

Lecture 8

ELE 301: Signals and Systems

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Properties of the Fourier Transform

- Properties of the Fourier Transform
 - ▶ Linearity
 - ▶ Time-shift
 - ▶ Time Scaling
 - ▶ Conjugation
 - ▶ Duality
 - ▶ Parseval
- Convolution and Modulation
- Periodic Signals
- Constant-Coefficient Differential Equations

Fourier series ഏവ്
property തന്മൂലം

$$x_1(t) \rightarrow X_1(f)$$

$$x_2(t) \rightarrow X_2(f)$$

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

multiple function এর জন্যেও ইতে পারে।

$$\sum_{k=1}^K a_k x_k(t) \Leftrightarrow \sum_{k=1}^K a_k X_k(f)$$

Linearity

Linear combination of two signals $x_1(t)$ and $x_2(t)$ is a signal of the form $ax_1(t) + bx_2(t)$.

Linearity Theorem: The Fourier transform is linear; that is, given two signals $x_1(t)$ and $x_2(t)$ and two complex numbers a and b , then

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega).$$

This follows from linearity of integrals:

$$\begin{aligned} & \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))e^{-j2\pi ft} dt \\ &= a \int_{-\infty}^{\infty} x_1(t)e^{-j2\pi ft} dt + b \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi ft} dt \\ &= aX_1(f) + bX_2(f) \end{aligned}$$

Finite Sums

This easily extends to finite combinations. Given signals $x_k(t)$ with Fourier transforms $X_k(f)$ and complex constants a_k , $k = 1, 2, \dots, K$, then

$$\sum_{k=1}^K a_k x_k(t) \Leftrightarrow \sum_{k=1}^K a_k X_k(f).$$

If you consider a system which has a signal $x(t)$ as its input and the Fourier transform $X(f)$ as its output, the system is linear!

Linearity Example

Find the Fourier transform of the signal

$$\underline{x(t)} = \begin{cases} \frac{1}{2} & \frac{1}{2} \leq |t| < 1 \\ 1 & |t| \leq \frac{1}{2} \end{cases}$$

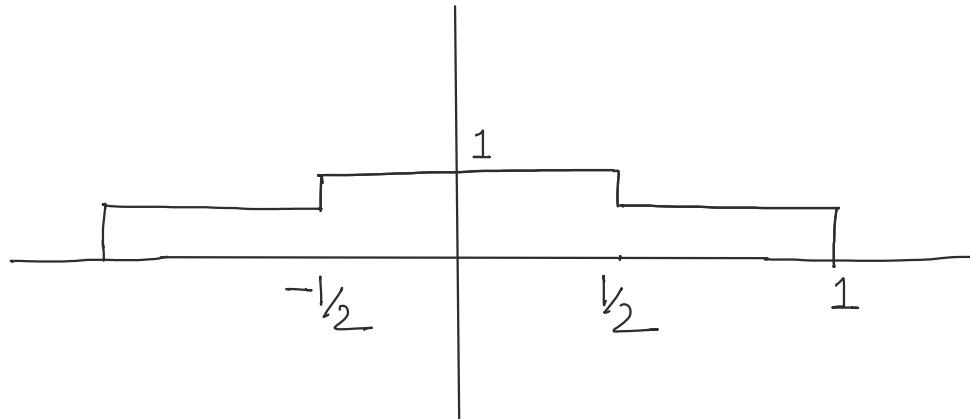
linear combination
হিমেত represent কৰবো!

This signal can be recognized as

$$x(t) = \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) + \frac{1}{2} \text{rect}(t)$$

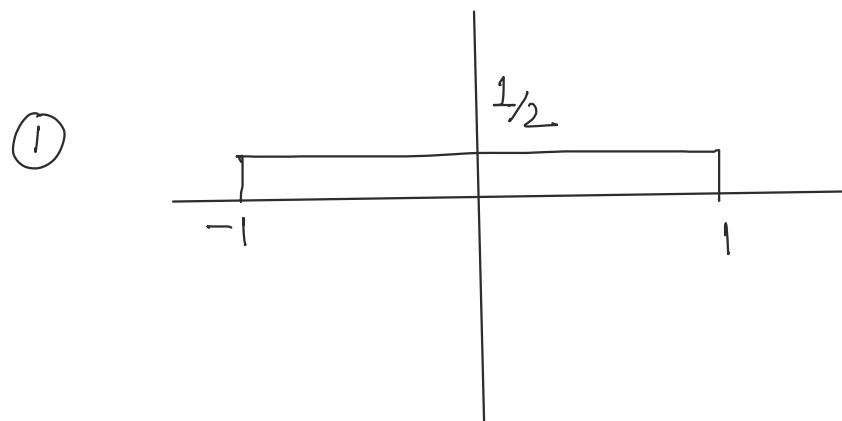
and hence from linearity we have

$$X(f) = \left(\frac{1}{2}\right) 2 \text{sinc}(2f) + \frac{1}{2} \text{sinc}(f) = \text{sinc}(2f) + \frac{1}{2} \text{sinc}(f)$$



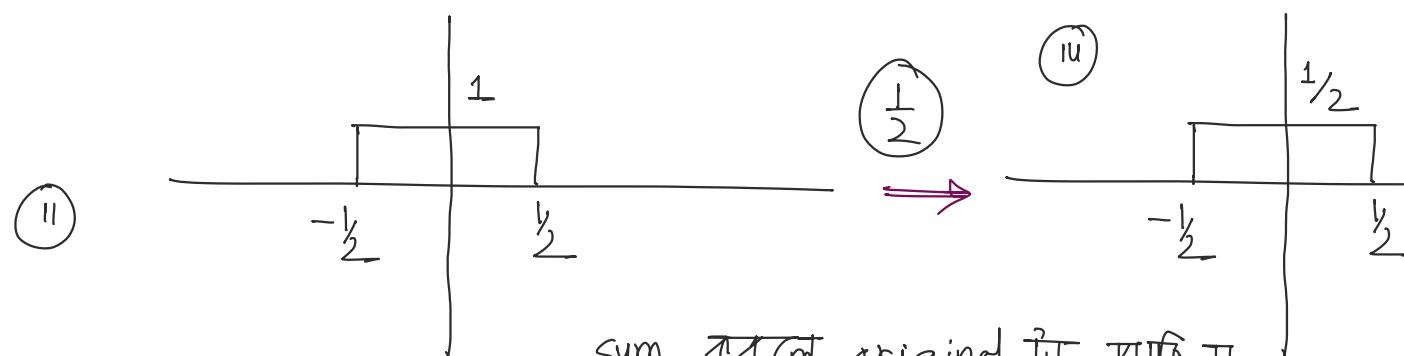
rectangular pulse আছে ২টা।

rectangular pulse এবং linear combination.

