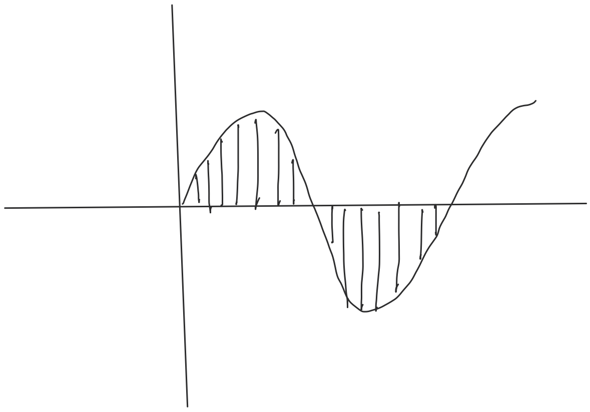
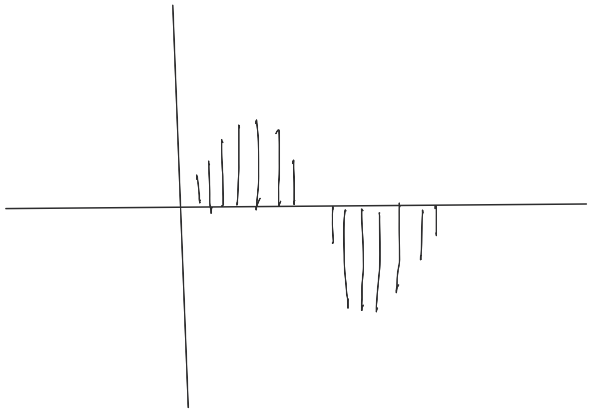
Lecture 9
ELE 301: Signals and Systems

Discrete Fourier Transform



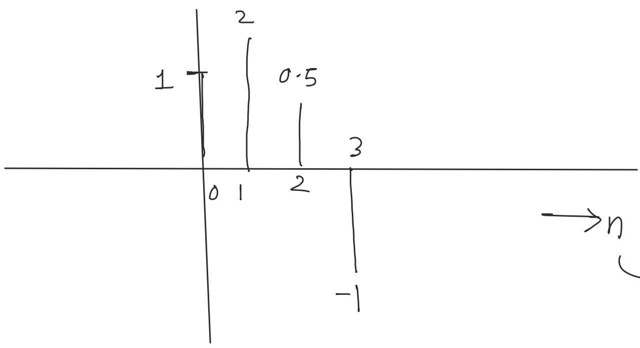
sample যত কাছে নিবো তত ভালো।
দূরে দূরে নিলে → let শুধু peak গুলো নিলাম তাহলে constant function এর shape পাবো। পুরো shape change হয়ে গেল।



যতগুলো point দেয়া থাকে → তত point DFT

sample

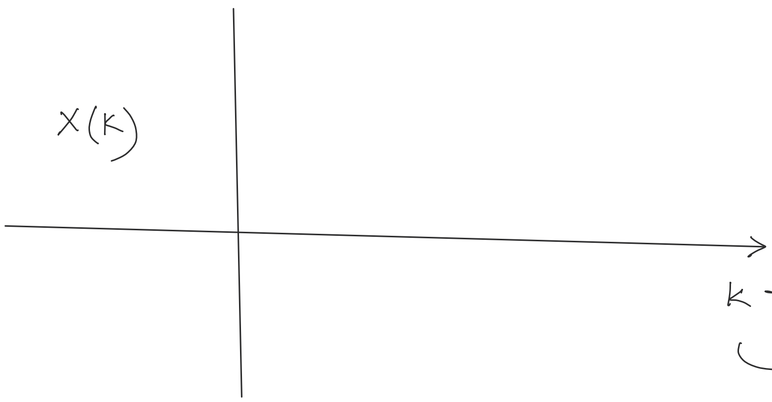
N দিয়ে বোঝায়



4 point DFT

→ n

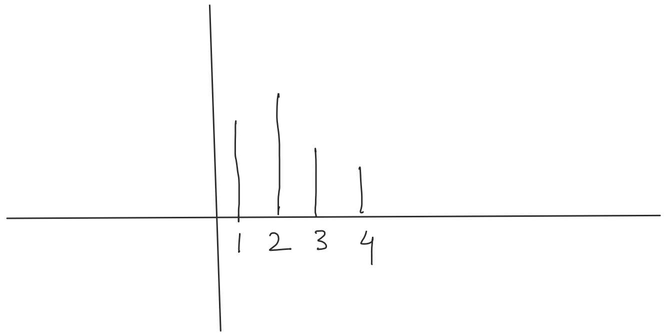
↪ 0 to N−1



K →

↪ 0 to N−1 range

K → freq. domain এর sample range same থাকে। যতগুলো time domain value freq. domain এও ততগুলো পাবো।



time domain এ sample given

corresponding freq domain amplitude:

$$X(k) = \sum x(n)\, e^{-j\frac{2\pi}{N}(kn)}$$

question এ থাকবে

sum এর range — যতগুলো point নিয়ে কাজ করছি

0 to N-1

⇓

শুরু হয় সবসময় 0 দিয়ে

K → freq domain কে যার সাপেক্ষে sample

ultimately K এর function থাকবে

$X(0) =$

$X(1) =$

↑

K এর value

0 to N-1 পর্যন্ত

$x[n]$ / $x(n)$ / $x(k)$ → discrete
$x(t)$ → continuous

Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T (i.e. a finite sequence of data). The transformation of discrete data between the time and frequency domain is quite useful in extracting information from the signal.

