CSE 219: Signals & Linear Systems

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CSE, BUET

Lect 1: Signals & Linear Systems

CFT: signals as LC of CX. exponentials e^{st}; $s = j\omega$
Laplace Transformation: Generalizes arbitrary s.

An LTI system: Impulse response $h(t)$, Input e^{st}
Output $y(t) = H(s)e^{st}$; $H(s)$ is the transfer function

where, $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$; for s imaginary $(s = j\omega)$
$H(s)$ corresponds to the CFT of $h(t)$.

For any general signal $x(t)$, the Laplace Transform is

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \; ; \; s = \sigma + j\omega$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(s)\big|_{s=j\omega} = \mathbb{F}\{x(t)\}$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$
$$= \mathbb{F}\{x(t)e^{-\sigma t}\}$$

<u>Ex 1:</u> Let $x(t) = e^{-at}u(t)$, CFT $X(j\omega)$ converges for $a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \frac{1}{j\omega + a}$$

Laplace Xform: $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt$

$$X(\sigma + j\omega) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}\,dt \quad [\text{CFT of } e^{-(\sigma+a)t}u(t)]$$

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0 \; ; \; \sigma = \mathbb{R}\{s\}$$

$$X(s) = \frac{1}{s+a} \; ; \; \mathbb{R}\{s\} > -a \; ; \; e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \; ; \; \mathbb{R}\{s\} > -a$$

Ex2: $x(t) = -e^{-at}u(-t)$

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)\,dt = -\int_{\infty}^{0} e^{-(s+a)}\,dt = \frac{1}{s+a}$$

$$-e^{-at}u(-t) \overset{\alpha}{\longleftrightarrow} \frac{1}{s+a}, \ \mathrm{Re}\{s\} < -a \ \text{(why?)}$$

Ex3: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}\,dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}e^{-st}u(t)\,dt - 2\int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)\,dt$$

$$= \frac{3}{s+2} - \frac{2}{s+1} \ ; \ e^{-t}u(t) \overset{\alpha}{\longleftrightarrow} \frac{1}{s+1} \ ; \ \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\alpha}{\longleftrightarrow} \frac{1}{s+2} \ ; \ \mathrm{Re}\{s\} > -2$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\alpha}{\longleftrightarrow} \frac{s-1}{s^2+3s+2}, \ \mathrm{Re}\{s\} > -1$$

Lect 1: Laplace Transformation   Sec. C

CFT: Signals as LC of CX. exponentials e^{st}, $s = j\omega$
Laplace Transformation: Generalizes CFT to arbitrary s.

An LTI system: Impulse response $h(t)$, input e^{st}
Output $y(t) = H(s)e^{st}$; $H(s)$ is the transfer function

where, $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$; for s imaginary $(s = j\omega)$
$H(s)$ corresponds to the CFT of $h(t)$

For any general signal $x(t)$, the Laplace Transform is

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \; ; \; s = \sigma + j\omega$$

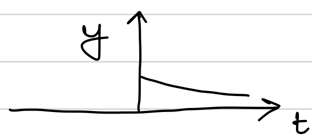
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \; ; \; X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Ex-1: Let $x(t) = e^{-at}u(t)$, CFT $X(j\omega)$ converges for $a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \frac{1}{j\omega + a}$$



Laplace Transform: $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} = \int_{0}^{\infty} e^{-(s+a)t}\,dt$

$$= \frac{1}{s+a}$$

$$X(\sigma + j\omega) = \int_{0}^{\infty} e^{-(\sigma + a)t}e^{-j\omega t}\,dt = \mathcal{F}(e^{-(\sigma+a)t}u(t))$$

$$= \frac{1}{(\sigma + a) + j\omega} \; ; \; \sigma + a > 0 \; ; \; \mathrm{Re}\{s\}$$

$$\Rightarrow \mathrm{Re}\{s\} > -a$$

Lect 1: Signals & Linear Systems Sec. A

CFT: Signals as LC of CX exponentials e^{st}; $s = j\omega$
Laplace Transformation: Generalizes to arbitrary s.

An LTI system: Impulse response $h(t)$, input e^{st}
Output $y(t) = H(s)e^{st}$; $H(s)$ is the transfer function.

where, $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$; for s imaginary $(s = j\omega)$
$H(s)$ corresponds to the CFT of $h(t)$

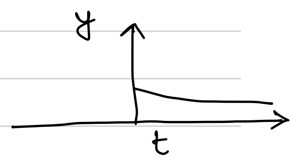
For any general signal $x(t)$, the Laplace Transform is

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt\,; \quad s = \sigma + j\omega \;\Big|\; x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = \int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-j\omega t}\,dt$$
$$= \mathbb{F}\{x(t)e^{-\sigma t}\}$$

Ex-1: Let $x(t) = e^{-at}u(t)$, CFT $X(j\omega)$ converges for $a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$
$$= \int_{0}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \frac{1}{j\omega + a}$$



Laplace Transform: $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt$
$$= \frac{1}{s+a}$$

$$X(\sigma + j\omega) = \int_{0}^{\infty}\left[e^{-(\sigma+a)t}\right]e^{-j\omega t}\,dt = \mathbb{F}\{e^{-(\sigma+a)t}u(t)\}$$

$$= \frac{1}{\sigma + j\omega + a} = \frac{1}{(\sigma+a) + j\omega}\,; \quad \sigma + a > 0,\ \sigma = \mathbb{R}\{s\}$$

$$X(s) = \frac{1}{s+a}\,; \quad \mathbb{R}\{s\} > -a$$

Lect.1: Laplace Transform Sec. B

CFT: signals as LC of cx. exponentials e^{st}; $s=j\omega$
Laplace Transform: Generalizes to arbitrary s.

An LTI system: Impulse response $h(t)$, Input e^{st}
Output $y(t) = H(s)e^{st}$; $H(s)$ is the transfer function.

where $H(s)=\int_{-\infty}^{\infty} h(t)e^{-st}dt$; for s imaginary ($s=j\omega$)
$H(s)$ corresponds to the CFT of $h(t)$.

