Hackaton: ICTP-SAIFR Second School of Quantum Computing

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1 Bose-Einstein Condensates and the Involvement in Advances for New Technologies

1.1 Interactions between atoms and the low-energy limit

1.1.1 a)

Traditionally, in a scattering problem, one deals with two bodies interacting via a finite-range two-body potential. Given that both particles have mass m, we have that, due to the conservation of energy, the particles have energy

$$E = \frac{\hbar^2 k^2}{m} \tag{1}$$

Which is equivalent to the energy of a free particle. In scattering theory, it is common to define a low-energy limit when the reduced wavelength of the particles is much smaller than the interaction range R, meaning

$$\frac{\lambda}{2\pi} \ll R$$

Or, since $k = \frac{2\pi}{\lambda}$,

$$kR \ll 1 \tag{2}$$

For spherically symmetric scattering potentials (as is often the case), it is customary to expand the scattering amplitude of the wave as a sum over angular momentum modes called partial waves. We can then study how each partial wave interacts with the scattering potential, which results in an effective potential including a repulsive centrifugal potential dependent on the angular momentum of the waves:

$$V_{eff}(r) = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} \tag{3}$$

For low enough energy ($E \approx 0$), unless the scattering potential is very strong (strong enough to enable $l \neq 0$ bound states at that energy range), the particle cannot overcome the centrifugal potential and partial waves with $l \neq 0$ do not influence the scattering process significantly. Thus, in this regime, the partial wave with $l = 0$, called the "s-wave"is most important.

1.1.2 b)

The s-wave scattering length a can be interpreted as the point at which the wave function outside the interaction range intercepts the r axis in a $\psi \times r$ plot. It is defined by $\lim_{k\to 0} k \cot \delta_0 = -\frac{1}{a}$ (where δ_0 is the scattering phase shift) and single-handedly determines the scattering cross section at the low-energy limit through $\sigma_{l=0} = 4\pi a^2$.

1.1.3 c)

In scattering theory, we can approximate the transition matrix T (by discarding higher order terms) as being equivalent to the scattering potential V. This is called the Born Approximation. In this approximation, we have that the scattering length a is given by

$$a = \frac{m_r}{2\pi\hbar^2} \int d^3 r U(\mathbf{r}) \tag{4}$$

Where $m_r = \frac{m}{2}$ is the reduced mass of the two-particle system. If we consider an effective scattering potential of the form $U_{eff}(\mathbf{r} - \mathbf{r}') = U_0 \delta(\mathbf{r} - \mathbf{r}')$, we can find the value of the coupling constant U_0 by substituting $U_e f f$ into (4):

$$a = \frac{m_r}{2\pi\hbar^2} \int d^3 r U_e f f(\mathbf{r}) = \frac{m}{4\pi\hbar^2} \int d^3 r U_0 \delta(\mathbf{r} - \mathbf{r}')$$

By the definition of the Dirac delta function, the integral is simply U_0, so that we can rearrange the above expression to obtain:

$$U_0 = \frac{4\pi\hbar^2 a}{m} \tag{5}$$

1.2 The Gross-Pitaevskii equation

1.2.1 a)

Consider the many body Hamiltonian for a gas of N interacting bosons,

$$H = \sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{r}_i) \right] + U_0 \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j) \tag{6}$$

Where $\frac{\mathbf{p}_i^2}{2m} = -\frac{\hbar^2}{2m}\nabla_i^2$ is the kinetic energy of the i-th boson (with m being their identical mass), $V(\mathbf{r}_i)$ is the external potential acting on each particle and U_0 is a coupling constant which sets the strength of the short-range interaction between them.

As in a Bose-Einstein Condensate most particles tend do be in the same quantum state, one may assume the wave function for the entire system to take the form of a symmetrized Hartree wave function:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \prod_{i=1}^{N} \phi(\mathbf{r}_i) \tag{7}$$

In the above expression, $\phi(\mathbf{r}_i)$ is the wave function of the i-th particle and is normalized to unity, such that

$$\int d^3\mathbf{r} |\phi(r)|^2 = 1 \tag{8}$$

To find the expectation value of the energy in this state, one must calculate the integral

$$E[\Psi] = \int d^3\mathbf{r} \Psi^* H \Psi \tag{9}$$

Substituting equations (6) and (7) in (9), we get

$$E[\phi] = \int d^3\mathbf{r}_1...d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) \left(\sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{r}_i) \right] + U_0 \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j) \right) \prod_{k=1}^{N} \phi(\mathbf{r}_k) \tag{10}$$

We can split the functional above in a kinetic energy term $K[\phi]$, an external potential term $V_{ext}[\phi]$ and an inter-particle interaction contribution, $U_{int}[\phi]$. For the kinetic energy, we have:

$$K[\phi] = \int d^3\mathbf{r}_1...d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \prod_{k=1}^{N} \phi(\mathbf{r}_k) = \int d^3\mathbf{r}_1...d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) \sum_{i=1}^{N} \frac{-\hbar^2}{2m} \nabla_i^2 \prod_{k=1}^{N} \phi(\mathbf{r}_k) \tag{11}$$

Since ∇_i^2 is the Laplacian operator with regard to the coordinates of the i-th particle and, in the Hartree mean-field approximation, we assume that the only way different particles affect each other is through a mean-field interaction, we can safely take $\nabla_i^2 \prod_{k=1}^{N} \phi(\mathbf{r}_k)$ to be $\nabla_i^2 \phi(\mathbf{r}_i) \prod_{k \neq i}^{N} \phi(\mathbf{r}_k)$, where the indices mean we take the product of all wave functions apart from the i-th one. We can thus rearrange equation (11) into the following form:

$$K[\phi] = \frac{-\hbar^2}{2m} \sum_{i=1}^{N} \int d^3\mathbf{r}_1...d^3\mathbf{r}_N \prod_{k \neq i}^{N} \phi^*(\mathbf{r}_k) \prod_{k \neq i}^{N} \phi(\mathbf{r}_k) \left[\phi^*(\mathbf{r}_i) \nabla_i^2 \phi(\mathbf{r}_i) \right]$$

The term $\prod_{k \neq i}^{N} \phi^*(\mathbf{r}_k) \prod_{k \neq i}^{N} \phi(\mathbf{r}_k)$ is evidently $\prod_{k \neq i}^{N} |\phi(\mathbf{r}_k)|^2$, so that applying the normalization condition (8) to the wave functions not describing the i-th particle gives

$$K[\phi] = \frac{-\hbar^2}{2m} \sum_{i=1}^{N} \int d^3\mathbf{r}_i \phi^*(\mathbf{r}_i) \nabla_i^2 \phi(\mathbf{r}_i)$$

