Lecture 7
ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

Fall 2011-12

Introduction to Fourier Transforms

→ এক domain থেকে অন্য domain এ

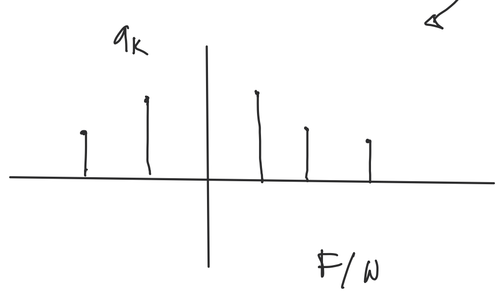
- Fourier transform
- Inverse Fourier transform: The Fourier integral theorem
- Example: the rect and sinc functions
- Cosine and Sine Transforms
- Symmetry properties
- Periodic signals and δ functions

subset of FT

optimized (periodic এর জন্য) ← FS (periodic)

signal periodic

FT করলে $\delta(f-f_0)$ আসবে।



• discrete value

integration

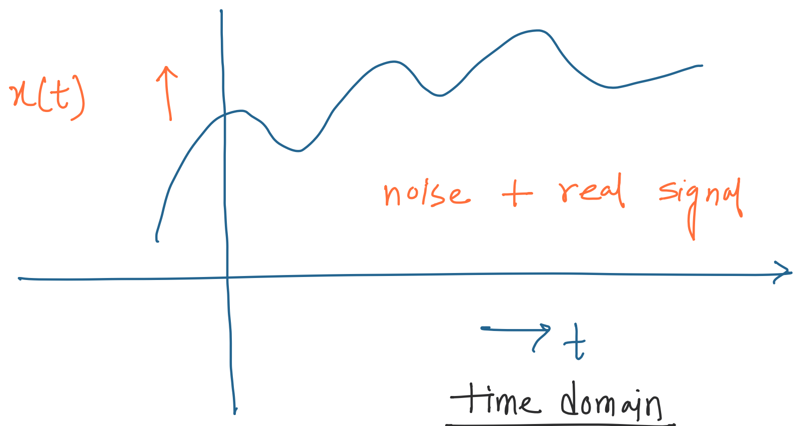
FT (all cases)

periodic না এমন signal

frequency domain এ

continuous

discrete/continuous 2 রকম value ই দিতে পারে। function dependent.



time domain

real life signal (irregular shape)

e.g. speech signal ↳ noise থাকে

application : datacom এ

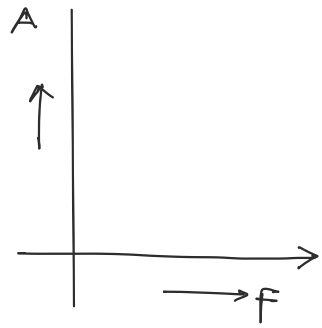
f →

freq. domain

এ transform

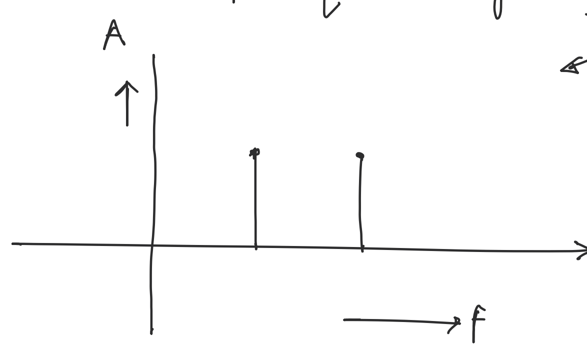
X axis → f

Y axis → amplitude



freq. distribution পাবো

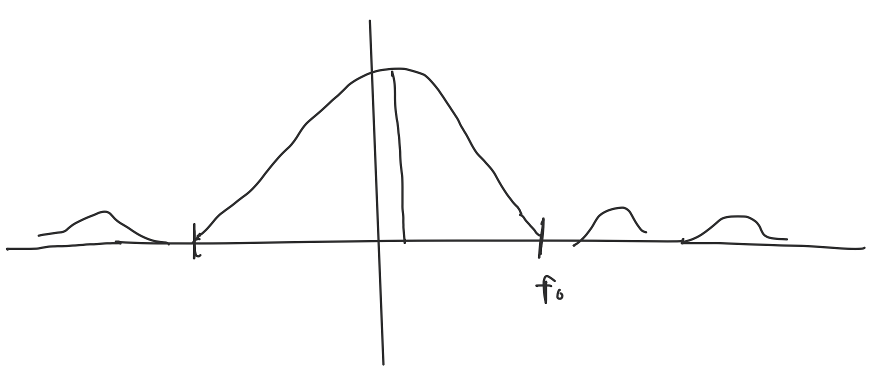
$\sin(2\pi t) + \cos(4\pi t) \rightarrow$ 2টা freq. র signal এর sum