For any general signal $x(t)$, the Laplace Transform is

$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$; $s=\sigma+j\omega$ | $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$
When $s=j\omega$, $X(s) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \mathcal{F}\{x(t)\}$

$$X(\sigma+j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty}[x(t)e^{-\sigma t}]e^{-j\omega t}dt$$
$$= \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Ex-1: Let $x(t)=e^{-at}u(t)$, CFT $X(j\omega)$ converges for $a>0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-at}e^{-j\omega t}dt$$
$$= \frac{1}{j\omega+a}$$

Laplace Transform: $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} = \int_0^{\infty} e^{-(s+a)t}dt = \frac{1}{s+a}$

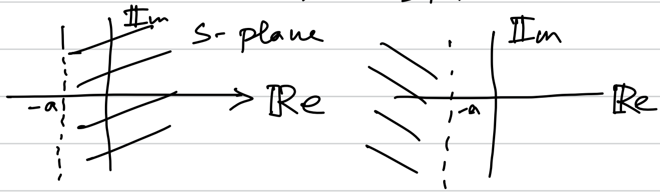
$$X(\sigma+j\omega) = \int_0^{\infty} e^{-at}e^{-\sigma t - j\omega t}dt = \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt = \mathcal{F}\{e^{-(\sigma+a)t}u(t)\}$$

$$X(\sigma+j\omega) = \frac{1}{(\sigma+a)+j\omega}\ ;\ \sigma+a>0 \Rightarrow \mathrm{Re}\{s\} > -a$$

Lect 2: Region of Convergence

Ex-1: $e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$; $\mathbb{R}\{s\} > -a$

Ex-2: $-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$; $a<0 \Rightarrow \mathbb{R}\{s\} < -a$



Properties of region of convergence

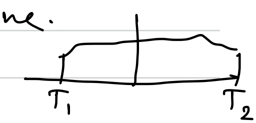
Property #1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane

$x(t) \xleftrightarrow{\mathcal{L}} X(s)$ converges iff $\mathbb{F}\{x(t)e^{-\sigma t}\}$ converges, which depends only on σ

Property #2: For rational Laplace transforms, the ROC does not contain any poles.

$$X(s) = \frac{Y(s)}{Z(s)} \begin{matrix} \rightarrow \text{roots } Y(s)=0 \\ \rightarrow \text{poles } Z(s)=0 \end{matrix}$$

Property #3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.



$$x(t) = \begin{cases} e^{-at}, & 0<t<T \\ 0, & \text{otherwise} \end{cases} ; \quad X(s) = \int_0^T e^{-at}e^{-st}\,dt$$

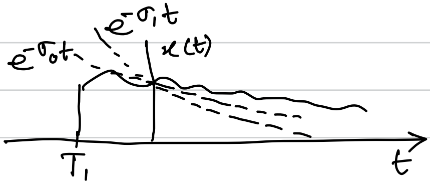
$$= \frac{1}{s+a}\left[1-e^{-(s+a)T}\right]$$

$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{\frac{d}{ds}\left(1-e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)}\right] = \lim_{s\to -a} Te^{-aT}e^{-sT} = T$$

$X(-a) \approx T$

Made with Goodnotes

Property 4: If $x(t)$ is right sided and if the line $\Re e\{s\}=\sigma_0$ is in the ROC, then all values of s for which $\Re e\{s\}>\sigma_0$ will also be in the ROC.

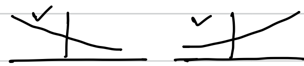


$x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ converges, then $x(t)e^{-\sigma_1 t}$ (where $\sigma_1>\sigma_0$) will also converge.

Property 5: If $x(t)$ is left sided and if the line $\Re e\{s\}=\sigma_0$ is in the ROC, then all values of s for which $\Re e\{s\}<\sigma_0$ will also be in the ROC.

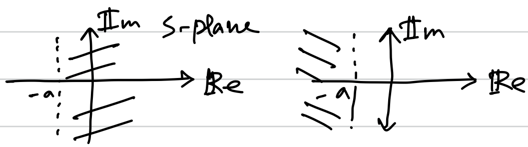
Lect-2: Region of Convergence

sec. A



Ex-1: $e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$; $\Re\{s\} > -a$

Ex-2: $-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$; $a<0 \Rightarrow \Re\{s\} < -a$



Properties of region of convergence

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ converges iff $\mathcal{F}\{x(t)e^{-\sigma t}\}$, which depends only on σ.

Property 2: For rational Laplace transform, the ROC does not contain any poles.

$$X(s) = \frac{Y(s) \rightarrow \text{roots } Y(s)=0}{Z(s) \rightarrow \text{poles } Z(s)=0}$$

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then ROC is the entire s-plane.

$x(t) = \begin{cases} e^{-at}, & 0<t<T \\ 0, & \text{otherwise} \end{cases}$; $X(s) = \int_0^T e^{-at}e^{-st}\,dt$

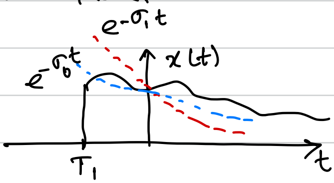
$= \frac{1}{s+a}\left[1-e^{-(s+a)T}\right]$

$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{d/ds\,(1-e^{-(s+a)T})}{d/ds\,(s+a)}\right] = \lim_{s\to -a} Te^{-aT}e^{-sT} = T$$

$$\lim_{s\to -a} X(s) = T \Rightarrow X(-a) \approx T$$

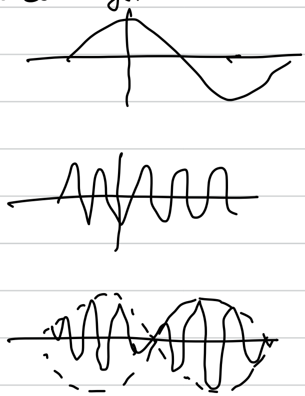
Made with Goodnotes

Property 4: If $x(t)$ is right sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$ will also be in the ROC.