Since the wave function is the same for all particles,

$$K[\phi] = \frac{-N\hbar^2}{2m} \int d^3\mathbf{r} \phi^*(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) \tag{12}$$

Where we omit the index i for notational simplicity. We can still develop the integral of equation (12) a little further by applying Green's First Identity:

$$\int d^3\mathbf{r} \phi^*(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) = \oint_{\partial V} \phi^*(\mathbf{r}) \nabla \phi(\mathbf{r}) \cdot d\mathbf{S} - \int d^3\mathbf{r} \nabla \phi^*(\mathbf{r}) \cdot \nabla \phi(\mathbf{r}) \tag{13}$$

Since the first integral on the RHS is over the boundary of the region we are integrating in, and that region is arbitrary, the equality must hold as the boundary goes to infinity. Naturally, the Bose-Einstein Condensate we mean to study does not extend to infinity, thus, ϕ^* and ϕ must go to 0 as $r \to \infty$, which implies that the surface integral itself is equal to 0. We have, then:

$$\int d^3\mathbf{r} \phi^*(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) = -\int d^3\mathbf{r} \nabla \phi^*(\mathbf{r}) \cdot \nabla \phi(\mathbf{r}) = \int d^3\mathbf{r} |\nabla \phi(\mathbf{r})|^2 \tag{14}$$

Finally, substituting (14) into (12), we have:

$$K[\phi] = N \int d^3\mathbf{r} \frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 \tag{15}$$

Now, for the external potential term $V_{ext}[\phi]$:

$$V_{ext}[\phi] = \int d^3\mathbf{r}_1...d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) \sum_{i=1}^{N} V(\mathbf{r}_i) \prod_{k=1}^{N} \phi(\mathbf{r}_k) \tag{16}$$

As we did before with the kinetic energy, we can rearrange (16) to give

$$\begin{aligned}
V_{ext}[\phi] &= \sum_{i=1}^{N} \int d^3\mathbf{r}_1 ... d^3\mathbf{r}_N \prod_{k\neq i}^{N} \phi^*(\mathbf{r}_k) \prod_{k\neq i}^{N} \phi(\mathbf{r}_k)\phi^*(\mathbf{r}_i)V(\mathbf{r}_i)\phi(\mathbf{r}_i) \\
&= \sum_{i=1}^{N} \int d^3\mathbf{r}_1 ... d^3\mathbf{r}_N \left[\prod_{k\neq i}^{N} |\phi(\mathbf{r}_k)|^2 \right] V(\mathbf{r}_i)|\phi(\mathbf{r}_i)|^2.
\end{aligned}$$

Applying the normalization condition (8),

$$V_{ext}[\phi] = \sum_{i=1}^{N} \int d^3\mathbf{r}_i V(\mathbf{r}_i)|\phi(\mathbf{r}_i)|^2.$$

Performing the sum and omitting the index, since we assume the wave function is the same for all bosons,

$$V_{ext}[\phi] = N \int d^3\mathbf{r} V(\mathbf{r})|\phi(\mathbf{r})|^2. \tag{17}$$

For the inter-particle interaction, U_{int}, we have:

$$\begin{aligned}
U_{int}[\phi] &= \int d^3\mathbf{r}_1 ... d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) U_0 \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j) \prod_{k=1}^{N} \phi(\mathbf{r}_k) \\
&= \sum_{i<j} \int d^3\mathbf{r}_1 ... d^3\mathbf{r}_N \prod_{k=1}^{N} \phi^*(\mathbf{r}_k) U_0 \delta(\mathbf{r}_i - \mathbf{r}_j) \prod_{k=1}^{N} \phi(\mathbf{r}_k)
\end{aligned}$$

Separating the product terms as we have with the kinetic and external potential energies,

$$U_{int}[\phi] = \sum_{i<j} \int d^3\mathbf{r}_1 ... d^3\mathbf{r}_N \left[\prod_{k\neq i, k\neq j}^{N} \phi^*(\mathbf{r}_k) \prod_{k\neq i, k\neq j}^{N} \phi(\mathbf{r}_k) \right] \phi^*(\mathbf{r}_i)\phi^*(\mathbf{r}_j) U_0 \delta(\mathbf{r}_i - \mathbf{r}_j)\phi(\mathbf{r}_i)\phi(\mathbf{r}_j).$$

Since $\left[\prod_{k\neq i,k\neq j}^{N} \phi^*(\mathbf{r}_k) \prod_{k\neq i,k\neq j}^{N} \phi(\mathbf{r}_k)\right] = \prod_{k\neq i,k\neq j}^{N} |\phi(\mathbf{r}_k)|^2$ we integrate out the products using the normalization condition (8) and get:

$$\begin{aligned}
U_{int}[\phi] &= \sum_{i<j} \int d^3\mathbf{r}_i d^3\mathbf{r}_j \phi^*(\mathbf{r}_i)\phi^*(\mathbf{r}_j) U_0 \delta(\mathbf{r}_i - \mathbf{r}_j)\phi(\mathbf{r}_i)\phi(\mathbf{r}_j) \\
&= \sum_{i<j} \int d^3\mathbf{r}_i d^3\mathbf{r}_j U_0 \delta(\mathbf{r}_i - \mathbf{r}_j)|\phi(\mathbf{r}_i)|^2|\phi(\mathbf{r}_j)|^2
\end{aligned}$$

Which by the definition of the Dirac delta function is simply

$$\begin{aligned}
U_{int}[\phi] &= \sum_{i<j} \int d^3\mathbf{r}_i U_0 |\phi(\mathbf{r}_i)|^4 \\
&= \sum_{j=1}^{N} \sum_{i=1}^{j-1} \int d^3\mathbf{r}_i U_0 |\phi(\mathbf{r}_i)|^4.
\end{aligned}$$

Performing the sum and omitting the index i as we have been doing so far, we have

$$U_{int}[\phi] = N \int d^3\mathbf{r} \frac{(N-1)}{2} U_0 |\phi(\mathbf{r})|^4. \tag{18}$$

Summing equations (15), (17) and (18), we get

$$E[\phi] = N \int d^3\mathbf{r} \frac{\hbar^2}{2m}|\nabla\phi(\mathbf{r})|^2 + V(\mathbf{r})|\phi(\mathbf{r})|^2 + \frac{N-1}{2} U_0 |\phi(\mathbf{r})|^4 \tag{19}$$

Which will be our final expression for the expectation value of the Hamiltonian.

1.2.2 b)

In order to get a more intuitive interpretation for (19), since in the BEC most of the particles are in the same quantum state, we can introduce the concept of a "wave function"of the entire BEC, $\psi(\mathbf{r}) = \sqrt{N}\phi(\mathbf{r})$, which is an order parameter that behaves as a wave function but is normalized to N instead of 1. This allows us to regard $|\psi(\mathbf{r}|^2$ as the particle density of the condensate.

