General storage models

Lévy processes and their applications: exam assignment

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Introduction

CLASSICAL QUEUEING THEORY

- Stochastic dynamical system
- Discrete 'events' occur randomly in time, require random service time to process
- Examples: customers queueing at a shop, requests arriving at a server, ...
- Models described by Kendall's notation A/B/C:
  - A = distribution of the inter-arrival times
  - B = distribution of service time
  - C = number of servers

THE M/G/1 QUEUE

- Exponential (‘Markovian’) inter-arrival times
- General service distribution
- Single server
- Can be described by two processes:
  - Incoming work A_t
  - Potential processed work B_t

THE M/G/1 QUEUE

- Incoming work descibed by compound Poisson process

$$A_t := \sum_{i=1}^{N_t} \xi_i$$

  - N_t Poisson process with rate λ
  - ξ_i i.i.d. service times
  - F service distribution, $\xi_1 \sim F$
- Work processed at linear rate: $B_t = t$

LÉVY-DRIVEN QUEUES & GENERAL STORAGE MODELS

- Extend the classical theory
- Input process not required to come from discrete events
- Can be any Lévy process
- Examples: reservoir of a dam, aggregate internet traffic

Workload

NAIVE DEFINITION

- Difference of input and processed work:

$$D_t := A_t - B_t$$

- Problem: can become negative!
- Look for a process L_t to 'compensate' D_t on $\{D_t < 0\}$
- L_t is called a regulator

REGULATOR PROCESS

It turns out that L_t is uniquely determined by this condition.

Theorem 1 (Existence & uniqueness of regulator)

Let L_t be any stochastic process with increasing right-continuous sample paths such that

(i) $W_t = D_t + L_t \geq 0$

(ii) $\int \mathbb{1}_{\{W_t > 0\}} dL_t = 0$.

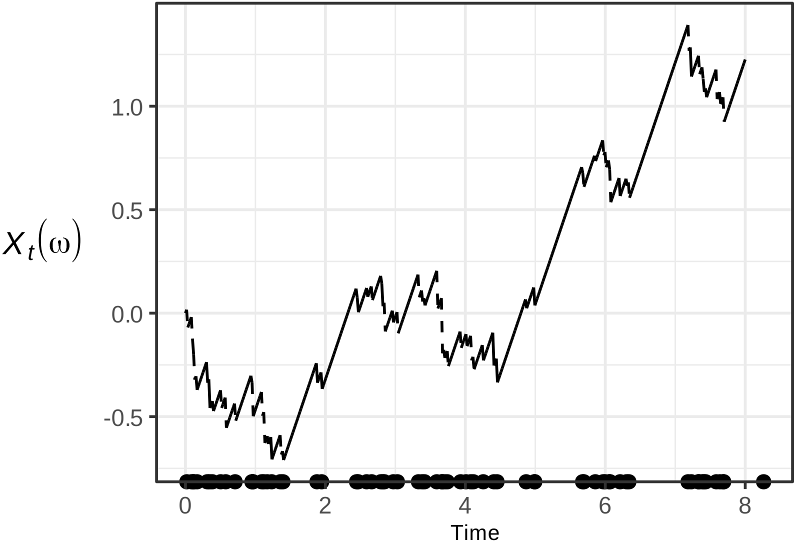
Then we have

$$L_t = -(\inf_{s \leq t} D_s \wedge 0) .$$

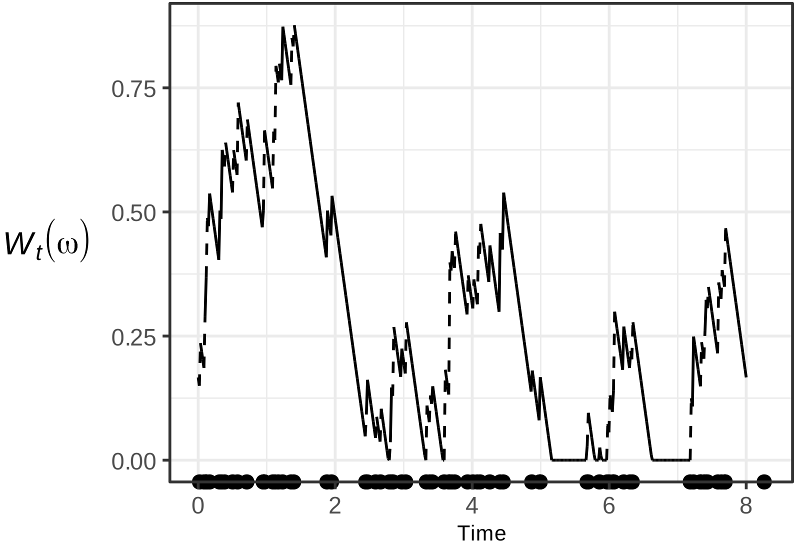
SIMULATED EXAMPLE: PARAMETERS

- discrete grid $t_1, \ldots, t_n$ with $t_1 = 0$, $t_n = 8$, $n = 480$ (samples every minute)
- $\lambda = 12$
- $\mu = 4/60 \approx 0.0667$
- $F \sim \Gamma(\frac{\mu^2}{\sigma^2}, \frac{\mu}{\sigma^2})$ with standard deviation $\sigma = 2.5/60 \approx 0.0417$