The DFT is denoted by $X(k)$ and given as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots N-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Here $X(k)$ is the DFT and it is computed at $k=0,1,2,\ldots.\ N-1$. "N" discrete points. Thus DFT $X(k)$ is the sequence of N samples. The sequence $x(n)$ can be obtained back from $X(k)$ by taking Inverse Discrete Fourier Transform, i.e. IDFT is given as,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad n = 0, 1, 2, \ldots. N-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Here $x(n)$ is sequence of N samples . thus $X(k)$ and $x(n)$ both contains N number of samples.

Let us define, $\boxed{W_N = e^{-j2\pi/N}}$

This is called twiddle factor. Hence DFT and IDFT equation can be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \qquad k = 0, 1, \ldots\ldots N-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

And

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \qquad n = 0, 1, \ldots.. N-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Example One

Calculate the four-point DFT of the aperiodic sequence $x[k]$ of length $N = 4$, which is defined as follows:

$$x[k] = \begin{cases} 2 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 1 & k = 3. \end{cases}$$

K দিয়ে time domain sample

$N=4$

Solution

r → freq. domain sample

$$X[r] = \sum_{k=0}^{3} x[k]e^{-j(2\pi kr/4)}$$

$$\sum_{K=0}^{3} x(K)\, e^{-j\frac{2\pi}{N}(Kr)}$$

$$= 2 + 3 \times e^{-j(2\pi r/4)} - 1 \times e^{-j(2\pi(2)r/4)} + 1 \times e^{-j(2\pi(3)r/4)},$$

for $0 \le r \le 3$. Substituting different values of r, we obtain

freq domain এর amplitude

$$r = 0 \quad X[0] = 2 + 3 - 1 + 1 = 5;$$

$$r = 1 \quad X[1] = 2 + 3e^{-j(2\pi/4)} - e^{-j(2\pi(2)/4)} + e^{-j(2\pi(3)/4)}$$

$$= 2 + 3(-j) - 1(-1) + 1(j) = 3 - 2j;$$

$$r = 2 \quad X[2] = 2 + 3e^{-j(2\pi(2)/4)} - e^{-j(2\pi(2)(2)/4)} + e^{-j(2\pi(3)(2)/4)}$$

$$= 2 + 3(-1) - 1(1) + 1(-1) = -3;$$

$$r = 3 \quad X[3] = 2 + 3e^{-j(2\pi(3)/4)} - e^{-j(2\pi(2)(3)/4)} + e^{-j(2\pi(3)(3)/4)}$$

$$= 2 + 3(j) - 1(-1) + 1(-j) = 3 + j2.$$

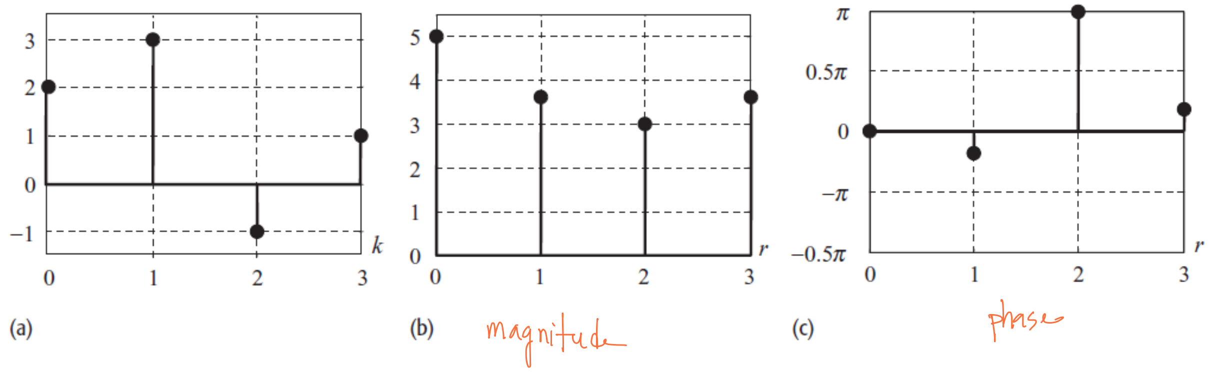


Fig. 1. (a) DT sequence $x[k]$;(b) magnitude spectrum and (c) phase spectrum for example one

Example Two

- Calculate the 4-point DFT of the sequence, x(n)={1,0,0,1}

4 point DFT

Sol^n.

