Quantum Computing Lecture 4

Recap

Basis $\{b \in V\}$ $\quad v = \sum_{i=1}^{n} c_i b_i$

Linear maps

$f(v)$ $\quad f(v+w) = f(v) + f(w)$

$f(5v) = 5f(v)$

$v = \sum_i c_i b_i \qquad c_i \in \mathbb{C} \quad b_i \in B$

$f(v) = f(\sum_i c_i b_i) = \sum_i f(c_i b_i)$

$= \sum c_i \underbrace{f(b_i)}$

$\mathbb{R}^2$ $\quad \hat{y}, \hat{x}$

$f(\hat{x}) = 4 \qquad f(\hat{y}) = 6$

v $\quad 2\hat{x} + 2\hat{y}$

$f(2\hat{x} + 2\hat{y}) = 2 \cdot f(\hat{x}) + 2f(\hat{y}) = 8 + 12 = 20$

Notation

$\underset{f}{\begin{bmatrix} 4 & 6 \end{bmatrix}} \underset{v}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} = 20$

$\begin{bmatrix} 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 + 14 = 17 \quad f(\hat{x}) = 3 \quad f(\hat{y}) = 7$

Extend this

$f : V \to V$

$f(\hat{x}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad f(\hat{y}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$f(3\hat{x} + 4\hat{y}) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 16 \end{bmatrix}$

$= \begin{bmatrix} 15 \\ 22 \end{bmatrix}$

f

$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

x goes to $\quad$ y goes to

$3y$ $\quad 2x$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

Linear maps $\Leftrightarrow$ maps

$f, g : V \to V$ $\quad$ linear

$g(f(v)) = (g \circ f)(v)$

$\underset{g}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \times \underset{f}{\begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix}}$

Special types/operations on matrices

Transpose of a matrix

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Conjugate transpose

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \qquad z = x + iy, \quad z^* = x - iy$

Hermitian matrices

$A = A^\dagger$

Unitary matrices

$AA^\dagger = \mathbb{1} \quad \Leftrightarrow \quad A^\dagger = A^{-1}$

Eigenvectors and Eigenvalues

$Av = \lambda v$

input $\quad$ output

v called an eigenvector

λ called an eigenvalue

$B\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

eigenvalue of B $\qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector of B

Quantum Mechanics

Quantum Coin

Two states $|H\rangle, |T\rangle$

In QM states form a vector space

$\frac{3}{8}|H\rangle + \frac{5}{8}|T\rangle$ is a state

And all linear combinations are

$a|H\rangle + b|T\rangle$ also a state

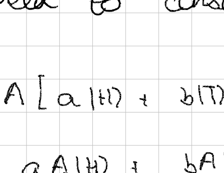
When I measure the coin it collapses to either $|H\rangle$ or $|T\rangle$

$P(|H\rangle) = aa^*$

$P(|T\rangle) = bb^*$

$aa^* + bb^* = 1$ (normalised)

Picture



State space is a vector space

Measurement collapses the state

Time evolution

Need to conserve probabilities

$A[a|H\rangle + b|T\rangle]$

$= aA|H\rangle + bA|T\rangle$

$(aA)(aA)^\dagger + (bA)(bA)^\dagger = 1$

$= aa^* AA^\dagger + bb^* AA^\dagger = 1$

$\Rightarrow AA^\dagger = \mathbb{1}$

$\Rightarrow A$ must be unitary

Example

$\frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|T\rangle$

$P(|H\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^* = \frac{1}{2}$

Phases

Any complex number c such that

$cc^* = 1$ can be written as

$c = e^{it} \quad t \in \mathbb{R}$

$e^{it}\left(a|H\rangle + b|T\rangle\right)$

$P(H) = e^{it}a\left(e^{it}a\right)^* = e^{it}ae^{-it}a^* = aa^*$

Braket Notation

$|H\rangle \quad |T\rangle$ $\quad$ kets

$(|H\rangle)^\dagger = \langle H|$ $\quad$ bra

$|H\rangle \circ |T\rangle = 0$

$\langle H|T\rangle = 0$

Quantum Computation

Qubit $\quad |0\rangle, |1\rangle$

$a|0\rangle + b|1\rangle \quad a, b \in \mathbb{C}$

$|0\rangle, |1\rangle$ form a basis

orthonormal basis

$\langle 0|0\rangle = \langle 1|1\rangle = 1$ $\quad |0\rangle$

$\langle 0|1\rangle = 0$ $\quad |1\rangle$

Basic Qubit Gates

NOT gate (X gate)

$|0\rangle \to |1\rangle$

$|1\rangle \to |0\rangle$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\text{NOT}[a|H\rangle + b|T\rangle] = a|T\rangle + b|H\rangle$

$= b|H\rangle + a|T\rangle$

Y, Z gates

$\underset{Y}{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}, \quad \underset{Z}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

$Y(a|0\rangle + b|1\rangle) = ia|1\rangle - ib|0\rangle$

$Z(a|0\rangle + b|1\rangle) = a|0\rangle - b|0\rangle$

Hadamard gate

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad$ Prepares a state that is equally likely to be 0 or 1

$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Multi qubits systems

4 basis states

$|00\rangle \quad |11\rangle \quad |01\rangle \quad |10\rangle$

$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

is a state

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Leftrightarrow a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

$\frac{|00\rangle + |01\rangle}{\sqrt{2}}$

Apply NOT gate to the first qubit

$\Rightarrow \frac{|10\rangle + |11\rangle}{\sqrt{2}}$

Or apply to second qubit

$\frac{|01\rangle + |00\rangle}{\sqrt{2}}$

Two qubit gates (CNOT)

$f(|00\rangle) = |00\rangle$ $\quad$ controlled NOT

$f(|01\rangle) = |01\rangle$

$f(|10\rangle) = |11\rangle$

$f(|11\rangle) = |10\rangle$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\text{CNOT}\left(\frac{|01\rangle + |11\rangle}{\sqrt{2}}\right) \quad \frac{|01\rangle + |10\rangle}{\sqrt{2}}$

Next Week:

- Entanglement
- Super-dense coding

Position/Momentum

$|x_1\rangle \quad |x_2\rangle \quad |x_3\rangle \quad |x_4\rangle$

$\underbrace{a|x_1\rangle + b|x_2\rangle}$ $\quad |s_1\rangle \quad |s_2\rangle$