MARKERBOARD 4.1

System Properties

$$x(t),\ x[n] \longrightarrow \boxed{} \longrightarrow y(t),\ y[n]$$

- Memory
- Invertibility
- Causality
- Stability
- Time Invariance
- Linearity

Time-Invariance

C-T:

$$x(t) \rightarrow y(t)$$

Then

$$x(t-t_0) \rightarrow y(t-t_0) \quad \text{any } t_0$$

D-T:

$$x[n] \rightarrow y[n]$$

$$x[n-n_0] \rightarrow y[n-n_0] \quad \text{any } n_0$$

Linearity

$$\phi_k \rightarrow \psi_k$$

Then

$$a_1\phi_1 + a_2\phi_2 + \dots \rightarrow a_1\psi_1 + a_2\psi_2 + \dots$$

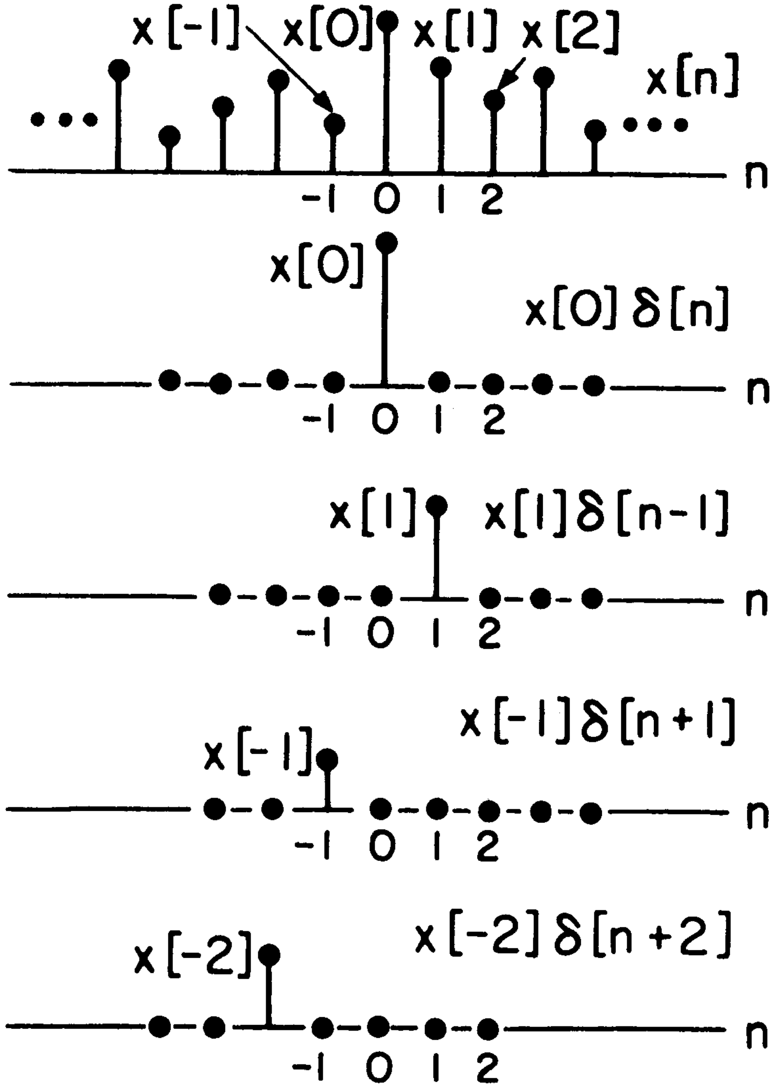
STRATEGY:

- decompose input signal into a linear combination of basic signals
- choose basic signals so that response easy to compute