For, k=0,

$$X(0)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 0n}{N}}$$

$$=x(0).e^{\frac{-j2\pi.0.0}{4}}+x(1).e^{\frac{-j2\pi.0.1}{4}}+x(2).e^{\frac{-j2\pi.0.2}{4}}+x(3).e^{\frac{-j2\pi.0.3}{4}}$$

$$=1.e^{\frac{-j2\pi.0.0}{4}}+0+0+1.e^{\frac{-j2\pi.0.3}{4}}$$

$$=1.e^{0}+0+0+1.e^{0}$$

$$=1+1$$

$$=2$$

For k=1,

$$X(1)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 1n}{N}}$$

$$=x(0).e^{\frac{-j2\pi.1.0}{4}}+x(1).e^{\frac{-j2\pi.1.1}{4}}+x(2).e^{\frac{-j2\pi.1.2}{4}}+x(3).e^{\frac{-j2\pi.1.3}{4}}$$

$$=1.e^{\frac{-j2\pi.1.0}{4}}+0+0+1.e^{\frac{-j2\pi.1.3}{4}}$$

$$=1+1.e^{\frac{-j2\pi.1.3}{4}}=1+[e^{(\frac{-j3\pi}{2})}]=1+[e^{-j270}]$$

$$[\because \pi\ rad=180°, \therefore \frac{3\pi}{2}rad=\frac{3x180}{2}=270°$$

$$=1+[\cos(270°)-j\sin(270°)]$$

$$[\therefore e^{-j\theta}=\cos\theta-j\sin\theta\ (Euler's Formula)$$

$$=1+[0-(-j1)]$$

$$=1+j$$

$$X(2)=\sum_{n=0}^{3}x(n)e^{\frac{-j2\pi 2n}{4}}$$

$$= x(0).\mathrm{e}^{\frac{-\mathrm{j}2\pi 2.0}{4}} + x(1)\mathrm{e}^{\frac{-\mathrm{j}2\pi 2.1}{4}} + x(2)\mathrm{e}^{\frac{-\mathrm{j}2\pi 2.2}{4}} + x(3)\mathrm{e}^{\frac{-\mathrm{j}2\pi.2.3}{4}}$$

$$= 1.e^{0} + 0.e^{-j\pi} + 0.e^{-j2\pi} + 1.e^{-j3\pi}$$

$$= 1 + \left[e^{-j3\pi}\right]$$

$$\therefore 3\pi = \pi$$

$$= 1 + [\cos\pi - j\sin\pi]$$

$$= 1 + (-1 - 0) = 0$$

$$\mathrm{X}(3) = \sum_{n=0}^{3} x(\mathrm{n})\mathrm{e}^{\frac{-\mathrm{j}2\pi 3\mathrm{n}}{4}}$$

$$= x(0).\mathrm{e}^{\frac{-\mathrm{j}2\pi 3.0}{4}} + x(1)\mathrm{e}^{\frac{-\mathrm{j}2\pi 3.1}{4}} + x(2)\mathrm{e}^{\frac{-\mathrm{j}2\pi 3.2}{4}} + x(3)\mathrm{e}^{\frac{-\mathrm{j}2\pi.3.3}{4}}$$

$$= 1.e^{0} + 0.e^{\frac{-j3\pi}{2}} + 0.e^{-j3\pi} + 1.e^{\frac{-j9\pi}{2}}$$

$$= 1 + \left[e^{-j\frac{\pi}{2}}\right]$$

$$1 + \left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$$

$$= 1 + (0 - j1) = 1 - j$$

$$Now \quad X(k) = [x(0), x(1), x(2), x(3)]$$

$$X(k) = [2, 1 + j, 0, 1 - j]$$

ث

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)(kn)}$$ N point sample

conjugate

Inverse

$X(k)$ to $x(n) \longrightarrow$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\left(\frac{2\pi}{N}\right)(kn)}$$

number of samples

$$X[r] = \begin{cases} 5 & r = 0 \\ 3 - j2 & r = 1 \\ -3 & r = 2 \\ 3 + j2 & r = 3. \end{cases}$$

$N = 4$

$$x(k) = \frac{1}{4} \sum_{r=0}^{\overset{3}{N-1}} x[r]\, e^{j(2\pi k r/4)}$$

$$= \frac{1}{4}\left[5 + (3-2j)\, e^{j2\pi k/4} - 3\, e^{j2\pi \cdot 2/4} + (3+2j)\, e^{j(2\pi(3)k/4)}\right]$$

K এর function পাবো

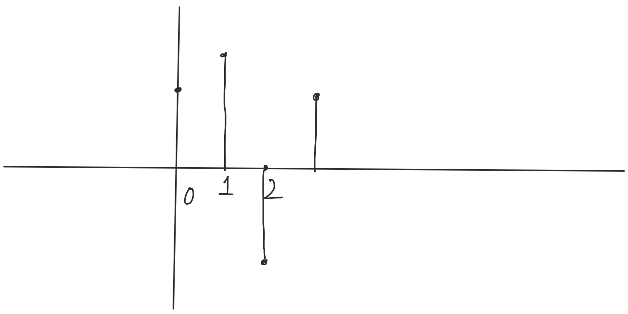
$0 \le K \le 3$

$$x[0] = \left(5 + (3-2j) - 3 + 3 + 2j\right) \times \frac{1}{4}$$

$$= 8 \times \frac{1}{4}$$

$$= 2$$

time domain এ সবগুলো real value পাবো। freq. domain এ complex আসতে পারে।



Example Three

Calculate the IDFT of

$$X[r] = \begin{cases} 5 & r = 0 \\ 3 - j2 & r = 1 \\ -3 & r = 2 \\ 3 + j2 & r = 3. \end{cases}$$

