2

Signals and Systems: Part I

In this lecture, we consider a number of basic signals that will be important building blocks later in the course. Specifically, we discuss both continuous-time and discrete-time sinusoidal signals as well as real and complex exponentials.

Sinusoidal signals for both continuous time and discrete time will become important building blocks for more general signals, and the representation using sinusoidal signals will lead to a very powerful set of ideas for representing signals and for analyzing an important class of systems. We consider a number of distinctions between continuous-time and discrete-time sinusoidal signals. For example, continuous-time sinusoids are always periodic. Furthermore, a time shift corresponds to a phase change and vice versa. Finally, if we consider the family of continuous-time sinusoids of the form $A \cos \omega_0 t$ for different values of ω_0, the corresponding signals are distinct. The situation is considerably different for discrete-time sinusoids. Not all discrete-time sinusoids are periodic. Furthermore, while a time shift can be related to a change in phase, changing the phase cannot necessarily be associated with a simple time shift for discrete-time sinusoids. Finally, as the parameter Ω_0 is varied in the discrete-time sinusoidal sequence $A\cos(\Omega_0 n + \phi)$, two sequences for which the frequency Ω_0 differs by an integer multiple of 2π are in fact indistinguishable.

Another important class of signals is exponential signals. In continuous time, real exponentials are typically expressed in the form ce^{at}, whereas in discrete time they are typically expressed in the form $c\alpha^n$.

A third important class of signals discussed in this lecture is continuous-time and discrete-time complex exponentials. In both cases the complex exponential can be expressed through Euler's relation in the form of a real and an imaginary part, both of which are sinusoidal with a phase difference of $\pi/2$ and with an envelope that is a real exponential. When the magnitude of the complex exponential is a constant, then the real and imaginary parts neither grow nor decay with time; in other words, they are purely sinusoidal. In this case for continuous time, the complex exponential is periodic. For discrete time the complex exponential may or may not be periodic depending on whether the sinusoidal real and imaginary components are periodic.

In addition to the basic signals discussed in this lecture, a number of additional signals play an important role as building blocks. These are introduced in Lecture 3.

Suggested Reading

Section 2.2, Transformations of the Independent Variable, pages 12–16

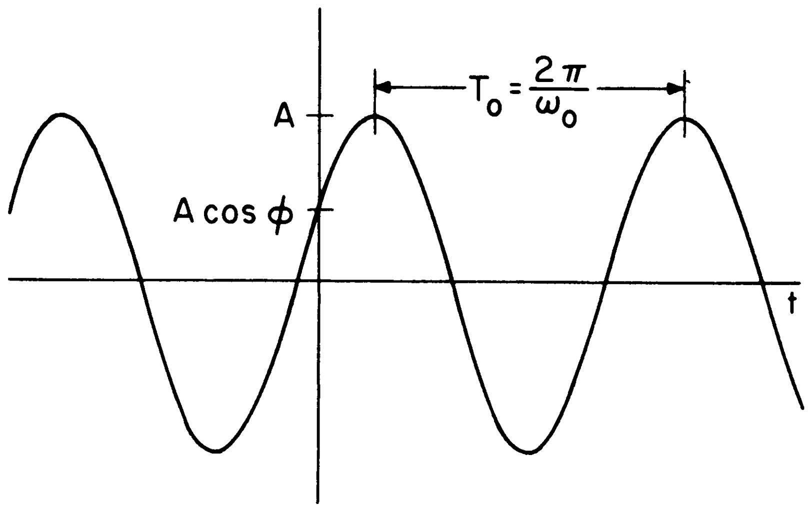
Section 2.3.1, Continuous-Time Complex Exponential and Sinusoidal Signals, pages 17–22

Section 2.4.2, Discrete-Time Complex Exponential and Sinusoidal Signals, pages 27–31

Section 2.4.3, Periodicity Properties of Discrete-Time Complex Exponentials, pages 31–35

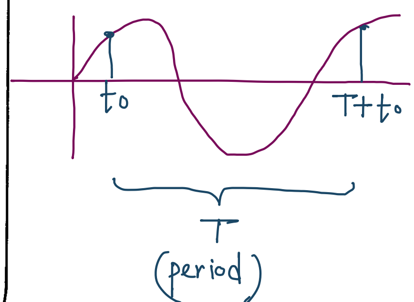
CONTINUOUS-TIME SINUSOIDAL SIGNAL

$$x(t) = A\cos(\omega_o t + \phi)$$



TRANSPARENCY 2.1
Continuous-time sinusoidal signal indicating the definition of amplitude, frequency, and phase.