Substituting $\psi(\mathbf{r})$ into (19), we get:

$$E[\psi] = N\int d^3\mathbf{r}\frac{\hbar^2}{2m}\frac{|\nabla\psi(\mathbf{r})|^2}{N} + V(\mathbf{r})\frac{|\psi(\mathbf{r})|^2}{N} + \frac{N-1}{2}U_0\frac{|\psi(\mathbf{r})|^4}{N^2} \tag{20}$$

$$\implies E[\psi] = \int d^3\mathbf{r}\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{U_0}{2}|\psi(\mathbf{r})|^4 - \frac{U_0}{2}\frac{|\psi(\mathbf{r})|^4}{N}$$

As we suppose N is very large, $\frac{U_0}{2}\frac{|\psi(\mathbf{r}|^4}{N} \approx 0$, which gives us

$$E[\psi] = \int d^3\mathbf{r}\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{U_0}{2}|\psi(\mathbf{r})|^4 \tag{21}$$

Now, if we want to find the ground state of the Bose-Einstein Condensate, we must find the wave function ψ which minimizes the functional (21) while conserving the number of bosons in the condensate. This can be done by applying techniques from variational calculus along with Lagrange multipliers. Namely, we must find the wave function ψ that minimizes $X[\psi] = E[\psi] - \mu N[\psi]$. Since $N = \int d^3\mathbf{r}|\psi(\mathbf{r})|^2$, we have

$$X[\psi] = \int d^3\mathbf{r}\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{U_0}{2}|\psi(\mathbf{r})|^4 - \mu|\psi(\mathbf{r})|^2 \tag{22}$$

Since $|\psi(\mathbf{r})|^2 = \psi^*(\mathbf{r})\psi(\mathbf{r})$, we have

$$X[\psi,\psi^*] = \int d^3\mathbf{r}\frac{\hbar^2}{2m}\nabla\psi^*(\mathbf{r})\cdot\nabla\psi(\mathbf{r}) + V(\mathbf{r})\psi^*(\mathbf{r})\psi(\mathbf{r}) + \frac{U_0}{2}\psi^*(\mathbf{r})^2\psi(\mathbf{r})^2 - \mu\psi^*(\mathbf{r})\psi(\mathbf{r}) \tag{23}$$

Since $\psi(\mathbf{r})$ and $\psi^*(\mathbf{r})$ are different functions, we must minimize in the functional in both of them separately. Let us call f the integrand of $X[\psi,\psi^*]$. Then, applying Green's First Identity (13) backwards on the kinetic energy term and taking the Euler-Lagrange equation for $\psi^*(\mathbf{r})$ we get:

$$\frac{\partial f}{\partial\psi^*} - \frac{d}{dt}\frac{\partial f}{\partial\dot{\psi^*}} = 0$$

Where $\dot{\psi^*}$ is the time-derivative of ψ^*. Since (23) does not depend on $\dot{\psi^*}$, $\frac{d}{dt}\frac{\partial f}{\partial\dot{\psi^*}} = 0$ and we get

$$\frac{\partial f}{\partial\psi^*} = \frac{\partial}{\partial\psi^*}\left[\frac{-\hbar^2}{2m}\frac{\partial\psi^*}{\partial\psi^*}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\frac{\partial\psi^*}{\partial\psi^*}\psi(\mathbf{r}) + \frac{U_0}{2}\frac{\partial(\psi^*)^2}{\partial\psi^*}\psi(\mathbf{r})^2 - \mu\frac{\partial\psi^*}{\partial\psi^*}\psi(\mathbf{r})\right] = 0$$

Applying the derivatives, that reduces to the Time-Independent Gross-Pitaevskii Equation:

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) + U_0|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}) \tag{24}$$

1.2.3 c)

Now, we will take the Time-Independent Gross-Pitaevskii Equation (24) and study it in the context of an uniform Bose gas.

An uniform Bose gas is, generally, a gas of uniformly distributed (constant density) interacting bosons under no action from external potentials ($V(\mathbf{r}) = 0$). Considering the fact that $|\psi(\mathbf{r})|^2 = n$ is the density of bosons in the gas, we can write the Time-Independent Gross-Pitaevskii Equation for an uniform Bose gas as follows:

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + nU_0\psi(\mathbf{r}) = \mu\psi(\mathbf{r}) \tag{25}$$

It can be easily seen that this corresponds to the Time-Independent Schrödinger Equation (TISE) for a free-particle of energy $\epsilon = \mu - nU_0$. This observation is in line with the notion that a Bose-Einstein Condensate behaves as one single macroscopic quantum object. In this case, if we consider constant density and no external potential, the entire BEC behaves as if it was a single free particle.

2 Prospects and Challenges for Quantum Machine Learning (October 14-16)

2.1 (1) [Exercise] Let $V = \mathbb{C}_2$ be the Hilbert space of a single qubit. Then, consider the set of objects $\{\mathbb{I}, \mathbb{X}\}$, where$\mathbb{I}$is the 2×2 identity matrix and $\mathbb{X}$ the Pauli-x matrix. Show that these objects, which represent bit-flips, form a group.

To show that this set, $\{\mathbb{I}, \mathbb{X}\}$, forms a group, G, with the usual product of matrices we need to show the following three characteristics:

1. Closure Property: All results of the group operation of any two elements in a group are also an element of that group;
2. Associative: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$;
3. Identity: There is an element $e \in G$ such that $e * g = g * e = g$ for all $g \in G$;
4. Inverse: For all $g \in G$, there exists $g^{-1} \in G$ such that $g^{-1} * g = g * g^{-1} = e$.

For the item number 1. Closure Property:
We can explicitly show all the composition possibilities in the group:

$$\begin{aligned} \mathbb{I} * \mathbb{I} &= \mathbb{I} \in G \\ \mathbb{X} * \mathbb{X} &= \mathbb{I} \in G \\ \mathbb{I} * \mathbb{X} &= \mathbb{X} \in G \\ \mathbb{X} * \mathbb{I} &= \mathbb{X} \in G \end{aligned}$$

For the item number 2. Associative: We have that the usual matrix operation that is associative so the group inherits this characteristic.
For the item number 3. Identity: We can show explicitly the element $\mathbb{I}$ is the identity of the group G:

$$\begin{aligned} \mathbb{I} * \mathbb{I} &= \mathbb{I} \\ \mathbb{I} * \mathbb{X} &= \mathbb{X} \\ \mathbb{X} * \mathbb{I} &= \mathbb{X}, \end{aligned}$$

so, i saw the element $\mathbb{I}$ is the identity of the group G.
For the number 4. Inverse: We can see that every element of the group has an inverse element, i.e., there is a element $\mathbb{I}^{-1}$ and $\mathbb{X}^{-1}$:

$$\mathbb{I} * \mathbb{I}^{-1} = \mathbb{I},$$

then we conclude that:

$$\mathbb{I}^{-1} = \mathbb{I}.$$

and, similarly,

$$\mathbb{X} * \mathbb{X}^{-1} = \mathbb{I},$$

the we conclude:

$$\mathbb{X}^{-1} = \mathbb{X}.$$

Finally we showed that set $\{\mathbb{X}, \mathbb{I}\}$ with the usual matrix operation forms a group.