$x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ converges, then $x(t)e^{-\sigma_1 t}$ (where $\sigma_1 > \sigma_0$) will also converge.

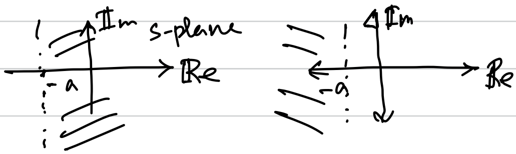
Property 5: If $x(t)$ is left sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ will also be in the ROC.



Lect. 2: Region of Convergence   Sec. B

Ex-1: $e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$; $\mathrm{Re}\{s\} > -a$

Ex-2: $-e^{-at}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$; $a<0 \Rightarrow \mathrm{Re}\{s\} < -a$



Properties of region of convergence

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.
$x(t) \xrightarrow{\mathcal{L}} X(s)$ converges iff $\mathcal{F}\{x(t)e^{-\sigma t}\}$ converges, which depends only on σ.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

$$X(s) = \frac{Y(s)}{Z(s)} \begin{array}{l} \rightarrow \text{root } Y(s)=0 \\ \rightarrow \text{poles } Z(s)=0 \end{array}$$

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then ROC is the entire s-plane.

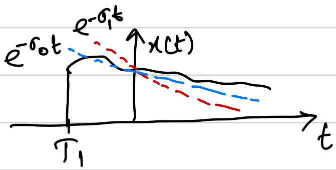
$$x(t) = \begin{cases} e^{-at}, & 0<t<T \\ 0, & \text{otherwise} \end{cases} ; \quad X(s) = \int_0^T e^{-at}e^{-st}\,dt$$

$$= \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

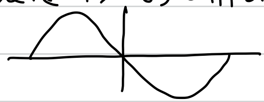
$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{\frac{d}{ds}(1-e^{-(s+a)T})}{\frac{d}{ds}(s+a)}\right] = \lim_{s\to -a} Te^{-aT}e^{-sT} = T$$

$X(-a) \simeq T$

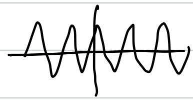
Property 4: If $x(t)$ is right sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$ will also be in the ROC.

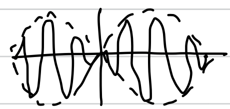


$x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ converges, then $x(t)e^{-\sigma_1 t}$ (where $\sigma_1 > \sigma_0$) will also converge.



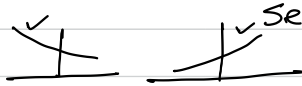
Property 5: If $x(t)$ is left sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ will also be in the ROC.





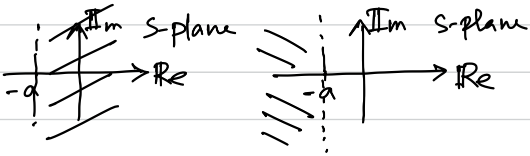
Lect. 2: Region of convergence

Sec. C



Ex-1: $e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$; $Re\{s\} > -a$

Ex-2: $-e^{-at}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$; $a<0 \Rightarrow R\{s\} < -a$



Properties of region of convergence

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

$x(t) \xrightarrow{\mathcal{L}} X(s)$ converge iff $\mathcal{F}\{x(t)e^{-\sigma t}\}$ converges, which depends only on σ.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

$$X(s) = \frac{Y(s)}{Z(s)} \quad \begin{matrix} \rightarrow \text{roots } Y(s)=0 \\ \rightarrow \text{poles } Z(s)=0 \end{matrix}$$

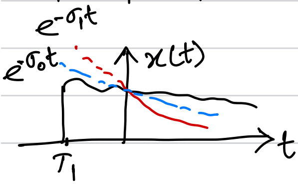
Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

$x = \begin{cases} e^{-at}, & 0<t<T \\ 0, & \text{otherwise} \end{cases}$; $X(s) = \int_0^T e^{-at}e^{-st}dt$

$= \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$

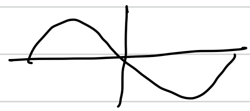
$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{\frac{d}{ds}\left(1-e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)}\right] = \lim_{s\to -a} T e^{-at}e^{-st} = T$$

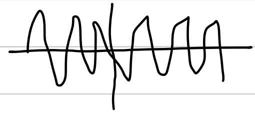
Property 4: If $x(t)$ is right sided and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.

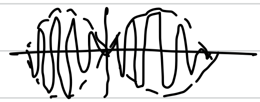


$x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ converges, then $x(t)e^{-\sigma_1 t}$ (where $\sigma_1 > \sigma_0$) will also converge.

Property 5: If $x(t)$ is left sided and if the line $R\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.

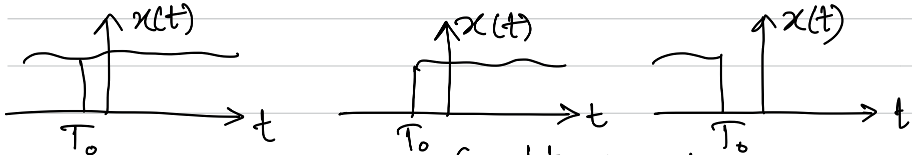




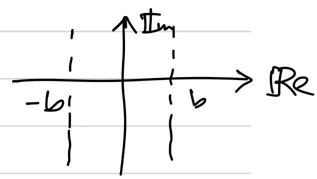


Lecture 3: Convergence (cont.) & Inverse LT

Property 6: If $x(t)$ is two sided, and if the line $\Re\{s\}=\sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re\{s\}=\sigma_0$



Ex-7: $x(t) = e^{-b|t|} = \begin{cases} e^{-bt}, & \text{when } t \geq 0 \\ e^{bt}, & \text{when } t < 0 \end{cases}$



$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$

$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \ \Re\{s\} > -b$

$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \ \Re\{s\} > +b$

$-b < \Re\{s\} < b$

Property 7: If the Laplace function $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole and analogous for the left-most pole.

