$.
2. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.

3. Give an equation of the tangent line to this curve at $t = \pi/2$.

Exercise 5.7

Consider the pebble from problem 5.1. The pebble's height was given by $y = 64 - 16t^2$. The pebble also has some horizontal velocity (it's moving at 3 ft/s to the right). If we let the origin be the base of the 64 ft building, then the position of the pebble at time t is given by $\vec{r}(t) = (3t, 64 - 16t^2)$.

See Sage or Wolfram Alpha. Stewart: The text has more practice in 13.4:3-6

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. At what time does the pebble hit the ground (the height reaches zero)?
3. Construct a graph of the pebble's path from when it leaves the top of the building until it hits the ground.
4. Find the pebble's velocity and acceleration vectors at $t = 1$? Draw these vectors on your graph with their base at the pebble's position at $t = 1$.

See Section 0.10 and Definition 3.3.

5. At what speed is the pebble moving when it hits the ground?

Exercise 5.8

Go back to the table you started in problem 5.4.

1. Add in the new problems you've completed.
2. What do you observe about the domain (n), co-domain (m) and the dimensions required for plotting?

In all the problems above, you should have noticed that in order to draw a function (provided you include arrows for direction, or use an animation to represent "time"), you can determine how many dimensions you need to graph a function by just summing the dimensions of the domain and codomain. This is true in general.

Challenge Exercise 5.1: More Parametric Surfaces

Consider the parametric surface $\vec{r}(u, v) = (u \cos v, u \sin v, u^2)$ for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. Construct a graph of this function. To do so, let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve. Then let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve until you can visualize the surface. [Hint: Think satellite dish.]

See Sage or Wolfram Alpha.

5.4 Functions of Several Variables:

Section Objectives:

- identify the domain and range of a multivariable or vector-valued function
- express curves and surfaces with parametric equations

In this section we'll focus on functions of the form $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^1$; we'll keep the output as a real number. In the next problem, you should notice that the input is a vector (x, y) and the output is a number z. There are two ways to graph functions of this type. The next two problems show you how.

5.4.1 3-D Surface Plots

Exercise 5.9

A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x, y) on the chip is measured. From these measurements, statistics is used to create a temperature function $T = z = f(x, y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $T = z = f(x, y) = 9 - x^2 - y^2$. Eventually we'll be creating a 3D model of this function in this problem, so you'll need to place all your graphs on the same x, y, z axes. However, to begin with you may find it easier to start with several 2-D plots.

See Sage or Wolfram Alpha.

1. What is the temperature at $(0, 0)$, $(1, 2)$, and $(-4, 3)$?

Stewart: See 14.1: 2, 7

2. Let $y = 0$, Substitute this value into the temperature function $T = z = f(x, y) = 9 - x^2 - y^2$. Plot the resulting function in the xz-plane. [Hint: Treat z as "y". It should be a (upside-down) parabola]

3. Now let $x = 0$. Substitute this value into the temperature function and simplify. Plot the result. This plot should appear in the yz plane and be another parabola.

4. Now let $z = 0$. Draw the resulting curve in the xy plane.
   If you drew the above curves in 2-D it is time to combine them. You should end up with 3 lines or a basic wire frame. To make a more complete wireframe we need to add more curves.

5. State the simplified equation, then plot and label the curves for:

   (a) $x = -2$
   (b) $x = -1$
   (c) $x = 1$
   (d) $x = 2$

   Note: You could have done this with y also.

6. Describe the shape. Add any extra features to your graph to convey the 3D image you are constructing.

Stewart: See 14.1: 23-31

5.4.2 2-D Contour Plots

Exercise 5.10

We'll be using the same function $T = z = f(x, y) = 9 - x^2 - y^2$ as the previous problem. However, this time we'll construct a graph of the function by only studying places where the temperature is constant. We'll do this by creating a graph in 2D of the surface (similar to a topographical map).

See Sage or Wolfram Alpha.

Stewart: See 14.1: Examples 9-13, for more practice problems 14.1:43-50

1. How can we find all the points which have zero temperature? [Hint: Think about what a temperature of zero means in the function]

2. Use your process to find the curve corresponding to a temperature of zero. Plot this curve in the xy-plane. Be sure to label it with $T = 0$
   This curve is called a level curve. As long as you stay on this curve, your temperature will remain level, it will not increase nor decrease.

3. Which points in the plane have temperature $z = 5$? Add this level curve to your 2D plot and write $z = 5$ next to it.

4. Repeat the above for $z = 8$, $z = 9$, and $z = 1$.

5. What's wrong with letting $z = 10$?

6. Using your 2D plot, construct a 3D image of the function by lifting each level curve to its corresponding height.

Definition 5.1. A level curve of a function $z = f(x, y)$ is a curve in the xy-plane found by setting the output z equal to a constant. Symbolically, a level curve of $f(x, y)$ is the curve $c = f(x, y)$ for some constant c. A 2D plot consisting of several level curves is called a contour plot of $z = f(x, y)$.

5.4.3 Parametric Surfaces Continued

Exercise 5.11

See Sage or Wolfram Alpha.
Stewart: More practice is in 14.1:51-52, 55-58

Consider the function $f(x, y) = x - y^2$.

1. Use the same process as problem 5.9 to construct a 3D surface plot of f. [So just graph in 3D the curves given by $x = 0$ and $y = 0$ and then try setting x or y equal to some other constants, like $x = 1$, $x = 2$, $y = 1$, $y = 2$, etc.]

2. Use the same process as problem 5.10 to construct a contour plot of f. [So just graph in 2D the curves given by setting z equal to a few constants, like $z = 0$, $z = 1$, $z = -4$, etc.]

3. Which level curve passes through the point $(2, 2)$? Draw this level curve on your contour plot.

Notice that when we graphed the previous two functions (of the form $z = f(x, y)$) we could either construct a 3D surface plot, or we could reduce the dimension by 1 and construct a 2D contour plot by letting the output z equal various constants.

The next function is of the form $w = f(x, y, z)$, so it has 3 inputs and 1 output. We could write $f \colon \mathbb{R}^3 \to \mathbb{R}^1$. We would need 4 dimensions to graph this function, but graphing in 4D is not an easy task. Instead, we'll reduce the dimension and create plots in 3D to describe the level surfaces of the function.

Exercise 5.12

See Sage.
Wolfram Alpha currently does not support drawing level surfaces. You could also use Mathematica or Wolfram Demonstrations.
Stewart: For more practice see 14.1:65-68

Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Now suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

1. Which points in space have a temperature of 99? Use algebra to simplify this to $x^2+y^2+z^2 = 1$. [Hint: What does the 99 replace in your function?]

2. Draw the resulting object/function.
3. Which points in space have a temperature of 96? of 84? Draw the surfaces.
4. What is your temperature at $(3, 0, -4)$? Draw the level surface that passes through $(3, 0, -4)$.
5. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?
6. As you move outwards, away from the origin, what happens to your temperature?

Exercise 5.13

See Sage.

Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function. This means that when $y = 0$ you can draw one curve. When $y = 1$, you should draw the same curve. When $y = 2$, again you draw the same curve, etc.

1. Draw a graph of the level surface $w = 4$.
2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).
3. Graph the surface $16 = x^2 + z^2$.

