VRIJE
UNIVERSITEIT
BRUSSEL

Graduation thesis submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

SENSITIVITY ANALYSIS OF STOCHASTIC RESERVING MODELS USING BOOTSTRAP SIMULATIONS

Othman El Hammouchi

June 2023

Promotors: dr. Robin Van Oirbeek prof. dr. Tim Verdonck prof. dr. Mark Sioen

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Acknowledgements

It has become almost obligatory to quote the immortal words of John Donne at the start of any paragraph of acknowledgements: "No man is an island." A shibboleth though it may be, yet it does have the considerable virtue of being true, which is why I have allowed myself to indulge in it here. The list of those to whom I have become indebted in the course of writing this thesis could fill a seperate bibliography, so that I can perforce only mention its most significant entries. Chief among these are my parents and sister, who have been my support and reliance throughout my life; words cannot express the gratitude I feel towards them. I would also like to extend a warm thanks to my active supervisor Dr. Robin Van Oirbeek, whose direction, encouragement and generosity have been absolutely wonderful; it was truly a joy to work with him.

Abstract

The problem of estimating the reserve needed to cover outstanding claims liabilities, and of quantifying the uncertainty of such estimates, has been the subject of much research within actuarial science. The insurance sector distinguishes itself from other model-heavy fields in the high degree of deference it accords to expert judgement. Reserving actuaries are reluctant to move away from established techniques, which are based on macro-level models that aggregate claims data over certain periods. The resulting datasets contain relatively few observations, limiting the efficacy of classical diagnostic methods, and making it difficult to verify model assumptions. We investigate the use of the statistical bootstrap to remedy this, studying how the simulatied predictive distribution of the reserve is impacted by the presence of observations which deviate from the assumptions of two major stochastic reserving models: the Mack chain ladder and the Overdispersed Poisson GLM. Our findings indicate that certain types of bootstraps are very promising for the purpose of distinguishing contaminated points from normal observations.

Samenvatting

Het schatten van de reserve die nodig is om openstaande schadeverplichtingen te dekken, evenals het kwantificeren van de onzekerheid van dergelijke schattingen, is het voorwerp van veel onderzoek binnen de actuariële wetenschap. De verzekeringssector onderscheidt zich van andere domeinen waarin wiskundige modellen een belangrijke rol spelen door de hoge achting die het toekent aan het oordeel van deskundigen. Actuarissen die verantwoordelijk zijn voor het schadereserveringsproces willen typisch niet afwijken van gevestigde technieken, en deze zijn gebaseerd op macro-modellen die schadegegevens aggregeren over bepaalde periodes. De datasets die men hieruit verkrijgt bevatten relatief weinig observaties, hetgeen de doeltreffendheid van klassieke diagnostische methodes beperkt en de verificatie van modelaannames beperkt. We onderzoeken het gebruik van de statistische bootstrap om deze moeilijkheid te remediëren. Dit doen we door te bestuderen hoe de gesimuleerde voorspellende verdeling van de reserve beïnvloed wordt door de aanwezigheid van observaties die afwijken van de aannames van twee belangrijke stochastische schadereserveringsmodellen: de Mack chain ladder en de Overdispersed Poisson GLM. Onze bevindingen geven aan dat bepaalde types bootstrap beloftevol zijn voor het onderscheiden van gecontamineerde datapunten van gewone waarnemingen.

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List of Symbols

The following list is intended to aid in cross-referencing the most commonly used symbols in the text. As a matter of general notation, an uppercase letter like X denotes a random variable and a lowercase letter such as c is used for a deterministic one. A bold lowercase letter like $\boldsymbol{\beta}$ is used for vectors and a bold uppercase one such as $\mathbf{W}$ denotes a matrix; both may or may not be random, which should be clear from the context. Any letter adorned with a hat denotes an estimate.

Probability & statistics

ε_{ij}, ε_i Regression errors

r_{ij}, r_i Regression residuals

$\mathcal{N}(\mu, \sigma^2)$ Normal distribution with mean μ and variance σ^2

$\Gamma(\alpha, \beta)$ Gamma distribution with shape parameter α and rate parameter β

$\text{Pois}(\lambda)$ Poisson distribution with rate parameter λ

I Fisher information matrix

l Log-likelihood function

Mack chain ladder

f_j Development factor

σ_j Variance parameter in the Mack chain ladder

ξ_{ij} Log-normal residuals shift parameter

m_{ij} Log-normal residuals mean parameter

s_{ij}^2 Log-normal residuals variance parameter

ODP model

μ_{ij} Incremental triangle means in the ODP model

c Intercept parameter

α_i Origin period parameters

β_j Development period parameters

ϕ	Dispersion parameter
V	Variance function
Q	Quasi-likelihood function

Other symbols

C_{ij}	Cumulative claim amount
X_{ij}	Incremental claim amount
$\mathcal{D}_I$	Claims triangle
$\mathcal{D}_I^c$	Lower triangle of future claims

Introduction

The most defining characteristic of the insurance industry is the inverted nature of its production cycle. In manufacturing, commerce, transport, etc., payment is usually received only upon delivery of goods or services. By contrast, insurance products are purchased long before the adverse events which they protect against have occured, if they ever do. Insurers therefore face the challenge of forecasting the amount and variability of funds needed to settle outstanding contracts, a process known as *claims reserving*. The reserving actuary must rely here on historical data, which is most often presented in the form of a *loss* or *run-off triangle* $\mathcal{D}_I$ consisting of either cumulative or incremental amounts of some actuarial variable (payments, number of claims, etc.), respectively denoted by C_{ij} and X_{ij}. The row index $1 \leq i \leq I$ denotes the *cohort*, *origin year* or *accident year*, and the column index $1 \leq j \leq J$ gives the *development year*, so that

$$\mathcal{D}_I = \{C_{ij} \mid 1 \leq j \leq J, i+j \leq I+1\} \quad \text{or} \quad \mathcal{D}_I = \{X_{ij} \mid 1 \leq j \leq J, i+j \leq I+1\}\ . \tag{1}$$

To simplify the formulas, we assume throughout this exposition that $I = J$. Embedding $\mathcal{D}_I$ into a matrix on and above the anti-diagonal, the actuary then seeks to predict the *total outstanding loss liabilities*

$$R = \sum_{i=2}^{I} (C_{i,I} - C_{i,I+1-i}) \tag{2}$$

by forecasting the values in the lower triangle $\mathcal{D}_I^{\mathsf{c}}$. A special difficulty arising in the actuarial context is the relatively small number of observations which is typically available.

One of the most frequently used loss reserving techniques in practice is the so-called *chain ladder* (CL), which predicts the cumulative claim in development year j by multiplying the previous year's amount by a so-called *age-to-age factor*, *link ratio* or *development factor*. It was originally conceived as a purely computational algorithm, but has since been framed as a stochastic model in a variety of ways. Its central assumption is that the pattern observed in earlier cohorts is applicable to later ones. In one sense, this is of course perfectly reasonable: all models ultimately use the past as a guide to the future. The dearth of data typically available to

C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_{21}	C_{22}	C_{23}	C_{24}	
C_{31}	C_{32}	C_{33}		
C_{41}	C_{42}			
C_{51}				

(a) Cumulative

X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_{21}	X_{22}	X_{23}	X_{24}	
X_{31}	X_{32}	X_{33}		
X_{41}	X_{42}			
X_{51}				

(b) Incremental

Table 1: General notation for a 5 by 5 claims triangle

the actuary makes it challenging to verify its validity, however, as it limits the efficacy of classical statistical techniques.

To illustrate this point, consider Table 2, which contains the dataset of cumulative payments for a motor insurance account from the UK given in [1] (this will serve as the running example throughout this text). It consists of a 7 by 7 claims triangle with a total of 28 observations. Fig. 1 shows the diagnostic plot of standardised residuals against fitted value for the Mack chain ladder, which will be discussed in Chapter 2. The details are not important at this point; all that matters at present is that it should be symmetric around the x-axis and exhibit no structural patterns if the model gives a good fit.

Origin	**Dev**						
	1	2	3	4	5	6	7
2007	3511	6726	8992	10704	11763	12350	12690
2008	4001	7703	9981	11161	12117	12746	
2009	4355	8287	10233	11755	12993		
2010	4295	7750	9773	11093			
2011	4150	7897	10217				
2012	5102	9650					
2013	6283						

Table 2: UK Motor cumulative claims triangle from Christofides [1]

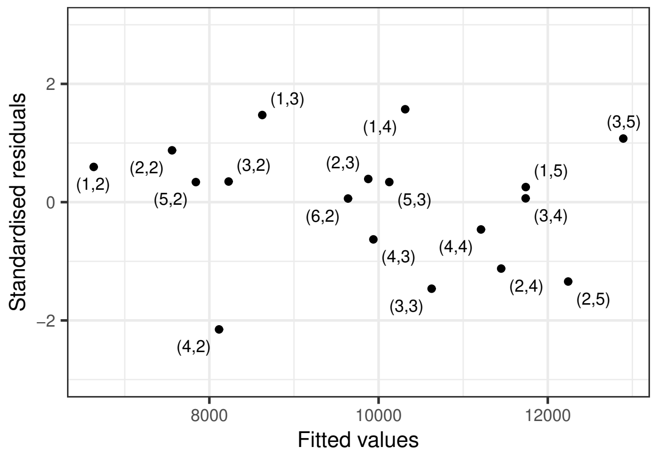
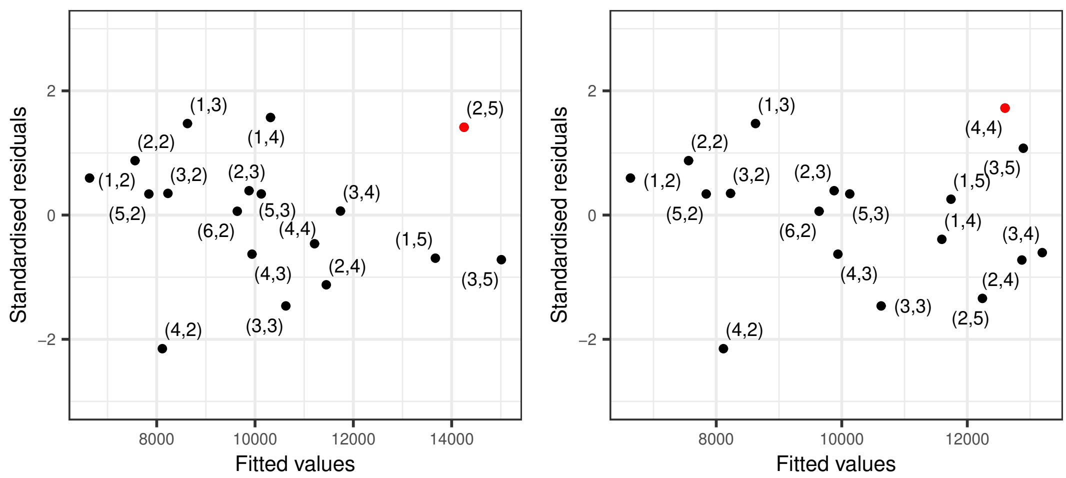


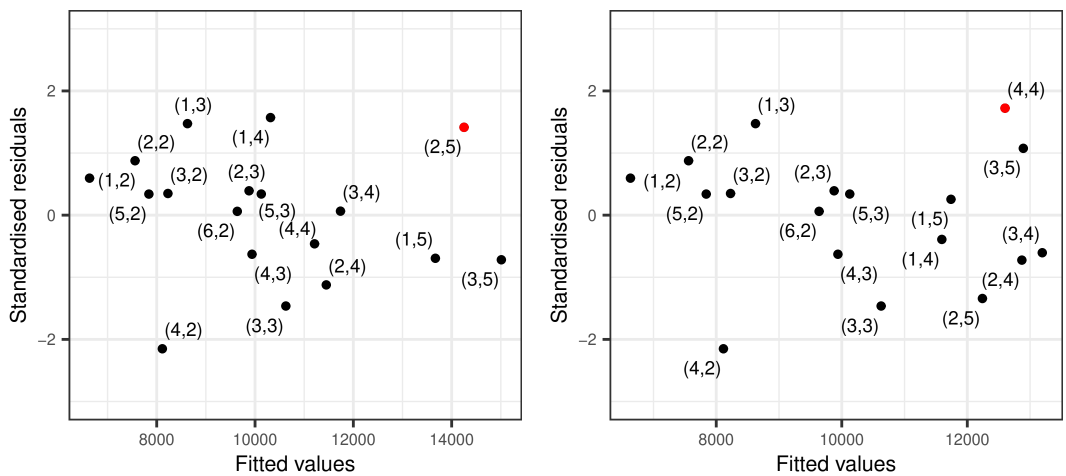
Figure 1: Diagnostic plot for original triangle

The same diagnostic plots are shown in Fig. 2 for the case where the points $(2, 5)$ and $(4, 4)$ have been perturbed by a factor 1.5, either by direct multiplication or through simulation from the underlying model. The residual corresponding to the pathological observation has been highlighted in red. As these examples demonstrate, it is not always feasible to identify deviations from the model assumptions by examining such plots, even for the trained eye.

Our aim in the present study is to investigate whether it is possible to use bootstrap simulations to remedy this problem. The next chapter introduces the bootstrap and explains how it can be used for inference on regression models, which is how we will frame the reserving



(a) Perturbed directly



(b) Perturbed according to model

Figure 2: Diagnostic plots for perturbed triangles

methods under consideration. In the following two chapters, we will then apply this theory to the most widespread stochastic claims reserving models, the Mack chain ladder and the Overdispersed Poisson GLM, in order to study whether it is possible to identify pattern breaks using bootstrap simulation. Specifically, we simulate claims triangles which perfectly follow the model assumptions, perturb these, and generate a bootstrap reserve from the resulting dataset. This will allow us to investigate how the simulated reserve is impacted by deviations from the model assumptions.

Chapter 1

The bootstrap method

When using a statistical model to describe a dataset in terms of a reduced number of parameters, we are not only interested in producing point estimates of these parameters, but also in quantifying their *uncertainty*. In classical statistics, the usual approach to achieve this is to start from the model assumptions and derive from them analytically the sampling distribution of the estimators. In most cases (the Gaussian distribution being a notable exception) this leads to intractible calculations, so that one is either forced to rely on approximations and asymptotic results, or make unrealistic simplifying assumptions. Moreover, estimates obtained in this way often heavily depend on their underlying assumptions, which can potentially lead to gross errors if these are violated.

The bootstrap method aims to remedy this problem by using numerical simulations to compute estimates of uncertainty. At its core, it is premised on the idea that the empirical distribution of the sample forms a good proxy for that the population distribution. Consequently, we can approximate sampling from the population by *resampling our data*, which, to the uncareful observer, can give the impression that we're 'magically' producing new information, using our single sample to 'pull ourselves up by our own bootstraps', which is where the procedure derives its name from. Let's see how this can be done concretely for a simple estimation problem.

1.1 Bootstrapping an estimator

Let $X_1, \ldots, X_n$ be an i.i.d. sample drawn from a distribution F, and consider an estimator $\widehat{h(F)} = g(X_1, \ldots, X_n)$ of some quantity $h(F)$ whose uncertainty we wish to estimate using e.g. the variance of the sampling distribution. Depending on the assumptions we are willing to make, we can choose between two broad approaches: *parameteric* methods and *nonparametric* ones.

In the nonparametric bootstrap, we use the data directly, drawing with replacement to simulate new samples $X_1^{(b)}, \ldots, X_n^{(b)}$. In other words, we approximate F using the *empirical cumulative distribution function*

$$\widehat{F}_n(x) \coloneqq \sum_{k=1}^{n} I_{\{X_k \leq x\}} \,, \tag{1.1}$$

which we use to generate new data. We then compute the statistic of interest on these pseudo-samples, yielding pseudo-observations $g^{(b)} = g(X_1^{(b)}, \ldots, X_n^{(b)})$ which approximate the sampling distribution of $\widehat{h(F)}$. Writing B for the total number of bootstrap samples, we can esimate the variance of $\widehat{h(F)}$ by

$$\frac{1}{B-1}\sum_{b=1}^{B}(g^{(b)}-\bar{g})^2\,, \tag{1.2}$$

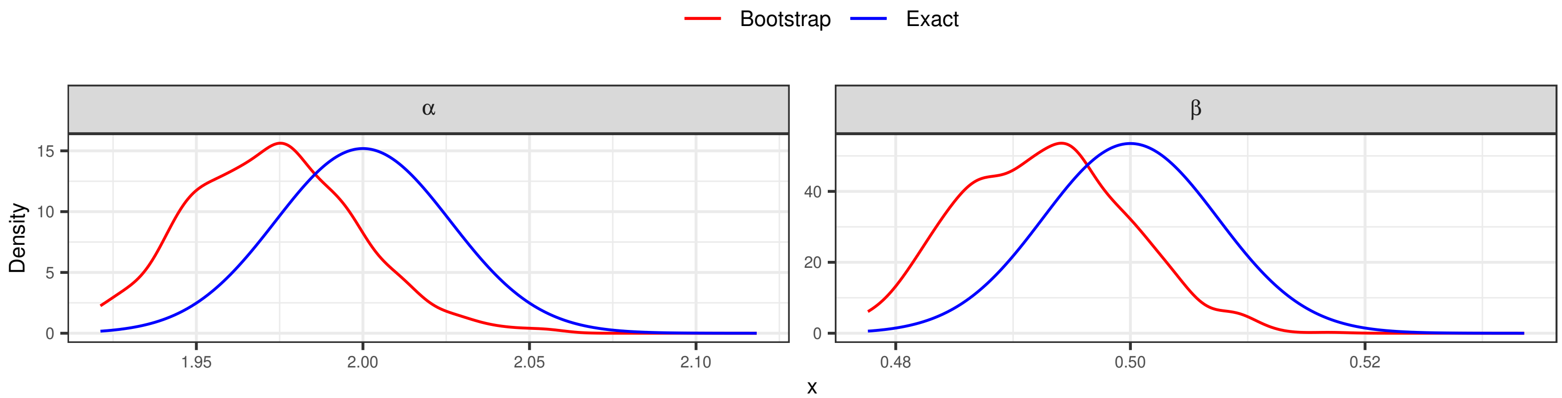
with $\bar{g} = \frac{1}{B}\sum_{b=1}^{B} g^{(b)}$. Provided $F \approx \widehat{F}_n$ holds with sufficient accuracy, this will yield a reasonable approximation to $\mathrm{Var}(\widehat{h(F)})$.

By contrast, in the parametric bootstrap, we first fit a model using the data, and then simulate samples from this with the help of a random number generator. As usual, the parametric approach offers the advantage of efficiency if its assumptions are met, at the risk of increased error when they are violated. If we assume that F belongs to some family $\{F_{\boldsymbol{\theta}} \mid \boldsymbol{\theta} \in \Theta\}$, then we can use the sample $X_1, \ldots, X_n$ to produce an estimate $\widehat{\boldsymbol{\theta}}$ of the parameter. Plugging this in then gives us $F_{\widehat{\boldsymbol{\theta}}}$, from which we can simulate $X_1^{(b)}, \ldots, X_n^{(b)}$ and $g^{(b)}$ as before. An estimate of the sampling variance can be obtained the same manner.

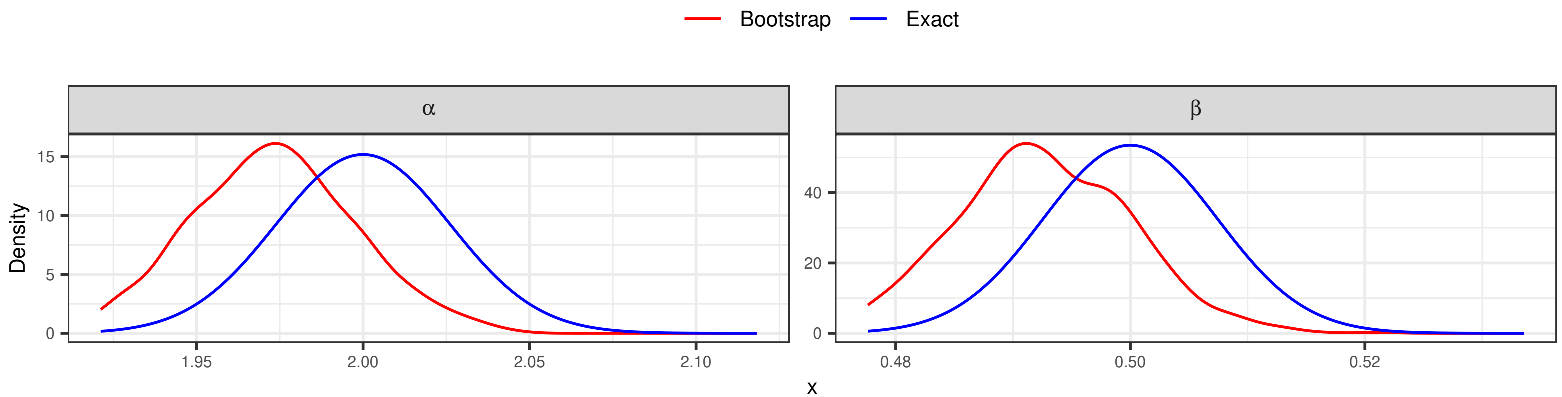
Although we have thus far only used it for the calculation of a single statistic, it is clear that the bootstrap produces a complete *simulated distribution* of the estimator, which can be used for many different forms of inference. This highlights its tremendous potential as a tool for statistical analysis, which explains the rise in popularity of bootstrap methods once personal computers powerful enough to carry out the requisite calculations became widely available. Let us give an example to illustrate this. Fig. 1.1 compares, for a $\Gamma(\alpha, \beta)$-distribution with $\alpha = 2$ and $\beta = 0.5$, the bootstrap distributions of the maximum likelihood estimators to their exact counterparts. It is clear that the bootstrap distribution approaches the analytic one. We use a simulated sample of size $n = 1000$ and compute . The analytic distributions are based on the well-known fact from likelihood theory that the asymptotic distribution of the MLE is given by the multivariate normal distribution $\mathcal{N}(\boldsymbol{\theta}, I(\boldsymbol{\theta})^{-1})$, where $\boldsymbol{\theta}$ is the parameter vector and $I(\boldsymbol{\theta})$ the Fisher information matrix. For the gamma distribution, the latter is given by

$$I(\alpha,\beta) = n\begin{pmatrix}\psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2\end{pmatrix} \tag{1.3}$$

where $\psi(x) \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ is the so-called digamma function.



(a) Parametric



(b) Nonparametric

Figure 1.1: Analytic and bootstrap distribution of MLE for a gamma distribution with $\alpha = 2$ and $\beta = 0.5$

1.2 Bootstrapping a regression model

Altough we have introduced the bootstrap in the context of a classical one-sample estimation problem, the same principles can be applied to data structures of arbitrary complexity, so long as we have a model for the probabilistic mechanism generating the observations (see [2, Chapter 8] for a general exposition of this methodology). In particular, bootstrap methods for regression models are well-established in the literature. We now turn our attention to these, as they will form the foundation for developing bootstrap methods for claims triangles.

Consider a set of covariates $X_1, \dots, X_p$ and a response variable Y whose relationship we model by a parametrised mapping $f(X_1, \dots, X_p; \boldsymbol{\beta})$. Given a sample of pairs $(\mathbf{x_1}, Y_1), \dots, (\mathbf{x_n}, Y_N)$ and a choice of loss function, we can fit this model to obtain an estimate $\widehat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$. For a new value $\mathbf{x}_+$ of the regressors, we can then predict the response Y_+ as $f(\mathbf{x}_+; \widehat{\boldsymbol{\beta}})$. It is worth emphasising these as two distinct operations, which correspond to different bootstrap procedures (see [3, Sections 6.3.3 and 7.2.4]). *Estimation* seeks to *identify* the value of a quantity which is *fixed but unknown*; *prediction* aims to *forecast* the value of a *random variable.*

Under the least squares criterion, for example, we know that the optimal predictor for Y is the conditional expectation $\mathbb{E}[Y \mid X = x]$. This is an ordinary function which returns a real number for any $x \in \mathbb{R}^p$, and which can therefore be estimated from a sample. Such an estimate will contain some error, which we have to take into account when doing inference. If we additionally want to measure the error we make when predicting Y using $\mathbb{E}[Y \mid X = x]$, we also have to incorporate the intrinsic randomness or *process variance* of the response variable. Prediction is therefore a two-stage procedure involving an intermediate estimation step.

Let's illustrate this in the case of the all-familiar linear regression model, which is given by

$$Y_i = \mathbf{x}_{\boldsymbol{i}}^T \boldsymbol{\beta} + \varepsilon_i \,, \qquad i \in 1, \dots, n \tag{1.4}$$

with $\mathbb{E}[\varepsilon_i] = 0$, $\mathrm{Var}(\varepsilon_i) = \sigma$ and $\mathbb{E}[\varepsilon_i \varepsilon_j] = 0$ for $i \neq j$. Considering the nonparametric bootstrap first, we need to identify a fundamental unit of resampling such that the resulting variables are interchangeable. One option would be to use some suitably standardised residuals which we obtain by fitting the model (see [3, Algorithm 6.1]). This approach is sometimes referred to as *semiparametric*, because it only uses the specification of certain aspects of the data distribution in terms of some parameters, but does not assume a specific form for it. Choosing, for example, the residuals

$$r_i \coloneqq \frac{Y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}}{\widehat{\sigma}\sqrt{1 - h_{ii}}} \,, \tag{1.5}$$

we resample these for B times to obtain pseudo-residuals $r_1^{(b)}, \dots, r_n^{(b)}$, which in turn yield pseudo-responses

$$Y_i^{(b)} \coloneqq \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \widehat{\sigma} r_i^{(b)} \,. \tag{1.6}$$

By refitting the model to this new pseudo-data, we then obtain bootstrapped regression parameter estimates $\widehat{\boldsymbol{\beta}}^{(b)}$ for $1, \dots, B$.

An alternative approach, which is fully nonparametric, is to resample the pairs $(\mathbf{x_{\boldsymbol{i}}}, Y_i)$ themselves (see [2, Section 9.5], [3, Algorithm 6.2]), which corresponds to approximating the multivariate distribution of $(X_1, \dots, X_n, Y)$ by the empirical distribution of the data. This has the significant benefit of parsimony, making no other assumption beside the i.i.d.-ness of the sample. The model is then fitted to the bootstrap samples $(\mathbf{x_1}^{(b)}, Y_1^{(b)}), \dots, (\mathbf{x_{\boldsymbol{n}}}^{(b)}, Y_n^{(b)})$ to produce pseudo-realisations $\widehat{\boldsymbol{\beta}}^{(b)}$ of the regression parameter estimator.

For the parametric case, we have to make an additional assumption about the distribution of ϵ, the classical choice being the Gaussian one. We then fit Eq. (1.4), which gives us estimates

$\widehat{\beta}$ for the regression parameters, and use a random number generator to produce bootstrap responses $Y_1^{(b)}, \ldots, Y_n^{(b)}$ by drawing from the estimated distribution $\mathcal{N}(\mathbf{x}^T\widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2)$. Finally, we refit the model to this pseudo-data to obtain bootstrap samples $\widehat{\boldsymbol{\beta}}^{(b)}$ of the regression parameter estimates.

1.3 Process variance and predictive distributions

If we want to do predictive inference on a regression model using the bootstrap, we additionally have to take into account the inherent variability of the response. Considering once again the example of the normal linear model, suppose we are interested in predicting the response Y_+ at new value $\mathbf{x}_+$ of the regressors. One way to quantify the accuracy of our forecast is to consider the *prediction error*

$$\delta := Y_+ - \widehat{Y}_+ \tag{1.7}$$

(see [3, Algorithm 6.4]). The second term in this expression can be bootstrapped using any one of the methods described in the previous section, as these yield replicates $\widehat{\boldsymbol{\beta}}^{(b)}$ of the parameter estimator and hence of the predictor $\widehat{Y}_+^{(b)} = \mathbf{x}_+{}^T\widehat{\boldsymbol{\beta}}^{(b)}$. It then remains for us to produce bootstrap simulations of the response Y_+ itself. In the semiparametric approach, this can be achieved by resampling the residuals a second time to obtain pseudo-realisations $r^{(s)}$ for $s = 1, \ldots, S$, and adding these (after correctly scaling them) to the value of the regression line at x_+. The resulting pseudo-responses $Y_+^{(s)}$ mimick the random fluctuations of Y_+, allowing us to approximate δ with the bootstrapped prediction errors

$$\widehat{\delta}^{(b,s)} := Y_+^{(s)} - \widehat{Y}_+^{(b)} = (\widehat{\mathbf{x}}_+^T\widehat{\boldsymbol{\beta}} + r^{(s)}) - \widehat{\mathbf{x}}_+^T\widehat{\boldsymbol{\beta}}^{(b)}\,. \tag{1.8}$$

A similar procedure can be outlined for the parametric bootstrap by appropriately adjusting the method of generating bootstrap response replicates. The method cannot be applied to the pairs bootstrap, however, as it lacks a mechanism for simulating new response values by themselves; we must therefore borrow this part from one of the alternative approaches, forsaking part of its parsimony and robustness properties in the process. We can then use Eq. (1.8) for different kinds of inference, e.g. obtaining prediction intervals or estimating the mean squared error of prediction.