$x(k) \longrightarrow X(r)$

$x(n) \longrightarrow X(k)$

time domain   freq. domain

Solution

$$x[k] = \frac{1}{4}\sum_{r=0}^{3} X[r]e^{j(2\pi kr/4)} = \frac{1}{4}\left[5 + (3 - j2) \times e^{j(2\pi k/4)} - 3 \times e^{j(2\pi(2)k/4)} + (3 + j2) \times e^{j(2\pi(3)k/4)}\right],$$

for $0 \le k \le 3$. On substituting different values of k, we obtain

$$x[0] = \frac{1}{4}[5 + (3 - j2) - 3 + (3 + j2)] = 2;$$

$$\begin{aligned} x[1] &= \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi/4)} - 3e^{j(2\pi(2)/4)} + (3 + j2)e^{j(2\pi(3)/4)}\right] \\ &= \frac{1}{4}[5 + (3 - j2)(j) - 3(-1) + (3 + j2)(-j)] = 3; \\ x[2] &= \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi(2)/4)} - 3e^{j(2\pi(2)(2)/4)} + (3 + j2)e^{j(2\pi(3)(2)/4)}\right] \\ &= \frac{1}{4}[5 + (3 - j2)(-1) - 3(1) + (3 + j2)(-1)] = -1; \\ x[3] &= \frac{1}{4}\left[5 + (3 - j2)e^{j(2\pi(3)/4)} - 3e^{j(2\pi(2)(3)/4)} + (3 + j2)e^{j(2\pi(3)(3)/4)}\right] \\ &= \frac{1}{4}[5 + (3 - j2)(-j) - 3(-1) + (3 + j2)(j)] = 1. \end{aligned}$$

Examples 12.1 and 12.2 prove the following DFT pair:

$$x[k] = \begin{cases} 2 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 1 & k = 3 \end{cases} \overset{\text{DFT}}{\longleftrightarrow} X[r] = \begin{cases} 5 & r = 0 \\ 3 - j2 & r = 1 \\ -3 & r = 2 \\ 3 + j2 & r = 3, \end{cases}$$

where both the DT sequence $x[k]$ and its DFT $X[r]$ have length $N = 4$.

Matrix multiplication:

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(Kn)}$$

N point হবে
0 to N-1

$0 \xrightarrow{n} N-1$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{array}{c} 0 \\ K \downarrow \\ N-1 \end{array} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ \vdots & W_N^2 & W_N^4 \cdots & W_N^{2N-2} \\ \vdots & & & \vdots \\ 1 & & & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

N×1 matrix থাবো   N×N   N×1

$$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

$$\therefore W_N^{Kn} = e^{-j\frac{2\pi}{N}\cdot Kn}$$

4 point বলা হলে → 4×4 matrix হবে।

যদি 4 point এর জন্য → matrix টা same থাকবে।
N×N

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

→ value গুলো constant হবে

$$W_N^{(nK)} = W_{\frac{N}{n}}^{K}$$

$$W_8^{4\times 3} = W_2^3$$

property-1

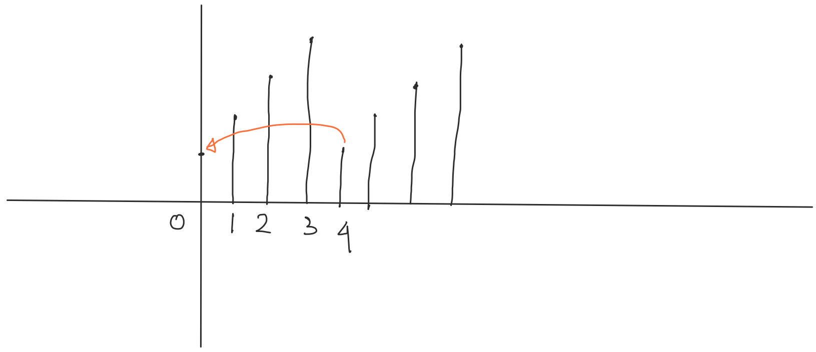
$$e^{-j\left(\frac{2\pi}{N}\right)(nK)}$$

$$= e^{-j\left(\frac{2\pi}{\frac{N}{n}}\right)(K)}$$

$$= W_{\frac{N}{n}}^{K}$$

property 2 : periodic property

DFT inherently periodic হিসেবে calculate করে।



$W_N^K = W_N^{K+N}$ N কে period হিসেবে চিন্তা করে

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \begin{array}{c} \\ 0 \\ \downarrow \\ N-1 \end{array} \begin{array}{c} 0 \longrightarrow N-1 \\ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-2(N-1)} \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \\ \vdots & & & & \\ 1 & & & & \end{bmatrix} \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

Complexity $O(N^2)$

→ multiplication টার

— runtime improvement হলো না। আগেও same ছিল

আগেরটার conjugate পাবো

$x(n) \longrightarrow X(K)$

power minus করে দিলেই হবে।

Matrix Multiplication — calculation parallaly করা যাবে।

An alternative representation for computing the DFT is matrix multiplication.

