Fourier Series

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Periodic Waveforms

- **A signal has period *T*,** $x(t) = x(t \pm nT)$ **if for all *t***

  Also periodic with periods $2T$, $3T$, etc., and $-T$, $-2T$ ...

  Smallest positive period T_0 is called the *fundamental period*

  Fundamental frequency f_0 is computed as $1 / T_0$

- **Harmonics**

  Harmonics are higher-frequency components whose frequencies are integer multiples of the fundamental frequency. For example, if the fundamental frequency is f_0, then the second harmonic will have a frequency of $2f_0$, the third harmonic will have a frequency of $3f_0$ and so on.

- **Finding fundamental frequency**

  Largest f_0 such that $f_k = k f_0$, i.e. $f_0 = \gcd\{ f_k \}$

  Consider notes A 440 Hz, E 660 Hz and F 880 Hz. $f_0 = 220$ Hz

Fourier Series

- ***Periodic* signals can be synthesized**

Periodic functions (like a signal that repeats itself over time) can be broken down into a sum of simple waveforms—specifically, sinusoids (sine and cosine functions).

These sinusoids have different frequencies, amplitudes, and phases. When added together, they reconstruct the original signal.

Conditions for the Existence of FS

- **Finite Number of minima and maxima in one period of time**

- **Finite number of discontinuities in one period of time**

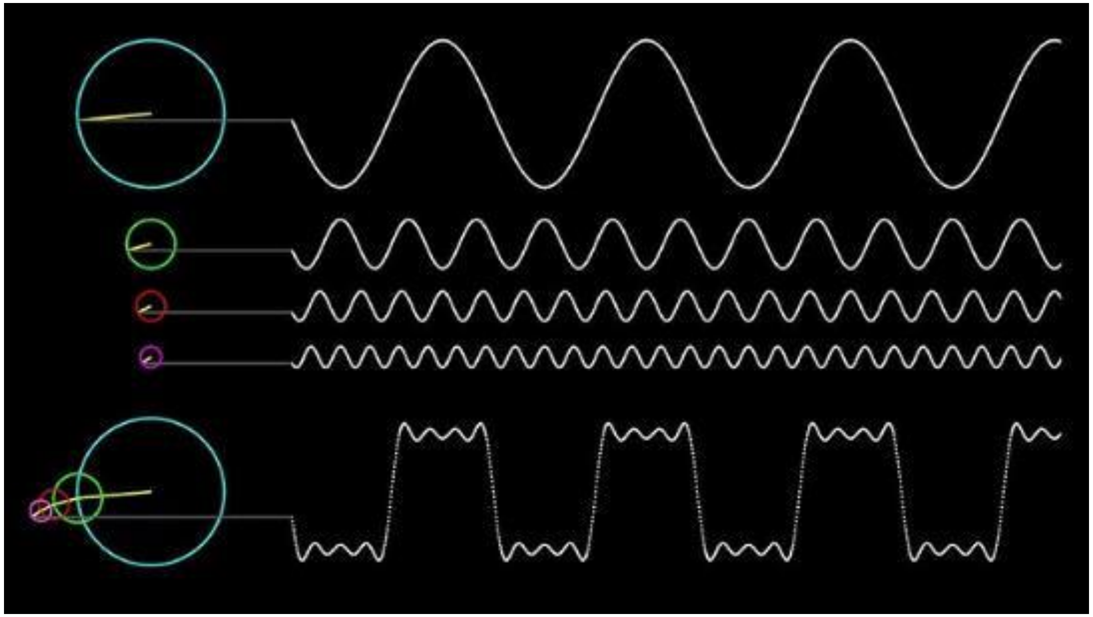
- **Absolutely integrable in one period**