This kind of graph is called a cylinder, and is important in manufacturing where you extrude an object through a hole.

Most of our examples of function of the form $w = f(x, y, z)$ can be drawn by using our knowledge about conic sections. We can graph ellipses and hyperbolas if there are only two variables. So the key idea is to set one of the variables equal to a constant and then graph the resulting curve. Repeat this with a few variables and a few constants, and you'll know what the surface is. Sometimes when you set a specific variable equal to a constant, you'll get an ellipse. If this occurs, try setting that variable equal to other constants, as ellipses are generally the easiest curves to draw.

To Valpo Students: If you feel a little lost and skipped the review unit on conic sections, now would be a good time to go back and go through it.

Exercise 5.14

Go back to the table you started in problem 5.4.

1. Add in the new problems you've completed.
2. What do you observe about the domain (n), co-domain (m) and the dimensions required for plotting?

Challenge Exercise 5.2: More Surfaces

See Sage.
Stewart: Remember you can find more practice in 14.1:65-68

Consider the function $w = f(x, y, z) = x^2 - y^2 + z^2$.

1. Draw a graph of the level surface $w = 1$. [You need to graph $1 = x^2 - y^2 + z^2$. Let $x = 0$ and draw the resulting curve. Then let $y = 0$ and draw the resulting curve. Let either x or y equal some more constants (whichever gave you an ellipse), and then draw the resulting ellipses.]

2. Graph the level surface $w = 4$. [Divide both sides by 4 (to get a 1 on the left) and the repeat the previous part.]

3. Graph the level surface $w = -1$. [Try dividing both sides by a number to get a 1 on the left. If $y = 0$ doesn't help, try $y = 1$ or $y = 2$.]

4. Graph the level surface that passes through the point $(3, 5, 4)$. [Hint: what is $f(3, 5, 4)$?]

5.4.4 Vector Fields and Transformations:

We will finish this section by considering vector fields and transformations.

- $\vec{T}(u, v) = (x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D transformation)
- $\vec{T}(u, v, w) = (x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D transformation)
- $\vec{F}(x, y) = (M, N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x, y, z) = (M, N, P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)

Notice that in all cases, the dimension of the input and output are the same. The difference between vector fields and transformations has to do with the application.

Transformations

In this section you will...

- identify the domain and range of multivariable or vector-valued functions

We've already seen examples of transformations with polar, cylindrical, and spherical coordinates.

Review

Consider the coordinate transformation

$$\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta).$$

1. Let $r = 3$ and then graph $\vec{T}(3, \theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.

Exercise 5.15

Consider the spherical coordinates transformation as defined in Stewart:

$$\vec{T}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi),$$

which could also be written as

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

Graphing this transformation requires $3 + 3 = 6$ dimensions. In this problem we'll construct parts of this graph by graphing various surfaces. We did something similar for the polar coordinate transformation in problem 4.5.

Recall that ϕ ("phi") is the angle down from the z axis, θ ("theta") is the angle counterclockwise from the x-axis in the xy-plane, and ρ ("rho") is the distance from the origin. Review problem 4.18 if you need a refresher.

1. Let $\rho = 2$ and graph the resulting surface. What do you get if $\rho = 3$?
2. Let $\phi = \pi/4$ and graph the resulting surface. What do you get if $\phi = \pi/2$?
3. Let $\theta = \pi/4$ and graph the resulting surface. What do you get if $\theta = \pi/2$?

See Sage or Wolfram Alpha.

See Sage or Wolfram Alpha.

Vector Fields

In this section you will...

- identify the domain and range of multivariable or vector-valued functions
- define and sketch two- or three- dimensional vector fields

We now explore a vector field example.

Exercise 5.16

Consider the vector field $\vec{F}(x, y) = (2x + y, x + 2y)$. In this problem, you will construct a graph of this vector field by hand.

1. Compute $\vec{F}(1, 0)$. Then draw the vector $F(1, 0)$ with its base at $(1, 0)$.
2. Compute $\vec{F}(1, 1)$. Then draw the vector $F(1, 1)$ with its base at $(1, 1)$.
3. Repeat the above process for the points $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$, and $(1, -1)$. Remember, at each point draw a vector.

See Sage or Wolfram Alpha. The computer will shrink the largest vector down in size so it does not overlap any of the others, and then reduce the size of all the vectors accordingly.
Stewart: See 16.1: 1-14 for more practice.

Exercise 5.17: The Spin field

Consider the vector field $\vec{F}(x, y) = (-y, x)$. Construct a graph of this vector field. Remember, the key to plotting a vector field is "at the point (x, y), draw the vector $\vec{F}(x, y)$ with its base at (x, y)." Plot at least 8 vectors (a few in each quadrant), so we can see what this field is doing.

Use the links above to see the computer plot this.
Stewart: See 16.1: 1-14 for more practice.

Sage can also help us visualize 3d vector fields, like $\vec{F}(x, y, z) = (y, z, x)$.

5.5 Constructing Functions

We now know how to draw a vector field provided someone tells us the equation. How do we obtain an equation of a vector field? The following problem will help you develop the gravitational vector field.

Exercise 5.18: Radial fields

Do the following:

1. Let $P = (x, y, z)$ be a point in space. At that point, what is the x, y, and z distance to the origin?
2. At the point P, let $\vec{F}(x, y, z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}(x, y, z)$ [Hint: An object following this vector would travel from the point to the origin].
3. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.

Use Sage to plot your vector fields.

4. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.

If someone gives us parametric equations for a curve in the plane, we know how to draw the curve. How do we obtain parametric equations of a given curve? In problem 5.2, we were given the parametric equation for the path of a horse, namely $x = 2\cos t, y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$. From those equations, we drew the path of the horse, and could have written a Cartesian equation for the path. How do we work this in reverse, namely if we had only been given the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$, could we have obtained parametric equations $\vec{r}(t) = (x(t), y(t))$ for the curve?

Exercise 5.19

Give a parametrization of the straight line from $(a, 0)$ to $(0, b)$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Remember to include bounds for t.

$$Hint : Review 3.3.$$

Exercise 5.20

Give a parametrization of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$. Remember the bounds for t.

Exercise 5.21

Give a parametrization of the function $y = f(x)$ for $x \in [a, b]$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Include bounds for t.

If someone gives us parametric equations for a surface, we can draw the surface. This is what we did in problems 5.5 and 5.1. How do we work backwards and obtain parametric equations for a given surface? This requires that we write an equation for x, y, and z in terms of two input variables (see 5.5 and 5.1 for examples). In vector form, we need a function $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$. We can often use a coordinate transformation $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ to obtain a parametrization of a surface.

The next three problems show how to do this.

Exercise 5.22

Consider the surface $z = 9 - x^2 - y^2$ plotted in problem 5.9.

Use Sage or Wolfram Alpha to plot your parametrization. Stewart: See section 16.6 for more examples

1. Using the rectangular coordinate transformation $\vec{T}(x, y, z) = (x, y, z)$, give a parametrization $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface.

   This is the same as saying

   $$x = x, y = y, z =?.$$

   [Hint: Use the surface equation to eliminate the input variable z in T.]