While this offers an avenue for incorporating the process variance into bootstrap procedures for predictive inference, it is not the one best suited to our purposes. Recall from the Introduction that we want to study how the reserve is impacted by violations of the assumptions of a given actuarial model which can be framed in terms of regression, as we shall see in Chapters 2 and 3. In other words, we would like to simulate the distribution of the response itself. In Eq. (1.8), this was done by generating fluctuations around $\widehat{\boldsymbol{\beta}}$ through resampling (or in the parametric case, with the help of a random number generator). The trouble with this approach is that it fails to account for the error in our parameter estimates, leading to an underestimation of the prediction uncertainty. Although we will endeavour in this exposition to remain agnostic with respect to philosophical questions about the interpretation of probability, it will be necessary in this case, for reasons which will become apparent shortly, to borrow a concept from Bayesian school of statistics in order to address this problem.

Recall that the Bayesian point of view is premised on the idea that the parameters $\boldsymbol{\theta}$ governing a statistical model $p(y \mid \boldsymbol{\theta}, x_1, \ldots, x_p)$ are themselves random variables. These are assumed to follow a so-called *prior distribution* $p(\theta)$, which is a probabilistic expression of the beliefs we have about them before observing any data[1]. When presented with a sample $D = \{(\mathbf{x_1}, Y_1), \ldots, (\mathbf{x_n}, Y_n)\}$, we are then led to *update our beliefs*; using the formula

$$p(\boldsymbol{\theta} \mid D) \propto p(D \mid \boldsymbol{\theta})p(\boldsymbol{\theta})\,, \tag{1.9}$$

which is known as *Bayes' rule*, we obtain the *posterior distribution* $p(\boldsymbol{\theta} \mid D)$ expressing the likelihood of different values of $\boldsymbol{\theta}$ given our observations. For any value of the parameters, the likelihood of the response at a new input, conditional on this value and the sample D, is now given by $p(y_+ \mid \boldsymbol{\theta}, D)$. By marginalising over the posterior distribution, we then incorporate all possible values of $\boldsymbol{\theta}$ in proportion to their likelihood under the data, resulting in the *posterior predictive distribution*.

$$p(y_+ \mid D) = \int p(y_+ \mid \boldsymbol{\theta})p(\boldsymbol{\theta} \mid D)\, d\boldsymbol{\theta}\,. \tag{1.10}$$

of Y_+ given D. This incorporates both the intrinsic variability of the response as well as our uncertainty regarding the parameters (or, in the classical view, their estimates). By contrast, *fitted distribution* $p(y_+ \mid \widehat{\boldsymbol{\theta}})$, obtained by plugging in some estimate of the parameters, does not have this virtue. Moreover, the predictive distribution can be very easily integrated into the bootstrap framework. Indeed, we can follow the same steps as when simulating the prediction errror, but instead of Eq. (1.8), we compute pseudo-realisations

$$Y_+^{(b,s)} := \mathbf{x}_+^T \widehat{\boldsymbol{\beta}}^{(b)} + \widehat{\sigma} r^{(s)}\,. \tag{1.11}$$

Eq. (1.11) also explains why it is difficult to fit this approach within a classical frequentist paradigm, as it is unclear in that case which theoretical quantity these bootstrap replicates would be approximating. We hasten to add that much work has been done to remedy this, leading, among other things, to the concept of a *confidence distribution* (see, for example, [4], [5]). It is true, however, that there has been a multiplicity of disparate frequentist versions of the predictive distribution, as remarked for instance by Dickson, Tedesco and Zehnwirth [6], lending credence to the idea that the notion fits more naturally into the Bayesian framework.

In order to illustrate the difference between fitted and predictive distributions, we also introduce a bootstrap definition for the former. This only differs from Eq. (1.11) in that it uses the original parameter estimate rather than the bootstrap one:

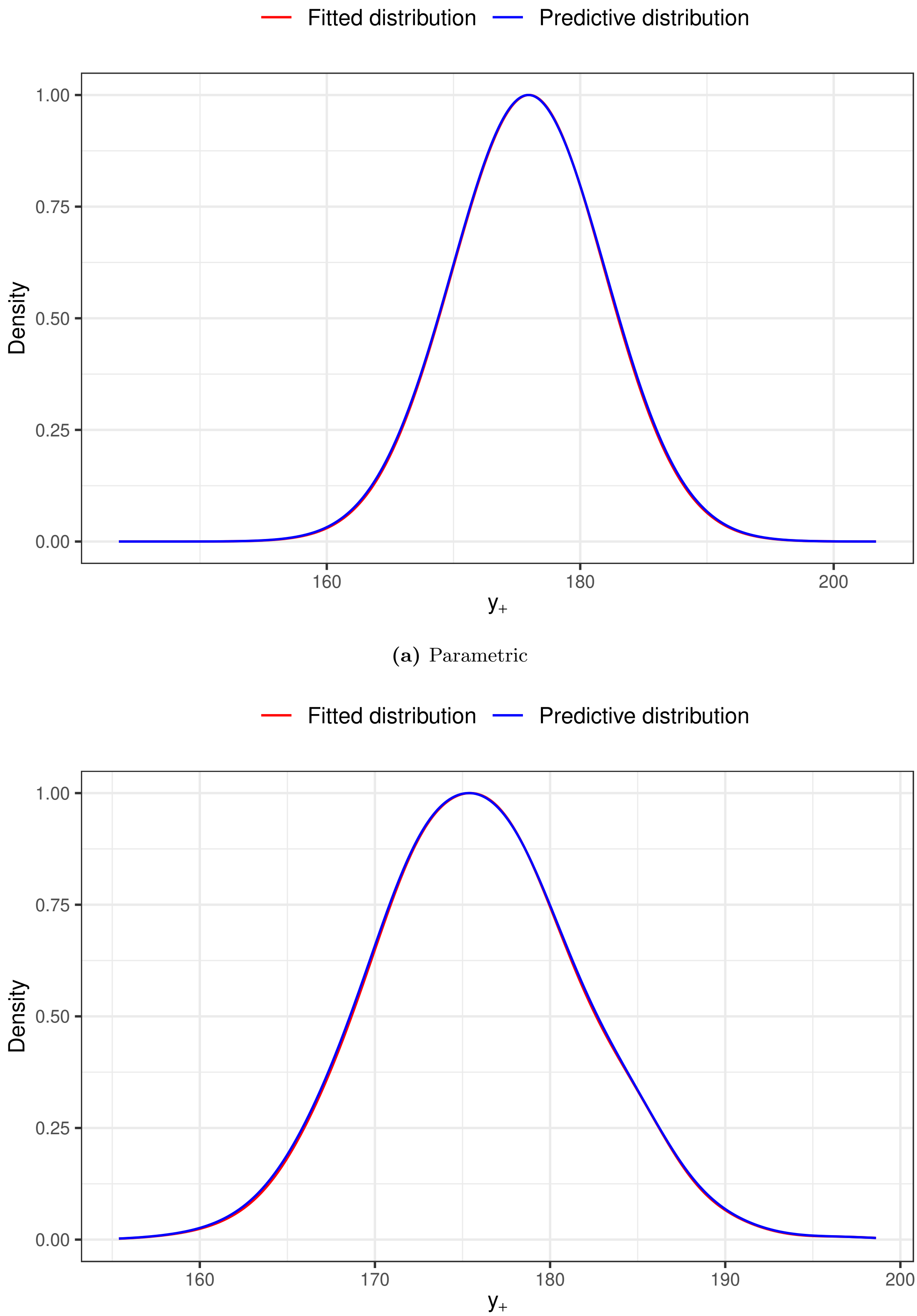
$$Y_+^{(s)} := \mathbf{x}_+^T \widehat{\boldsymbol{\beta}} + \widehat{\sigma} r^{(s)}\,. \tag{1.12}$$

We also note that the notions of predictive and fitted distribution originate in mathematical statistics, where the goal is to derive analytical expressions for them; in this text, we shall always understand them to refer to their bootstrap equivalents. For the following example, we consider a normal linear model with $p = 10$, $n = 1000$, $\sigma = 5.7$ and

$$\boldsymbol{\beta} = \begin{bmatrix} 4.1 & 4.4 & -2.1 & 3.3 & 1.4 & 0.2 & 2.4 & -3.7 & 1.6 & 2.1 \end{bmatrix}\,.$$

The predictive and fitted distributions for both the parametric as well as non-parametric bootstrap are is shown in Fig. 1.2. We can see in this case that the two are nigh indistinguishable, which is presumably due to the fact that the data are simulated and thus follow the model perfectly. As we shall see in Section 2.4, we will not encounter such perfect agreement with the claims reserving models.

[1] Strictly speaking, this is only one possible paradigm within Bayesian probability, known as subjective Bayes. Objective Bayes dispenses with the subjective aspect of the method through the use of so-called *uninformative priors*. Since this distinction is not important for our exposition, however, we will not pursue it further.



(a) Parametric

(b) Semiparametric

Figure 1.2: Comparison of the fitted and predictive bootstrap distributions for a normal linear model

Chapter 2

Mack's model

2.1 Introduction

In his seminal paper [7], Mack proposed the following model for cumulative claims triangles, which remains among the most influencial in actuarial reserving.

Model 1 (Mack chain ladder).

(Mack1) *There exist development factors $f_1, \dots, f_{I-1}$ such that*

$$\mathbb{E}[C_{ij} \parallel C_{i,j-1}, \dots, C_{i1}] = \mathbb{E}[C_{ij} \parallel C_{i,j-1}] = f_{j-1} C_{i,j-1} \tag{2.1}$$

for $1 \leq i \leq I$.

(Mack2) *There exist variance parameters $\sigma_1, \dots, \sigma_{I-1}$ such that*

$$\mathrm{Var}[C_{ij} \parallel C_{i,j-1}, \dots, C_{i1}] = \mathrm{Var}[C_{ij} \parallel C_{i,j-1}] = \sigma_{j-1}^2 C_{i,j-1}\,, \tag{2.2}$$

for $1 \leq i \leq I$.

(Mack3) *The cumulative claims processes $(C_{ij})_j, (C_{i'j})_j$ are independent for $i \neq i'$.*

The development factors are estimated by

$$\widehat{f}_j(\mathcal{D}_I) = \widehat{f}_j(C_{1j}, \dots, C_{I-j,j}, \dots, C_{1,j+1}, \dots, C_{I-j,j+1}) \coloneqq \frac{\sum_{i=1}^{I-j} C_{i,j+1}}{\sum_{i=1}^{I-j} C_{i,j}}\,. \tag{2.3}$$

If we define the *single* or *individual* development factors as

$$F_{i,j+1} \coloneqq \frac{C_{i,j+1}}{C_{ij}}\,, \tag{2.4}$$

then $\widehat{f}_j$ can be obtained as the weighted average

$$\widehat{f}_j = \frac{\sum_{i=1}^{I-j} C_{ij} F_{i,j+1}}{\sum_{i=1}^{I-j} C_{ij}}\,. \tag{2.5}$$

The variance parameters are estimated by

$$\widehat{\sigma}_j \coloneqq \sqrt{\frac{1}{I-j-1}\sum_{i=1}^{I-j} C_{ij}\left(F_{i,j+1}-\widehat{f}_j\right)^2} \tag{2.6}$$

for $j < I-1$. This formula does not work for $j = I-1$, as we only have a single pair of observations in the last two columns of the triangle. To remedy this, Mack proposed a simple extrapolation from the previous development years, leading to the estimate

$$\widehat{\sigma}_{I-1} = \sqrt{\min\left\{\frac{\widehat{\sigma}^4_{I-2}}{\widehat{\sigma}^2_{I-3}}, \widehat{\sigma}^2_{I-2}, \widehat{\sigma}^2_{I-3}\right\}}\,, \tag{2.7}$$

which appears to be the most widely adopted solution in the literature.

Under the assumptions of the Mack chain ladder, it can be shown (see [8, pp. 17 sqq.]) that $\widehat{f}_j$ and $\widehat{\sigma}_j$ are (conditionally) unbiased; moreover, the $\widehat{f}_j$ are uncorrelated. Predictions for the ultimate claim amounts C_{iI} are obtained by substituting these estimates for the unknown development factors f_j in the conditional expectation. In other words, we predict the ultimate loss using the conditional mean $\mathbb{E}[C_{iI} \parallel C_{i,I+1-i}]$, and estimate the latter by plugging in $\widehat{f}_j$, yielding

$$\widehat{C}_{iI} \coloneqq \widehat{\mathbb{E}}[C_{i,I} \parallel C_{i,I+1-i}] = C_{i,I+1-i}\prod_{j=I+1-i}^{I-1}\widehat{f}_j\,. \tag{2.8}$$

From this, we then finally obtain the reserve predictor

$$\widehat{R} = g(\mathcal{D}_I) \coloneqq \sum_{i=2}^{I}(\widehat{C}_{i,I} - C_{i,I+1-i})\,. \tag{2.9}$$

The Mack chain ladder is often referred to as "distribution-free" because it only makes assumptions about the first two moments of the claims triangle observations. Indeed, we will show that the Mack chain ladder can be viewed as a series of linear regressions through the origin (i.e. without intercept term), hence these are same assumptions as for the Gauss-Markov theorem, and the minimal ones[1] required to guarantee optimality. Introduce, for any development year $j \in \{1, \ldots, I-1\}$, the notation

$$\mathbf{c_j} \coloneqq \begin{bmatrix} C_{1,j} \\ \vdots \\ C_{I-j,j}\end{bmatrix}\,, \tag{2.10}$$

then the first two assumptions of the Mack chain ladder can be equivalently stated as

$$\mathbf{c_{j+1}} = f_j\mathbf{c_j} + \boldsymbol{\varepsilon}\,, \tag{2.11}$$

with $\boldsymbol{\varepsilon}$ a random vector satisfying

$$\mathbb{E}[\boldsymbol{\varepsilon} \parallel C_{1,j}, \ldots, C_{i,I-j}] = \mathbf{0} \qquad \operatorname{Var}(\boldsymbol{\varepsilon} \parallel C_{1,j}, \ldots, C_{i,I-j}) = \sigma_j^2\begin{bmatrix} C_{1j} & & \\ & \ddots & \\ & & C_{I-j,j}\end{bmatrix}\,. \tag{2.12}$$

[1]If we want to be completely precise, the third assumption is slightly stronger than needed, as Gauss-Markov only requires the errors to be uncorrelated.

Consequently, it follows (see [9, Proposition 1.7]) that the weighted least squares method with weights matrix

$$\mathbf{W} = \begin{bmatrix} 1/C_{1j} & & \\ & \ddots & \\ & & 1/C_{I-j,j} \end{bmatrix}, \tag{2.13}$$

leads to an estimator for f_j which has minimal variance in the class of linear unbiased estimators. This estimator is given by

$$\widehat{f}_j^{\mathrm{WLS}} = (\mathbf{c}_{\boldsymbol{j}}^T \mathbf{W} \mathbf{c}_{\boldsymbol{j}})^{-1} \mathbf{c}_{\boldsymbol{j}}^T \mathbf{W} = \frac{\sum_{i=1}^{I-j} C_{i,j+1}}{\sum_{i=1}^{I-j} C_{i,j}}, \tag{2.14}$$

which is the same expression as Eq. (2.3).

2.2 A challenging simulation

Owing to its recursive nature, Mack's model does not readily lend itself to application of the theory from Chapter 1. The actuarial literature on bootstrap methods is not very helpful in this regard either, as it has mostly tended to focus on generalised linear models—even papers such as [10] which address the Mack chain ladder do so by reframing it in this way. As will become clear shortly, this passes over some subtleties related to the particular structure of Mack's model, and we therefore choose to take a different approach. In particular, our starting point will be the problem of deriving a closed-form estimate of the so-called conditional *mean square error of prediction* (MSEP) for the Mack predictor. While this might appear at first glance to be unrelated to the bootstrap, we will see that it furnishes us with the necessary theoretical framework to understand the special issues involved in resampling a recursive model.

The MSEP is a measure for the total uncertainty associated with a given predictive model. It is defined as the Euclidean distance between the predictor and the response in the underlying filtered probability space, i.e.

$$\underset{R \,|\, \mathcal{D}_I}{\mathrm{MSEP}}(\widehat{R}) := \mathbb{E}\Big[(\widehat{R} - R)^2 \;\Big\|\; \mathcal{D}_I\Big] \tag{2.15}$$

for our special case of predicting the reserve. The MSEP admits a decomposition, similar to the familiar bias-variance decomposition from classical statistics, into so-called *parameter* or *estimation error* and *process error*:

$$\begin{aligned}
\mathbb{E}\Big[(\widehat{R} - R)^2 \;\Big\|\; \mathcal{D}_I\Big] &= \mathbb{E}\big[(R - \mathbb{E}[R \parallel \mathcal{D}_I])^2 \parallel \mathcal{D}_I\big] + \mathbb{E}\Big[(\mathbb{E}[R \parallel \mathcal{D}_I] - \widehat{R})^2 \;\Big\|\; \mathcal{D}_I\Big] \\
&\quad - 2\mathbb{E}\Big[(R - \mathbb{E}[R \parallel \mathcal{D}_I])(\mathbb{E}[R \parallel \mathcal{D}_I] - \widehat{R}) \;\Big\|\; \mathcal{D}_I\Big]
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&= \mathrm{Var}(R \parallel \mathcal{D}_I) + (\mathbb{E}[R \parallel \mathcal{D}_I] - \widehat{R})^2 \\
&\quad - 2(\mathbb{E}[R \parallel \mathcal{D}_I] - \widehat{R})(\mathbb{E}[R - \mathbb{E}[R \parallel \mathcal{D}_I] \parallel \mathcal{D}_I])
\end{aligned} \tag{2.17}$$

$$= \underbrace{\mathrm{Var}(R \parallel \mathcal{D}_I)}_{\text{process error}} + \underbrace{(\mathbb{E}[R \parallel \mathcal{D}_I] - \widehat{R})^2}_{\text{estimation error}}, \tag{2.18}$$

corresponding to the two stages of bootstrapping a predictor which we discussed in Section 1.2. Consider now, for any accident year $i \in \{1, \dots, I\}$, the MSEP for the associated ultimate

$$\underset{C_{iI} \,|\, \mathcal{D}_I}{\mathrm{MSEP}}(\widehat{C}_{iI}) = (\mathbb{E}[C_{iI} \parallel \mathcal{D}_I] - \widehat{C}_{iI})^2 + \mathrm{Var}(C_{iI} \parallel \mathcal{D}_I), \tag{2.19}$$

and suppose we are interested in obtaining a closed-form estimator for it. Such an expression can be derived relatively straightforwardly for the process error from the assumptions of the Mack chain ladder in the following way. We begin by applying the law of total variance in conjunction with (Mack1) and (Mack2) to obtain

$$\begin{aligned}
\mathrm{Var}(C_{iI} \parallel \mathcal{D}_I) &= \mathrm{Var}(C_{iI} \parallel C_{i,I+1-i}) && (2.20)\\
&= \mathbb{E}[\mathrm{Var}(C_{iI} \parallel C_{i,I-1}) \parallel C_{i,I+1-i}] + \mathrm{Var}(\mathbb{E}[C_{iI} \parallel C_{i,I-1}] \parallel C_{i,I+1-i}) && (2.21)\\
&= \sigma_{I-1}^2 \mathbb{E}[C_{i,I-1} \parallel C_{i,I+1-i}] + f_{I-1}^2 \mathrm{Var}(C_{i,I-1} \parallel C_{i,I+1-i}) && (2.22)\\
&= \sigma_{I-1}^2 C_{i,I+1-i} \prod_{j=I+1-i}^{I-2} f_j + f_{I-1}^2 \mathrm{Var}(C_{i,I-1} \parallel C_{i,I+1-i})\,, && (2.23)
\end{aligned}$$

which is a linear recurrence equation of the form

$$x_n = a_{n-1} x_{n-1} + g_{n-1} \tag{2.24}$$

with $x_n = \mathrm{Var}(C_{in} \parallel C_{i,I+1-i})$ and

$$g_{n-1} = \sigma_{n-1}^2 C_{i,I+1-i} \prod_{j=I+1-i}^{n-1} f_j\,, \qquad a_{n-1} = f_{n-1}^2\,. \tag{2.25}$$

The general solution is given by

$$x_n = \left(\prod_{j=n_0}^{n-1} a_j\right)\left(x_{n_0} + \sum_{k=n_0}^{n-1} \frac{g_k}{\prod_{l=n_0}^{k} a_l}\right) \tag{2.26}$$

where n_0 denotes the first index of the sequence x_n, in our case $I+1-i$. Using the initial condition $x_{I+1-i} = \mathrm{Var}(C_{i,I+1-i} \parallel C_{i,I+1-i}) = 0$, we finally obtain

$$\begin{aligned}
\mathrm{Var}(C_{iI} \parallel \mathcal{D}_I) &= \left(\prod_{j=I+1-i}^{I-1} f_j^2\right)\left(\sum_{k=I+1-i}^{I-1} \frac{\sigma_k^2 C_{i,I+1-i} \prod_{j=I+1-i}^{k-1} f_j}{\prod_{j=I+1-i}^{k} f_j^2}\right) && (2.27)\\
&= \left(\prod_{j=I+1-i}^{I-1} f_j^2\right) C_{i,I+1-i}^2 \left(\sum_{k=I+1-i}^{I-1} \frac{\sigma_k^2/f_k^2}{\prod_{j=I+1-i}^{k-1} f_j C_{i,I+1-i}}\right) && (2.28)\\
&= \mathbb{E}[C_{iI} \parallel C_{I+1-i}]^2 \sum_{k=I+1-i}^{I-1} \frac{\sigma_k^2/f_k^2}{\mathbb{E}[C_{ik} \parallel C_{i,I+1-i}]}\,, && (2.29)
\end{aligned}$$

which we can estimate by plugging in $\widehat{f}_j$ and $\widehat{\sigma}_j$ for f_j and σ_j, respectively.

For the parameter error, if we use the definitions from the previous section to rewrite it as

$$\begin{aligned}
(\mathbb{E}[C_{iI} \parallel \mathcal{D}_I] - \widehat{C}_{iI})^2 &= C_{i,I+1-i}^2 \left(\prod_{j=I+1-i}^{I-1} f_j - \prod_{j=I+1-i}^{I-1} \widehat{f}_j\right)^2 && (2.30)\\
&= C_{i,I+1-i}^2 \left(\prod_{j=I+1-i}^{I-1} f_j^2 + \prod_{j=I+1-i}^{I-1} \widehat{f}_j^2 - 2 \prod_{j=I+1-i}^{I-1} f_j \widehat{f}_j\right)\,, && (2.31)
\end{aligned}$$

it becomes clear that things are more complicated than with process error. Indeed, we cannot simply subtitute the $\widehat{f}_j$ for the unknown parameters in this expression as that would cause it to vanish, yielding an estimate which will generally not be accurate. This problem was recognised by Mack himself in [11], and is caused by the fact that the claims triangle observations are used for both estimation and forecasting (see [12, Section 2] for a more general discussion). His suggested solution was to apply some kind of conditional averaging to the $\widehat{f}_j$. Ideally, one would like to condition on all available observations in $\mathcal{D}_I$, but the $\mathcal{D}_I$-measurability of the $\widehat{f}_j$ would then bring us right back where we started. We must therefore use a smaller set in order to allow $\widehat{f}_{I+1-i}, \ldots, \widehat{f}_{I-1}$ to fluctuate around $f_{I+1-i}, \ldots, f_{I-1}$. This corresponds to asking which other values $\widehat{f}_j$ could have taken, given that we fix a certain subset of the data—in other words, it's a resampling scheme on the parameter estimates. Thus, one can obtain an estimate of the parameter error by specifying a mechanism for generating new realisations of $\widehat{f}_j$ (see [13], [8, pp. 44 sqq.]), with different mechanisms yielding different estimates. The literature uses this mostly as a theoretical device to facilitate analytical calculations; for the specific approach developed by Mack, it leads to the estimator

$$\widehat{\mathrm{MSEP}}(\widehat{R}_i) \coloneqq \widehat{C}_{iI} \sum_{j=I+1-i}^{I-1} \frac{\widehat{\sigma}_j^2}{\widehat{f}_j} \left(\frac{1}{\widehat{C}_{ij}} + \frac{1}{\sum_{i=1}^{I-j} C_{ij}} \right) \tag{2.32}$$

(see [11, p. 11]). In this case, however, the theory happens to fit in perfectly with the resampling framework, and we can therefore employ it as a basis for bootstrap procedures. In the remainder of this section, we outline two approaches for estimating Eq. (2.30) and indicate the corresponding resampling methods.

Denote the subset of observations in $\mathcal{D}_I$ up to and including development year k by

$$\mathcal{B}_k \coloneqq \{C_{ij} \in \mathcal{D}_I \mid j \leq k\} , \tag{2.33}$$

and write

$$\mathcal{D}_{I,k}^O \coloneqq \{C_{ij} \in \mathcal{D}_I \mid j > I + 1 - k\} \tag{2.34}$$

for its complement. One option would then be to take the conditional expectation of $\widehat{f}_j$ with respect to $\mathcal{B}_{I+1-i}$, leading to the estimate

$$\mathbb{E}[(\mathbb{E}[C_{iI} \parallel \mathcal{D}_I] - \widehat{C}_{iI})^2 \parallel \mathcal{B}_{I+1-i}] = C_{i,I+1-i}^2 \mathbb{E}\left[\left(\prod_{j=I+1-i}^{I-1} f_j - \prod_{j=I+1-i}^{I-1} \widehat{f}_j \right)^2 \middle\| \mathcal{B}_{I+1-i} \right] \tag{2.35}$$

$$= C_{i,I+1-i}^2 \left(\mathbb{E}\left[\prod_{j=I+1-i}^{I-1} \widehat{f}_j^2 \middle\| \mathcal{B}_{I+1-i} \right] - \prod_{j=I+1-i}^{I-1} f_j^2 \right) , \tag{2.36}$$

where we used the fact that the $\widehat{f}_j$ are uncorrelated. This corresponds to averaging over the distribution of $\mathcal{D}_{I,k}^O$, or, expressed in terms of resampling, to generating new pseudo-observations in the upper right triangle. Borrowing the nomenclature from [13], we call this the *unconditional approach*. Alternatively, we could average each $\widehat{f}_j$ only over the observations after j. This is equivalent to fixing the denominator $\sum_{i=1}^{I-j} C_{ij}$ in the development factor estimator Eq. (2.3) and allowing the numerator $\sum_{i=1}^{I-j} C_{i,j+1}$ to vary. Formally, it corresponds to taking the expectation with respect to the probability measure defined on $\mathcal{D}_{I,i}^O$ by

$$\mathbb{P}^*_{\mathcal{D}_I}(\{dz_{ij}\}_{i+j\leq I+1}) \coloneqq \prod_{j=1}^{I-1}\prod_{i=1}^{I-j} \mathbb{P}_{C_{i,j+1}}(dz_{i,j+1} \mid C_{ij} = c_{ij})\,, \tag{2.37}$$

yielding the estimate

$$\mathbb{E}_{\mathbb{P}^*_{\mathcal{D}_I}}\left[(\mathbb{E}[C_{iI} \parallel \mathcal{D}_I] - \widehat{C}_{iI})^2\right] = C_{iI}^2\, \mathbb{E}_{\mathbb{P}^*_{\mathcal{D}_I}}\left[\left(\prod_{j=I+1-i}^{I-1} f_j - \prod_{j=I+1-i}^{I-1} \widehat{f}_j\right)^2\right] \tag{2.38}$$

$$= C_{i,I+1-i}^2\left(\mathbb{E}_{\mathbb{P}^*_{\mathcal{D}_I}}\left[\prod_{j=I+1-i}^{I-1} \widehat{f}_j^2\right] - \prod_{j=I+1-i}^{I-1} f_j^2\right) \tag{2.39}$$

$$= C_{i,I+1-i}^2\left(\prod_{j=I+1-i}^{I-1} \mathbb{E}\left[\widehat{f}_j^2 \;\middle\|\; \mathcal{B}_j\right] - \prod_{j=I+1-i}^{I-1} f_j^2\right) \tag{2.40}$$

(see [8, p. 46]). This is referred to as the *conditional approach*, and it corresponds to a scheme in which only the observations from the next period are resampled to produce a new pseudo-realisation of the parameter estimate for the current period.