LTI Systems:

delayed impulses $\Longleftrightarrow$ Convolution

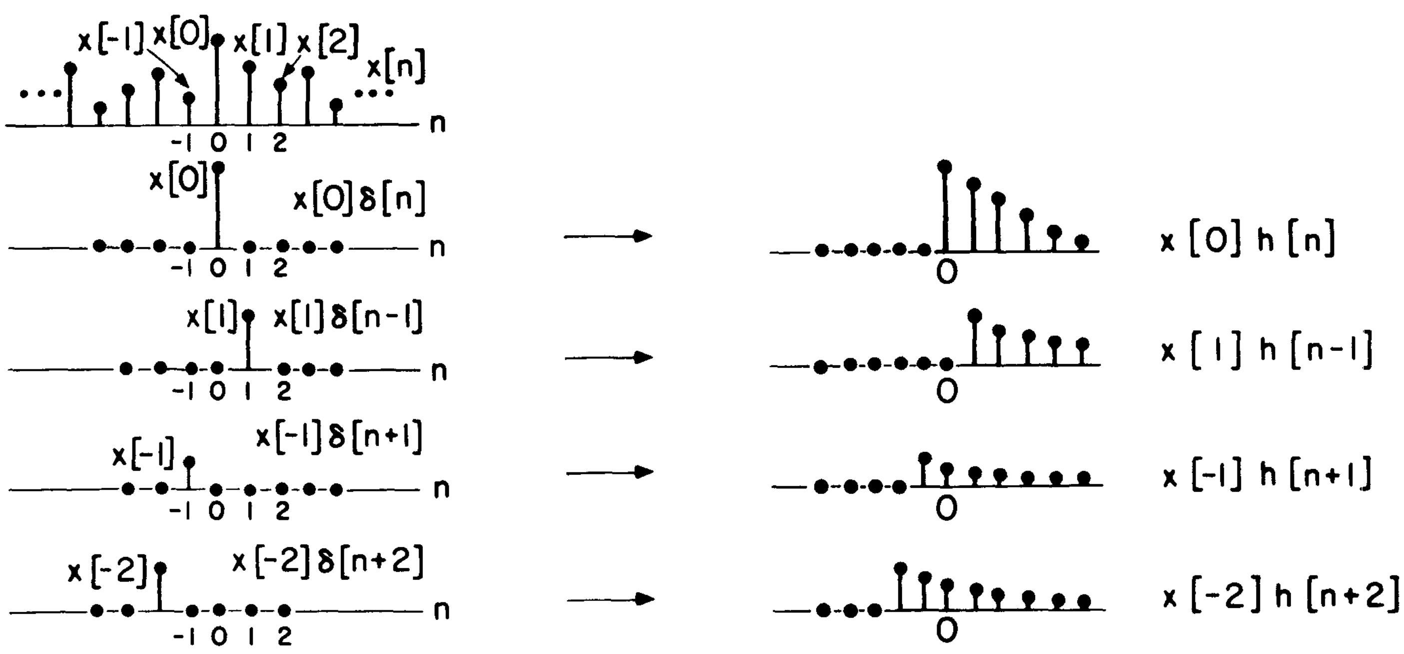
complex exponentials $\Longleftrightarrow$ Fourier Analysis



$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[-1]\delta[n+1] + \cdots = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

TRANSPARENCY 4.1
A general discrete-time signal expressed as a superposition of weighted, delayed unit impulses.

TRANSPARENCY 4.2
The convolution sum for linear, time-invariant discrete-time systems expressing the system output as a weighted sum of delayed unit impulse responses.



TRANSPARENCY 4.3
One interpretation of the convolution sum for an LTI system. Each individual sequence value can be viewed as triggering a response; all the responses are added to form the total output.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\ \delta[n-k]$$

