Quantum Computing Lecture 2

Recap
- Sets
- Maps
- Complex numbers

Linear Algebra

Language of QM

What is a vector?

- Vectors are arrows $\longrightarrow$
- Vectors are members of a vector space

Vector spaces $(V, S, +, \cdot)$

V is a set (the set of vectors)

S is also a set (the set of scalars)

$+ : V \times V \to V$ $\qquad +(x,y) = z$, $\quad x + y = z$

$\cdot : S \times V \to V$ $\qquad \cdot(s, v) = w$, $\quad s \cdot v = w$, $\quad sv = w$

Examples

$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$

$V = \left\{ \begin{bmatrix}a\\b\end{bmatrix} : a, b \in \mathbb{R} \right\}$ $\quad \mathbb{R}^2$

$+ : V \times V \to V$

$\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}a+c\\d+e\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

$S = \mathbb{R}$

$\cdot : S \times V \to V$

$a \cdot \begin{bmatrix}b\\c\end{bmatrix} = \begin{bmatrix}ab\\ac\end{bmatrix}$

$5 \cdot \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}5\\15\end{bmatrix}$

$(\mathbb{R}^2, \mathbb{R}, +, \cdot)$ is a vector space

$\mathbb{R}^n$ $\quad \begin{bmatrix}a\\ \vdots\\ n\end{bmatrix}$

Properties of $+$

Associativity $\quad (v + w) + u = v + (w + u)$

$(1 + 3) + 4 = 1 + (3 + 4)$

Commutativity $\quad u + v = v + u$

$1 + 3 = 3 + 1$

Zero vector $\quad v = 0$

$v + w = w$ $\quad$ for all w

$= w + v$

Inverses

$v \in V$ $\qquad w \in V$

$v + w = 0$ $\qquad w = -v$

$v - v = 0$

Example

$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$

$\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}\right) + \begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}9\\12\end{bmatrix}$

$\underline{\begin{bmatrix}0\\0\end{bmatrix}} + \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0+a\\0+b\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

$\begin{bmatrix}1\\7\end{bmatrix} + \begin{bmatrix}-1\\-7\end{bmatrix} = \begin{bmatrix}1+-1\\7+-7\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}-a\\-b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Properties of $\cdot$

Identity $\quad a \in S$ $\quad a \cdot v = v$

"1"

acts like 1

Associativity $\quad a, b \in S$ $\quad v \in V$

$a \cdot (b \cdot v) = (ab) \cdot v$

$5 \cdot \left(3 \cdot \begin{bmatrix}2\\2\end{bmatrix}\right) = 5 \cdot \begin{bmatrix}6\\6\end{bmatrix} = \begin{bmatrix}30\\30\end{bmatrix}$

$(5 \times 3) \cdot \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}30\\30\end{bmatrix}$

Zeros $\quad 0 \in S$ $\quad \hat{0}, v \in V$

$0 \cdot v = \hat{0}$

$0 \cdot \begin{bmatrix}3\\4\end{bmatrix} = \overset{\hat{0}}{\begin{bmatrix}0\\0\end{bmatrix}}$

Distributivity $\quad a, b \in S$

$(a + b) \cdot v = a \cdot v + b \cdot v$

$(5 + 3)\begin{bmatrix}4\\2\end{bmatrix} = 8 \cdot \begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}32\\16\end{bmatrix}$

$5 \cdot \begin{bmatrix}4\\2\end{bmatrix} + 3 \cdot \begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}20\\10\end{bmatrix} + \begin{bmatrix}12\\6\end{bmatrix}$

$= \begin{bmatrix}32\\16\end{bmatrix}$

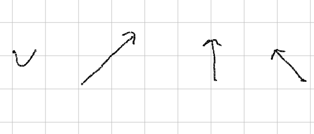
Distributivity $\quad s \in S$ $\quad u, w \in V$

$s \cdot (v + u) = s \cdot v + s \cdot u$

$5\left(\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}4\\6\end{bmatrix}\right)$

$= 5 \cdot \begin{bmatrix}3\\2\end{bmatrix} + 5 \cdot \begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}35\\40\end{bmatrix}$

Arrows



$S : \mathbb{R}$

NB: S has to be a field "nice addition, multiplication, division, subtraction"

$S = \mathbb{R}, \mathbb{C}$

$+ : V \times V \to V$

$\cdot : S \times V \to V$

$\circ : V \times V \to S$ $\qquad$ inner product, dot product, $a + bi \to a - bi$

Conjugate symmetry $\quad v \circ w = (w \circ v)^*$

Linearity in the <u>first argument</u>

$(v + u) \circ w = v \circ w + u \circ w$

Positive definiteness

$v \circ v \geq 0$ and real

Example

$\begin{bmatrix}5\\3\end{bmatrix} \circ \begin{bmatrix}2\\4\end{bmatrix} = 5 \times 2 + 3 \times 4 = 22$

$\begin{bmatrix}a\\b\end{bmatrix} \circ \begin{bmatrix}c\\d\end{bmatrix} = ac + bd$

$\begin{bmatrix}5\\3\end{bmatrix} \circ \begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \circ \begin{bmatrix}5\\3\end{bmatrix}$

$\left(\begin{bmatrix}12\\4\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix}\right) \circ \begin{bmatrix}6\\7\end{bmatrix} = \begin{bmatrix}12\\4\end{bmatrix} \circ \begin{bmatrix}6\\7\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix} \circ \begin{bmatrix}6\\7\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} \circ \begin{bmatrix}a\\b\end{bmatrix} = a^2 + b^2 \geq 0$

Example 2 $\qquad S = \mathbb{C}$

$\begin{bmatrix}a+bi\\c+di\end{bmatrix} \circ \begin{bmatrix}d+ei\\f+gi\end{bmatrix}$

$= (a + bi)(d + ei)^* + (c + di)(f + gi)^*$

$\begin{bmatrix}3+4i\\7+2i\end{bmatrix} \circ \begin{bmatrix}2+3i\\4\end{bmatrix} =$

$(3 + 4i)(2 - 3i) + (7 + 2i)4 =$

$18 - i + 28 + 8i = 46 + 7i$

$\begin{bmatrix}2+3i\\4\end{bmatrix} \circ \begin{bmatrix}3+4i\\7+2i\end{bmatrix}$

$= (2 + 3i)(3 - 4i) + 4(7 - 2i)$

$(v + w) \circ u = [u \circ (v + w)]^*$

$\parallel$

$v \circ u + w \circ u$

Size of a vector

$\begin{bmatrix}a\\b\end{bmatrix} \circ \begin{bmatrix}a\\b\end{bmatrix} = aa^* + bb^* \geq 0$

$|v| = \sqrt{v \circ v}$

$\begin{bmatrix}5\\3\end{bmatrix} \circ \begin{bmatrix}5\\3\end{bmatrix} = 25 + 9 = 34$

$\left|\begin{bmatrix}5\\3\end{bmatrix}\right| = \sqrt{34}$

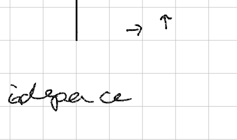
Geometry / Angles

$\cos(t) = \dfrac{x \circ y}{|x|\,|y|}$ $\qquad x, y \in V$

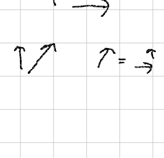
$t = \cos^{-1}\left(\dfrac{x \circ y}{|x|\,|y|}\right)$

Next Week

- Building vectors out of vectors
- Are there some basis vector I can build everything out of?



- Linear independence



- Matrices