

# 5

## Properties of Linear, Time-Invariant Systems

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In this lecture we continue the discussion of convolution and in particular explore some of its algebraic properties and their implications in terms of linear, time-invariant (LTI) systems. The three basic properties of convolution as an algebraic operation are that it is commutative, associative, and distributive over addition. The commutative property means simply that  $x$  convolved with  $h$  is identical with  $h$  convolved with  $x$ . The consequence of this property for LTI systems is that for a system with a specified input and impulse response, the output will be the same if the roles of the input and impulse response are interchanged. The associative property specifies that while convolution is an operation combining two signals, we can refer unambiguously to the convolution of three signals without concern about how they are grouped pairwise.

As demonstrated in the lecture, the associative property combined with the commutative property leads to an extremely important property of LTI systems. Specifically, if we have several LTI systems cascaded together, the output generated by an input to the overall cascade combination does not depend on the order in which the systems are cascaded. This property of LTI systems plays an extremely important role in system design, implementation, and analysis. It is generally not true for arbitrary systems that are not linear and time-invariant, and it represents one very important consequence of exploiting the properties of linearity and time invariance. The distributive property states that a signal convolved with the sum of two signals is identical to the result of carrying out the convolution with each signal in the sum individually and then summing the result. The consequence of this for the interconnection of LTI systems is that a parallel combination can be collapsed into a single system whose impulse response is the sum of the two individual ones. In looking at and understanding the algebraic properties of convolution, it is worthwhile to recognize that convolution as an algebraic property relates to addition in exactly the same way that multiplication relates to addition: that is, multiplication is commutative, associative, and distributive over addition.

In Lecture 3 we defined system properties in addition to linearity and time invariance, specifically properties of memory, invertibility, stability, and causality. While these properties are independent of linearity and time invariance, for LTI systems they can be related to properties of the system impulse response. For example, if an LTI system is memoryless, then the impulse response must be a scaled impulse. If a system with impulse response  $h$  is invertible, then the impulse response  $h_i$  of the inverse system has the property that  $h$  convolved with  $h_i$  is an impulse. For LTI systems an equivalent condition to stability is that the impulse response be absolutely summable (discrete time) or absolutely integrable (continuous time). If an LTI system is causal, then its impulse response must be zero for  $t$  (or  $n$ )  $< 0$ ; furthermore, if the impulse response has this property, then the system is guaranteed to be causal.

In the process of discussing these properties for LTI systems, we discuss another very important consequence of linearity—the fact that for a linear system (whether or not it is time-invariant), if the input is zero for all  $t$  (or  $n$ ), then the output is zero also.

In this lecture we illustrate the properties discussed above with some systems. The problems associated with this lecture provide the opportunity to explore these properties further.

In Lecture 3 in discussing the continuous-time impulse function, we indicated some inherent difficulty with defining the impulse simply as the limiting form of a rectangular pulse. We also suggested that the important properties of impulses relate not to what they are at each instant of time but to how they behave under integration. Specifically, in the context of signal and system analysis, the important property of impulses and what we call higher-order singularity functions is not what they are but what they do under convolution. This operational definition of impulses and derivatives of impulses is briefly touched on at the end of this lecture.

### **Suggested Reading**

Section 3.2, Discrete-Time LTI Systems: The Convolution Sum, pages 84–87

Section 3.3, Continuous-Time LTI Systems: The Convolution Integral, pages 90–95

Section 3.4, Properties of Linear Time-Invariant Systems, pages 95–101

Section 3.7, Singularity Functions, pages 120–124

**Convolution Sum:**

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k] = x[n] * h[n]$$

**Convolution Integral:**

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t-\tau) d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t)$$

**TRANSPARENCY**

**5.1**

The convolution sum and convolution integral.

**Commutative:**

$$x[n] * h[n] = h[n] * x[n]$$

$$x(t) * h(t) = h(t) * x(t)$$

$$u[n] * \alpha^n u[n] = \alpha^n u[n] * u[n]$$

$$u(t) * e^{-at} u(t) = e^{-at} u(t) * u(t)$$

**TRANSPARENCY**

**5.2**

The commutativity property of convolution.