Inverse Laplace Transform

Let $s=\sigma+j\omega$, then the LT of signal $x(t)$ is

$$X(\sigma+j\omega)=\mathbb{F}\{x(t)e^{-\sigma t}\}=\int_{-\infty}^{+\infty}x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$\Rightarrow x(t)e^{-\sigma t}=\mathbb{F}^{-1}\{X(\sigma+j\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\sigma+j\omega)e^{j\omega t}\,d\omega$$

$$\Rightarrow x(t)=\mathbb{F}^{-1}\{X(\sigma+j\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\sigma+j\omega)e^{(\sigma+j\omega)t}\,d\omega$$

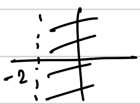
$$\Rightarrow x(t)=\mathbb{F}^{-1}\{X(s)\}=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}\,ds$$

$X(s)=\sum_{i=1}^{m}\frac{A_i}{s+a_i}$. If ROC is to the right of $s=-a_i$, then the inverse FT is $A_ie^{-a_it}u(t)$ and $-A_ie^{-a_it}u(-t)$ if ROC is to the left.

Ex. 9: $X(s)=\frac{1}{(s+1)(s+2)}$; $\mathrm{Re}\{s\}>-1$

$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}\Rightarrow A(s+2)+B(s+1)=1\Rightarrow A=1, B=-1$

$X(s)=\frac{1}{s+1}-\frac{1}{s+2}$



$e^{-t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+1}$ $\mathrm{Re}\{s\}>-1$

$e^{-2t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+2}$ $\mathrm{Re}\{s\}>-2$

Ex-10: $X(s)=\frac{1}{(s+1)(s+2)}$; $\mathrm{Re}\{s\}<-2$

$-e^{-t}u(-t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+1}$; $\mathrm{Re}\{s\}<-1$

$-e^{-2t}u(-t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+2}$; $\mathrm{Re}\{s\}<-2$

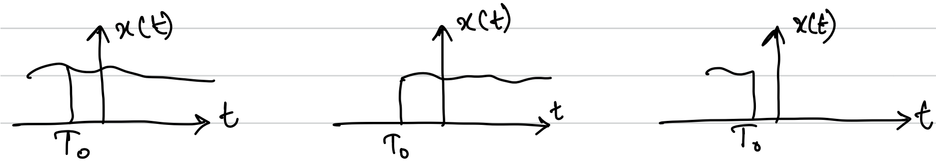
$\Rightarrow x(t)=[e^{-t}+e^{-2t}]u(-t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{(s+1)(s+2)}$

Ex-11: $X(s)=\frac{1}{(s+1)(s+2)}$; $-2<\mathrm{Re}\{s\}<-1$

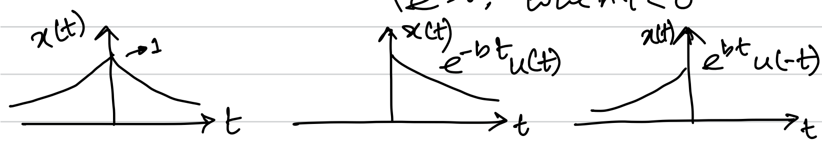
$x(t)=-e^{-t}u(t)-e^{-2t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{(s+1)(s+2)}$, $-2<\mathrm{Re}\{s\}<-1$

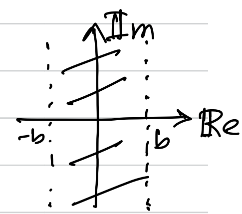
Lecture 3: Converge (cont.) Sec. B

Property 6: If $x(t)$ is two sided, and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC, then ROC will consist of a strip in the s-plane that includes the line $\mathrm{Re}\{s\}=\sigma_0$.



Ex-7: $x(t)=e^{-b|t|}=\begin{cases} e^{-bt}, & \text{when } t\geq 0 \\ e^{bt}, & \text{when } t<0 \end{cases}$





$x(t)=e^{-bt}u(t)+e^{bt}u(-t)$

$e^{-bt}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+b}$; $\mathrm{Re}\{s\}>-b$

$e^{+bt}u(-t) \xrightarrow{\mathcal{L}} \frac{-1}{s-b}$; $\mathrm{Re}\{s\}<b$

$-b<\mathrm{Re}\{s\}<b$

Property 7: If the Laplace function $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace Transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole and analogous for the left most pole.

Inverse Laplace Transform

Let $s=\sigma+j\omega$, then the LT of the signal $x(t)$ is

$$X(\sigma+j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$\Rightarrow x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma+j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega \qquad s=\sigma+j\omega$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds \qquad ds=jd\omega$$

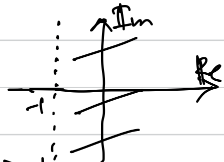
$X(s)=\sum_{i=1}^{m}\frac{A_i}{s+a_i}$. If ROC is to the right of $s=-a_i$, then the inverse FT is $A_i e^{-a_i t}u(t)$ and $-A_i e^{-a_i t}u(-t)$ if ROC is to the left.

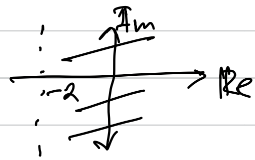
Ex. 9: $X(s)=\frac{1}{(s+1)(s+2)}$; $\mathrm{Re}\{s\}>-1$

$A=1,\ B=-1$

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2} \Rightarrow A(s+2)+B(s+1)=1$$

$$\Rightarrow X(s)=\frac{1}{s+1}-\frac{1}{s+2}$$



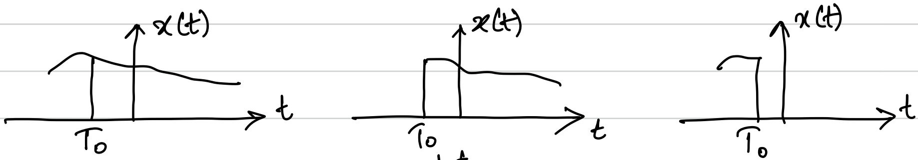


$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1};\ \mathrm{Re}\{s\}>-1$$

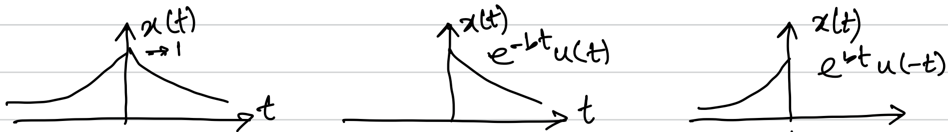
$$e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2};\ \mathrm{Re}\{s\}>-2 \quad \Rightarrow x(t)=[e^{-t}-e^{-2t}]u(t)$$

Ex-10, 11

Property 6: If $x(t)$ is two sided, and if the line $\Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re\{s\} = \sigma_0$.