2. What bounds must you place on x and y to obtain the portion of the surface above the plane $z = 0$?

3. If $z = f(x, y)$ is any surface, give a parametrization of the surface (i.e., $x =?, y =?, z =?$ or $\vec{r}(?, ?) = (?, ?, ?)$.)

Exercise 5.23

Again consider the surface $z = 9 - x^2 - y^2$.

Use Sage or Wolfram Alpha to plot your parametrization with your bounds (see 5.22 for examples).

1. Using cylindrical coordinates, $\vec{T}(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$, obtain a parametrization $\vec{r}(r, \theta) = (?, ?, ?)$ of the surface using the input variables r and θ. In other words, if we let $x = r\cos\theta, y = r\sin\theta, z = z$, write $z = 9 - x^2 - y^2$ in terms of r and θ.

2. What bounds must you place on r and θ to obtain the portion of the surface above the plane $z = 0$?

Exercise 5.24

We did very similar things in problem 5.15.

Recall the spherical coordinate transformation as given in Stewart:

$$\vec{T}(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

This is a function of the form $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$. If we hold one of the three inputs constant, then we have a function of the form $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$, which is a parametric surface.

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

1. Give a parametrization of the sphere of radius 2, using ϕ and θ as your input variables.

2. What bounds should you place on ϕ and θ if you want to hit each point on the sphere exactly once? [Hint: There are two possible answers here]

3. What bounds should you place on ϕ and θ if you only want the portion of the sphere above the plane $z = 1$? [Hint: There are also two possible answers here]

Sometimes you'll have to invent your own coordinate system when constructing parametric equations for a surface. If you notice that there are lots of circles parallel to one of the coordinate planes, try using a modified version of cylindrical coordinates. Instead of circles in the xy plane ($x = r\cos\theta, y = r\sin\theta, z = z$), maybe you need circles in the yz-plane ($x = x, y = r\sin\theta, z = r\sin\theta$) or the xz plane. Just look for lots of circles, and then construct your parametrization accordingly.

Challenge Exercise 5.3

Find parametric equations for the surface $x^2 + z^2 = 9$. [Hint: read the paragraph above.]

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

1. What bounds should you use to obtain the portion of the surface between $y = -2$ and $y = 3$?

2. What bounds should you use to obtain the portion of the surface above $z = 0$?

3. What bounds should you use to obtain the portion of the surface with $x \geq 0$ and $y \in [2, 5]$?

Challenge Exercise 5.4

Construct a graph of the surface $z = x^2 - y^2$. Do so in 2 ways. (1) Construct a 3D surface plot. (2) Construct a contour plot, which is a graph with several level curves. Which level curve passes through the point $(3, 4)$? Use Wolfram Alpha to know if you're right. Just type "plot z=x2̂-y2̂."

Challenge Exercise 5.5

Construct a plot of the vector field

$$\vec{F}(x, y) = (x + y, -x + 1)$$

by graphing the field at many integer points around the origin (I generally like to get the 8 integer points around the origin, and then a few more). Then explain how to modify your graph to obtain a plot of the vector field

$$\hat{F}(x, y) = \frac{(x + y, -x + 1)}{\sqrt{(x + y)^2 + (1 - x)^2}}.$$

5.6 Summary of Functions

In this unit we've covered a lot of different function types. Be sure you can recognize each one both from its functional form and its name. The following list provides a summary of the unit.

- $y = f(x)$ or $f\colon \mathbb{R} \to \mathbb{R}$ (functions of a single variable)
- $\vec{r}(t) = (x, y)$ or $f\colon \mathbb{R} \to \mathbb{R}^2$ (parametric curves)
- $\vec{r}(t) = (x, y, z)$ or $f\colon \mathbb{R} \to \mathbb{R}^3$ (space curves)
- $\vec{r}(u, v) = (x, y, z)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces)
- $z = f(x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}$ (functions of two variables)
- $z = f(x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$ (functions of three variables)
- $\vec{T}(u, v) = (x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D transformation)
- $\vec{T}(u, v, w) = (x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D transformation)
- $\vec{F}(x, y) = (M, N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x, y, z) = (M, N, P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)

To Valpo Students:

5.7 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.
- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.

Unit 6

Derivatives

In this unit you will learn how to...

1. Compute partial derivatives.
2. Explain how to obtain the total derivative from the partial derivatives (using a matrix).
3. Find equations of tangent lines and tangent planes to surfaces. (We'll do this three ways.)
4. Find derivatives of composite functions, using the chain rule (matrix multiplication).

6.1 Introduction

In this section you will...

- connect Calculus I/II to Calculus III: Multivariable via vectors

We'll find that throughout this course, the key difference between first-semester calculus and multivariate calculus is that we replace the input x and output y of functions with the vectors $\vec{x}$ and $\vec{y}$. For the function $f(x, y) = z$, we can write f in the vector notation $\vec{y} = \vec{f}(\vec{x})$ if we let $\vec{x} = (x, y)$ and $\vec{y} = (z)$. Notice that $\vec{x}$ is a vector of inputs, and $\vec{y}$ is a vector of outputs.

Exercise 6.1

For each of the functions below, state what $\vec{x}$ and $\vec{y}$ should be so that the function can be written in the form $\vec{y} = \vec{f}(\vec{x})$. In addition, identify what type of function each is from the list in 5.6.

The point to this problem is to help you learn to recognize the dimensions of the domain and codomain of the function. If we write $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x}$ is a vector in $\mathbb{R}^n$ with n components, and $\vec{y}$ is a vector in $\mathbb{R}^m$ with m components.

1. $f(x, y, z) = w$
2. $\vec{r}(t) = (x, y, z)$
3. $\vec{r}(u, v) = (x, y, z)$
4. $\vec{F}(x, y) = (M, N)$
5. $\vec{F}(\rho, \phi, \theta) = (x, y, z)$

6.2 The (multi-dimensional) Derivative

In this section we will...

- understand differentials in matrix notation
- learn to compute partial and total derivatives

Before we introduce multi-dimensional derivatives, let's recall the definition of a differential. If $y = f(x)$ is a function, then we say the differential dy is the expression $dy = f'(x)dx$. We could also write this as $dy = \frac{dy}{dx}dx$. Similarly, if $w = g(t)$ then we have the derivative as $\frac{dw}{dt} = g'(t)$ and the differential as $dw = g'(t)dt$.

Observation 6.1. Here's the key. Think of differential notation $dy = f'(x)dx$ in the following way:

> A change in the output y equals the derivative multiplied by a change in the input x. To get dy, we just need the derivative times dx.

To get the derivative in all dimensions, we just substitute in vectors to obtain the differential notation $d\vec{y} = f'(\vec{x})d\vec{x}$. The derivative is precisely the thing that tells us how to get $d\vec{y}$ from $d\vec{x}$. We'll quickly see that $f'(\vec{x})$ must be a matrix, and then we'll start calling it Df instead of f'.

Let's now examine some problems you have seen before.

Exercise 6.2

The volume of a right circular cylinder is $V(r, h) = \pi r^2 h$. Imagine that each of V, r, and h depends on t (we might be collecting rain water in a can, or crushing a cylindrical concentrated juice can, etc.).