SIMULATED EXAMPLE: INPUT PROCESS



SIMULATED EXAMPLE: WORKLOAD PROCESS



Idle time

TOTAL IDLE TIME

Definition 2

The total idle time of a general storage model is the integral

$$I := \int_{(0,+\infty)} \mathbb{1}_{\{W_t=0\}} dt\,.$$

TOTAL IDLE TIME

- Measures the efficiency of the system
- Too large => resources not fully utilised
- Example: parallel computation with race conditions => worker threads in spinlock
- Distribution determined by $\lambda\mu$ (mean incoming work per unit time):
  - $\lambda\mu \leq 1$ => queue repeatedly becomes empty
  - $\lambda\mu > 1$ => eventually the queue never empties

DISTRIBUTION OF I

Theorem 3

Let $\{W_t\}$ be the workload process of an $M/G/1$ queue with arrival rate λ and service distribution F satisfying $\mathbb{E}_F[X] = \mu$, and consider the function

$$\psi(\theta) := \theta - \lambda \int_{(0,\infty)} \left(1 - e^{-\theta x}\right) F(dx),$$

defined for $\theta \geq 0$. Then the following hold:

(i) *If $\mu\lambda \leq 1$, then $I = \infty$ a.s.*

(ii) *If $\mu\lambda > 1$ and θ^* is the largest root of ψ, then*

$$\mathbb{P}_I = (1 - e^{-\theta^* w})\delta_0 + \theta^* e^{-\theta^*(w+x)}\mathrm{Leb}$$

where δ_0 denotes the Dirac measure at 0 and Leb is the Lebesgue measure.

SIMULATED EXAMPLE WITH $\lambda\mu \leq 1$: PARAMETERS

- discrete grid $t_1, \ldots, t_n$ with $t_1 = 0$, $t_n = 8$, $n = 480$ (samples every minute)
- $\lambda = 12$
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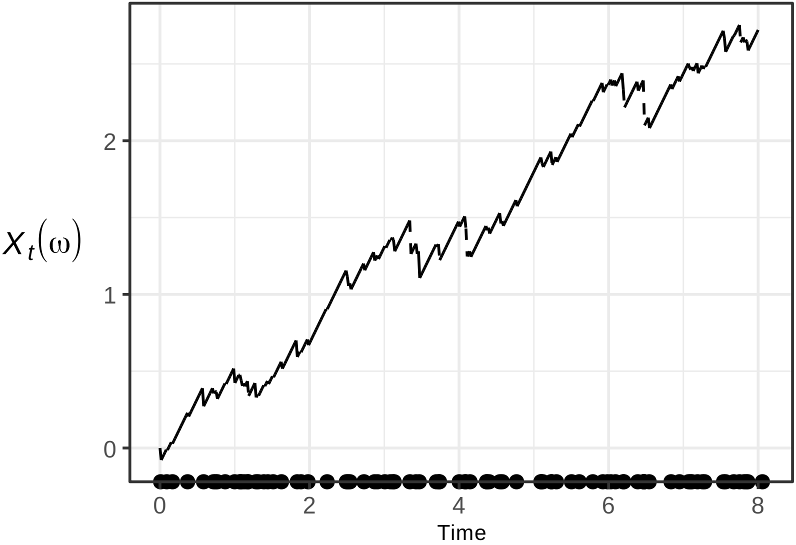
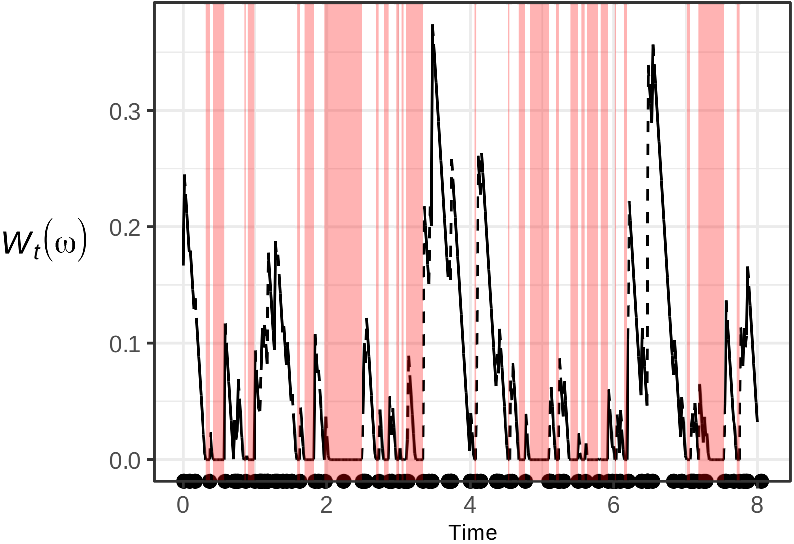