Linear System:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h_k[n]$$

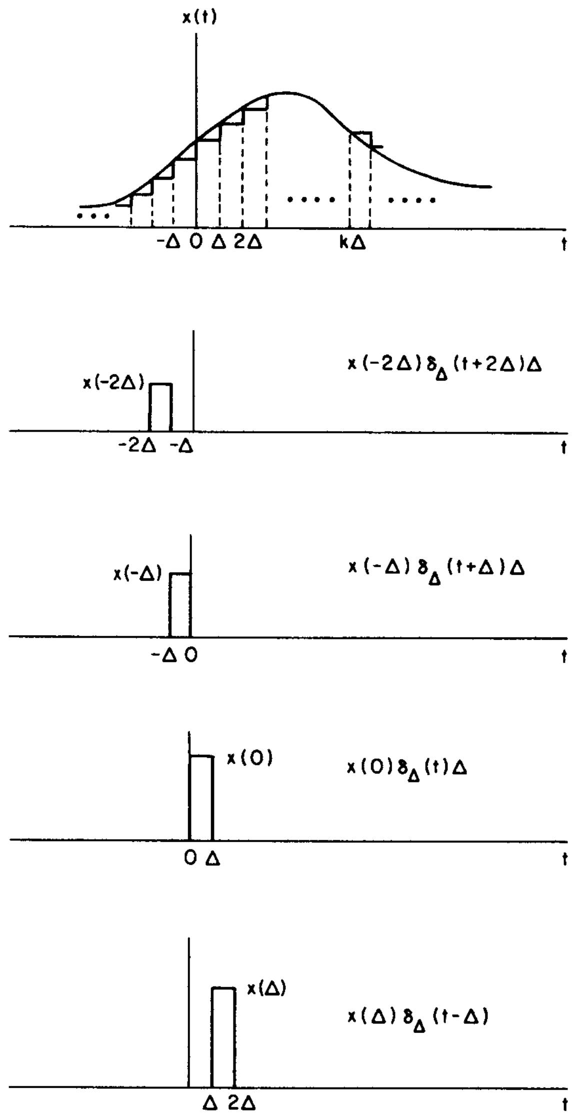
$$\delta[n-k] \rightarrow h_k[n]$$

If Time-Invariant:

$$h_k[n] = h_o[n-k]$$

$$\text{LTI:}\ \boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k]}$$

Convolution Sum



TRANSPARENCY 4.4
Approximation of a continuous-time signal as a linear combination of weighted, delayed, rectangular pulses. [The amplitude of the fourth graph has been corrected to read $x(0)$.]

$$x(t) \cong x(o)\,\delta_{\triangle}(t)\,\Delta + x(\Delta)\,\delta_{\triangle}(t-\Delta)\,\Delta$$

$$+\ x(-\Delta)\,\delta_{\triangle}(t+\Delta)\,\Delta + \ldots$$

$$x(t) \cong \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_{\triangle}(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\triangle \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_{\triangle}(t-k\Delta)\,\Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

TRANSPARENCY 4.5
As the rectangular pulses in Transparency 4.4 become increasingly narrow, the representation approaches an integral, often referred to as the sifting integral.

TRANSPARENCY 4.6
Derivation of the convolution integral representation for continuous-time LTI systems.

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_{\Delta}(t - k\Delta)\, \Delta$$

Linear System:

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, h_{k\Delta}(t)\, \Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, h_{\tau}(t)\, d\tau$$

If Time-Invariant:

$$h_{k\Delta}(t) = h_o(t - k\Delta)$$

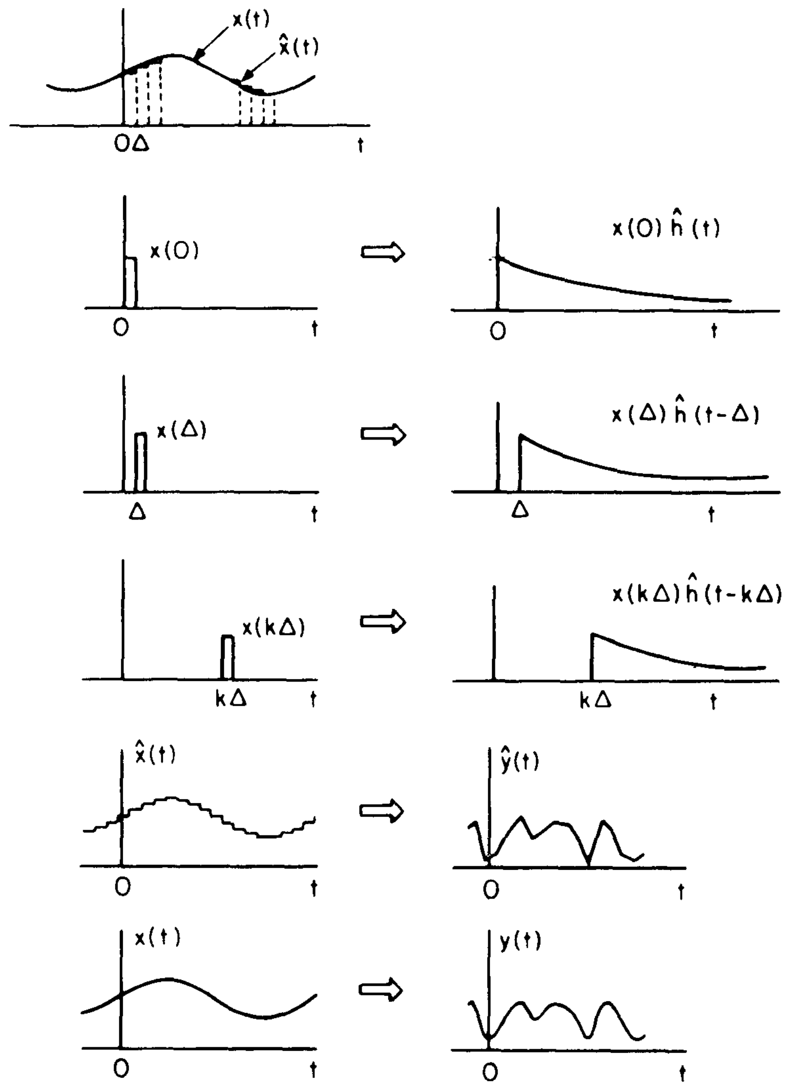
$$h_{\tau}(t) = h_o(t - \tau)$$

LTI:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau}$$

Convolution Integral

TRANSPARENCY 4.7
Interpretation of the convolution integral as a superposition of the responses from each of the rectangular pulses in the representation of the input.



Convolution Sum:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\ \delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k] = x[n] * h[n]$$

Convolution Integral:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\ \delta(t-\tau)\ d\tau$$

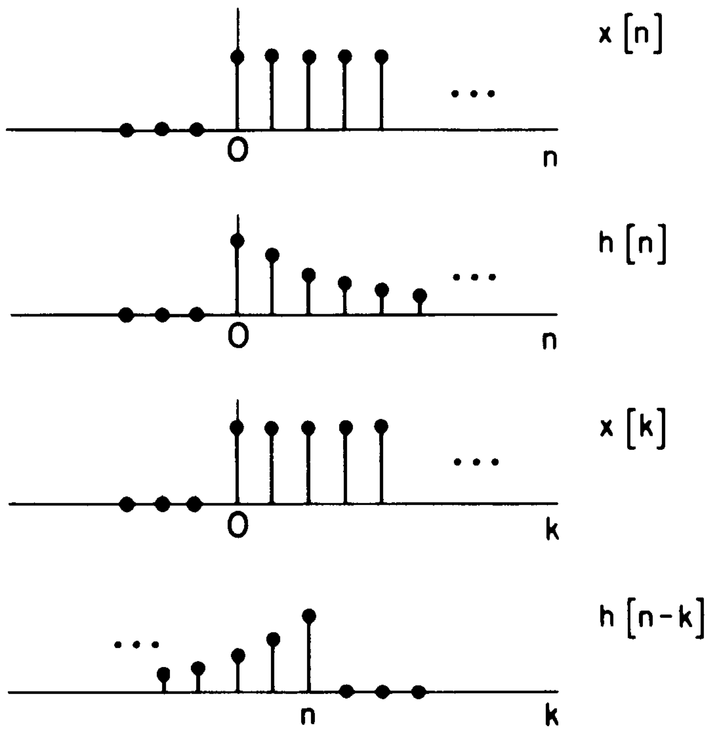
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\ h(t-\tau)\ d\tau = x(t) * h(t)$$

TRANSPARENCY 4.8
Comparison of the convolution sum for discrete-time LTI systems and the convolution integral for continuous-time LTI systems.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$x[n] = u[n]$