There has been some controversy about which of these approaches should be preferred, leading to the vigorous discussion found in [13]–[16]. As we shall see in Section 2.3, the difference between the results which they produce is negligeable, and so the question is mainly of theoretical interest. Nevertheless, based on the previous exposition, it seems reasonable to prefer whichever method produces resampled parameter estimates approximating the original $\widehat{f}_j$ most closely. In particular, we note that these posses the following property, the proof of which can be found in [14].

Theorem 1. *The squares of two successive development factor estimates in the Mack chain ladder are negatively correlated:*

$$\mathrm{Cov}(\widehat{f}_j, \widehat{f}_{j-1}) < 0\,. \tag{2.41}$$

In the conditional approach, the resampled parameter estimates are independent by construction, and so they cannot incorporate this covariance structure. In light of this, it would appear that the unconditional scheme has slightly better theoretical properties. As the empirical difference between the two is minimal, however, the conditional version is a reasonable approximation to fall back on when needed. In the next section, we will see how both approaches give rise to a variety of different bootstrap methods.

2.3 Bootstrap methodology

In Section 1.2, we introduced a taxonomy for the different kinds of bootstrap, distinguishing between the semiparameteric, nonparametric and parametric type. We now consider how each of these can be applied to the Mack chain ladder. For comparison, Table 2.1 shows the results of applying the Mack chain ladder with the estimator Eq. (2.32) to the dataset from Table 2.

For the semiparametric bootstrap, the crucial step is to find a suitable definition for the residuals which ensures that they are interchangeable. The distribution-free nature of the model makes this difficult, however, as it limits the statements we can make about the errors to the first

$i\,/\,j$	$\widehat{f}_j^{\mathrm{CL}}$	$\widehat{\sigma}_i^{\mathrm{CL}}$	$\widehat{R}_i^{\mathrm{CL}}$	$\widehat{\mathrm{MSEP}}(\widehat{R}_j)$
2	1.89	2.83	350.9	3.62
3	1.28	3.34	1037.54	22.9
4	1.15	2.98	2044.86	141.98
5	1.1	1.07	3663.4	426.7
6	1.05	0.16	7162.15	692.39
7	1.03	0.02	14396.92	900.58

Table 2.1: Mack CL results for UK Motor triangle

two moments. We can resolve this in one of two ways. The first option would be to extrapolate from homogeneity of the first two moments to homogeneity of the distributions. In that case, the *raw residuals*

$$e_{i,j+1} \coloneqq C_{i,j+1} - \widehat{C}_{i,j+1} = C_{i,j+1} - \widehat{f}_j C_{ij} \tag{2.42}$$

are not an option, as these suffer from heteroscedasticity,

$$\mathrm{Var}(e_{i,j+1} \parallel C_{ij}) = \sigma_j^2 \left(C_{ij} - \frac{C_{ij}^2}{\sum_{i=1}^{I-j} C_{ij}} \right) . \tag{2.43}$$

We can address this by dividing out this variance, i.e. we consider the errors

$$\varepsilon_{i,j+1} \coloneqq \frac{C_{i,j+1} - f_j C_{ij}}{\sigma_j \sqrt{C_{ij}} \sqrt{1 - \frac{C_{ij}}{\sum_{i=1}^{I-j} C_{ij}}}} , \tag{2.44}$$

which satisfy $\mathbb{E}[\varepsilon_{i,j+1} \parallel C_{ij}] = 0$ and $\mathrm{Var}(\varepsilon_{i,j+1} \parallel C_{ij}) = 1$. Provided the sampling variability of the $\widehat{f}_j$ and $\widehat{\sigma}_j$ is not too bad (which is not obvious given the small sample sizes we're usually dealing with), the same should hold approximately for the corresponding residuals

$$r_{i,j+1} \coloneqq \frac{C_{i,j+1} - \widehat{f}_j C_{ij}}{\widehat{\sigma}_j \sqrt{C_{ij}} \sqrt{1 - \frac{C_{ij}}{\sum_{i=1}^{I-j} C_{ij}}}} , \tag{2.45}$$

obtained by substituting these estimators. Note that the factor $\sqrt{1 - \frac{C_{ij}}{\sum_{i=1}^{I-j} C_{ij}}}$ in the denominator corresponds to the leverage adjustment, as can be seen by computing the hat matrix:

$$\mathbf{H} = \mathbf{c_j} (\mathbf{c_j}^T \mathbf{W} \mathbf{c_j})^{-1} \mathbf{c_j}^T \mathbf{W} \tag{2.46}$$

$$= \frac{1}{\sum_{i=1}^{I-j} C_{ij}} \begin{bmatrix} C_{1j} & \dots & C_{1j} \\ \vdots & \ddots & \vdots \\ C_{I-j,j} & \dots & C_{I-j,j} \end{bmatrix} . \tag{2.47}$$

It's worth emphasising, however, that the above extrapolation should not be made lightly, as it is perfectly possible for the error distribution to exhibit heterogeneity in other ways than through its mean and variance (see [2, p. 114] for an example where the *percentiles* vary with the value of the regressor). In light of this, an alternative approach would be to augment our model with some explicit distributional assumptions, which is more transparent and allows us to make precise statements about errors and residuals. One such augmentation that has been studied in the literature (see [8, p. 49]) is the autoregressive Gaussian time series model

$$C_{i,j+1} = f_j C_{ij} + \sigma_j \sqrt{C_{ij}}\, \varepsilon_{i,j+1}, \qquad \varepsilon_{i,j+1} \sim \mathcal{N}(0,1)\,, \tag{2.48}$$

which can easily be seen to be compatible with the Mack chain ladder. Because the Mack chain ladder can be viewed as a series of weighted linear regressions, as we say in Section 2.1, this has the benefit of making available to us the results of classical regression theory. We know, for example, that the *externally studentised residuals*

$$r_{i,j+1} \coloneqq \frac{e_{i,j+1}}{\widehat{\sigma}_{j(i)}\sqrt{1-\mathbf{H}_{ii}}}\sqrt{\mathbf{W}_{ii}} = \frac{C_{i,j+1} - \widehat{f}_j C_{ij}}{\widehat{\sigma}_{j(i)}\sqrt{C_{ij}}\sqrt{1 - \frac{C_{ij}}{\sum_{i=1}^{I-j} C_{ij}}}}\,, \tag{2.49}$$

with $\widehat{\sigma}_{j(i)}$ denoting the leave-i-out estimator of σ_j, follow a t_{I-j-1} distribution. Another option are the *standardised* or *internally studentised* residuals

$$r_{i,j+1} \coloneqq \frac{C_{i,j+1} - \widehat{f}_j C_{ij}}{\widehat{\sigma}_j\sqrt{C_{ij}}\sqrt{1 - \frac{C_{ij}}{\sum_{i=1}^{I-j} C_{ij}}}} \tag{2.50}$$

which also share the same distribution, albeit a more complicated one (see [17, pp. 267 sqq.]).

The Gaussian model has a major shortcoming: it makes it possible to have a negative realisation in the next step of the time series, in which case all future observations from that point on are undefined, because of the square root factor appearing in the variance. This is not merely a theoretical problem: we have sometimes observed this phenomenon in our numerical implementation, where the resampling produces negative pseudo-realisations of certain claim amounts, particularly when the model has been severely perturbed. An obvious way of avoiding it would be to simply discard the current bootstrap iteration as soon as a negative value is produced. This has the drawback of requiring more computational power, sometimes beyond the realm of what is reasonable. Moreover, we have seen cases in which the probability of having no negative replicates was so vanishingly small as to cause the program to get stuck indefinitely. It is therefore imperative that we develop a different approach, if only as a fall-back in case of problems with Eq. (2.48).

England and Verrall [10, p. 238] discuss an analogous difficulty in the narrower context of generating bootstrap realisations of future claim amounts in order to simulate a predictive claims distribution (see Section 2.4). Here as well, one needs to ensure that the model does not produce negative draws, and the authors achieve this by subtituting for the normal distribution a gamma distribution with the same mean and variance. If we write $C_{ij} \sim \Gamma(\alpha, \beta)$, this means that α, β must satisfy

$$\frac{\alpha}{\beta} = f_{j-1} C_{i,j-1} \quad \text{and} \quad \frac{\alpha}{\beta^2} = \sigma_{j-1}^2 C_{i,j-1}\,, \tag{2.51}$$

hence

$$\alpha = \frac{f_{j-1}^2 C_{i,j-1}}{\sigma_{j-1}^2} \quad \text{and} \quad \beta = \frac{f_{j-1}}{\sigma_{j-1}^2}\,. \tag{2.52}$$

While this does not directly address our present concern, namely of bootstrapping the parameters in the Mack chain ladder, it suggests that our woes may be remedied through replacement of the Gaussian distribution in Eq. (2.48) with a different one capable of guaranteeing

$$\varepsilon_{i,j+1} > -\frac{f_j\sqrt{C_{ij}}}{\sigma_j}\,. \tag{2.53}$$

Clearly, the support of such a distribution has to be bounded from below, and it must moreover allow for normalisation, so that the resulting residuals can be made interchangeable by an algebraic transformation. A good candidate satisfying these conditions which readily comes to mind is the *shifted log-normal distribution*

$$\log(\varepsilon_{i,j+1} - \xi_{ij}) \sim \mathcal{N}(m_{ij}, s_{ij}^2) \tag{2.54}$$

for certain parameters ξ_{ij}, m_{ij} and s_{ij}. his has support $(\xi_{ij}, +\infty)$, leading us to choose $\xi_{ij} = -\frac{f_j\sqrt{C_{ij}}}{\sigma_j}$, and again we must determine m_{ij} and s_{ij} to satisfy

$$\begin{cases} \mathbb{E}[\varepsilon_{i,j+1} \parallel C_{ij}] = \exp\left(m_{ij} + \dfrac{s_{ij}^2}{2}\right) + \xi_{ij} = 0 \\ \operatorname{Var}(\varepsilon_{i,j+1} \parallel C_{ij}) = (\exp s_{ij}^2 - 1)\exp(s_{ij}^2 + 2m_{ij}) = 1\,. \end{cases} \tag{2.55}$$

Solving these equations, we obtain

$$s_{ij} = \sqrt{\log\left(1 + \frac{1}{\xi_{ij}^2}\right)}, \qquad m_{ij} = \log(-\xi_{ij}) - \frac{s_{ij}^2}{2}\,. \tag{2.56}$$

Provided the sampling variability of $\widehat{f}_j$ and $\widehat{\sigma}_j$ is not too severe, this means that the residuals

$$r_{i,j+1} \coloneqq \frac{\log\left(C_{i,j+1} - \widehat{f}_j C_{ij} + \widehat{f}_j\sqrt{C_{ij}}/\widehat{\sigma}_j\right) - \widehat{m}_{ij}}{\widehat{s}_{ij}} \tag{2.57}$$

$$= \frac{\log\left(C_{i,j+1} - \widehat{f}_j C_{ij} + \widehat{f}_j\sqrt{C_{ij}}/\widehat{\sigma}_j\right) - \log(\widehat{f}_j\sqrt{C_{ij}}/\widehat{\sigma}_j) + \frac{1}{2}\log\left(1 + \widehat{\sigma}_j^2/\widehat{f}_j^2 C_{ij}\right)}{\sqrt{\log\left(1 + \widehat{\sigma}_j^2/\widehat{f}_j^2 C_{ij}\right)}} \tag{2.58}$$

are approximately $\mathcal{N}(0,1)$-distributed.

After selecting a particular type of residual, the next step is to fit the model and compute the residuals from it. We then resample these to generate bootstrap residuals $r_{ij}^{(b)}$, from which a bootstrap triangle is obtained by inverting the appropriate residuals formula. In the conditional approach, the inversion is based on the original triangle, whereas the unconditional version uses the previously generated bootstrap observations. Finally, the model is refitted to the generated triangle to obtain bootstrap development factor and dispersion parameter estimators $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{\sigma}}$. The entire procedure is outlined in Algorithms 1 and 2 for the case of standardised residuals, and the results for the example data from Table 2 are given in Tables 2.2 to 2.4.

Next, we consider the fully nonparametric bootstrap, in which we resample the pairs $(C_{ij}, C_{i,j+1})$ at every development year index j. For this procedure, we have no choice regarding the resampling scheme which is used: the only possibility is conditional resampling. To see why this is the case, consider what it would mean to implement unconditional resampling. If the resampled pairs for the first two columns are denoted by

$$\{(C_{11}^*, C_{12}^*), \ldots, (C_{I-j,1}^*, C_{I-j,2}^*)\}\,,$$

this would mean using the generated C_{i2}^* as the regressor column in the second step. However, as the pairs are fundamental i.i.d. unit for this method, we have to ensure that these remain paired to the same response from the third column. In effect, this means that we are forced to permute

Input: Cumulative claims triangle $\mathcal{D}_I$, number of bootstrap samples B
  $(\{r_{ij} \mid i+j \leq I+1\}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\sigma}}) \leftarrow \textsc{fit}(\mathcal{D}_I)$
  for $b \leftarrow 1, B$ **do**
    $\{r_{ij}^{(b)} \mid i+j \leq I+1\} \leftarrow \textsc{resample}(\{r_{ij} \mid i+j \leq I+1\})$
    for $j \leftarrow 1, I-1$ **do**
      for $i \leftarrow 1, I-j$ **do**
        $C_{i,j+1}^{(b)} \leftarrow \widehat{f}_j C_{ij} + \widehat{\sigma}_j \sqrt{C_{ij}} r_{i,j+1}^{(b)}$
        $F_{i,j+1}^{(b)} \leftarrow C_{i,j+1}^{(b)} / C_{ij}$
      end for
      $\widehat{f}_j^{(b)} \leftarrow \sum_{i=1}^{I-j} C_{i,j+1}^{(b)} / \sum_{i=1}^{I-j} C_{ij}$
      if $j < I-1$ **then**
        $$\widehat{\sigma}_j^{(b)} \leftarrow \sqrt{\frac{1}{I-j-1} \sum_{i=1}^{I-j} C_{ij} \left(F_{i,j+1}^{(b)} - \widehat{f}_j^{(b)}\right)^2}$$
      else
        $$\widehat{\sigma}_{I-1}^{(b)} \leftarrow \sqrt{\min\left\{\frac{(\widehat{\sigma}_{I-2}^{(b)})^4}{(\widehat{\sigma}_{I-3}^{(b)})^2}, (\widehat{\sigma}_{I-2}^{(b)})^2, (\widehat{\sigma}_{I-3}^{(b)})^2\right\}}$$
      end if
    end for
  end for
  return $\{(\widehat{\boldsymbol{f}}^{(b)}, \widehat{\boldsymbol{\sigma}}^{(b)}) \mid b = 1, \ldots, B\}$

Algorithm 1: Conditional semiparametric bootstrap for the Mack chain ladder

Input: Cumulative claims triangle $\mathcal{D}_I$, number of bootstrap samples B
  $(\{r_{ij} \mid i+j \leq I+1\}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\sigma}}) \leftarrow \textsc{fit}(\mathcal{D}_I)$
  for $b \leftarrow 1, B$ **do**
    for $i \leftarrow 1, I$ **do**
      $C_{i1}^{(b)} \leftarrow C_{i1}$
    end for
    for $j \leftarrow 1, I-1$ **do**
      for $i \leftarrow 1, I-j$ **do**
        $C_{i,j+1}^{(b)} \leftarrow \widehat{f}_j C_{ij}^{(b)} + \widehat{\sigma}_j \sqrt{C_{ij}^{(b)}} r_{i,j+1}^{(b)}$
        $F_{i,j+1}^{(b)} \leftarrow C_{i,j+1}^{(b)} / C_{ij}^{(b)}$
      end for
      $\widehat{f}_j^{(b)} \leftarrow \sum_{i=1}^{I-j} C_{i,j+1}^{(b)} / \sum_{i=1}^{I-j} C_{ij}^{(b)}$
      if $j < I-1$ **then**
        $$\widehat{\sigma}_j^{(b)} \leftarrow \sqrt{\frac{1}{I-j-1} \sum_{i=1}^{I-j} C_{ij}^{(b)} \left(F_{i,j+1}^{(b)} - \widehat{f}_j^{(b)}\right)^2}$$
      else
        $$\widehat{\sigma}_{I-1}^{(b)} \leftarrow \sqrt{\min\left\{\frac{(\widehat{\sigma}_{I-2}^{(b)})^4}{(\widehat{\sigma}_{I-3}^{(b)})^2}, (\widehat{\sigma}_{I-2}^{(b)})^2, (\widehat{\sigma}_{I-3}^{(b)})^2\right\}}$$
      end if
    end for
  end for
  return $\{(\widehat{\boldsymbol{f}}^{(b)}, \widehat{\boldsymbol{\sigma}}^{(b)}) \mid b = 1, \ldots, B\}$

Algorithm 2: Unconditional semiparametric bootstrap for the Mack chain ladder

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.69	338.57	22.28
3	1.28	4.86	967.69	118.79
4	1.15	5.58	2009.03	240.38
5	1.10	4.48	3601.84	450.97
6	1.05	1.20	7091.87	661.19
7	1.03	0.54	14305.17	822.05

(a) Conditional

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.70	350.79	3.72
3	1.28	3.19	1036.88	22.91
4	1.15	2.76	2043.54	141.99
5	1.10	0.96	3660.72	425.09
6	1.05	0.12	7155.61	694.83
7	1.03	0.03	14385.37	905.82

(b) Unconditional

Table 2.2: Semiparameteric bootstrap results for standardised residuals

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	5.11	338.79	41.04
3	1.28	8.88	961.64	209.89
4	1.15	9.43	2009.08	497.71
5	1.10	6.70	3620.87	950.44
6	1.05	1.87	7112.64	1378.98
7	1.03	0.81	14318.64	1715.18

(a) Conditional

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	5.11	350.64	7.91
3	1.28	6.01	1037.24	49.44
4	1.15	5.27	2045.67	297.21
5	1.10	1.66	3651.50	895.15
6	1.05	0.23	7155.71	1446.47
7	1.03	0.08	14395.86	1887.46

(b) Unconditional

Table 2.3: Semiparameteric bootstrap results for studentised residuals

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.38	338.74	20.35
3	1.28	4.47	969.44	110.29
4	1.14	5.15	2023.23	217.67
5	1.10	4.16	3607.15	397.80
6	1.05	1.16	7100.02	586.11
7	1.03	0.52	14313.18	725.15

(a) Conditional

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.38	350.95	3.26
3	1.28	2.74	1037.37	20.27
4	1.15	2.45	2044.65	125.25
5	1.10	0.84	3660.69	378.25
6	1.05	0.11	7159.14	615.27
7	1.03	0.03	14393.80	799.03

(b) Unconditional

Table 2.4: Semiparameteric bootstrap results for log-normal residuals

$$\begin{array}{cccc} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & \\ C_{41} & C_{42} & & \\ C_{51} & & & \end{array} \longrightarrow \begin{array}{cccc} C_{41} & C_{42} & & \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{11} & C_{12} & C_{13} & C_{14} \\ C_{31} & C_{32} & C_{33} & \\ C_{51} & & & \end{array} \longrightarrow ?$$

Figure 2.1: Failure of unconditional pairs resampling

the rows of the triangle at every stage. But this creates a problem: the last point in the second column does not have a successor in the triangle, and we therefore become stuck in the second step if we had previously drawn it, as illustrated in Fig. 2.1. Hence the only option is to carry out the resampling of each column pairs independently of the others, using the original data, and compute bootstrapped development factor and dispersion parameter estimates from these. The entire procedure is outlined in Algorithm 3, and the results for the UK Motor dataset are given in Table 2.5.

Input: Cumulative claims triangle $\mathcal{D}_I$, number of bootstrap samples B, parameter CONDITIONAL specifying the resampling approach

for $b \leftarrow 1, B$ **do**

  for $j \leftarrow 1, I-1$ **do**

    $\{(C_{i,j}^{(b)}, C_{i,j+1}^{(b)}) \mid i = 1, \dots, I-j\} \leftarrow \text{RESAMPLE}(\{(C_{i,j}, C_{i,j+1}) \mid i = 1, \dots, I-j\})$

    $\widehat{f}_j^{(b)} \leftarrow \sum_{i=1}^{I-j} C_{i,j+1}^{(b)} / \sum_{i=1}^{I-j-1} C_{ij}^{(b)}$

    for $i \leftarrow 1, I-j$ **do**

      $F_{i,j+1}^{(b)} \leftarrow C_{i,j+1}^{(b)} / C_{ij}^{(b)}$

    end for

    $\widehat{\sigma}_j^{(b)} \leftarrow \sqrt{\frac{1}{I-j} \sum_{i=1}^{I-j-1} C_{ij}^{(b)} \left(F_{i,j+1}^{(b)} - \widehat{f}_j^{(b)}\right)^2}$

  end for

end for

return $\{(\widehat{\boldsymbol{f}}^{(b)}, \widehat{\boldsymbol{\sigma}}^{(b)}) \mid b = 1, \dots, B\}$

Algorithm 3: Pairs bootstrap for the Mack chain ladder

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	6.75	350.90	2.54
3	1.28	8.80	1037.58	20.72
4	1.15	6.86	2038.59	136.14
5	1.10	0.77	3663.60	415.48
6	1.05	0.01	7158.19	675.49
7	1.03	0.00	14396.28	878.12

(a) Semiparametric

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	6.74	350.90	2.54
3	1.28	8.81	1037.16	20.69
4	1.15	6.59	2045.14	135.51
5	1.10	0.78	3664.09	414.79
6	1.05	0.01	7164.13	682.63
7	1.03	0.00	14399.01	884.44

(b) Parametric

Table 2.5: Pairs bootstrap results for different simulation methods

Finally, we discuss the parametric bootstrap, in which we simulate directly from the fitted model. Based on Eq. (2.48), we might be tempted to simply substitute $\mathcal{N}(0,1)$-distributed draws from a random number generator for the residuals in Algorithms 1 and 2. The problem with this approach, is that it doesn't extend easily to other distributions, because it is not possible, in general, to write these as the sum of a mean and a scaled error term. A better idea, then, is to generate bootstrap responses directly from the fitted distribution. Depending on whether the conditional or the unconditional scheme is used, this will either depend on the original triangle observations or the ones generated at the previous step. The Mack chain ladder is then refitted to the bootstrapped triangle in order to obtain $\widehat{\boldsymbol{f}}^{(b)}$ and $\widehat{\boldsymbol{\sigma}}^{(b)}$. The algorithms and results are given in Algorithms 4 and 5 and Tables 2.6 and 2.7.

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.70	337.81	97.30
3	1.28	4.97	968.55	202.59
4	1.14	5.43	2026.69	583.82
5	1.10	4.42	3612.92	916.39
6	1.05	1.24	7088.42	1227.17
7	1.03	0.55	14313.72	1416.73

(a) Conditional

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.72	350.77	6.90
3	1.28	3.19	1037.25	19.24
4	1.15	2.82	2048.48	121.06
5	1.10	0.94	3667.20	389.40
6	1.05	0.12	7161.25	619.39
7	1.03	0.03	14393.47	789.32

(b) Unconditional

Table 2.6: Parametric bootstrap results for the normal distribution

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.70	337.75	96.58
3	1.28	4.94	962.76	200.40
4	1.15	5.54	2022.25	593.79
5	1.10	4.53	3615.22	936.06
6	1.05	1.21	7095.24	1237.38
7	1.03	0.54	14313.25	1434.88

(a) Conditional

j	$\widehat{f}_j^B$	$\widehat{\sigma}_j^B$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	1.89	2.70	350.93	7.30
3	1.28	3.18	1037.30	19.84
4	1.15	2.73	2042.37	122.93
5	1.10	0.95	3653.75	376.26
6	1.05	0.13	7156.47	610.36
7	1.03	0.03	14394.57	781.67

(b) Unconditional

Table 2.7: Parametric bootstrap results for the gamma distribution

2.4 Incorporating the process error

We end this chapter by discussing how the process error can be incorporated into these bootstrap procedures. As described in Section 1.3, our aim is to obtain a predictive distribution of the reserve which incorporates both parameter and process error. Following the procedure outlined there, we can achieve this by simulating the lower triangle $\mathcal{D}_I^\mathsf{c} = \{C_{ij} \mid i + j > I + 1\}$, giving us pseudo-realisations of the future claim amounts. This will then in turn yield bootstrap replicates

$$R^{(b)} := \sum_{i=2}^{I} (C_{iI}^{(b)} - C_{i,I+1-i}) \tag{2.59}$$

for the reserve. The method proposed in [10] is based on (2.48): starting from the antidiagonal of $\mathcal{D}_I$, we recursively sample

$$C_{i,j+1}^{(b)} \sim \mathcal{N}(\widehat{f}_j^{(b)}\, C_{ij}^{(b)}, \widehat{\sigma}_j^{(b)}\, C_{ij})\,, \tag{2.60}$$

Input: Cumulative claims triangle $\mathcal{D}_I$, number of bootstrap samples B

$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\sigma}}) \leftarrow \text{FIT}(\mathcal{D}_I)$

for $b \leftarrow 1, B$ **do**

  for $j \leftarrow 1, I-1$ **do**

    for $i \leftarrow 1, I-j$ **do**

      $C_{i,j+1}^{(b)} \leftarrow \text{SAMPLE}(\,\mathcal{N}(\widehat{f}_j C_{ij}, \widehat{\sigma}_j^2 C_{ij})\,)$

      $F_{i,j+1}^{(b)} \leftarrow C_{i,j+1}^{(b)} / C_{ij}$

    end for

    $\widehat{f}_j^{(b)} \leftarrow \sum_{i=1}^{I-j} C_{i,j+1}^{(b)} / \sum_{i=1}^{I-j} C_{ij}$

    if $j < I-1$ **then**

      $$\widehat{\sigma}_j^{(b)} \leftarrow \sqrt{\frac{1}{I-j-1} \sum_{i=1}^{I-j} C_{ij} \left(F_{i,j+1}^{(b)} - \widehat{f}_j^{(b)} \right)^2}$$

    else

      $$\widehat{\sigma}_{I-1}^{(b)} \leftarrow \sqrt{\min\left\{ \frac{(\widehat{\sigma}_{I-2}^{(b)})^4}{(\widehat{\sigma}_{I-3}^{(b)})^2}, (\widehat{\sigma}_{I-2}^{(b)})^2, (\widehat{\sigma}_{I-3}^{(b)})^2 \right\}}$$

    end if

  end for

end for

return $\{(\widehat{\boldsymbol{f}}^{(b)}, \widehat{\boldsymbol{\sigma}}^{(b)}) \mid b = 1, \ldots, B\}$

Algorithm 4: Conditional parametric bootstrap for the Mack chain ladder

Input: Cumulative claims triangle $\mathcal{D}_I$, number of bootstrap samples B

$(\{r_{ij} \mid i+j \leq I+1\}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\sigma}}) \leftarrow \text{FIT}(\mathcal{D}_I)$

for $b \leftarrow 1, B$ **do**

  for $i \leftarrow 1, I$ **do**

    $C_{i1}^{(b)} \leftarrow C_{i1}$

  end for

  for $j \leftarrow 1, I-1$ **do**

    for $i \leftarrow 1, I-j$ **do**

      $C_{i,j+1}^{(b)} \leftarrow \text{SAMPLE}(\,\mathcal{N}(\widehat{f}_j C_{ij}^{(b)}, \widehat{\sigma}_j^2 C_{ij}^{(b)})\,)$