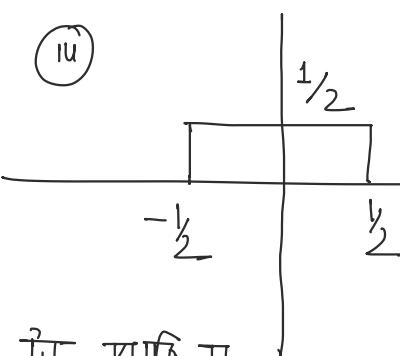


(I), (III) sum

— original signal



$\frac{1}{2}$



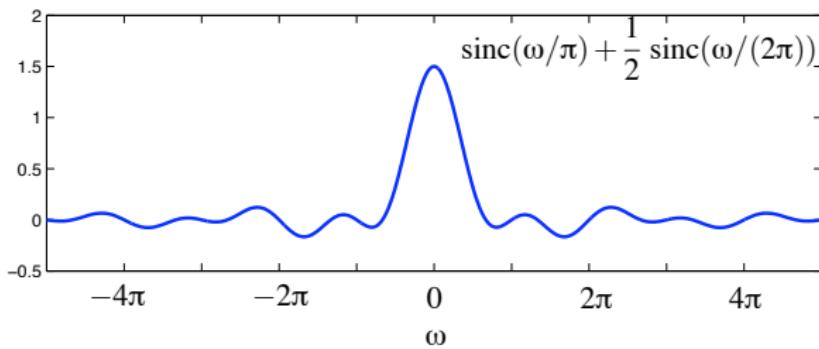
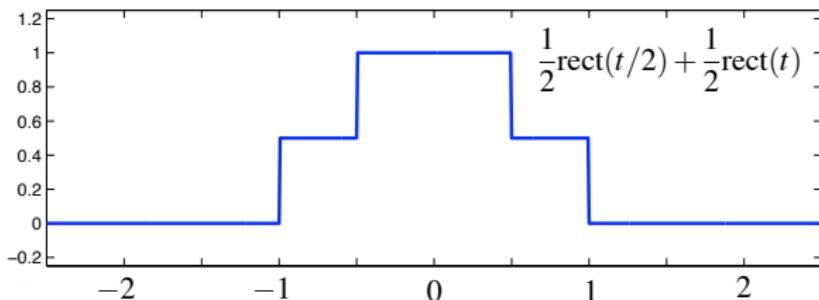
sum এর লক্ষণে original টা পাওয়া যাবে।
→ (I), (II)

$$x(t) = \frac{1}{2} \operatorname{rect}\left(\frac{t-t_0}{\tau}\right) + \frac{1}{2} \operatorname{rect}\left(\frac{t+t_0}{\tau}\right)$$

$$A \operatorname{rect}\left(\frac{t}{\tau}\right) = A \tau \operatorname{sinc}(ft)$$

$$X(f) = \frac{1}{2} \times 2 \operatorname{sinc}(2f) + \frac{1}{2} \operatorname{sinc}(f)$$

$$= \operatorname{sinc}(2f) + \frac{1}{2} \operatorname{sinc}(f)$$



Scaling Theorem

time domain \downarrow compress freq. domain \downarrow expand
expand compress

Stretch (Scaling) Theorem: Given a transform pair $x(t) \Leftrightarrow X(f)$, and a real-valued nonzero constant a ,

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

B a_k এ আর
change \Rightarrow
harmonic হিসেবে নথি,
 $X(f)$ টিল। $X\left(\frac{f}{a}\right)$ রয়ে।

Proof: Here consider only $a > 0$. (negative a left as an exercise) Change variables $\tau = at$

$$\int_{-\infty}^{\infty} x(at)e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau/a} \frac{d\tau}{a} = \frac{1}{a} X\left(\frac{f}{a}\right).$$

$\tau = at$
limit change \Rightarrow অন্তর্বর্ণ কোণ থাবায়,

If $a = -1 \Rightarrow$ "time reversal theorem:"

$$X(-t) \Leftrightarrow X(-f)$$

— time domain \downarrow reverse \Rightarrow
frequency domain \downarrow (3) reverse \Rightarrow

$$X(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau$$

$$X\left(\frac{f}{a}\right) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{a} \tau} d\tau$$

$$\int_{-\infty}^{\infty} x(-at) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(\tau) e^{+j2\pi f\tau/a} \frac{d\tau}{-a}$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{+j2\pi f \tau/a} \frac{d\tau}{a}$$

$\Rightarrow \frac{1}{a} \times \left(-\frac{f}{a} \right)$

$a (+ve)/(-ve)$ হলেও এই part
positive।

$\frac{1}{|a|}$ দিয়ো।

এর sign যেকোনো আছবে।

$$x(at) = \frac{1}{|a|} \times \left(\frac{f}{a} \right)$$

Scaling Examples

$$\text{rect}(t) = \text{sinc}(f) \quad \underline{T=1}$$

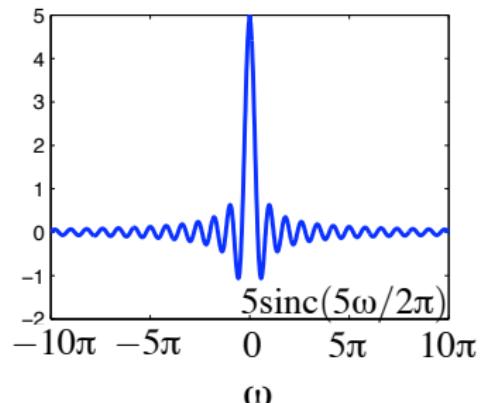
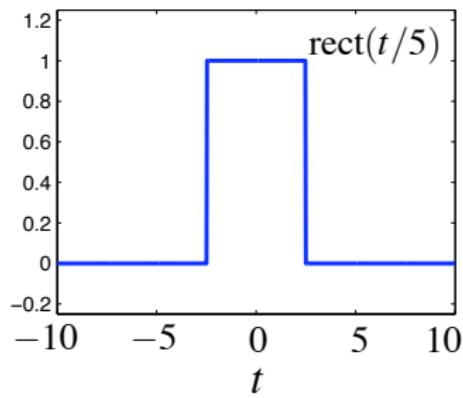
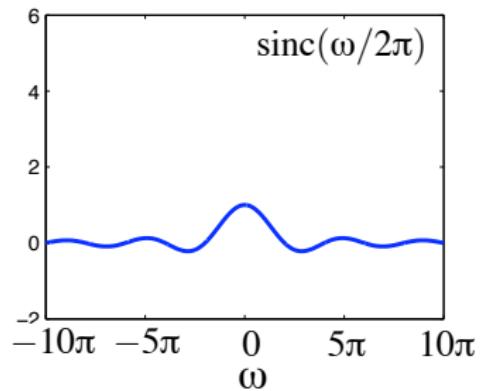
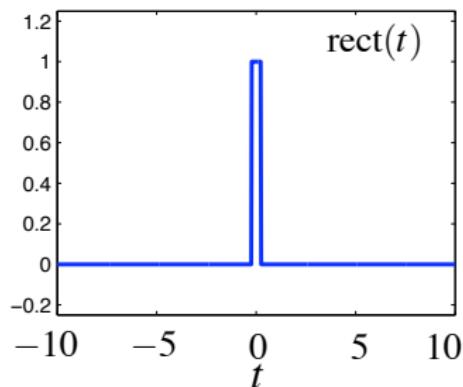
$$\rightarrow A=1, \alpha = \frac{1}{T}$$