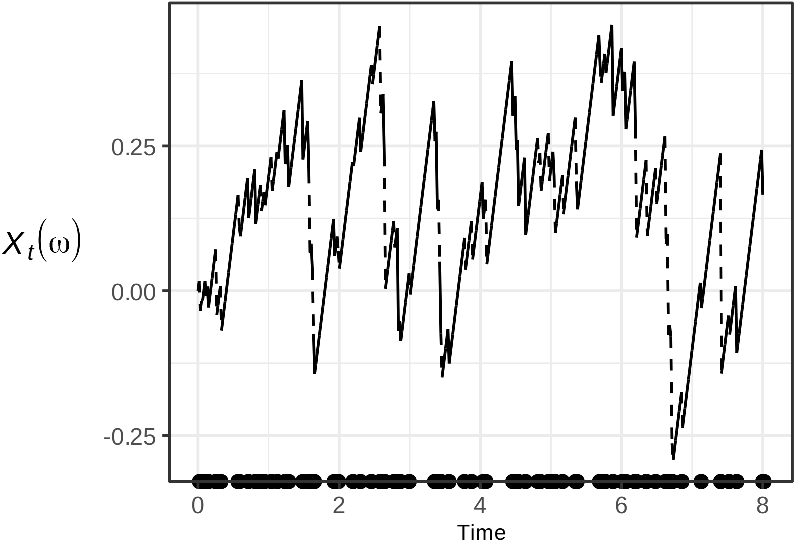
SIMULATED EXAMPLE WITH $\lambda\mu \leq 1$: WORKLOAD PROCESS



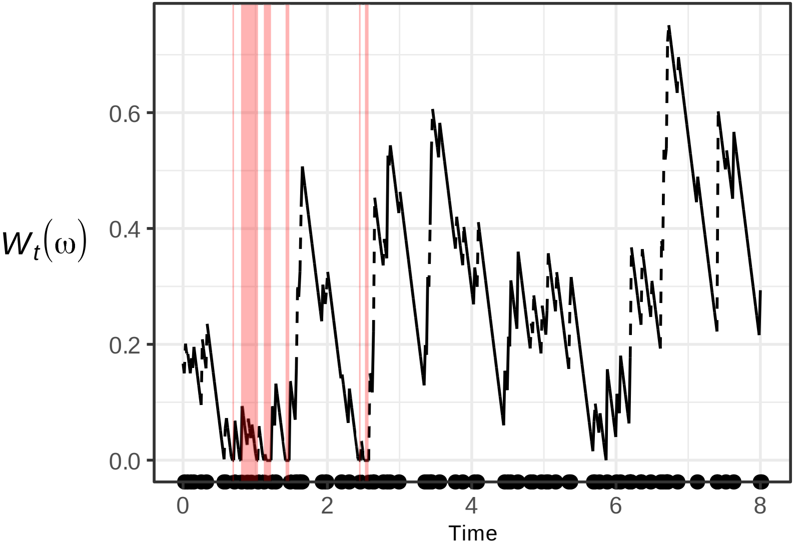
SIMULATED EXAMPLE WITH $\lambda\mu > 1$: PARAMETERS

- discrete grid $t_1, \ldots, t_n$ with $t_1 = 0$, $t_n = 8$, $n = 480$ (samples every minute)
- $\lambda = 12$
- $\mu = 6/60 = 0.1$
- $F \sim \Gamma(\frac{\mu^2}{\sigma^2}, \frac{\mu}{\sigma^2})$ with standard deviation $\sigma = 2.5/60 \approx 0.0417$

SIMULATED EXAMPLE WITH $\lambda\mu > 1$: INPUT PROCESS



SIMULATED EXAMPLE WITH $\lambda\mu > 1$: WORKLOAD PROCESS



PROOF OUTLINE

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[1] Lars Nørvang Andersen et al. "Lévy Processes with Two-Sided Reflection". In: Lévy Matters V: Functionals of Lévy Processes. Cham: Springer International Publishing, 2015, pp. 67–182. ISBN: 978-3-319-23138-9. DOI: 10.1007/978-3-319-23138-9_2. URL: https://doi.org/10.1007/978-3-319-23138-9_2.

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