periodic না হলে f এর function হবে।

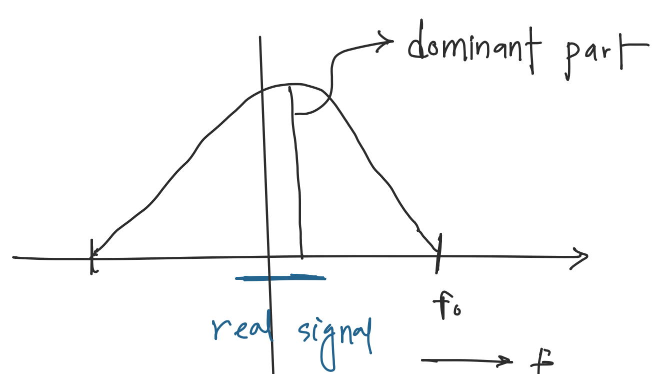
মানে continuous

$x(t) \xrightarrow[FT]{} X(F)$



real life এ এভাবে থাকে

high freq র pulse — noise



maximum frequency (f_0)

low freq. → dominant

Low Pass Filter — low freq যেতে দেয়

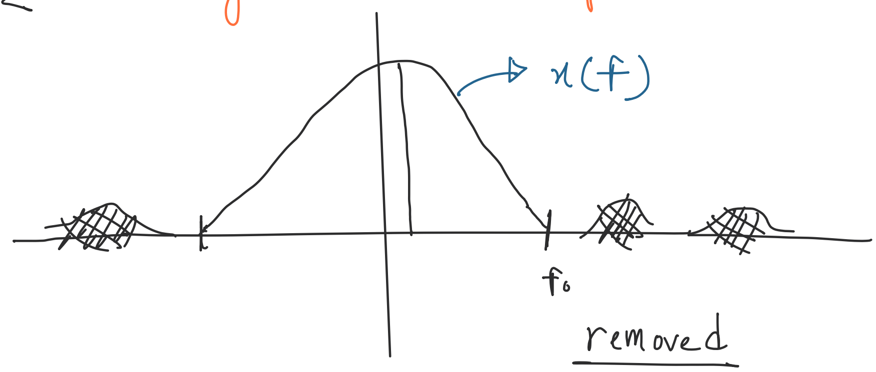
High " " — cut off freq এর উপরের গুলো

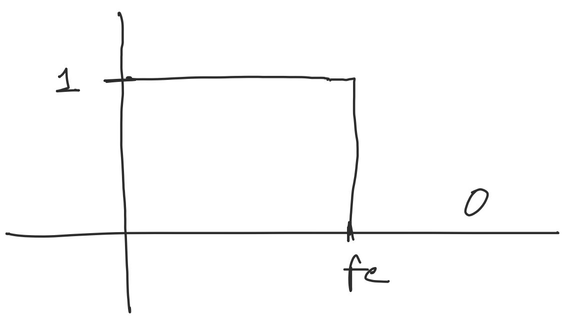
এখানে f_0 cut off এ low pass filter use.

f_0 এর উপরে গুলো 0।

$(0 - f_0)$ ← নিচের গুলো 1 এর সাথে গুণ হয়ে যা আছে তাই থাকবে।

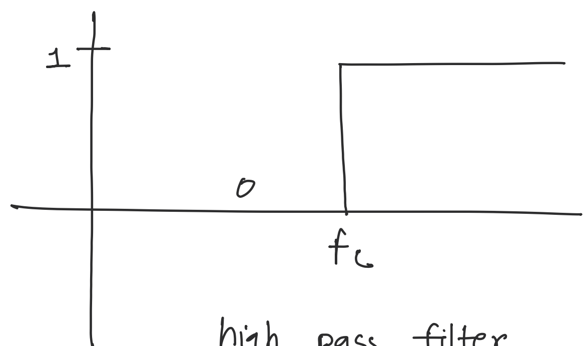
— noise part remove হয়ে যাচ্ছে। real signal পাবো। most of the noise 0 হয়ে যাবে।





low pass filter

$(0 - f_0)$ এর সাথে 1 গুণ



high pass filter

$> f_0$ এর সাথে 1 গুণ।

$x(t)$ — non periodic

Fourier Transforms

Given a continuous time signal $x(t)$, define its *Fourier transform* as the function of a real f:

