Lecture 9
ELE 301: Signals and Systems

Discrete Fourier Transform

Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T (i.e. a finite sequence of data). The transformation of discrete data between the time and frequency domain is quite useful in extracting information from the signal.

The DFT is denoted by $X(k)$ and given as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad k = 0,1,2,\ldots N-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Here $X(k)$ is the DFT and it is computed at $k=0,1,2,\ldots.\ N-1$. "N" discrete points. Thus DFT $X(k)$ is the sequence of N samples. The sequence $x(n)$ can be obtained back from $X(k)$ by taking Inverse Discrete Fourier Transform, i.e. IDFT is given as,

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad n = 0,1,2,\ldots.N-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Here $x(n)$ is sequence of N samples . thus $X(k)$ and $x(n)$ both contains N number of samples.

Let us define, $W_N = e^{-j2\pi/N}$

This is called twiddle factor. Hence DFT and IDFT equation can be written as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad k = 0,1,\ldots\ldots N-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

And

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \quad n = 0,1,\ldots..N-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Example One

Calculate the four-point DFT of the aperiodic sequence $x[k]$ of length $N = 4$,

which is defined as follows:

$$x[k] = \begin{cases} 2 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 1 & k = 3. \end{cases}$$

Solution

$$X[r] = \sum_{k=0}^{3} x[k] e^{-j(2\pi kr/4)}$$
$$= 2 + 3 \times e^{-j(2\pi r/4)} - 1 \times e^{-j(2\pi(2)r/4)} + 1 \times e^{-j(2\pi(3)r/4)},$$

for $0 \le r \le 3$. Substituting different values of r, we obtain

$$r = 0 \quad X[0] = 2 + 3 - 1 + 1 = 5;$$
$$r = 1 \quad X[1] = 2 + 3e^{-j(2\pi/4)} - e^{-j(2\pi(2)/4)} + e^{-j(2\pi(3)/4)}$$
$$= 2 + 3(-j) - 1(-1) + 1(j) = 3 - 2j;$$
$$r = 2 \quad X[2] = 2 + 3e^{-j(2\pi(2)/4)} - e^{-j(2\pi(2)(2)/4)} + e^{-j(2\pi(3)(2)/4)}$$
$$= 2 + 3(-1) - 1(1) + 1(-1) = -3;$$
$$r = 3 \quad X[3] = 2 + 3e^{-j(2\pi(3)/4)} - e^{-j(2\pi(2)(3)/4)} + e^{-j(2\pi(3)(3)/4)}$$
$$= 2 + 3(j) - 1(-1) + 1(-j) = 3 + j2.$$

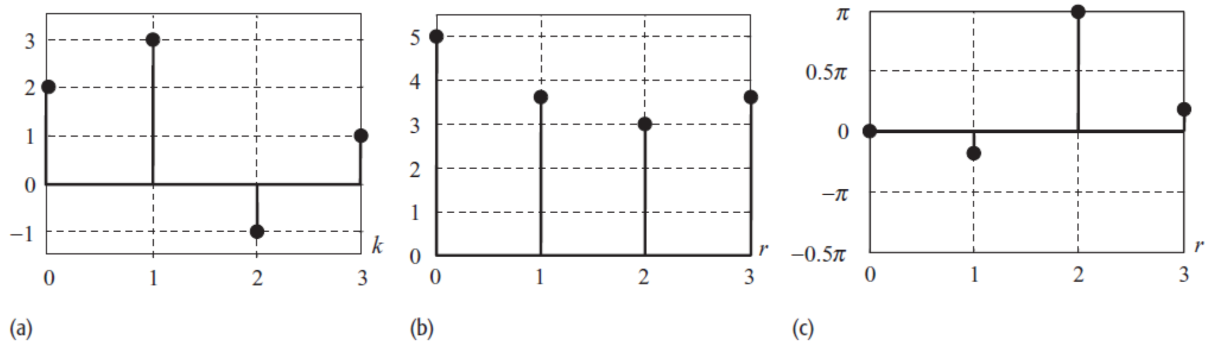

Fig. 1. (a) DT sequence *x*[*k*];(b) magnitude spectrum and (c) phase spectrum for example one

Example Two

- Calculate the 4-point DFT of the sequence, x(n)={1,0,0,1}

Sol^n.