We have already seen that $\text{rect}(t/T) \Leftrightarrow T \text{sinc}(Tf)$ by brute force integration. The scaling theorem provides a shortcut proof given the simpler result $\text{rect}(t) \Leftrightarrow \text{sinc}(f)$.

This is a good point to illustrate a property of transform pairs. Consider this Fourier transform pair for a small T and large T , say $T = 1$ and $T = 5$. The resulting transform pairs are shown below to a common horizontal scale:

$$\begin{array}{ccc} \text{rect}(at) & \text{rect}(t/\tau) & \xrightarrow{\quad \frac{1}{|a|} \times \left(\frac{f}{\tau}\right)} \tau \text{sinc}(f\tau) \\ \downarrow & \downarrow & \\ \frac{1}{|a|} \text{sinc}\left(\frac{f}{a}\right) & T \text{sinc}(Tf) & \end{array}$$

Compress in time - Expand in frequency



Scaling Example 2

$$\begin{aligned}|t| &= t & t > 0 \\ &= -t & t < 0\end{aligned}$$

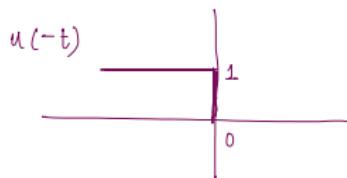
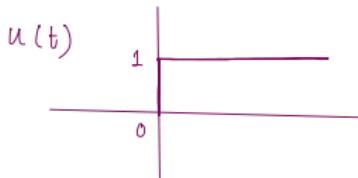
$$e^{-at|t|} = \begin{cases} e^{-at}, & t > 0 \\ e^{at}, & t < 0 \end{cases}$$

As another example, find the transform of the time-reversed exponential

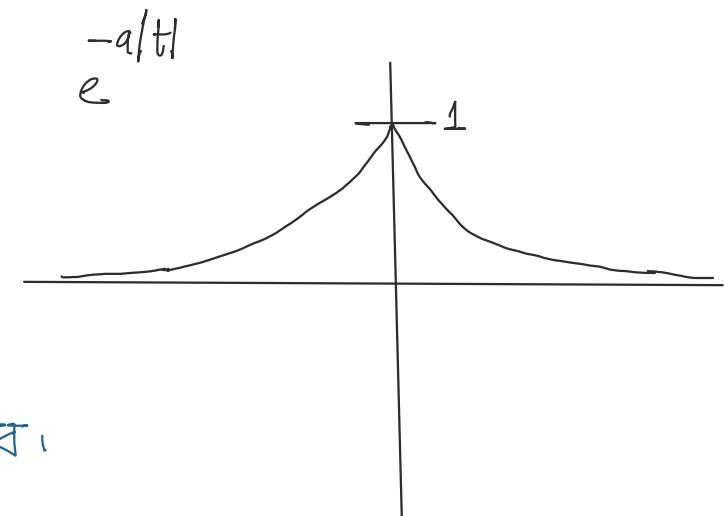
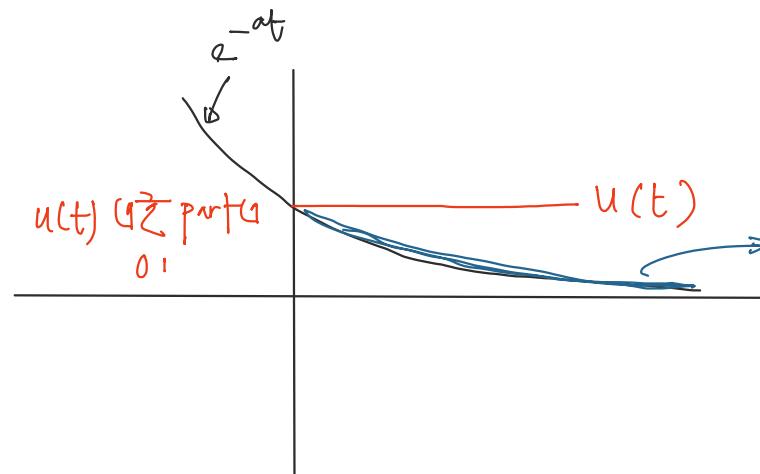
$$x(t) = e^{at} u(-t).$$

This is the exponential signal $y(t) = e^{-at} u(t)$ with time scaled by -1, so the Fourier transform is