$$\int_{T_0} |x(t)| dt < \infty$$

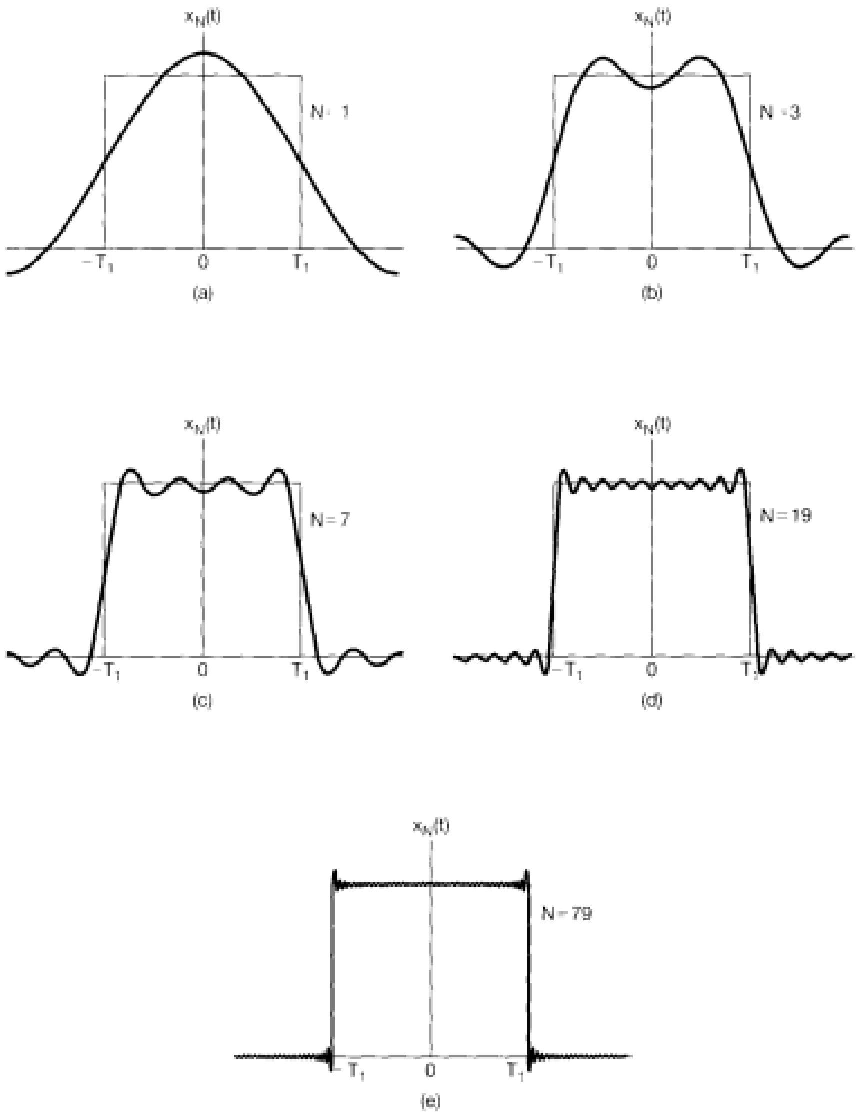
Fourier Series (Trigonometric Form)

- Represents a periodic signal

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$



Convergence



Expressions for the coefficients

- **Coefficients**

$$a_0, a_n \ and \ b_n$$

Expressions for the coefficients

- **Coefficients**

$$a_0, a_n \ and \ b_n$$

- **Equations**

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t) dt$$

Helpful Properties for Proof

$$\int_{-L}^{L} \cos(nx)\cos(mx)dx =$$

{
0; when n != m
L; when n=m
}

$$\int_{-L}^{L} \sin(nx)\sin(mx)dx =$$

{
0; when n != m
L; when n=m
}

Exercises

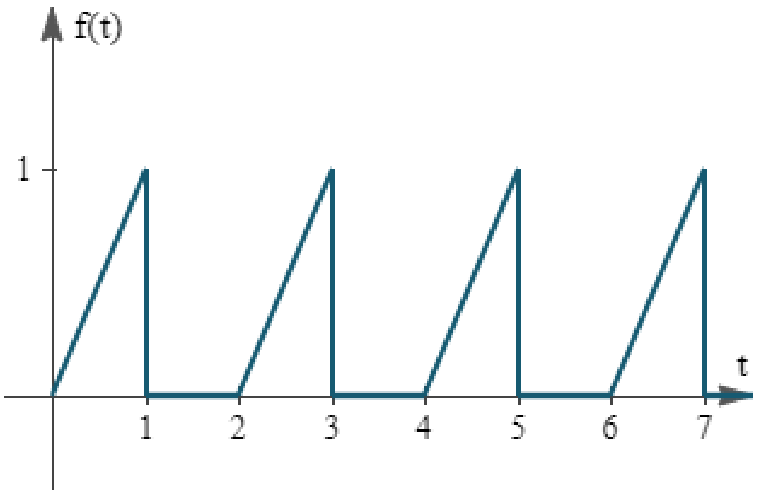
(1) $x(t) = \{\ 1.5$ **when** $0 \le t < 1$

$\quad -1.5$ **when** $1 \le t < 2$

$\}$

x(t) is periodic. Find the coefficients

(2) Determine the Fourier series representations for the following signal:



Solution (1)

$$Here, T = 2$$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{1}{2}\int_0^2 x(t)dt = \frac{1}{2}\left(\int_0^1 (1.5)dt + \int_1^2 (-1.5)dt\right)$$

$$= 0$$

$$a_n = \frac{2}{2}\int_0^1 1.5\cos(n\pi t)dt + \int_1^2 (-1.5)\cos(n\pi t)dt$$

$$= 1.5\frac{\sin(n\pi t)}{n\pi}\Big]_0^1 - 1.5\frac{\sin(n\pi t)}{n\pi}\Big|_1^2$$

$$= 0$$

Contd.

$$b_n = \frac{2}{2}\int_0^1 1.5\sin(n\pi t)dt - \int_1^2 \; 1.5\sin(n\pi t)dt$$

$$= -\frac{1.5}{n\pi}\cos(n\pi t)\,|_0^1 + \frac{1.5}{n\pi}\cos(n\pi t)\,|_1^2$$

$$= -\frac{1.5}{n\pi}(\cos(n\pi)-1) + \frac{1.5}{n\pi}(\cos(2n\pi)-\cos(n\pi))$$

When n is even, $b_n = 0$. But when n is odd $b_n = \dfrac{6}{n\pi}$

Fourier Series (Complex Form)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Intuition Behind the Coefficients:

•Each a_k represents the amplitude and phase of a particular frequency component $k\omega_0$ within the signal.

•If a_k is large, the frequency component $k\omega_0$ has a significant contribution to the signal. If a_k is small, the contribution is minimal.

•The complex nature of a_k encodes both the amplitude (magnitude) and the phase (angle) of the corresponding sinusoidal component.

Question

How is the complex form equivalent to trigonometric form?

Fourier Series

- ***Analysis:* start with $x(t)$ and compute $\{ a_k \}$**

Integrate $x(t)$ over fundamental period T_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_0 t}\,dt$$

Calculation of a_0 simplifies to average value of $x(t)$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

Example #1: With $x(t) = \cos(2\pi f_0 t)$, what is a_0 ?

Example #2: With $x(t) = \cos^2(2\pi f_1 t)$, what is a_0 ?

Spectrum of the Fourier Series

- **Find Fourier series coefficients for $x(t) = \cos^3(3\pi t)$**

Approach **#1** $$a_k = \frac{1}{T_0}\int_0^{T_0} \cos^3(3\pi t)\, e^{-j2\pi k f_0 t}\, dt$$

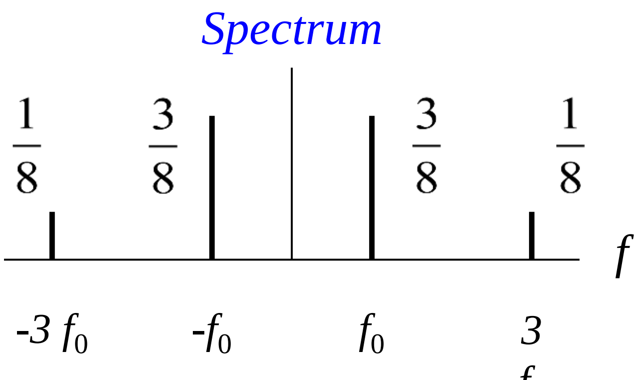
Approach #2: Expand into complex exponentials

$$x(t) = \left(\frac{e^{j3\pi t} + e^{-j3\pi t}}{2}\right)^3 = \frac{1}{8}\left(e^{j9\pi t} + 3e^{j3\pi t} + 3e^{-j3\pi t} + e^{-j9\pi t}\right)$$

