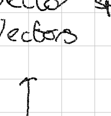
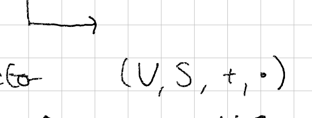
QC Lecture 3

Recap
- Vector spaces
- Vectors are not arrows



Vector $(V, S, +, \cdot)$



Linear Combinations

$v, w \in V$

$5v + 7w$

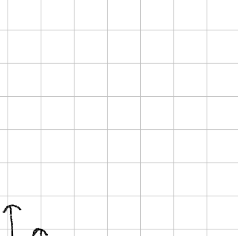
$8v - 4w$

$6\begin{bmatrix}1\\2\end{bmatrix} + 7\begin{bmatrix}4\\6\end{bmatrix}$

Span

$\alpha\begin{bmatrix}1\\0\end{bmatrix}$

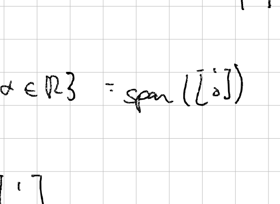
$\{\alpha\begin{bmatrix}1\\0\end{bmatrix} \mid \alpha \in \mathbb{R}\} = \text{span}(\begin{bmatrix}1\\0\end{bmatrix})$



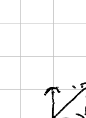
$\alpha\begin{bmatrix}0\\1\end{bmatrix} + \beta\begin{bmatrix}1\\0\end{bmatrix}$

What is the span?

The plane



$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}$



$\begin{bmatrix}\alpha\\0\end{bmatrix}$

Orthogonality
When do vectors point in the same direction?

Inner product

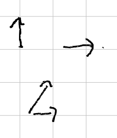
$\circ : V \times V \to S$

$\begin{bmatrix}0\\1\end{bmatrix} \circ \begin{bmatrix}1\\0\end{bmatrix} = 0 + 0 = 0$

$\begin{bmatrix}0\\1\end{bmatrix} \circ \begin{bmatrix}0\\4\end{bmatrix} = 0 + 4 = 4$

Two vectors v, w are orthogonal if $v \circ w = 0$.

orthogonal don't point in the direction at all



orthogonal

not orthogonal

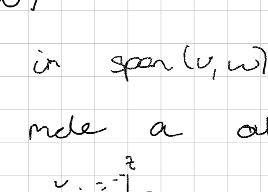
Linear independence

$v, w \in V$

span (v, w)

Is a in span (v, w)? e.g.

can I make a out of v and w?



$a \notin \text{span}(v, w)$ then we say a is linearly independent of v and w

Basis

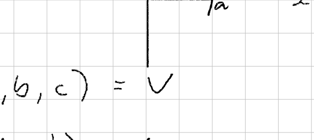
Intuition: Vectors spaces - really complicated

Can we build a vector space out of simpler parts?

Some vectors we can build the whole space out of?

$\text{span}(v_1, v_2, v_3 \ldots) = V$

minimal e.g. no unnecessary vectors

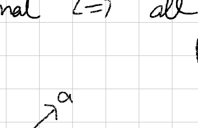


$\text{span}(a, b, c) = V$

$\text{span}(a, b) = V$

Basis: A minimal spanning set for a vector space

minimal $\Leftrightarrow$ all are linearly independent pairwise



linearly independent

not be

Matrices

Defn $n \times m$ matrix

$\begin{bmatrix}2 & 2\\1 & 3\end{bmatrix}$ 2x2 matrix

Addition of matrices

If two matrices are of the same size then we can then add them

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix} = \begin{bmatrix}3 & 6\\9 & 12\end{bmatrix}$

$[\quad] + [\quad]$

Multiplication of matrices

2x2 case

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e & f\\g & h\end{bmatrix} = \begin{bmatrix}ae + bg & af + bh\\ce + dg & cf + dh\end{bmatrix}$

$\begin{bmatrix}3 & 4\\1 & 7\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} = \begin{bmatrix}3\times1 + 1\times4 & 3\times2 + 4\times0\\1\times1 + 7\times1 & 1\times2 + 7\times0\end{bmatrix}$

$= \begin{bmatrix}7 & 6\\8 & 2\end{bmatrix}$

Linear Maps

Maps from one vector space to another

Act "linearly"

V vector space $\qquad \mathcal{B}$ basis

$v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 \ldots \alpha_n b_n$

$v = \sum_{i=1}^{n} \alpha_i b_i$

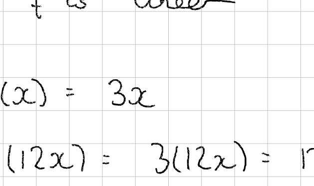
$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

Then f is linear

$f(x) = 3x$

$f(12x) = 3(12x) = 12 \cdot 3x = 12 f(x)$

$f(v) = f(\sum \alpha_i b_i) = \sum \alpha_i f(b_i)$



Example

$f(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}2\\3\end{bmatrix} \qquad f(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}4\\5\end{bmatrix}$

$f(\begin{bmatrix}7\\6\end{bmatrix}) = f(7\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix})$

$= 7f(\begin{bmatrix}1\\0\end{bmatrix}) + 6f(\begin{bmatrix}0\\1\end{bmatrix})$

$= 7 \cdot \begin{bmatrix}2\\3\end{bmatrix} + 6 \cdot \begin{bmatrix}4\\5\end{bmatrix}$

Special matrices

Identity I or $\mathbb{1}$

$A\mathbb{1} = A$

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\begin{bmatrix}2 & 4\\6 & 8\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2\cdot1 + 4\cdot0 & 0\cdot2 + 4\cdot1\\6\cdot1 + 8\cdot0 & 6\cdot0 + 8\cdot1\end{bmatrix}$

$= \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix}$

Inverse

$AB = \mathbb{1}$

$B = A^{-1}$