$$X(f) = Y(-f) = \frac{1}{a - j2\pi f}.$$

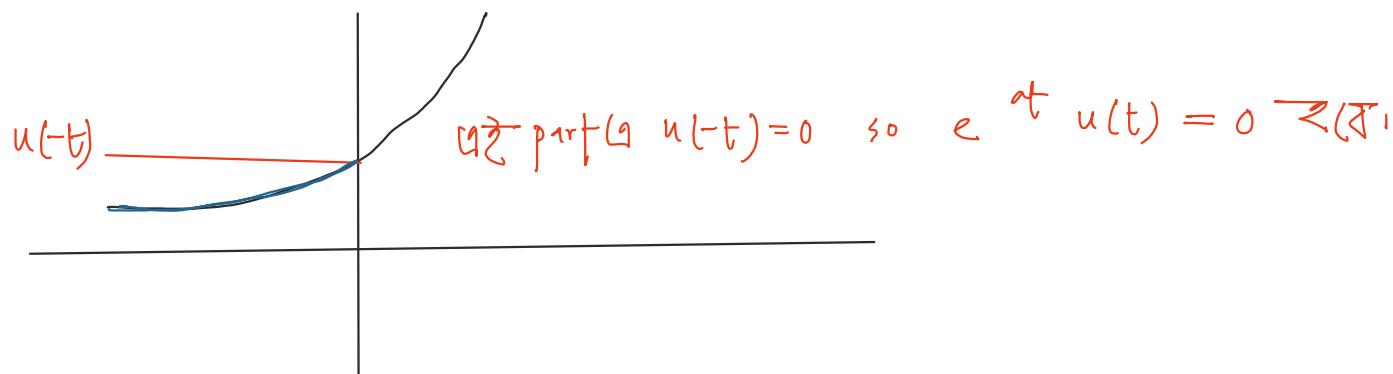


$$e^{-\alpha|t|} = e^{-\alpha t} u(t) + e^{\alpha t} u(-t)$$



এই প্রতি স্থানে।

$e^{-\alpha t} u(t)$ করা হল।



$$X_1(f) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-j2\pi f t} dt$$

$$= \int_0^{\infty} e^{-\alpha t} e^{-j2\pi f t} dt$$

$$= \int_0^{\infty} e^{-(at + j2\pi f t)} dt$$

$$= \int_0^{\infty} e^{-(a + j2\pi f)t} dt$$

$$= - \frac{1}{a + j2\pi f} [0 - 1]$$

$$= \frac{1}{a + j2\pi f}$$

$$\therefore X_2(f) = \frac{1}{a - j2\pi f} \quad (f \text{ এর জ্যাগাম } -f \text{ দিলেই রয়ে)$$

$$x(-t) = x(-f) \rightarrow \text{property use}$$

$$x_1(-t) = x_1(-f)$$

$$x_2(t) = x_1(-f)$$

Scaling Example 3

Important

As a final example which brings two Fourier theorems into use, find the transform of

$$x(t) = e^{-a|t|}.$$

This signal can be written as $e^{-at}u(t) + e^{at}u(-t)$. Linearity and time-reversal yield

$$\begin{aligned} X(f) &= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} \\ &= \frac{2a}{a^2 - (j2\pi f)^2} \\ &= \frac{2a}{a^2 + (2\pi f)^2} \end{aligned}$$

Much easier than direct integration!

Complex Conjugation Theorem

$$x(t) = a_k$$
$$x^*(t) = a_k^*$$

Complex Conjugation Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$x^*(t) \Leftrightarrow X^*(-f)$$

Proof: The Fourier transform of $x^*(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi ft} dt &= \left(\int_{-\infty}^{\infty} x(t) e^{j2\pi ft} dt \right)^* \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{-(j2\pi f)t} dt \right)^* = X^*(-f) \end{aligned}$$

Duality Theorem

$e^{-|f|}$ → freq. domain'ର ଦେଉଁଥା। ଆମେ FT କରେ ଦେଉଁଥା ଆଛେ। inverse କରାଲାଗାଏ

We discussed duality in a previous lecture.

Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

This result effectively gives us two transform pairs for every transform we find.

- *Exercise* What signal $x(t)$ has a Fourier transform $e^{-|f|}$?

অমরা তানি

$$e^{-at} \xleftrightarrow{\text{FT}} \frac{2a}{a^2 + (2\pi f)^2}$$

time domain to freq. domain

$$e^{-|f|t} \xrightarrow{a=1} e^{-|f|t}$$

এটির মাথে
কিলোহার্ট

$$\frac{2}{1+(2\pi t)^2} \xrightarrow[\text{(duality use)}]{\text{consider}} e^{-|-f|t} = e^{-|f|t}$$

$$e^{-|f|t} \text{ পর time domain } \& \text{ represent } \frac{2}{1+(2\pi t)^2}$$

$\text{rect}(t) \leftrightarrow \text{sinc}(f)$

$$\text{sinc}(t) = ?$$

Shift Theorem

The Shift Theorem:

$$x(t - \tau) \Leftrightarrow e^{-j2\pi f \tau} X(f)$$

Proof:

যা পরিমাণ shift এবং phase change এ যাব।

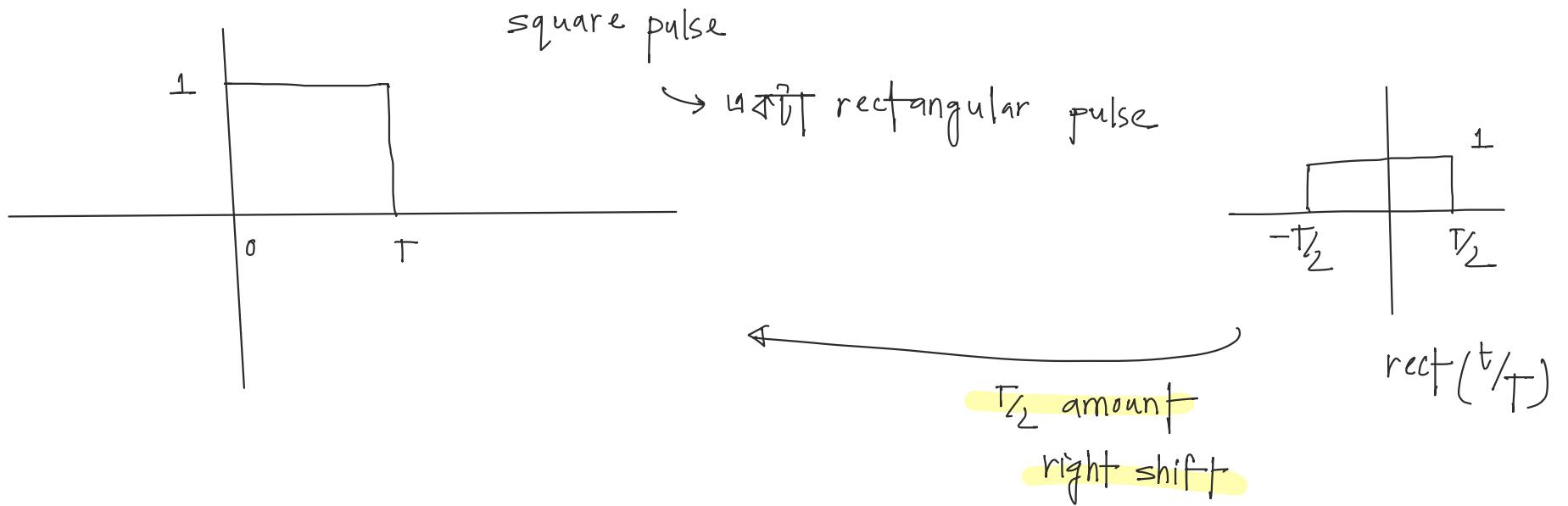
Example: square pulse

Consider a causal square pulse $p(t) = 1$ for $t \in [0, T)$ and 0 otherwise.
We can write this as