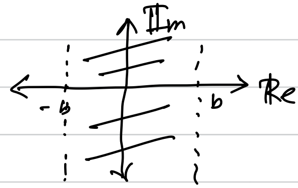
Ex. 7: $x(t) = e^{-b|t|} = \begin{cases} e^{-bt}, & \text{when } t \geq 0 \\ e^{bt}, & \text{when } t < 0 \end{cases}$



$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$

$e^{-bt}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+b}$; $\Re\{s\} > -b$

$e^{bt}u(-t) \xrightarrow{\mathcal{L}} \frac{-1}{s-b}$; $\Re\{s\} < b$



$-b < \Re\{s\} < b$

Property 7: If the Laplace Transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole and analogous for the left pole.

Inverse Laplace Transform

Let $s = \sigma + j\omega$, then the Laplace transform of signal $x(t)$ is

$$X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

$$\Rightarrow x(t) e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega \qquad s = \sigma + j\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds \qquad \Rightarrow ds = j d\omega$$

$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$. If ROC is to the right of $s = -a_i$, then the inverse FT is $A_i e^{-a_i t} u(t)$ and $-A_i e^{-a_i t} u(-t)$ if ROC is to the left.

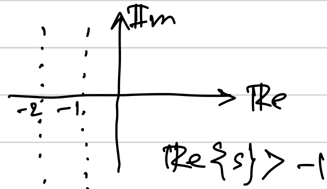
Ex. 9: $X(s) = \frac{1}{(s+1)(s+2)}$; $\Re\{s\} > -1$

$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow A(s+2) + B(s+1) = 1 \Rightarrow A = 1, B = -1$

$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$

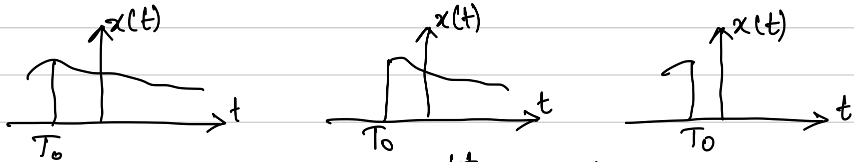
$e^{-t} u(t) \xrightarrow{\mathcal{L}} 1/(s+1)$; $\Re\{s\} > -1$

$e^{-2t} u(t) \xrightarrow{\mathcal{L}} 1/(s+2)$; $\Re\{s\} > -2$

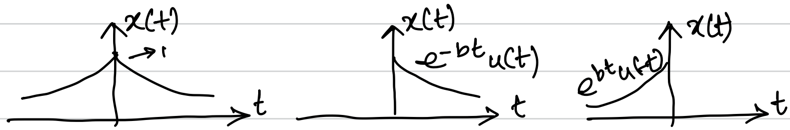


Ex: 10, 11

Property 6: If $x(t)$ is two sided, and if the line $\Re\{s\}=\sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re\{s\}=\sigma_0$



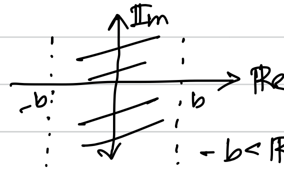
Ex-7: $x(t) = e^{-b|t|} = \begin{cases} e^{-bt}, & \text{when } t \geq 0 \\ e^{bt}, & \text{when } t<0 \end{cases}$



$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$

$e^{-bt}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+b}, \ \Re\{s\} > -b$

$e^{+bt}u(-t) \xrightarrow{\mathcal{L}} \frac{-1}{s-b}, \ \Re\{s\} < b$



$-b < \Re\{s\} < b$

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no pole of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. [Analogous for the left most pole.

Inverse Laplace Transform

Let $s = \sigma + j\omega$, then the Laplace transform of signal $x(t)$ is

$$X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t)\, e^{-\sigma t} e^{-j\omega t}\, dt$$

$$\Rightarrow x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

$s = \sigma + j\omega$

$\Rightarrow ds = j\,d\omega$

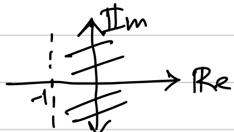
$\Rightarrow d\omega = ds/j$

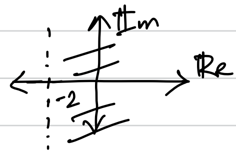
$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$. If ROC is to the right of $s = -a_i$, then the inverse LT is $A_i e^{-a_i t} u(t)$ and $-A_i e^{-a_i t} u(-t)$ if ROC is to the left.

Ex.-9: $X(s) = \frac{1}{(s+1)(s+2)}$, $\Re\{s\} > -1$

$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow A(s+2) + B(s+1) = 1 \Rightarrow A = 1, B = -1$

$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$





$e^{-t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+1}$; $\Re\{s\} > -1$

$e^{-2t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+2}$; $\Re\{s\} > -2$

$\Re\{s\} > -1$

$x(t) = e^{-t}u(t) - e^{-2t}u(t)$

Ex. 10, 1

Lecture 4: Properties of the Laplace Transform

Property 1: Linearity

If $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$ with ROC R_1 and $x_2(t) \xrightarrow{\mathcal{L}} X_2(s)$ with ROC R_2
then $ax_1(t) + bx_2(t) \xrightarrow{\mathcal{L}} aX_1(s) + bX_2(s)$ with ROC $R_1 \cap R_2$.