      $F_{i,j+1}^{(b)} \leftarrow C_{i,j+1}^{(b)} / C_{ij}^{(b)}$

    end for

    $\widehat{f}_j^{(b)} \leftarrow \sum_{i=1}^{I-j} C_{i,j+1}^{(b)} / \sum_{i=1}^{I-j} C_{ij}^{(b)}$

    if $j < I-1$ **then**

      $$\widehat{\sigma}_j^{(b)} \leftarrow \sqrt{\frac{1}{I-j-1} \sum_{i=1}^{I-j} C_{ij}^{(b)} \left(F_{i,j+1}^{(b)} - \widehat{f}_j^{(b)} \right)^2}$$

    else

      $$\widehat{\sigma}_{I-1}^{(b)} \leftarrow \sqrt{\min\left\{ \frac{(\widehat{\sigma}_{I-2}^{(b)})^4}{(\widehat{\sigma}_{I-3}^{(b)})^2}, (\widehat{\sigma}_{I-2}^{(b)})^2, (\widehat{\sigma}_{I-3}^{(b)})^2 \right\}}$$

    end if

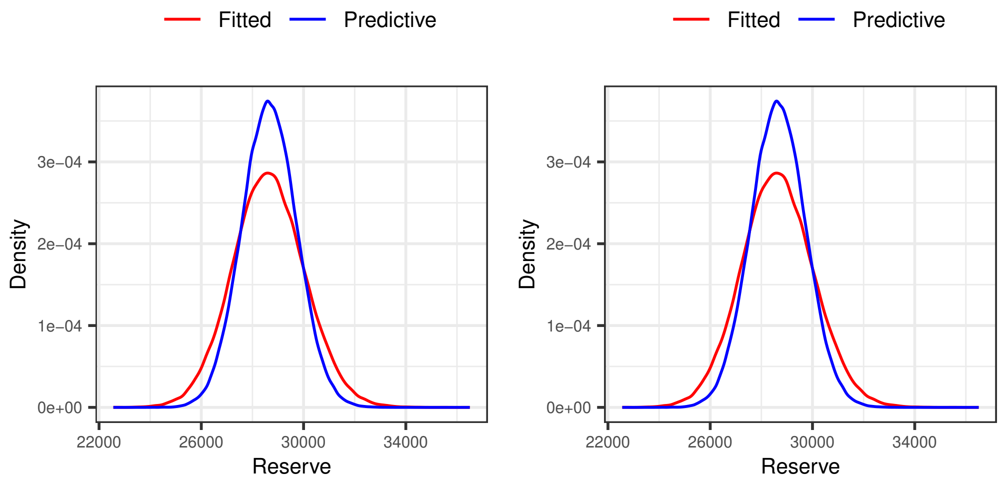
  end for

end for

return $\{(\widehat{\boldsymbol{f}}^{(b)}, \widehat{\boldsymbol{\sigma}}^{(b)}) \mid b = 1, \ldots, B\}$

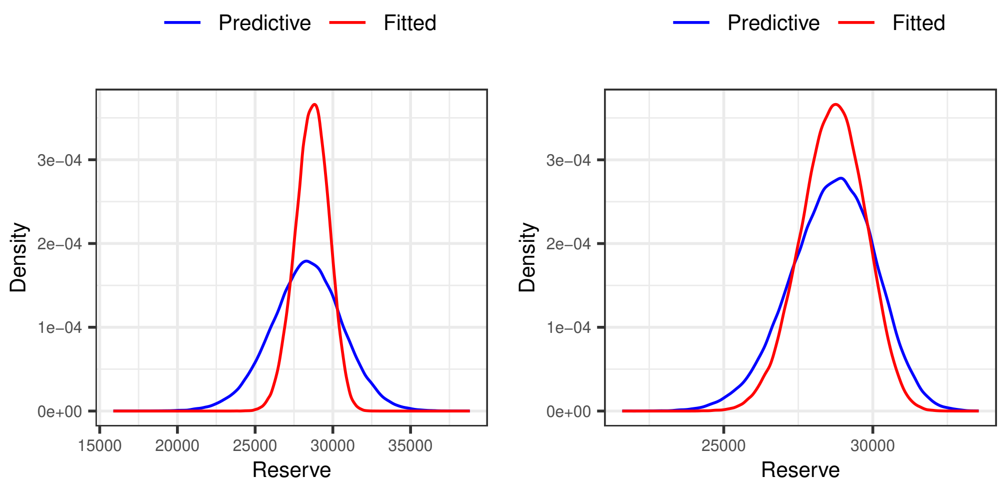
Algorithm 5: Unconditional parametric bootstrap for the Mack chain ladder until the final development year I is reached. As noted in Section 2.3 when discussing the semiparametric bootstrap, however, this approach makes it possible to draw negative samples, which is problematic. The remedies available to us here depend on the type of bootstrap employed. It is possible to avoid this difficulty with the semiparametric bootstrap by using the residuals from Eq. (2.57), which are based on the shifted log-normal distribution. For the parametric bootstrap, we can follow the suggestion from [10], discussed in the previous section, and use Eq. (2.52) to simulate future claim amounts. The solution adopted for the pairs bootstrap will depend on which of the two other variants is used in the simulation step.

In the synthetic example we gave in Section 1.3, not much difference could be discerned between the fitted and predictive distributions. This stands in stark contrast to Figs. 2.2 to 2.5, which compare these for the reserve under different bootstrap configurations. We can clearly see that fitted distribution severely underestimates the uncertainty of the reserve in all cases.



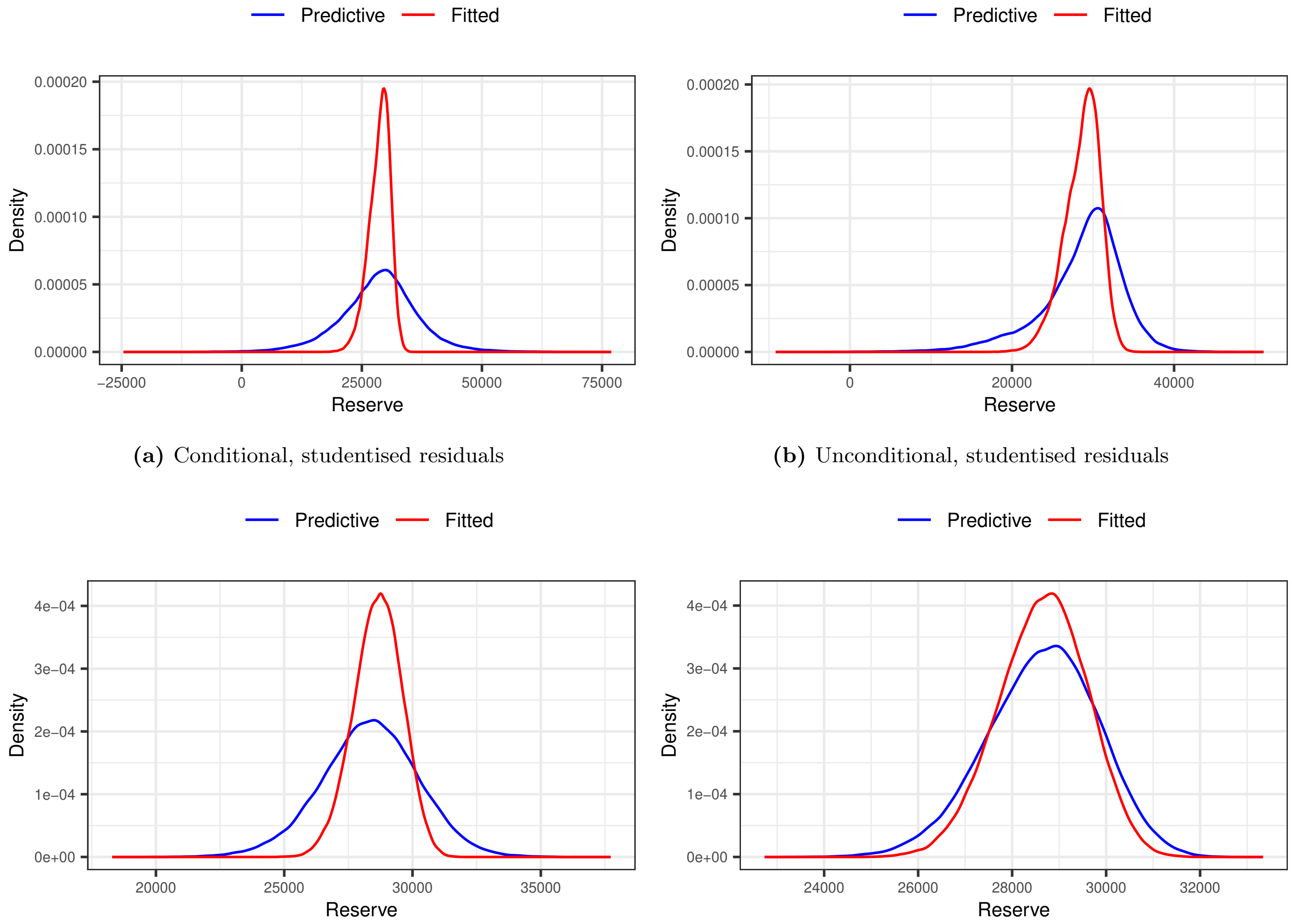
(a) Parametric **(b)** Semiparametric

Figure 2.2: Comparison of the fitted and predictive distribution for the pairs bootstrap with different simulation methods



(a) Conditional **(b)** Unconditional

Figure 2.3: Comparison of the fitted and predictive distribution for the semiparametric bootstrap with standardised residuals



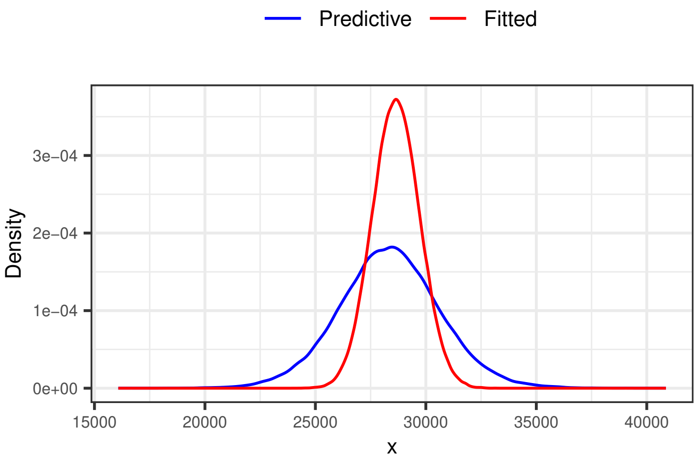
(a) Conditional, studentised residuals

(b) Unconditional, studentised residuals

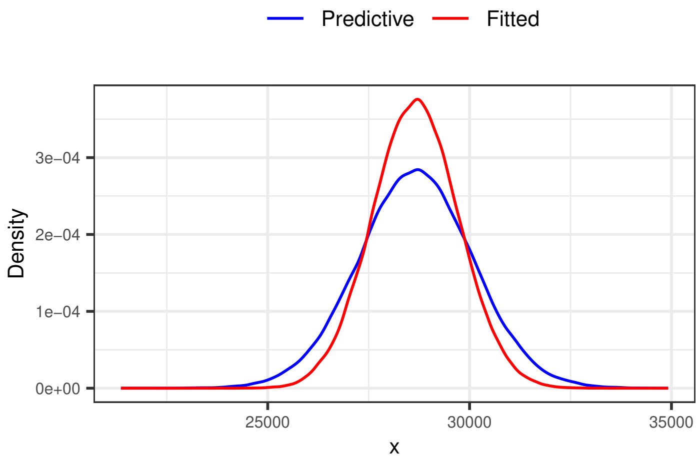
(c) Conditional, log-normal residuals

(d) Unconditional, log-normal residuals

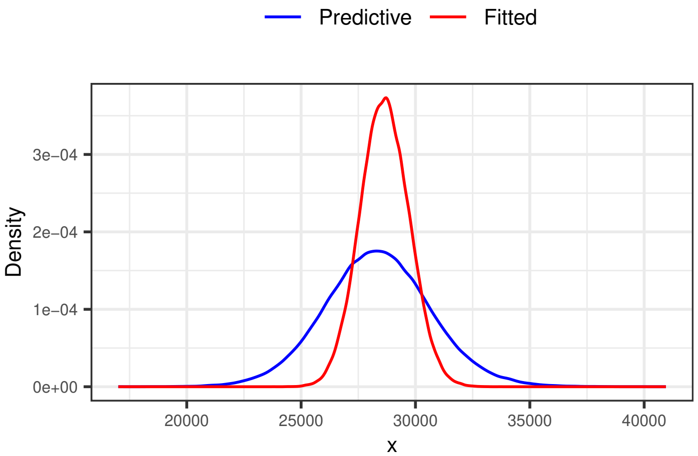
Figure 2.4: Comparison of the fitted and predictive distribution for the semiparametric bootstrap



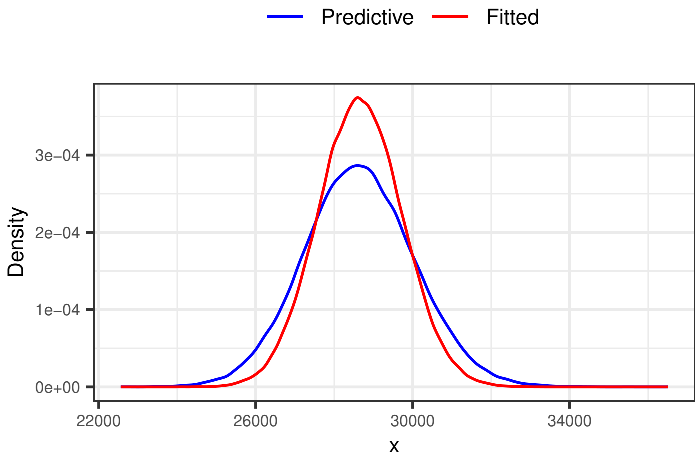
(a) Conditional, normal distribution



(b) Unconditional, normal distribution



(c) Conditional, gamma distribution



(d) Unconditional, gamma distribution

Figure 2.5: Comparison of the fitted and predictive distribution for the parametric bootstrap

Chapter 3

Overdispersed Poisson GLM

3.1 Introduction

The overdispersed Poisson model (ODP), proposed by Renshaw and Verrall in [18], belongs to the family of so-called *generalised linear models* (GLM). In contrast to the Mack chain ladder, it describes the incremental claims X_{ij}, and is non-recursive: the diffferent observations in the triangle are modelled independently from each other. To ease the exposition, we will only state the assumptions for the ordinary Poisson model (i.e. without overdispersion) at this point; after this concept has been introduced in Section 3.2.2, we will then adjust the model accordingly.

Model 2 (Poisson GLM).

(Pois1) *The incremental claims are independent from each other.*

(Pois2) *There exist parameters c, $a_1, \dots, a_I$ and $b_1, \dots, b_I$ such that*

$$\log(\mathbb{E}[X_{ij}]) = c + a_i + b_j \,, \tag{3.1}$$

with $a_1 = b_1 = 0$.

(Pois3) *The incremental claims follow a Poisson distribution:*

$$X_{ij} \sim \mathrm{Pois}(e^{c+a_i+b_j})\,. \tag{3.2}$$

The condition $a_1 = b_1 = 0$ is necessary to obtain an identifiable model; without it, we could derive, from any set of parameters $c, a_1, \dots, a_I, b_1, \dots, b_I$ satisfying the assumptions, an infinite number of alternatives $c + a' + b', a_1 - a', \dots, a_I - a', b_1 - b', \dots, b_I - b'$ for $a', b' \in \mathbb{R}$. We therefore have two superfluous degrees of freedom, which can be eliminated by imposing the same number of conditions on the parameters.

By defining $\xi_i := e^{c+a_i}$ and $\gamma_j := e^{b_j}$, we get a different parametrisation of the model which has a multiplicative structure for the mean, i.e.

$$\mathbb{E}[X_{ij}] = \xi_i \gamma_j \,. \tag{3.3}$$

This is sometimes preferred to the additive one for reasons of interpretability. Indeed, it is clear that the multiplicative form has one fewer degree of freedom than the linear one, and if we remove this by imposing the constraint

$$\sum_{j=1}^{I} \gamma_j = 1 \tag{3.4}$$

then we can view the ξ_i as expected ultimate claim amounts, and the γ_j as the expected development pattern.

As mentioned in the introduction, stochastic claims reserving models have to reproduce the chain ladder point predictions in order to be acceptable to practitioners. While less obvious than for the Mack chain ladder, it can be shown that the Poisson GLM also satisfies this requirement (see [8, Lemma 2.16]).

3.2 Generalised linear models

GLMs were first conceived by Nelder and Wedderburn in [19] as a way of unifying the many disparate generalisations of linear regression with Gaussian errors which were then in existence. These sought to extend the classical model by allowing the use of different functional forms for the conditional mean and different distributions for the response, thus making it suited to modelling counts data (Poisson regression) or the probability of binary events (logistic regression), among others. For a set of covariates $X_1, \ldots, X_p$ and a response variable Y, a GLM consists of three parts:

1. The *random component*, a distribution for response Y belonging to the so-called *exponential dispersion model* family (EDM), which consists of all probability distributions whose density (with respect either to the Lebesgue or counting measure) has the form
$$p(y \mid \theta, \phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right\}, \tag{3.5}$$
where a, b and c are known functions, and b is at least twice differentiable. We call θ the *canonical parameter* of the distribution and ϕ the *dispersion parameter*.

2. The *systematic component*, a predictor $\eta := \mathbf{x}^T\boldsymbol{\beta}$ which is a linear function of the covariates.

3. A monotonic differentiable link function $g : \mathbb{R} \to \mathbb{R}$ giving the relation between the conditional expectation and the linear predictor,
$$\mu := \mathbb{E}[Y \parallel X_1, \ldots, X_p] = g^{-1}(\eta). \tag{3.6}$$

The Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ can be seen to belong to the EDM family by rewriting its density as
$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\} = \exp\left\{-\frac{y^2}{2\sigma^2} + \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log(\sqrt{2\pi}\sigma)\right\} \tag{3.7}$$
$$= \exp\left\{\frac{y\mu - \mu^2/2}{\sigma^2} - \frac{y^2}{2\sigma^2} - \log(\sqrt{2\pi}\sigma)\right\}, \tag{3.8}$$
which is of the form Eq. (3.5) with $\theta = \mu$, $b(\theta) = \frac{\theta^2}{2}$, $\phi = \sigma^2$, $a(\phi) = \phi$ and $c(y, \sigma) = -\frac{y^2}{2\sigma^2} - \log(\sqrt{2\pi}\sigma)$. Thus, the familiar normal linear model can be obtained from the GLM framework with response distribution $\mathcal{N}(\mu, \sigma^2)$ and identity link $g(\mu) = \mu$.

3.2.1 Estimation

The EDM family has a number of properties which greatly facilitate the computations involved in estimation. Recall from likelihood theory that $l(\theta \mid y, \phi) := \log p(y \mid \theta, \phi)$ satisfies
$$\mathbb{E}\left[\frac{\partial l(\theta \mid Y)}{\partial \theta}\right] = 0, \qquad \mathrm{Var}\left(\frac{\partial l(\theta \mid Y)}{\partial \theta}\right) = -\mathbb{E}\left[\frac{\partial^2 l(\theta \mid Y)}{\theta^2}\right], \tag{3.9}$$

where $\frac{\partial l(\theta|Y)}{\partial \theta}$ is known as the *score function.* Using Eq. (3.5), we then find that

$$\mathbb{E}\left[\frac{Y-b'(\theta)}{a(\phi)}\right]=0\,,\qquad \mathrm{Var}\left(\frac{Y-b'(\theta)}{a(\phi)}\right)=-\mathbb{E}\left[\frac{-b''(\theta)}{a(\phi)}\right],\tag{3.10}$$

from which we obtain the elegant relations

$$\mu=b'(\theta)\,,\qquad \mathrm{Var}(Y)=a(\phi)b''(\theta)\,.\tag{3.11}$$

Observe that this implies that $\frac{d\mu}{d\theta}=b''(\theta)>0$ (because the variance is always positive), which means that $\theta\mapsto\mu(\theta)$ is one-to-one and therefore invertible. In particular, we can always write the likelihood as function of the mean. The function $V(\mu):=b''((b')^{-1}(\mu))$ is called the *variance function* and determines how the scale of the response varies as a function of its mean.

Special care has to be taken with the parameter ϕ, as it occupies a rather awkward position in GLM theory.

The problem arises from the desire of the progenitors of the GLM framework to incorporate two-parameter distributions such as the normal and gamma distribution into a paradigm which can fundamentally only handle a single parameter gracefully (the more flexible approach of *vector GLMs* is an attempt to remedy this; see [20, Chapter 2] for a general discussion). The dispersion is therefore regelated to the role of nuisance parameter and subjected to severe (and often unrealistic) constraints. In essence, one would like ϕ to be constant as a function of the covariates, but this precludes certain special cases such as binomial regression with a different number of trials for each observation in the sample which practitioners wanted to accommodate. In the end, the compromise which seems to have gained the widest acceptance in the literature is to allow the function a in the denominator of Eq. (3.5) to vary across different sample responses as $a_i(\phi)=\phi/w_i$, where w_i is a known weight. The parameter ϕ itself is then considered as known, and estimated outside of the GLM framework, most commonly using the Pearson statistic

$$\widehat{\phi}:=\frac{1}{n-p}\sum_{i=1}^{N}\frac{(Y_i-\widehat{\mu_i})^2}{V(\widehat{\mu_i})}\,.\tag{3.12}$$

Given a sample $(\mathbf{x_1},Y_1),\ldots,(\mathbf{x_N},Y_N)$, the standard way to fit a GLM is by means of maximum likelihood estimation (MLE). The joint log-likelihood of the sample is given by

$$l(\boldsymbol{\beta}\mid\mathbf{y},\phi)=\sum_{i=1}^{N}\frac{y_i\theta_i-b(\theta_i)}{a_i(\phi)}+c(y_i,\phi)\,,\tag{3.13}$$

which we must differentiate with respect to β_j to obtain the likelihood equations. An application of the chain rule gives us

$$\frac{\partial l(\boldsymbol{\beta}\mid\mathbf{y},\phi)}{\partial\beta_j}=\sum_{i=1}^{N}\frac{\partial l(\boldsymbol{\beta}\mid y_i,\phi)}{\partial\theta_i}\frac{\partial\theta_i}{\partial\mu_i}\frac{\partial\mu_i}{\partial\eta_i}\frac{\partial\eta_i}{\partial\beta_j}\tag{3.14}$$

$$=\sum_{i=1}^{N}\frac{y_i-b_i'(\theta)}{a_i(\phi)}\frac{1}{b_i''(\theta_i)}\frac{\partial\mu_i}{\partial\eta_i}x_{ij}\tag{3.15}$$

$$=\sum_{i=1}^{N}\frac{y_i-\mu_i}{\mathrm{Var}(Y_i)}\frac{\partial\mu_i}{\partial\eta_i}x_{ij}\,,\tag{3.16}$$

and setting this equal to 0 yields a system of p (usually nonlinear) equations. It is generally impossible to solve these analytically, and so we must resort to numerical methods. In particular, we use a modified version of the Newton-Raphson algorithm known as *Fisher scoring*, which replaces the negative Hessian of the log-likelihood, called the *observed information*, by its expectation

$$\mathcal{I}_{jk} := \mathbb{E}\left[-\frac{\partial^2 l(\boldsymbol{\beta} \mid \mathbf{y}, \phi)}{\partial\beta_j \partial\beta_k}\right], \tag{3.17}$$

which is known as the *Fisher information matrix*. Thus, starting from an initial guess $\widehat{\boldsymbol{\beta}}^{(0)}$ for the parameters, we compute a successive approximations via

$$\widehat{\boldsymbol{\beta}}^{(k+1)} = \widehat{\boldsymbol{\beta}}^{(k)} + \mathcal{I}(\widehat{\boldsymbol{\beta}}^{(k)})^{-1} \nabla l(\widehat{\boldsymbol{\beta}}^{(k)} \mid \mathbf{y}, \phi). \tag{3.18}$$

Similarly to Eq. (3.9), it can be shown that

$$\mathbb{E}\left[\frac{\partial^2 l(\boldsymbol{\beta} \mid \mathbf{y}, \phi)}{\partial\boldsymbol{\beta}\,\partial\boldsymbol{\beta}^T}\right] = -\mathrm{Var}\big(\nabla l(\boldsymbol{\beta} \mid \mathbf{y}, \phi) \nabla l(\boldsymbol{\beta} \mid \mathbf{y}, \phi)^T\big) < 0, \tag{3.19}$$

from which we also see that the log-likelihood is concave, and will therefore have a global maximum. Using the fact that the Y_i are independent, so that $\mathbb{E}[(Y_i - \mu_i)(Y_l - \mu_l)] = 0$ for $i \neq l$, we then obtain

$$\begin{aligned}
I_{jk} &= \mathbb{E}\left[\left(\sum_{i=1}^{N} \frac{Y_i - \mu_i}{\mathrm{Var}(Y_i)} \frac{\partial\mu_i}{\partial\eta_i} x_{ij}\right)\left(\sum_{l=1}^{N} \frac{Y_l - \mu_l}{\mathrm{Var}(Y_l)} \frac{\partial\mu_l}{\partial\eta_l} x_{lk}\right)\right] && (3.20)\\
&= \sum_{i=1}^{N} \left(\frac{\partial\mu_i}{\partial\eta_i}\right)^2 \frac{\mathbb{E}\big[(Y_i - \mu_i)^2\big]}{\mathrm{Var}(Y_i)^2} x_{ij} x_{ik} && (3.21)\\
&= \sum_{i=1}^{N} \left(\frac{\partial\mu_i}{\partial\eta_i}\right)^2 \frac{x_{ij} x_{ik}}{\mathrm{Var}(Y_i)} && (3.22)\\
&= \mathbf{x}_{\boldsymbol{j}}^T \mathbf{W} \mathbf{x}_{\boldsymbol{k}} && (3.23)
\end{aligned}$$

where $\mathbf{W}^{(k)}$ is a diagonal matrix with

$$\mathbf{W}_{ii}^{(k)} = \frac{1}{\mathrm{Var}(Y_i)} \left(\frac{\partial\mu_i}{\partial\eta_i}\right)^2_{\widehat{\boldsymbol{\beta}}^{(k)}}. \tag{3.24}$$

Hence we have $\mathcal{I}(\widehat{\boldsymbol{\beta}}^{(k)}) = \mathbf{X}^T \mathbf{W}^{(k)} \mathbf{X}$ and we see from Eq. (3.16) that

$$\nabla l(\boldsymbol{\beta} \mid \mathbf{y}, \phi) = \mathbf{X}^T \mathbf{W} \tilde{\mathbf{z}} \tag{3.25}$$

with $\tilde{\mathbf{z}}_i = (y_i - \mu_i)\left(\frac{\partial\eta_i}{\partial\mu_i}\right)$. Multiplying both sides of 3.18 by $\mathcal{I}(\boldsymbol{\beta}^{(k)})$ and using Eqs. (3.23) to (3.25), we finally obtain

$$\mathbf{X}^T \mathbf{W}^{(k)} \mathbf{X} \widehat{\boldsymbol{\beta}}^{(k+1)} = \mathbf{X}^T \mathbf{W}^{(k)} \mathbf{z} \tag{3.26}$$

with $\mathbf{z} = \mathbf{X}\widehat{\boldsymbol{\beta}}^{(k)} + \tilde{\mathbf{z}}$ and all quantities evaluated at the current estimate $\boldsymbol{\beta}^{(k)}$ of the parameter vector. In other words, the Fisher scoring is equivalent to a series of weighted least squares problems, where the new parameter estimates are obtained by regressing the vector $\mathbf{z}$ on the original covariates $\mathbf{x_1}, \ldots, \mathbf{x_N}$ using weight matrix $\mathbf{W}$, and $\mathbf{z}$ and $\mathbf{W}$ are determined by the current estimate $\boldsymbol{\beta}^{(k)}$—hence why the algorithm is called *iteratively reweighted least squares* (IRWLS).