$x(t) = x(t+T)$



→ smallest যে time পরে signal repeat হয়।

- **Periodic:** time origin shifted

$$x(t) = x(t+T_o) \qquad \text{period} \triangleq \text{smallest } T_o$$

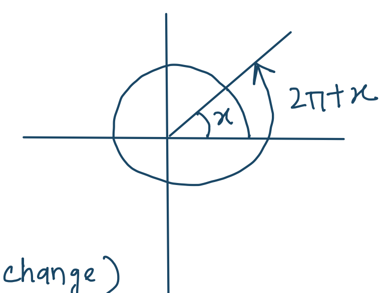
$$A\cos[\omega_o t + \phi] = A\cos[\omega_o t + \underbrace{\omega_o T_o}_{2\pi m} + \phi]$$

m → integer

$$T_o = \frac{2\pi m}{\omega_o} \Rightarrow \text{period} = \frac{2\pi}{\omega_o}$$

$\cos x = \cos(2\pi + x)$
$= \cos(4\pi + x)$
$= \cos(2\pi m + x)$

TRANSPARENCY 2.2
Relationship between a time shift and a change in phase for a continuous-time sinusoidal signal.



t_o

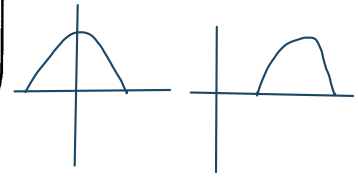
- **Time Shift <=> Phase Change**

fraction/ integer হতে পারে।

→ φ (phase change) Δφ

$$A\cos[\omega_o(t+t_o)] = A\cos[\omega_o t + \omega_o t_o]$$

$$A\cos[\omega_o(t+t_o)+\phi] = A\cos[\omega_o t + \omega_o t_o + \phi]$$



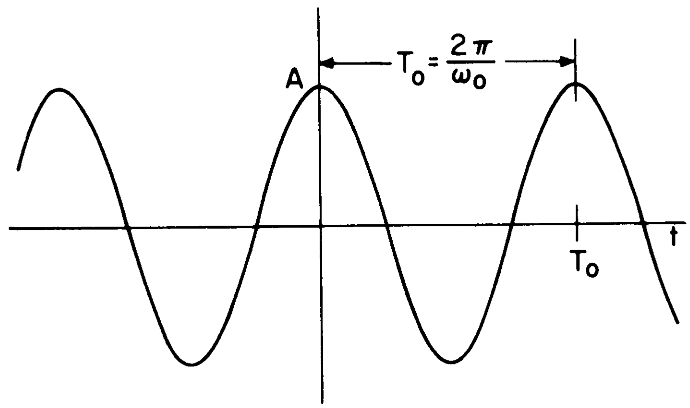
time shift

TRANSPARENCY 2.3
Illustration of the signal $A \cos \omega_0 t$ as an even signal.

$\boxed{\phi = 0}$ $\qquad x(t) = A \cos \omega_0 t$

even function

$\cos x = \cos(-x)$
↳ even signal



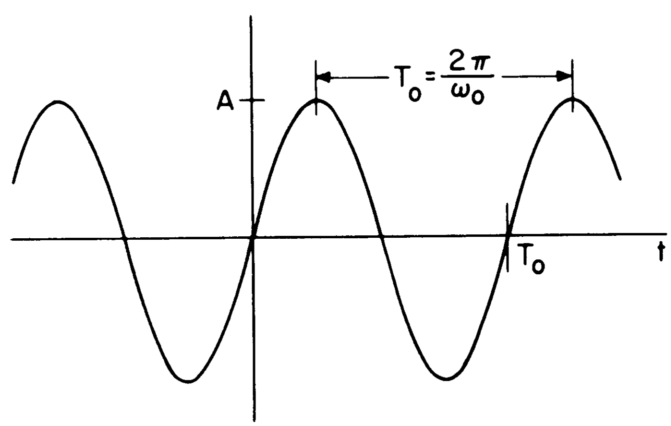
Periodic: $\quad x(t) = x(t + T_0)$

Even: $\quad x(t) = x(-t)$ mirror image (about the origin)

TRANSPARENCY 2.4
Illustration of the signal $A \sin \omega_0 t$ as an odd signal.

$\boxed{\phi = -\frac{\pi}{2}}$ $\quad x(t) = \begin{cases} A \cos(\omega_0 t - \frac{\pi}{2}) \\ A \sin \omega_0 t \\ A \cos[\omega_0 (t - \frac{T_0}{4})] \end{cases}$

$\frac{T_0}{4}$ time shift ↔
$\pi/2$ phase change



Periodic: $\quad x(t) = x(t + T_0)$

Odd: $\quad x(t) = -x(-t)$ mirror image flipped over

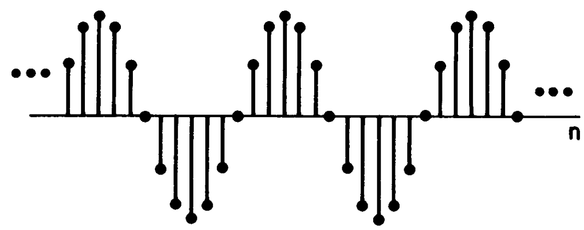
↗ নির্দিষ্ট time পরপর sampling নিবে। $n \rightarrow$ integer

DISCRETE-TIME SINUSOIDAL SIGNAL

$$x[n] = A\cos(\Omega_o n + \phi) \quad , n \in \mathbb{Z}$$

$\Omega_o = \frac{2\pi}{12}$

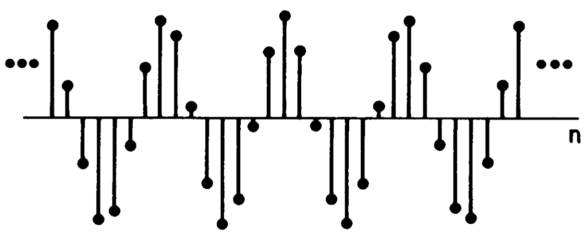
$\phi = 0$



frequency, period diff.