For, k=0,

$$X(0)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 0n}{N}}$$

$$=x(0).e^{\frac{-j2\pi.0.0}{4}}+x(1).e^{\frac{-j2\pi.0.1}{4}}+x(2).e^{\frac{-j2\pi.0.2}{4}}+x(3).e^{\frac{-j2\pi.0.3}{4}}$$

$$=1.e^{\frac{-j2\pi.0.0}{4}}+0+0+1.e^{\frac{-j2\pi.0.3}{4}}$$

$$=1.e^{0}+0+0+1.e^{0}$$

$$=1+1$$

$$=2$$

For k=1,

$$X(1)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 1n}{N}}$$

$$=x(0).e^{\frac{-j2\pi.1.0}{4}}+x(1).e^{\frac{-j2\pi.1.1}{4}}+x(2).e^{\frac{-j2\pi.1.2}{4}}+x(3).e^{\frac{-j2\pi.1.3}{4}}$$

$$=1.e^{\frac{-j2\pi.1.0}{4}}+0+0+1.e^{\frac{-j2\pi.1.3}{4}}$$

$$=1+1.e^{\frac{-j2\pi.1.3}{4}}=1+[e^{(\frac{-j3\pi}{2})}]=1+[e^{-j270}]$$

$$[\because \pi\ rad=180°, \therefore \frac{3\pi}{2}rad=\frac{3x180}{2}=270°$$

$$=1+[\cos(270°)-j\sin(270°)]$$

$$[\therefore e^{-j\theta}=\cos\theta-j\sin\theta\ (Euler's Formula)$$

$$=1+[0-(-j1)]$$

$$=1+j$$

$$X(2)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 2n}{4}}$$

$$= x(0).\mathrm{e}^{\frac{-j2\pi 2.0}{4}} + x(1)\mathrm{e}^{\frac{-j2\pi 2.1}{4}} + x(2)\mathrm{e}^{\frac{-j2\pi 2.2}{4}} + x(3)\mathrm{e}^{\frac{-j2\pi .2.3}{4}}$$

$$= 1.e^{0} + 0.e^{-j\pi} + 0.e^{-j2\pi} + 1.e^{-j3\pi}$$

$$= 1 + \left[e^{-j3\pi}\right]$$

$$\therefore 3\pi = \pi$$

$$= 1 + [\cos\pi - j\sin\pi]$$

$$= 1 + (-1 - 0) = 0$$

$$X(3) = \sum_{n=0}^{3} x(\mathrm{n})\mathrm{e}^{\frac{-j2\pi 3n}{4}}$$

$$= x(0).\mathrm{e}^{\frac{-j2\pi 3.0}{4}} + x(1)\mathrm{e}^{\frac{-j2\pi 3.1}{4}} + x(2)\mathrm{e}^{\frac{-j2\pi 3.2}{4}} + x(3)\mathrm{e}^{\frac{-j2\pi .3.3}{4}}$$

$$= 1.e^{0} + 0.e^{\frac{-j3\pi}{2}} + 0.e^{-j3\pi} + 1.e^{\frac{-j9\pi}{2}}$$

$$= 1 + \left[e^{-j\frac{\pi}{2}}\right]$$

$$1 + \left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$$

$$= 1 + (0 - j1) = 1 - j$$

$$Now \quad X(k) = [x(0), x(1), x(2), x(3)]$$

$$X(k) = [2, 1 + j, 0, 1 - j]$$

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Example Three

Calculate the IDFT of
$$X[r] = \begin{cases} 5 & r = 0 \\ 3 - j2 & r = 1 \\ -3 & r = 2 \\ 3 + j2 & r = 3. \end{cases}$$

Solution

$$x[k] = \frac{1}{4}\sum_{r=0}^{3} X[r]e^{j(2\pi kr/4)} = \frac{1}{4}\left[5 + (3 - j2) \times e^{j(2\pi k/4)} - 3 \times e^{j(2\pi(2)k/4)} + (3 + j2) \times e^{j(2\pi(3)k/4)}\right],$$