- **Resulting spectrum**

$\omega_0 = \gcd(3\pi, 9\pi) = 3\pi$

$f_0 = 1.5$ Hz

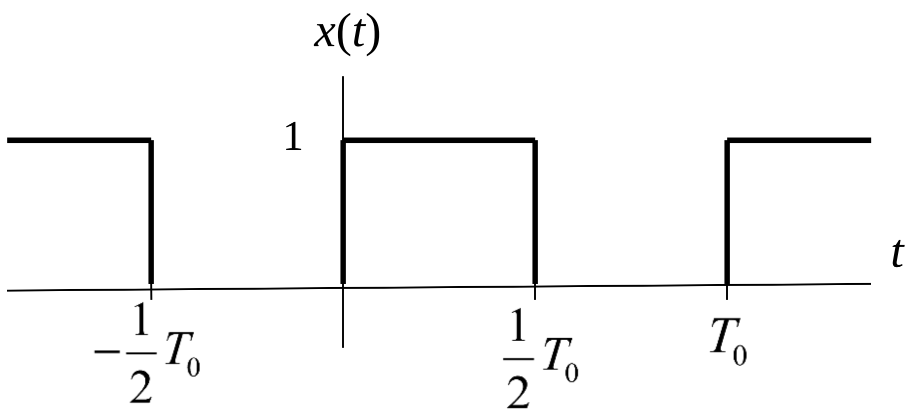


Fourier Analysis of a Square Wave

- **Periodic square wave with 50% duty cycle**

Defined for one period as

$$s(t)=\begin{cases}1 \text{ for } 0\le t<\frac{1}{2}T_0\\[2ex] 0 \text{ for } \frac{1}{2}T_0\le t<T_0\end{cases}$$



- **Fourier coefficients**

1. a_0 = ½ because $x(t)$ is 1 half the time and 0 half the time
2. Then, $a_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kf_0t}dt=\frac{1}{T_0}\int_0^{\frac{1}{2}T_0}e^{-j2\pi kf_0t}dt=\left(\frac{1}{T_0}\right)\frac{e^{-j2\pi kf_0t}}{-j2\pi kf_0}\bigg|_0^{\frac{1}{2}T_0}$

$\boxed{For\ k\neq 0}$

$$a_k=\left(\frac{1}{T_0}\right)\frac{e^{-j2\pi kf_0(T_0/2)}-e^{-j2\pi kf_0(0)}}{-j2\pi kf_0}=-\frac{e^{-j\pi k}-1}{j2\pi k}=\frac{1-(-1)^k}{j2\pi k}$$