2.2 (2) [Exercise] Proves that the set of all unitaries of the form $U = e^{-i\phi_3 \mathbb{Y}} e^{-i\phi_2 \mathbb{X}} e^{-i\phi_1 \mathbb{Y}}$ constitutes a representation of the unitary Lie group SU(2).

Proof Consider the first order approximation:

$$\begin{aligned}
U &\approx (\mathbb{I} - i\phi_3 \mathbb{Y})(\mathbb{I} - i\phi_2 \mathbb{X})(\mathbb{I} - i\phi_1 \mathbb{Y}) \\
&= (\mathbb{I} - i\phi_2 \mathbb{X} - i\phi_3 \mathbb{Y} - \phi_3\phi_2 \mathbb{Y}\mathbb{X})(\mathbb{I} - i\phi_1 \mathbb{Y}) \\
&= \mathbb{I} - i\phi_1 \mathbb{Y} - i\phi_2 \mathbb{X} - \phi_2\phi_1 \mathbb{X}\mathbb{Y} - i\phi_3 \mathbb{Y} - \phi_3\phi_1 \mathbb{I} - \phi_3\phi_2 \mathbb{Y}\mathbb{X} + i\phi_3\phi_2\phi_1 \mathbb{Y}\mathbb{X}\mathbb{Y} \\
&= \mathbb{I}(1 - \phi_3\phi_1) - i\mathbb{Y}(\phi_1 + \phi_3) - i\mathbb{X}(\phi_2 + \phi_3\phi_2\phi_1) - i\mathbb{Z}(\phi_2\phi_1 - \phi_3\phi_2),
\end{aligned}$$

and we see that we have access to all directions of the Lie Group and the Lie Algebra. Now, let us consider the exponential with all orders. First we note that, since $\sigma_k^2 = \mathbb{I}$, so $\sigma_k^{2n} = \mathbb{I}$ and $\sigma_k^{2n+1} = \sigma_k$ and therefore

$$\begin{aligned}
e^{-i\alpha\sigma_i} &= \sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{n!} \sigma_i^n \\
&= \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n}}{(2n)!} \sigma_i^{2n} + \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n+1}}{(2n+1)!} \sigma_i^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} \mathbb{I} - i \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} \sigma_i \\
&= \cos(\alpha)\mathbb{I} - i\sin(\alpha)\sigma_i.
\end{aligned}$$

Now,

$$\begin{aligned}
U &= e^{-i\phi_3 \mathbb{Y}} e^{-i\phi_2 \mathbb{X}} e^{-i\phi_1 \mathbb{Y}} \\
&= [\cos(\phi_3)\mathbb{I} - i\sin(\phi_3)\mathbb{Y}]\,[\cos(\phi_2)\mathbb{I} - i\sin(\phi_2)\mathbb{X}]\,[\cos(\phi_1)\mathbb{I} - i\sin(\phi_1)\mathbb{Y}] \\
&= [\cos(\phi_3)\cos(\phi_2)\mathbb{I} - i\cos(\phi_3)\sin(\phi_2)\mathbb{X} - i\sin(\phi_3)\cos(\phi_2)\mathbb{Y} - \sin(\phi_3)\sin(\phi_2)\mathbb{Y}\mathbb{X}]\,[\cos(\phi_1)\mathbb{I} - i\sin(\phi_1)\mathbb{Y}] \\
&= [\cos(\phi_3)\cos(\phi_2)\mathbb{I} - i\cos(\phi_3)\sin(\phi_2)\mathbb{X} - i\sin(\phi_3)\cos(\phi_2)\mathbb{Y} + i\sin(\phi_3)\sin(\phi_2)\mathbb{Z}]\,[\cos(\phi_1)\mathbb{I} - i\sin(\phi_1)\mathbb{Y}] \\
&= \cos(\phi_3)\cos(\phi_2)\cos(\phi_1)\mathbb{I} - i\cos(\phi_3)\cos(\phi_2)\sin(\phi_1)\mathbb{Y} - i\cos(\phi_3)\sin(\phi_2)\cos(\phi_1)\mathbb{X} \\
&- \cos(\phi_3)\sin(\phi_2)\sin(\phi_1)\mathbb{X}\mathbb{Y} - i\sin(\phi_3)\cos(\phi_2)\cos(\phi_1)\mathbb{Y} - \sin(\phi_3)\cos(\phi_2)\sin(\phi_1)\mathbb{Y}^2 \\
&+ i\sin(\phi_3)\sin(\phi_2)\cos(\phi_1)\mathbb{Z} + \sin(\phi_3)\sin(\phi_2)\sin(\phi_1)\mathbb{Z}\mathbb{Y} \\
&= \mathbb{I}\cos(\phi_2)[\cos(\phi_3)\cos(\phi_1) - \sin(\phi_3)\sin(\phi_1)] - i\mathbb{X}\sin(\phi_2)[\cos(\phi_3)\cos(\phi_1) + \sin(\phi_3)\sin(\phi_1)] \\
&- i\mathbb{Y}\cos(\phi_2)[\cos(\phi_3)\sin(\phi_1) + \sin(\phi_3)\cos(\phi_1)] - i\mathbb{Z}\sin(\phi_2)[\cos(\phi_3)\sin(\phi_1) - \sin(\phi_3)\cos(\phi_1)] \\
&= \mathbb{I}\cos(\phi_2)\cos(\phi_3 + \phi_1) - i\mathbb{X}\sin(\phi_2)\cos(\phi_3 - \phi_1) - i\mathbb{Y}\cos(\phi_2)\sin(\phi_1 + \phi_3) - i\mathbb{Z}\sin(\phi_2)\sin(\phi_3 - \phi_1).
\end{aligned}$$