The above DFT and IDFT are obtained by putting $e^{-j2\pi/N} = W_N$ in equation (1)

x_N

and equation (2). Let us represents sequence *x(n)* as vector of N samples

$$x_N = \begin{array}{r} n=0 \\ n=1 \\ \vdots \\ \vdots \\ n=N-1 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

And *X(k)* can be represented as a vector of x_N N samples

$$X_N = \begin{array}{r} k=0 \\ k=1 \\ \vdots \\ \vdots \\ k=N-1 \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1}$$

$[W_N]$

The values of can be represented as a matrix of size $N \times N$ as follows:

$$[W_N] = \begin{array}{r} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N} \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Here the individual elements are written as with "k" rows and "n" columns. Then N – point DFT of equation (3) can be represented as

$$\boxed{X_N = [W_N]\, x_N} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Similarly IDFT of equation (4) can be expressed in matrix form as,

$$\boxed{x_N = \frac{1}{N}\left[W_N^*\right]X_N}$$

$W_N^{kn} = [W_N]$, hence $\left[W_N^*\right] = W_N^{-kn}$ as written above.

Or in other expression matrix vector format are given by:

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT vector } \vec{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overset{W_N}{e^{-j(2\pi/N)}} & \overset{W_N^2}{e^{-j(4\pi/N)}} & \cdots & e^{-j(2(N-1)\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & \cdots & e^{-j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(2(N-1)\pi/N)} & e^{-j(4(N-1)\pi/N)} & \cdots & e^{-j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{DFT matrix } F} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{signal vector } \vec{x}}.$$

Similary, the expression for IDFT given by:

$$\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{signal vector } x} = \underbrace{\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j(2\pi/N)} & \overset{W_N^{-2}}{e^{j(4\pi/N)}} & \cdots & e^{j(2(N-1)\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & \cdots & e^{j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(2(N-1)\pi/N)} & e^{j(4(N-1)\pi/N)} & \cdots & e^{j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{DFT matrix } G=F^{-1}} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT vector } X},$$

Periodicity property of W_N

Let us see the values of W_N for N=8.

We know that W_N is given as,

$$W_N = e^{-j\frac{2\pi}{N}}$$

With N=8 above equation becomes

$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

Table below shows values of $W_8^0, W_8^1, W_8^2, \dots W_8^{15}$

The values of these phasors are observe that,

$$W_8^1 = W_8^9 = \dots = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Example Four

Calculate the four-point DFT of the aperiodic signal $x[k]$ considered in Example one.

Solution

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/N)} & e^{-j(4\pi/N)} & e^{-j(6\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & e^{-j(12\pi/N)} \\ 1 & e^{-j(6\pi/N)} & e^{-j(12\pi/N)} & e^{-j(18\pi/N)} \end{bmatrix}}_{\text{DFT matrix: } F} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/4)} & e^{-j(4\pi/4)} & e^{-j(6\pi/4)} \\ 1 & e^{-j(4\pi/4)} & e^{-j(8\pi/4)} & e^{-j(12\pi/4)} \\ 1 & e^{-j(6\pi/4)} & e^{-j(12\pi/4)} & e^{-j(18\pi/4)} \end{bmatrix}}_{\text{DFT matrix: } F} \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3-j2 \\ -3 \\ 3+j2 \end{bmatrix}.$$

Example Five

Calculate the inverse DFT of $X[r]$ considered in Example two.

Solution

Arranging the values of the DFT coefficients in the DFT vector X, we obtain

$X = [5\ 3-j2\ -3\ \ 3+j2]$.

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/N)} & e^{j(4\pi/N)} & e^{j(6\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & e^{j(12\pi/N)} \\ 1 & e^{j(6\pi/N)} & e^{j(12\pi/N)} & e^{j(18\pi/N)} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/4)} & e^{j(4\pi/4)} & e^{j(6\pi/4)} \\ 1 & e^{j(4\pi/4)} & e^{j(8\pi/4)} & e^{j(12\pi/4)} \\ 1 & e^{j(6\pi/4)} & e^{j(12\pi/4)} & e^{j(18\pi/4)} \end{bmatrix}\begin{bmatrix} 5 \\ 3-j2 \\ -3 \\ 3+j2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 12 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

The above values for the DT sequence x[k] are the same as the ones obtained in Example two.

Properties of the DFT

- **Linearity**

If $x1[k]$ and $x2[k]$ are two DT sequences with the following M-point DFT pairs:

$x1[k] \underset{DFT}{\longleftrightarrow} X1[r]$ and $x2[k] \underset{DFT}{\longleftrightarrow} X2[r]$,

$A x_1(n) + B x_2(n)$

$\rightarrow A\ X_1(K) + B\ X_2(K)$ } DFT

then the linearity property states that

$a1x1[k] + a2x2[k] \underset{DFT}{\longleftrightarrow} a1X1[r] + a2X2[r]$.