$X(i\omega)$ দিয়েও লেখা যায়

2π difference

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$$

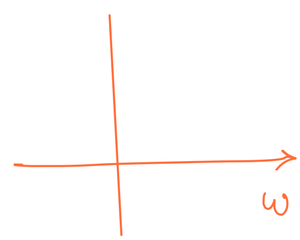
$$X(\omega) = \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt$$

$$X(f) = X(i2\pi f)$$

formula change হবে না।

This is similar to the expression for the Fourier series coefficients.

Note: Usually $X(f)$ is written as $X(i2\pi f)$ or $X(i\omega)$. This corresponds to the Laplace transform notation which we encountered when discussing transfer functions $H(s)$.



Continuous-time Fourier Transform

Which yields the *inversion formula* for the Fourier transform, the *Fourier integral theorem*:

$$
\boxed{
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt, \\
x(t) &= \int_{-\infty}^{\infty} X(f) e^{+j2\pi f t}\, df.
\end{aligned}
}
$$

IFT → F to t

Inverse Fourier Transform

→ f এর limit

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} \frac{d\omega}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(F) = X\left(\frac{\omega}{2\pi}\right)$$

2π constant so $X(\omega)$ লেখা যাবে।

Comments:

- There are usually technical conditions which must be satisfied for the integrals to converge – forms of smoothness or Dirichlet conditions.
- The intuition is that Fourier transforms can be viewed as a limit of Fourier series as the period grows to infinity, and the sum becomes an integral.
- $\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df$ is called the *inverse Fourier transform* of $X(f)$. Notice that it is identical to the Fourier transform except for the sign in the exponent of the complex exponential.
- If the inverse Fourier transform is integrated with respect to ω rather than f, then a scaling factor of $1/(2\pi)$ is needed.

Cosine and Sine Transforms

Assume $x(t)$ is a possibly complex signal.

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)\left(\cos(2\pi ft) - j\sin(2\pi ft)\right) dt \\ &= \underbrace{\boxed{\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt}}_{\text{cosine transform}} - j\underbrace{\boxed{\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt.}}_{\text{sine transform}} \end{aligned}$$

$F_c[x(t)]$

$\underbrace{F(x(t))}_{X(F)} = F_c[x(t)] - j\,F_s[x(t)]$

$x(t) \rightarrow$ even $\qquad F_s$ — odd function

(even x odd)

$= 0$

$\therefore F[x(t)] = F_c[x(t)]$

$x(t) \rightarrow$ odd $\qquad F[x(t)] = 0 - j\,F_s[x(t)] \qquad F_c \rightarrow$ odd

Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $x(t)$ as

$$\mathcal{F}\left[x(t)\right] = X(f)$$

and the inverse Fourier transform of $X(f)$ as

$$\mathcal{F}^{-1}\left[X(f)\right] = x(t).$$

Note that

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[x(t)\right]\right] = x(t)$$

and at points of continuity of $x(t)$.

$\mathcal{F}[x(t)]$

$\uparrow$

fourier transform

$\mathcal{F}^{-1}$

$\hookrightarrow$ inverse fourier transform

Rectangular pulse

Rect Example — Time limited function (finite)

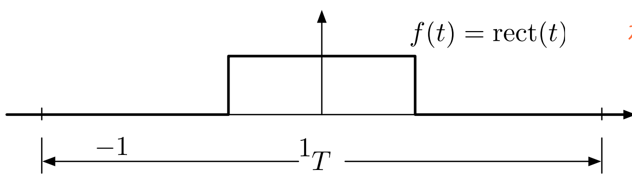
$\square(t)$

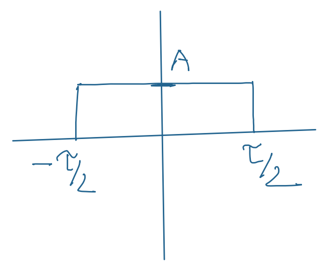
$\mathrm{Rect}(t)$

For example, assume $x(t) = \mathrm{rect}(t)$, and that we are computing the Fourier series over an interval T,