Stewart: ??

1. Let's rewrite $V(r,h)$. For each of the following start with $V(r,h) = \pi r^2 h$
   (a) Substitute $r(t)$ for r
   (b) Substitute $h(t)$ for h
   (c) Now replace both r and t with $r(t)$ and $h(t)$ respectively
2. In which of the equations from part 1 does h NOT change as time changes?
3. If the height remains constant, what is dV/dt in terms of dr/dt? Times both sides by dt to obtain a formula for dV when h is constant.
4. If the radius remains constant, what is dV/dt in terms of dh/dt? What is dV when r is constant?
5. If neither the radius nor height remains constant, what is dV/dt in terms of dh/dt? Solve for dV.
6. Show that we can write dV as the matrix product

$$dV = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix} \begin{bmatrix} dr \\ dh \end{bmatrix}.$$

   How do the columns of this matrix relate to the previous portions of the problem.

The matrix $\begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}$ is the derivative. The columns we'll call the partial derivatives. The partial derivatives make up the whole.

$\boxed{\textbf{Exercise 6.3}}$
The volume of a box is $V(x,y,z) = xyz$. Imagine that each variable depends on t.

1. If both y and z remain constant we only need to replace x with $x(t)$
   (a) State the new equation for V
   (b) What is dV/dt?
   (c) Times both sides by dt to obtain a formula for dV when all bur x is constant.
2. Repeat Part 1 for when y is the only non constant variable, and then for when z is the only non constant variable.
3. What is dV/dt in terms of dx/dt, dy/dt, and dz/dt when all three variables are not constant.
4. Show that we can write dV as the matrix product (fill in the blanks)

$$dV = \begin{bmatrix} yz & ? & ? \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

   How do the columns of this matrix relate to the previous portions of the problem.

The matrix $\begin{bmatrix} yz & ? & ? \end{bmatrix}$ is the derivative. The columns we'll call the partial derivatives. The partial derivatives make up the whole.

Part 4 in each problem above is the KEY idea, let me repeat, THE KEY IDEA, to the rest of this course. It all goes back to differentials. We can compute a small change in volume, if we know how much the radius and height have changed, or if we know how much the length, width, and height will change.

Exercise 6.4
Use matrix multiplication to answer the following questions.

1. In 6.2 we showed that a change in the volume of a cylinder is approximately
$$dV = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix} \begin{bmatrix} dr \\ dh \end{bmatrix}.$$
If we know that $r = 3$ and $h = 4$, and we know that r could going to increase by about .1 and h could increase by about .2, then by about how much will V increase by?

2. The volume of a box is given by $V = xyz$. From 6.3 we know the differential of the volume is $V = \begin{bmatrix} yz & xz & xy \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$. If the current measurements are $x = 2$, $y = 3$, and $z = 5$, and we know that $dx = .01$, $dy = .02$, and $dz = .03$, then by about how much will the volume increase.

In more general terms, we can compute the change in a function $f(x, y)$ if we know how much x and y will change.

Exercise 6.5
Consider the function $f(x, y) = x^2y + 3x + 4\sin(5y)$.

1. If both x and y depend on t, then use implicit differentiation to obtain a formula for df/dt in terms of dx/dt and dy/dt. This will be the last time we use implicit differentiation.

2. Solve for df, and write your answer as the matrix product (fill in the blank)
$$df = \begin{bmatrix} ? & x^2 + 20\cos(5y) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

3. If you hold y constant, then what is df/dx?

4. If you hold x constant, then what is df/dy?

Problem 6.5 is precisely the content to this chapter. We just need to add some vocabulary to make it easier to talk about what we just did. Let's introduce the vocabulary in terms of the problem above, and then make a formal definition.

- The derivative of f in the previous problem is the matrix
$$Df(x, y) = \begin{bmatrix} 2xy + 3 & x^2 + 20\cos(5y) \end{bmatrix}.$$
Some people call this the total derivative, as it's made up of two parts, called partial derivatives.

- The first column of this matrix is just part of the whole derivative. We can get the first column by holding y constant, and then differentiating with respect to x. This is precisely a partial derivative. We'll write this as $\frac{\partial f}{\partial x} = 2xy + 3$, or sometimes just $f_x = 2xy + 3$.

- The second column of the derivative is the partial of f with respect to y. We can get the second column by holding x constant, and then differentiating with respect to y. We'll write this as $\frac{\partial f}{\partial y} = x^2 + 20\cos(5y)$, or $f_y = x^2 + 20\cos(5y)$.

- Remember, the derivative of f is a matrix. The columns of the matrix are the partial derivatives with respect to the input variables.

Definition 6.2: Derivatives and Partial Derivatives. Let f be a function.

- The partial derivative of f with respect to x is the regular derivative of f, provided we hold every every input variable constant except x. (This is what we did in the first parts of problems 6.2 and 6.3. We'll use the notations
$$\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}[f], \quad f_x, \quad \text{and } D_x f$$
to mean the partial of f with respect to x.

- The partial of f with respect to y, written $\dfrac{\partial f}{\partial y}$ or f_y, is the regular derivative of f, provided we hold every input variable constant except y. A similar definition holds for partial derivatives with respect to any variable.

- The derivative of f is a matrix. The columns of the derivative are the partial derivatives. When there's more than one input variable, we'll use Df rather than f' to talk about derivatives. The order of the columns must match the order you list the variables in the function. If the function is $f(x,y)$, then the derivative is $Df(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$. If the function is $V(x,y,z)$, then the derivative is $DV(x,y,z) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix}$.

It's time to practice these new words on some problems. Remember, we're doing the exact same thing as before the definitions above. Now we just have some vocabulary which makes it much easier to talk about differentiation.

Exercise 6.6

Compute the requested partial and total derivatives.

Stewart: See 14.3:15-40 for more practice.
I strongly suggest you practice a lot of this type of problem until you can compute partial derivatives with ease.

1. For $f(x,y) = x^2 + 2xy + 3y^2$, compute both $\dfrac{\partial f}{\partial x}$ and f_y. Then state $Df(x,y)$.

2. For $f(x,y,z) = x^2y^3z^4$, compute all three of f_x, $\dfrac{\partial f}{\partial y}$, and $D_z f$. Then state $Df(x,y,z)$.

Remember, the partial derivative of a function with respect to x is just the regular derivative with respect to x, provided you hold all other variables constant. We put the partials into the columns of a matrix to obtain the (total) derivative.

Please take a moment and practice computing partial and total derivatives. Your textbook has lots of examples to help you with partial derivatives. However, the textbook leaves out the actual derivative. This handwritten file (follow the link) has 6 problems, together with solutions, that you can use as extra practice for total derivatives. Please open the file before moving on.

Exercise 6.7

Compute the requested partial and total derivatives.

1. Consider the parametric surface $\vec{r}(u,v) = (u, v, v\cos(uv))$. Compute both $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$. Thenm state $D\vec{r}(u,v)$. If you end up with a 3 by 2 matrix, you did this correctly.

2. Consider the vector field $\vec{F}(x,y) = (-y, xe^{3y})$. Compute both $\dfrac{\partial \vec{F}}{\partial x}$ and $\dfrac{\partial \vec{F}}{\partial y}$. Then state $D\vec{F}(x,y)$.