for $0 \le k \le 3$. On substituting different values of k, we obtain

$$x[0] = \frac{1}{4}[5 + (3 - j2) - 3 + (3 + j2)] = 2;$$

$$x[1] = \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi/4)} - 3e^{j(2\pi(2)/4)} + (3 + j2)e^{j(2\pi(3)/4)}\right]$$
$$= \frac{1}{4}[5 + (3 - j2)(j) - 3(-1) + (3 + j2)(-j)] = 3;$$
$$x[2] = \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi(2)/4)} - 3e^{j(2\pi(2)(2)/4)} + (3 + j2)e^{j(2\pi(3)(2)/4)}\right]$$
$$= \frac{1}{4}[5 + (3 - j2)(-1) - 3(1) + (3 + j2)(-1)] = -1;$$
$$x[3] = \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi(3)/4)} - 3e^{j(2\pi(2)(3)/4)} + (3 + j2)e^{j(2\pi(3)(3)/4)}\right]$$
$$= \frac{1}{4}[5 + (3 - j2)(-j) - 3(-1) + (3 + j2)(j)] = 1.$$

Examples 12.1 and 12.2 prove the following DFT pair:

$$x[k] = \begin{cases} 2 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 1 & k = 3 \end{cases} \xleftrightarrow{\text{DFT}} X[r] = \begin{cases} 5 & r = 0 \\ 3 - j2 & r = 1 \\ -3 & r = 2 \\ 3 + j2 & r = 3, \end{cases}$$

where both the DT sequence $x[k]$ and its DFT $X[r]$ have length $N = 4$.

Matrix Multiplication

An alternative representation for computing the DFT is matrix multiplication.

The above DFT and IDFT are obtained by putting $e^{-j2\pi/N} = W_N$ in equation (1)

x_N

and equation (2). Let us represents sequence *x(n)* as vector of N samples

$$x_N = \begin{array}{r} n=0 \\ n=1 \\ \vdots \\ \vdots \\ n=N-1 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

And *X(k)* can be represented as a vector of x_N N samples

$$X_N = \begin{array}{r} k=0 \\ k=1 \\ \vdots \\ \vdots \\ k=N-1 \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1}$$

$[W_N]$

The values of can be represented as a matrix of size $N \times N$ as follows:

$$[W_N] = \begin{array}{r} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N} \end{array} \qquad \text{......................} \ (5)$$

Here the individual elements are written as with "k" rows and "n" columns. Then N – point DFT of equation (3) can be represented as

$$X_N = [W_N]\, x_N \qquad \text{......................} \ (6)$$

Similarly IDFT of equation (4) can be expressed in matrix form as,

$$x_N = \frac{1}{N}\left[W_N^*\right]X_N$$

$W_N^{kn} = [W_N]$, hence $\left[W_N^*\right] = W_N^{-kn}$ as written above.

Or in other expression matrix vector format are given by:

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT vector } \vec{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j(2\pi/N)} & e^{-j(4\pi/N)} & \cdots & e^{-j(2(N-1)\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & \cdots & e^{-j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(2(N-1)\pi/N)} & e^{-j(4(N-1)\pi/N)} & \cdots & e^{-j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{DFT matrix } F} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{signal vector } \vec{x}}.$$

Similary, the expression for IDFT given by:

$$\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{signal vector } x} = \frac{1}{N}\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j(2\pi/N)} & e^{j(4\pi/N)} & \cdots & e^{j(2(N-1)\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & \cdots & e^{j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(2(N-1)\pi/N)} & e^{j(4(N-1)\pi/N)} & \cdots & e^{j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{DFT matrix } G=F^{-1}} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT vector } x},$$

Periodicity property of W_N

Let us see the values of W_N for N=8.

We know that W_N is given as,

$$W_N = e^{-j\frac{2\pi}{N}}$$

With N=8 above equation becomes

$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

Table below shows values of $W_8^0, W_8^1, W_8^2, \ldots W_8^{15}$

The values of these phasors are observe that,

$$W_8^1 = W_8^9 = \dots = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Example Four

Calculate the four-point DFT of the aperiodic signal $x[k]$ considered in Example one.

Solution

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/N)} & e^{-j(4\pi/N)} & e^{-j(6\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & e^{-j(12\pi/N)} \\ 1 & e^{-j(6\pi/N)} & e^{-j(12\pi/N)} & e^{-j(18\pi/N)} \end{bmatrix}}_{\text{DFT matrix: } F} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/4)} & e^{-j(4\pi/4)} & e^{-j(6\pi/4)} \\ 1 & e^{-j(4\pi/4)} & e^{-j(8\pi/4)} & e^{-j(12\pi/4)} \\ 1 & e^{-j(6\pi/4)} & e^{-j(12\pi/4)} & e^{-j(18\pi/4)} \end{bmatrix}}_{\text{DFT matrix: } F} \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 - j2 \\ -3 \\ 3 + j2 \end{bmatrix}.$$

Example Five

Calculate the inverse DFT of $X[r]$ considered in Example two.

