Estimation of the linear fractional stable motion. Numerical results.

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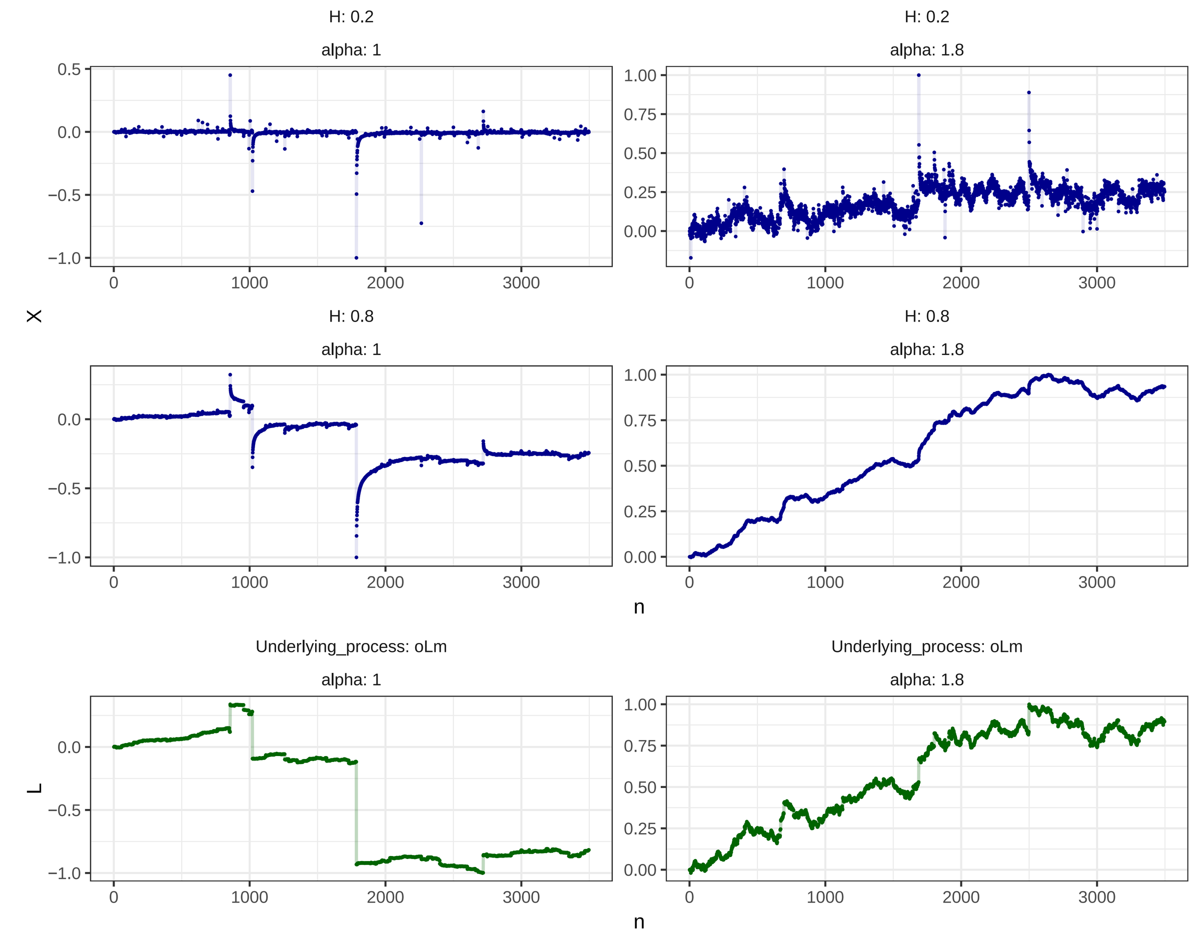
The model

$$X_t = \int_R \left\{ (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right\} dL_s, \qquad (1)$$

where L is a symmetric α-stable Lévy motion, $\alpha \in (0,2)$, with scale parameter $\sigma > 0$ and $H \in (0,1)$.