As you completed the problems above, did you notice any connections between the size of the matrix and the size of the input and output vectors? Make sure you ask in class about this. We'll make a connection.

We've now seen that the derivative of $z = f(x,y)$ is a matrix $Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$. This is a function itself that has inputs x and y, and outputs f_x and f_y. This means it has 2 inputs and 2 outputs, so it's a vector field. What does the vector field tell us about the original function?

Challenge Exercise 6.1

Consider the function $f(x,y) = y - x^2$.

1. In the xy plane, please draw several level curves of f (maybe $z = 0$, $z = 1$, $z = -4$, etc.) Write the height on each curve (so you're making a topographical map).

2. Compute the derivative of f. (Remember this is now a vector field.)

3. Pick several points in the xy plane that lie on the level curves you already drew. At these points, add the vector given by the derivative. (So at (0,0), you'll need to draw the vector (0,1). At (1,1), you'll need to draw the vector (-2,1).) Add 8 vectors to your picture, and then write down to share with the class any observations you make.

We'll come back to this problem more in chapter 9 as we discuss optimization. There are lots of connections between the derivative and level curves.

Let's now explain geometrically what a partial derivative is. The next two problems will help with this.

Exercise 6.8

Consider the change of coordinates $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$.

1. Compute the partial derivatives $\dfrac{\partial \vec{T}}{\partial r}$ and $\dfrac{\partial \vec{T}}{\partial \theta}$

2. State the derivative $D\vec{T}(r,\theta)$.

   *Hint : Ifyougeta2by2matrix, thenyou′reontherighttrack.Eachpartialderivativeisavector.(Thisoneisinthe*handwritte

3. Consider the polar point $(r,\theta) = (4, \pi/2)$.

   (a) Compute $T(4,\pi/2)$ (the Cartesian coordinate)

   (b) Compute both partial derivatives at $(4,\pi/2)$. [Hint: You should get a point and two vectors.]

   (c) At the point, draw both vectors.

4. If you were standing at the polar point $(4,\pi/2)$ and someone said, "Hey you, keep your angle constant, but increase your radius," then which direction would you move? What if someone said, "Hey you, keep your radius constant, but increase your angle"?

5. Now change the polar point to $(r, \theta) = (2, 3\pi/4)$. Try, without doing any computations, to repeat part 2 (at the point draw both partial derivatives). Explain.

If your answers to the 2nd and 3rd part above were the same, then you're doing this correctly. The partial derivatives, when vectors, tell us precisely about motion. The next problem reinforces this concept.

Review

If you know that a line passes through the point $(1, 2, 3)$ and is parallel to the vector $(4, 5, 6)$, give a vector equation, and parametric equations, of the line. See [1] for an answer.

Exercise 6.9

Consider the parametric surface $\vec{r}(a, t) = (a \cos t, a \sin t, t)$ for $2 \leq a \leq 4$ and $0 \leq t \leq 4\pi$. We encountered this parametric surface in 5.5 when we considered a smoke screen left by multiple jets.

1. How many inputs does this function have? How many outputs?
2. What dimensions does that make the derivative?
3. Compute the partial derivatives $\vec{r}_a$ and $\vec{r}_t$ (they are vectors), and state the total derivative.
4. Look at a plot of the surface (use one of the links to the right). Now, suppose an object is on this surface at the point $\vec{r}(3, \pi) = (-3, 0, \pi)$. At that point, please draw the partial derivatives $\vec{r}_a(3, \pi)$ and $\vec{r}_t(3, \pi)$.
5. If you were standing at $\vec{r}(3, \pi)$ and someone told you, "Hey you, hold t constant and increase a," then in which direction would you move. What if f someone told you, "Hey you, hold a constant and increase t"?
6. Give vector equations for two tangent lines to the surface at $\vec{r}(3, \pi)$.

   [Hint: You've got the point by plugging $(3, \pi)$ into $\vec{r}$, and you've got two different direction vectors from $D\vec{r}$. Once you have a point and a vector, we know 1.5 how to get an equation of a line.]

Please see Sage or Wolfram Alpha for a plot of the surface. Click on either link.

In the previous problem, you should have noticed that the partial derivatives of $\vec{r}(a, t)$ are tangent vectors to the surface. Because we have two tangent vectors to the surface, we should be able to use them to construct a normal vector to the surface (recall **??** , and from that, a tangent plane (recall 1.26. That's just cool and leads us into the next section...

Review

If you know that a plane passes through the point $(1, 2, 3)$ and has normal vector $(4, 5, 6)$, then give an equation of the plane. See [2] for an answer.

---

[1] A vector equation is $\vec{r}(t) = (4, 5, 6)t + (1, 2, 3)$ or $\vec{r}(t) = (4t+1, 5t+2, 6t+3)$. Parametric equations for this line are $x = 4t + 1$, $y = 5t + 2$, and $z = 6t + 3$.

[2] An equation of the plane is $4(x - 1) + 5(y - 2) + 6(y - 3) = 0$. If (x, y, z) is any point in the plane, then the vector $(x - 1, y - 2, z - 3)$ is a vector in the plane, and hence orthogonal to $(4, 5, 6)$. The dot product of these two vectors should be equal to zero, which is why the plane's equation is $(4, 5, 6) \cdot (x - 1, y - 2, z - 3) = 0$.

Exercise 6.10

Consider again the parametric surface $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $2 \le a \le 4$ and $0 \le t \le 4\pi$. We'd like to obtain an equation of the tangent plane to this surface at the point $\vec{r}(3,\pi)$. To find a plane we need a point and a normal (orthogonal) vector to the [tangent] plane.

1. State the coordinates for $\vec{r}(3,\pi)$

Since $\vec{r}_a(3,\pi)$ and $\vec{r}_t(3,\pi)$ (the two partial derivatives) are tangent to the surface they must be in our tangent plane also. So a vector orthogonal to $\vec{r}_a$ and $\vec{r}_t$ will be normal to the tangent plane.

2. Find this normal vector.
3. Give an equation for the tangent plane.

[Hint: How do I obtain a vector orthogonal to both? Is it the scalar, dot, or cross product?]

Since a partial derivative is a function, we can take partial derivatives of that function as well. If we want to first compute a partial with respect to x, and then with respect to y, we would write one of

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

The shorthand notation f_{xy} is easiest to write. In upper-level courses, we will use subscripts to mean other things. At that point, we'll have to use the fractional partial notation to avoid confusion.

Challenge Exercise 6.2: Mixed Partials Agree

Complete the following:

1. Let $f(x,y) = 3xy^3 + e^x$. Compute the four second partials
$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and } \frac{\partial^2 f}{\partial x \partial y}.$$
2. For $f(x,y) = x^2\sin(y) + y^3$, compute both f_{xy} and f_{yx}.
3. Make a conjecture about a relationship between f_{xy} and f_{yx}. Then use your conjecture to quickly compute f_{xy} if
$$f(x,y) = 3xy^2 + \tan^2(\cos(x))(x^{49} + x)^{1000}.$$

6.3 Tangent Planes

This section will cover how to...

- Find equations of tangent lines and tangent planes to surfaces.