This procedure can be specialised to the particular case of the Poisson GLM in the following way. First, in order to obtain the matrix-vector form used above, we must flatten the tabular

Origin	**Dev**	**Value**
2007	1	3511
2008	1	4001
2009	1	4355
2010	1	4295
2011	1	4150
2012	1	5102
2013	1	6283
2007	2	3215
2008	2	3702
2009	2	3932
2010	2	3455
2011	2	3747
2012	2	4548
2007	3	2266
2008	3	2278
2009	3	1946
2010	3	2023
2011	3	2320
2007	4	1712
2008	4	1180
2009	4	1522
2010	4	1320
2007	5	1059
2008	5	956
2009	5	1238
2007	6	587
2008	6	629
2007	7	340

Table 3.1: UK Motor incremental claims triangle in long format

response (using, for example, the colexicographical ordering $(i,j) \mapsto jI+i$, i.e. column-major order). This gives us a triangle in long format, as illustrated in Table 3.1 for the incremental form of the UK Motor dataset from Christofides [1]. If we define the parameter vector

$$\boldsymbol{\beta} := \begin{bmatrix} c & a_2 & \cdots & a_I & b_2 & \cdots & b_I \end{bmatrix}^T , \tag{3.27}$$

then Eq. (3.1) can be rewritten as

$$\log(\mu_{ij}) = c + a_i + b_j = (\mathbf{e_1} + \mathbf{e}_{\boldsymbol{i}} + \mathbf{e}_{\boldsymbol{I+j-1}})^T \boldsymbol{\beta} \tag{3.28}$$

where $\mathbf{e}_{\boldsymbol{k}}$ denotes the kth standard basis vector in $\mathbb{R}^{(2I-1)}$. Hence we see that the covariates are binary vectors of length $2I-1$, with the position of the nonzero entries determined by the indices of the observation in the triangle, forming the rows of a very sparse design matrix. As the Poisson GLM uses the log link, we have $\mu_{ij} = e^{\eta_{ij}}$ and

$$\frac{\partial \mu_{ij}}{\partial \eta_{ij}} = e^{\eta_{ij}} , \tag{3.29}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{bmatrix}}_{\mathbf{X}^T} \underbrace{\begin{bmatrix} e^{\widehat{c}^{(k)}} & & & \\ & e^{\widehat{c}^{(k)}+\widehat{a}_1^{(k)}} & & \\ & & \ddots & \\ & & & e^{\widehat{c}^{(k)}+\widehat{a}_I^{(k)}+\widehat{b}_I^{(k)}} \end{bmatrix}}_{\mathbf{W}^{(k)}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} \widehat{c}^{(k)} \\ \widehat{a}_2^{(k)} \\ \vdots \\ \widehat{a}_I^{(k)} \\ \widehat{b}_2^{(k)} \\ \vdots \\ \widehat{b}_I^{(k)} \end{bmatrix}}_{\widehat{\boldsymbol{\beta}}^{(k)}}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{bmatrix}}_{\mathbf{X}^T} \underbrace{\begin{bmatrix} e^{\widehat{c}^{(k)}} & & & \\ & e^{\widehat{c}^{(k)}+\widehat{a}_1^{(k)}} & & \\ & & \ddots & \\ & & & e^{\widehat{c}^{(k)}+\widehat{a}_I^{(k)}+\widehat{b}_I^{(k)}} \end{bmatrix}}_{\mathbf{W}^{(k)}} \mathbf{z}$$

Figure 3.1: IRWLS equation for Poisson GLM in matrix form

from which, using Eq. (3.24), we finally obtain

$$\mathbf{W}_{ii} = \frac{1}{e^{\eta_{ij}}}(e^{\eta_{ij}})^2 = e^{\eta_{ij}}\,, \tag{3.30}$$

giving us all the components of the IRWLS algorithm. Fig. 3.1 shows in matrix form.

3.2.2 Quasi-likelihood methods

We have assumed up to this point that a GLM requires us to specify an exact distribution for the response variable. In many practical situations, however, this is either infeasible or leads to unrealistic models. An example which is particularly common with count data is a phenomenon known as *overdispersion*, where the variability of the data is greater than would be suggested by e.g. the Poisson or binomial distribution. Recall that the variance of a Pois(λ) distribution is λ, and that of a $B(n,p)$ distribution is $np(1-p)$; in both cases, it is fully determined by the mean, and we have no degree of freedom with which to adjust it in order to obtain a better fit to the data, as would be the case with the normal distribution, for example.

To remedy this, an extension can be made to the GLM framework, which only relies on the specification of a relation between mean and variance. Recall from above that the MLE works by setting the score equal to 0. If we write the likelihood in terms of $\boldsymbol{\mu}$, this will have components

$$\frac{\partial l(\boldsymbol{\mu} \mid \mathbf{y}, \phi)}{\partial \mu_j} = \frac{y_j - \mu_j}{\phi V(\mu_j)}\,, \tag{3.31}$$

and is therefore completely determined by $V(\cdot)$. Suppose now, conversely, that we start from

$V(\cdot)$. We could then define functions

$$Q_i(\mu \mid y_i, \phi) \coloneqq \int_{y_i}^{\mu} \frac{y_i - u}{\phi V(u)}\, du\,, \tag{3.32}$$

and estimate $\boldsymbol{\mu}$ (and therefore $\boldsymbol{\beta}$) by minimising

$$Q(\boldsymbol{\mu} \mid \mathbf{y}, \phi) \coloneqq \sum_{i=1}^{N} Q_i(\mu_i \mid y_i, \phi)\,. \tag{3.33}$$

It must be stressed that Q has no probabilistic significance: it does not, in general, correspond to the log-likelihood of any distribution. Rather, it functions a device to obtain estimates of the desired parameters, fulfilling in this a similar role to that of the log-likelihood, which is why we refer to it as a *quasi-likelihood function*[1]. This extension of the classical GLM framework allows for more flexibility in our modelling, since we only have to assume a particular mean-variance relation for the response, not a distribution. This finally allows us to state the main model of this chapter.

Model 3 (Overdispersed Poisson GLM).

(ODP1) *The incremental claims are independent from each other.*

(ODP2) *There exist parameters c, $a_1, \ldots, a_I$ and $b_1, \ldots, b_I$ such that*

$$\log(\mu_{ij}) = c + a_i + b_j\,, \tag{3.34}$$

with $\mu_{ij} \coloneqq \mathbb{E}[X_{ij}]$ and $a_1 = b_1 = 0$.

(ODP3) *There exists a parameter ϕ such that*

$$\mathrm{Var}(X_{ij}) = \phi\mu_{ij}\,. \tag{3.35}$$

It is usual to identify quasi-likelihood models derived from specific distributions (i.e. using the same variance function) by prefixing 'quasi' to their name, e.g. quasi-Poisson or quasi-binomial. The quasi-model will yield the same point estimates for the parameters as the ordinary variant, provided the observations belong to the support of the latter. For data consisting entirely of positive integers (like the triangle from Table 2), it thus follows from the remark at the end of Section 3.1 that the Overdispersed Poisson GLM yields the same predictions as the chain ladder, although it differs significantly when it comes to the error/variance, as can be seen from Table 3.2

More generally, it can be shown (see [18, Section 2]) that the chain ladder results will be reproduced as long as the additional condition

$$\sum_{i=1}^{I} X_{ij} \geq 0 \tag{3.36}$$

is satisfied for $j \in \{1, \ldots, I\}$. The Overdispersed Poisson GLM is therefore robust to the presence of a limited number of negative claim amounts, which is sometimes observed in practice. Moreover, it lifts the unrealistic restriction that the response values must be integers, and gives us a way of accounting for overdispersion, which is a feature of many run-off triangles.

[1] It would actually be more correct to call Q a quasi-*log*-likelihood, but the current nomenclature has been widely adopted and the literature seems to have resigned itself to it.

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j$	$\widehat{\mathrm{MSEP}}(\widehat{R}_j)$	
				Poisson	ODP
2	0.03	-0.12	350.9	27.07	125.81
3	0.1	-0.63	1037.54	44.12	205.08
4	0.03	-1.03	2044.86	60	278.85
5	0.09	-1.31	3663.4	83.22	386.79
6	0.28	-1.86	7162.15	130.22	605.27
7	0.49	-2.43	14396.92	249.17	1158.12
$\widehat{c}$					8.26
$\widehat{\phi}$					21.6

Table 3.2: ODP and Poisson GLM results for UK Motor triangle

3.3 Bootstrap methodology

Developing a bootstrap procedure for the Overdispersed Poisson GLM is in some respects easier than for the Mack chain ladder, because the absence of a recursive structure makes it more straightforward to reason about resampling. Furthermore, bootstrap methods for claims triangle GLMs have seen more discussion in the literature (see e.g. [21] and [10]), and so we can draw upon this material for our exposition. As in Chapter 2, we shall take Section 1.2 as our starting point, where we distinguished between nonparametric, semiparametric and parametric bootstraps. Of these, the nonparametric variant, which involves resampling predictor-response pairs from the original data, is the most difficult to apply to the Poisson GLM. Because the size of claims triangles is typically limited, this can easily lead to a situation in which there are no observations for a given row or column. In that case, no estimates can be obtained for the corresponding parameters, and it becomes impossible to simulate pseudo-responses in (part of) the future triangle. This is especially likely for the lower left and upper right corner points, which are the only observations in their row and column, respectively.

In order to avoid this difficulty, we therefore need to find a way of guaranteeing that every row and column contains at least one observation. The crudest option would be to simply discard any resampled triangles which are defective, but this is quite inefficient; the same result can be achieved at far lower computational cost by constructing resampled triangles directly in such a way as to satisfy the desired property. We experimented with three ways of doing this: preserving the corner points, preserving the first row and column, and randomly selecting an observation in every row and column to preserve; the results are shown in Table 3.3. However, none of these turn out to be satisfactory, as seen from the comparison with Table 3.2. We will therefore exclude the pairs bootstrap from consideration in the remainder of our investigation.

For the semiparameteric bootstrap, the essential step is to find a satisfactory definition for the residuals such that they are i.i.d. , exactly as in Section 2.3. Things are more complicated here than for Mack's model, as there generally exists no natural seperation of the response into mean and additive error for a non-Gaussian response (this problem was recognised early on in the literature on GLM bootstrapping, see [22]). Consequently, a multitude of different residual types are available. We will look at two of these in particular.

The *Pearson residuals*

$$r_{ij} := \frac{X_{ij} - \widehat{\mu}_{ij}}{\sqrt{V(\widehat{\mu}_{ij})}}\,, \tag{3.37}$$

attempt to deal with the inherent heteroscedasticity of the GLM response by dividing out the

Origin	$\widehat{\mathbf{MSEP}}(\widehat{R}_j^B)$		
	First row and column	Corner points	Randomised
2	88.37	85.99	88.40
3	153.71	155.55	174.22
4	282.34	211.79	270.11
5	436.18	272.94	370.45
6	744.95	389.42	548.55
7	978.89	548.92	907.98

Table 3.3: Pairs bootstrap results for different strategies

component of the variance which is specific to each observation. In this, they resemble the standardised residuals in the context of weighted linear regression. Extending this analogy further, we can adjust Eq. (3.37) for the leverage of the observation, i.e.

$$\tilde{r}_{ij} := \frac{X_{ij} - \widehat{\mu}_{ij}}{\sqrt{V(\widehat{\mu}_{ij})(1 - h_{ij})}}\,, \tag{3.38}$$

where h_{ij} is the appropriate diagonal element in the hat matrix

$$\mathbf{H} = \mathbf{X}(\mathbf{X}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{z} \tag{3.39}$$

corresponding to the final iteration of the IRWLS algorithm.

Another kind of residuals are based on a goodness-of-fit measure for GLMs known as the *deviance*. It can be derived from Eq. (3.31) by noticing that the mean parametrisation of the log-likelihood is maximised at $\boldsymbol{\mu} = \mathbf{y}$, so that the quantity

$$D(\mathbf{y}, \boldsymbol{\mu}) := \sum_{i=1}^{N} d(y_i, \mu_i) := 2\sum_{i=1}^{N}(l(y_i \mid y_i) - l(\widehat{\mu}_i \mid y_i)) \tag{3.40}$$

expresses the departure of our model from a perfect fit. The functions $D(\mathbf{y}, \boldsymbol{\mu})$ and $d(y_i, \mu_i)$ are called the *total* and *unit deviance*, respectively. The *deviance residuals* are then defined as

$$r_{ij} := \operatorname{sign}(x_{ij} - \widehat{\mu}_{ij})\sqrt{d(x_{ij}, \widehat{\mu}_{ij})}\,. \tag{3.41}$$

Similarly to the Pearson residuals, we can also define leverage-adjusted version of these:

$$\tilde{r}_{ij} := \frac{\operatorname{sign}(x_{ij} - \widehat{\mu}_{ij})\sqrt{d(x_{ij}, \widehat{\mu}_{ij})}}{\sqrt{1 - h_{ij}}}\,, \tag{3.42}$$

where H is defined as in Eq. (3.39). Under certain assumptions (see [23, Section 7.5]), it can be shown that both the Pearson and deviance residuals have an asymptotic normal distribution.

The semiparametric bootstrap procedure for the Overdispersed Poisson GLM is outlined in Algorithm 6. Here, the notation $\delta(x_{ij}, \mu_{ij})$ refers to one of the residual types discussed above. Note that inversion of the residuals (in order to obtain pseudo-responses) requires us to solve

$$\delta(x, \widehat{\mu}_{ij}) = r_{ij}^{(k)}\,, \tag{3.43}$$

Input: Incremental claims triangle $\mathcal{D}_I = \{X_{ij} \mid i+j \leq I+1\}$, number of bootstrap samples B

$\{(r_{ij}, \widehat{\mu}_{ij}) \mid i+j \leq I+1\} \leftarrow \textsc{fit}(\mathcal{D}_I)$
for $k \leftarrow 1, B$ **do**
  $\{r_{ij}^{(k)} \mid i+j \leq I+1\} \leftarrow \textsc{resample}(\{r_{ij} \mid i+j \leq I+1\})$
  for $j \leftarrow 2, I$ **do**
    for $i \leftarrow I+2-j, I$ **do**
      $X_{ij}^{(k)} \leftarrow \textsc{solve}(\delta(x, \widehat{\mu}_{ij}) = r_{ij}^{(k)})$
    end for
  end for
  $\widehat{\boldsymbol{\beta}}^{(k)} \coloneqq (\widehat{c}^{(k)}, \widehat{a}_2^{(k)}, \ldots, \widehat{a}_I^{(k)}, \widehat{b}_2^{(k)}, \ldots, \widehat{b}_I^{(k)}) \leftarrow \textsc{fit}(\mathcal{D}_I^{(k)})$
end for
return $\{\widehat{\boldsymbol{\beta}}^{(k)} \mid k = 1, \ldots, B\}$

Algorithm 6: Semiparametric bootstrap for the Overdispersed Poisson GLM

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	0.03	−0.12	353.41	121.96
3	0.10	−0.63	1040.90	196.06
4	0.03	−1.03	2040.92	267.20
5	0.09	−1.32	3662.56	368.93
6	0.28	−1.87	7149.55	577.77
7	0.49	−2.45	14420.05	1116.74

(a) Pearson

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j^B$	$\widehat{\text{MSEP}}(\widehat{R}_j^B)$
2	0.03	−0.11	417.16	85.25
3	0.10	−0.61	1201.75	136.75
4	0.03	−1.00	2321.53	186.85
5	0.09	−1.26	4084.83	258.45
6	0.27	−1.78	7841.82	404.07
7	0.48	−2.32	15452.66	757.83

(b) Deviance

Table 3.4: Semiparametric bootstrap results for different residual types

which, for the deviance residuals, yields a set of nontrivial equations. It should also be noted that the Pearson and deviance residuals corresponding to the lower left and upper right corner will always have a value of zero by construction; we follow the general trend of the literature (see [21, p. 706], [24, p. 6]) in excluding these from the resampling process.

Table 3.4 shows the results of applying the semiparameteric bootstrap to the data in Table 2. Comparing it to Table 3.2, we clearly see that the deviance variant severely underestimates the uncertainty of the reserve. The reason for this is a technical limitation of the deviance residuals which was pointed out in [25, pp. 13–14]: if the response data is positive, we can only resample residuals which satisfy the condition

$$r_{ij} > -\sqrt{2\min(\mathcal{D}_I)}\,. \tag{3.44}$$

Larger residuals are more likely to violate this, but discarding them introduces bias in the bootstrap, causing the predictive distribution to become narrower than would be warranted by the model. Consequently, we will not consider deviance residuals in the remainder of this study.

Finally, we consider the parametric bootstrap. Here again, the development proceeds in a very similar way to Section 2.3: we fit the model to the original data, giving us estimates for c, a_i, b_j and ϕ, and then use these to simulate a pseudo-realisation of the triangle. We can choose any distribution, as long as it satisfies

$$\mathrm{Var}(X_{ij}) = \widehat{\phi}\, V(\widehat{\mu}_{ij}) = \phi\, \widehat{\mu}_{ij}\,. \tag{3.45}$$

An obvious candidate would be the normal distribution $\mathcal{N}(\widehat{\mu}_{ij}, \phi\widehat{\mu}_{ij})$. Alternatives which have been proposed in the literature (see [26]) are the gamma distribution $\Gamma(\alpha, \beta)$ with

$$\alpha = \frac{\widehat{\mu}_{ij}}{\phi}, \qquad \beta = \frac{1}{\phi}\,, \tag{3.46}$$

and the scaled Poisson distribution

$$Y = \phi X\,, \quad X \sim \mathrm{Pois}(\widehat{\mu}_{ij}/\phi)\,, \tag{3.47}$$

which satisfies $\mathbb{E}[Y] = \widehat{\mu}_{ij}$ and $\mathrm{Var}(Y) = \phi\widehat{\mu}_{ij}$. The model is then refitted to the simulated triangle to obtain $\widehat{c}^{(k)}$, $\widehat{a}_i^{(k)}$, $\widehat{b}_j^{(k)}$ and $\widehat{\phi}^{(k)}$. The algorithm is outlined in Algorithm 7, and the results for the three choices of distribution are shown in Tables 3.5 and 3.6

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j^B$	$\widehat{\mathrm{MSEP}}(\widehat{R}_j^B)$
2	0.03	−0.12	350.26	125.91
3	0.10	−0.63	1038.71	207.26
4	0.04	−1.03	2053.19	278.32
5	0.09	−1.31	3676.36	385.16
6	0.28	−1.86	7187.67	607.74
7	0.49	−2.47	14428.44	1161.72

(a) Normal distribution

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j^B$	$\widehat{\mathrm{MSEP}}(\widehat{R}_j^B)$
2	0.03	−0.12	345.95	123.28
3	0.10	−0.63	1035.70	202.69
4	0.03	−1.03	2045.65	280.66
5	0.09	−1.32	3662.17	389.09
6	0.28	−1.87	7164.46	604.11
7	0.49	−2.47	14395.83	1143.86

(b) Gamma distribution

Table 3.5: Parametric bootstrap results

i/j	$\widehat{a}_i$	$\widehat{b}_j$	$\widehat{R}_j^B$	$\widehat{\mathrm{MSEP}}(\widehat{R}_j^B)$
2	0.03	−0.12	347.70	126.24
3	0.10	−0.63	1030.26	204.74
4	0.03	−1.03	2039.72	274.15
5	0.09	−1.31	3658.26	383.81
6	0.28	−1.88	7169.73	604.00
7	0.49	−2.47	14390.55	1173.80

Table 3.6: Parametric bootstrap results for scaled Poisson distribution

3.4 Incorporating the process error

The only question that remains is how to incorporate the intrinsic variability of the response in our bootstrap simulations. Just as in Section 1.3 and Section 2.4, we will use the concept of the predictive distribution to achieve this. Depending on the type of bootstrap under consideration,

Input: Incremental claims triangle $\mathcal{D}_I = \{X_{ij} \mid i + j \leq I + 1\}$, number of bootstrap samples B, boolean QUASI indicating the GLM type

$\{\widehat{\mu}_{ij} \mid i + j \leq I + 1\} \leftarrow \text{FIT}(\mathcal{D}_I)$
for $k \leftarrow 1, B$ **do**
  for $j \leftarrow 2, I$ **do**
    for $i \leftarrow I + 2 - j, I$ **do**
      if QUASI **then**
        $X_{ij}^{(k)} \leftarrow \mathcal{N}(\widehat{\mu}_{ij}, \widehat{\phi}\,\widehat{\mu}_{ij})$
      else
        $X_{ij}^{(k)} \leftarrow \text{Pois}(\widehat{\mu}_{ij})$
      end if
    end for
  end for
  $\widehat{\boldsymbol{\beta}}^{(k)} \coloneqq (\widehat{c}^{(k)}, \widehat{a}_2^{(k)}, \ldots, \widehat{a}_I^{(k)}, \widehat{b}_2^{(k)}, \ldots, \widehat{b}_I^{(k)}) \leftarrow \text{FIT}(\mathcal{D}_I^{(k)})$
end for
return $\{\widehat{\boldsymbol{\beta}}^{(k)} \mid k = 1, \ldots, B\}$

Algorithm 7: Parametric bootstrap for (overdispersed) Poisson GLM

this will result in different procedures. For the semiparametric bootstrap, we resample the residuals a second time and invert the appropriate formula to obtain bootstrap responses; with the parametric bootstrap, we achieve this through direct simulation. The results are shown in Figs. 3.2 and 3.3.

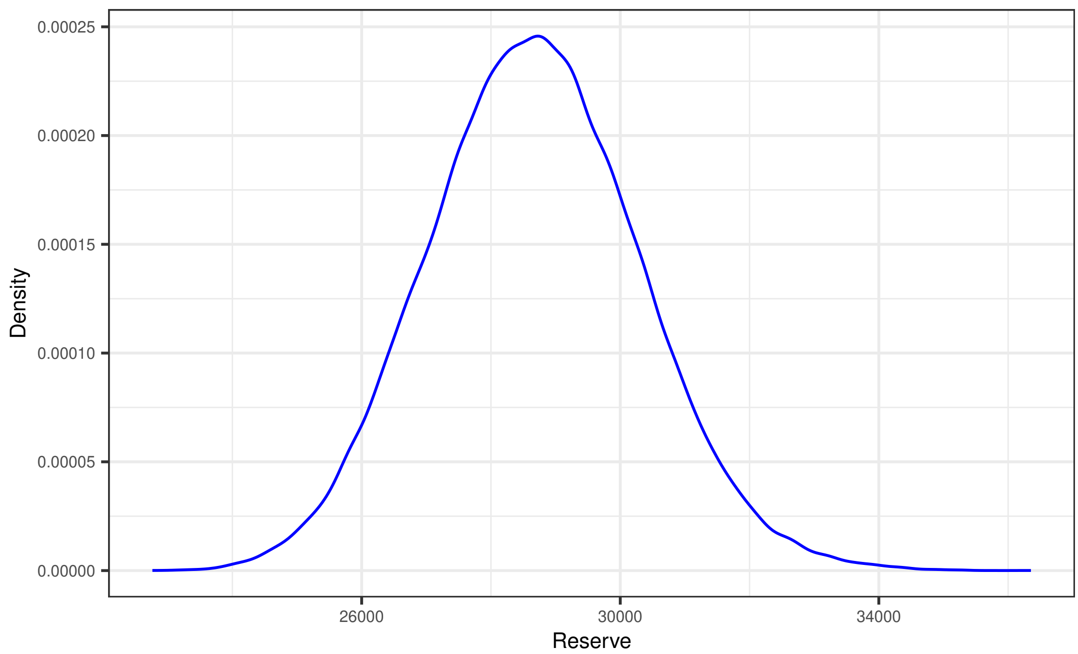


Figure 3.2: Predictive distribution for the semiparametric bootstrap

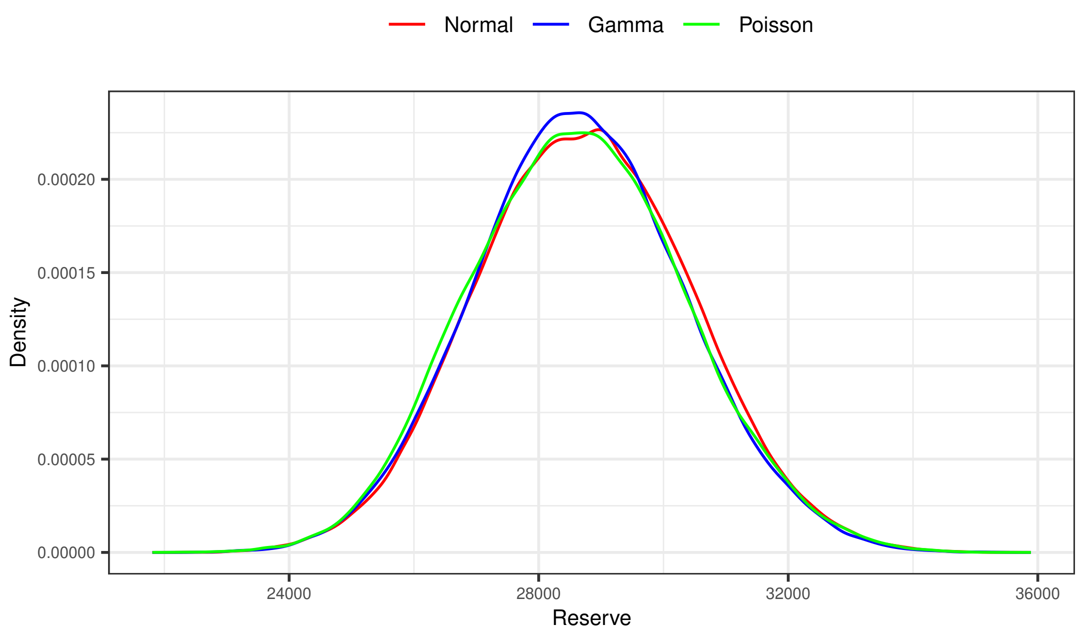


Figure 3.3: Predictive distribution for the parametric bootstrap with different distributions

Chapter 4

Simulation study

As explained in the Introduction and Section 1.3, we want to study the impact of deviations from the model assumptions of the Mack chain ladder and the Overdispersed Poisson GLM on the predictive distribution of the total outstanding claims. To this end, we will simulate a large number of run-off triangles, following a similar approach to the one described in [27], which satisfy these assumptions by construction, except for a number of deviating observations (we will call such data *contaminated*). We will then use the bootstrap methods described in Chapters 2 and 3 to obtain the predictive distribution while excluding a given subset of points from the resampling process. Our aim is to find out whether predictive distributions computed from data in which the correct points have been discarded (which we will refer to as the *cleaned* data) differ significantly from those obtained after removing arbitrary observations. This would then suggest a way of detecting irregularities in new triangles: compare for every point (resp. calendar, origin period) the predictive bootstrap distribution obtained with and without this point; if a significant difference can be detected, then this subset constitutes a deviation from the model.

4.1 Mack's model

Recall from Section 2.1 that the Mack chain ladder is recursive in nature, modelling the cumulative claim amounts in the next development period as a function of those in the previous one. Hence, it requires that observations for the first development period be given initially. This could be done using a separate model, but we choose here the simpler approach of borrowing the first column from the UK Motor example data in Table 2. Similarly, we choose as the development factors $f_1, \ldots, f_{I-1}$ and dispersion parameters $\sigma_1, \ldots, \sigma_{I-1}$ of the simulated triangles the fitted ones from this dataset.

In order to obtain a synthetic triangle which satisfies (Mack1) to (Mack3) except at a single point (i^*, j^*) (where $i^* + j^* \leq I+1$ and $j^* > 1$), we can then start from the initial column and use any suitable data-generating process (such as Eq. (2.48)) to recursively generate the next anti-diagonal of the triangle. For the non-contaminated rows, we continue this process until reaching the current calendar year. In the contaminated row, we stop at column $j^* - 1$ and use c_μ and c_σ to draw an observation from the perturbed distribution $\mathcal{N}(c_\mu f_{j^*-1} C^*_{i^*,j^*-1}, c_\sigma \sigma^2_{j^*-1} C^*_{i^*,j^*-1})$, after which we resume the previous process. The whole procedure is outlined in Algorithm 8. Except for some additional bookkeeping, simulating triangles where an entire origin or calendar period has been perturbed can be done in exactly the same way.

Input: Development factors $f_1, \ldots, f_{I-1}$, dispersion parameters $\sigma_1, \ldots, \sigma_{I-1}$, perturbed point (i^*, j^*) with $i^* + j^* \leq I + 1$ and $j^* > 1$, perturbation factors c_μ, c_σ

```
for all 1 ≤ i, j ≤ I do
    C_ij ← NA
end for
for i ← 1, I do
    C*_i1 ← C_i1
    if i ≠ i* then
        for j ← 2, I + 1 − i do
            C*_ij ← N(f_j C*_{i,j−1}, σ_j^2 C*_{i,j−1})
        end for
    else
        for j ← 1, j* − 1 do
            C*_{i*j} ← N(f_j C*_{i*,j−1}, σ_j^2 C*_{i*,j−1})
        end for
        C*_{i*j*} ← N(c_μ f_j C*_{i*,j*−1}, c_σ σ_j^2 C*_{i*,j*−1})
        for j ← j* + 1, I + 1 − i do
            C*_{i*j} ← N(f_j C*_{i*,j−1}, σ_j^2 C*_{i*,j−1})
        end for
    end if
end for
return (C*_ij)_{1≤i,j≤I}
```

Algorithm 8: Simulating a claims triangle with a single perturbed point for the Mack chain ladder

4.2 Overdispersed Poisson GLM

Simulating triangles conforming to the Overdispersed Poisson GLM is significantly easier than for the Mack chain ladder, as we don't have to deal with recursion. Using the fitted parameters from Table 3.1 for ϕ, c, $a_1, \ldots, a_I$ and $b_1, \ldots, b_I$, we generate an observation for every cell (i, j) in the upper triangle by drawing from an appropriate distribution. Triangles with a single contaminated point can then be obtained by applying the perturbation before calling the random number generator. Again, similar algorithms can be used for perturbed origin or calendar periods.