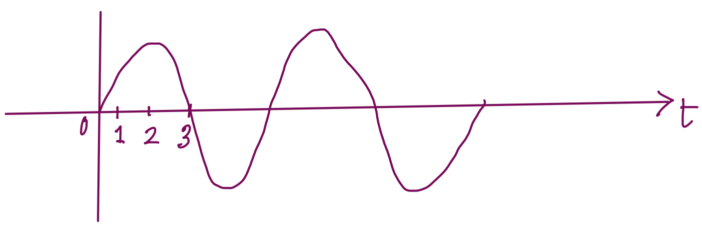
$\Omega_o = \frac{8\pi}{31}$

$\phi = 0$



TRANSPARENCY 2.5
Illustration of discrete-time sinusoidal signals.

Discrete এ value অবশ্যই integer হবে। → uniform distance এ sample নিচ্ছি তাই।



Time Shift => Phase Change → integer নাও হতে পারে। তাই ⇐ নেই।

$$A\cos[\Omega_o(n + n_o)] = A\cos[\Omega_o n + \underbrace{\Omega_o n_o}_{\text{constant}}]$$

$\Delta\phi$

$$x[n] = A\cos(\Omega_o n + \phi)$$

$$x[n + n_o] = A\cos(\Omega_o n + \underbrace{\Omega_o n_o}_{\text{phase change}} + \phi)$$

$$A\cos \Omega_o n$$

$$A\cos(\Omega_o n + \phi)$$

$\phi = \Omega_o n_o$ [n_o must be an integer]

phase change $\Leftrightarrow$ time shift

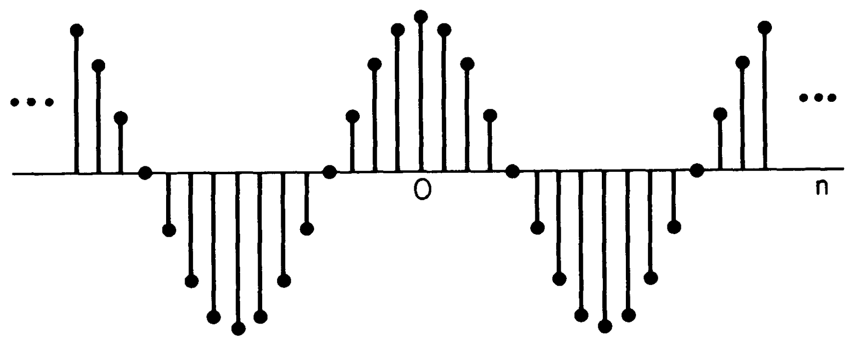
TRANSPARENCY 2.6
Relationship between a time shift and a phase change for discrete-time sinusoidal signals. In discrete time, a time shift always implies a phase change.

হতে হলে।

otherwise phase change হলেও time shift হবে না।
(n_0 integer না)

TRANSPARENCY 2.7
The sequence $A \cos \Omega_0 n$ illustrating the symmetry of an even sequence.

$\phi = 0$ $\quad x[n] = A \cos \Omega_o n$



$\Omega_o = \frac{\pi}{8}$

even: $\quad x[n] = x[-n]$ $\quad$ even symmetry

TRANSPARENCY 2.8
The sequence $A \sin \Omega_0 n$ illustrating the antisymmetric property of an odd sequence.

$\phi = -\frac{\pi}{2}$

$$x[n] = \begin{cases} A \cos(\Omega_o n - \frac{\pi}{2}) \\ A \sin \Omega_o n \\ A \cos[\Omega_o(n - n_o)] \end{cases}$$

$n_o = ?$ $\quad \frac{1}{4} \times$ period

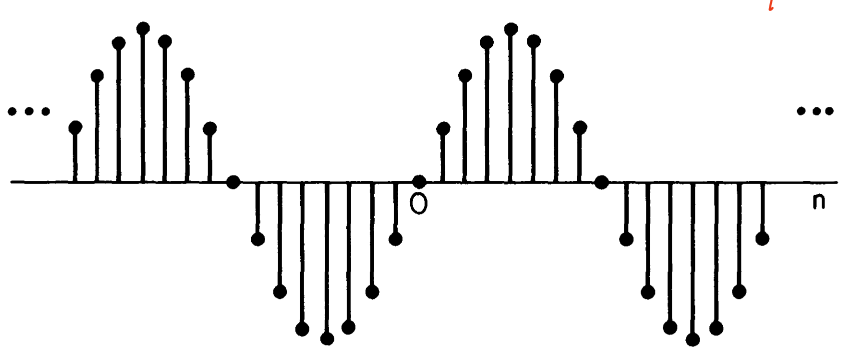
$\Omega_o = \frac{\pi}{8}$

$\Omega_0 n_0 = -\frac{\pi}{2}$

$\rightarrow \frac{\pi}{8} n_0 = -\frac{\pi}{2}$

$\therefore \boxed{n_0 = -4}$

$\therefore$ time shift হবে -4 amount.



odd: $\quad x[n] = -x[-n]$ $\quad$ odd symmetry

Time Shift => Phase Change

$$A\cos[\Omega_o(n+n_o)] = A\cos[\Omega_o n + \Omega_o n_o]$$

Time Shift $\overset{?}{<=}$ Phase Change → no

$$A\cos[\Omega_o(n+n_o)] \overset{?}{=} A\cos[\Omega_o n + \phi]$$

$\underbrace{-\Omega_o n_o}_{\phi}$

integer হতে হবে

⇐ হওয়ার জন্য।

TRANSPARENCY 2.9
For a discrete-time sinusoidal sequence a time shift always implies a change in phase, but a change in phase might not imply a time shift.

in continuous time, the amount of time shift didn't have to be an integer amount. But in discrete time — it must be an integer.

$$x[n] = A\cos(\Omega_o n + \phi)$$

Periodic?