I promised earlier in this chapter that you can obtain most of the results in multivariate calculus by replacing the x and y in $dy = f'dx$ with $\vec{x}$ and $\vec{y}$. The last problem asked you to obtain a tangent plane to a parametric surface. You've never had to find a tangent plane to a function of the form $z = f(x,y)$. Let's review how to do it for functions of the form $y = f(x)$, and then generalize.

Exercise 6.11: Tangent Lines

Consider the function $y = f(x) = x^2$.

1. The derivative is $f'(x) =?$.
   (a) At the point $x = 3$ the derivative is $f'(3) =?$
   (b) and the output y is $y = f(3) =?$.

If we move from the point $(3, f(3))$ to the point (x, y) along the tangent line, then a small change in x is $dx = x - 3$.

2. What is dy in terms of y?

Differential notation states that a change in the output dy equals the derivative times a change in the input dx, which gives us the equation $dy = f'(3)dx$.

3. Replace dx, dy, and $f'(3)$ with what we know they equal, to obtain an equation $y-? =?(x-?)$.
4. What does this equation represent?
5. Draw both f and the equation from the previous part on the same axes.

In first semester calculus, differential notation says $dy = f'dx$. A change in the output equals the derivative times a change in the inputs. For the next problem, the output is z, and input is (x, y), which means differential notation says $dz = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$.

Exercise 6.12: Tangent Planes

See Sage for a picture.

Stewart: See 14.4:1-6 for more practice

Consider the function $z = f(x, y) = 9 - x^2 - y^2$.

1. The derivative is $Df(x, y) = \begin{bmatrix} -2x & ? \end{bmatrix}$.
2. At the point $(x, y) = (2, 1)$, the derivative is $Df(2, 1) = \begin{bmatrix} -4 & ? \end{bmatrix}$ and the output z is $z = f(2, 1) =?$.
3. If we move from the point $(2, 1, f(2, 1))$ to the point (x, y, z) along the tangent plane, then a small change in x is $dx = x - 2$. What are dy and dz in terms of y and z?
4. Explain why an equation of the tangent plane is

We'll construct a graph of f and it's tangent plane in class.

$$z - 4 = \begin{bmatrix} -4 & -2 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix} \quad \text{or} \quad z - 4 = -4(x - 2) - 2(y - 1).$$

   [Hint: What does differential notation tell us?]

Look back at the previous two problems. The first semester calculus tangent line equation, with differential notation, generalized immediately to the tangent plane equation for functions of the form $z = f(x, y)$. Let's try this on another problem. We just used the differential notation $dy = f'dx$ in 2D, and generalized to $dz = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$. Let's repeat this on another problem.

Challenge Exercise 6.3

Let $f(x, y) = x^2 + 4xy + y^2$. Give an equation of the tangent plane at $(3, -1)$.

[Hint: Just as in the previous problem, find $Df(x, y)$, dx, dy, and dz. Then use differential notation.]

See Sage.

Stewart: See 14.4:1-6 for more practice

Let's look again at the function $z = 9 - x^2 - y^2$, and show how parametric surfaces can add more light to unlocking the derivative and its geometric meaning. With a parametrization, partial derivatives are vectors, instead of just numbers. Once we have vectors, we can describe motion. This makes it easier to visualize.

Exercise 6.13

Let $z = f(x, y) = 9 - x^2 - y^2$. We'll parameterize this function by writing $x = x, y = y, z = 9 - x^2 - y^2$, or in vector notation we'd write

$$\vec{r}(x, y) = (x, y, f(x, y)).$$

See Sage for a picture.

Stewart: See 16.6:33-38 for more practice

1. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
2. Next evaluate these partials at $(x, y) = (2, 1)$. You should have $f_x = -4$.
3. What does the number -4 mean? What does the number f_y mean?
4. Compute $\dfrac{\partial \vec{r}}{\partial x}$ and $\dfrac{\partial \vec{r}}{\partial y}$.
5. Now evaluate these partials at $(x, y) = (2, 1)$.
6. What do these vectors mean? [Hint: Draw the surface, and at the point $(2, 1, 4)$, draw these vectors. See the Sage plot.]
7. The vectors above are tangent to the surface. Use them to obtain a normal vector to the tangent plane, and then given an equation of the tangent plane. (You should compare it to your equation from problem 6.12.)

The next problem generalizes the tangent plane and normal vector calculations above to work for any parametric surface $\vec{r}(u, v)$.

Challenge Exercise 6.4

See Sage.

Stewart: See 16.6:33-38 for more practice

Consider the cone parametrized by $\vec{r}(u, v) = (u \cos v, u \sin v, u)$.

1. Give vector equations of two tangent lines to the surface at $\vec{r}(2, \pi/2)$ (so $u = 2$ and $v = \pi/2$).
2. Give a normal vector to the surface at $\vec{r}(2, \pi/2)$.
3. Give an equation of the tangent plane at $\vec{r}(2, \pi/2)$.

We now have two different ways to compute tangent planes. One way generalizes differential notation $dy = f'dx$ to $dz = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$ and then uses matrix multiplication. This way will extend to tangent objects in EVERY dimension. It's the key idea needed to work on really large problems.

The other way requires that we parametrize the surface $z = f(x, y)$ as $\vec{r}(x, y) = (x, y, f(x, y))$ and then use the cross product on the partial derivatives. Both give the same answer. The next problem has you give a general formula for a tangent plane. To tackle this problem, you'll need to make sure you can use symbolic notation. The review problem should help with this.

Review

Joe wants to to find the tangent line to $y = x^3$ at $x = 2$. He knows the derivative is $y = 3x^2$, and when $x = 2$ the curve passes through 8. So he writes an equation of the tangent line as $y - 8 = 3x^2(x - 2)$.

a What's wrong?

b What part of the general formula $y - f(c) = f'(c)(x - c)$ did Joe forget?

See [3] for an answer.

Exercise 6.14: Tangent Plane General Formula

Consider the function $z = f(x, y)$. Explain why an equation of the tangent plane to f at $(x, y) = (a, b)$ is given by

$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - a).$$

Then give an equation of the tangent plane to $f(x, y) = x^2 + 3xy$ at $(3, -1)$. [Hint: Use either differential notation or a parametrization. Try both ways.]

6.4 The Chain Rule

In this section you will learn how to...

- Compute partial and total derivatives of multivariable and vector functions:
  - Find derivatives of composite functions, using the chain rule (matrix multiplication).

We'll now see how the chain rule generalizes to all dimensions. Just as before, we'll find that the first semester calculus rule will generalize to all dimensions, if we replace f' with the matrix Df. Let's recall the chain rule from first-semester calculus.

Theorem 6.3 (The Chain Rule)**.** *Let x be a real number and f and g be functions of a single real variable. Suppose f is differentiable at $g(x)$ and g is differentiable at x. The derivative of $f \circ g$ at x is*

$$(f \circ g)'(x) = \frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x).$$

[3] Joe forgot to replace x with 2 in the derivative. The equation should be $y - 8 = 12(x - 2)$. The notation $f'(c)$ is the part he forgot. He used $f'(x) = 3x^2$ instead of $f'(2) = 8$.