**TRANSPARENCY**  
5.3  
Three algebraic  
properties of  
convolution.

**Commutative:**

$$x * h = h * x$$

**Associative:**

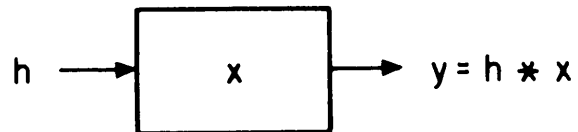
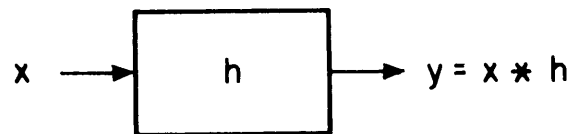
$$x * \{h_1 * h_2\} = \{x * h_1\} * h_2$$

**Distributive:**

$$x * \{h_1 + h_2\} = x * h_1 + x * h_2$$

**TRANSPARENCY**  
5.4  
An interpretation of  
the commutativity  
property.

Commutative



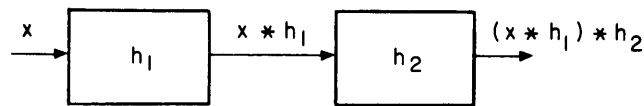
**LTI System:** output the same  
if input and impulse  
response interchanged

**TRANSPARENCY**

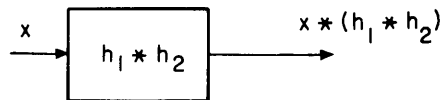
5.5

One consequence of the algebraic properties of convolution is that LTI systems can be cascaded in any order.

LTI Systems can be cascaded in any order



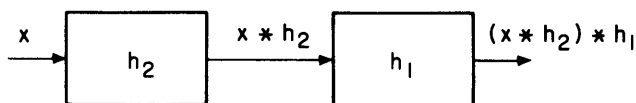
Associative:



Commutative:

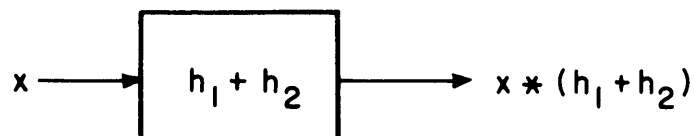
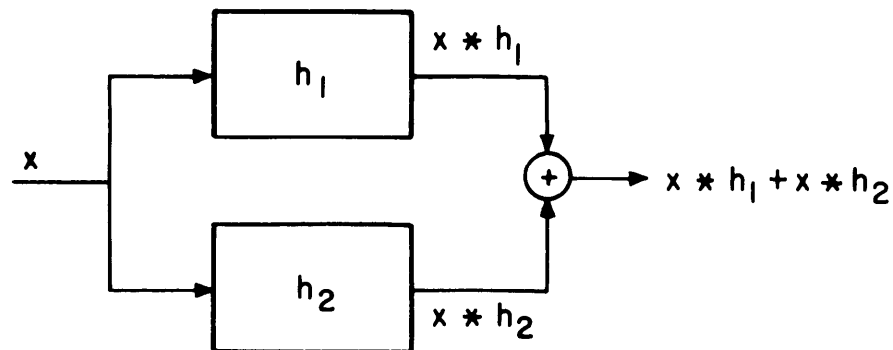


Associative:

**Distributive****TRANSPARENCY**

5.6

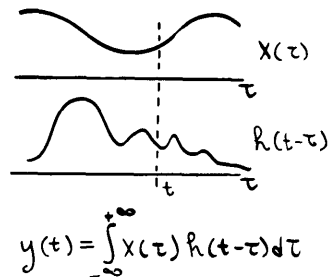
One consequence of the distributive property of convolution is that LTI systems in parallel can be collapsed into a single LTI system.



MARKERBOARD

5.1

MEMORY



MEMORYLESS  $\Rightarrow$

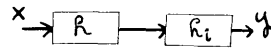
$h(t-\tau)$  nonzero only at  $\tau=t$

$$\Rightarrow h(t) = k\delta(t) \quad \text{C-T}$$

$$h[n] = k\delta[n] \quad \text{D-T}$$

$$y(t) = kx(t) \quad y[n] = kx[n]$$

INVERTIBILITY



$$y = x * (\underbrace{R * R_i}_{\delta}) = x$$

$$h_i = h^{-1}$$

STABILITY

Every Bounded input  $\Rightarrow$   
Bounded Output (BIBO)

$$\text{STABLE} \Rightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Zero Input Response

$$x(t) = 0 \quad \text{for all } t$$

$$\downarrow$$

$$y(t) = 0 \quad \text{for all } t$$

$$x[n] = 0 \quad \text{for all } n$$

$$\downarrow$$

$$y[n] = 0 \quad \text{for all } n$$

$$x(t) \rightarrow y(t)$$

Then

$$ax(t) \rightarrow ay(t)$$

$a=0$

MARKERBOARD

5.2

Causality

$$\text{If } x_1(t) = x_2(t) \quad t < t_0$$

$$\text{Then } y_1(t) = y_2(t) \quad t < t_0$$

Same for D-T

Linear System:

If

$$x(t) = 0 \quad t < t_0$$

$$\text{Then } y(t) = 0 \quad t < t_0$$

Same for D-T

why?

zero in  $\nearrow$   
zero out

For LTI Systems:

$$\text{Causality} \Leftrightarrow \begin{matrix} h(t) = 0 & t < 0 \\ h[n] = 0 & n < 0 \end{matrix}$$

$$\Rightarrow \begin{matrix} \delta(t) = 0 & t < 0 \\ \delta[n] = 0 & n < 0 \end{matrix}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

Example:

Accumulator:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

LTI -

$$h[n] = u[n]$$

$$h[n] \neq k\delta[n] \Rightarrow \text{memory}$$

$$h[n] = 0 \quad n < 0 \Rightarrow \text{causal}$$

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \infty \Rightarrow \text{not stable}$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$= y[n-1] + x[n]$$

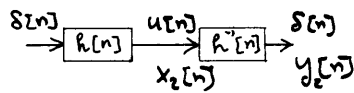
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5.3

Inverse of  
Accumulator

$$R[n] = u[n]$$

$$R^{-1}[n] = ?$$



$$u[n] - u[n-1] = \delta[n]$$

so

$$x_2[n] - x_2[n-1] = y_2[n]$$

$$\delta[n] - \delta[n-1] = h^{-1}[n]$$

Example

$$y[n] - ay[n-1] = x[n]$$

initial rest  $\Rightarrow$  LTI

$$x[n] = \delta[n]$$

$$h[n] = a^n u[n]$$

memory ?

causal ?

Stable ?

Example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

initial rest  $\Rightarrow$  LTI

$$h(t) = e^{-2t} u(t)$$

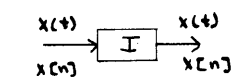
memory ?

causal ?

Stable ?

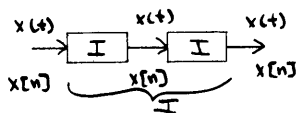
## MARKERBOARD

5.4



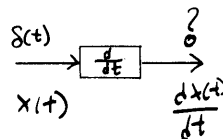
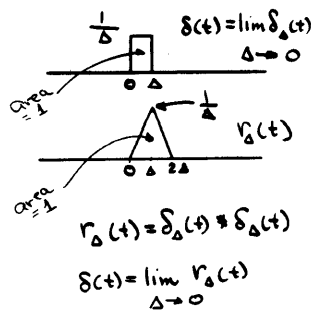
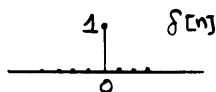
$$h(t) = \delta(t)$$

$$h[n] = \delta[n]$$



$$\delta(t) * \delta(t) = \delta(t)$$

$$\delta[n] * \delta[n] = \delta[n]$$



Operational Definition

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt}[\delta(t)]$$

$$\frac{x(t)}{\delta(t)} \rightarrow \frac{d}{dt} \rightarrow u_1(t)$$

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

$$\frac{x(t)}{\delta(t)} \rightarrow \frac{d}{dt} \rightarrow \frac{d}{dt} \rightarrow u_2(t)$$

$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2x(t)}{dt^2}$$

**MARKERBOARD**

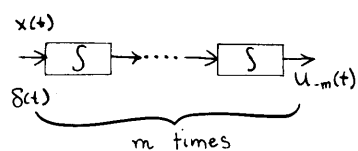
5.5

Note that  $\frac{d^k x(t)}{dt}$  should read  $\frac{d^k x(t)}{dt^k}$  in both places.