Property 2: Time Shifting

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, with ROC $= R$, then $x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$ with ROC $= R$

$$y(t) = x(t-t_0) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s(t-t_0)-st_0} d\tau = e^{-st_0} X(s)$$

Property 3: Shifting in the s-domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, with ROC $= R$, then $e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)$ with ROC $= R + \mathrm{Re}\{s_0\}$

Special case $s_0 = j\omega_0 \Rightarrow$ shift in the s-plane parallel to the $j\omega$-axis.

Property 4: Time scaling

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC $= R$, then $x(at) \xrightarrow{\mathcal{L}} \frac{1}{|a|} X(s/a)$ with ROC $R_1 = aR$

Property 5: Conjugation

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R, $x^*(t) \xrightarrow{\mathcal{L}} X^*(s^*)$, with ROC $= R$.

Property 6: Convolution

If $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$, with ROC $= R_1$, and $x_2 \xrightarrow{\mathcal{L}} X_2(s)$ with ROC R_2,
then $x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) X_2(s)$, with ROC $R_1 \cap R_2$.

Property 7: Differentiation in the time domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, with ROC $= R$, then $dx(t)/dt \xleftrightarrow{\mathcal{L}} sX(s)$ with ROC R.

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{st} ds \Rightarrow dx(t)/dt = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} sX(s) e^{st} ds = sX(s)$$

Property 8: Differentiation in the s-domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then $-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} dX(s)/ds$ with ROC $=R$

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \Rightarrow dX(s)/ds = \int_{-\infty}^{\infty} (-t) x(t) e^{-st} dt \Rightarrow -tx(t) \overset{\mathcal{L}}{\longleftrightarrow} dX(s)/ds$

Ex-14: $x(t) = te^{-at} u(t)$

$e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} 1/(s+a) \cdot \Re\{s\} > -a$

$te^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} -d/ds\left[1/(s+a)\right] = 1/(s+a)^2, \Re\{s\} > -a$

$t^2/2\, e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} 1/(s+a)^3, \Re\{s\} > -a \Rightarrow t^{n-1}/(n-1)!\, e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} 1/(s+a)^n, \Re\{s\} > -a$

Property 9: Integration in the time domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $=R$, then $\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{L}}{\longleftrightarrow} 1/s\, X(s)$ with ROC $R \cap \{\Re\{s\} > 0\}$

Self-study: 9.6

Lect. 4: Properties of Laplace Transform Sec. C

Property 1: Linearity

If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then $ax_1(t)+bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s)+bX_2(s)$ with ROC $R_1 \cap R_2$.

Property 2: Time shifting

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $=R$, then $x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s)$ with ROC $=R$.

$$y(t)=x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{+\infty} x(t-t_0)e^{-st}dt = \int_{-\infty}^{+\infty} x(t-t_0)e^{-st+st_0-st_0}dt$$
$$= e^{-st_0}\int_{-\infty}^{\infty} x(t-t_0)e^{-s(t-t_0)}dt = e^{-st_0}\int_{-\infty}^{+\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

Property 3: Shifting in the s-domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then $e^{s_0t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0)$ with ROC $=R+\Re e\{s_0\}$

Property 4: Time scaling

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $=R$, then $x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X(s/a)$ with $R_1=aR$

Property 5: Conjugation

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $=R$, then $x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$, with ROC $=R$.

Property 6: Convolution

If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, with ROC $=R_1$ and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, with ROC $=R_2$, then $x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s)$ with ROC $=R_1 \cap R_2$

Property 7: Differentiation in the time domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, with ROC $=R$, then $dx(t)/dt \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$ with ROC $=R$.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \Rightarrow dx(t)/dt = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds \Rightarrow \frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$$

Property 8: Differentiation in the s-domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R, then $-tx(t) \xleftrightarrow{\mathcal{L}} dX(s)/ds$ with ROC = R

$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt \Rightarrow dX(s)/ds = \int_{-\infty}^{+\infty} -tx(t)e^{-st}dt \Rightarrow -tx(t) \xleftrightarrow{\mathcal{L}} dX(s)/ds$

Ex-14: $x(t) = te^{-at}u(t)$

$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} 1/(s+a)$; $\Re\{s\} > -a$

$\Rightarrow -te^{-at}u(t) \xleftrightarrow{\mathcal{L}} d/ds\left(1/(s+a)\right) = -1/(s+a)^2 \Rightarrow te^{-at}u(t) = 1/(s+a)^2, \Re\{s\} > -a$

$\Rightarrow \frac{t^2}{2}e^{-at}u(t) \longleftrightarrow 1/(s+a)^3$; $\Re\{s\} > -a$

$\Rightarrow t^{n-1}/(n-1)!\, e^{-at}u(t) \longleftrightarrow 1/(s+a)^n$; $\Re\{s\} > -a$

Property 9: Integration in the time domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC = R, then $\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$ with ROC $\supset R \cap \{\Re\{s\} > 0\}$

Self-study: 9.6

Lect. 4: Properties of Laplace transform   Sec. A

Property 1: Linearity

If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC $= R_1$ and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC $= R_2$, then
$a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_2(s)$ with ROC $R_1 \cap R_2$.

Property 2: Time shifting

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, with ROC $= R$, then $x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)$ with ROC $= R$.

$y(t) = x(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{+\infty} y(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-st+st_0-st_0} dt \,/\, \tau = t - t_0$
$= e^{-st_0} \int_{-\infty}^{+\infty} x(t-t_0) e^{-s(t-t_0)} dt = e^{-st_0} \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} d\tau = e^{-st_0} X(s)$

Property 3: Shifting in the s-domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0)$ with ROC $= R + \mathrm{Re}\{s_0\}$
Special case, $s_0 = j\omega_0 \Rightarrow$ Shift in the s-plane parallel to the $j\omega$-axis.

Property 4: Time scaling

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X(s/a)$ with ROC $R_1 = aR$

Property 5: Conjugation

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $x^*(t) \overset{\mathcal{L}}{\longrightarrow} X^*(s^*)$ with ROC $= R$.