In the matrix form we have

$$U = \begin{pmatrix} \cos(\phi_2)\cos(\phi_3 + \phi_1) - i\sin(\phi_2)\sin(\phi_3 - \phi_1) & -\cos(\phi_2)\sin(\phi_1 + \phi_3) - i\sin(\phi_2)\cos(\phi_3 - \phi_1) \\ \cos(\phi_2)\sin(\phi_1 + \phi_3) - i\sin(\phi_2)\cos(\phi_3 - \phi_1) & \cos(\phi_2)\cos(\phi_3 + \phi_1) + i\sin(\phi_2)\sin(\phi_3 - \phi_1) \end{pmatrix}.$$

The determinant is

$$\begin{aligned}
\det(U) &= \cos^2(\phi_2)\cos^2(\phi_3+\phi_1) + \sin^2(\phi_2)\sin^2(\phi_3-\phi_1) + \cos^2(\phi_2)\sin^2(\phi_1+\phi_3) + \sin^2(\phi_2)\cos^2(\phi_3-\phi_1) \\
&= \sin^2(\phi_2)[\cos^2(\phi_3-\phi_1) + \sin^2(\phi_2)] + \cos^2(\phi_2)[\cos^2(\phi_3+\phi_1) + \sin^2(\phi_3+\phi_1)] \\
&= \cos^2(\phi_2) + \sin^2(\phi_2) \\
&= 1.
\end{aligned}$$

We also know that any $U \in \mathrm{SU}(2)$ is of the type:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where $a, b \in \mathbb{C}$, with:

$$|a|^2 + |b|^2 = 1.$$

So, we found another way of parameterizing the $SU(2)$. For example, if we want to reach the identity operator on the group with $a = 1$ and $b = 0$, we see that we need to choose $\phi_2 = \phi_3 + \phi_1 = 0$. Now, to finish the proof, we note that the usual Lie Algebra parametrization gives

$$\exp\left(-i\sum_{k=1}^{3} x^k \sigma_k\right) = \cos(r)\,\mathbb{I} - i\frac{\sin(r)}{r}\sum_{k=1}^{3} x^k \sigma_k,$$

where

$$r \equiv \left[\sum_{k=1}^{3} \left(x^k\right)^2\right]^{1/2},$$

and in its matrix form

$$\exp\left(-i\sum_{k=1}^{3} x^k \sigma_k\right) = \begin{pmatrix} \cos(r) + i\,\dfrac{\sin(r)}{r}\,z & i\,\dfrac{\sin(r)}{r}\,(x-iy) \\ i\,\dfrac{\sin(r)}{r}\,(x+iy) & \cos(r) - i\,\dfrac{\sin(r)}{r}\,z \end{pmatrix}.$$

2.3 (3) [Challenge] Show that the dimension D of the commutant $C(k)(G)$, outlined in Definition 8 of the lecture notes, is determined by $D = \sum_\lambda m_\lambda^2$.

Let us consider a linear representation R of a group G, which is a group homomorphism that maps the group to the invertible linear maps on some vector space V. That is

$$R : G \to GL(V),$$

with

$$R(g_1.g_2) = R(g_1)R(g_2).$$

The operator R can be decomposed into irreducible representations of the group G. If R is not irreducible, then R can be written in its block-diagonal form as:

$$R(g) = \bigoplus_\lambda \mathbb{I}_{m_\lambda} \otimes R_\lambda,$$

where R_λ is the $d_\lambda \times d_\lambda$ matrix representing the irreducible representation and m_λ is the multiplicity of λ in the decomposition, $\mathbb{I}_{m_\lambda}$ is the $m_\lambda \times m_\lambda$ identity matrix.

Similarly, the elements of the commutant $A \in C(k)(G)$, also have a block-diagonal structure compatible with that of R, in such a way that they must be written as:

$$A = \bigoplus_\lambda A_\lambda \otimes \mathbb{I}_{d_\lambda},$$

where A_λ is a $m_\lambda \times m_\lambda$ matrix as mentioned in the lecture notes. So now it is clear that for each block of the representation, A_λ has m_λ^2 degrees of freedom and by collecting all blocks we have all degrees of freedom of the commutant $C(k)(G)$. The dimension D of the commutant is the sum of the dimensions of the blocks A_λ. Therefore, the total dimension of the commutant is:

$$\boxed{D = \sum_\lambda m_\lambda^2.}$$

3 High Dimensional Quantum Communication with Structured Light (October 14, 16-17)

3.1 (1) [Exercise] Let $HG_\theta(x, y)$ be the first order Hermite-Gauss mode rotated counterclockwise by θ. Show that:

a) $HG_\theta(x, y) = \cos\theta HG_{10}(x, y) + \sin\theta HG_{01}(x, y)$

Considering the following definition for Hermite-Gaussian modes:

$$HG_{nm}(x, y) = \frac{A_{nm}}{w(z)} H_n\left(\frac{\sqrt{2}}{w(z)}x\right) H_m\left(\frac{\sqrt{2}}{w(z)}y\right) \exp\left(-\frac{x^2 + y^2}{w^2(z)}\right) \exp\left[ik\frac{x^2 + y^2}{2R(z)} - i\phi_{nm}(z)\right],$$

where $H_n(u)$ and $H_m(u)$ are Hermite polynomials of order n and m, A_{nm} is the normalization factor, $w(z)$ is the beam waist as a function of the axial distance z, $R(z)$ is the radius of curvature of the beam, and

$$\phi_{nm}(z) = (n + m + 1)\arctan(z/z_R)$$

is the Gouy phase. For the first modes, we have:

$$HG_{10}(x, y) = \frac{A_{10}}{\omega} H_1\left(\frac{\sqrt{2}}{\omega}x\right) H_0\left(\frac{\sqrt{2}}{\omega}y\right) \exp\left(-\frac{x^2 + y^2}{\omega^2}\right) \exp\left[ik\frac{x^2 + y^2}{2R(z)} - i\phi_{10}(z)\right],$$

$$HG_{01}(x, y) = \frac{A_{01}}{\omega} H_0\left(\frac{\sqrt{2}}{\omega}x\right) H_1\left(\frac{\sqrt{2}}{\omega(z)}y\right) \exp\left(-\frac{x^2 + y^2}{\omega^2(z)}\right) \exp\left[ik\frac{x^2 + y^2}{2R(z)} - i\phi_{01}(z)\right],$$

but $H_1(u) = 2u$ and $H_0(u) = 1$ and on the other hand $\phi_{01}(z) = \phi_{10}(z) = 2\arctan(z/z_R)$, so

$$\begin{cases} HG_{10}(x, y) = \dfrac{A_{10}}{\omega(z)} 2\left(\dfrac{\sqrt{2}}{\omega(z)}x\right) \exp\left(-\dfrac{x^2 + y^2}{\omega^2(z)}\right) \exp\left[ik\dfrac{x^2 + y^2}{2R(z)} - 2i\arctan(z/z_R)\right] & (26) \\ HG_{01}(x, y) = \dfrac{A_{01}}{\omega(z)} 2\left(\dfrac{\sqrt{2}}{\omega}y\right) \exp\left(-\dfrac{x^2 + y^2}{\omega^2(z)}\right) \exp\left[ik\dfrac{x^2 + y^2}{2R(z)} - 2i\arctan(z/z_R)\right]. & (27) \end{cases}$$