$x(t) = A \; \mathrm{Rect}(t/\tau)$ → range

rect(t) → $A = 1$, $\tau = 1$





→ only 1 pulse

— periodic πT

The Fourier Transform is: sinc(f)

where $\mathrm{sinc(f)} = \frac{\sin(\pi f)}{\pi f}$

Fourier series πT

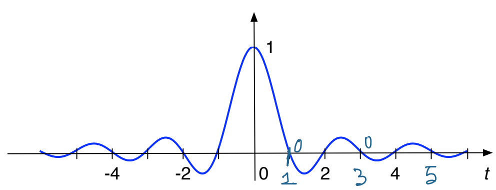
Fourier transform

even function

The Sinc Function

$\text{sinc}(t) = \underbrace{\frac{\sin t}{t} = \frac{\sin(\pi t)}{\pi t}}_{\text{shape same থাকে, value different হয়।}}$

rect function $\xrightarrow{FT}$ sinc function



$-\pi$ $\quad$ π $\quad 2\pi$ $\quad 3\pi$

infinite

$\text{sinc}\, t = \frac{\sin(\pi t)}{\pi t}$

$\lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} = 1$ (0 তে)

or $\lim_{t \to 0} \frac{\sin t}{t} = 1$

$\sin c\, t = 0$

$\rightarrow \frac{\sin \pi t}{\pi t} = 0$ / $\frac{\sin t}{t} = 0$

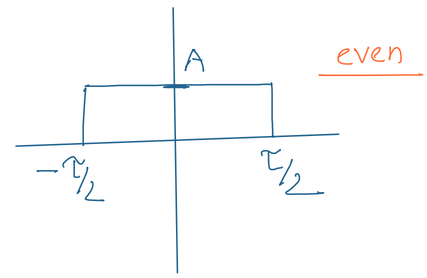
$\sin(\pi t) = \sin(n\pi)$ $\qquad t = n\pi$

$t = n$ $\qquad = \pi, 2\pi, 3\pi, -\pi, -2\pi, -3\pi \ldots$

$n \in \mathbb{Z}$ (not zero) $\qquad$ shape এখানে same, value গুলো different

$n = 1, 2, 3, -1, -2, -3 \ldots$ তে $\text{sinc}\, t = 0$ হবে।

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$



$$= \int_{-\tau/2}^{\tau/2} A e^{-j2\pi f t}\, dt$$

$$= A \left. \frac{e^{-j2\pi f t}}{-j2\pi f} \right|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j2\pi f} \left[e^{-j\pi f \tau} - e^{j\pi f \tau} \right]$$

↳ euler (sine)

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{A}{j2\pi f} \left[e^{j\pi f \tau} - e^{-j\pi f \tau} \right] \quad \theta$$

$$= \frac{A}{\pi f} \frac{\left[e^{j\pi f \tau} - e^{-j\pi f \tau} \right]}{2j}$$

$$= \frac{A\tau}{\pi f \tau} \sin(\pi f \tau)$$

$$= A\tau \operatorname{sinc}(f\tau)$$

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$\tau \to$ constant

$f \to$ variable

spectrum graph: A Vs f

↳ continuous graph হবে (FT)

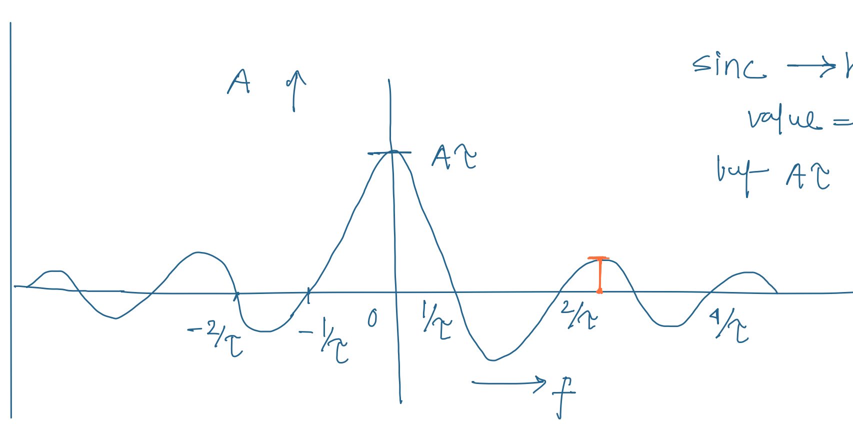
a_k Vs kf_0 $\longrightarrow$ discrete fourier series এ

$A\tau \operatorname{sinc}(f\tau) = 0$

$$\frac{\sin(\pi f \tau)}{\pi f \tau} = 0$$

$\pi f \tau = n\pi$

$f\tau = n \to f = \frac{n}{\tau}$



sinc $\to$ highest value $= 1$

but $A\tau$ এখানে গুণ আছে।

$$\text{rect} \xrightarrow{FT} \text{sinc}$$

rule

time domain এ limited

freq. domain এ infinite

কখনো যদি time domain এ infinite হয় → freq domain এ limited.