$x[n] = x[n+N]$ smallest integer $N \overset{\Delta}{=}$ period

$$A\cos[\Omega_o(n+N)+\phi] = A\cos[\Omega_o n + \underbrace{\Omega_o N}_{} + \phi]$$

integer multiple of 2π ?

Periodic $=> \Omega_o N = 2\pi m$

$$\boxed{N = \frac{2\pi m}{\Omega_o}}$$

N,m must be integers

smallest N (if any) = period

TRANSPARENCY 2.10
The requirement on Ω_0 for a discrete-time sinusoidal signal to be periodic.

✓ all continuous time sinusoids are periodic. In discrete case — not necessarily true.

xm এ আসতে পারে

TRANSPARENCY 2.11
Several sinusoidal sequences illustrating the issue of periodicity.

$\Omega_0 = \frac{2\pi}{12}$

$\rightarrow 12 = \frac{2\pi}{\Omega_0}$

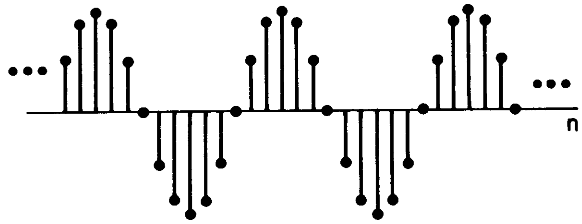
$m = 1$

$N = 12$

$\therefore$ signal periodic.

$\Omega_0 = \frac{2\pi}{12}$

$\phi = 0$

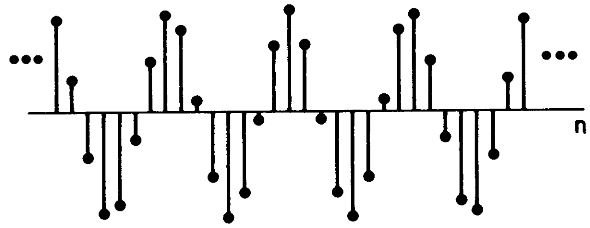


$N = \frac{2\pi m}{\Omega_0}$

format এ নিয়ে N, m এর value বের করবো।

$\Omega_0 = \frac{8\pi}{31}$

$\phi = 0$



periodic $\begin{cases} N = 31 \\ m = 4 \end{cases}$

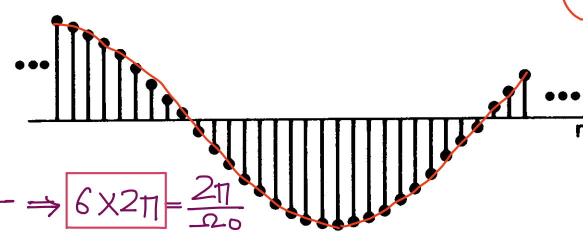
continuous time equiv to this - periodic

envelop looks periodic but signal is not periodic.

discrete time

$\Omega_0 = \frac{1}{6}$

$\phi = 0$



$6 = \frac{1}{\Omega_0} \Rightarrow \boxed{6 \times 2\pi} = \frac{2\pi}{\Omega_0}$

integer হবে না, so periodic না, — Aperiodic

TRANSPARENCY 2.12
Some important distinctions between continuous-time and discrete-time sinusoidal signals.

Continuous	Discrete
$A \cos(\omega_0 t + \phi)$	$A \cos(\Omega_0 n + \phi)$
Distinct signals for distinct values of ω_0	Identical signals for values of Ω_0 separated by 2π
Periodic for any choice of ω_0	Periodic only if $\Omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0$ and m

$A\cos(\Omega_1 n + \phi)$ $\quad$ Ω_1, Ω_2 seperated by 2π.

$A\cos(\Omega_2 n + \phi)$ $\quad$ so $= \Omega_2 = 2\pi + \Omega_1$

$= A\cos\{(2\pi + \Omega_1)n + \phi\}$

$= A\cos\{\Omega_1 n + \phi + 2\pi n\}$

$= A\cos(\Omega_1 n + \phi)$, $[n \in \mathbb{Z}]$

↳ discrete এর জন্য hold করবে but
continuous " " hold " না। cz continuous এ n → t
যেকোনো কিছু হতে পারে।

→ 2π দিয়ে যদি seperated থাকে তাহলে signal দুটো identical.

