Fourier Series

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Periodic Waveforms

- **A signal has period *T*,** $x(t) = x(t \pm nT)$ **if for all *t***
  - Also periodic with periods $2T$, $3T$, etc., and $-T$, $-2T$ ...
  - Smallest positive period T_0 is called the *fundamental period*
  - *Fundamental frequency* f_0 is computed as $1 / T_0$
- **Harmonics**
  - Harmonics are higher-frequency components whose frequencies are integer multiples of the fundamental frequency. For example, if the fundamental frequency is f_0, then the second harmonic will have a frequency of $2f_0$, the third harmonic will have a frequency of $3f_0$ and so on.
- **Finding fundamental frequency**
  - Largest f_0 such that $f_k = k f_0$, i.e. $f_0 = \gcd\{f_k\}$
  - Consider notes A 440 Hz, E 660 Hz and F 880 Hz. $f_0 = 220$ Hz

Fourier Series

- ***Periodic* signals can be synthesized**

Periodic functions (like a signal that repeats itself over time) can be broken down into a sum of simple waveforms—specifically, sinusoids (sine and cosine functions).

These sinusoids have different frequencies, amplitudes, and phases. When added together, they reconstruct the original signal.

Conditions for the Existence of FS

- **Finite Number of minima and maxima in one period of time**

- **Finite number of discontinuities in one period of time**

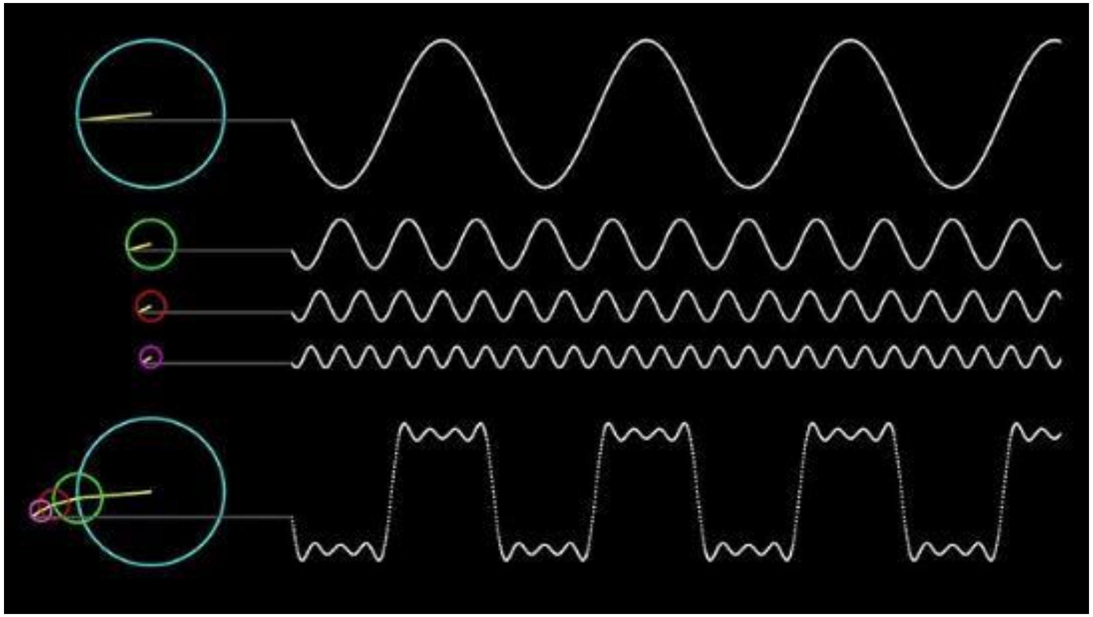
- **Absolutely integrable in one period**

$$\int_{T_0} |x(t)| dt < \infty$$

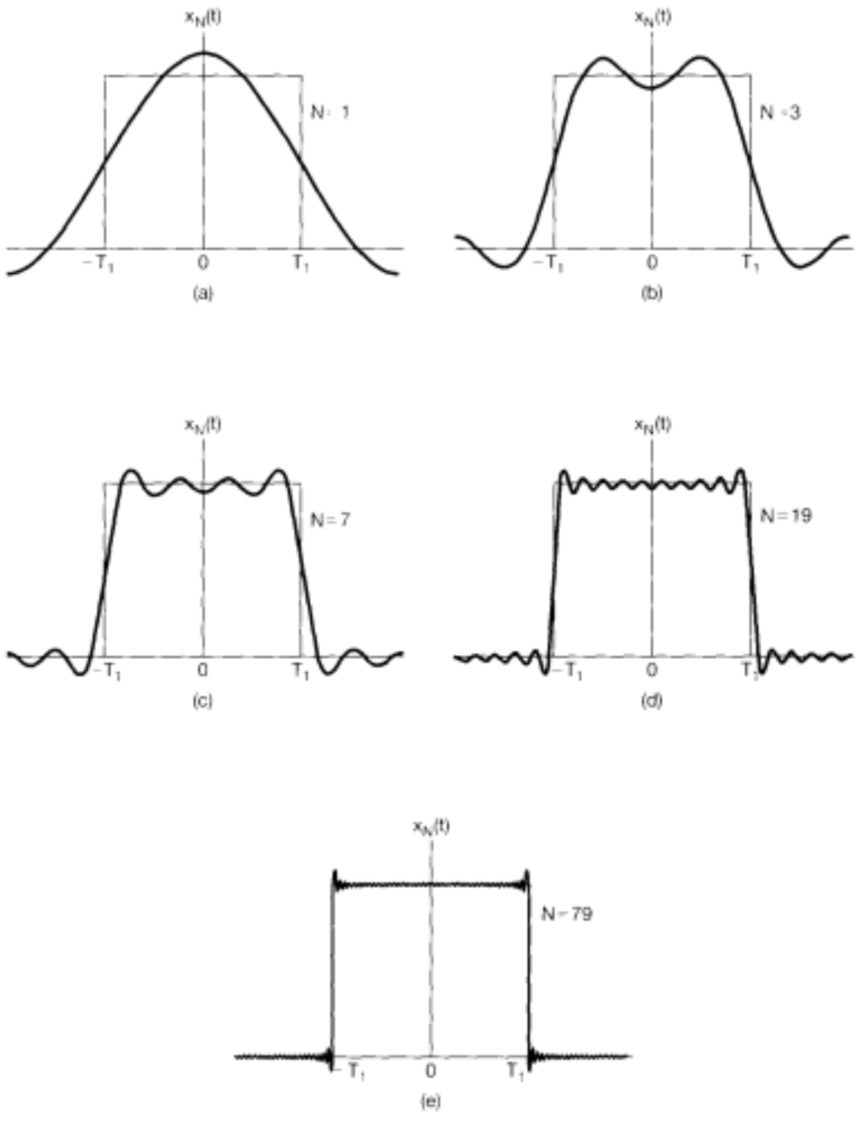
Fourier Series (Trigonometric Form)