Some people remember the theorem above as "the derivative of a composition is the derivative of the outside (evaluated at the inside) multiplied by the derivative of the inside." If $u = g(x)$, we sometimes write $\dfrac{df}{dx} = \dfrac{df}{du}\dfrac{du}{dx}$. The following problem should help us master this notation.

Exercise 6.15

Suppose we know that $f'(x) = \dfrac{\sin(x)}{2x^2+3}$ and $g(x) = \sqrt{x^2+1}$. Notice we don't know $f(x)$.

Not knowing a function f is actually quite common in real life. We can often measure how something changes (a derivative) without knowing the function itself.

1. State $f'(x)$ and $g'(x)$.
2. State $f'(g(x))$, and explain the difference between $f'(x)$ and $f'(g(x))$.
3. Use the chain rule to compute $(f \circ g)'(x)$.

We now generalize to higher dimensions. If I want to write $\vec{f}(\vec{g}(\vec{x}))$, then $\vec{x}$ must be a vector in the domain of g. After computing $\vec{g}(\vec{x})$, we must get a vector that is in the domain of f.

Since the chain rule in first semester calculus states $(f(g(x))' = f'(g(x))g'(x)$, then in high dimension it should state $D(f(g(x)) = Df(g(x))Dg(x)$, the product of two matrices.

Exercise 6.16

In problem 6.2, we showed that for a circular cylinder with volume $V = \pi r^2 h$, the derivative is

$$DV(r,h) = \begin{bmatrix} 2\pi r h & \pi r^2 \end{bmatrix}.$$

Recall that to get this derivative we assumed that the radius and height are both changing with respect to time. Now we actually use functions for each letting $r = 3t$ and $h = t^2$. We'll write this parametrically as $\vec{x}(t) = (r,h)(t) = (3t, t^2)$.

1. In $V = \pi r^2 h$, replace r and h with what they are in terms of t.
2. Compute $\dfrac{dV}{dt}$.
3. Find the derivative of $\vec{x}(t)$, i.e. the derivative of $(r,h)(t)$. [Hint: The output should be a 2x1 matrix.]
4. We know $DV(r,h)$ and $D(r,h)(t)$ In first semester calculus, the chain rule was the product of derivatives. Multiply these matrices together to get find $\frac{dV}{dt}$. I.E. computer:

   $$\frac{dV}{dt} = DV((r,h)(t)) \cdot D(r,h)(t).$$

   (Did you get the same answer as the first part?)

For the results in part 1 and 4 to match you had to replace r and h with what they equaled in terms of t.

5. What part of the notation $\dfrac{dV}{dt} = DV((r,h)(t)) \cdot D(r,h)(t)$ tells you to replace r and h with what they equal in terms of t?

Let's look at some physical examples involving motion and temperature, and try to connect what we know should happen to what the chain rule states.

Exercise 6.17
Consider $f(x,y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$. Imagine the following scenario. A horse runs around outside in the cold. The horse's position at time t is given parametrically by the elliptical path $\vec{r}(t)$. The function $T = f(x,y)$ gives the temperature of the air at any point (x,y).

1. At time $t = 0$, what is the horse's position $\vec{r}(0)$, and what is the temperature $f(\vec{r}(0))$ at that position?
2. Find the temperatures at $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ as well.
3. In the plane, draw the path of the horse for $t \in [0, 2\pi]$.
4. On the same 2D graph, include a contour plot of the temperature function f. Make sure you include the level curves that pass through the points in part 1, and write the temperature on each level curve you draw.
5. As the horse runs around, the temperature of the air around the horse is constantly changing. At which t does the temperature around the horse reach a maximum? A minimum? Explain, using your graph.
6. As the horse moves past the point at $t = \pi/4$, is the temperature of the surrounding air increasing or decreasing? In other words, is $\frac{df}{dt}$ positive or negative? Use your graph to explain.
7. We'll complete this part in class, but you're welcome to give it a try yourself. Draw the 3D surface plot of f. In the xy-plane of your 3D plot (so $z = 0$) add the path of the horse. In class, we'll project the path of the horse up into the 3D surface.

If you end up with an ellipse and several concentric circles, then you've done this right.

This idea leads to an optimization technique, Lagrange multipliers, later in the semester.

Exercise 6.18
Consider again $f(x,y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$, which means $x = 2\cos t$ and $y = 3\sin t$.

1. At the point $\vec{r}(t)$, we'd like a formula for the temperature $f(\vec{r}(t))$. What is the temperature of the horse at any time t? [In $f(x,y)$, replace x and y with what they are in terms of t.]
2. Compute df/dt (the derivative as you did in first-semester calculus).
3. Construct a graph of $f(t)$ (use software to draw this if you like). From your graph, at what time values do the maxima and minima occur?
4. What is df/dt at $t = \pi/4$?
5. Compare your work with the previous problem.

Exercise 6.19
Consider again $f(x,y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$.

1. Compute both $Df(x,y)$ and $D\vec{r}(t)$ as matrices. One should have two columns. The other should have one column (but two rows).

We can write the temperature at any time t symbolically as $f(r(t))$. First semester calculus suggests the derivative should be the product $(f(\vec{r}(t)))' = f'(\vec{r}(t))\vec{r}'(t)$.

2. Write this using D notation instead of prime notation.

3. Compute the matrix product $DfD\vec{r}$, and then substitute $x = 2\cos t$ and $y = 3\sin t$.

4. What is the change in temperature with respect to time at $t = \pi/4$? Is it positive or negative? Compare with the previous problem.

The previous three problems all focused on exactly the same concept. The first looked at the concept graphically, showing what it means to write $(f \circ \vec{r})(t) = f(\vec{r}(t))$. The second reduced the problem to first-semester calculus. The third tackled the problem by considering matrix derivatives. In all three cases, we wanted to understand the following problem:

> If $z = f(x, y)$ is a function of x and y, and both x and y are functions of t ($\vec{r}(t) = (x(t), y(t))$), then how do we discover how quickly f changes as we change t. In other words, what is the derivative of f with respect to t. Notationally, we seek $\frac{df}{dt}$ which we formally write as $\frac{d}{dt}[f(x(t), y(t))]$ or $\frac{d}{dt}[f(\vec{r}(t))]$.

To answer this problem, we use the chain rule, which is just matrix multiplication.

Theorem 6.4 (The Chain Rule)**.** *Let $\vec{x}$ be a vector and $\vec{f}$ and $\vec{g}$ be functions so that the composition $\vec{f}(\vec{g}(\vec{x}))$ makes sense (we can use the output of g as an input to f). Suppose $\vec{f}$ is differentiable at $\vec{g}(\vec{x})$ and that $\vec{g}$ is differentiable at $\vec{x}$. Then the derivative of $\vec{f} \circ \vec{g}$ at $\vec{x}$ is*

$$D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x})) \cdot D\vec{g}(\vec{x}).$$

The derivative of a composition is equal to the derivative of the outside (evaluated at the inside), multiplied by the derivative of the inside.

This is exactly the same as the chain rule in first-semester calculus. The only difference is that now we have vectors above every variable and function, and we replaced the one-by-one matrices f' and g' with potentially larger matrices Df and Dg. If we write everything in vector notation, the chain rule in all dimensions is the EXACT same as the chain rule in one dimension.