Spectrum for a Square Wave

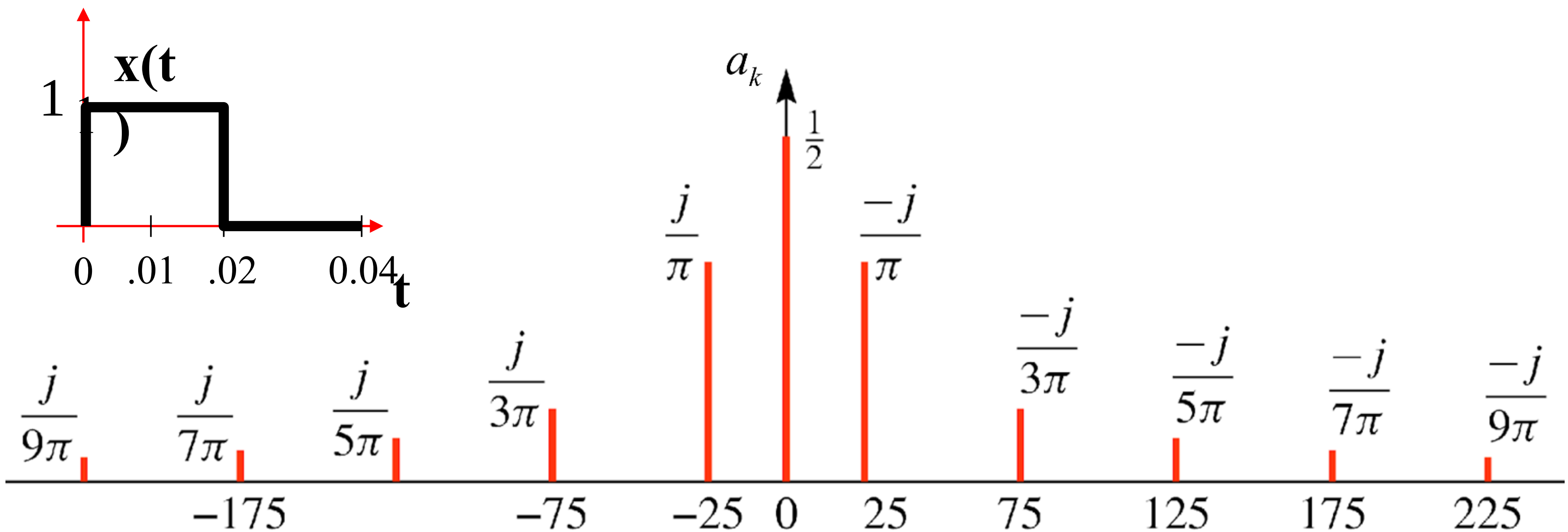
- **Fourier coefficients**

  Independent of T_0

- **Example**

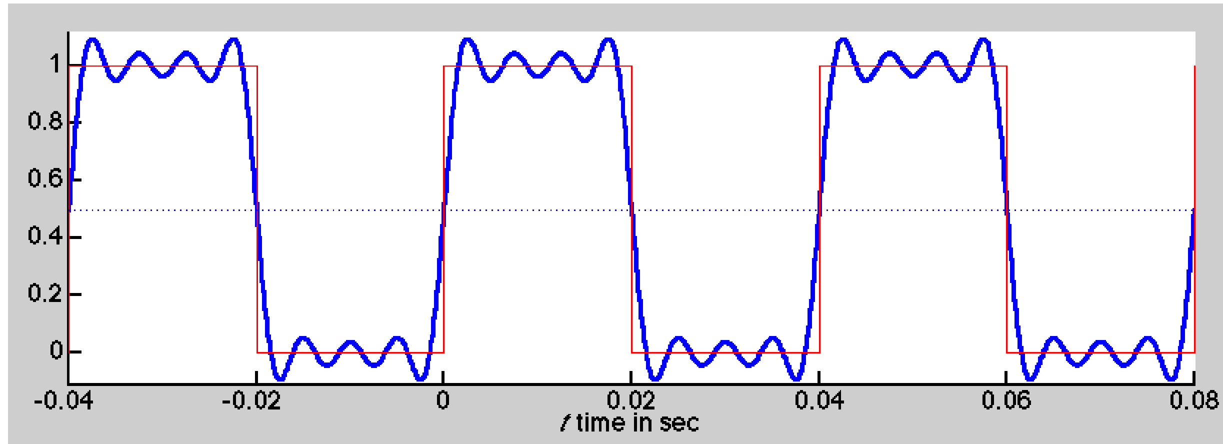
  $T_0 = 0.04$ s □ $f_0 = 25$ Hz

$$a_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ 0 & \text{for } k \text{ even but not zero} \\ -\frac{j}{\pi k} & \text{for } k \text{ odd} \end{cases}$$

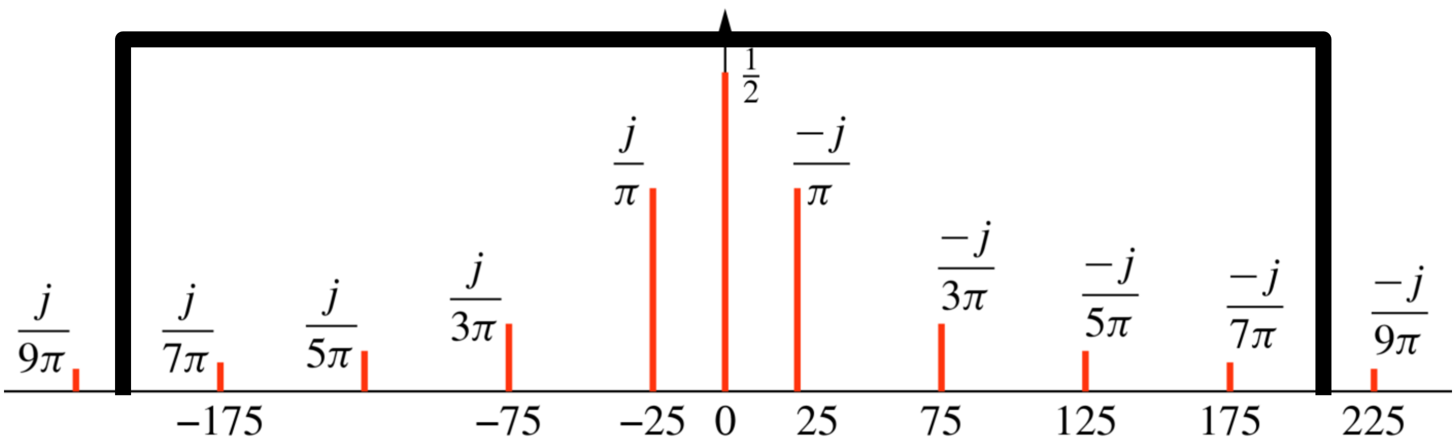


Fourier Synthesis of a Square Wave

- **Synthesis using up to 7th harmonic**



$$y(t)=\frac{1}{2}+\frac{2}{\pi}\sin(50\pi t)+\frac{2}{3\pi}\sin(150\pi t)+\frac{2}{5\pi}\sin(250\pi t)+\frac{2}{7\pi}\sin(350\pi t)$$



demo

Properties of CTFS

- **Linearity**

Let x(t) and y(t) denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k respectively. That is,

x(t) $\rightarrow a_k$

y(t) $\rightarrow b_k$

Now,

z(t) = Ax(t) + By(t)

z(t) $\rightarrow Aa_k + Bb_k$

Contd.