Property 6: Convolution

If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC $= R_1$ and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC $= R_2$, then
$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s)$ with ROC $R_1 \cap R_2$.

Property 7: Differentiation in the time domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $dx(t)/dt \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$ with ROC $= R$.

$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{st} ds \Rightarrow \frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} s X(s) e^{st} ds \Rightarrow \frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$

Property 8: Differentiation in the s-domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC=R, then $-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s)$ with ROC=R.

$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt \Rightarrow \frac{d}{ds}X(s) = \int_{-\infty}^{+\infty} -tx(t) e^{-st} dt \Rightarrow -tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s)$

Ex-14: $x(t) = te^{-at}u(t)$

$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$; $\Re e\{s\} > -a$

$-te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{(s+a)^2} \Rightarrow te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^2} \Rightarrow \frac{t^2}{2} e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^3}$

$\Rightarrow \frac{t^{n-1}}{(n-1)!} e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}$; $\Re e\{s\} > -a$

Property 9: Integration in the time domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC=R, then $\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)$ with ROC $R \cap \{\Re e\{s\} > 0\}$

Self-study: 9.6

Lecture 4: Properties of the Laplace transform

Property 1: Linearity

If $x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC $= R_1$ and $x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC $= R_2$,
then $a x_1(t) + b x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_2(s)$ with ROC $= R_1 \cap R_2$

Property 2: Time shifting

If $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then $x(t-t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)$ with ROC $= R$

$$y(t) = x(t-t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{+\infty} y(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-st+st_0-st_0} dt \quad \Big| \begin{matrix} \tau = t - t_0 \\ \Rightarrow d\tau = dt \end{matrix}$$

$$= e^{-st_0} \int_{-\infty}^{+\infty} x(t-t_0) e^{-s(t-t_0)} dt = e^{-st_0} \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{-st_0} X(s)$$

Property 3: Shifting in the s-domain

If $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $e^{s_0 t} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s - s_0)$ with ROC $= R + \mathrm{Re}\{s_0\}$

Special case: $s_0 = j\omega_0 \Rightarrow$ shift in the plane parallel to the $j\omega$-axis.

Property 4: Time scaling

If $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $x(at) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X(s/a)$ with ROC $= R_1$.

Property 5: Conjugation

If $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $x^*(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$ with ROC $= R$.

Property 6: Convolution

If $x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC $= R_1$, and $x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC $= R_2$, then
$x_1(t) * x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s)$, with ROC $R_1 \cap R_2$.

Property 7: Differentiation in the time domain

If $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then $\frac{d}{dt} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s)$ with ROC R.

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{st} ds \Rightarrow \frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} s X(s) e^{st} ds \Rightarrow \frac{d}{dt} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s)$$

Property 8: Differentiation in the s-domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R, then $-tx(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s)$ with ROC = R

$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \Rightarrow \frac{d}{ds}X(s) = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt \Rightarrow -tx(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s)$

Ex.14: $x(t) = te^{-at}u(t)$

$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$, $\Re\{s\} > -a$

$-te^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{-1}{(s+a)^2}$

$\Rightarrow te^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}$; $\Re\{s\} > -a$

$\Rightarrow \frac{t^2}{2}e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^3}$, $\Re\{s\} > -a \Rightarrow \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n}$, $\Re\{s\} > -a$

Property 9: Integration in the time domain

If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC = R, then $\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$ with

ROC = $R \cap \{\Re\{s\} > 0\}$

self-study: 9.6

Lect. 5: The z-transform

z-transform is the discrete-time counterpart of the Laplace transform.
For a discrete-time LTI system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is $y[n] = H[z]z^n$ where $H[z] = \sum_{n=-\infty}^{n=+\infty} h[n]z^{-n}$

For $z=e^{j\omega}$ with $|z|=1$, the sum corresponds to the DFT of $h[n]$. When $|z| \neq 1$, then the sum is referred to as the z-transform of $h[n]$.

The z-transform of a general DFT signal $x[n]$ is defined as

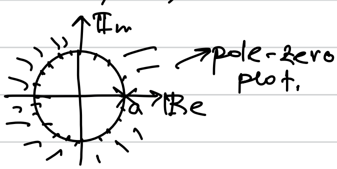
$$X(z) = \sum_{n=-\infty}^{n=\infty} x[n]z^{-n}, \quad x[n] \xleftrightarrow{z} X(z)$$

Let $z = re^{j\omega}$, then $X(re^{j\omega}) = \sum_{n=-\infty}^{n=\infty} x[n](re^{j\omega})^{-n}$

$$= \sum_{n=-\infty}^{n=+\infty} \{x[n]r^{-n}\}e^{-j\omega n} = \mathbb{F}\{x[n]r^{-n}\}$$

Ex-1: $x[n] = a^n u[n]$, then $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n$
For convergence, we require $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty \Leftrightarrow |az^{-1}| < 1 \Leftrightarrow |z| > |a|$

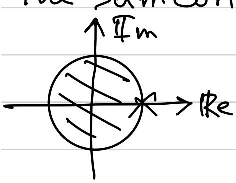
Then $X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \; |z| > |a|$



Ex-2: $x[n] = -a^n u[-n-1]$, then $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} a^n z^{-n}$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n, \; |a^{-1}z| < 1$$

The sum converges to $X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}, \; |z| < |a|$



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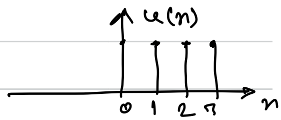
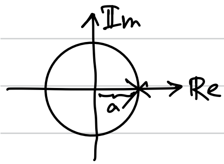
For $z = e^{jw}$ with $|z| = 1$, the sum corresponds to the DFT of $h[n]$. When $|z|$ is not necessarily 1, then the sum is referred to as the z-transfom of $h[n]$

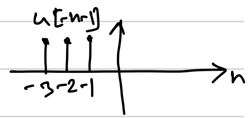
The z-transform of a general DT signal $x[n]$ is defined as
$$X(z) = \sum_{n=-\infty}^{n=+\infty} x[n] z^{-n}, \quad x[n] \xleftrightarrow{z} X(z)$$