- Represents a periodic signal

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$



Convergence



Expressions for the coefficients

- **Coefficients**

$$a_0, a_n \; and \; b_n$$

Expressions for the coefficients

- **Coefficients**

$$a_0, a_n \; and \; b_n$$

- **Equations**

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t) dt$$

Helpful Properties for Proof

$$\int_{-L}^{L} \cos(nx)\cos(mx)\,dx =$$

{
0; when n != m
L; when n=m
}

$$\int_{-L}^{L} \sin(nx)\sin(mx)\,dx =$$

{
0; when n != m
L; when n=m
}

Exercises

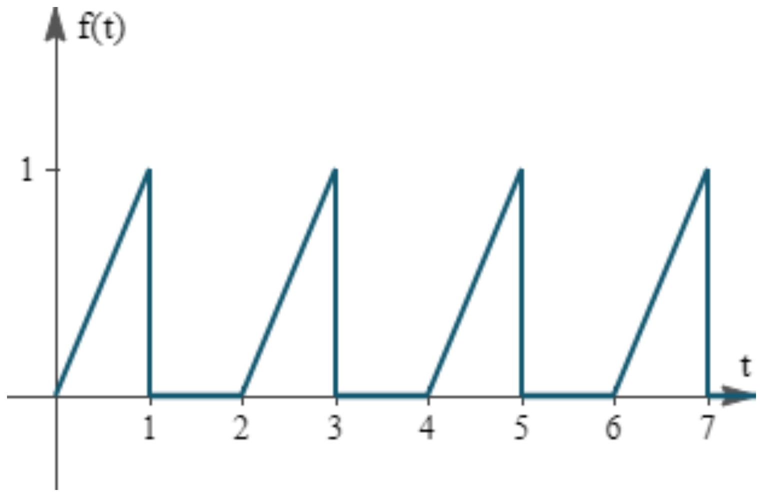
(1) $x(t) = \{$ 1.5 when $0 \le t < 1$

-1.5 when $1 \le t < 2$

$\}$

x(t) is periodic. Find the coefficients

(2) Determine the Fourier series representations for the following signal:



Solution (1)

$$Here, T = 2$$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{1}{2}\int_0^2 x(t)dt = \frac{1}{2}(\int_0^1 (1.5)dt + \int_1^2 (-1.5)dt)$$

$$= 0$$

$$a_n = \frac{2}{2}\int_0^1 1.5\cos(n\pi t)dt + \int_1^2 (-1.5)\cos(n\pi t)dt$$

$$= 1.5\frac{\sin(n\pi t)}{n\pi}\Big]_0^1 - 1.5\frac{\sin(n\pi t)}{n\pi}\Big|_1^2$$

$$= 0$$

Contd.

$$b_n = \frac{2}{2}\int_0^1 1.5\sin(n\pi t)dt - \int_1^2 \ 1.5\sin(n\pi t)dt$$

$$= -\frac{1.5}{n\pi}\cos(n\pi t)\,|_0^1 + \frac{1.5}{n\pi}\cos(n\pi t)\,|_1^2$$

$$= -\frac{1.5}{n\pi}(\cos(n\pi) - 1) + \frac{1.5}{n\pi}(\cos(2n\pi) - \cos(n\pi))$$

When n is even, $b_n = 0$. But when n is odd $b_n = \dfrac{6}{n\pi}$

Fourier Series (Complex Form)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Intuition Behind the Coefficients:

•Each a_k represents the amplitude and phase of a particular frequency component $k\omega_0$ within the signal.

•If a_k is large, the frequency component $k\omega_0$ has a significant contribution to the signal. If a_k is small, the contribution is minimal.

•The complex nature of a_k encodes both the amplitude (magnitude) and the phase (angle) of the corresponding sinusoidal component.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$ [trigonometric form]

$$= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + \underbrace{\sum_{n=1}^{\infty} b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}}_{\sum_{n=1}^{\infty} b_n \frac{-je^{jn\omega_0 t} + je^{-jn\omega_0 t}}{2}}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} \frac{a_{-n} + jb_{-n}}{2} e^{jn\omega_0 t}$$

$$\sum_{n=1}^{100} e^{n} = e^{1} + e^{2} + e^{3} + \cdots\cdots + e^{100}$$

$$= \sum_{n=-100}^{-1} e^{-n}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad , \; n = 0 \Rightarrow C_0$$

$$n = -2 \Rightarrow \sum_{n=\infty}^{\infty} C_{-2} e^{-2j\omega_0 t}$$

→ complex form

C_n can be real or complex

Question

How is the complex form equivalent to trigonometric form?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_{T_0} x(t)\, e^{-jm\omega_0 t} = \sum_{k=-\infty}^{\infty} \int_{T_0} a_k\, e^{j(k-m)\omega_0 t}\, dt$$

$k \neq m \Longrightarrow$ integration $= 0$

$$k = m \Rightarrow \sum_{k=-\infty}^{\infty} \int_{T_0} a_k \, dt$$

sum রাখার দরকার
নেই। কারণ only একটা
term থাকে। যখন $k=m$।
বাকি সব 0।

$$= a_k T_0$$

$$\therefore a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-j\cancel{m}k\omega_0 t}$$

$$e^{j(2\pi f_0)3t}, \quad k = 3$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{1}{2j}\left(\quad \ldots\ldots \quad \right)$$

$\therefore a_k$ $a + jb$ form এ থাকবে।

sin, cos বা এর combination থাকলে integrate না করেও coeff বের করা যাবে।

Fourier Series

- ***Analysis:* start with $x(t)$ and compute $\{ a_k \}$**

Integrate $x(t)$ over fundamental period T_0

$$c_n \quad a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_0 t}\,dt \qquad 2\pi f_0 = \omega_0$$

Calculation of a_0 simplifies to average value of $x(t)$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

cos এর জন্য a_0 থাকেই না। So, $a_0 = 0$।

Example #1: With $x(t) = \cos(2\pi f_0 t)$, what is a_0?

Example #2: With $x(t) = \cos^2(2\pi f_1 t)$, what is a_0?

cos, sin থাকলে euler দিয়েই বের করতে পারি।

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} = \frac{1}{2}e^{\overset{K=1}{j2\pi f_0 t}} + \frac{1}{2}e^{\overset{K=-1}{-j2\pi f_0 t}}$$

a_1, a_{-1} থাকবে cos এর জন্য।

important

Spectrum of the Fourier Series

a_k vs frequency (kf_0)

- **Find Fourier series coefficients for $x(t) = \cos^3(3\pi t)$**

Approach #1 $\qquad a_k = \dfrac{1}{T_0}\displaystyle\int_0^{T_0} \cos^3(3\pi t)\, e^{-j2\pi k f_0 t}\, dt$

$a_k \rightarrow$ complex ও হতে পারে

Approach #2: Expand into complex exponentials

$$x(t) = \left(\frac{e^{j3\pi t} + e^{-j3\pi t}}{2}\right)^3 = \frac{1}{8}\left(\underbrace{e^{j9\pi t}}_{K=3} + \underbrace{3e^{j3\pi t}}_{K=1} + \underbrace{3e^{-j3\pi t}}_{K=-1} + \underbrace{e^{-j9\pi t}}_{K=-3}\right)$$