$$p(t) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

From shift and scaling theorems

$$P(f) = Te^{-j\pi fT} \text{sinc}(Tf).$$



$$\text{rect}(t) \rightarrow \text{rect}\left(\frac{t - T/2}{T}\right)$$

↳ scaling $\text{rect}(t/T)$
then shifting $\frac{T}{2}$

$$\text{rect}(t) \rightarrow \text{sinc}(ft)$$

$$\text{rect}\left(\frac{t}{T}\right) \rightarrow T \text{sinc}(ftT)$$

scaling $\frac{1}{T} \uparrow$

$$\text{shifting } \frac{T}{2} \text{ amount} \rightarrow e^{-j2\pi f \cdot \frac{T}{2}} T \text{sinc}(ftT) = e^{-j\pi f T} T \text{sinc}(ftT)$$

The Derivative Theorem

The Derivative Theorem: Given a signal $x(t)$ that is differentiable almost everywhere with Fourier transform $X(f)$,

$$x'(t) \Leftrightarrow j2\pi f X(f)$$

Similarly, if $x(t)$ is n times differentiable, then

$$\frac{d^n x(t)}{dt^n} \Leftrightarrow (j2\pi f)^n X(f)$$

$$x(t) \xleftrightarrow{\text{FT}} X(f)$$

differentiation $\underline{x'(t)} \longleftrightarrow j2\pi f X(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$\rightarrow x'(t) = \int_{-\infty}^{\infty} \underbrace{j2\pi f X(f)}_{f \text{ এর function}} e^{j2\pi ft} df$$

differentiating w.r.f. t.

$x'(t)$ এবং $X(f)$ এবে

$$\underline{j2\pi f X(f)}$$

Fourier transform $x(t) \rightarrow X(f)$ then multiply with $j2\pi f$ ($j2\pi f X(f)$)
use করে differentiation:

then Inverse Fourier transform

result : $x'(t)$

differentiation

Dual Derivative Formula

There is a dual to the derivative theorem, i.e., a result interchanging the role of t and f . Multiplying a signal by t is related to differentiating the spectrum with respect to f .

$$(-j2\pi t)x(t) \Leftrightarrow X'(f)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

differentiating w.r.t. f

$$x'(f) = \int_{-\infty}^{\infty} (-j2\pi t) x(t) e^{-j2\pi f t} dt$$

$$(-j2\pi t) x(t) \xrightleftharpoons[\text{F.T.}]{\quad} x'(f)$$

constant এর F.T. \rightarrow প্রটো

$$t x(t) \xrightleftharpoons[-j2\pi]{\quad} \frac{x'(f)}{-j2\pi}$$

$t x(t)$ এর F.T. কোনো বো।

$$x(t) \xrightleftharpoons[\text{F.T.}]{\quad} X(f)$$

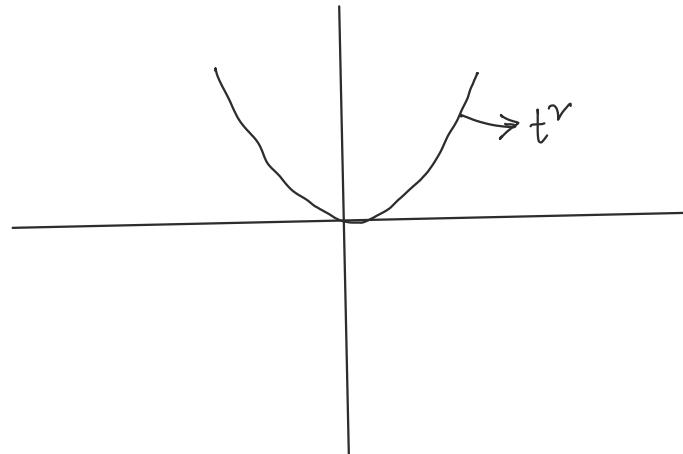
then $x'(f)$

then $x'(f) / j2\pi$

$$t \ x(t) \rightarrow x(f)$$

$$\int_{-\infty}^{\infty} t^r e^{-2\pi f t} dt$$

$$t^r x(t) \xrightleftharpoons[\text{F.T.}]{\quad} \frac{x'(f)}{-j 2\pi}$$



Fourier Transform of the Unit Step Function

How do we know the derivative of the unit step function?

The unit step function does not converge under the Fourier transform.
But just as we use the delta function to accommodate periodic signals, we can handle the unit step function with some sleight-of-hand.

Use the approximation that $u(t) \approx e^{-at} u(t)$ for small a .

A symmetric construction for approximating $u(t)$

Example: Find the Fourier transform of the signum or sign signal

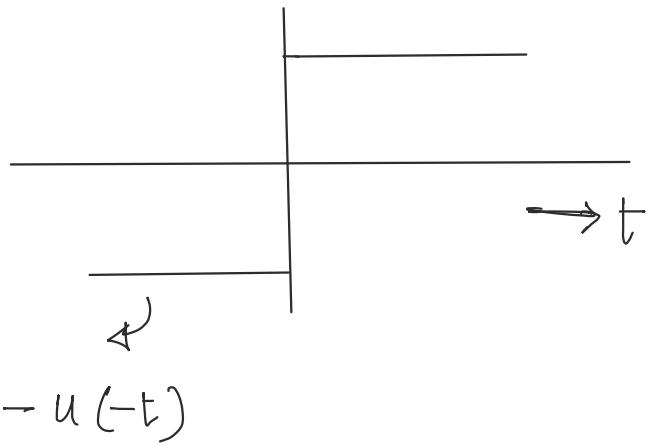
$$f(t) = \text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}.$$

We can approximate $f(t)$ by the signal

$$f_a(t) = \underbrace{e^{-at}}_1 u(t) - \underbrace{e^{at}}_1 u(-t)$$

as $a \rightarrow 0$.