Let $z = re^{jw}$, then $X(re^{jw}) = \sum_{n=-\infty}^{n=+\infty} x[n] (re^{jw})^{-n}$
$$= \sum_{n=-\infty}^{n=+\infty} x[n] r^{-n} e^{-jwn} = \mathcal{F}\{x[n] r^{-n}\}$$

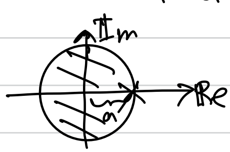


Ex-1: $x[n] = a^n u[n]$, $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[n] z^{-n}$
$$= \sum_{n=0}^{n=\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |az^{-1}| < 1 \Rightarrow |z| > |a|$$





Ex-2: $x[n] = -a^n u[-n-1]$, then $X(z) = -\sum_{n=-\infty}^{n=+\infty} a^n u[-n-1] z^{-n}$
$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{n=-\infty}^{n=0} a^n z^{-n} = 1 - \sum_{n=0}^{n=\infty} (a^{-1}z)^n$$
$$= 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{-z}{a-z} = \frac{z}{z-a} ; \quad |a^{-1}z| < 1 \Rightarrow |z| < |a|$$



Lect. 5: The z-transform Sec. B

z-transform is the discrete-time counterpart of the Laplace transform. For a discrete-time LTI system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is $y[n] = H[z]\, z^n$ where, $H[z] = \sum_{n=-\infty}^{n=+\infty} h[n] z^{-n}$.

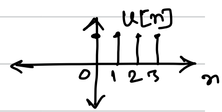
For $z = e^{j\omega}$ with $|z| = 1$, the sum $H[z]$ corresponds to the DFT of $h[n]$. When $|z|$ is not necessarily 1, then the sum is referred to as the z-transform of $h[n]$.

The z-transform of a general DFT signal $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x[n] z^{-n} \quad [r=1,\ \underbrace{X(z)}_{\hookrightarrow \text{DFT}} = \sum_{n=-\infty}^{n=+\infty} x[n] e^{-j\omega n}]$$

Let $z = re^{j\omega}$, then $X(re^{j\omega}) = \sum_{n=-\infty}^{n=+\infty} x[n] (re^{j\omega})^{-n}$

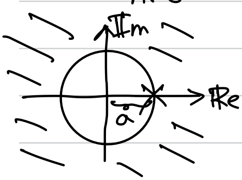
$$= \sum_{n=-\infty}^{n=+\infty} x[n] r^{-n} e^{-j\omega n} = \mathcal{F}\{x[n] r^{-n}\}$$



Ex-1: $x[n] = a^n u[n]$, $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[n] z^{-n}$

$$= \sum_{n=0}^{n=+\infty} a^n z^{-n} = \sum_{n=0}^{n=+\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}; \ |az^{-1}| < 1$$

$\Rightarrow |a| < |z| \Rightarrow |z| > |a|$

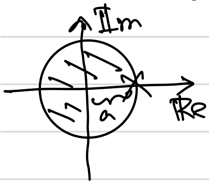


Ex-2: $x[n] = -a^n u[-n-1]$, then $X(z) = -\sum_{n=-\infty}^{n=+\infty} a^n u[-n-1] z^{-n}$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{n=-\infty}^{n=0} a^n z^{-n} = 1 - \sum_{n=0}^{n=\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{-z}{a-z} = \frac{z}{z-a}; \ |a^{-1}z| < 1$$

$\Rightarrow |z| < |a|$



Made with Goodnotes

z-transform is the discrete-time countepart of the Laplace transform. For a discrete-time signal with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is $y[n] = H[z]\, z^n$ where $H[z] = \sum_{n=-\infty}^{n=+\infty} h[n]\, z^{-n}$

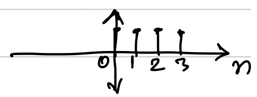
For $z = e^{j\omega}$ with $|z| = 1$, the sum corresponds to the DFT of $h[n]$. When $|z|$ is not necessarily 1, then the sum is referred to as the z-transform of $h[n]$.

The z-transform of a general DT signal $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x[n]\, z^{-n} \; ; \quad x[n] \xleftrightarrow{z} X(z) \quad \Big| \quad X(e^{j\omega}) = \sum_{n=-\infty}^{n=+\infty} x[n]\, e^{-j\omega n}$$

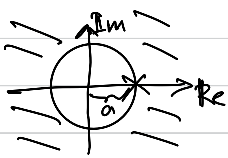
Let $z = re^{j\omega}$, then $X(re^{j\omega}) = \sum_{n=-\infty}^{n=+\infty} x[n]\, (re^{j\omega})^{-n}$

$$= \sum_{n=-\infty}^{n=+\infty} x[n]\, r^{-n} e^{-j\omega n} = \mathcal{F}\{x[n]\, r^{-n}\}$$

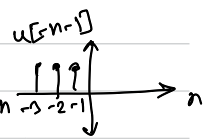


Ex-1: $x[n] = a^n u[n]$, $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[n]\, z^{-n}$

$$= \sum_{n=0}^{n=+\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} \; ; \quad |az^{-1}| < 1 \Rightarrow |a| < |z| \Rightarrow |z| > |a|$$



$$= \frac{z}{z-a}$$



Ex-2: $x[n] = -a^n u[-n-1]$, $X(z) = \sum_{n=-\infty}^{n=+\infty} a^n u[-n-1]\, z^{-n}$

$$= -\sum_{n=-\infty}^{n=-1} (az^{-1})^n = 1 - \sum_{n=-\infty}^{n=0} (az^{-1})^n = 1 - \sum_{m=0}^{m=\infty} (az^{-1})^{-m}$$

$[m = -n]$

$$= 1 - \sum_{m=0}^{m=\infty} (a^{-1}z)^m = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{a^{-1}z}{a^{-1}z-1} = \frac{z}{z-a}$$

$|a^{-1}z| < 1 \Rightarrow |z| < |a|$