$a_3 = \frac{1}{8}$

$a_1 = \frac{3}{8}$

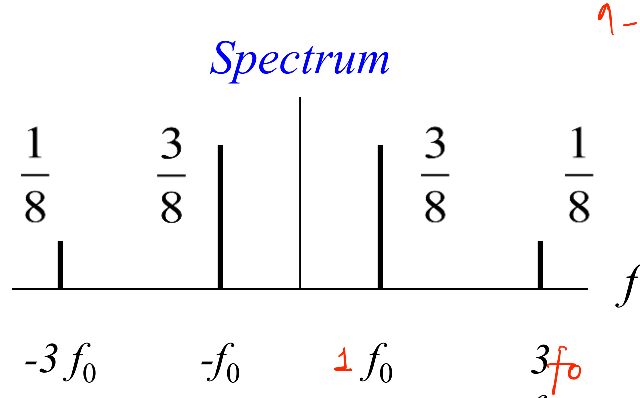
$a_{-1} = \frac{3}{8}$

$a_{-3} = \frac{1}{8}$

- **Resulting spectrum**

$\omega_0 = \gcd(3\pi, 9\pi) = 3\pi$

$f_0 = 1.5$ Hz



K এর জায়গাতে f change হবে।

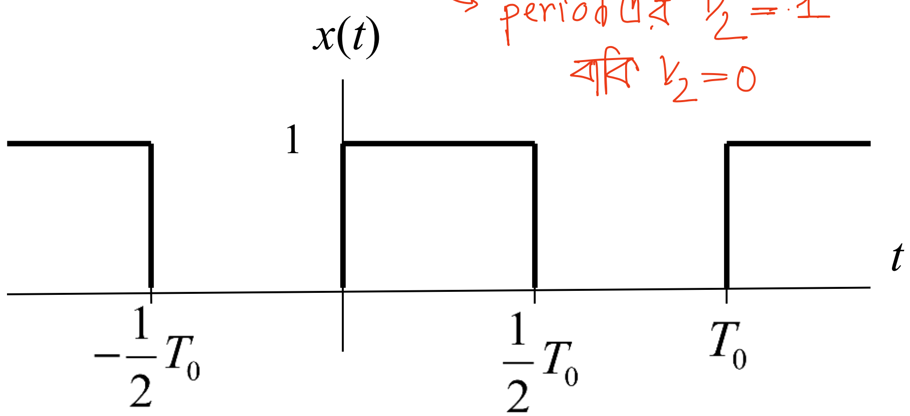
Fourier Analysis of a Square Wave

- **Periodic square wave with 50% duty cycle**

period এর ½ = 1
বাকি ½ = 0

Defined for one period as

$$s(t)=\begin{cases} 1 \text{ for } 0 \le t < \frac{1}{2}T_0 \\ 0 \text{ for } \frac{1}{2}T_0 \le t < T_0 \end{cases}$$



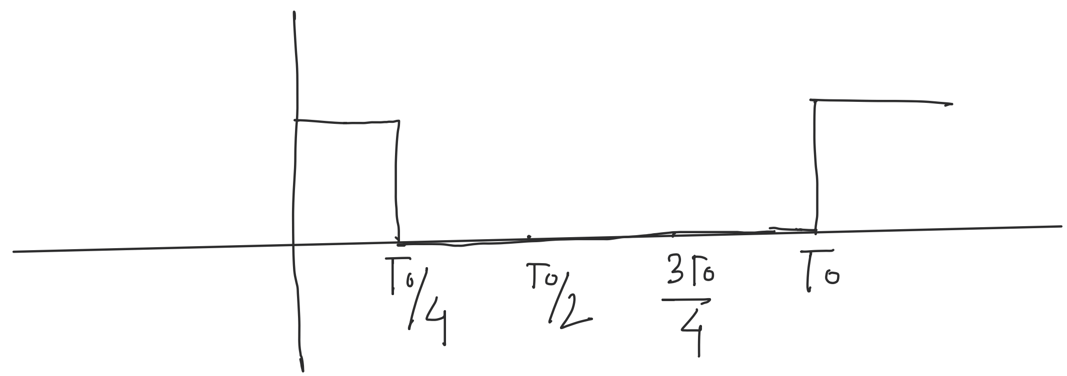
- **Fourier coefficients**

1. $a_0 = ½$ because $x(t)$ is 1 half the time and 0 half the time
2. Then, $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_0 t}dt = \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} e^{-j2\pi k f_0 t}dt = \left(\frac{1}{T_0}\right)\frac{e^{-j2\pi k f_0 t}}{-j2\pi k f_0}\Bigg|_0^{\frac{1}{2}T_0}$

For $k \neq 0$

$$a_k = \left(\frac{1}{T_0}\right)\frac{e^{-j2\pi k f_0 (T_0/2)} - e^{-j2\pi k f_0 (0)}}{-j2\pi k f_0} = -\frac{e^{-j\pi k}-1}{j2\pi k} = \frac{1-(-1)^k}{j2\pi k}$$

duty cycle 25%



$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jw_0kt}\, dt$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0/2} (1 \cdot 1)\, dt$$

$$= \frac{1}{T_0} \times \frac{T_0}{2}$$

$$= \frac{1}{2}$$

$a_0 = 1/2$ (average value) $\Rightarrow$ duty cycle 50%

$$a_k = \frac{1-(-1)^k}{j2\pi k}$$

$$k=0 \Rightarrow \frac{0}{0}$$

so 0 এর জন্য আলাদাভাবে বের করা লাগবে। generalize করলে কোনো value পাচ্ছিনা।

$$a_K = \frac{1}{T_0} \int x(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} e^{-jK\omega_0 t}\, dt \qquad x(t) = 1$$

$$= \frac{1}{T_0} \left[\frac{e^{-jK\omega_0 t}}{-jK\omega_0} \right]_0^{T_0/2}$$

$$= \frac{1}{T_0} \frac{e^{-j2\pi f_0 K T_0/2} - e^{-j2\pi K f_0 (0)}}{-jK2\pi f_0} = -\frac{e^{-j\pi K} - 1}{j2\pi K f_0} \times \frac{1}{T_0}$$

$$= \frac{-(-1)^K + 1}{2\pi K j f_0} \times \frac{1}{T_0}$$

$$a_K = \frac{-(-1)^K + 1}{2\pi K j}$$ → T_0 এর উপর depend করেনা।

শুধু K এর উপর depend করে। ⟹ duty cycle 50%

$K = \text{even} \Longrightarrow a_K = 0$

$\text{odd} \Longrightarrow \frac{1}{\pi K j}$

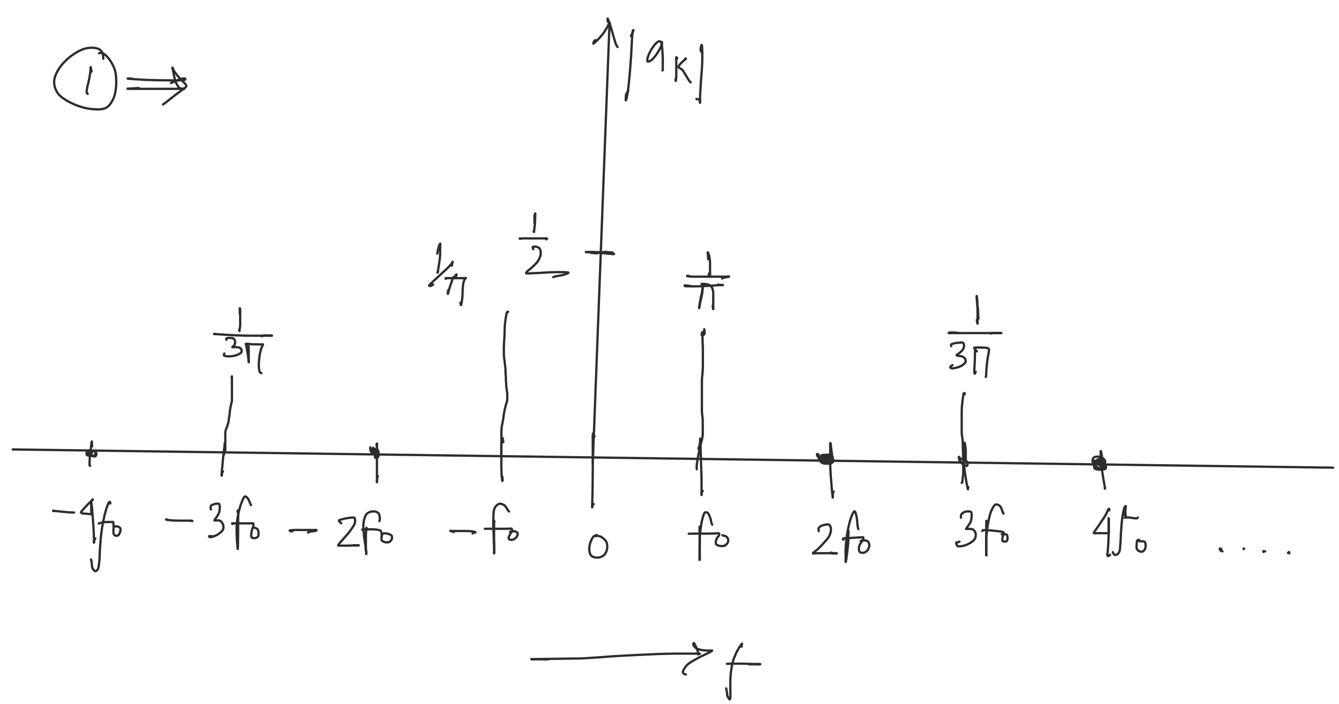
$$a_k = \begin{cases} \frac{1}{2} & \text{for } k=0 \\ 0 & \text{for } k \text{ even but not zero} \\ -\frac{j}{\pi k} & \text{for } k \text{ odd} \end{cases}$$