$h[n] = \alpha^n u[n]$



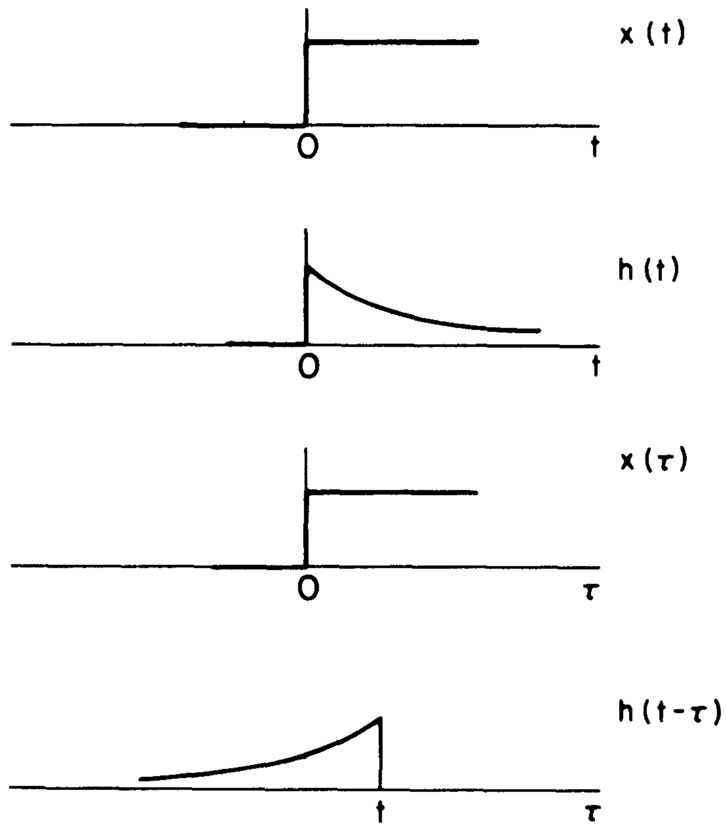
TRANSPARENCY 4.9
Evaluation of the convolution sum for an input that is a unit step and a system impulse response that is a decaying exponential for $n > 0$.

TRANSPARENCY 4.10
Evaluation of the convolution integral for an input that is a unit step and a system impulse response that is a decaying exponential for $t > 0$.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$x(t) = u(t)$

$h(t) = e^{-at}\, u(t)$

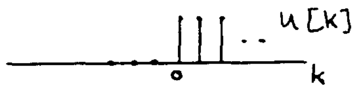


MARKERBOARD
4.2

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} u[k]\, \alpha^{n-k} u[n-k]$$

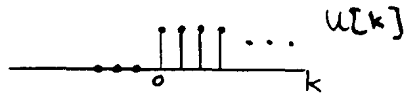
Interval 1: $n<0$

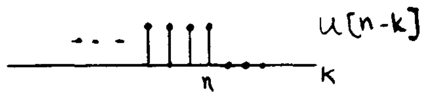


No overlap $\Longrightarrow$

$y[n] = 0 \quad n<0$

Interval 2: $n>0$





overlap for $k = 0, 1, \ldots n$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

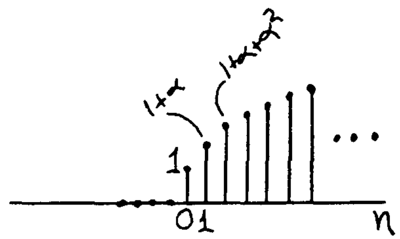
$$= \alpha^{n} \sum_{k=0}^{n} (\alpha^{-1})^{k}$$

$$\sum_{k=0}^{r} \beta^{k} = \frac{1-\beta^{r+1}}{1-\beta}$$

$$y[n] = \alpha^{n} \underbrace{\sum_{k=0}^{n} (\alpha^{-1})^{k}}_{\frac{1-(\alpha^{-1})^{n+1}}{1-\alpha^{-1}}}$$

$\vdots$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \quad n>0$$



MARKERBOARD 4.3

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} u(\tau)\, e^{-a(t-\tau)}\, u(t-\tau)\, d\tau$$

Interval 1: $t<0$

No overlap between

$u(\tau)$ & $u(t-\tau) \Longrightarrow$

$y(t) = 0 \quad t<0$

Interval 2: $t>0$

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)\, e^{-a(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau)u(t-\tau) = 1$

for $\quad 0 \le \tau \le t$

$$y(t) = \int_0^t e^{-a(t-\tau)}\, d\tau$$

$$= e^{-at} \underbrace{\int_0^t e^{a\tau}\, d\tau}_{\frac{1}{a}\left[e^{at}-1\right]}$$

$$= \frac{1}{a}\left[1 - e^{-at}\right] \quad t>0$$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{a}\left[1-e^{-at}\right] & t>0 \end{cases}$$