Lecture 7
ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

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Introduction to Fourier Transforms

- Fourier transform
- Inverse Fourier transform: The Fourier integral theorem
- Example: the rect and sinc functions
- Cosine and Sine Transforms
- Symmetry properties
- Periodic signals and δ functions

Fourier Transforms

Given a continuous time signal $x(t)$, define its *Fourier transform* as the function of a real f:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt$$

This is similar to the expression for the Fourier series coefficients.

Note: Usually $X(f)$ is written as $X(i2\pi f)$ or $X(i\omega)$. This corresponds to the Laplace transform notation which we encountered when discussing transfer functions $H(s)$.

Continuous-time Fourier Transform

Which yields the *inversion formula* for the Fourier transform, the *Fourier integral theorem*:

$$\boxed{\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt, \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df. \end{aligned}}$$

Comments:

- There are usually technical conditions which must be satisfied for the integrals to converge – forms of smoothness or Dirichlet conditions.
- The intuition is that Fourier transforms can be viewed as a limit of Fourier series as the period grows to infinity, and the sum becomes an integral.
- $\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$ is called the *inverse Fourier transform* of $X(f)$. Notice that it is identical to the Fourier transform except for the sign in the exponent of the complex exponential.
- If the inverse Fourier transform is integrated with respect to ω rather than f, then a scaling factor of $1/(2\pi)$ is needed.

Cosine and Sine Transforms

Assume $x(t)$ is a possibly complex signal.

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)\left(\cos(2\pi ft) - j\sin(2\pi ft)\right) dt \\ &= \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt. \end{aligned}$$

Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $x(t)$ as

$$\mathcal{F}\left[x(t)\right] = X(f)$$

and the inverse Fourier transform of $X(f)$ as

$$\mathcal{F}^{-1}\left[X(f)\right] = x(t).$$

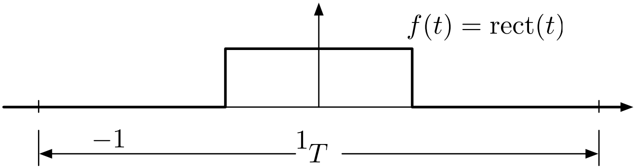
Note that

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[x(t)\right]\right] = x(t)$$

and at points of continuity of $x(t)$.

Rect Example

For example, assume $x(t) = \mathrm{rect}(t)$, and that we are computing the Fourier series over an interval T,



The Fourier Transform is: sinc(f)

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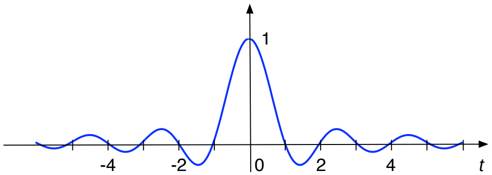
where $\mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$

The Sinc Function



Duality

Notice that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are almost the same.

Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

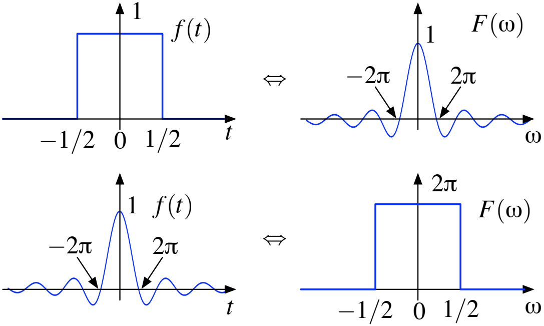
In other words, $\mathcal{F}\left[\mathcal{F}\left[x(t)\right]\right] \; = \; x(-t)$.

Example of Duality

- Since $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$ then

$$\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(-f) = \mathrm{rect}(f)$$

(Notice that if the function is even then duality is very simple)



Generalized Fourier Transforms: δ Functions

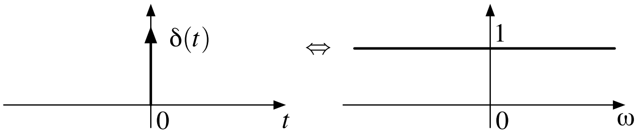
A unit impulse $\delta(t)$ is not a signal in the usual sense (it is a generalized function or distribution). However, if we proceed using the sifting property, we get a result that makes sense:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = 1$$

so

$$\delta(t) \Leftrightarrow 1$$

This is a *generalized Fourier transform*. It behaves in most ways like an ordinary FT.