→ ২টা graph এ (complex থাকলে)

(i) magnitude $|a_k|$ Vs f_0

(ii) phase $\angle a_k$ Vs f_0

(i) ⟹



magnitude vs f

a_1

$K=1$

$\hookrightarrow \frac{1}{j\pi}$

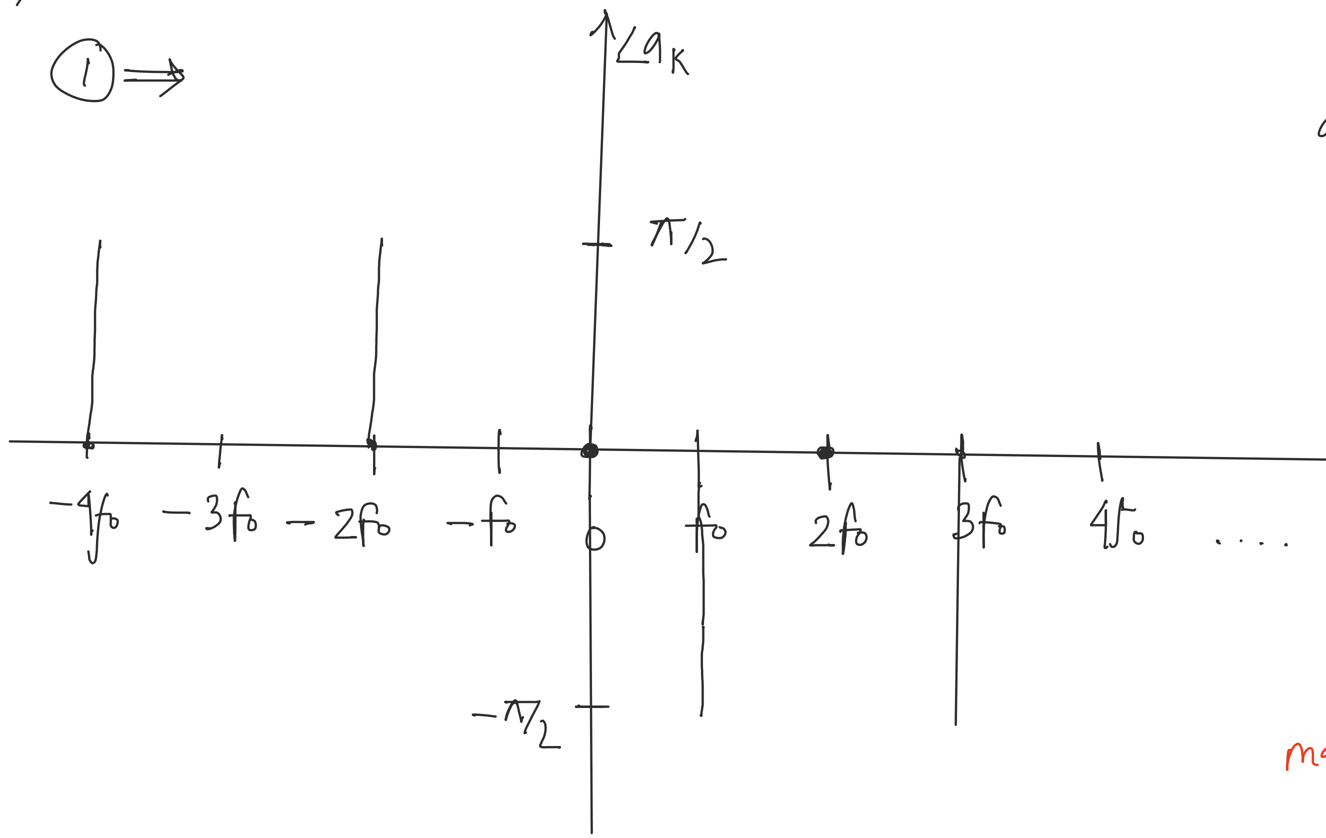
$= \frac{-j}{\pi}$

$\text{magnitude} = \frac{1}{\pi}$

$a_1, a_{-1} \to$ value same but phase different আসবে।

(ii) $\Rightarrow$

(i) $\Rightarrow$



$$a_1 = \frac{1}{\pi j}$$

$$= -\frac{j}{\pi}$$

phase $= -\pi/2$

$a = 0$

a থাকলে

$\tan^{-1}\frac{b}{a}$ করে

magnitude $= \sqrt{a^2+b^2}$

phase vs frequency

Spectrum for a Square Wave

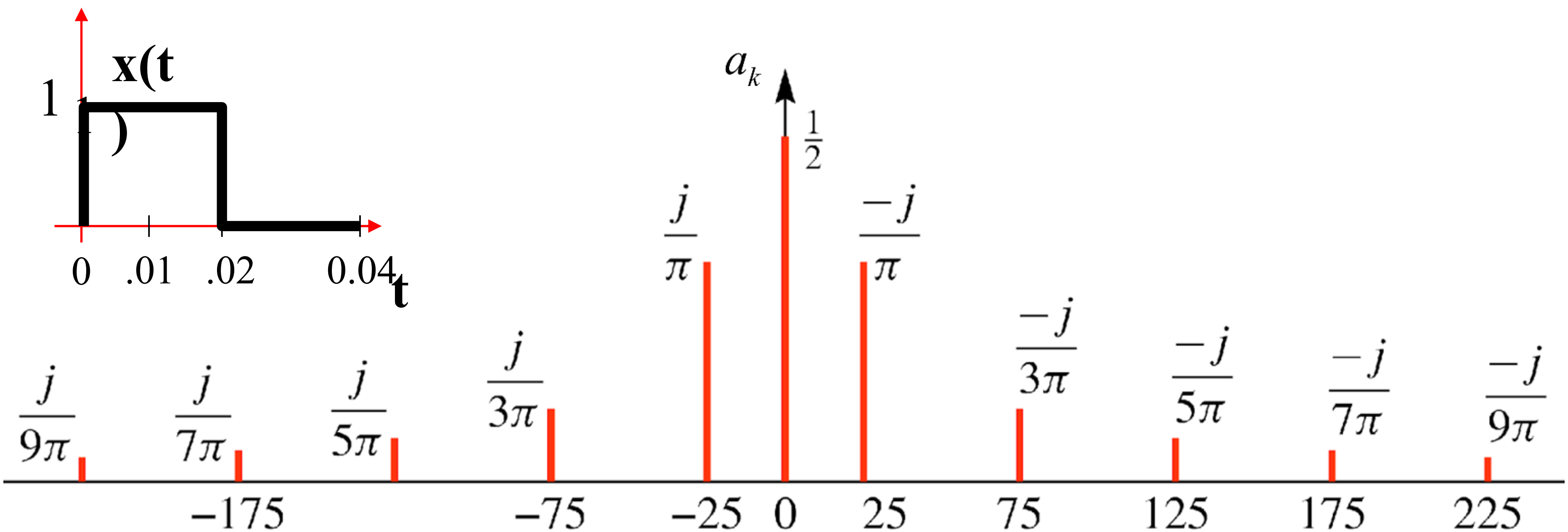
- **Fourier coefficients**

  Independent of T_0

- **Example**

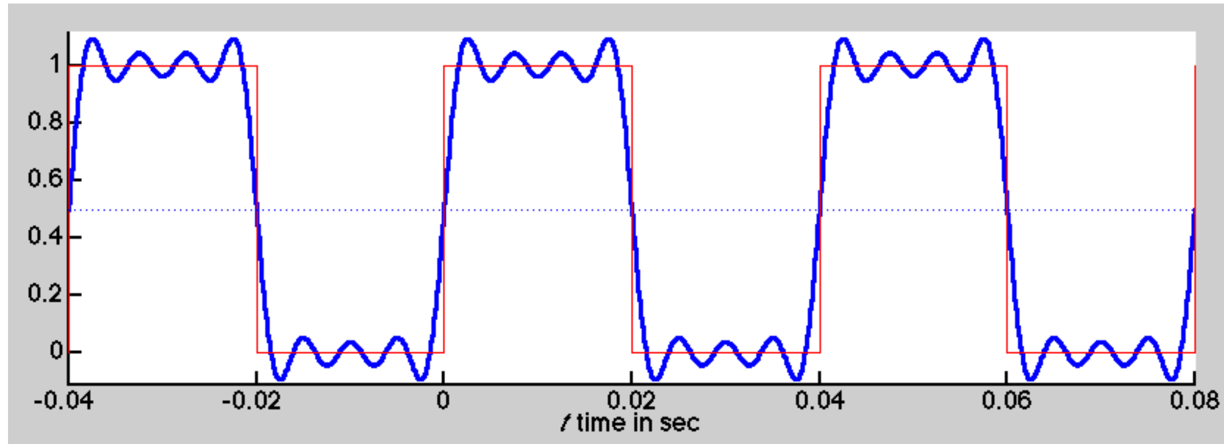
  $T_0 = 0.04$ s □ $\quad f_0 = 25$ Hz

$$a_k = \begin{cases} \dfrac{1}{2} & \text{for } k = 0 \\ 0 & \text{for } k \text{ even but not zero} \\ -\dfrac{j}{\pi\kappa} & \text{for } k \text{ odd} \end{cases}$$

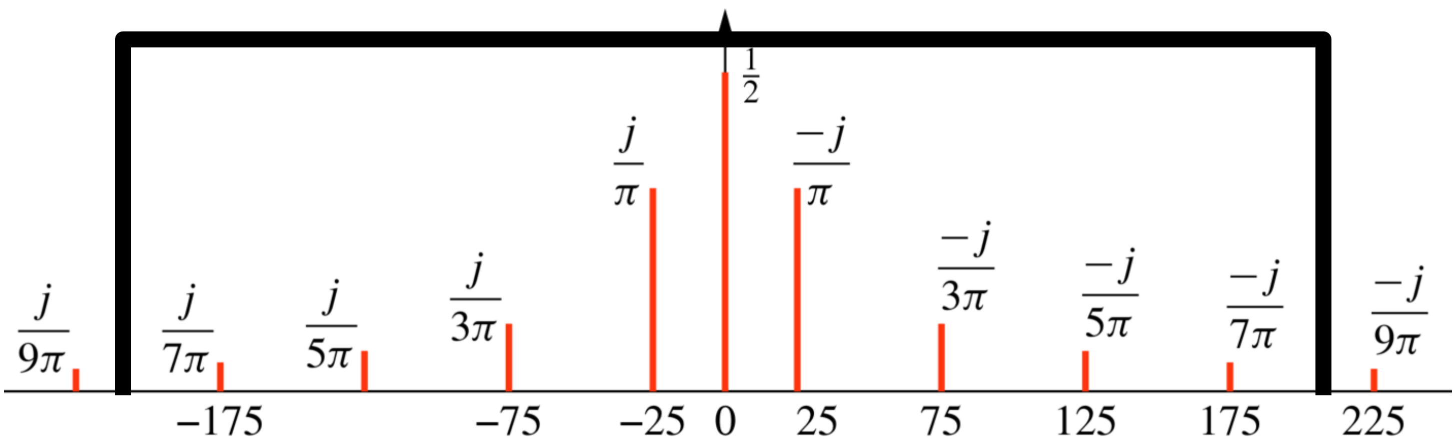


Fourier Synthesis of a Square Wave

- **Synthesis using up to 7th harmonic**



$$y(t)=\frac{1}{2}+\frac{2}{\pi}\sin(50\pi t)+\frac{2}{3\pi}\sin(150\pi t)+\frac{2}{5\pi}\sin(250\pi t)+\frac{2}{7\pi}\sin(350\pi t)$$



demo

Properties of CTFS

continuous time fourier series

- **Linearity**

Let x(t) and y(t) denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k respectively. That is,

same freq. ধরে নিচ্ছি

$x(t) \rightarrow a_k$

$y(t) \rightarrow b_k$

c_k

Now,

$$z(t) = A \underbrace{x(t)}_{a_k} + B \underbrace{y(t)}_{b_k}$$