SINUSOIDAL SIGNALS AT DISTINCT FREQUENCIES:

Continuous time:

$x_1(t) = A\cos(\omega_1 t + \phi)$ If $\omega_2 \neq \omega_1$

$x_2(t) = A\cos(\omega_2 t + \phi)$ Then $x_2(t) \neq x_1(t)$

Discrete time:

$x_1[n] = A\cos[\Omega_1 n + \phi]$ If $\Omega_2 = \Omega_1 + 2\pi m$

$x_2[n] = A\cos[\Omega_2 n + \phi]$ Then $x_2[n] = x_1[n]$

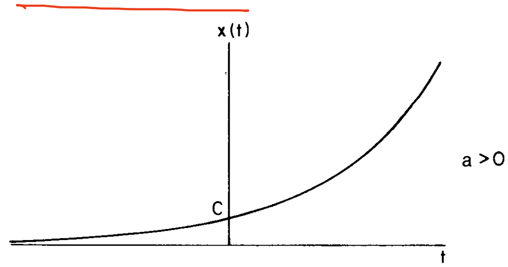
$\Omega_1, \Omega_2 \rightarrow$ two diff. frequencies

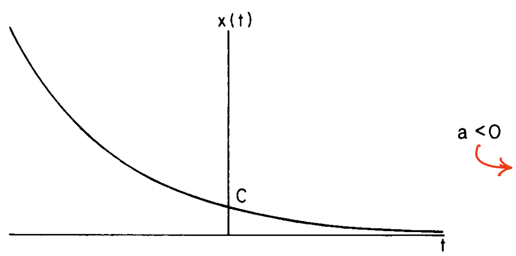
TRANSPARENCY 2.13
Continuous-time sinusoidal signals are distinct at distinct frequencies. Discrete-time sinusoidal signals are distinct only over a frequency range of 2π.

REAL EXPONENTIAL: CONTINUOUS-TIME

$x(t) = Ce^{at}$

C and a are real numbers





↳ decaying exponential function

Time Shift <=> Scale Change

$Ce^{a(t+t_0)} = Ce^{at_0}e^{at}$ ↳ Ce^{at_0} পরিমাণ scale change হবে।

TRANSPARENCY 2.14
Illustration of continuous-time real exponential signals.

$\alpha < 0 \rightarrow \beta$ imaginary number
discrete time case এ possible

TRANSPARENCY 2.15
Illustration of discrete-time real exponential sequences.

REAL EXPONENTIAL: DISCRETE-TIME

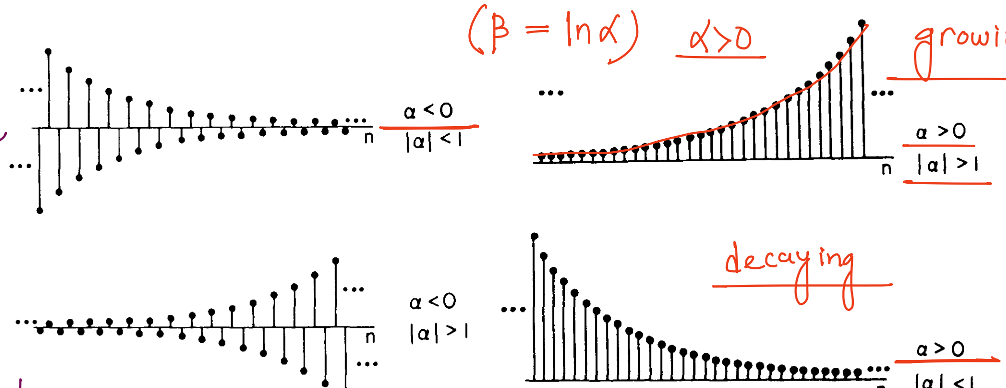
$$x[n] = Ce^{\beta n} = C\alpha^n$$

C, α are real numbers → complex number ও হতে পারে। এখানে IR ধরে নিচ্ছি।

α negative হলে β complex হবে।

$\boxed{e^{\beta} = \alpha}$

$(\beta = \ln \alpha)$ $\alpha > 0$ growing



decaying

↳ $x[n] = (-2)^n$

↳ $x[n] = \left(\frac{1}{2}\right)^n$

↳ decay করবে

যখন $n = 0, 2, 4 \ldots$ (even)
$\alpha < 0 \rightarrow \alpha^n > 0$
$n = 1, 3, 5 \ldots$ (odd)
$\alpha < 0 \rightarrow \alpha^n < 0$

$(-2)^n$

n	$x[n]$
0	1
1	-2
2	4
3	-8

→ (+ve)/(-ve) alternate হচ্ছে

TRANSPARENCY 2.16
Continuous-time complex exponential signals and their relationship to sinusoidal signals.

COMPLEX EXPONENTIAL: CONTINUOUS-TIME

$$x(t) = Ce^{at}$$

C and a are $\boxed{\text{complex numbers}}$

exponential part and sinusoidal part আলাদা করতে →

$$C = |C|\, e^{j\theta}$$ (polar form)

$$a = r + j\omega_o$$

$$x(t) = |C|\, e^{j\theta}\, e^{(r + j\omega_o)t}$$

$$= \underbrace{|C|\, e^{rt}}_{\text{time varying amplitude}}\, e^{j(\omega_o t + \theta)}$$

Euler's Relation: $\cos(\omega_o t + \theta) + j\sin(\omega_o t + \theta) = e^{j(\omega_o t + \theta)}$

$$x(t) = \underbrace{|C|\, e^{rt}\cos(\omega_o t + \theta)}_{\text{Real}} + j\, \underbrace{|C|\, e^{rt}\sin(\omega_o t + \theta)}_{\text{imaginary}}$$

2 sinu. signal 90° out of phase.