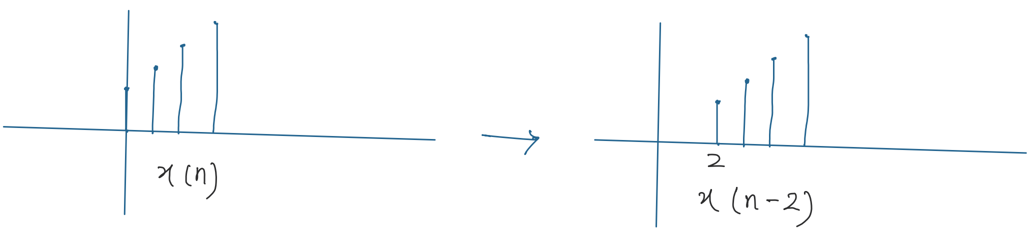
$A x_1(t) + B x_2(t)$

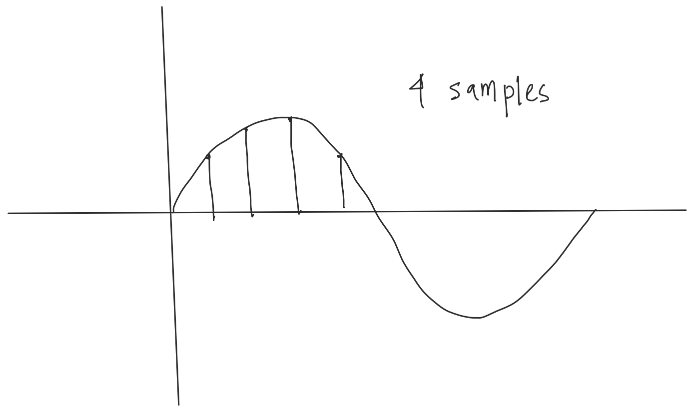
$\rightarrow A\ X_1(f) + B\ X_2(f)$

for any arbitrary constants $a1$ and $a2$.

- **Time shifting**

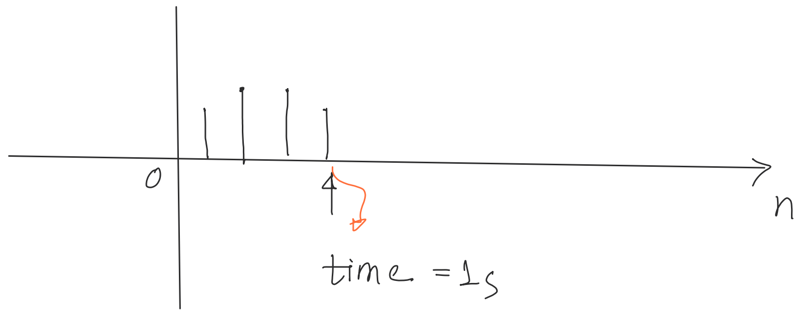
— sample shifting

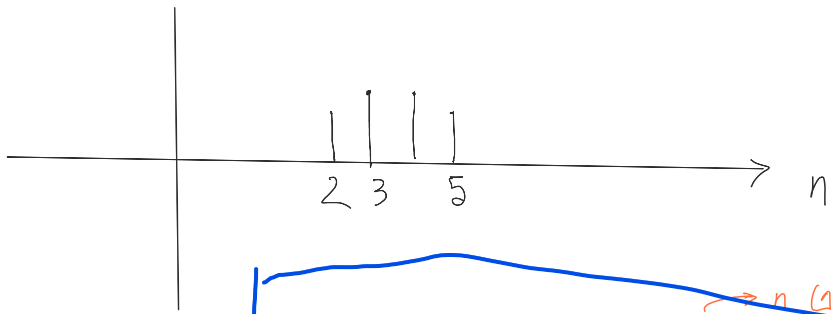




$\frac{2}{4} = \frac{1}{2}s$

time length





$$\boxed{\text{time difference} = \frac{2}{4}}$$

n의 shift

sample size

If $x[k] \underset{DFT}{\longleftrightarrow} X[r]$, then

$$x(t-t_0) = e^{-j2\pi f t_0} X(f)$$

DFT: $x(n-n_0) = e^{-j\frac{2\pi}{N}(k n_0)} X(k)$

$x(n+n_0) = e^{j\frac{2\pi}{N}(k n_0)} X(k)$

$$x[k-k0] \underset{DFT}{\longleftrightarrow} e^{-j2\pi k_0 r/M} X[r]$$

for an M-point DFT and any arbitrary integer $k0$.

Circular convolution

If $x1[k]$ and $x2[k]$ are two DT sequences with the following M-point DFT pairs:

$x1[k] \underset{DFT}{\longleftrightarrow} X1[r]$ and $x2[k] \underset{DFT}{\longleftrightarrow} X2[r]$,

then the circular convolution property states that

$$\otimes\ x1[k]\ x2[k] \underset{DFT}{\longleftrightarrow} X1[r]X2[r]\ (12.27)$$

and

$$x1[k]x2[k] \underset{DFT}{\longleftrightarrow} [X1[r] \otimes X2[r]],$$

where $\otimes$ denotes the circular convolution operation. Note that the two sequences must have the same length in order to compute the circular convolution.

Find the circular convolution between

x[n]=[1,2,3,4]

x[n]=[4,3,2,1]

$$y[n] = \sum_{m=0}^{3} x[m]h[n-m]$$

$$y[0] = \sum_{m=0}^{3} x[m]h[-m]$$

h[3] h[2] h[1] ⟹ value বসাবে

$$= x[0]h[0] + x[1]h[-1] + x[2]h[-2] + x[3]h[-3]$$

$$= 1\times 4 + 2\times 1 + 3\times 2 + 4\times 3$$

$$= 4 + 2 + 6 + 12 = 24$$

→ sample এ এই value গুলো ছিল না।
circular থেকে আসে।
-1 মানে একদম শেষেরটা।
0 to 3 তে map করে বসাবো।