$b_k \Longrightarrow$ fourier coefficient

$z(t) = Ax(t) + By(t)$

$z(t) \rightarrow Aa_k + Bb_k$

$$c_k = A a_k + B b_k$$

$$c_k = \frac{1}{T_0} \int_{T_0} z(t) e^{-j\omega_0 t} dt$$

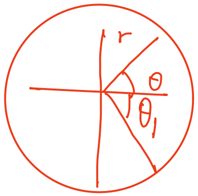
$$\frac{1}{T_0}\int_{T_0}\left(A\,x(t)+B\,y(t)\right)e^{-j\omega_0 t}\,dt$$

$$=\frac{1}{T_0}\int_{T_0} A\,x(t)\,e^{-j\omega_0 t}\,dt+\frac{1}{T_0}\int_{T_0} B\,y(t)\,e^{-j\omega_0 t}\,dt$$

$$=A a_K+B b_K$$

important

Contd.



- **Time Shifting**

x(t) → a_k $\quad$ $x(t)$ দেওয়া a_k

$x(t-t_0) \rightarrow ?$ $\quad$ $x(t-3) = ?$ $\quad$ $a_k \cdot e^{-jk\omega_0 3}$

Then, if the signal is delayed $t_0 s$,

এটা extra multiply হয় মানে phase change হয়।

$$x(t-t_0) \rightarrow \boxed{e^{-jk\omega_0 t_0}} a_k = \mathrm{b_k}$$

magnitude same থাকে

$x(t+t_0) \rightarrow e^{jk\omega_0 t_0} a_k$

One consequence of this property is that, when a periodic signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered. That is,

$$|b_k| = |a_k|$$

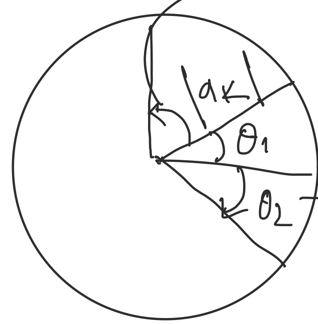
$$x(t-t_0) = e^{-j\omega_0 k t_0} a_k$$

clockwise change

$|a_k| = |b_k|$

magnitude change হবে না।

phase chase হবে।



anticlockwise phase change

clockwise phase change

__Time Reversal:__

$x(t) \rightarrow a_{-3}$

$$a_k = \frac{1}{\pi k} = -\frac{1}{3\pi}$$

$x(-t) \rightarrow a_3$

__even__ $\quad x(t) = x(-t)$

$$a_k = a_{-k}$$

$a_k = a_{-k}$

$x(t)$ কেমন?

ans: __even__

$$x(t) = -x(-t)$$

$$a_K = -a_{-K}$$

$\therefore \; a_K = -a_{-K} \longrightarrow x(t)$ odd function.