Now, let us consider the definition:

$$HG_\theta(x, y) \equiv HG_{10}(x', y'),$$

where x' and y' are given by a clockwise rotation of the x and y coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

that preserves the length of vectors

$$x'^2 + y'^2 = x^2 + y^2,$$

which can be easily explicitly verified. Then it follows from the definition that:

$$
\begin{aligned}
HG_{10}(x',y') &= \frac{A_{10}}{\omega(z)} H_1\left(\frac{\sqrt{2}}{\omega(z)}(x\cos\theta + y\sin\theta)\right) H_0\left(\frac{\sqrt{2}}{\omega(z)}(-x\cos\theta + y\sin\theta)\right) \exp\left(-\frac{x^2+y^2}{\omega^2(z)}\right) \\
&\times \exp\left[ik\frac{x^2+y^2}{2R(z)} - 2i\arctan(z/z_R)\right] \\
&= \frac{A_{10}}{\omega(z)} 2\left(\frac{\sqrt{2}}{\omega(z)}(x\cos\theta + y\sin\theta)\right) \exp\left(-\frac{x^2+y^2}{\omega^2(z)}\right) \exp\left[ik\frac{x^2+y^2}{2R(z)} - 2i\arctan(z/z_R)\right] \\
&= \frac{A_{10}}{\omega} 2\left(\frac{\sqrt{2}}{\omega}x\right) \exp\left(-\frac{x^2+y^2}{\omega^2}\right) \exp\left[ik\frac{x^2+y^2}{2R(z)}\cos\theta - 2i\arctan(z/z_R)\right] \cos(\theta) \\
&+ \frac{A_{10}}{\omega} 2\left(\frac{\sqrt{2}}{\omega}y\right) \exp\left(-\frac{x^2+y^2}{\omega^2}\right) \exp\left[ik\frac{x^2+y^2}{2R(z)} - 2i\arctan(z/z_R)\right] \sin\theta.
\end{aligned}
$$

Now comparing with equantions (26) and (27) we can recognize the $HG_{10}(x,y)$ and $HG_{01}(x,y)$, and we have:

$$
\boxed{HG_\theta(x,y) = HG_{10}(x,y)\cos\theta + HG_{01}(x,y)\sin\theta.}
$$

- Show that

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} HG_\theta^*(x,y) HG_\theta(x,y) dxdy = 1.
$$

Using the item a), the integral becomes:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} HG_\theta^*(x,y) HG_\theta(x,y) dxdy &= \int_{-\infty}^{\infty} (HG_{10}^*\cos(\theta) + HG_{01}^*\sin(\theta))(HG_{10}\cos(\theta) + HG_{01}\sin(\theta))dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{10}|^2\cos^2(\theta)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{01}|^2\sin^2(\theta)dxdy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(\theta)\sin(\theta)(HG_{10}HG_{01}^* + HG_{01}HG_{01}^*)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{10}|^2\cos^2(\theta)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{01}|^2\sin^2(\theta)dxdy
\end{aligned}
\tag{28}
$$

where we used that the cross terms are zero because if $n \neq m$ the orthogonality of the Hermite polynomials yields to

$$
\int_{-\infty}^{\infty} H_n(u)H_m(u)\exp(-u^2)du = 0.
$$

The first integral in (28), is:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{10}|^2\cos^2\theta dxdy &= \cos^2\theta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{A_{10}}{\omega}H_1\left(\frac{\sqrt{2}x}{\omega}\right) H_0\left(\frac{\sqrt{2}x}{\omega}\right)\exp\left(-\frac{x^2+y^2}{\omega^2}\right)\right]^2 dxdy \\
&= \left(\frac{\cos\theta A_{10}}{\omega}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{2\sqrt{2}}{\omega}\right)^2 x^2 \exp\left[-2\left(\frac{x^2+y^2}{\omega^2}\right)\right] dxdy \\
&= \left(\frac{\cos\theta A_{10}}{\omega}\right)^2 \left(\frac{8}{\omega^2}\right) \int_{-\infty}^{\infty} x^2\exp\left(-2\frac{x^2}{\omega^2}\right)dx \int_{-\infty}^{\infty} \exp\left(-2\frac{y^2}{\omega^2}\right)dy.
\end{aligned}
\tag{29}
$$

Now, the Gaussian integral in y is

$$\int_{-\infty}^{\infty} \exp\left(-\frac{2y^2}{\omega^2}\right) dy = \omega\sqrt{\frac{\pi}{2}}, \tag{30}$$

and the considering $\alpha = \frac{2}{\omega^2}$, the x integral is:

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2 \exp\left(-2\frac{x^2}{\omega^2}\right) dx &= \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2) dx \\
&= \int_{-\infty}^{\infty} -\frac{d}{d\alpha} \exp(-\alpha x^2) dx \\
&= -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} \\
&= \frac{\sqrt{\pi}}{2}\alpha^{-3/2} \\
&= \frac{\sqrt{\pi}}{2}\left(\frac{2}{\omega^2}\right)^{-3/2} \\
&= \frac{\sqrt{\pi}}{2}\left(\frac{\omega^3}{2\sqrt{2}}\right).
\end{aligned} \tag{31}$$

So, replacing (30) and (31) in equation (29)

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{10}|^2 \cos^2(\theta) dxdy &= \left(\frac{\cos(\theta)A_{10}}{\omega}\right)^2 \left(\frac{8}{\omega^2}\right)\left(\frac{\omega^3\sqrt{\pi}}{4\sqrt{2}}\right)\left(\omega\sqrt{\frac{\pi}{2}}\right) \\
&= \pi\cos^2(\theta)A_{10}^2.
\end{aligned}$$