$$y[1] = \sum_{m=0}^{3} x[m]h[1-m]$$

$$= x[0]h[1] + x[1]h[0] + x[2]h[-1] + x[3]h[-2]$$

$x_1(n) \otimes x_2(n)$

$X_1(K) \cdot X_2(K)$

$x_1(n)$, $x_2(n)$ } different signal

sample size diff

$[1 \quad 2 \quad 3 \quad 4]$

$[1 \quad 2 \quad 3] \longrightarrow [1 \quad 2 \quad 3 \quad 0]$

এখন 4 sample হবে।

$$= 1 \times 3 + 2 \times 4 + 3 \times 1 + 4 \times 2$$

$$= 3 + 8 + 3 + 8 = 22$$

$$y[2] = \sum_{m=0}^{3} x[m]h[2-m]$$

$$= x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[-1]$$

$$= 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 1$$

$$= 2 + 6 + 12 + 4 = 24$$

$$y[3] = \sum_{m=0}^{3} x[m]h[3-m]$$

$$= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$= 1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 = 30$$

Cross Correlation Property — offline

২টা signal থাকবে highest alignment কোথায় পাচ্ছি বের করা লাগবে।

Cross-correlation between two discrete signals measures the similarity between them as a function of the time shift applied to one of the signals. It helps identify how well one signal matches with another when one of them is shifted by various amounts. If x(n) and y(n) ae two discrete signals, cross coreelation can be calculated :

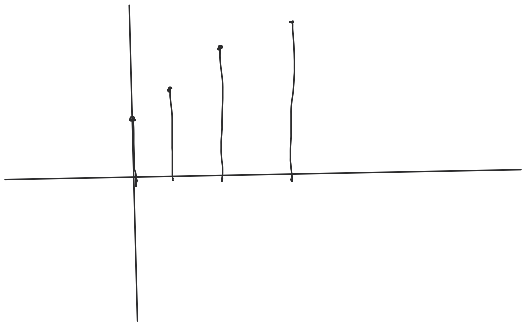
r(n)= IDFT(X(k)Y*(k))

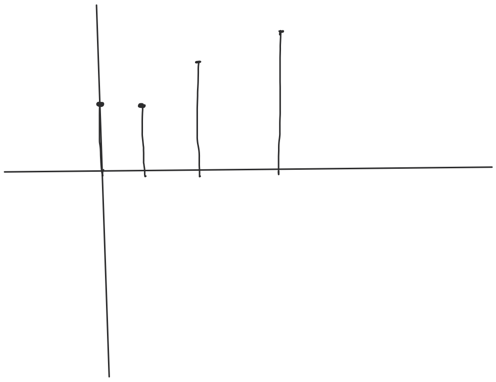
In r(n) , find the index that has the highest positive amplitude. That index represents the shift amount. In time domain,

r(n) is the circular convolution of x(n) and y*(-n)

$x_1(n)$ $x_2(n)$

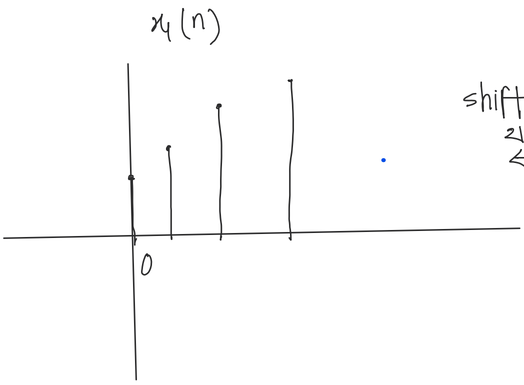
$x_3(n) \rightarrow x_1(n), x_2(n)$ এর মধ্যে similarity কেমন সেটা define করবে।



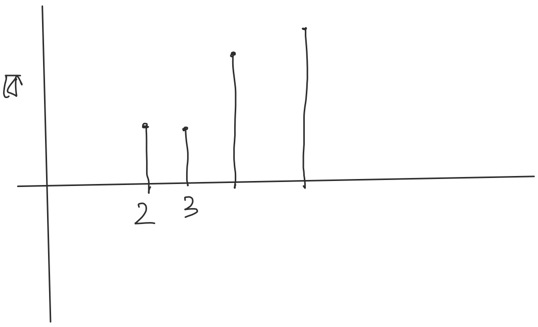


same না। but ২টাই non-decreasing

similarity



shift করতে থাকবো 1কে



signal এর alignment কতটুকু lag এ similar হবে সেটা বের করে — cross correlation.

↳ 2 lag পাবো

— time lag পাবো → difference in time

time length multiply করে ↳ distance পাবো

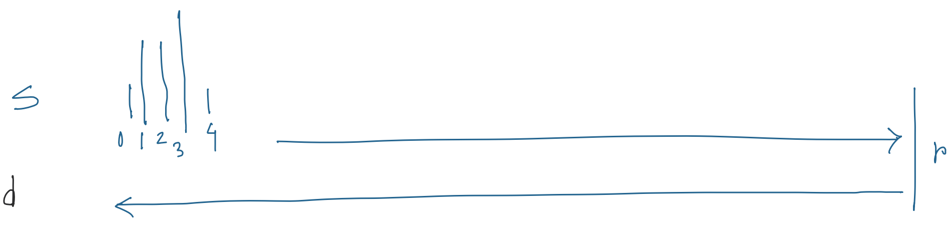
FFT → fast fourier transform

→ efficient algo.