Continuous case

Low frequency kth order increments of X:

$$\Delta_{i,k}^r X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-rj)}, \qquad i \geq rk \qquad (2)$$

$$\varphi_{\text{low}}(t,k)_n := \frac{1}{n} \sum_{i=k}^{n} \cos\left(t \Delta_{i,k}^{r=1} X\right) \xrightarrow{\mathbb{P}} \qquad (3)$$

$$\xrightarrow{\mathbb{P}} \varphi(t,k) := \exp\left(-|\sigma\|h_k\|_\alpha t|^\alpha\right) \qquad (4)$$

$$R_{\text{low}}(p,k)_n := \frac{\sum_{i=2k}^{n} \left|\Delta_{i,k}^2 X\right|^p}{\sum_{i=k}^{n} \left|\Delta_{i,k}^1 X\right|^p} \xrightarrow{\mathbb{P}} 2^{pH} \qquad (5)$$

When $H - 1/\alpha > 0$, X_t is continuous and we can take $k = 2, p \in (0, 1/2)$ and use the estimators

$$\widehat{H}_{\text{low}}(p,k)_n := \frac{1}{p} \log_2 \left(R_{\text{low}}(p,k)_n\right)$$

$$\widehat{\alpha}_{\text{low}} := \frac{\log|\log \varphi_{\text{low}}(t_2;k)_n| - \log|\log \varphi_{\text{low}}(t_1;k)_n|}{\log t_2 - \log t_1}$$

$$\widehat{\sigma}_{\text{low}} := \left(-\log \varphi_{\text{low}}(t_1;k)_n\right)^{1/\widehat{\alpha}_{\text{low}}} / \left(t_1 \|h_{k,1}\|_{\widehat{\alpha}_{\text{low}}}\right)$$

General case

$\varphi_{\text{low}}(t,k)_n$, $R_{\text{low}}(p,k)_n$ have normal asymptotic distributions and convergence rates $\sqrt{n}$ if $k > H + 1/\alpha$ (=> the same holds for hats). Therefore, in the general case we consider the preliminary estimator of α with $k = 1$, that is consistent, given by

$$\widehat{\alpha}^0_{\text{low}}(t_1,t_2)_n = \widehat{\alpha}_{\text{low}}(t_1,t_2,k_0 = 1)_n.$$

then, estimate k_{low}

$$\hat{k}_{\text{low}}(t_1,t_2)_n := 2 + \left\lfloor \widehat{\alpha}^0_{\text{low}}(t_1,t_2)_n^{-1} \right\rfloor.$$

and use it to obtain the parameters

$$\widetilde{H}_{\text{low}} = \widehat{H}_{\text{low}}(-p, \hat{k}_{\text{low}})_n, \qquad (6)$$