Analogously, the second integral in (28), is:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |HG_{01}|^2 \sin^2\theta dxdy &= \sin^2\theta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{A_{01}}{\omega} H_0\left(\frac{\sqrt{2}x}{\omega}\right) H_1\left(\frac{\sqrt{2}y}{\omega}\right)\exp\left(-\frac{x^2+y^2}{\omega^2}\right)\right]^2 dxdy \\
&= \left(\frac{\sin\theta A_{01}}{\omega}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{2\sqrt{2}}{\omega}\right)^2 y^2 \exp\left[-2\left(\frac{x^2+y^2}{\omega^2}\right)\right] dxdy \\
&= \left(\frac{\sin\theta A_{01}}{\omega}\right)^2 \left(\frac{8}{\omega^2}\right)\int_{-\infty}^{\infty} y^2\exp\left(-2\frac{y^2}{\omega^2}\right)dy \int_{-\infty}^{\infty}\exp\left(-2\frac{x^2}{\omega^2}\right)dx \\
&= \pi\sin^2(\theta)A_{01}^2,
\end{aligned} \tag{32}$$

since, by relabeling the variables, we have the same integral as above. So, it follows that replacing (29) and (32) in equation (28), it follows that, by choosing

$$A_{10} = A_{01} = \frac{1}{\sqrt{\pi}},$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} HG_\theta^*(x,y)HG_\theta(x,y)dxdy &= \pi[\cos^2(\theta)A_{10}^2 + \sin^2(\theta)A_{01}^2] \\
&= 1.
\end{aligned}$$

- Show that $\int HG^*_\theta(x,y)HG_{\theta+\frac{\pi}{2}}(x,y)dxdy = 0$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} HG^*_\theta(x,y)HG_{\theta+\frac{\pi}{2}}(x,y)dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [HG^*_{10}\cos\theta + HG^*_{01}\sin\theta][-HG_{10}\sin\theta + HG_{01}\cos\theta] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos^2\theta HG^*_{10}HG_{01}dxdy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\cos\theta\sin\theta|HG_{10}|^2dxdy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\theta\sin\theta|HG_{01}|^2dxdy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\sin^2\theta HG^*_{10}HG^*_{01}dxdy
\end{aligned}
$$

where we used that the cross terms are zero because if $n \neq m$ the orthogonality of the Hermite polynomials yields to

$$\int_{-\infty}^{\infty} H_n(u)H_m(u)\exp\bigl(-u^2\bigr)du = 0.$$

Then, the integral becomes:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} HG^*_\theta(x,y)HG_{\theta+\frac{\pi}{2}}(x,y)dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\cos\theta\sin\theta|HG_{10}|^2dxdy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\theta\sin\theta|HG_{01}|^2dxdy \\
&= \cos\theta\sin\theta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|HG_{01}|^2 - |HG_{10}|^2)dxdy \\
&= 0.
\end{aligned}
$$

since from equations (29) and (32) we see both integrals are equal, if we just relabel the coordinates.

- $LG_\pm(r,\varphi) = \frac{1}{\sqrt{2}}[HG_{10}(x,y) \pm iHG_{01}(x,y)]$

The Laguerre-Gauss functions are defined as

$$LG_{pl}(r,\phi,z) = \frac{A_{pl}}{w(z)}\left(\frac{\sqrt{2}\rho}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp(il\phi)\exp\left(ik\frac{\rho^2}{2R(z)} - i\psi_{pl}(z)\right),$$

where $L_p^{|l|}$ are the generalized Laguerre polynomials, the normalization is

$$A_{pl} = \sqrt{\frac{2p!}{\pi(p+|l|)!}},$$

and in the complex phase we have

$$\psi_{pl}(z) = (2p+|l|+1)\arctan(z/z_R).$$

Now consider the following definition

$$LG_\pm(r,\phi,z) \equiv LG_{0,\pm1}(r,\phi,z),$$

and using $LG_0^\alpha = 1$, explicitly we have

$$\begin{aligned} LG_\pm(r,\phi,z) &= \sqrt{\frac{2}{\pi}}\left(\frac{1}{\omega}\right)\left(\frac{\sqrt{2}\rho}{\omega}\right)\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left[ik\frac{\rho^2}{2R(z)} - 2i\arctan(z/z_R)\right]\exp\left(\pm i\phi\right) \\ &= \frac{2}{\omega^2\sqrt{\pi}}\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left[ik\frac{\rho^2}{2R(z)} - 2i\arctan(z/z_R)\right][\rho\cos(\phi) \pm i\rho\sin(\phi)] \\ &= \frac{2}{\omega^2\sqrt{\pi}}\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left[ik\frac{\rho^2}{2R(z)} - 2i\arctan(z/z_R)\right](x \pm iy) \\ &= \frac{1}{\sqrt{2}}\frac{1}{\omega\sqrt{\pi}}\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left[ik\frac{\rho^2}{2R(z)} - 2i\arctan(z/z_R)\right]\left(2\sqrt{2}\frac{x}{\omega} \pm i2\sqrt{2}\frac{y}{\omega}\right), \end{aligned}$$

where we used that $x = \rho\cos(\phi)$ and $y = \rho\sin(\phi)$. Now comparing with the definitions of HG_{10} and HG_{01} of equations (26) and (27) respectively, we see that since $\rho^2 = x^2 + y^2$ and $A_{10} = A_{01} = 1/\sqrt{\pi}$, we have

$$\boxed{LG_\pm(\rho,\phi,z) = \frac{1}{\sqrt{2}}[HG_{10}(x,y,z) \pm iHG_{01}(x,y,z)].}$$

3.2 [Exercise] Let $u_{nm}(x, y)$ be a basis set of the square integrable functions in $\mathbb{R}^2$. Show that:

$$\sum_{n,m=0}^{\infty} u_{nm}(x,y)u^*_{nm}(x',y') = \delta(x-x')\delta(y-y'). \tag{33}$$

H_{int}: Use bra-ket notation to write $u_{n,m}(x,y) = \langle x,y|u_{nm}\rangle$.

Using the bra-ket notation, it follows that

$$\begin{aligned} \sum_{n,m=0}^{\infty} u_{nm}(x,y)u^*_{nm}(x',y') &= \sum_{n,m=0}^{\infty} \langle x,y|u_{n,m}\rangle \langle u_{n,m}|x',y'\rangle \\ &= \langle x,y|\left(\sum_{n,m=0}^{\infty}|u_{n,m}\rangle\langle u_{n,m}|\right)|x',y'\rangle \\ &= \langle x,y|x',y'\rangle \\ &= \delta(x-x')\delta(y-y'), \end{aligned}$$

where we used the closure relation for this basis, given vy

$$\sum_{n,m=0}^{\infty}|u_{n,m}\rangle\langle u_{n,m}| = \mathbb{I}.$$

3.3 (3) [Challenge] Consider the linear polarization unit vectors rotated counter-clockwise by θ:

$$\hat{\mathbf{e}}_\theta = \cos\theta\hat{\mathbf{e}}_H + \sin\theta\hat{\mathbf{e}}_V \tag{34}$$

- Show that the vector structures used for alignment-free quantum communication,

$$\Psi_\theta(x,y) = HG_\theta(x,y)\hat{\mathbf{e}}_\theta + HG_{\theta+\frac{\pi}{2}}(x,y)\hat{\mathbf{e}}_{\theta+\frac{\pi}{2}} \tag{35}$$

$$\Phi_\theta(x,y) = HG_\theta(x,y)\hat{\mathbf{e}}_{\theta+\frac{\pi}{2}} + HG_{\theta+\frac{\pi}{2}}(x,y)\hat{\mathbf{e}}_\theta, \tag{36}$$

are rotation invariant.