$$f(t) = u(t) - u(-t) \quad \underline{a=0}$$



$$u(t) - u(-t)$$

$$f(t) = \operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$\frac{1}{\downarrow} \longrightarrow \delta(f)$

converge করে না, duality দিয়ে কৈ
কারি।

$e^{-at} u(t) + e^{-at} u(-t)$

$$\frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f}$$

$$u(t) - u(-t)$$

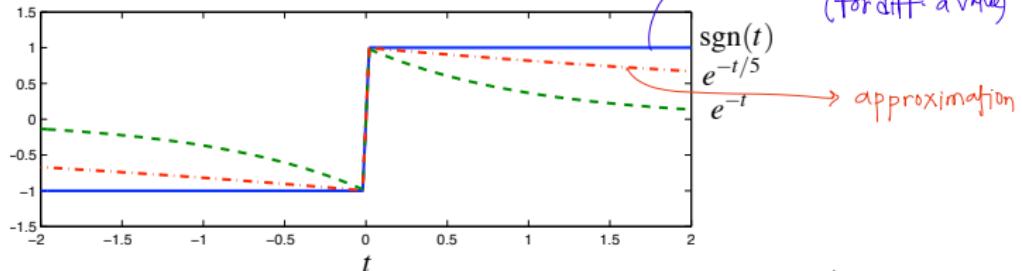
→ F.T. জ্ঞান।

approximate কর্য লাগব।

$$e^{-at} u(t) - e^{-at} u(-t)$$

$$\Rightarrow \frac{1}{a + j 2\pi f} - \frac{1}{a - j 2\pi f}$$

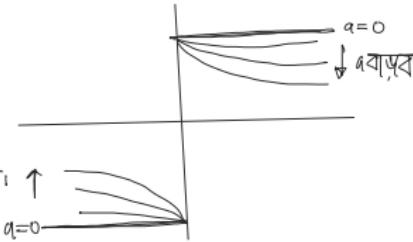
This looks like



As $a \rightarrow 0$, $f_a(t) \rightarrow \text{sgn}(t)$.

The Fourier transform of $f_a(t)$ is

$$\begin{aligned} F_a(f) &= \mathcal{F}[f_a(t)] \\ &= \mathcal{F}[e^{-at}u(t) - e^{at}u(-t)] \\ &= \mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)] \\ &= \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \\ &= \frac{-j4\pi f}{a^2 + (2\pi f)^2} \end{aligned}$$



Therefore,

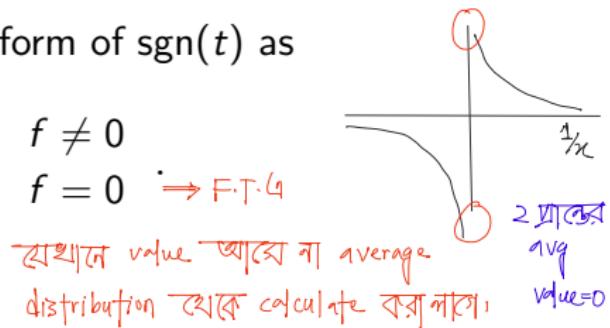
$$\begin{aligned}\lim_{a \rightarrow 0} F_a(f) &= \lim_{a \rightarrow 0} \frac{-j4\pi f}{a^2 + (2\pi f)^2} \\ &= \frac{-j4\pi f}{(2\pi f)^2} \\ &= \frac{1}{j\pi f}.\end{aligned}$$

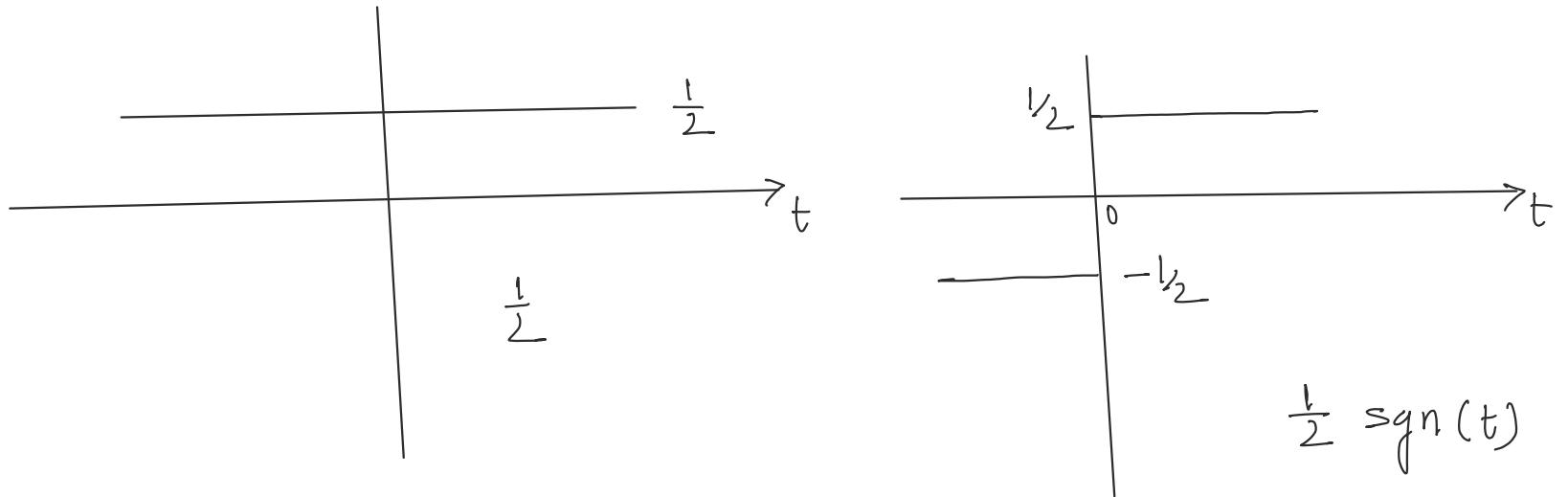
graph রাখ $\frac{1}{j\pi f}$ এর মাত্রা (magnitude)