$\boxed{\text{real exponential part} \times \text{real sinusoidal part}}$

$r = 0$ হলে → pure sinusoidal signal পাবো।

$\rightarrow |C| e^{rt}$

$$x(t) = C e^{at}$$

$C, a \rightarrow$ complex

$C = |C| e^{j\theta}$ (polar format এ)

$a = r + j\boxed{\omega_0}$ (cartesian format এ)

$$x(t) = |C| e^{j\theta} \cdot e^{(r + j\omega_0)t}$$

$$= |C| e^{rt} \cdot e^{j(\theta + \omega_0 t)}$$

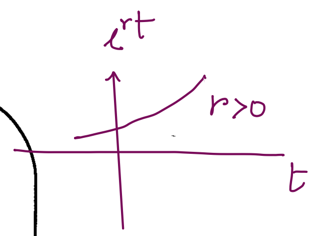
$$= |C| e^{rt} \cdot \{\cos(\theta + \boxed{\omega_0} t) + j \sin(\theta + \omega_0 t)\}$$

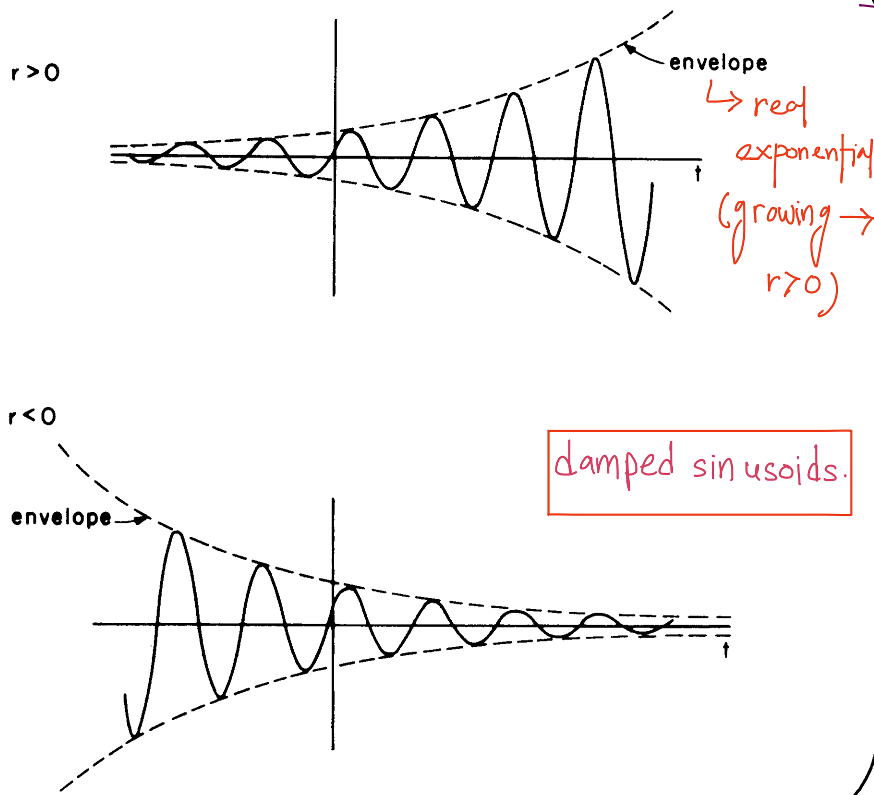
$\rightarrow$ frequency

$$C = |C| e^{j\theta}$$

Complex number C এর argument $\rightarrow$ phase $\rightarrow \theta$

$|c| = e^{rt} \cos(\omega_0 t + \theta)$





TRANSPARENCY 2.17
Sinusoidal signals with exponentially growing and exponentially decaying envelopes.

COMPLEX EXPONENTIAL: DISCRETE-TIME

$$x[n] = C\alpha^n$$

C and α are complex numbers

$$C = |C|\, e^{j\theta}$$

$$\alpha = |\alpha|\, e^{j\Omega_0}$$

$$x[n] = |C|\, e^{j\theta}\, (|\alpha|\, e^{j\Omega_0})^n$$

$$= \underbrace{|C|\, |\alpha|^n}_{\text{real exponential (amplitude)}}\ \underbrace{e^{j(\Omega_0 n + \theta)}}_{\text{purely imaginary exponential}}$$

real exponential (amplitude) → purely imaginary exponential

Euler's Relation: $\cos(\Omega_0 n + \theta) + j \sin(\Omega_0 n + \theta)$

$$x[n] = |C|\, |\alpha|^n \cos(\Omega_0 n + \theta) + j\, |C|\, |\alpha|^n \sin(\Omega_0 n + \theta)$$