$x(t) \xrightarrow{F(t)} X(f)$

$X(t) \xrightarrow{FT} x(-f)$

$x(t) \xrightarrow{FT} X(f) \longrightarrow$ function of f

$A\,\mathrm{rect}(t/\tau)$ $\quad$ $A\tau\,\mathrm{sinc}(f\tau)$

$A\tau\,\mathrm{sinc}(t\tau) \xrightarrow{FT} A\,\underline{\mathrm{rect}}(-f/\tau)$

even function so

$= A\,\mathrm{rect}(f/\tau)$

$A\tau\ \mathrm{sinc}(ft)$

$x(t) = A\tau\,\mathrm{sinc}(t\tau)$

Duality

Notice that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are almost the same.

Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

In other words, $\mathcal{F}[\mathcal{F}[x(t)]] = x(-t)$.

$F[x(t)]$

$\Downarrow$

$F[X(f)] = x(-t)$

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, dt$$

FT of $X(t)$

$x(t) \xrightarrow{F(t)} X(f)$

$X(t) \xrightarrow{FT} x(-f)$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi fT}\, df$$

(inverse fourier transform)

$X(F) \xrightarrow{IFT} x(t)$

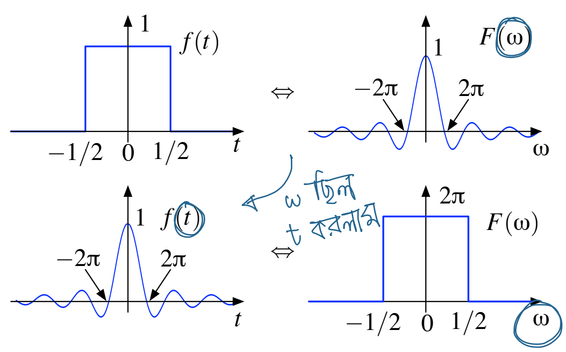
$$x(f) = \int_{-\infty}^{\infty} X(t)\, e^{j2\pi ft}\, dt \longrightarrow x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, dt$$

Example of Duality

- Since $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$ then $\tau = 1 \, , \, A = 1$

$$\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(-f) = \mathrm{rect}(f)$$

(Notice that if the function is even then duality is very simple)



$x(\omega)$

ω ছিল
t করলাম

ω domain এ দেয়া (2π) গুণ হবে f domain এর সাথে।

Generalized Fourier Transforms: δ Functions

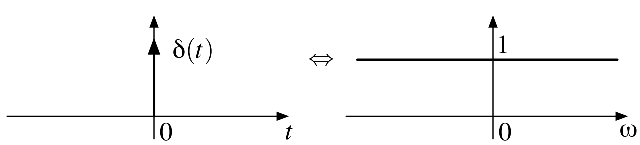
A unit impulse $\delta(t)$ is not a signal in the usual sense (it is a generalized function or distribution). However, if we proceed using the sifting property, we get a result that makes sense:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = 1$$

so

$$\delta(t) \Leftrightarrow 1$$

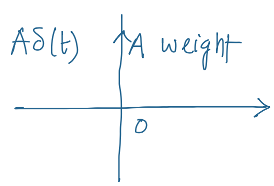
This is a *generalized Fourier transform*. It behaves in most ways like an ordinary FT.