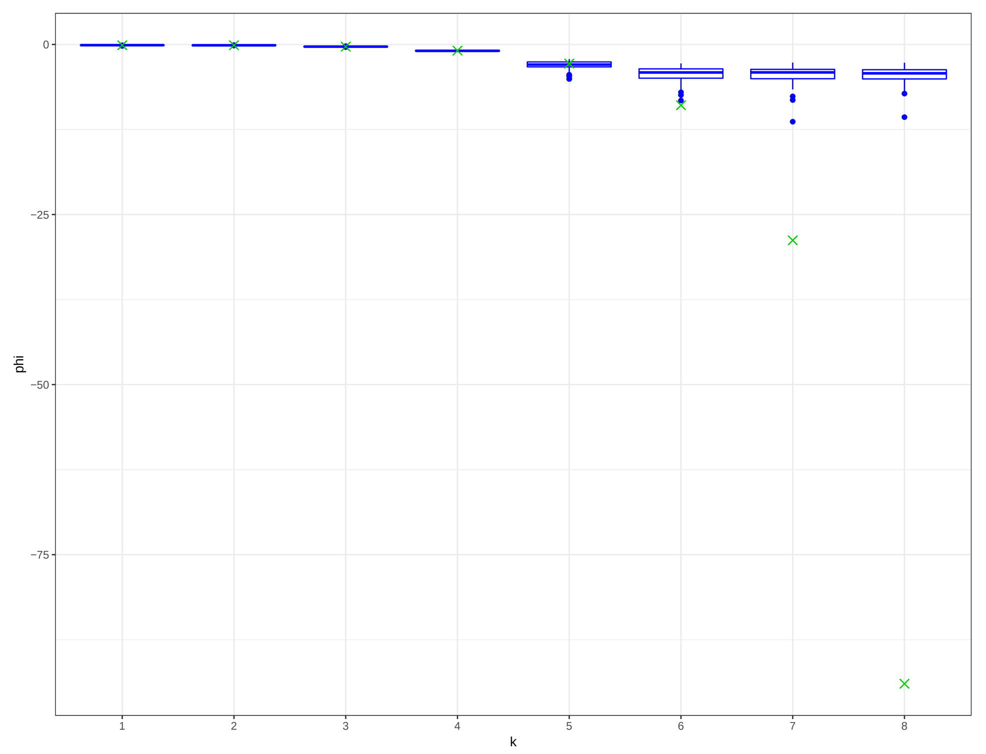
$$\widetilde{\alpha}_{\text{low}} = \widehat{\alpha}_{\text{low}}(\hat{k}_{\text{low}}; t_1, t_2)_n, \qquad (7)$$

$$\widetilde{\sigma}_{\text{low}} = \widehat{\sigma}_{\text{low}}(\hat{k}_{\text{low}}; t_1, t_2)_n \qquad (8)$$

When $\alpha^{-1} \notin \mathbb{N}$ tilde estimators have the same convergence rate $(\sqrt{n}, \sqrt{n}, \sqrt{n})$ and normal asymptotic distribution.

Convergence of $\varphi_{\text{Low}}(t,k)_n$

Function $\varphi(t,k)$ used to extract α and σ from data is very sensitive to k. As k increases $|\varphi_{\text{low}}(t,k)_n - \varphi(t,k)|$ grows, which leads to generally worse performance of the tilde estimators compared to that of the hat estimators.

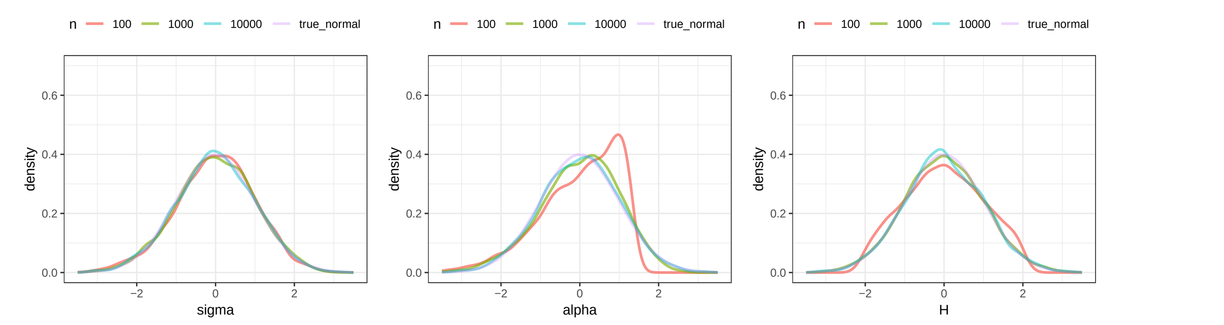


Logarithmic scale. $n = 1000, (\sigma, \alpha, H) = (0.3, 1.8, 0.8)$. Dependences of $\varphi_{\text{low}}(t;k)_n$ (blue box plot) and $\varphi(t;k)$ (green crosses) on k. For the former, a Monte Carlo experiment with $2 * 10^2$ trials was setup.