This suggests we *define* the Fourier transform of $\text{sgn}(t)$ as

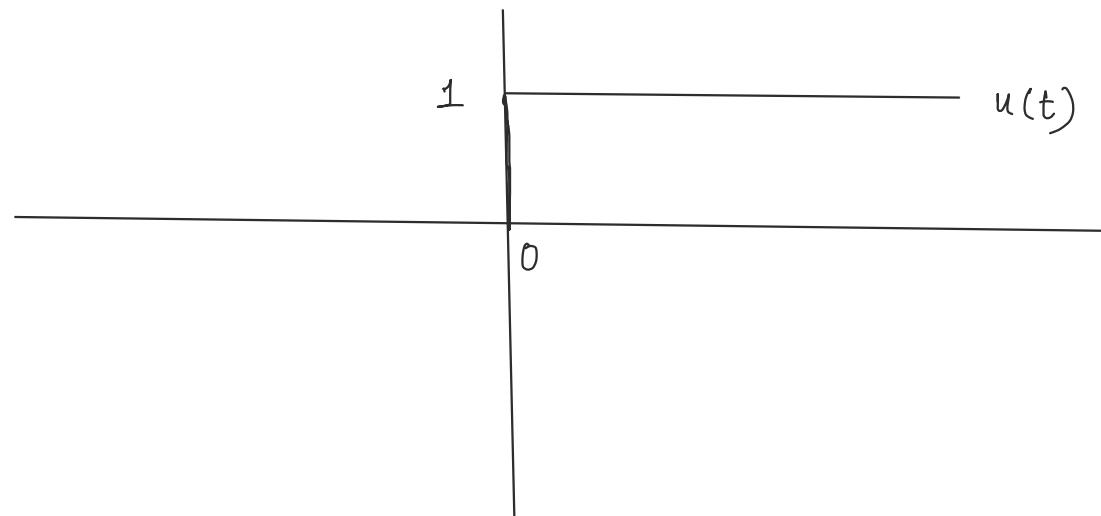
$$\frac{\text{sgn}(t)}{\sqrt{2}} \Leftrightarrow \begin{cases} \frac{2}{j2\pi f} & f \neq 0 \\ 0 & f = 0 \end{cases}.$$

\uparrow
 $u(t)$ কে এখন ঝর্ণায়
represent করতে চাই।





$$u(t) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t)$$



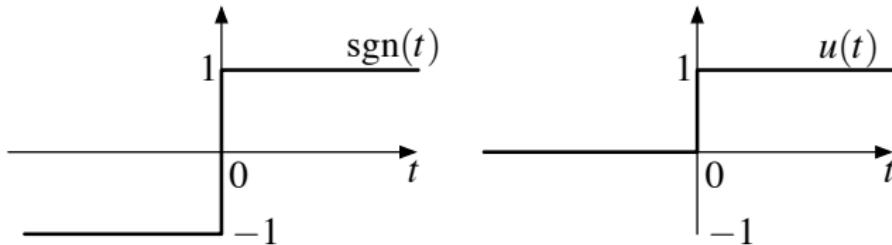
$$1 \leftrightarrow \delta(f)$$

$$U(f) = \frac{1}{2} \delta(f) + \frac{1}{2} \frac{1}{j\pi f}$$

With this, we can find the Fourier transform of the unit step,

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

as can be seen from the plots

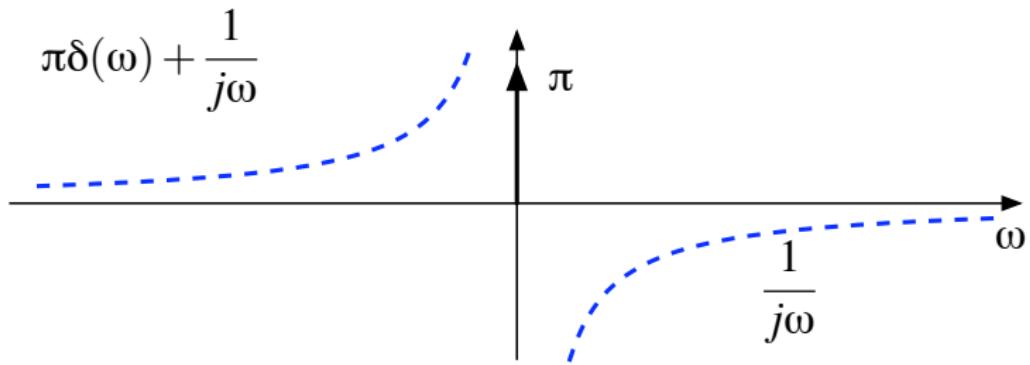


The Fourier transform of the unit step is then

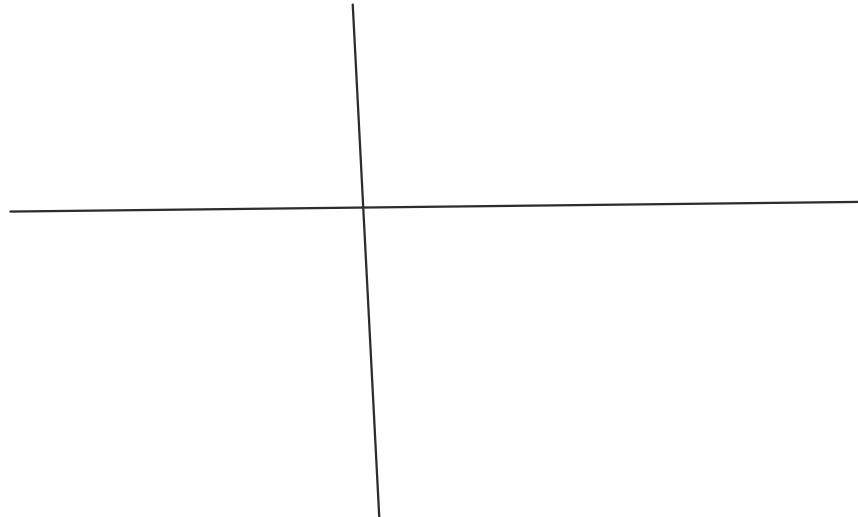
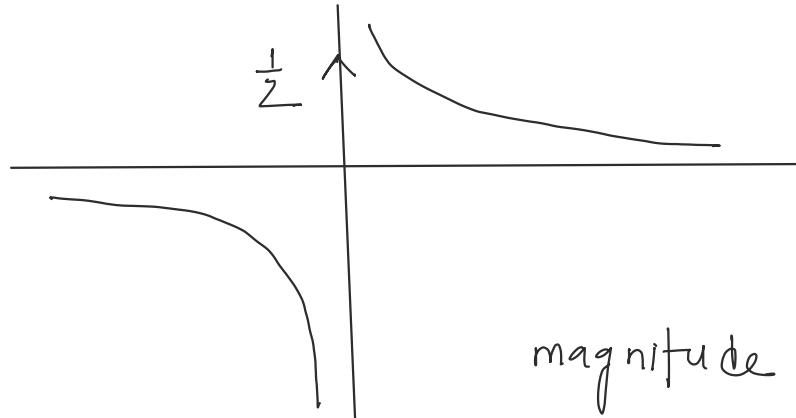
$$\begin{aligned}\mathcal{F}[u(t)] &= \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right] \\ &= \frac{1}{2}\delta(f) + \frac{1}{2}\left(\frac{1}{j\pi f}\right).\end{aligned}$$

The transform pair is then

$$u(t) \Leftrightarrow \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$



phase
magnitude } zt-ally graph



phase

Parseval's Theorem

(Parseval proved for Fourier series, Rayleigh for Fourier transforms. Also called Plancherel's theorem)

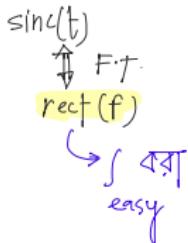
Recall signal energy of $x(t)$ is

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Interpretation: energy dissipated in a one ohm resistor if $x(t)$ is a voltage.
Can also be viewed as a measure of the size of a signal.