4.3 Setup

We are now ready to discuss the general setup of the simulation study. Using Algorithms 8 and 9, we generate claims triangles which perfectly satisfy the assumptions of the Mack chain ladder or the Overdispersed Poisson GLM, except for a selected number of points. In particular, we consider the three cases of (i) a single perturbed observation (ii) a perturbed calendar period and (iii) a perturbed origin period. In each instance, we carry out one of the bootstrap procedures outlined in Chapters 2 and 3 while excluding a subset of observations of the same type as the

Input: Intercept c, development period parameters $a_2, \ldots, a_I$, origin period parameters $b_2, \ldots, b_I$, dispersion parameter ϕ, perturbation factor c_λ

$a_1 \leftarrow 0$, $b_1 \leftarrow 0$
for all $1 \leq i, j \leq I$ **do**
  $\mu_{ij} \leftarrow \exp(c + a_i + b_j)$
  $X_{ij} \leftarrow 0$
end for
for $i \leftarrow 1, I$ **do**
  for $j \leftarrow 1, I + 1 - i$ **do**
    $X^*_{ij} \leftarrow \mathcal{N}(\mu_{ij}, \phi\mu_{ij})$
  end for
end for
return $(X^*_{ij})_{1 \leq i,j \leq I}$

Algorithm 9: Simulating triangle with single perturbed point for the Overdispersed Poisson GLM

contaminated one, and then compare the resulting predictive distributions. Our hope is that we can detect a significant difference between the predictive distribution obtained when excluding the right subset and the ones where a different subset has been removed.

The way in which observations can be excluded when carrying out the bootstrap is specific to each method, and raises a number of practical issues. For the semiparametric bootstraps, the most obvious way is to simply remove the corresponding residuals from the resampling pool. In the case of the Mack chain ladder, this implies that the first column cannot contain contaminated points; this makes sense, as this model does not make any statements regarding the behaviour of the initial observations. While odd points in the triangle may be considered *outliers* in a general sense, they do not necessarily constitute deviations from the model assumptions. To minimise the impact of this extraneous factor on the results, we have taken the approach of borrowing this initial column from the benchmark dataset in Table 2, as explained in Section 4.1.

For the parametric bootstrap, it is not as clear how we should go about excluding certain points. Obviously, our goal is to nullify the impact of these observations on the bootstrap result, which suggests removing them from the initial fitting process when obtaining the parametric model used to simulate new pseudo-data. We have to be careful here, however, as this can sometimes lead to situations where the model cannot be fit at all. In particular, for the Mack chain ladder, we cannot exclude the upper right corner point, because we cannot estimate the final development factor otherwise. The approach outlined here is therefore not suitable to detecting deviations for this point. This is no great loss, however: the final development year requires special attention in any case, as it is very tied up with issues of tail factors and extrapolation. For the Poisson GLM, the restriction is even more stringent; here, we cannot exclude the lower left corner either, because the resulting model would have no parameter corresponding to the final origin period, and consequently cannot be used for projection.

Finally, for the pairs bootstrap, we simply exclude the affected points from the resampling process.

4.4 Implementation and results

The code for carrying out the simulations and generating the examples from previous chapters has been mostly implemented in the `R` language [28]. The computational power required to execute the large number of possible configurations exceeds the capacity of this high-level scripting language, however: the results dataframe for the Mack chain ladder contains 37,602,000 rows, for example. It was therefore decided to write the most numerically intensive parts of the simulation in `Fortran` and link it to `R` through a `C++` wrapper using the `Rcpp` package [29]. For reproducibility and convenience, all software has been bundled together into a package, which is available at `https://github.com/OthmanElHammouchi/claimsBoot`. Instructions for installing are provided there as well.

The results are mostly presented as faceted density plots comparing, for different configurations, the predictive density of the reserve when the contaminated observations have been excluded (shown in red) with the predictive densities obtained after excluding model-conforming points. As it would be impractical to show all of them, we have therefore endeavoured to select a representative sample for inclusion. The reader is encouraged to download the code and experiment with it for him- or herself.

4.4.1 Mack's model

In experiments where a single point has been perturbed, the semiparametric bootstrap performs poorly when used with standardised and log-normal residuals, as shown by Figs. 4.1 to 4.4. The predictive distribution does not differ significantly between the cleaned and contaminated data; this holds across different perturbation factors, and regardless of whether the conditional or unconditional approach is used.

Closer inspection of the simulation reveals the reason for this: standardised and log-normal residuals are not very sensitive to anomalies in a single observation. Consider for instance the triangle in Table 4.1, which is perturbed in point $(1, 2)$ with $c_\mu = 100$ and $c_\sigma = 1$, leading to hugely inflated values in the first row. The corresponding residual in Tables 4.2a and 4.2b does not exhibit a deviation of commensurate severity, however, and so excluding it does.

This stands in stark contrast with the studentised residuals, which are strongly affected by single point deviations in the simulated triangle, as can be seen in Table 4.3. Consequently, the cleaned triangle stands out from the contaminated ones. This is illustrated in Table 4.4, which contains the means of the predictive distribution based on different outlier and excluded points with $c_\mu = 2$ and $c_\sigma = 1$. The minimum in each row is highlighted in red. This lines up almost perfectly with diagonal, which means that the predictive reserve mean of the cleaned triangle tends to stand out when using studentised residuals.

The parametric bootstrap shows good performance across the board. The choice of distribution does not appear to have a noticeable impact, although we do observe a substantial difference between the conditional and unconditional approaches, with the latter generally outperforming the former, often drastically, as shown in Fig. 4.5. Lastly, the pairs bootstrap is better at distinguishing contaminated from cleaned data than the semiparametric bootstrap with log-normal or standardised residuals, but does not outperform the one using studentised residuals or the parametric bootstrap. As Table 4.5 shows, the mean of the predictive distribution simulated on the basis of the cleaned data stands out for every outlier point, but often only barely. In all cases, differences tend to be noticeable for later development periods.

The same general trends are observed in a more pronounced form when a calendar or origin period is perturbed, owing to the larger number of deviating points. Consequently, even the semiparametric bootstrap with log-normal or standardised residuals shows serviceable results, as can be seen in Figs. 4.7a, 4.7b, 4.8a and 4.8b. Later periods universally show stronger effects, because they contain more observations; for the first period, no bootstrap method can distinguish between clean and contaminated data.

Origin	**Dev**						
	1	2	3	4	5	6	7
1	3511	663172.28	847468.65	972420.52	1069036.05	1123357.24	1154289.54
2	4001	7455.21	9760.10	11230.45	12303.74	12949.43	
3	4355	8291.40	10770.20	12376.93	13605.41		
4	4295	7850.70	9992.00	10968.64			
5	4150	7842.25	9869.06				
6	5102	9389.75					
7	6283						

Table 4.1: Simulated triangle where observation C_{12} has been perturbed, with $c_\mu = 100$ and $c_\sigma = 1$

Origin	**Dev**					
	2	3	4	5	6	7
2	2.24	−0.63	0.88	1.00	−1	
3	−0.39	1.43	0.13	−1.41	1	
4	−0.41	1.01	0.08	−0.02		
5	−0.40	−0.26	−1.73			
6	−0.40	−0.94				
7	−0.45					

(a) Standardised

Origin	**Dev**					
	2	3	4	5	6	7
2	2.04	−0.13	0.16	0.15	−0.11	
3	−1.16	1.42	0.14	−1.41	0.99	
4	−1.19	1.01	0.09	−0.02		
5	−1.21	−0.25	−1.74			
6	−1.17	−0.93				
7	−1.31					

(b) Log-normal

Table 4.2: Residuals corresponding to the triangle in Table 4.1

Origin	**Dev**					
	2	3	4	5	6	7
2	4834.13	−0.58	0.83	0.99		
3	−0.35	1.78	0.11	−36.03		
4	−0.37	1.02	0.07	−0.02		
5	−0.37	−0.23	−19.38			
6	−0.36	−0.92				
7	−0.41					

Table 4.3: Studentised residuals corresponding to the triangle in Table 4.1

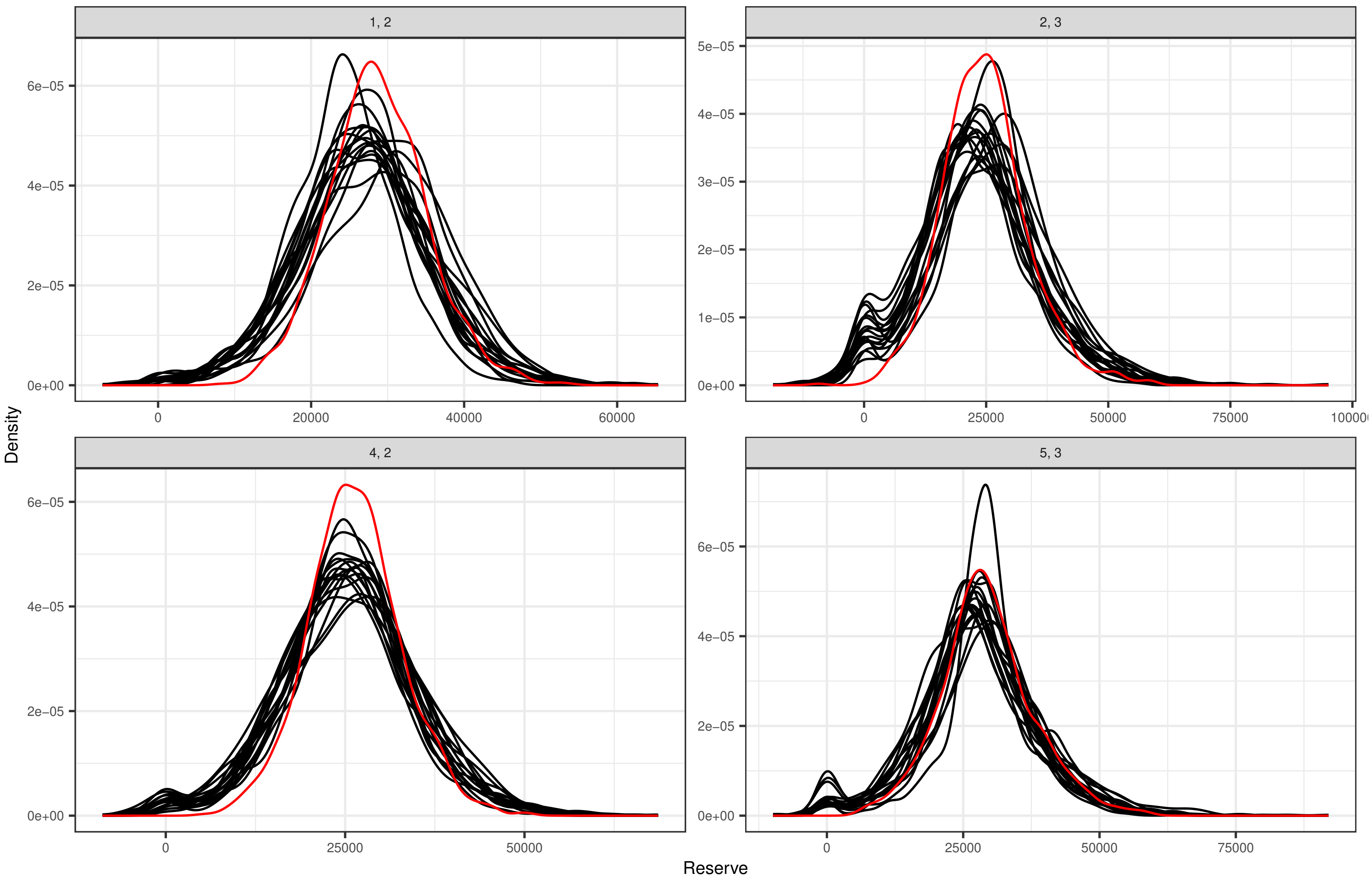


Figure 4.1: Density plots of predictive distributions obtained using the conditional semiparametric bootstrap of Mack's model with standardised residuals for different deviating points. The curves in each facet correspond to densities obtained while excluding a different point in the resampling procedure, with the result from the cleaned dataset highlighted in red. The perturbation factors are $c_\mu = c_\sigma = 0.5$.

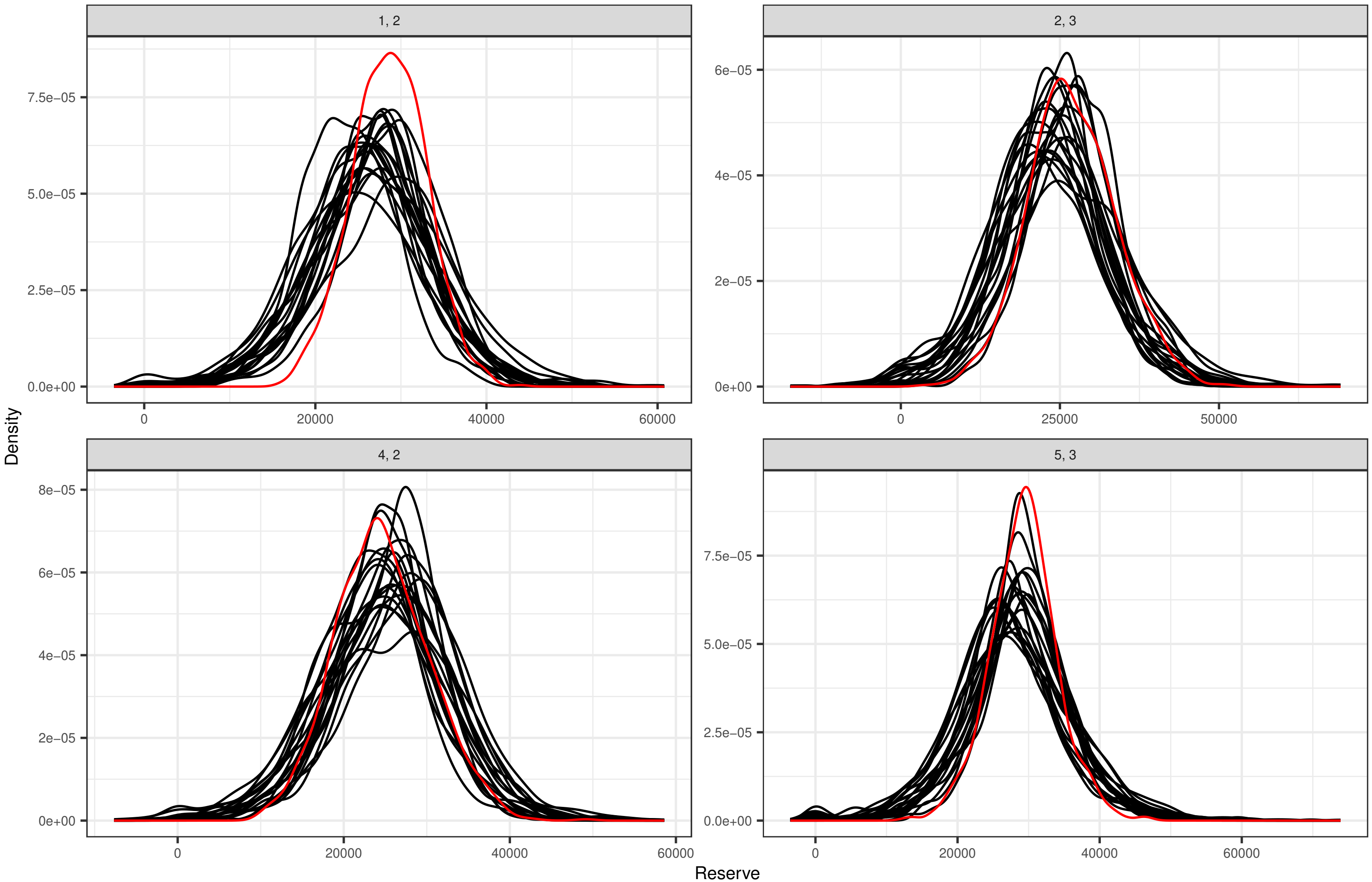


Figure 4.2: Density plots of predictive distributions obtained using the conditional semiparametric bootstrap of Mack's model with log-normal residuals for different deviating points. The curves in each facet correspond to densities obtained while excluding a different point in the resampling procedure, with the result from the cleaned dataset highlighted in red. The perturbation factors are $c_\mu = 0.5$ and $c_\sigma = 2$.

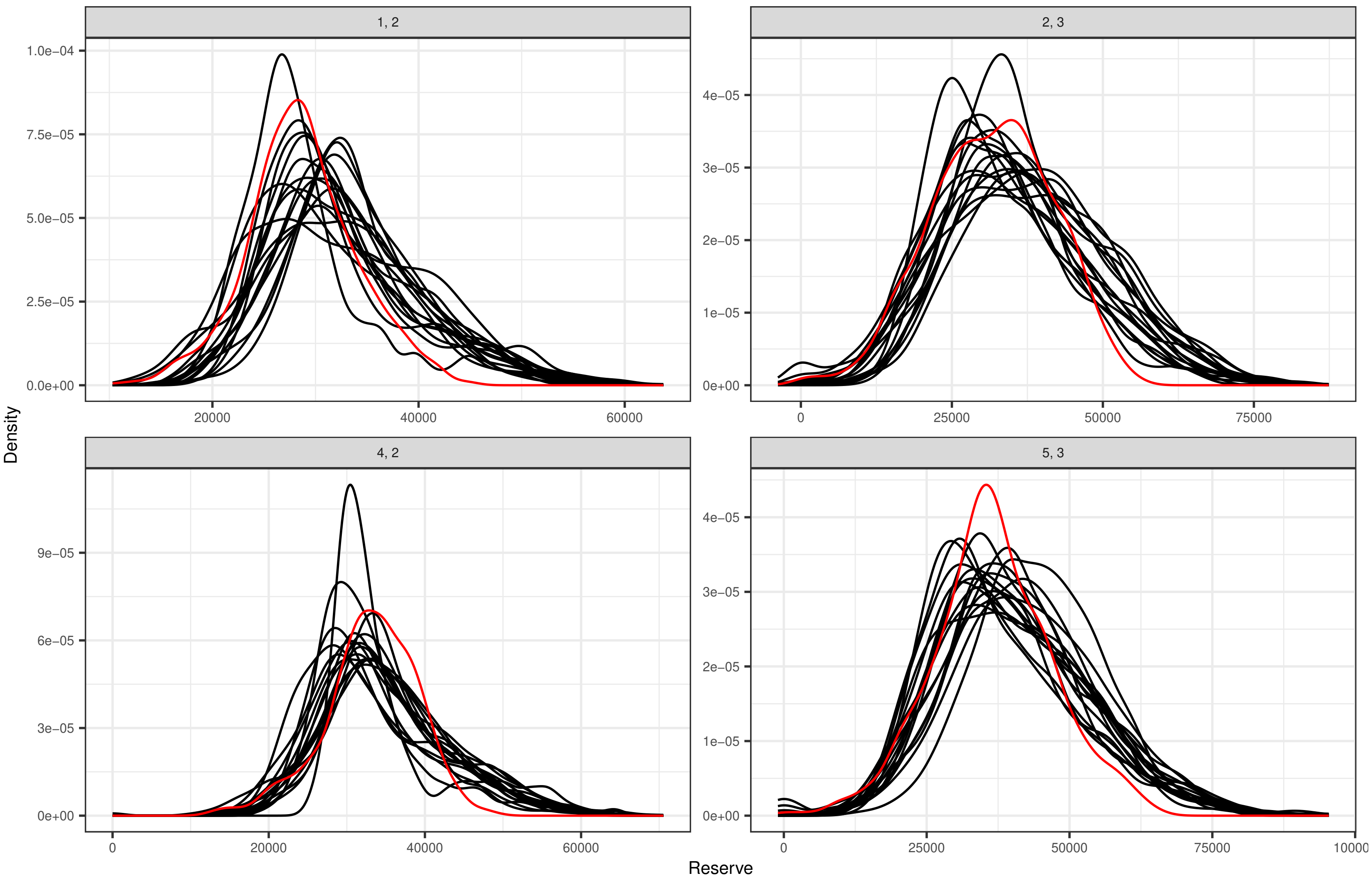


Figure 4.3: Density plots of predictive distributions obtained using the unconditional semiparametric bootstrap of Mack's model with standardised residuals for different deviating points. The curves in each facet correspond to densities obtained while excluding a different point in the resampling procedure, with the result from the cleaned dataset highlighted in red. The perturbation factors are $c_\mu = 2$ and $c_\sigma = 0.5$.

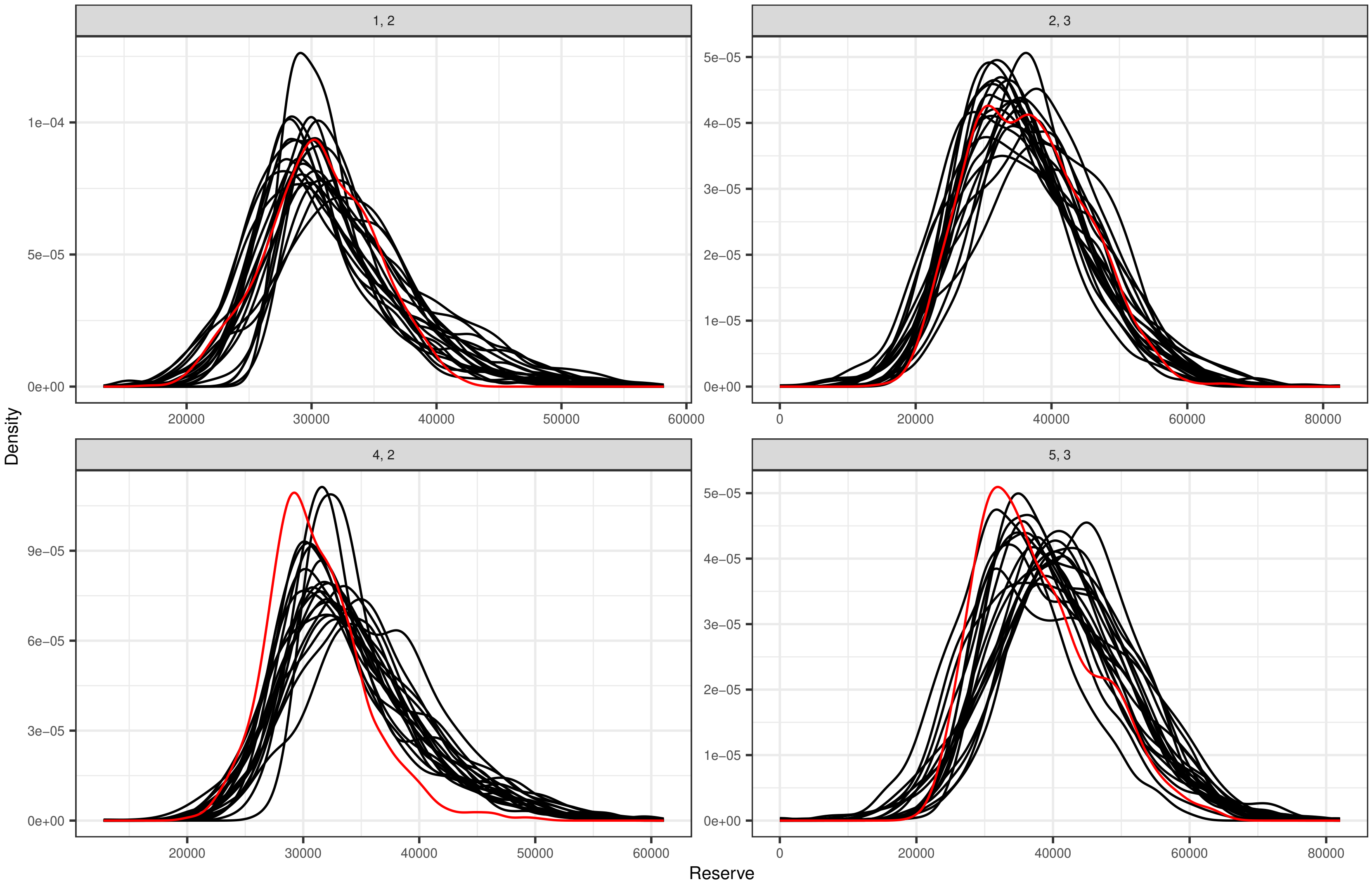
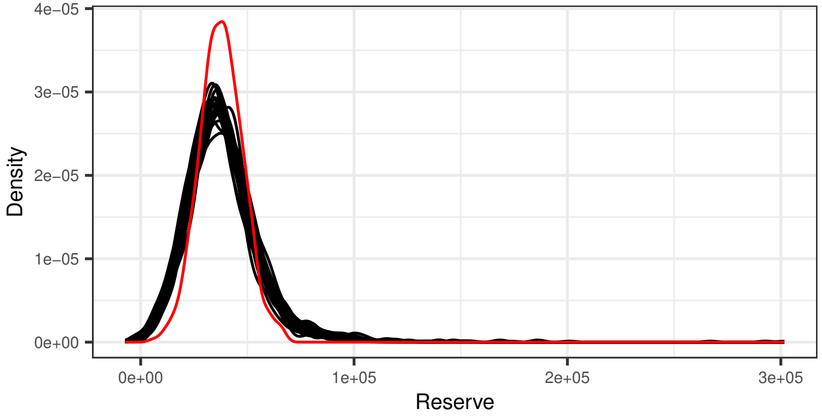
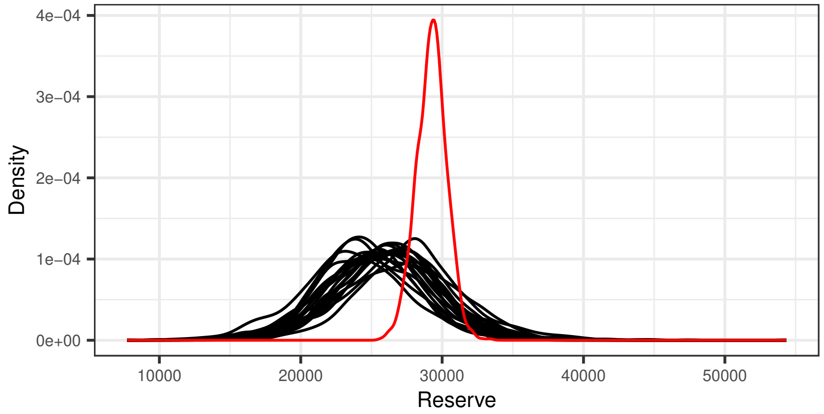


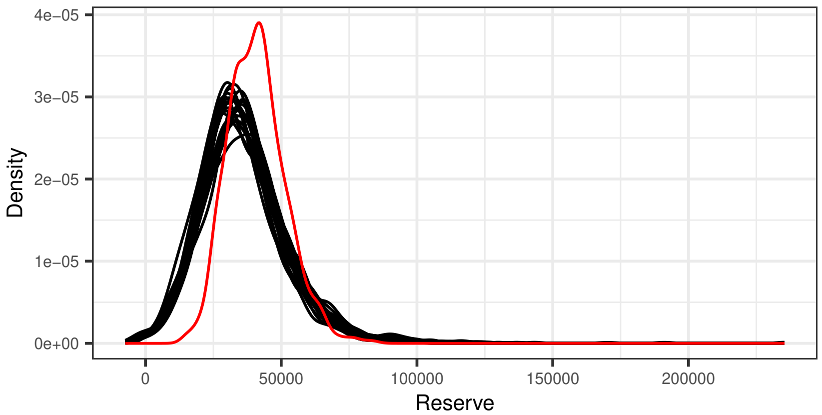
Figure 4.4: Density plots of predictive distributions obtained using the unconditional semiparametric bootstrap of Mack's model with log-normal residuals for different deviating points. The curves in each facet correspond to densities obtained while excluding a different point in the resampling procedure, with the result from the cleaned dataset highlighted in red. The perturbation factors are $c_\mu = c_\sigma = 2$.