- **Time Shifting**

x(t) → a_k

Then, if the signal is delayed $t_0 s$,

$$x(t - t_0) \rightarrow e^{-jk\omega_0 t_0} a_k = \mathrm{b_k}$$

One consequence of this property is that, when a periodic signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered. That is,

$$|b_k| = |a_k|$$

Contd.

- **Time Reversal**

$$x(t) \rightarrow a_k$$
$$x(-t) \rightarrow a_{-k}$$

- **Time scaling**

$$x(t) \rightarrow \mathrm{a_k}$$
$$x(at) \rightarrow a_k \text{ (No change in coefficients)}$$

But time period and frequency will change

$$T_0' = \frac{\mathrm{T_0}}{\mathrm{a}}$$
$$\omega_0' = \omega_0 a$$

Contd.

- **Conjugation**

$$\boldsymbol{x(t) \rightarrow a_k}$$
$$\boldsymbol{x(t)^* \rightarrow a^*_{-k}}$$

- **Differentiation in Time**

$$x(t) \rightarrow a_k$$

$$\frac{d}{dt}(x(t)) \rightarrow jk\omega_0 a_k$$

Generalized form,

$$\frac{d^n}{dt^n}(x(t)) \rightarrow (jk\omega_0)^n a_k$$

Contd.

- **Integration in Time**

$$x(t) \rightarrow a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \rightarrow \frac{a_k}{jk\omega_0}$$

- **Convolution**

$$x_1(t) \rightarrow a_{k1}$$

$$x_2(t) \rightarrow a_{k2}$$

$$x(t) = x_1(t) * x_2(t), then$$

$$x(t) \rightarrow T_0(a_{k1}a_{k2})$$

Here, x(t) is convolution of x1(t) and x2(t)

Symmetricities in FS

- **Even Symmetry**

  If the signal is even, then FS expansion will have harmonics of even s ignals. That means b_n will be 0

- **Odd Symmetry**

  If the signal is even, then FS expansion will have harmonics of even s ignals. That means b_n will be 0

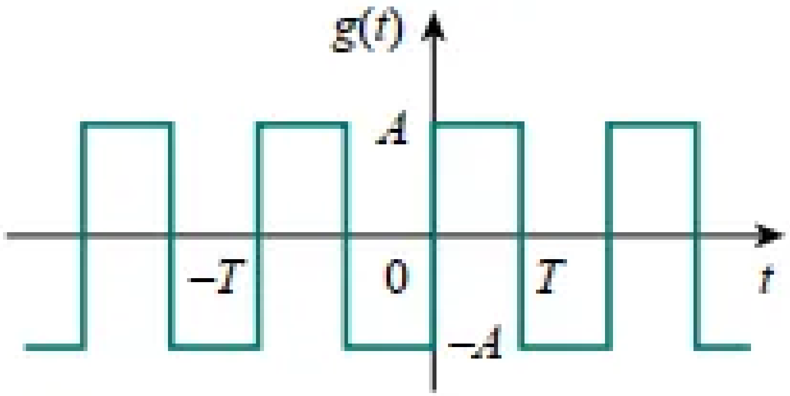
- **Half wave symmetry**

  If the signal has half wave symmetry, then FS expansion will only have odd harmonics

Half wave symmetry

- **Condition**

$$x(t) = -x(t \pm T_0 / 2)$$



(b)

Proof

- **If the signal has half wave symmetry, then FS expansion will only have odd harmonics**

$$x(t) \xrightarrow{\text{FS}} a_k$$

$$so,\ x(t+\frac{T_0}{2}) \xrightarrow{\text{FS}} a_k e^{jk\omega_0 \frac{T_0}{2}}$$

$$\Rightarrow -x(t+\frac{T_0}{2}) \xrightarrow{\text{FS}} -a_k e^{jk\omega_0 \frac{T_0}{2}}$$

$$\Rightarrow a_k = -a_k e^{jk\omega_0 \frac{T_0}{2}}$$

$$\Rightarrow 1 + e^{jk\omega_0 \frac{T_0}{2}} = 0$$

$$\Rightarrow 1 + e^{jk\pi} = 0$$

$$\Rightarrow 1 + (-1)^k = 0$$

The above equation can only be true when k is odd.