$$u_k(t) = \underbrace{u_1(t) * u_1(t) \dots}_{k \text{ times}}$$

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

$$u_0(t) = \delta(t)$$



$$u_{-1}(t) = \text{unit step}$$

$$u_{-2}(t) = \text{unit ramp}$$

$$x(t) * u_{-m}(t) = m^{\text{th}} \text{ running integral}$$

$u_k(t)$  defined by

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

$$u_0(t) = \delta(t)$$

$$u_{-1}(t) = u(t)$$

$$u_k(t) * u_p(t) = u_{k+p}(t)$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

system memoryless, causal

বিনা question  
সাবধানে

$$h[n] = u[n] \quad \left. \begin{array}{l} \text{then } y[n] = \sum_{k=-\infty}^n x[k] \end{array} \right\} \text{Accumulator}$$

→ discrete time system এর জন্য integrator

impulse response = scaled impulse  $\Rightarrow$  system memoryless

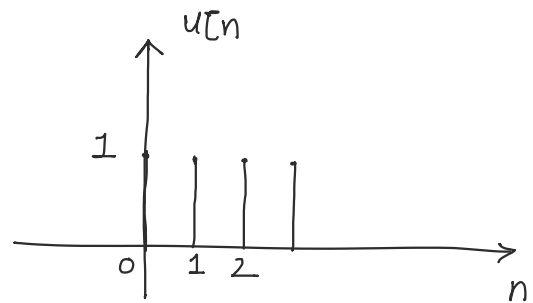
$$h[n] \neq k \delta[n] \quad \text{particular point এ impulse থাকে}$$

unit step function

$\therefore$  has memory

$$h[n] = 0 \text{ if } n < 0$$

$\therefore$  causal



$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \rightarrow \text{not stable}$$

$\rightarrow$  o/p infinity

$$1 + 1 + \dots + 1 \quad \infty = \infty$$

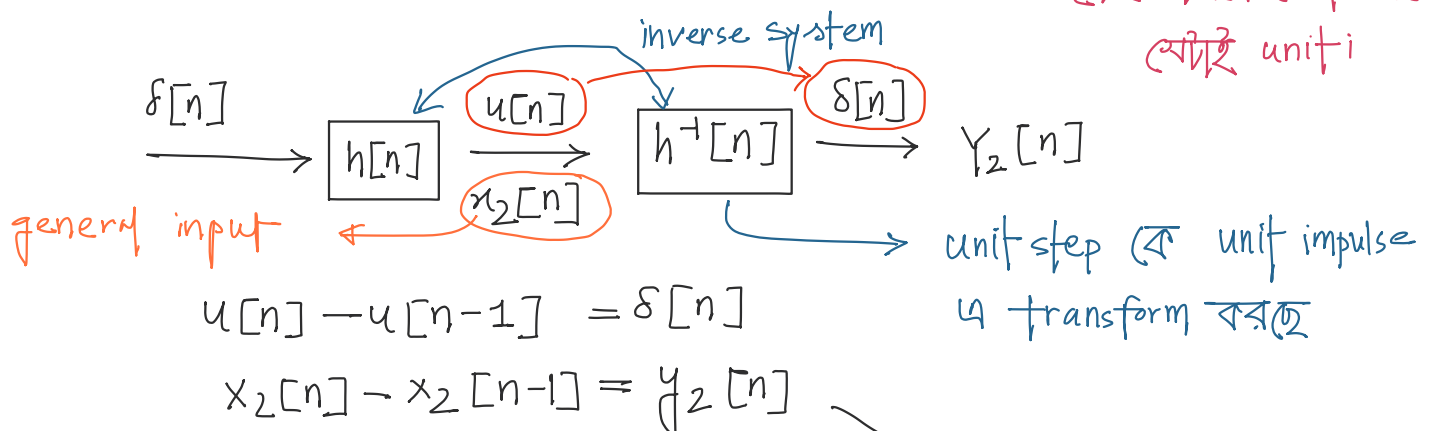
$\therefore$  bounded না।

### Inverse Accumulator:

$$h[n] = u[n]$$

$$h^{-1}[n] = ?$$

$x_2[n]$  এর জায়গায়  
যদি unit impulse  
দেই অহলে যেটা পাওয়া  
যেটাই unit i



$$\delta[n] \longrightarrow \boxed{h^{-1}[n]} \longrightarrow h^{-1}[n] \quad x_1 = \delta$$

$$\delta[n] - \delta[n-1] = h^{-1}[n]$$

Identity:

$$x(t) \longrightarrow \boxed{I} \longrightarrow x(t)$$

$$h(t) = \delta(t)$$

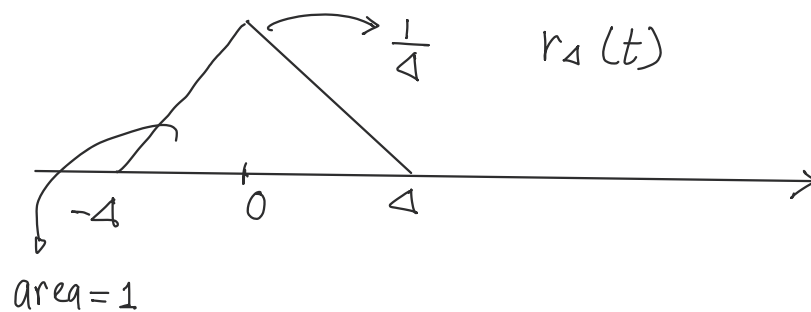
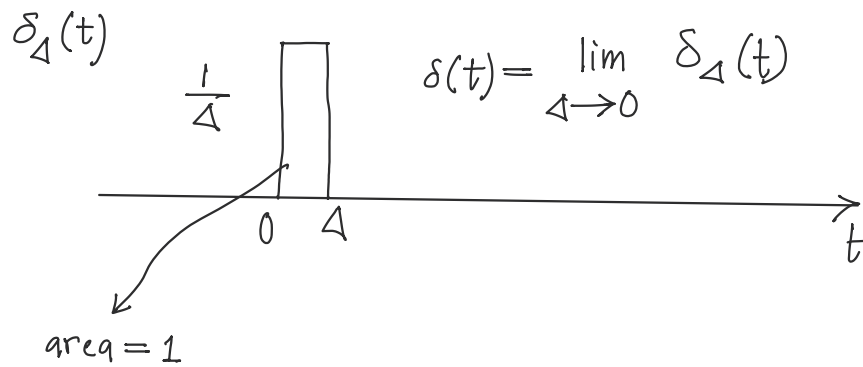
$$\frac{x(t)}{\delta(t)} \longrightarrow \boxed{I} \xrightarrow[\delta(t)]{x(t)} \boxed{I} \xrightarrow[\delta(t)]{x(t)}$$

$$\delta(t) * \delta(t) = \delta(t)$$

$$\delta[n] \times \delta[n] = \delta[n]$$

একটি system এর unit impulse দিয়ে convolve

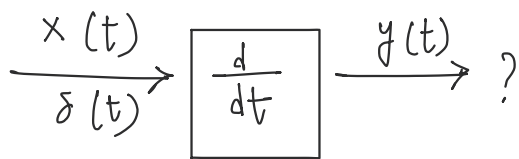
→ যেটা দিবে সেটাই return (identity)



$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t)$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} r_{\Delta}(t)$$

## Operational definition:



$x(t)$  কে differentiate করে o/p  
impulse response func. জানি না

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt} \{ \delta(t) \} = u_1(t)$$

$$x(t) * \underline{u_1(t)} = \frac{d}{dt} x(t) \rightarrow \text{operational defn}$$

mathematically বি ভেটা জানি না

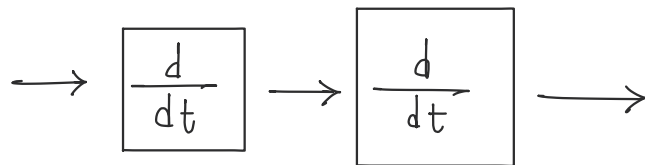
$$u_0(t) = \delta(t)$$

$$u_1(t) = \int \delta(t) = u(t)$$

$$u_2(t) = \underline{\hspace{2cm}}$$

$u_1(t) \rightarrow$  differentiator এর impulse response function.

আরেকটা differentiator cascade করলে — 2nd order derivative বের  
করতে পারবো।



$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2}{dt^2} \{ x(t) \}$$

$$u_k(t) = u_1(t) * u_1(t) * \dots * u_1(t) \quad (k \text{ times})$$

$$x(t) * u_k(t) = \frac{d^k}{dt^k} \{ x(t) \}$$

$$x(t) * u_{-m}(t) \\ = m\text{-th running integral}$$

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