Exercise 6.20

Suppose that $f(x, y) = x^2 + xy$ and that $x = 2t + 3$ and $y = 3t^2 + 4$.

Stewart: See 14.5:1-12 for more practice. Don't use the formulas in the chapter, rather practice using matrix multiplication. The formulas are just a way of writing matrix multiplication without writing down the matrices, and only work for functions from $\mathbb{R}^n \to \mathbb{R}$. Our matrix multiplication method works for any function from $\mathbb{R}^n \to \mathbb{R}^m$.

1. Rewrite the parametric equations $x = 2t + 3$ and $y = 3t^2 + 4$ in vector form, so we can apply the chain rule. This means you need to create a function $\vec{r}(t) = ($____________, ____________$)$.

2. Compute the derivatives $Df(x, y)$ and $D\vec{r}(t)$, and then multiply the matrices together to obtain $\frac{df}{dt}$.

3. How can you make your answer only depend on t (not x or y)? Do so.

4. The chain rule states that $D(f \circ \vec{r})(t) = Df(\vec{r}(t))D\vec{r}(t)$. Explain why we write $Df(\vec{r}(t))$ instead of $Df(x, y)$.

If you'd like to make sure you are correct, try the following. Replace x and y in $f = x^2 + xy$ with what they are in terms of t, and then just use first-semester calculus to find df/dt. Is it the same?

Exercise 6.21

Suppose $f(x, y, z) = x + 2y + 3z^2$ and $x = u + v$, $y = 2u - 3v$, and $z = uv$. Our goal is to find how much f changes if we were to change u (so $\partial f/\partial u$) or if we were to change v (so $\partial f/\partial v$). Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problem on your own.

Stewart: See 14.5:1-12 for more practice.

1. Rewrite the equations for x, y, and z in vector form $\vec{r}(u, v) = (x, y, z)$.
2. Compute the derivatives $Df(x, y, z)$ and $D\vec{r}(u, v)$, and then multiply them together. Notice that since this composite function has 2 inputs, namely u and v, we should expect to get two columns when we are done.
3. What are $\partial f/\partial u$ and $\partial f/\partial v$? [Hint: remember, each input variable gets a column.]

Exercise 6.22

Let $\vec{F}(s, t) = (2s + t, 3s - 4t, t)$ and $s = 3pq$ and $t = 2p + q^2$. This means that changing p and/or q should cause $\vec{F}$ to change. Our goal is to find $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$. Note that since $\vec{F}$ is a vector-valued function, the two partial derivatives should be vectors. Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problems.

1. Rewrite the parametric equations for s and t in vector form.
2. Compute $D\vec{F}(s, t)$ and the derivative of your vector function from part 1, and then multiply them together to find the derivative of $\vec{F}$ with respect to p and q. How many columns should we expect to have when we are done multiplying matrices?
3. What are $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$?

Review

Suppose $f(x, y) = x^2 + 3xy$ and $(x, y) = \vec{r}(t) = (3t, t^2)$. Compute both $Df(x, y)$ and $D\vec{r}(t)$. Then explain how you got your answer by writing what you did in terms of partial derivatives and regular derivatives. See [4] for an answer.

Exercise 6.23: General Chain Rule Formulas

Complete the following:

[4] We have $Df(x, y) = \begin{bmatrix} 2x + 3y & 3y \end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix}$. We just computed f_x and f_y, and dx/dt and dy/dt, which gave us $Df(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}$.

1. Suppose that $w = f(x, y, z)$ and that x, y, z are all function of one variable t (so $x = g(t), y = h(t), z = k(t)$). Use the chain rule with matrix multiplication to explain why
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt}.$$
which is equivalent to writing
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$
[Hint: Rewrite the parametric equations for x, y, and z in vector form $\vec{r}(t) = (x, y, z)$ and compute $Dw(x, y, z)$ and $D\vec{r}(t)$.]

2. Suppose that $R = f(V, T, n, P)$, and that V, T, n, P are all functions of x. Give a formula (similar to the above) for $\dfrac{dR}{dx}$.

Challenge Exercise 6.5

Suppose $z = f(s, t)$ and s and t are functions of u, v and w. Use the chain rule to give a general formula for $\partial z/\partial u$, $\partial z/\partial v$, and $\partial z/\partial w$.

Review

If $w = f(x, y, z)$ and x, y, z are functions of u and v, obtain formulas for $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$. See [5] for an answer.

You've now got the key ideas needed to use the chain rule in all dimensions. You'll find this shows up many places in upper-level math, physics, and engineering courses. The following problem will show you how you can use the general chain rule to get an extremely quick way to perform implicit differentiation from first-semester calculus.

Exercise 6.24

Suppose $z = f(x, y)$. If z is held constant, this produces a level curve. As an example, if $f(x, y) = x^2 + 3xy - y^3$ then $5 = x^2 + 3xy - y^3$ is a level curve. Our goal in this problem is to find dy/dx in terms of partial derivatives of f.

Suppose $x = x$ and $y = y(x)$, so y is a function of x. We can write this in parametric form as $\vec{r}(x) = (x, y(x))$. We now have $z = f(x, y)$ and $\vec{r}(x) = (x, y(x))$.

To practice the idea developed in this problem, show that if $w = F(x, y, z)$ is held constant at $w = c$ and we assume that $z = f(x, y)$ depends on x and y, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

[5] We have $Df(x, y, z) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{bmatrix}$.

The parametrization $\vec{r}(u, v) = (x, y, z)$ has derivative $D\vec{r} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{bmatrix}$.

The product is $D(f(\vec{r}(u, v))) = \begin{bmatrix} \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial u} & \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial v} \end{bmatrix}$.

The first column is $\dfrac{\partial f}{\partial u}$, and the second column is $\dfrac{\partial f}{\partial v}$.

1. Compute both $Df(x,y)$ and $D\vec{r}(x)$ symbolically. Don't use the function $f(x,y) = x^2 + 3xy - y^3$ until the last step.

2. Use the chain rule to compute $D(f(\vec{r}(x)))$. What is dz/dx (i.e., df/dx)?

3. Since z is held constant, we know that $dz/dx = 0$. Use this fact, together with part 2 to explain why $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.

4. For the curve $5 = x^2 + 3xy - y^3$, use this formula to compute dy/dx.

Contents

To Valpo Students:

6.5 Wrap Up

This concludes the Unit. Look at the objectives at the beginning of the unit. Can you now do all the things you were promised?

Review Guide Creation

Your assignment: organize what you've learned into a small collection of examples that illustrates the key concepts. I'll call this your unit review guide. I'll provide you with a template which includes the unit's key concepts from the objectives at the beginning. Once you finish your review guide, scan it into a PDF document (use any scanner on campus or photo software) and upload it to Gradescope.

As you create this review guide, consider the following:

- Before each Celebration of Knowledge we will devote a class period to review. With well created lesson plans, you will have 4-8 pages(for 2-4 Chapters) to review for each, instead of 50-100 problems.

- Think ahead 2-5 years. If you make these lesson plans correctly, you'll be able to look back at your lesson plans for this semester. In about 20-25 pages, you can have the entire course summarized and easy for you to recall.