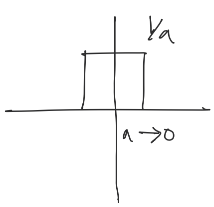


$\delta(t) \longrightarrow$ unit impulse

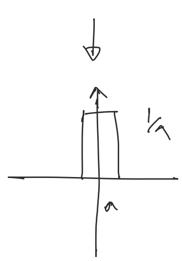
Delta function

$= \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$





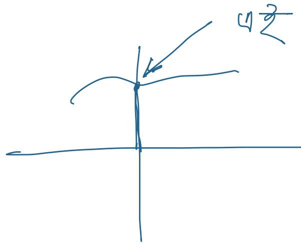
$a \times \frac{1}{a} = 1$

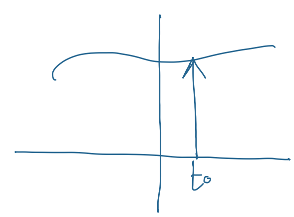


$\delta(t)\ (x(t))$

$= \delta(t)\ x(0)$

এই value টা পাবো





$\delta(t-t_0)\ (x(t))$

$= x(t_0)\ \delta(t-t_0)$

$\sum_{n=1}^{100} \delta(t - nT_0)\ x(t)$

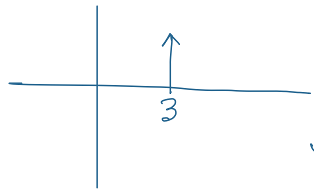
$\longrightarrow$ 100 টা sample পাচ্ছি

→ 2টা sample এর মাঝে যত কম distance নেয়া যায়।

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

area পাবো। area = 1

$\delta(t-3) =$



$$\int_{4}^{\infty} \delta(t-3)\, dt = 0$$

δ function limit এর বাইরে।

যদি δ function range এর মধ্যে থাকে o/p হবে 1।

$$\int_{-\infty}^{\infty} \delta(t-3)\, dt = 1$$

$$X(f) = \int_{-\infty}^{\infty} \underline{\delta(t)}\, \underline{e^{-j2\pi ft}}\, dt$$

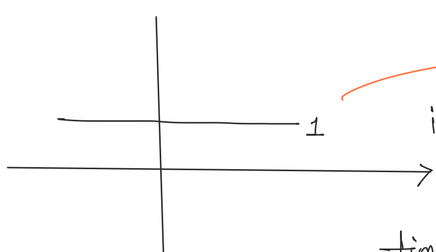
0 তে আছে

$\delta(t)x(t)$ format এ, so একটা value sample হবে $t=0$ এর।

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f \times 0}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt$$

$X(F) = 1$ [2nd property, $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$]

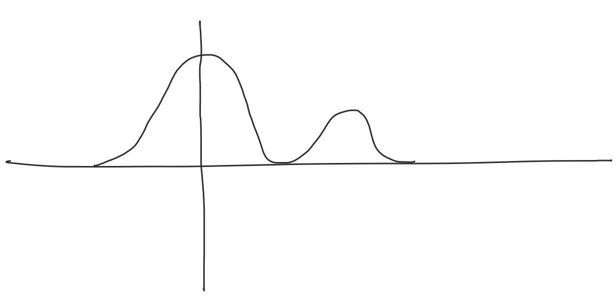


infinite

দেওয়া থাকলে, এর inverse হবে delta function

time domain এ finite ছিল

frequency domain এ infinite হবে।



$x(t)$ $\quad \delta(t) \xrightarrow[FT]{} 1 \quad X(f)$

$1 \longleftrightarrow x(-f) \quad \delta(-f)$

$= \delta(f)$

\# $x(t) = 1$

infinite। integration এর result infinite আসে। converge করে না।

$$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi ft}\,dt = \left.\frac{e^{-j2\pi ft}}{-j2\pi ft}\right|_{-\infty}^{\infty}$$

converge না করলে formula দিয়ে পারবো না। duality দিয়ে করা লাগবে।

$$= e^{-\infty} - e^{\infty}$$

$$= 0 - \infty$$

$\Rightarrow$ formula দিয়ে করা যায় না।

duality principle দিয়ে বের করবো। $\delta(f)$ আসবে।

Shifted δ

A shifted delta has the Fourier transform

$$
\begin{aligned}
\mathcal{F}[\delta(t-t_0)] &= \int_{-\infty}^{\infty} \delta(\underline{t-t_0}) e^{-j2\pi ft}\,dt \\
&= e^{-j2\pi t_0 f}
\end{aligned}
$$