90° out of phase

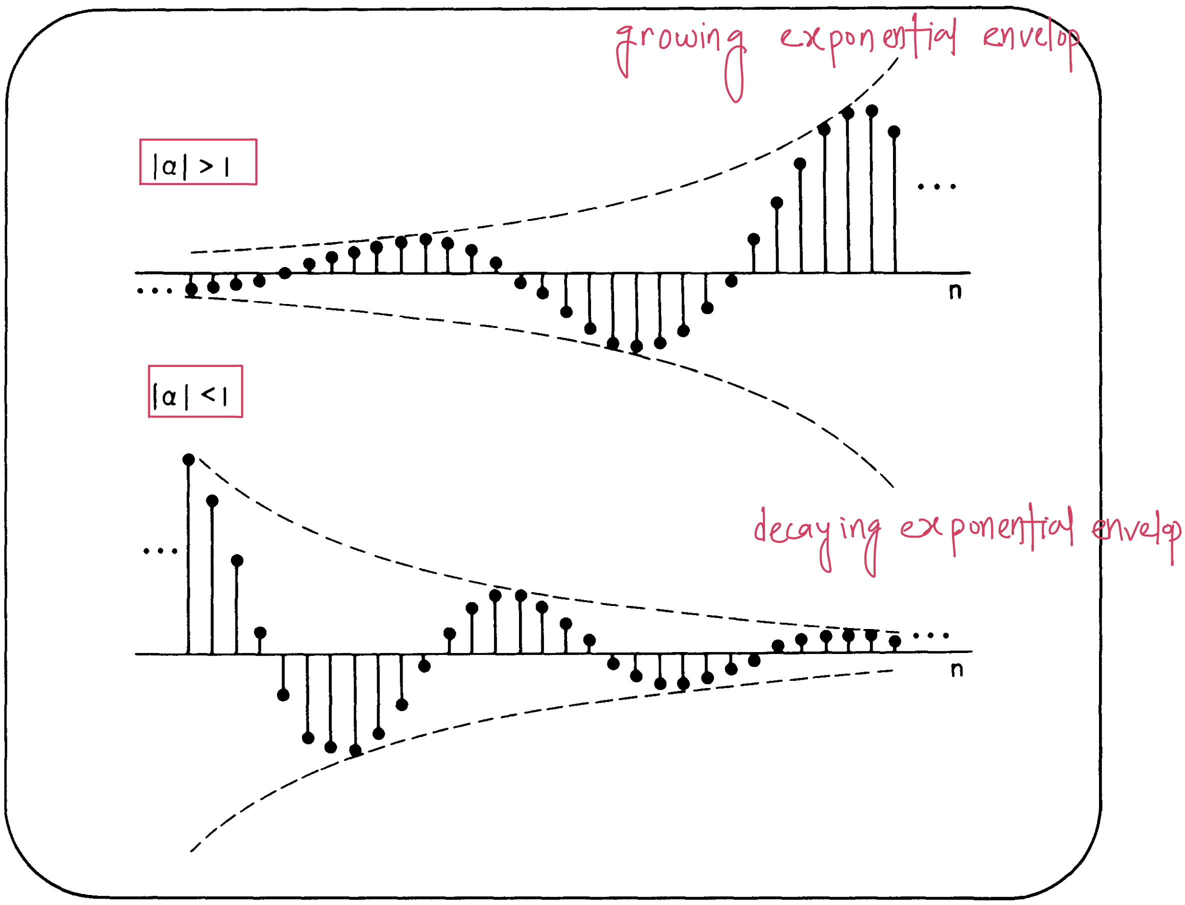
$|\alpha| = 1$ => sinusoidal real and imaginary parts

$Ce^{j\Omega_0 n}$ periodic ? — may or may not be periodic depending on the value of Ω_0

sinusoidals

TRANSPARENCY 2.18
Discrete-time complex exponential signals and their relationship to sinusoidal signals.

TRANSPARENCY 2.19
Sinusoidal sequences with geometrically growing and geometrically decaying envelopes.

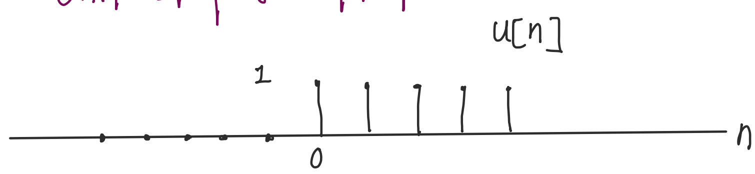


$\alpha = 1$, both real and imaginary part এ pure sinusoidal signal থাকবে।

In a continuous time case when we have a pure complex exponential $\longrightarrow$ those exponentials are always periodic. Because the real and imaginary components (sinusoidal) are always periodic.

Resource: Signals and Systems
Professor Alan V. Oppenheim

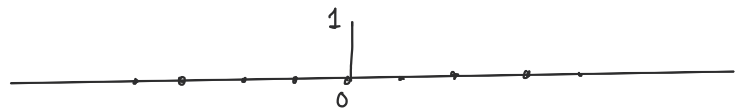
1. Unit step function



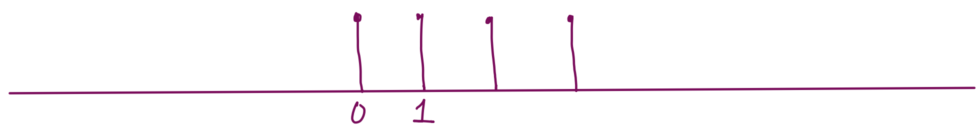
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

2. Unit Impulse function

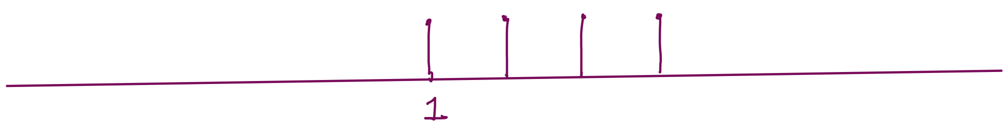
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