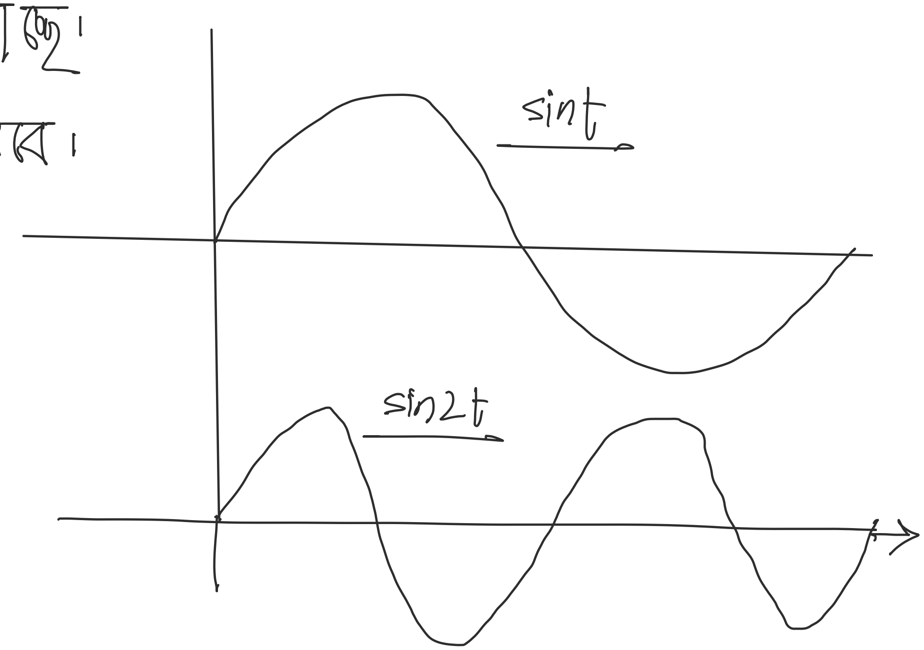
Time scaling:

$x(t) = a_K$ sin t

$x(at) \longrightarrow a_K$ (no change)   sin 2t

$\hookrightarrow$ frequency বেড়ে যাচ্ছে।

time period কমবে।



Proof: $x(t) = a_k$

$x(at) = b_k(let)$

$$b_k = \frac{1}{\frac{T_0}{a}}$$

$$b_k = \frac{1}{\frac{T_0}{a}} \int_{-\frac{T_0}{2a},\ 0}^{\frac{T_0}{2a},\ \frac{T_0}{a}} x(at) \cdot e^{-jk(a\omega_0)t}\, dt$$

b_k কে a_k এর মাধ্যমে represent করতে চাই।

তাই at থেকে t তে যাওয়া লাগবে।

$$\therefore b_k = \frac{1}{\frac{T_0}{a}} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-jk(\omega_0 a)\frac{t'}{a}} \cdot \frac{dt'}{a}$$

$at = t'$

$a\,dt = dt'$

$dt = \frac{dt'}{a}$

$$= \frac{a}{T_0} \times \frac{1}{a} \int_{-T_0/2}^{T_0/2} x(t')\, e^{-jk\omega_0 t'}\, dt'$$

T_0, w_0 change হয়ে a_k এর value আগের মতই রয়েছে।

Conjugation:

$$z = a + jb$$

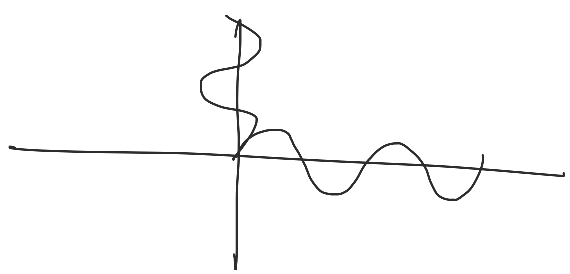
$$z^* = a - jb$$

$$z = a$$

$$\bar{z} = a$$

$x(t) = \sin t + jt^2$ — complex signal

j part না থাকলে — real signal



২টো signal থাকবে। both periodic. Period T

Conjugate করা হলে time period change হয় না।

Contd.

- **Time Reversal**

$$x(t) \rightarrow a_k$$
$$x(-t) \rightarrow a_{-k}$$

Time period : T_0

" " : T_0

- **Time scaling**

$$x(t) \rightarrow \mathrm{a_k}$$
$$x(at) \rightarrow a_k \text{ (No change in coefficients)}$$

But time period and frequency will change

$$T'_0 = \frac{\mathrm{T_0}}{\mathrm{a}}$$
$$\omega'_0 = \omega_0 a$$

Contd.

- **Conjugation**

$$x(t) \rightarrow a_k$$

b_k

$$x(t)^* \rightarrow a^*_{-k}$$

- **Differentiation in Time**

$x(t) \longrightarrow a_k$

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(t) \rightarrow a_k$$

$$\frac{d}{dt}(x(t)) \rightarrow jk\omega_0 a_k$$

$x'(t) = \sum a_k e^{jk\omega_0 t} (jk\omega_0)$

$= a_k (jk\omega_0)$

Generalized form,

$$\frac{d^n}{dt^n}(x(t)) \rightarrow (jk\omega_0)^n a_k$$

$\underline{n=3}$ $\qquad j^3 = -j$

$$x(t) \rightarrow a_k$$
$$x(t)^* \rightarrow a^*_{-k} \quad (b_K)$$

$$a_K = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jKw_0t}\, dt$$

$$b_K = \frac{1}{T_0}\int_{T_0} x^*(t)\, e^{-jw_0Kt}\, dt$$

j/-j
বাদে বাকি
part same

$$a_K^* = \frac{1}{T_0}\int_{T_0} x^*(t)\, e^{jKw_0t}\, dt$$

$$a_{-K}^* = \frac{1}{T_0}\int_{T_0} x^*(t)\, e^{-jKw_0t}\, dt$$

Question: $x(t) \rightarrow T_0$

property থেকে ছোট question থাকবে।

$$a_k = -j\left(\frac{1}{2}\right)^{|k|}$$

$x(t)$ real or complex ?

$$x(t) \rightarrow a_k$$

$$x^*(t) \rightarrow a^*_{-k}$$

$$a^*_{-k} = j\left(\frac{1}{2}\right)^{|-k|} = j\left(\frac{1}{2}\right)^{|k|}$$

$a_k \neq a^*_{-k} \rightarrow \therefore x(t)$ complex signal

$a_k = a^*_{-k}$ হলে $x(t)$ real signal

Contd.

- **Integration in Time**

$$x(t) \to a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \to \frac{a_k}{jk\omega_0}$$

- **Convolution**

$$x_1(t) \to a_{k1}$$

$$x_2(t) \to a_{k2}$$

$$x(t) = x_1(t) * x_2(t), then$$

$$x(t) \to T_0(a_{k1}a_{k2})$$

Here, x(t) is convolution of x1(t) and x2(t)

Convolution: $z(t)=x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$b_K \qquad T_0 \qquad T_0$

$a_{K_1} \qquad a_{K_2}$

$$= \int_{\cancel{-T_0/2}\ 0}^{\cancel{T_0/2}\ T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

convolution এর coefficient কি হবে?

convolution এ time period change হয় না। এক্ষেত্রে T_0 ই হবে।

$$b_K = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} z(t)\, e^{-j\omega_0 t k}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\int_{-T_0/2}^{+T_0/2} x(\tau)\, x_2(t-\tau)\, d\tau\ e^{-j\omega_0 t k}\, dt$$