→ t_0 value টা sampled হবে।

→ complex signal. ⟶ a+jb format এ নিবো

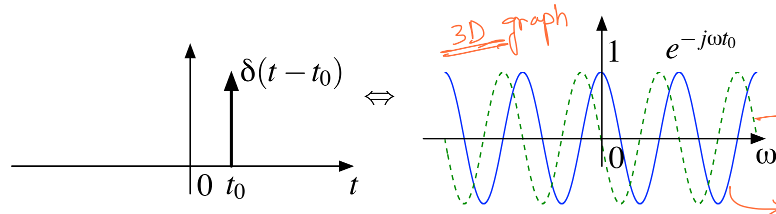
so we have the transform pair

$$\delta(t-t_0) \Leftrightarrow e^{-j2\pi t_0 f}$$

$= \cos(2\pi t_0 f) - j\sin(2\pi t_0 f)$

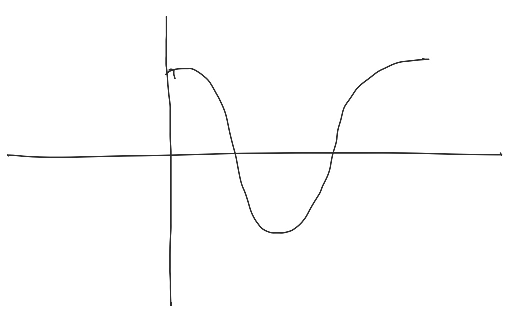
→ 2 টা part এর আলাদা আলাদা graph হবে।

3D graph



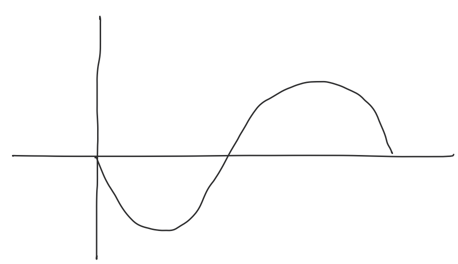
imaginary.

real



real part

(cosine)



imaginary part

(−sine)

Constant

Next we would like to find the Fourier transform of a constant signal $x(t) = 1$. However, direct evaluation doesn't work:

$$\begin{aligned} \mathcal{F}[1] &= \int_{-\infty}^{\infty} e^{-j2\pi ft} dt \\ &= \left. \frac{e^{-j2\pi ft}}{-j2\pi f} \right|_{-\infty}^{\infty} \end{aligned}$$

and this doesn't converge to any obvious value for a particular f.

We instead use duality to guess that the answer is a δ function, which we can easily verify.

$$\begin{aligned} \mathcal{F}^{-1}[\delta(f)] &= \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df \\ &= 1. \end{aligned}$$

$x(t) = \frac{1}{2\pi}$

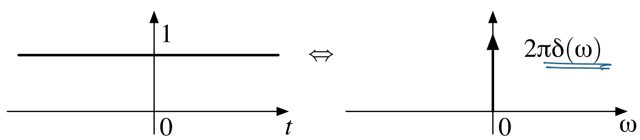
↪ ω domain এ থাকে $\Longrightarrow 2\pi x(-\omega)$

So we have the transform pair

$$1 \Leftrightarrow \delta(f)$$

$\delta(t) \longrightarrow 1$

$1 \longrightarrow \delta(f)$



This also does what we expect – a constant signal in time corresponds to an impulse a zero frequency.

Sinusoidal Signals

If the δ function is shifted in frequency,

+ হচ্ছে

$$\mathcal{F}^{-1}[\delta(f - f_0)] = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} df$$
$$= e^{j2\pi f_0 t}$$

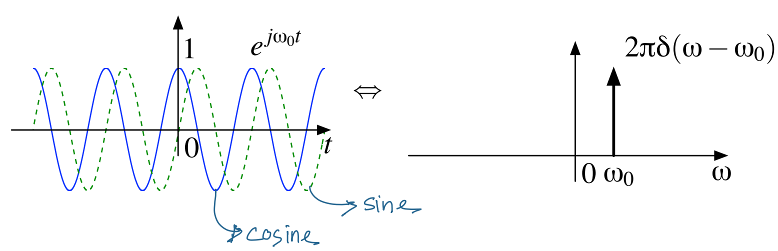
$\delta(F - F_0)$ এর inverse $e^{j2\pi f_0 t}$