Performance of $(\widehat{\sigma}_{\text{low}}, \widehat{\alpha}_{\text{low}}, \widehat{H}_{\text{low}})$

n	$\widehat{\sigma}_{\text{low}}$	$\widehat{\alpha}_{\text{low}}$	$\widehat{H}_{\text{low}}$
100	-0.024/0.06	-0.038/0.18	-0.05/0.12
1000	-8e-4/0.02	0.012/0.068	-0.012/0.05
10000	1.4e-4/6e-3	5e-4/0.022	-5e-3/0.016

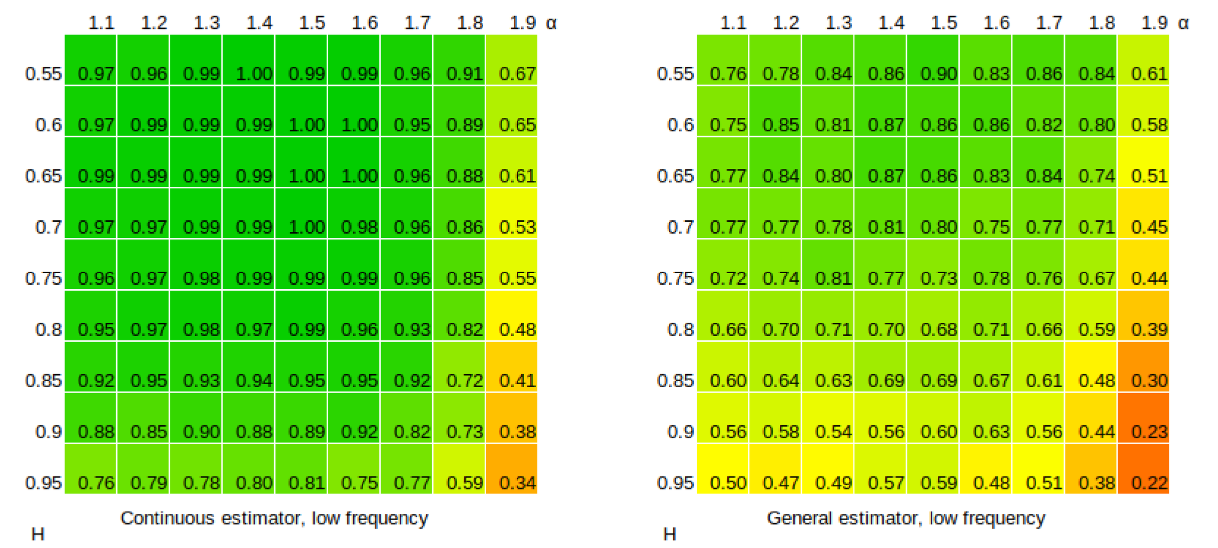
Bias/standard deviation of the estimators $(\widehat{\sigma}_{\text{low}}, \widehat{\alpha}_{\text{low}}, \widehat{H}_{\text{low}})$. $p = 0.4$, $k = 2$, $(\sigma, \alpha, H) = (0.3, 1.8, 0.8)$.



Empirical distributions of estimates $(\widehat{\sigma}_{\text{low}}, \widehat{\alpha}_{\text{low}}, \widehat{H}_{\text{low}})$. $p = 0.4$, $k = 2$, $(\sigma, \alpha, H) = (0.3, 1.8, 0.8)$.

Deterioration of $(\widetilde{\sigma}_{\text{low}}, \widetilde{\alpha}_{\text{low}}, \widetilde{H}_{\text{low}})$

Due to the nature of plug-in estimators, the low-frequency continuous case estimator encounters (much) less numerical errors than the general low-frequency one. Here, a Monte-Carlo experiment was performed to compute probabilities of obtaining an estimate. The picture below shows the comparison of success rates for ContinEstim and GenLowEstim.