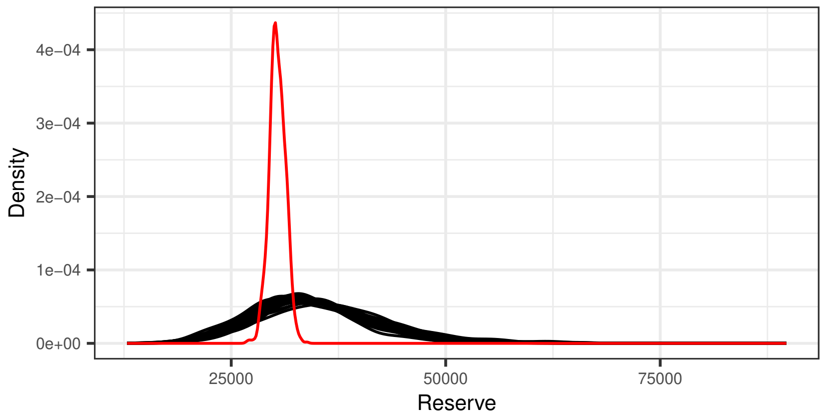
(a) Conditional, normal distribution, $c_\mu = 2$



(b) Unconditional, normal distribution, $c_\mu = 0.5$



(c) Conditional, gamma distribution, $c_\mu = 0.5$



(d) Unconditional, gamma distribution, $c_\mu = 2$

Figure 4.5: Density plots of predictive distributions obtained using the conditional parametric bootstrap of Mack's model for different deviating points. The curves in each facet correspond to densities obtained while excluding a different point in the resampling procedure, with the result from the cleaned dataset highlighted in red. We set $c_\sigma = 1$ and $i^* = 4$, $j^* = 2$ for all subfigures.

Outlier	Excluded																	
	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(1, 5)	(2, 5)	(3, 5)
(1, 2)	2e+03	219.29	1e+03	275.55	258.79	125.66	150.80	280.92	243.98	299.74	312.52	719.08	175.85	162.29	212.13	219.21	428.30	193.71
(2, 2)	543.74	34.96	514.50	437.14	2e+03	230.12	396.41	764.36	209.35	282.48	320.86	804.80	131.92	584.82	286.52	6e+03	297.47	2e+03
(3, 2)	182.99	3e+03	37.31	483.43	467.83	241.62	545.59	726.44	272.80	177.99	279.90	359.43	153.36	278.10	141.93	428.18	284.43	165.91
(4, 2)	366.53	243.56	234.10	38.32	146.77	311.62	399.59	339.10	283.71	136.05	330.86	665.72	498.99	175.34	148.15	999.43	384.48	349.31
(5, 2)	402.16	147.21	145.60	341.70	37.76	597.28	141.24	227.13	186.35	476.86	724.56	776.12	1e+03	190.78	176.31	360.09	315.54	378.15
(6, 2)	384.64	498.07	177.62	377.37	132.25	68.30	182.33	376.80	335.74	215.74	497.38	103.79	335.67	243.03	192.55	277.35	260.58	229.31
(1, 3)	209.29	255.28	127.41	422.40	365.59	251.60	38.68	504.08	266.62	328.41	575.94	2e+03	238.50	331.84	168.43	140.68	235.28	288.63
(2, 3)	263.77	166.83	416.34	283.49	134.10	170.86	333.84	44.77	297.97	183.68	231.22	230.65	145.56	321.50	329.59	94.05	183.79	466.67
(3, 3)	171.25	288.31	335.53	292.72	238.50	326.28	282.13	287.98	63.27	313.81	538.01	345.52	1e+03	563.04	284.47	1e+03	2e+03	214.76
(4, 3)	303.37	138.11	170.32	208.45	234.14	424.47	593.07	434.98	540.24	41.19	870.30	1e+03	255.15	806.90	196.63	263.68	280.08	425.28
(5, 3)	223.84	549.37	484.85	140.58	159.31	124.67	326.74	170.80	310.05	385.67	34.99	811.91	122.26	247.51	197.32	172.24	127.31	214.01
(1, 4)	326.15	513.85	895.42	256.74	8e+03	278.81	231.64	3e+03	199.07	228.24	579.38	56.55	394.18	367.45	1e+03	403.11	520.22	229.81
(2, 4)	850.73	756.43	848.70	894.14	2e+03	384.94	584.89	500.55	280.97	286.67	400.13	223.68	67.76	772.88	1e+03	316.27	468.90	2e+03
(3, 4)	304.79	213.68	419.24	376.67	864.85	2e+03	180.49	764.56	2e+03	428.15	364.83	537.76	2e+04	56.89	912.01	389.27	228.16	266.18
(4, 4)	139.09	649.76	208.80	244.51	140.37	893.08	1e+03	204.89	385.52	200.13	427.89	355.60	194.15	362.30	28.66	348.71	317.82	217.54
(1, 5)	1e+03	3e+03	4e+04	2e+03	4e+03	830.36	761.26	1e+04	2e+04	4e+04	2e+03	8e+03	4e+03	1e+03	2e+04	71.46	424.43	6e+04
(2, 5)	2e+03	2e+03	2e+03	4e+03	1e+03	1e+03	4e+03	1e+03	1e+04	1e+05	824.15	1e+03	2e+05	6e+04	2e+03	2e+03	109.64	925.88
(3, 5)	2e+03	4e+03	6e+06	2e+04	7e+03	3e+05	650.20	5e+04	2e+03	425.55	2e+05	1e+03	3e+03	3e+03	1e+05	1e+03	3e+03	35.83

Table 4.4: Predictive reserve mean for different contaminated and excluded points (thousands)

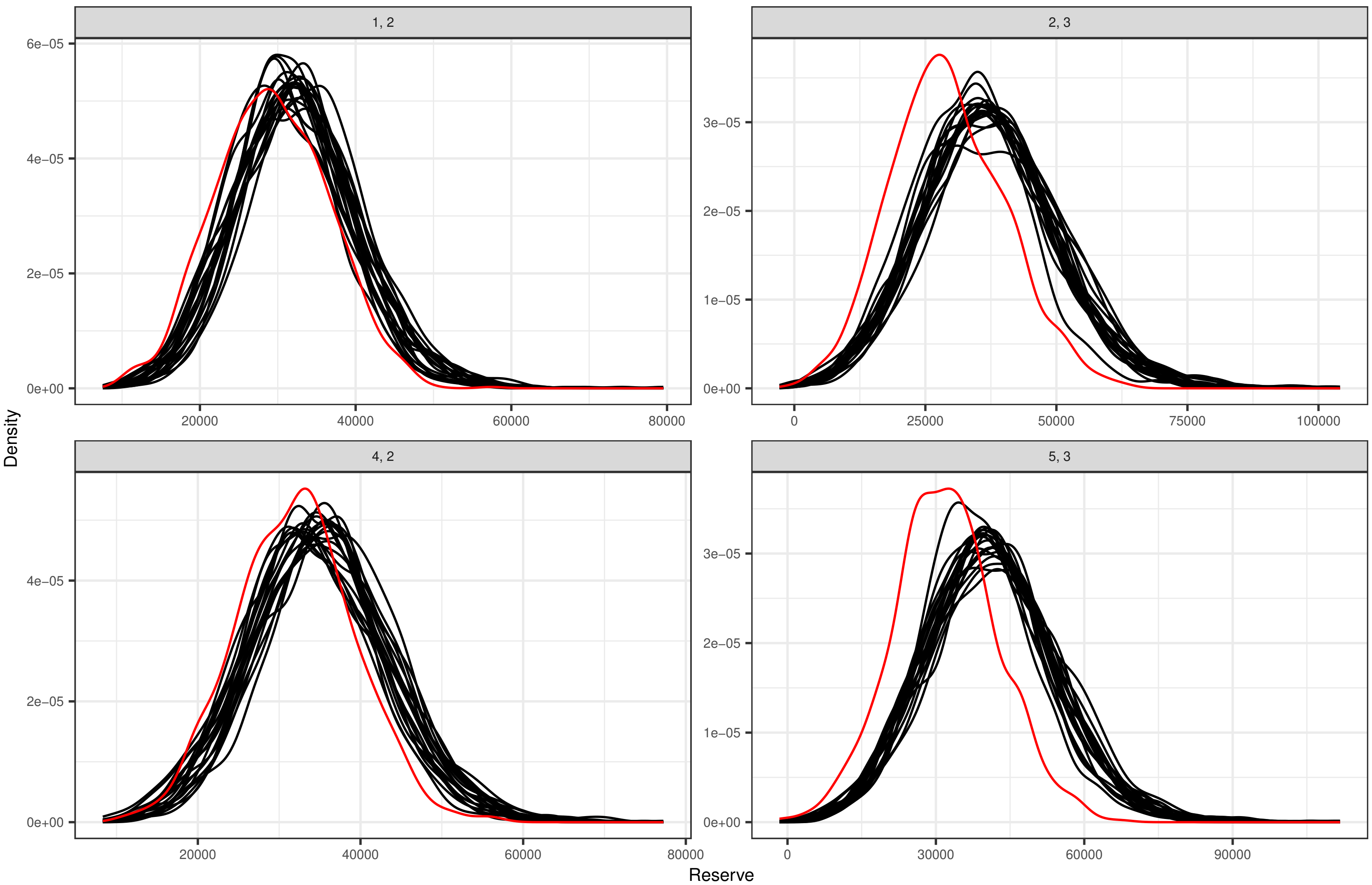
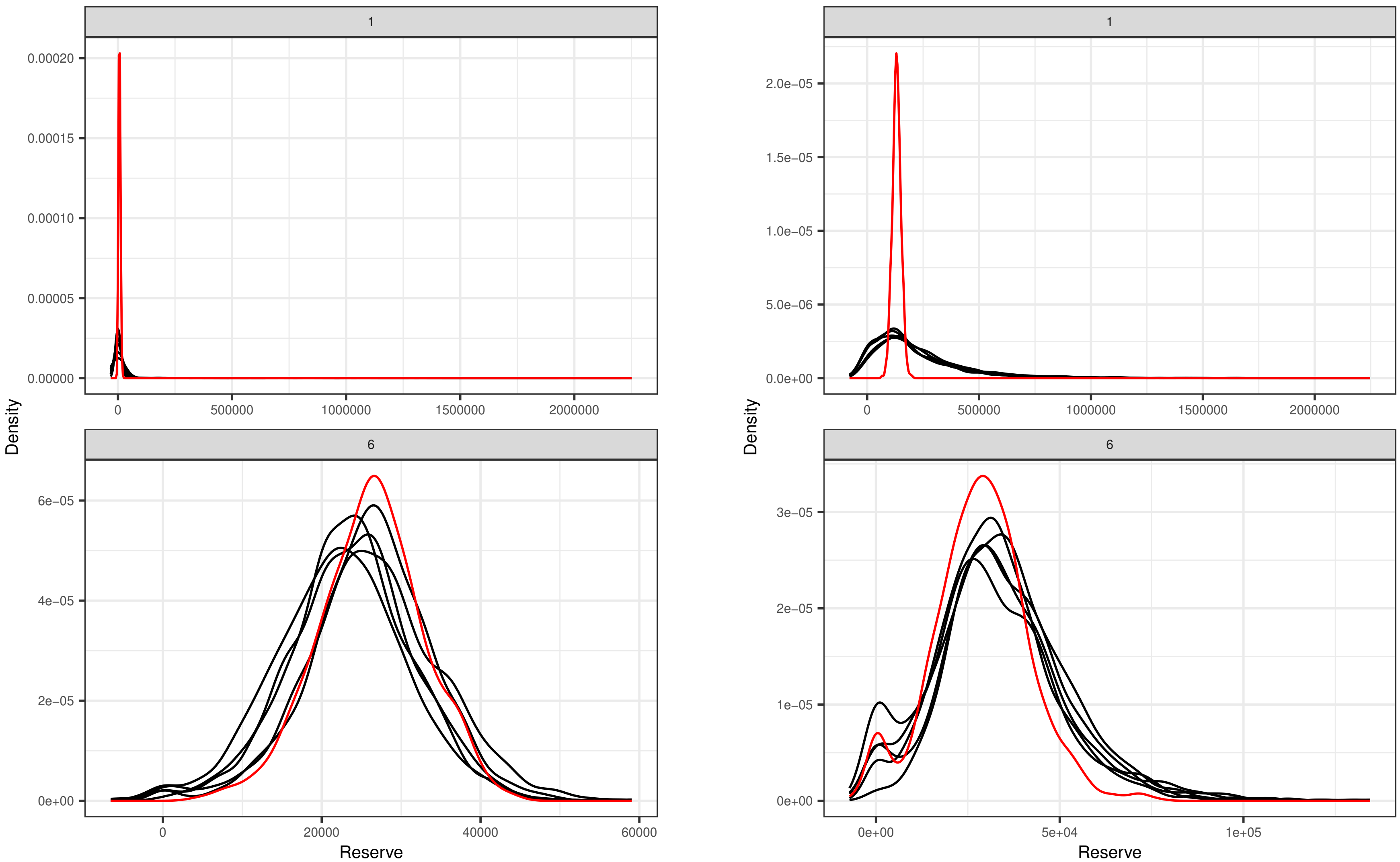


Figure 4.6: Density plots of predictive distributions obtained using the pairs bootstrap of Mack's model for different deviating points.

Outlier	Excluded																			
	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(1, 5)	(2, 5)	(3, 5)	(1, 6)	(2, 6)
(1, 2)	29.28	32.66	32.79	33.59	33.30	33.18	32.03	33.14	30.58	32.51	32.30	31.27	31.04	34.36	30.58	31.94	32.24	30.30	30.96	31.48
(2, 2)	35.23	27.12	33.94	37.45	33.72	32.12	31.98	30.82	32.54	31.63	33.53	33.17	33.15	32.61	32.40	32.87	33.08	31.65	31.05	32.62
(3, 2)	32.68	32.99	29.76	34.34	33.59	34.02	34.37	32.70	31.03	33.84	33.51	30.92	32.38	34.67	33.33	33.76	33.38	33.72	33.72	31.74
(4, 2)	35.33	33.88	36.23	31.99	34.38	36.40	34.37	36.97	33.61	33.35	33.71	34.47	36.60	35.57	36.33	34.39	34.16	32.83	35.98	35.46
(5, 2)	35.80	36.12	35.39	36.90	32.70	36.71	34.34	36.35	35.51	33.12	34.61	37.88	36.75	38.60	33.11	37.66	36.08	35.10	38.09	35.41
(6, 2)	40.77	39.97	39.23	42.90	39.25	34.86	42.56	39.52	39.68	42.03	39.83	37.78	39.00	40.28	40.80	41.01	42.64	41.42	41.82	40.66
(1, 3)	34.92	35.58	35.47	34.82	35.50	35.37	27.92	36.36	35.76	35.64	37.11	37.68	34.69	34.16	36.29	33.68	37.10	36.01	35.02	35.36
(2, 3)	36.07	37.02	37.13	36.40	35.78	36.51	37.19	29.27	39.36	38.44	38.94	36.38	38.54	35.73	36.99	36.49	37.78	33.34	36.64	37.29
(3, 3)	36.37	40.24	39.15	37.17	36.94	37.02	38.12	39.90	30.94	40.36	39.07	38.88	37.28	37.38	39.56	37.82	39.70	38.15	37.69	38.84
(4, 3)	38.30	40.83	39.54	38.59	39.68	38.91	41.41	41.44	43.08	31.65	43.06	39.29	37.98	42.38	38.53	39.73	38.57	38.48	39.68	39.38
(5, 3)	41.20	40.82	38.08	40.54	41.33	38.84	40.10	42.86	41.16	43.69	31.98	41.23	41.37	41.57	41.88	41.37	41.23	40.59	38.80	39.09
(1, 4)	41.86	42.98	40.50	42.82	39.97	38.79	40.46	40.26	42.17	41.35	39.50	30.94	45.39	45.33	44.26	40.19	41.18	41.71	39.56	41.27
(2, 4)	42.13	40.03	44.93	40.57	44.02	42.89	43.81	43.58	43.52	44.77	41.24	45.93	30.50	46.56	47.74	43.53	43.43	43.13	45.52	42.03
(3, 4)	45.10	43.38	45.58	43.95	44.12	43.94	41.02	42.93	44.15	43.20	46.13	49.22	48.02	31.21	49.61	43.46	44.91	44.93	44.20	44.30
(4, 4)	45.99	46.90	44.42	45.02	44.67	43.31	44.96	47.11	43.62	44.69	45.81	50.26	50.84	47.89	32.98	47.17	47.52	44.22	44.94	45.56
(1, 5)	51.29	51.65	52.82	51.13	50.95	50.36	50.92	50.48	55.08	50.75	52.39	51.14	52.07	50.89	53.53	30.85	58.80	58.99	53.92	51.67
(2, 5)	54.64	54.70	52.29	54.70	50.49	56.70	55.62	51.87	56.59	53.49	50.90	55.32	52.85	55.61	54.29	65.36	34.06	66.17	53.52	53.91
(3, 5)	57.07	56.63	53.98	58.70	55.85	54.30	54.03	60.96	61.07	55.24	55.24	55.43	58.33	57.23	55.95	62.04	66.74	36.22	58.83	57.54
(1, 6)	80.39	74.36	76.74	80.77	75.70	69.91	74.78	78.02	79.15	76.06	78.59	73.50	76.88	78.62	74.89	74.78	73.72	72.30	44.02	109.13
(2, 6)	80.68	78.98	82.33	77.46	78.31	83.51	80.84	83.42	81.07	80.07	79.70	78.50	75.68	80.95	84.15	78.92	79.79	80.07	108.42	45.57

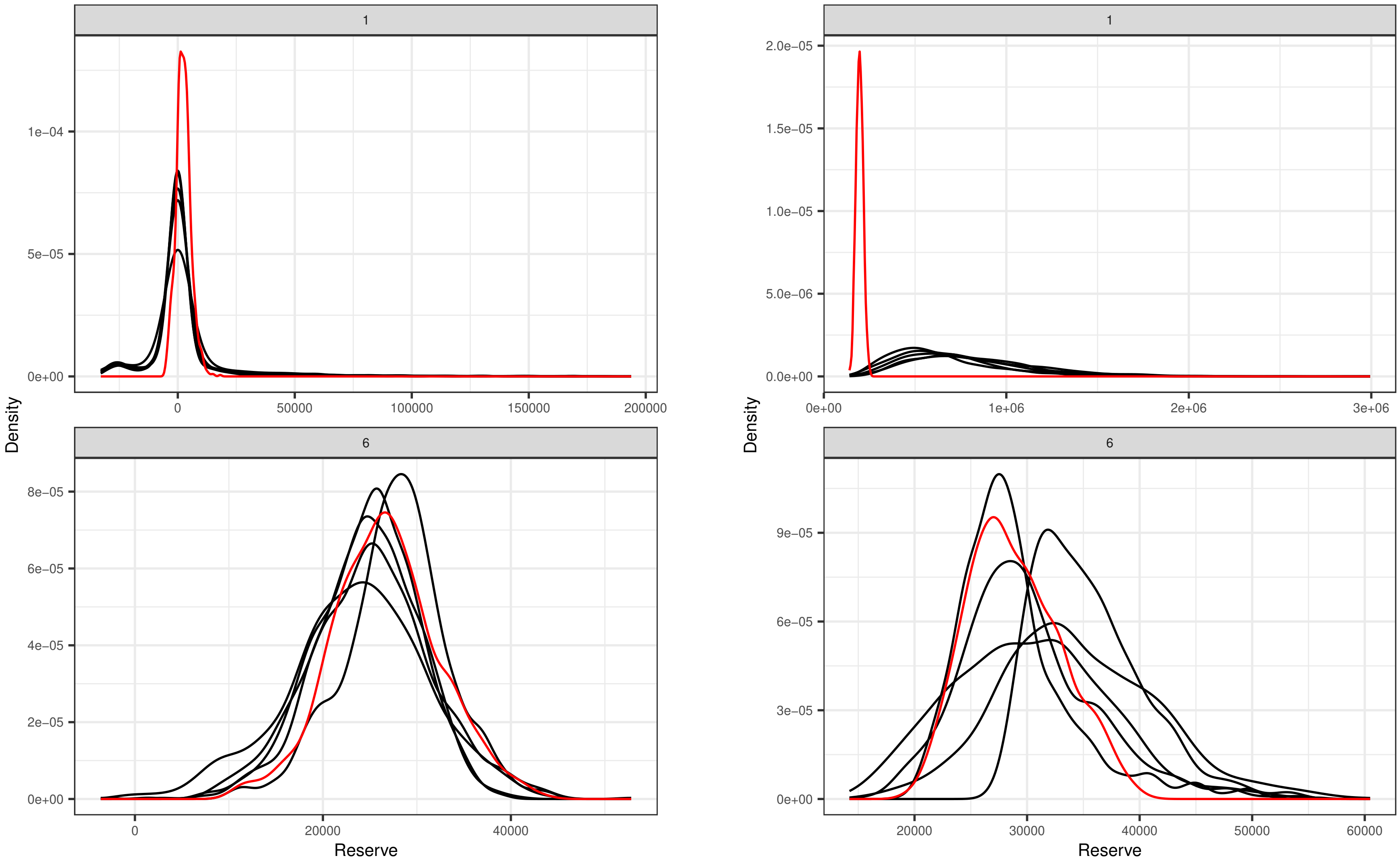
Table 4.5: Predictive reserve mean for different contaminated and excluded points (thousands)



(a) Conditional, $c_\mu = c_\sigma = 0.5$

(b) Unconditional, $c_\mu = 2$ and $c_\sigma = 0.5$

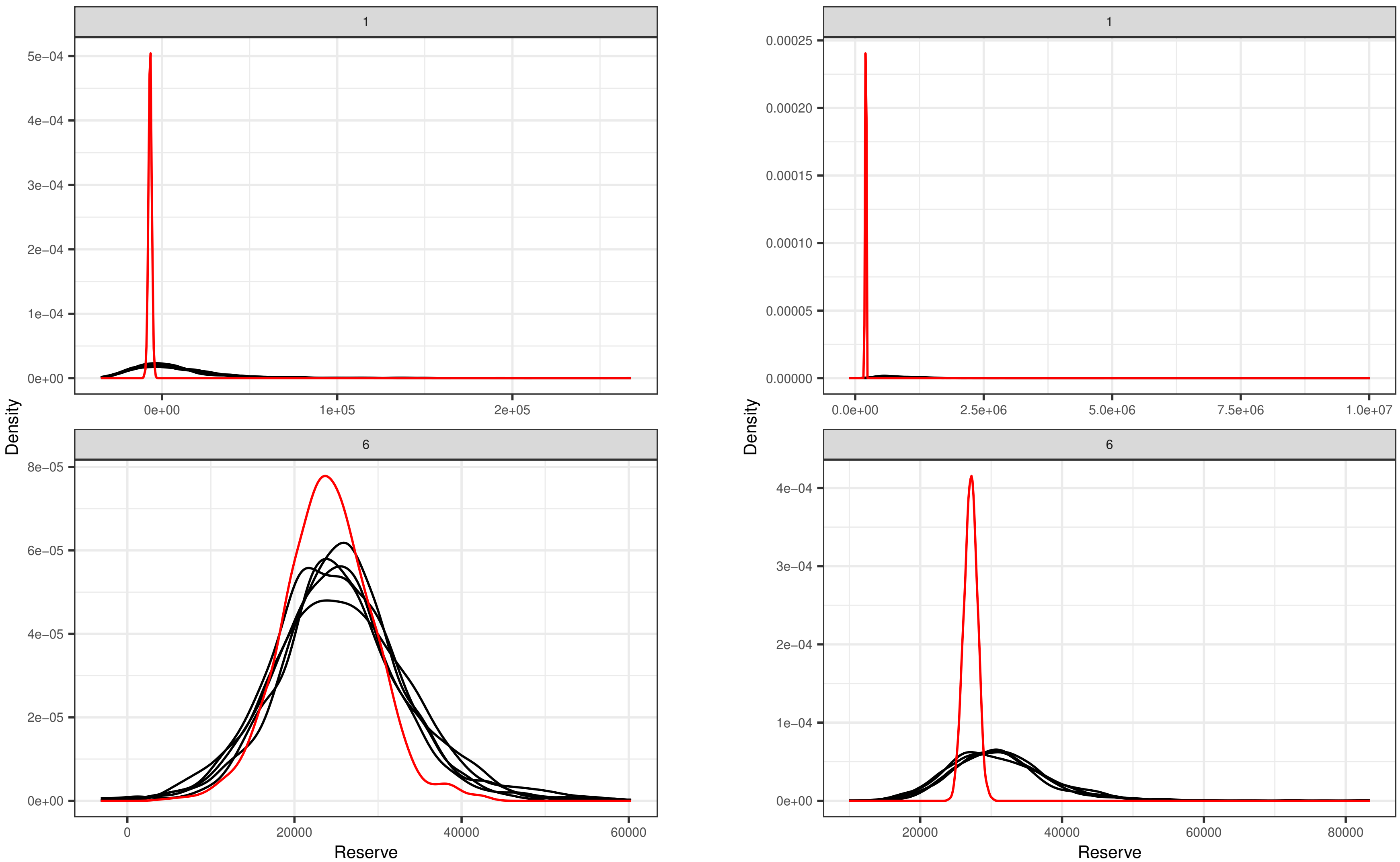
Figure 4.7: Density plots of predictive distributions obtained using the semiparametric bootstrap of Mack's model with standardised residuals for different deviating calendar periods



(a) Conditional, $c_\mu = 0.5$ and $c_\sigma = 2$

(b) Unconditional, $c_\mu = c_\sigma = 2$

Figure 4.8: Density plots of predictive distributions obtained using the semiparametric bootstrap of Mack's model with log-normal residuals for different deviating calendar periods



(a) Conditional, normal distribution, $c_\mu = 0.5$ and $c_\sigma = 1$

(b) Unconditional, gamma distribution, $c_\mu = 2$ and $c_\sigma = 1$

Figure 4.9: Density plots of predictive distributions obtained using of Mack's model the parametric bootstrap for different deviating calendar periods

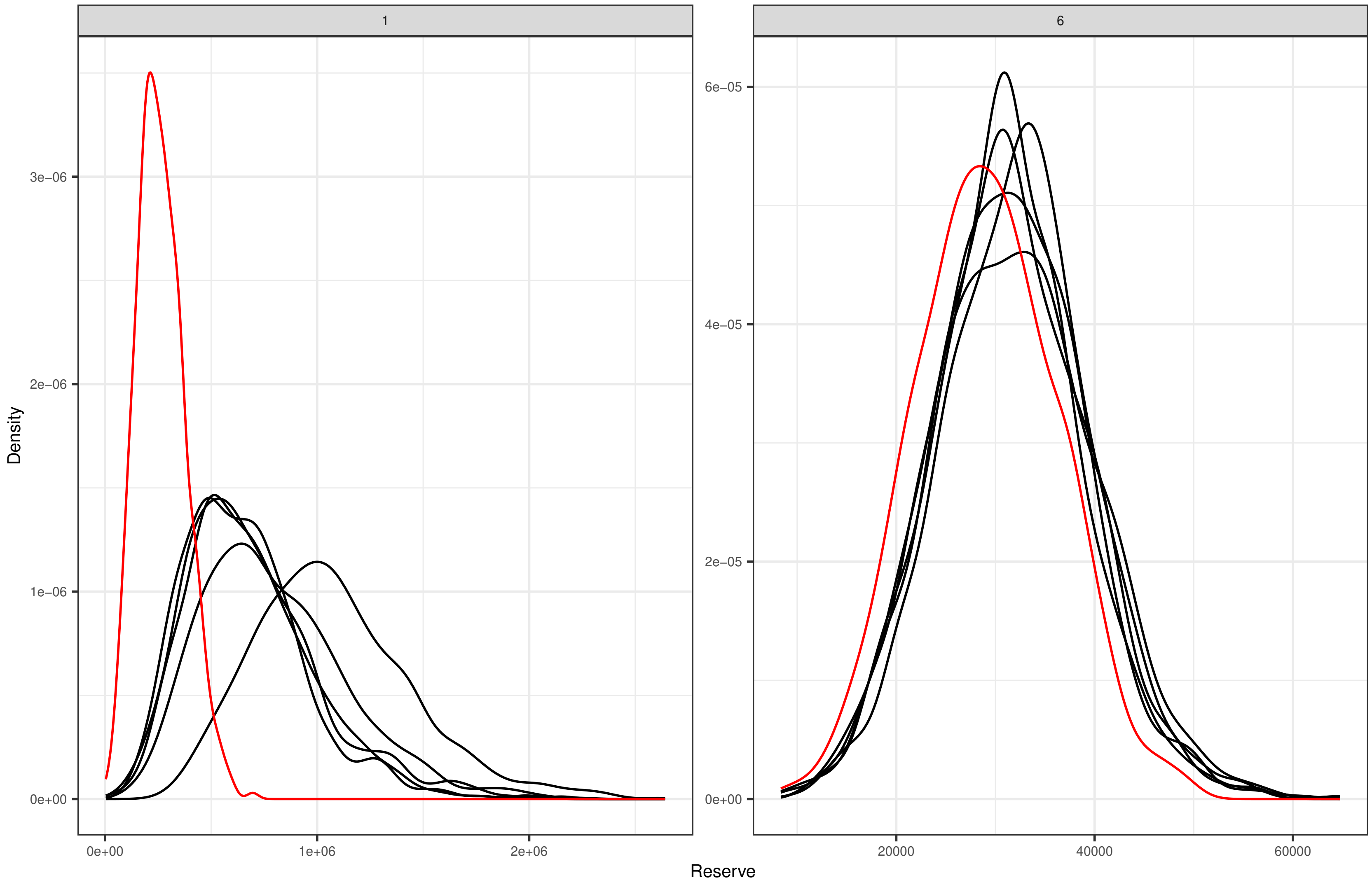


Figure 4.10: Density plots of predictive distributions obtained using of Mack's model the pairs bootstrap for different deviating calendar periods

4.4.2 Overdispersed Poisson GLM

In the single point simulation, the semiparametric bootstrap performs rather poorly compared to the parametric one, regardless of which distribution is used in the latter. As in Section 4.4.1, we can trace the cause of this by analysing how the residuals are affected by perturbations. Table 4.6 shows a simulated triangle containing a perturbation at the point $(1, 1)$. The Pearson residuals for the original and perturbed triangles are shown in Table 4.7. As we can see, the contamination does not remain localised, but spreads across the whole triangle of residuals, leading to a situation in which the affected point no longer stands out.