$e^{j2\pi f_0 t}$ এর FT $\delta(F - F_0)$

so

time domain এ f_0 constant

$$e^{j2\pi f_0 t} \Leftrightarrow \delta(f - f_0)$$



ω domain এ graph আঁকাতে বলা হলে → 2π গুণ হবে।

$\xrightarrow{FS} \quad a_k e^{j2\pi f_0 k t}$

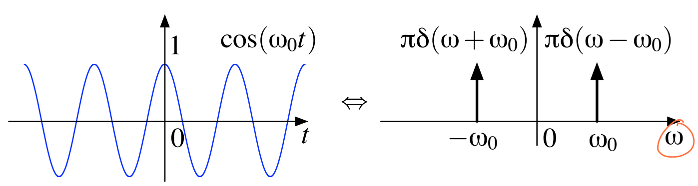
Cosine

With Euler's relations we can find the Fourier transforms of sines and cosines

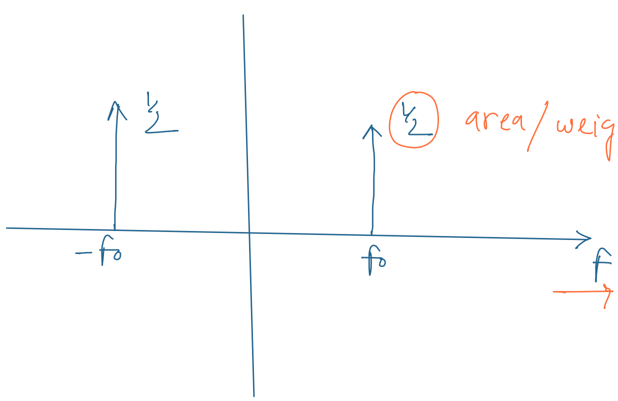
$$
\begin{aligned}
\mathcal{F}\left[\cos(2\pi f_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}\left(\overset{a=1}{e^{j2\pi f_0 t}} + \overset{a=-1}{e^{-j2\pi f_0 t}}\right)\right] \quad \text{FS done} \quad a_1 = a_{-1} = \frac{1}{2} \\
&= \frac{1}{2}\left(\mathcal{F}\left[e^{j2\pi f_0 t}\right] + \mathcal{F}\left[e^{-j2\pi f_0 t}\right]\right) \\
&= \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right). \quad \mathcal{F}\left[e^{j2\pi f_0 t}\right] = \delta(f - f_0)
\end{aligned}
$$

so

$$\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right).$$



ω domain 2π গুণ হচ্ছে



½ area/weight বোঝাচ্ছে

FS → optimized

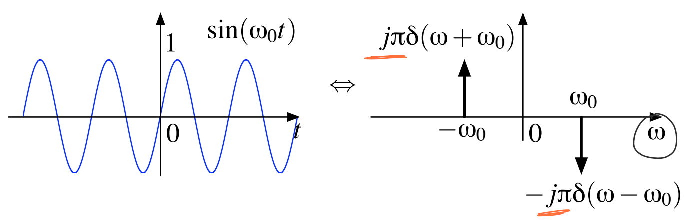
কম timeএ করা যায়।

FT, FS → same ans আসবে।

Sine

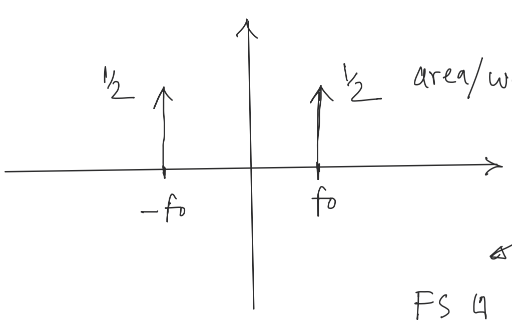
Similarly, since $\sin(f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t})$ we can show that

$$\sin(f_0 t) \Leftrightarrow \frac{j}{2}(\delta(f + f_0) - \delta(f - f_0)) .$$



আলাদা ২ টা graph।

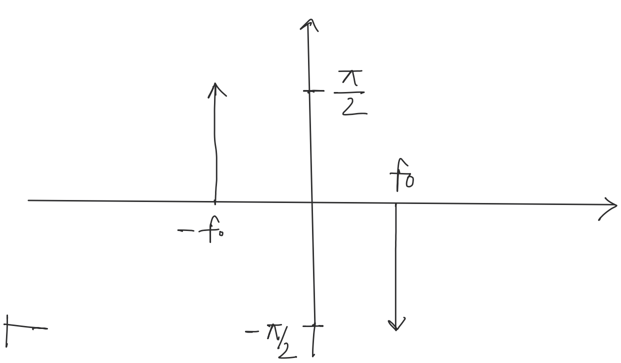
The Fourier transform of a sine or cosine at a frequency f_0 only has energy exactly at $\pm f_0$, which is what we would expect.



area/weight

FS এ coefficient

magnitude diagram



phase diagram