Theorem:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$



time domain } freq. domain } energy বের করলে same value দিবে।

Proof:

$$|x(t)|^v = x(t) x^*(t)$$

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^v dt$$

$$= \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(f) e^{j 2\pi f t} df \right) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(f) \int_{-\infty}^{\infty} (x^*(t) e^{-j 2\pi f t} dt) df$$

$$= \int_{-\infty}^{\infty} x(f) x^*(f) df = \int_{-\infty}^{\infty} |x(f)|^v df$$

$x(t)$ এর F.T. দিয়ে
replace

Example of Parseval's Theorem

Parseval's theorem provides many simple integral evaluations. For example, evaluate

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2(t) dt$$

We have seen that $\operatorname{sinc}(t) \Leftrightarrow \operatorname{rect}(f)$.

Parseval's theorem yields

✓

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}^2(t) dt &= \int_{-\infty}^{\infty} \operatorname{rect}^2(f) df \\ &= \int_{-1/2}^{1/2} 1 df \\ &= 1.\end{aligned}$$

Try to evaluate this integral directly and you will appreciate Parseval's shortcut.

* The Convolution Theorem *

Convolution in the time domain \Leftrightarrow multiplication in the frequency domain

This can simplify evaluating convolutions, especially when cascaded.

This is how most simulation programs (e.g., Matlab) compute convolutions, using the FFT.

The Convolution Theorem: Given two signals $x_1(t)$ and $x_2(t)$ with Fourier transforms $X_1(f)$ and $X_2(f)$,

$$(x_1 * x_2)(t) \Leftrightarrow X_1(f)X_2(f)$$

Proof: The Fourier transform of $(x_1 * x_2)(t)$ is

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau \right) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t - \tau) e^{-j2\pi ft} dt \right) d\tau. \end{aligned}$$

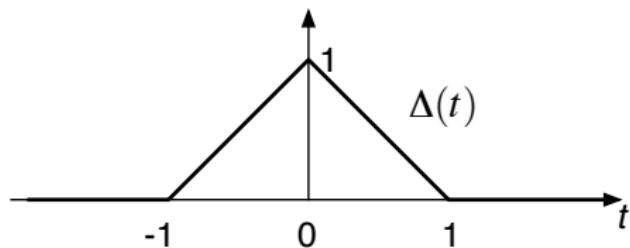
Using the shift theorem, this is

$$\begin{aligned} &= \int_{-\infty}^{\infty} x_1(\tau) \left(e^{-j2\pi f\tau} X_2(f) \right) d\tau \\ &= X_2(f) \int_{-\infty}^{\infty} x_1(\tau) e^{-j2\pi f\tau} d\tau \\ &= X_2(f) X_1(f). \end{aligned}$$

Examples of Convolution Theorem

Unit Triangle Signal $\Delta(t)$

$$\begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



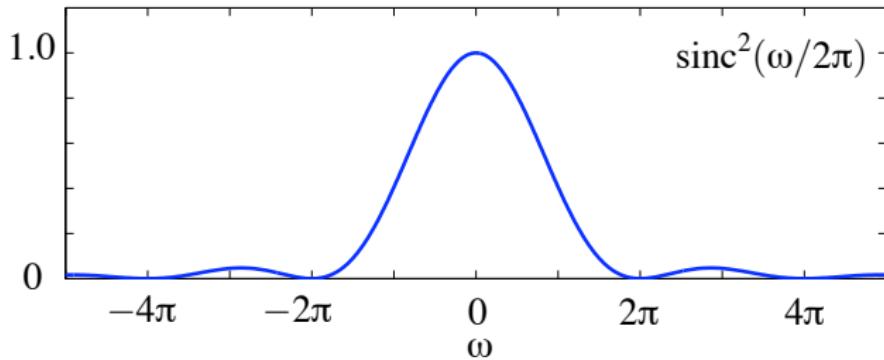
Easy to show $\Delta(t) = \text{rect}(t) * \text{rect}(t)$.

Since

$$\text{rect}(t) \Leftrightarrow \text{sinc}(f)$$

then

$$\Delta(t) \Leftrightarrow \text{sinc}^2(f)$$



Trans

Multiplication Property

If $x_1(t) \Leftrightarrow X_1(f)$ and $x_2(t) \Leftrightarrow X_2(f)$,

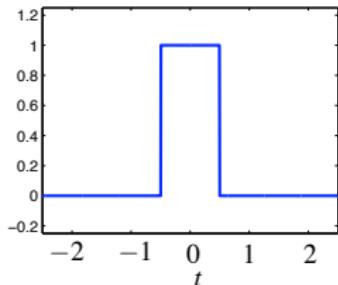
$$x_1(t)x_2(t) \Leftrightarrow (X_1 * X_2)(f).$$

This is the dual property of the convolution property.

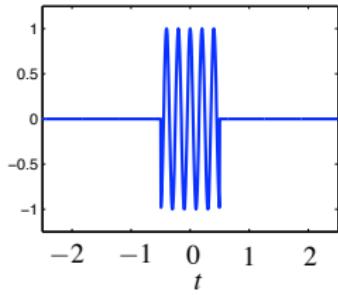
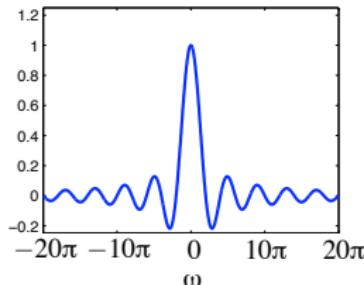
Note: If ω is used instead of f , then a $1/2\pi$ term must be included.

Examples of Modulation Theorem

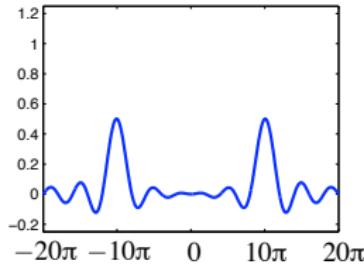
rect(t)



sinc($\omega/2\pi$)



rect(t) $\cos(10\pi t)$



$$\frac{1}{2}\text{sinc}\left(\frac{\omega - 10\pi}{2\pi}\right) + \frac{1}{2}\text{sinc}\left(\frac{\omega + 10\pi}{2\pi}\right)$$