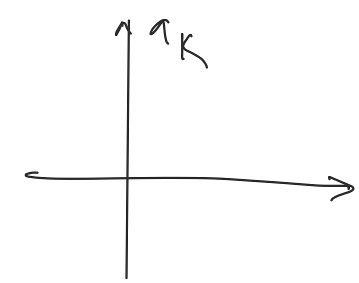
fourier series এর formula থেকে আছে।

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\int_{-T_0/2}^{+T_0/2} x_1(\tau)\, x_2(t-\tau)\, d\tau\; e^{-j\omega_0 t k}\, dt \cdot \frac{e^{-j\omega_0 k\tau}}{e^{-j\omega_0 k\tau}}\, dt$$

$$= \int_{-T_0/2}^{T_0/2} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(\tau)\, e^{-jk\omega_0\tau}\, d\tau\; x_2(t-\tau)\, e^{-j\omega_0 k(t-\tau)}\, dt$$

$$= a_{k_1}\int_{-T_0/2-\tau}^{T_0/2-\tau} x_2(t')\, e^{-jk\omega_0 t'}\, dt'$$

$$t-\tau = t'$$



$$= T_0 \frac{1}{T_0}\, a_{k_1}\int_{-T_0/2-\tau}^{T_0/2-\tau} x_2(t')\, e^{-jk\omega_0 t'}\, dt'$$

a_K freq. dependent

$$= T_0\, a_{k_1}\, a_{k_2}$$

Time domain এ convolution হলে freq domain এ product হবে।

Time domain এ গুণ হলে freq domain এ convolution

$$z(t) = x_1(t) \quad x_2(t)$$

$$a_{k_1} \qquad a_{k_2}$$

$$= a_{k_1} * a_{k_2}$$

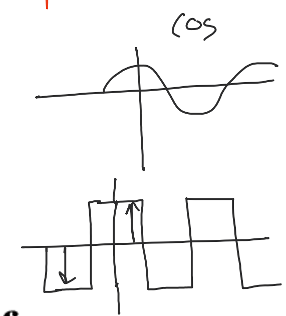
$$\frac{T_0}{2} - \tau + \frac{T_0}{2} + \tau$$

upper − lower limit

$$= T_0 \text{ (Period)}$$

a_0, a_n, b_n → sine এর সাথে

Symmetricities in FS



- **Even Symmetry**

  If the signal is even, then FS expansion will have harmonics of even s ignals. That means b_n will be 0

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt$$

sin — odd sym.

- **Odd Symmetry**

  If the signal is even, then FS expansion will have harmonics of even s ignals. That means $\cancel{b_n}$ will be 0

  a_n

a_n cos ... → even sym

a_n → 0

- **Half wave symmetry**

  If the signal has half wave symmetry, then FS expansion will only have odd harmonics

Half wave symmetry

- **Condition**

shift করছি।

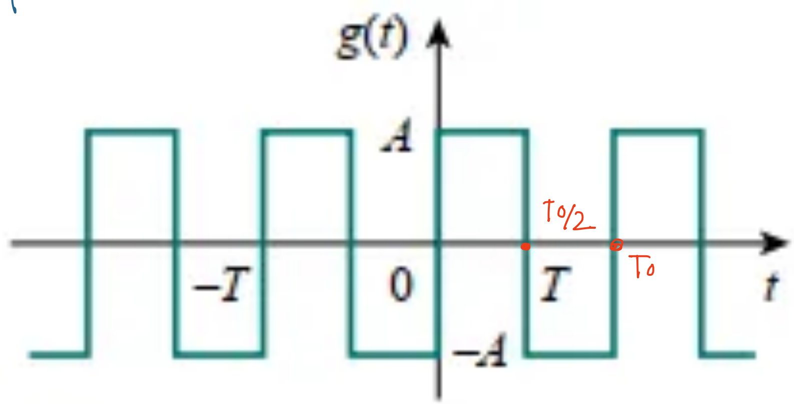
$$x(t) = -x(t \pm T_0 / 2)$$

original signal এর inverted

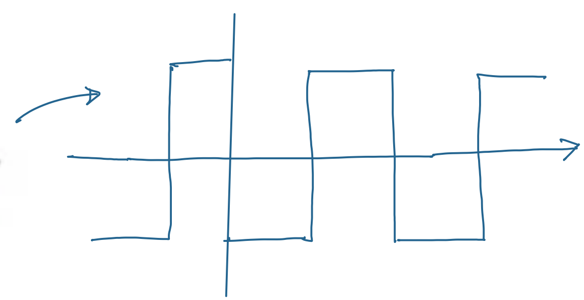
half wave

$x(-t) = x(t)$

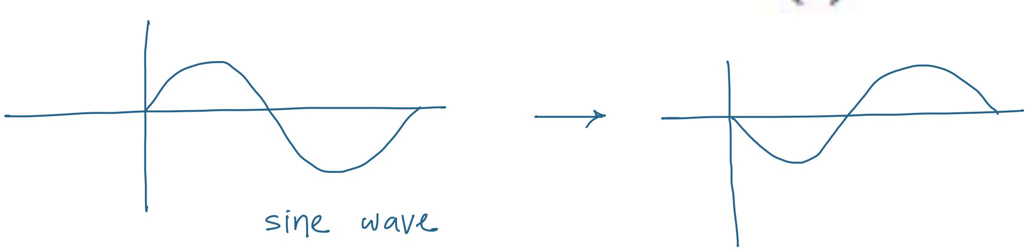
$x(-t) = -x(t)$



(b)



shifted signal



sine wave

shifted

Proof

- If the signal has half wave symmetry, then FS expansion will only have odd harmonics

time shifting property:

$x(t+t_0)$

$= a_k e^{jk\omega_0 t_0}$

$$x(t) \xrightarrow{FS} a_k$$

$$so,\ x(t + \frac{T_0}{2}) \xrightarrow{FS} a_k e^{jk\omega_0 \frac{T_0}{2}}$$

(−) দিয়েও করা যাবে

$$\Rightarrow -x(t + \frac{T_0}{2}) \xrightarrow{FS} -a_k e^{jk\omega_0 \frac{T_0}{2}}$$

$x(t) =$

যেহেতু half wave symmetric ধরে নিচ্ছি।

$$\Rightarrow a_k = -a_k e^{jk\omega_0 \frac{T_0}{2}}$$

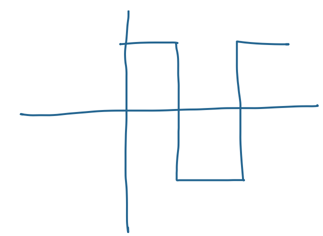
$$\Rightarrow 1 + e^{jk\omega_0 \frac{T_0}{2}} = 0$$

$$\Rightarrow 1 + e^{jk\pi} = 0$$

$\cos(k\pi) + i\sin(k\pi)$

$= (-1)^k$

$$\Rightarrow 1 + (-1)^k = 0$$



$a_n = 0$

$b_1, b_3, \ldots$ থাকবে

b_2, b_4 থাকবে না।

$b_{100} = ?$ $\boxed{= 0}$

The above equation can only be true when k is odd. $a_{100} = 0$

K even এর value গুলো exist করে না। যেগুলো 0 হবে।

Parseval's Power Theorem

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Where C_n *is the fourier series coefficient of* $x(t)$

- **Basic Equation of avg power**

$$P_{x(t)} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Example

- **Find the avg power of the signal $\cos^3(3\pi t)$**

$$x(t) = \left(\frac{e^{j3\pi t} + e^{-j3\pi t}}{2}\right)^3 = \frac{1}{8}\left(e^{j9\pi t} + 3e^{j3\pi t} + 3e^{-j3\pi t} + e^{-j9\pi t}\right)$$

Coefficients are,

$$C_1 = C_{-1} = \frac{3}{8}$$

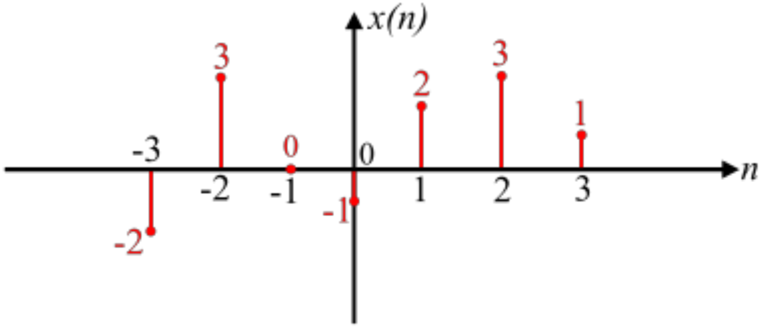
$$C_3 = C_{-3} = \frac{1}{8}$$

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2 = 2.(|\frac{3}{8}|)^2 + 2.(|\frac{1}{8}|)^2 = 5/16 \text{ watts}$$

DTFS

- **Discretete Time Signal**

The signals which are defined only at discrete instants of time are known as discrete time signals. The discrete time signals are represented by x[n] where n is the independent variable in time domain.