কোনো lag ছাড়াই similarity বেশি → প্রায় same signal

3 lag → shift 3

sender
যেটা send
করে



receiver

d →
↳ distance আছে
so time delay থাকবে

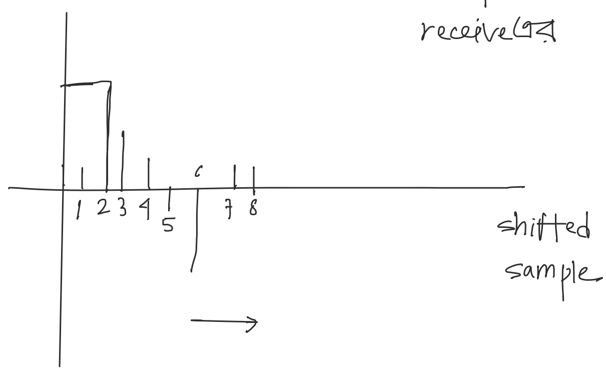
receiver
থেকে যেটা
পাই

↳ shape same
(noise নাই ধরে নিচ্ছি)

sample shift = 2

$$\frac{\text{sample shift}}{\text{sample rate}} = \text{time delay}$$

correlation score → send আর
receive এর



peak পাচ্ছি 2 এ

cross correlation
costly operation

$x_1(n) \otimes x_2^*(-n)$ circular convolution (cross correlation of $x_1(n)$ and $x_2(n)$)

send received

↳ সব real so
conjugate করা না করা same

$X_1(K) \times X_2^*(K)$

multiplication

conjugation
property

circular conjugation
property

$x(t) \longrightarrow X(f)$

$x^*(t) \longrightarrow X^*(-f)$

$X(K) = X_1(K) \times X_2^*(K)$

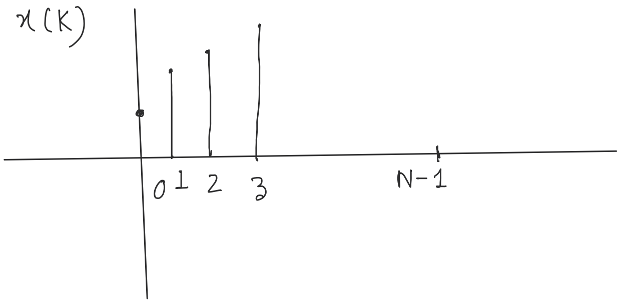
Conjugation
property

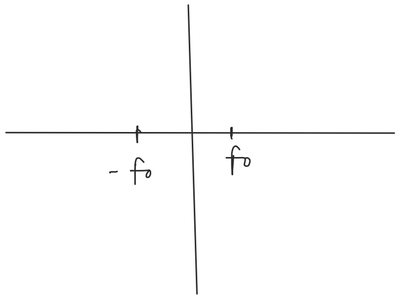
$x(n) \longrightarrow X(K)$

$x^*(n) \longrightarrow X^*(-K)$

$\underline{X(K)} \xrightarrow[\text{IDFT}]{}$ cross correlation পাবো

↳ cross correlation array
(N সংখ্যক value থাকবে)





(negative এও থাকা উচিত)

circular →

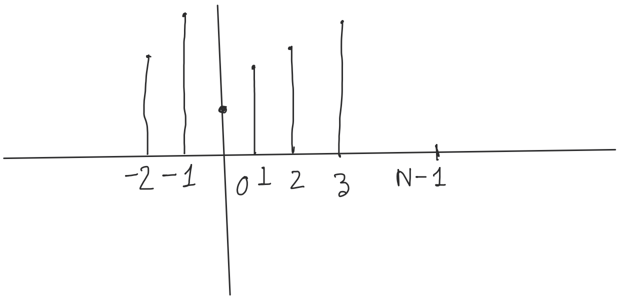
−1 এ থাকবে 3 এর value

frequency সবসময় pair এ থাকে

একটা positive side এ

অন্যটা negative side এ

0 এর respect এ symmetric হয়



$0 - \frac{N}{2}$ → positive হিসাবে treat

$\frac{N}{2} - N$ → negative

প্রতিটা index থেকে N minus করে যে index পাবো সেখানে ঐ value বসবে।

current index − total → negative side এ map হবে

$0 - \frac{N}{2} \longrightarrow i$

$\frac{N}{2}$ to $N \longrightarrow i-N$

- minus এ peak পাওয়া মানে advance

  → $x(t+t_0)$

1	2	3	4	5	6

2 ঘর right rotate   5 6 1 2 3 4

4 ঘর left rotate   same

(6 − 2)

minimum → 2

$(t+t_0)$ → delay

$[0 \quad 1 \quad \ldots \quad N-1]$ → সবসময় + হিসাবে treat হচ্ছে − আসবে না। advance টা পাচ্ছি না।

N − 2 ঘর right rotate

N − (N−2) = 2 ঘর left rotate

6
→ $\frac{1}{2}$ করলে 3

3 এর বেশি হলে
left shift.

$\frac{N}{2}$ < N−2

N−2 −N করবো
= −2
↳ 2ঘর left shift.

$\frac{N}{2}$ > N−2
→ right shift.

Positive এর highest frequency এর value গুলোকে (−ve) হিসাবে treat করছি। নাহলে ঝামেলায় delay লেগেছে advance পাচ্ছি না।

O(nlogn) divide and conquer

O(n^2)