Origin	**Dev**						
	1	2	3	4	5	6	7
1	376.31	6.73	8.99	10.70	11.76	12.35	12.69
2	4.00	7.70	9.98	11.16	12.12	12.75	
3	4.36	8.29	10.23	11.76	12.99		
4	4.30	7.75	9.77	11.09			
5	4.15	7.90	10.22				
6	5.10	9.65					
7	6.28						

Table 4.6: Simulated triangle where observation X_{11} has been perturbed, with $c_\lambda = 100$ (thousands)

Origin	**Dev**						
	1	2	3	4	5	6	7
1	−4.06	−4.47	−1.21	2.68	2.38	1.94	0
2	−0.21	1.36	2.59	0.22	−1.54	−1.87	
3	1.27	2.40	−1.15	−0.90	−0.73		
4	3.53	0.69	−0.87	−1.87			
5	−0.64	−0.24	0.63				
6	−0.05	0.04					
7	0.00						

(a) Original

Origin	**Dev**						
	1	2	3	4	5	6	7
1	195.62	−144.73	−139.01	−124.02	−102.99	−65.62	0
2	−171.65	48.45	76.31	99.05	129.01	178.47	
3	−155.27	70.60	95.29	124.99	161.95		
4	−127.27	90.19	121.01	151.03			
5	−102.47	123.92	166.30				
6	−70.27	202.37					
7	0.00						

(b) Perturbed

Table 4.7: Example of Pearson residuals obtained in the simulation

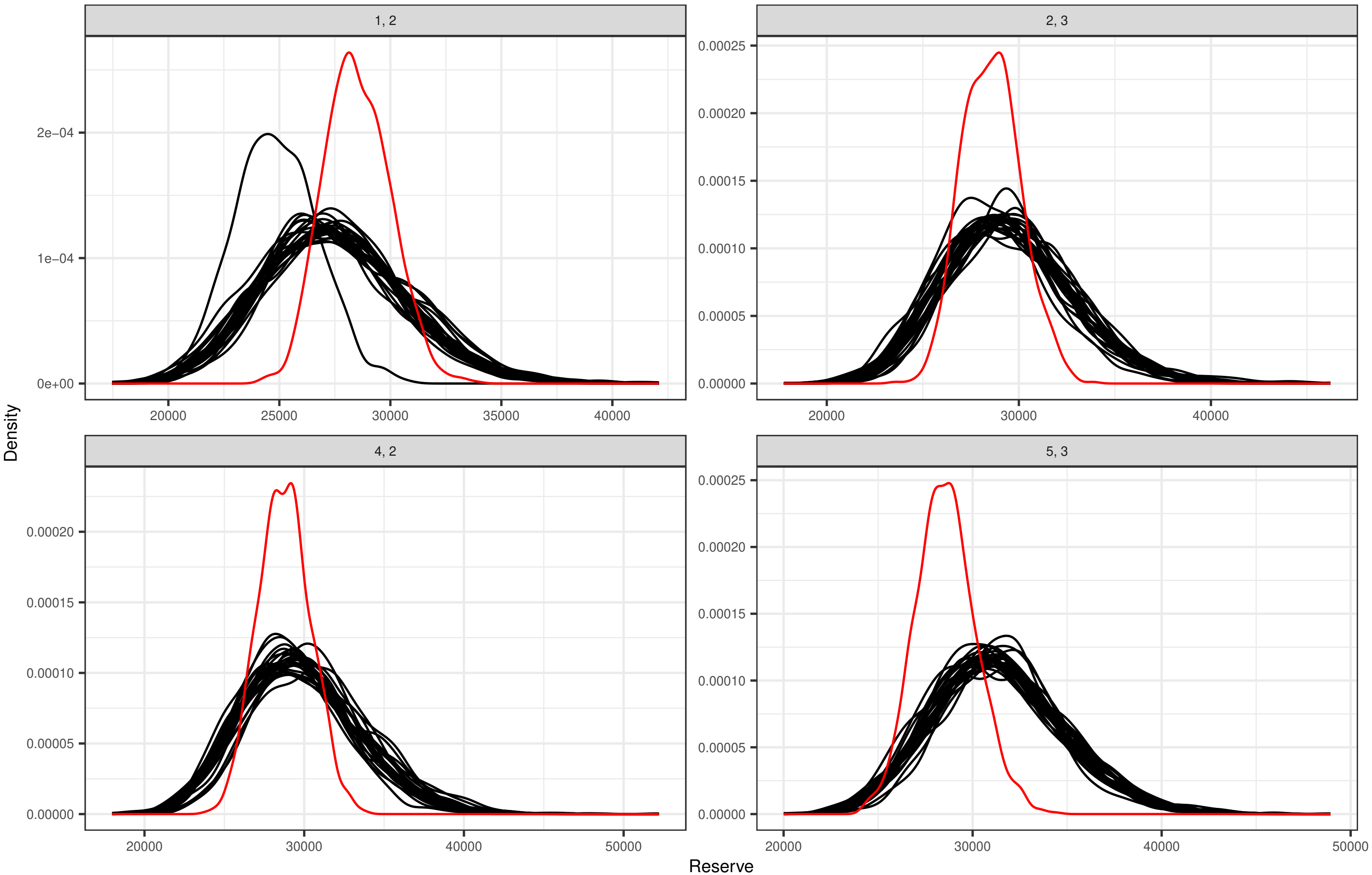


Figure 4.11: Single point contamination ODP model simulation results for the parametric bootstrap with gamma distribution, $c_\lambda = 2$

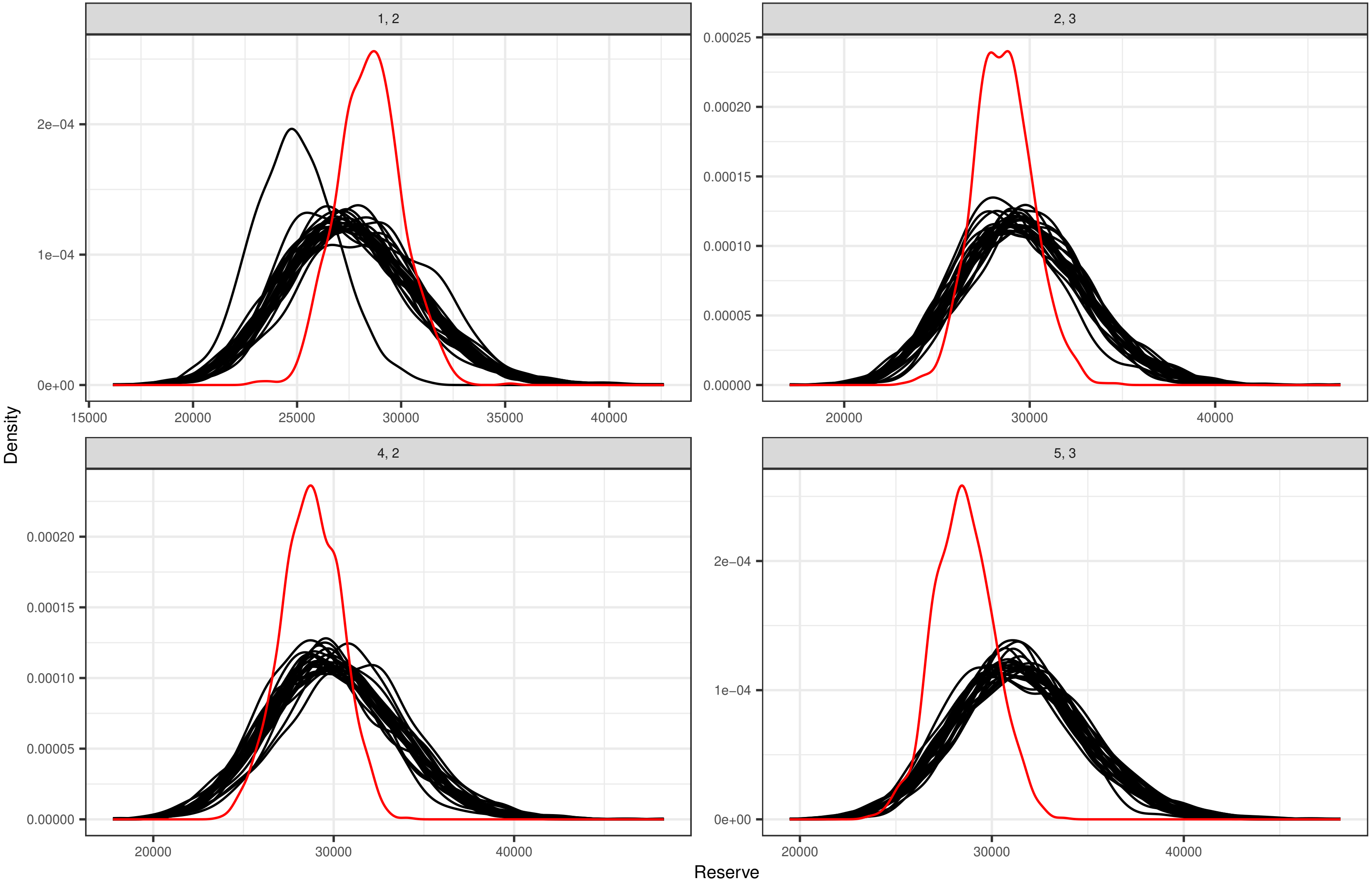


Figure 4.12: Single point contamination ODP model simulation results for the parametric bootstrap with normal distribution, $c_\lambda = 2$

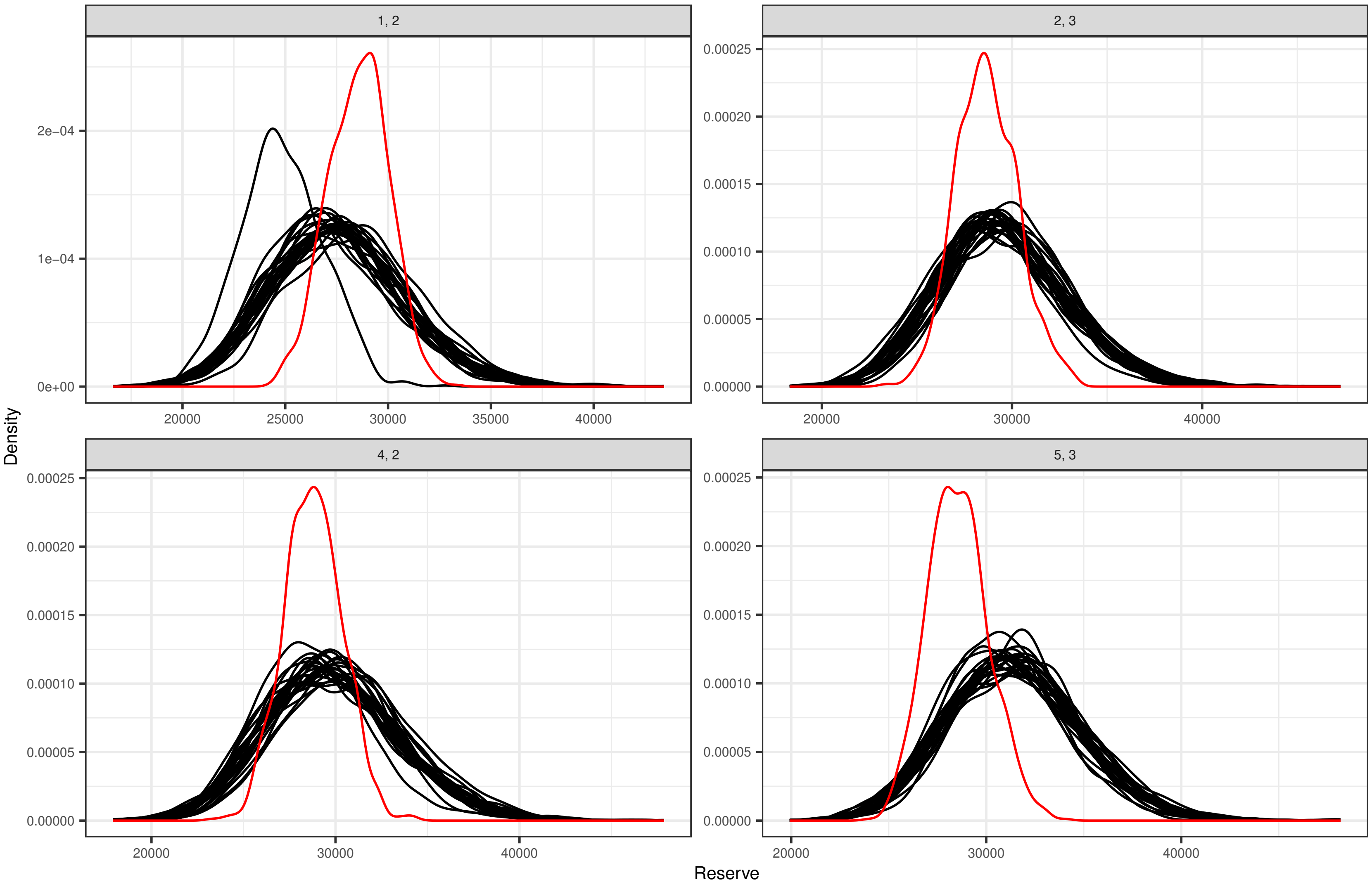


Figure 4.13: Single point contamination ODP model simulation results for the parametric bootstrap with Poisson distribution, $c_\lambda = 2$

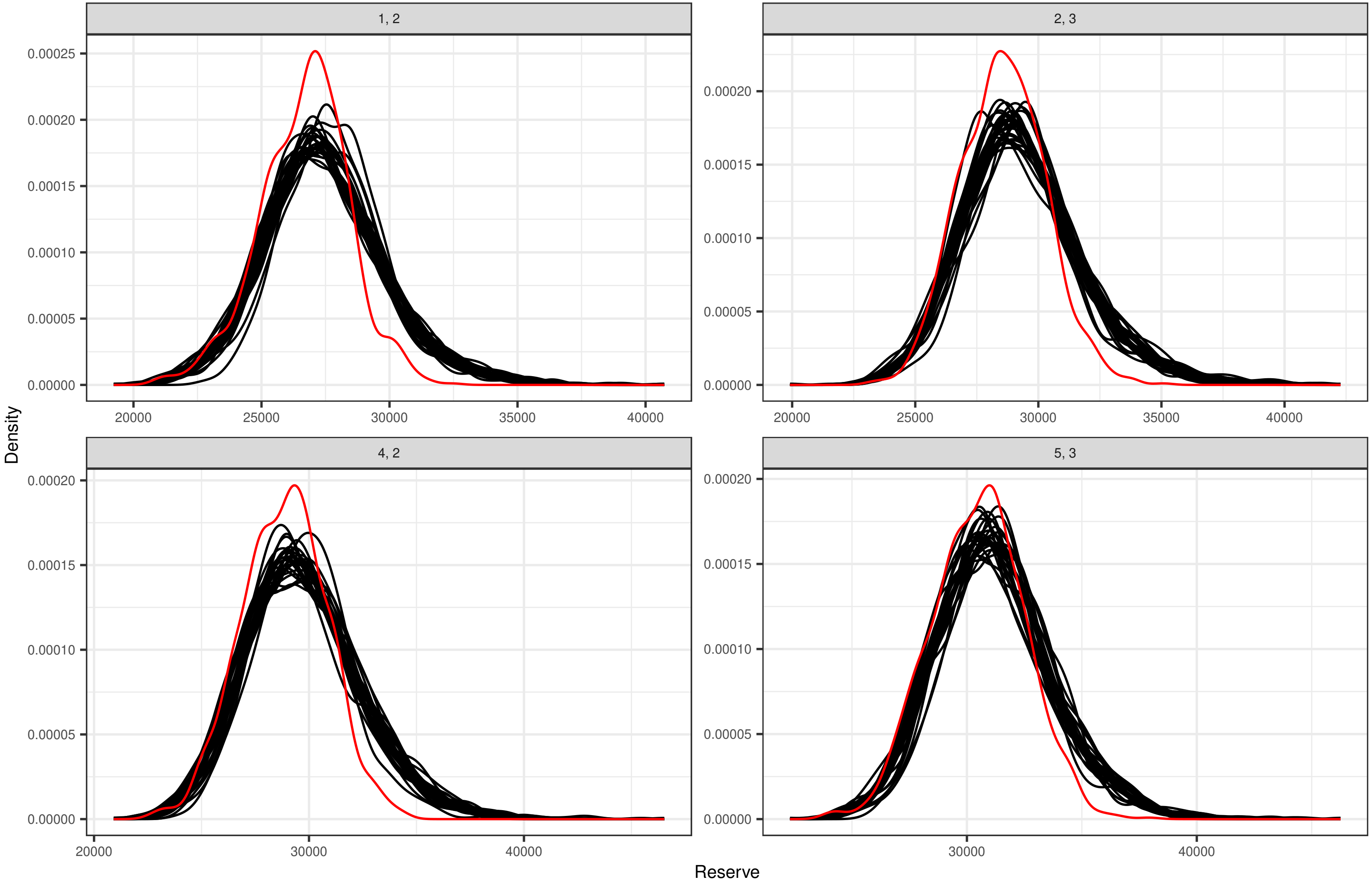
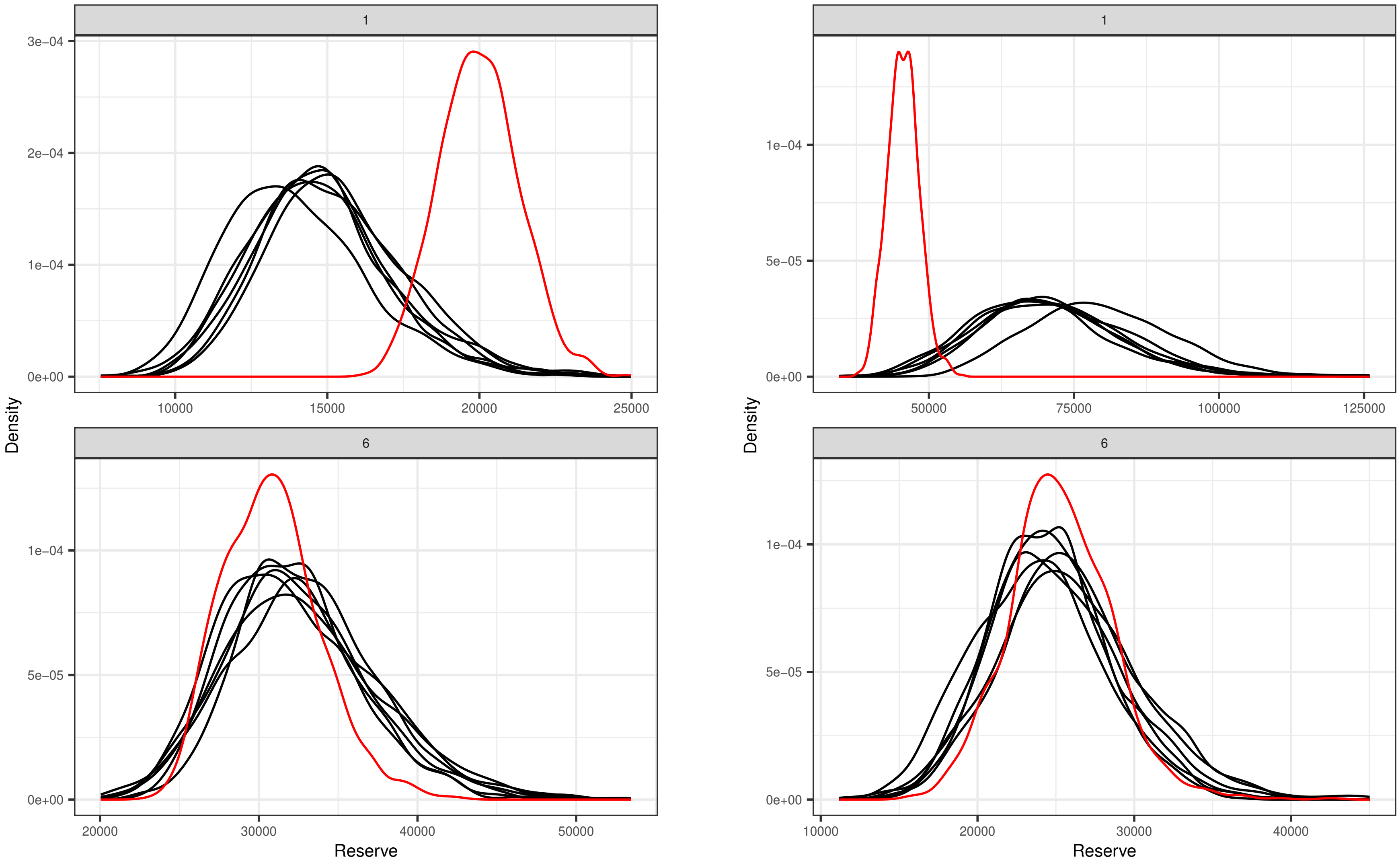


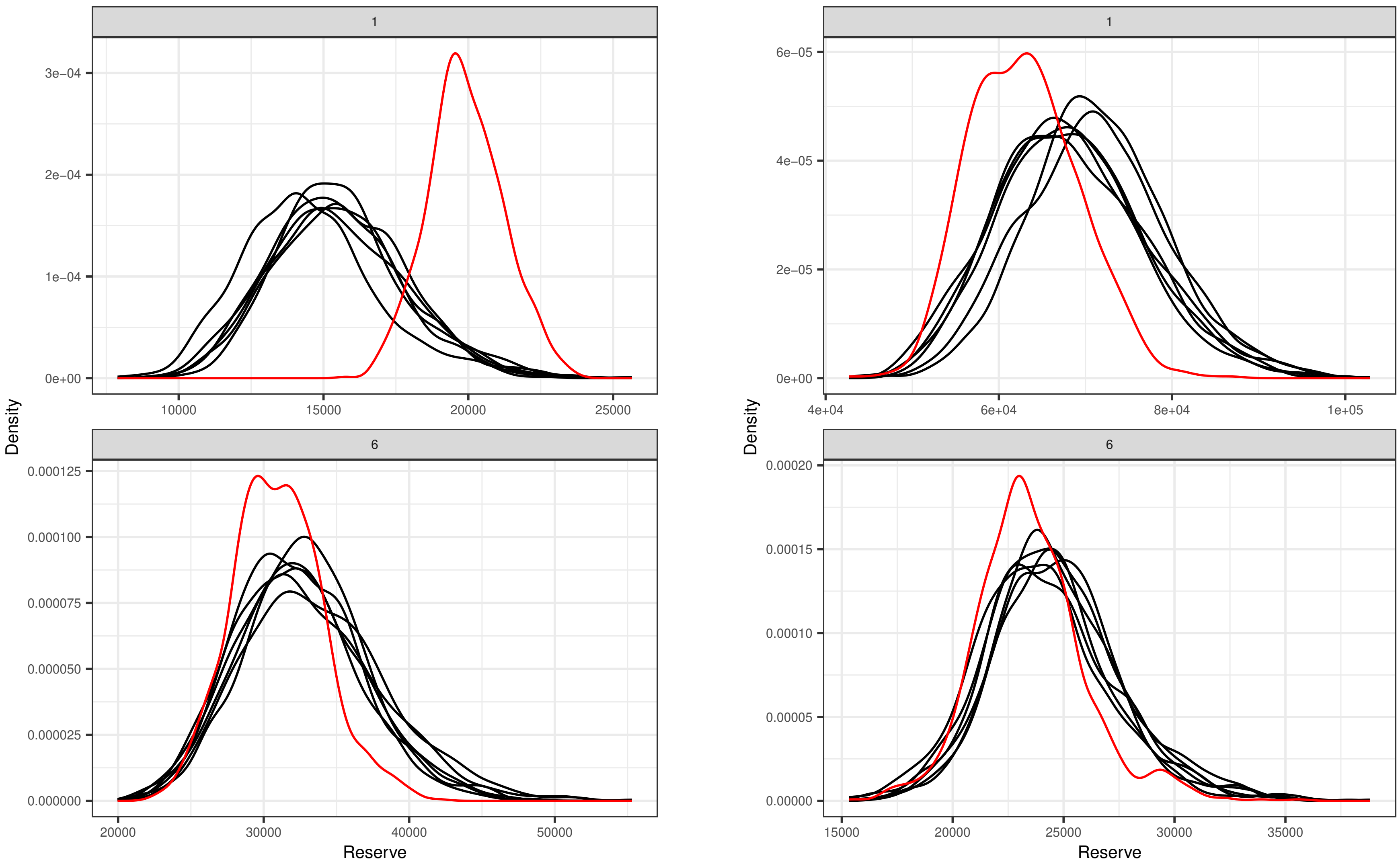
Figure 4.14: Single point contamination ODP model simulation results for the semiparametric bootstrap with Pearson residuals, $c_\lambda = 2$



(a) Gamma distribution, $c_\lambda = 0.5$

(b) Normal distribution, $c_\lambda = 2$

Figure 4.15: Density plots of predictive distributions obtained using the parametric bootstrap of the ODP model for different deviating calendar periods



(a) Parametric with scaled Poisson distribution, $c_\lambda = 0.5$

(b) Semiparametric, $c_\lambda = 2$

Figure 4.16: Density plots of predictive distributions obtained using the bootstrap of the ODP model for different deviating calendar periods

Conclusion

We have investigated the application of bootstrap methods to the problem of detecting deviations from the model assumptions for two major actuarial reserving models, the Mack chain ladder and the Overdispersed Poisson GLM. By simulating datasets which exhibit a myriad of different irregularities and studying their effect on the predictive distribution, we managed to identify a number of bootstrap configurations which appear promising for this purpose. The strength of the observed effect is tied to the sensitivity of a bootstrap procedure to the presence of contamination in the data. Parametric bootstraps were found to perform very well: even deviations in a single point caused a significant change in the predictive distribution. For semiparametric bootstraps, the results strongly depend on the type of residuals employed. Finally, the fully nonparametric pairs bootstrap, which can only be used with Mack's model, showed a reasonable performance, though not as good as the parametric variant. These results suggest a way of flagging suspicious datapoints by reverse-engineering the simulation process: instead of starting from synthetic triangles, we use real data and generate a predictive distribution while excluding all observations by turn; the ones for which effect is observed can then be identified as requiring special attention. Although this falls outside the purview of our investigation, it suggests an interesting avenue for future research.

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