Parseval's Power Theorem

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Where C_n *is the fourier series coefficient of* $x(t)$

- **Basic Equation of avg power**

$$P_{x(t)} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Example

- **Find the avg power of the signal $\cos^3(3\pi t)$**

$$x(t)=\left(\frac{e^{j3\pi t}+e^{-j3\pi t}}{2}\right)^3=\frac{1}{8}\left(e^{j9\pi t}+3e^{j3\pi t}+3e^{-j3\pi t}+e^{-j9\pi t}\right)$$

Coefficients are,

$$C_1=C_{-1}=\frac{3}{8}$$

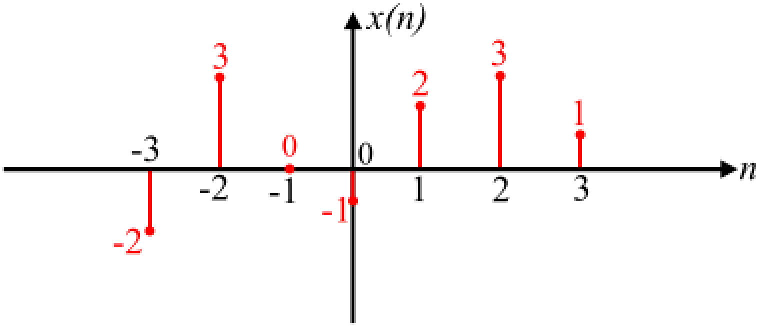
$$C_3=C_{-3}=\frac{1}{8}$$

$$P_{x(t)}=\sum_{n=-\infty}^{\infty}|C_n|^2=2.(|\frac{3}{8}|)^2+2.(|\frac{1}{8}|)^2={}^5\!/_{16}\ \text{watts}$$

DTFS

- **Discretete Time Signal**

The signals which are defined only at discrete instants of time are known as discrete time signals. The discrete time signals are represented by x[n] where n is the independent variable in time domain.



DTFS

- **Representation**

$$x[n] = \sum_{k=<N>} x(k) e^{jk\Omega_0 n}$$

Where

$$\Omega_0 = 2\pi / N$$

N = fundamental frequency
x(k) = kth coefficient

DTFS

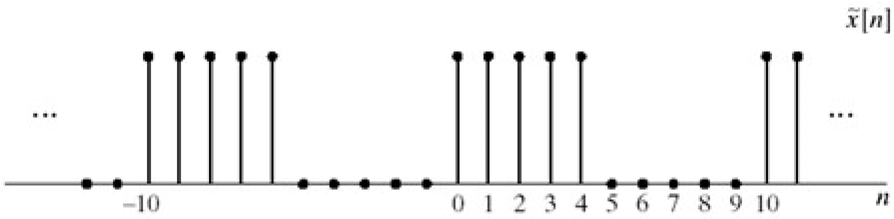
- **Calculating the coefficient**

$$x(k) = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\Omega_0 n}$$

Practice Problems

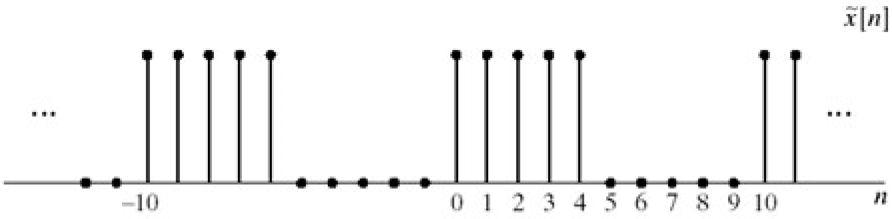
(1) $x[n] = \cos(\frac{\pi}{3} n)$

(2)



Find coefficients and draw the spectrums

Solution



From the graph,

N=10

$$x(k) = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\Omega_0 n}$$

$$= \frac{1}{10} \sum_{n=0-4} x[n] e^{-jk\Omega_0 n}$$

$$= \frac{1}{10} (1 + e^{-jk\frac{\pi}{5}} + e^{-jk\frac{2\pi}{5}} + e^{-jk\frac{3\pi}{5}} + e^{-jk\frac{4\pi}{5}})$$