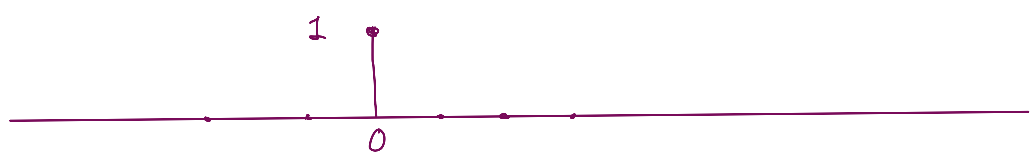
$\delta[n] = u[n] - u[n-1]$



$u[n]$



$u[n-1]$
right shift



$u[n] - u[n-1]$

↓

শুধু 0 তে value 1

বাকি সব জায়গায় value 0।

impulse function. → $n=0$ তে একটা pulse আছে।

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

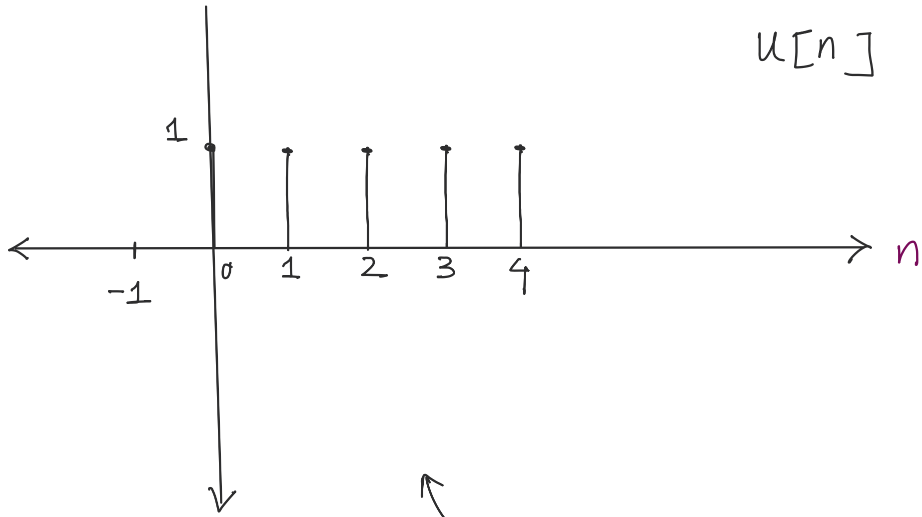
$$\left.\begin{array}{l} m = -\infty \text{ to } 1 \\ m = 0 \text{ to } n \end{array}\right\}$$

unit impulse function এর value গুলো add করলেই $u[n]$ পাবো।

↓

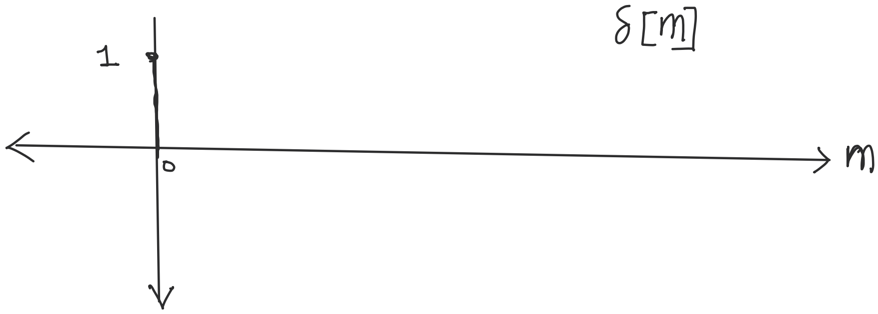
unit step function

$u[n]$



এটা বানাতে চাই → running sum

$\delta[m]$



unit impulse function to unit step function →

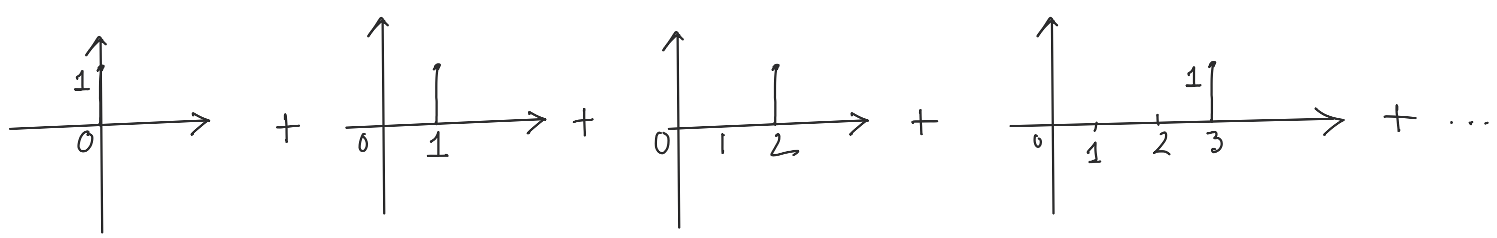
অনেকগুলো pulse লাগবে।

বিভিন্ন point in time এ pulse বের করে add করে।

↓

unit step function এ right shift করে

$u[n-1]$



$\delta[n]$ $\quad$ $\delta[n-1]$ $\quad$ $\delta[n-2]$ $\quad$ $\delta[n-3]$

$n-0$

$$\sum_{0}^{\infty} \delta[n-k] \qquad \underline{n-k=m}$$

$$\hookrightarrow \sum_{-\infty}^{n} \delta[m] = u[n]$$