Solution

Arranging the values of the DFT coefficients in the DFT vector X, we obtain

$X = [5\ 3 - j2\ {-3}\ \ 3 + j2]$.

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/N)} & e^{j(4\pi/N)} & e^{j(6\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & e^{j(12\pi/N)} \\ 1 & e^{j(6\pi/N)} & e^{j(12\pi/N)} & e^{j(18\pi/N)} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/4)} & e^{j(4\pi/4)} & e^{j(6\pi/4)} \\ 1 & e^{j(4\pi/4)} & e^{j(8\pi/4)} & e^{j(12\pi/4)} \\ 1 & e^{j(6\pi/4)} & e^{j(12\pi/4)} & e^{j(18\pi/4)} \end{bmatrix}\begin{bmatrix} 5 \\ 3 - j2 \\ -3 \\ 3 + j2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 12 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

The above values for the DT sequence x[k] are the same as the ones obtained in Example two.

Properties of the DFT

- **Linearity**

If $x1[k]$ and $x2[k]$ are two DT sequences with the following M-point DFT pairs:

$x1[k] \underset{DFT}{\longleftrightarrow} X1[r]$ and $x2[k] \underset{DFT}{\longleftrightarrow} X2[r]$,

then the linearity property states that

$a1x1[k] + a2x2[k] \underset{DFT}{\longleftrightarrow} a1X1[r] + a2X2[r$.

for any arbitrary constants $a1$ and $a2$.

- **Time shifting**

If $x[k] \overset{DFT}{\longleftrightarrow} X[r]$, then

$$x[k-k_0] \overset{DFT}{\longleftrightarrow} e^{-j2\pi k_0 r/M} X[r]$$

for an M-point DFT and any arbitrary integer k_0.

Circular convolution

If $x_1[k]$ and $x_2[k]$ are two DT sequences with the following M-point DFT pairs:

$x_1[k] \overset{DFT}{\longleftrightarrow} X_1[r]$ and $x_2[k] \overset{DFT}{\longleftrightarrow} X_2[r]$,

then the circular convolution property states that

$$x_1[k] \otimes x_2[k] \overset{DFT}{\longleftrightarrow} X_1[r]X_2[r] \quad (12.27)$$

and

$$x_1[k]x_2[k] \overset{DFT}{\longleftrightarrow} [X_1[r] \otimes X_2[r]],$$

where $\otimes$ denotes the circular convolution operation. Note that the two sequences must have the same length in order to compute the circular convolution.

Find the circular convolution between

x[n]=[1,2,3,4]

x[n]=[4,3,2,1]

$$y[n] = \sum_{m=0}^{3} x[m]h[n-m]$$

$$y[0] = \sum_{m=0}^{3} x[m]h[-m]$$

$$= x[0]h[0] + x[1]h[-1] + x[2]h[-2] + x[3]h[-3]$$

$$= 1 \times 4 + 2 \times 1 + 3 \times 2 + 4 \times 3$$

$$= 4 + 2 + 6 + 12 = 24$$

$$y[1] = \sum_{m=0}^{3} x[m]h[1-m]$$

$$= x[0]h[1] + x[1]h[0] + x[2]h[-1] + x[3]h[-2]$$

$$= 1 \times 3 + 2 \times 4 + 3 \times 1 + 4 \times 2$$

$$= 3 + 8 + 3 + 8 = 22$$

$$y[2] = \sum_{m=0}^{3} x[m]h[2-m]$$

$$= x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[-1]$$

$$= 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 1$$

$$= 2 + 6 + 12 + 4 = 24$$

$$y[3] = \sum_{m=0}^{3} x[m]h[3-m]$$

$$= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$= 1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 = 30$$

Cross Correlation Property

Cross-correlation between two discrete signals measures the similarity between them as a function of the time shift applied to one of the signals. It helps identify how well one signal matches with another when one of them is shifted by various amounts. If x(n) and y(n) ae two discrete signals, cross coreelation can be calculated :

r(n)= IDFT(X(k)Y*(k))

In r(n) , find the index that has the highest positive amplitude. That index represents the shift amount. In time domain,

r(n) is the circular convolution of x(n) and y*(-n)