$$\Omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$$

Solution

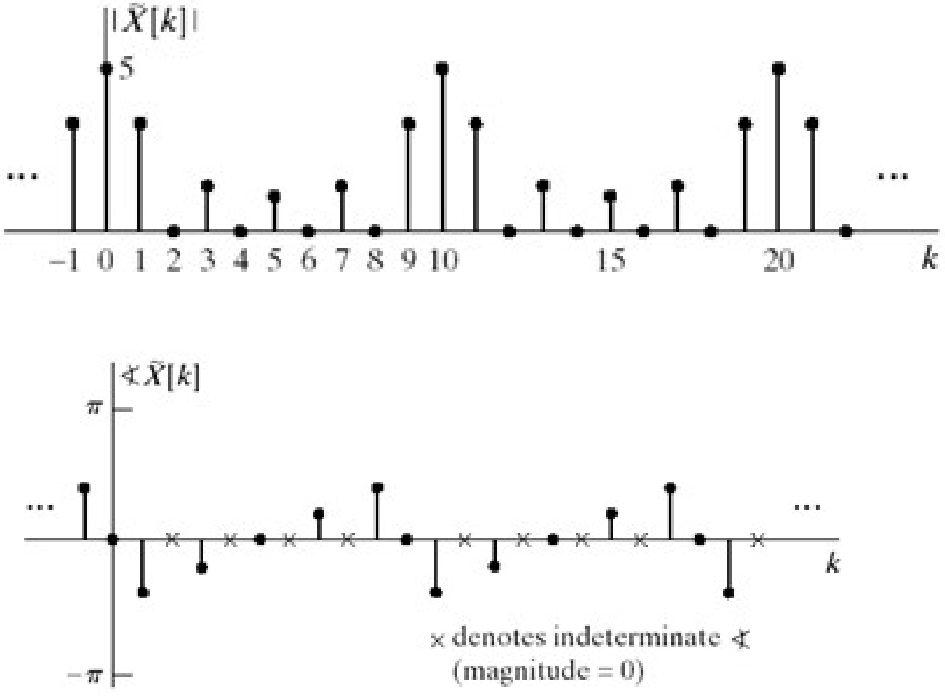
- **Now, for k=0**

  x(0) = 5/10 = 0.5

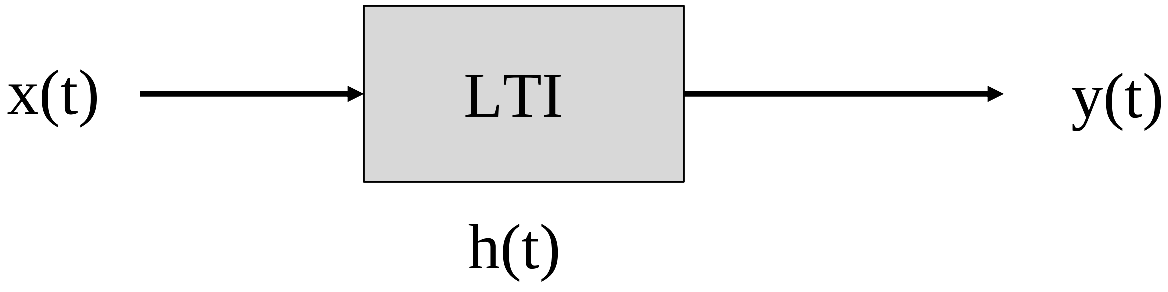
  x(1) = 0.310 $\angle -1.7$

  x(2)=0

Similarly, determine x(3) x(4) ….. (Do yourself)



Fourier Series for LTI System



$$x(t) \rightarrow a_k$$
$$y(t) \rightarrow H(kj\omega_0)a_k$$

Where h(t) is the impulse response $H(kj\omega_0)$ is the frequency response of the LTI system

Example

- **Input x(t)=cos(2πt)+sin(πt). Impulse Response**

H(s) = $\frac{1}{4+s}$ **. y(t) is the output of the LTI system. Find the FS coefficients of y(t).**

Solution:

$$x(t) = \cos(2\pi t) + \sin(\pi t)$$

$$= \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$

$$a_1 = \frac{1}{2j}$$

So, coefficients of input x(t)

$$a_2 = \frac{1}{2}$$

$$a_{-1} = -\frac{1}{2j}$$

$$a_{-2} = \frac{1}{2}$$

Contd

- Let, b_k be the coefficients for the output signal y(t)

$$b_1 = H(j\omega_0)a_1 = \frac{1}{4+j\omega_0}\frac{1}{2j} = \frac{1}{4+j\pi}.\frac{1}{2j}$$

$$b_{-1} = H(-j\omega_0)a_{-1} = \frac{1}{4-j\omega_0}\frac{1}{(-2j)} = \frac{1}{4-j\pi}.\frac{1}{(-2j)}$$

$$b_2 = H(2j\omega_0)a_2 = \frac{1}{4+2j\omega_0}\frac{1}{2} = \frac{1}{4+2j\pi}.\frac{1}{2}$$

$$b_{-2} = H(-2j\omega_0)a_{-2} = \frac{1}{4-2j\omega_0}\frac{1}{2} = \frac{1}{4-2\pi j}.\frac{1}{2}$$