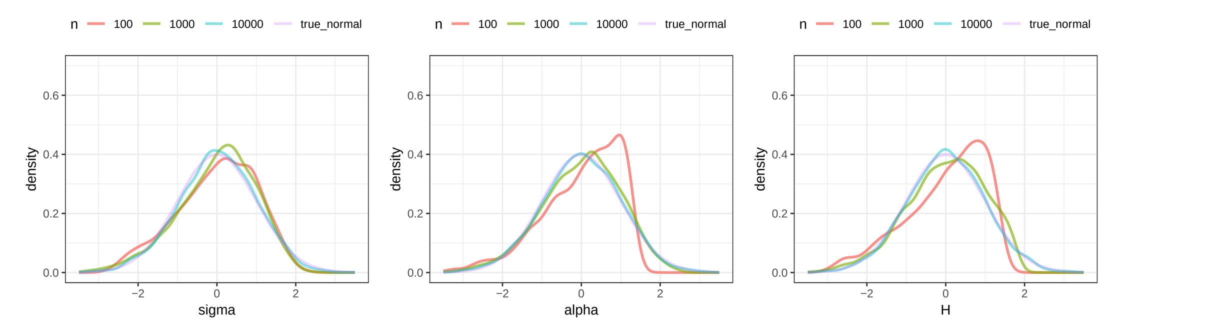


Path length $N = 200$, number of sample paths $N_{MC} = 300$; $t_1 = 1, t_2 = 2, k = 2, p = 0.4, \sigma = 0.3$.

Performance of $(\widetilde{\sigma}_{\text{low}}, \widetilde{\alpha}_{\text{low}}, \widetilde{H}_{\text{low}})$

n	$\widetilde{\sigma}_{\text{low}}$	$\widetilde{\alpha}_{\text{low}}$	$\widetilde{H}_{\text{low}}$
100	-0.05/0.09	-0.031/0.18	-0.12/0.23
1000	-4e-3/0.04	0.01/0.068	-0.018/0.12
10000	3e-4/0.015	1e-3/0.022	-3e-3/0.05

Bias/standard deviation of the estimator $(\widetilde{\sigma}_{\text{low}}, \widetilde{\alpha}_{\text{low}}, \widetilde{H}_{\text{low}})$. $p = -0.4$, $(\sigma, \alpha, H) = (0.3, 1.8, 0.8)$.



Empirical distributions of estimates $(\widetilde{\sigma}_{\text{low}}, \widetilde{\alpha}_{\text{low}}, \widetilde{H}_{\text{low}})$. $p = 0.4$, $(\sigma, \alpha, H) = (0.3, 1.8, 0.8)$.

n	$\widetilde{\sigma}_{\text{low}}$	$\widetilde{\alpha}_{\text{low}}$	$\widetilde{H}_{\text{low}}$
100	-0.06/0.31	-0.003/0.41	-0.15/0.24
1000	-0.05/0.27	-0.08/0.31	0.003/0.13
10000	0.03/0.26	0.008/0.27	0.04/0.05

Bias/standard deviation of the estimator $(\widetilde{\sigma}_{\text{low}}, \widetilde{\alpha}_{\text{low}}, \widetilde{H}_{\text{low}})$. $p = -0.4$, $(\sigma, \alpha, H) = (0.3, 0.8, 0.8)$.

Supplementary materials

References

[1] A. Basse-O'Connor, R. Lachièze-Rey and M. Podolskij (2016): Limit theorems for stationary increments Lévy driven moving averages. *Annals of Probability.*

[2] S. Mazur, D. Otryakhin and M. Podoslkij (2018): Parameter estimation of the linear fractional stable motion. Submitted.

[3] S. Mazur, D. Otryakhin (2018): Linear fractional stable motion: The rlfsm package. Working paper.