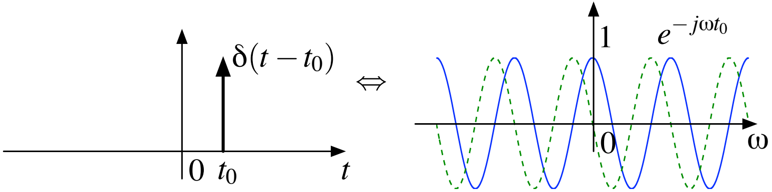
Shifted δ

A shifted delta has the Fourier transform

$$\begin{aligned}\mathcal{F}[\delta(t-t_0)] &= \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j2\pi ft}dt \\ &= e^{-j2\pi t_0 f}\end{aligned}$$

so we have the transform pair

$$\delta(t-t_0) \Leftrightarrow e^{-j2\pi t_0 f}$$



Constant

Next we would like to find the Fourier transform of a constant signal $x(t) = 1$. However, direct evaluation doesn't work:

$$\begin{aligned}\mathcal{F}[1] &= \int_{-\infty}^{\infty} e^{-j2\pi ft} dt \\ &= \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_{-\infty}^{\infty}\end{aligned}$$

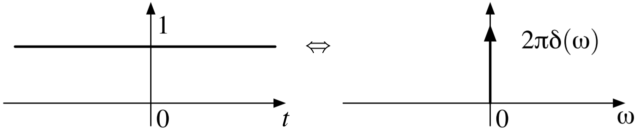
and this doesn't converge to any obvious value for a particular f.

We instead use duality to guess that the answer is a δ function, which we can easily verify.

$$\begin{aligned}\mathcal{F}^{-1}[\delta(f)] &= \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df \\ &= 1.\end{aligned}$$

So we have the transform pair

$$1 \Leftrightarrow \delta(f)$$



This also does what we expect – a constant signal in time corresponds to an impulse a zero frequency.

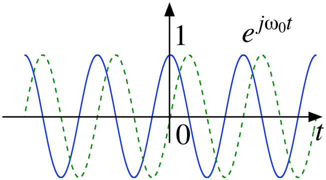
Sinusoidal Signals

If the δ function is shifted in frequency,

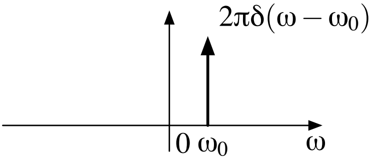
$$\begin{aligned} \mathcal{F}^{-1}\left[\delta(f - f_0)\right] &= \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} df \\ &= e^{j2\pi f_0 t} \end{aligned}$$

so

$$e^{j2\pi f_0 t} \Leftrightarrow \delta(f - f_0)$$



$\Leftrightarrow$



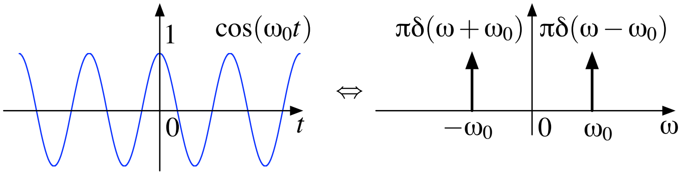
Cosine

With Euler's relations we can find the Fourier transforms of sines and cosines

$$\begin{aligned}\mathcal{F}\left[\cos(2\pi f_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\right] \\ &= \frac{1}{2}\left(\mathcal{F}\left[e^{j2\pi f_0 t}\right] + \mathcal{F}\left[e^{-j2\pi f_0 t}\right]\right) \\ &= \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right).\end{aligned}$$

so

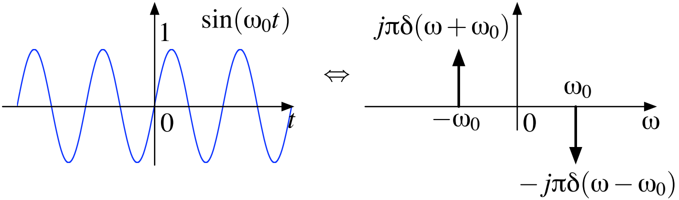
$$\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right).$$



Sine

Similarly, since $\sin(f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t})$ we can show that

$$\sin(f_0 t) \Leftrightarrow \frac{j}{2}\left(\delta(f + f_0) - \delta(f - f_0)\right).$$



The Fourier transform of a sine or cosine at a frequency f_0 only has energy exactly at $\pm f_0$, which is what we would expect.