DTFS

- **Representation**

$$x[n] = \sum_{k=<N>} x(k) e^{jk\Omega_0 n}$$

Where

$$\Omega_0 = 2\pi / N$$

N = fundamental frequency
x(k) = kth coefficient

DTFS

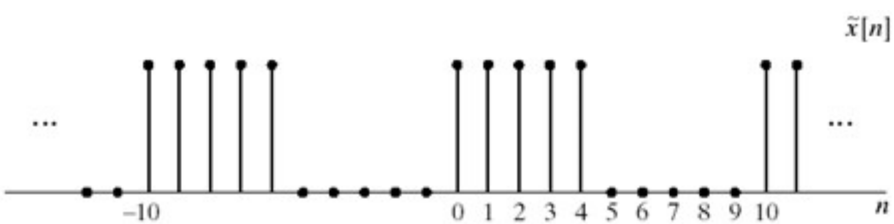
- **Calculating the coefficient**

$$x(k) = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\Omega_0 n}$$

Practice Problems

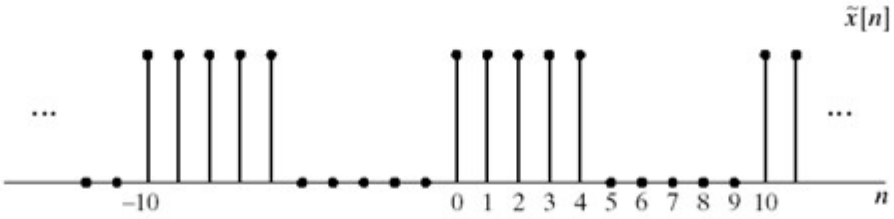
(1) $x[n] = \cos(\frac{\pi}{3} n)$

(2)



Find coefficients and draw the spectrums

Solution



From the graph,

$$N=10$$

$$x(k)=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\Omega_0 n}$$

$$=\frac{1}{10}\sum_{n=0-4}x[n]e^{-jk\Omega_0 n}$$

$$=\frac{1}{10}(1+e^{-jk\frac{\pi}{5}}+e^{-jk\frac{2\pi}{5}}+e^{-jk\frac{3\pi}{5}}+e^{-jk\frac{4\pi}{5}}) \qquad \Omega_0=\frac{2\pi}{10}=\frac{\pi}{5}$$

Solution

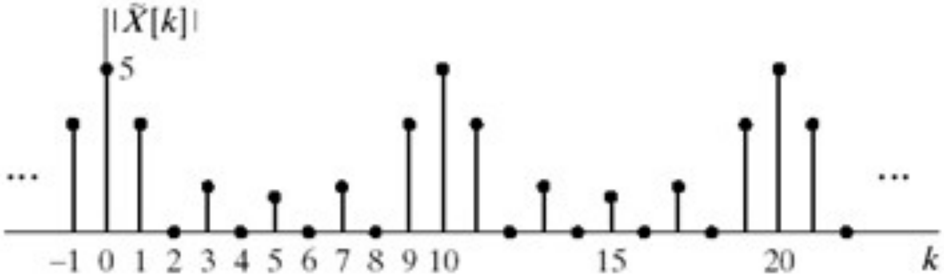
- **Now, for k=0**

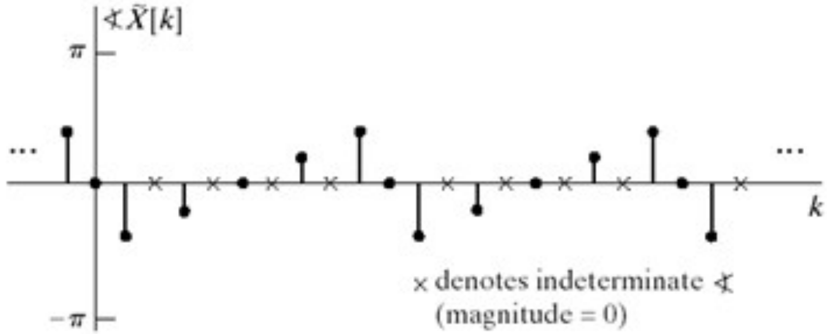
  x(0) = 5/10 = 0.5

  x(1) = 0.310 $\angle -1.7$

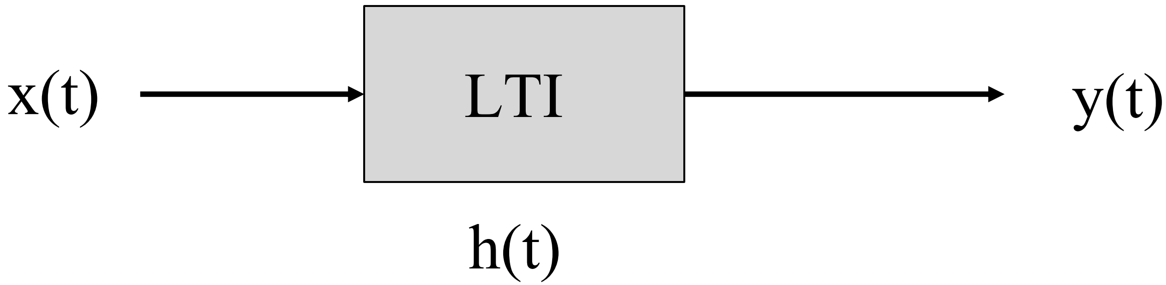
  x(2)=0

Similarly, determine x(3) x(4) ….. (Do yourself)





Fourier Series for LTI System



$$x(t) \rightarrow a_k$$

$$y(t) \rightarrow H(kj\omega_0)a_k$$

Where h(t) is the impulse response $H(kj\omega_0)$ is the frequency response of the LTI system

Example

- **Input x(t)=cos(2πt)+sin(πt). Impulse Response H(s) = $\frac{1}{4+s}$. y(t) is the output of the LTI system. Find the FS coefficients of y(t).**

Solution:

$$x(t) = \cos(2\pi t) + \sin(\pi t)$$

$$= \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$

So, coefficients of input x(t)

$$a_1 = \frac{1}{2j}$$

$$a_2 = \frac{1}{2}$$

$$a_{-1} = -\frac{1}{2j}$$

$$a_{-2} = \frac{1}{2}$$

Contd

- **Let, b_k be the coefficients for the output signal y(t)**

$$b_1 = H(j\omega_0)a_1 = \frac{1}{4 + j\omega_0}\frac{1}{2j} = \frac{1}{4 + j\pi}\cdot\frac{1}{2j}$$

$$b_{-1} = H(-j\omega_0)a_{-1} = \frac{1}{4 - j\omega_0}\frac{1}{(-2j)} = \frac{1}{4 - j\pi}\cdot\frac{1}{(-2j)}$$

$$b_2 = H(2j\omega_0)a_2 = \frac{1}{4 + 2j\omega_0}\frac{1}{2} = \frac{1}{4 + 2j\pi}\cdot\frac{1}{2}$$

$$b_{-2} = H(-2j\omega_0)a_{-2} = \frac{1}{4 - 2j\omega_0}\frac{1}{2} = \frac{1}{4 - 2\pi j}\cdot\frac{1}{2}$$