We start remembering that $HG_\theta = HG_{10}\cos(\theta) + \sin(\theta)HG_{01}$ and since $\cos(\theta+\pi/2) = -\sin(\theta)$ and $\sin(\theta+\pi/2) = \cos(\theta)$, it follows that

$$\begin{aligned}
\Psi_\theta(x,y) &= [\cos(\theta)HG_{10} + \sin(\theta)HG_{01}]\,[\cos(\theta)\hat{\mathbf{e}}_H + \sin(\theta)\hat{\mathbf{e}}_V] \\
&\quad + [-\sin(\theta)HG_{10} + \cos(\theta)HG_{01}]\,[-\sin(\theta)\hat{\mathbf{e}}_H + \cos(\theta)\hat{\mathbf{e}}_V] \\
&= \cos^2(\theta)HG_{10}\hat{\mathbf{e}}_H + \cos(\theta)\sin(\theta)HG_{10}\hat{\mathbf{e}}_V + \sin(\theta)\cos(\theta)HG_{01}\hat{\mathbf{e}}_H + \sin^2(\theta)HG_{01}\hat{\mathbf{e}}_V \\
&\quad + \sin^2(\theta)HG_{10}\hat{\mathbf{e}}_H - \sin(\theta)\cos(\theta)HG_{10}\hat{\mathbf{e}}_V - \cos(\theta)\sin(\theta)HG_{01}\hat{\mathbf{e}}_H + \cos^2(\theta)HG_{01}\hat{\mathbf{e}}_V \\
&= \cos^2(\theta)HG_{10}\hat{\mathbf{e}}_H + \sin^2(\theta)HG_{01}\hat{\mathbf{e}}_V + \sin^2(\theta)HG_{10}\hat{\mathbf{e}}_H + \cos^2(\theta)HG_{01}\hat{\mathbf{e}}_V. \\
&= HG_{10}[\cos^2(\theta) + \sin^2(\theta)]\hat{\mathbf{e}}_H + HG_{01}[\cos^2(\theta) + \sin^2(\theta)]\hat{\mathbf{e}}_V \\
&= HG_{10}\hat{\mathbf{e}}_H + HG_{01}\hat{\mathbf{e}}_V,
\end{aligned}$$

which is invariant of the angle θ. Analogously

$$\begin{aligned}
\Phi_\theta(x,y) &= HG_\theta(x,y)\hat{\mathbf{e}}_{\theta+\frac{\pi}{2}} + HG_{\theta+\frac{\pi}{2}}(x,y)\hat{\mathbf{e}}_\theta, \\
&= [\cos(\theta)HG_{10} + \sin(\theta)HG_{01}]\,[-\sin(\theta)\hat{\mathbf{e}}_H + \cos(\theta)\hat{\mathbf{e}}_V] \\
&\quad + [-\sin(\theta)HG_{10} + \cos(\theta)HG_{01}]\,[\cos(\theta)\hat{\mathbf{e}}_H + \sin(\theta)\hat{\mathbf{e}}_V] \\
&= -\sin(\theta)\cos(\theta)HG_{10}\hat{\mathbf{e}}_H + \cos^2(\theta)HG_{10}\hat{\mathbf{e}}_V - \sin^2(\theta)HG_{01}\hat{\mathbf{e}}_H + \sin(\theta)\cos(\theta)HG_{01}\hat{\mathbf{e}}_V \\
&\quad - \sin(\theta)\cos(\theta)HG_{10}\hat{\mathbf{e}}_H - \sin^2(\theta)HG_{10}\hat{\mathbf{e}}_V + \cos^2(\theta)HG_{01}\hat{\mathbf{e}}_H + \cos(\theta)\sin(\theta)HG_{01}\hat{\mathbf{e}}_V,
\end{aligned}$$

which does not seems to be invariant, because the sum signs do not match. Let us consider instead

$$\begin{aligned}
\Phi_\theta(x,y) &= HG_\theta(x,y)\hat{\mathbf{e}}_{\theta+\frac{\pi}{2}} + HG_{\theta-\frac{\pi}{2}}(x,y)\hat{\mathbf{e}}_\theta, \\
&= [\cos(\theta)HG_{10} + \sin(\theta)HG_{01}]\,[-\sin(\theta)\hat{\mathbf{e}}_H + \cos(\theta)\hat{\mathbf{e}}_V] \\
&\quad + [\sin(\theta)HG_{10} - \cos(\theta)HG_{01}]\,[\cos(\theta)\hat{\mathbf{e}}_H + \sin(\theta)\hat{\mathbf{e}}_V] \\
&= -\sin(\theta)\cos(\theta)HG_{10}\hat{\mathbf{e}}_H + \cos^2(\theta)HG_{10}\hat{\mathbf{e}}_V - \sin^2(\theta)HG_{01}\hat{\mathbf{e}}_H + \sin(\theta)\cos(\theta)HG_{01}\hat{\mathbf{e}}_V \\
&\quad + \sin(\theta)\cos(\theta)HG_{10}\hat{\mathbf{e}}_H + \sin^2(\theta)HG_{10}\hat{\mathbf{e}}_V - \cos^2(\theta)HG_{01}\hat{\mathbf{e}}_H - \cos(\theta)\sin(\theta)HG_{01}\hat{\mathbf{e}}_V \\
&= HG_{10}[\cos^2(\theta) + \sin^2(\theta)]\hat{\mathbf{e}}_V - HG_{01}[\cos^2(\theta) + \sin^2(\theta)]\hat{\mathbf{e}}_H \\
&= HG_{10}\hat{\mathbf{e}}_V - HG_{01}\hat{\mathbf{e}}_H,
\end{aligned}$$

which does not depend on the angle and is invariant under rotations, so maybe occurred some sort of typo.

- Show that the polarization Stokes parameters for these vector structures are all equal to zero, if measured with large area detectors.

Consider

$$\Psi_\theta(x,y) = HG_{10}\hat{\mathbf{e}}_H + HG_{01}\hat{\mathbf{e}}_V,$$

so

$$\begin{cases} E_x = HG_{10} \\ E_y = HG_{01} \end{cases}$$

The intensity measured in the horizontal direction I_H and in the vertical direction I_V can be expressed as:

$$\begin{aligned} S_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|HG_{10}|^2 - |HG_{01}|^2)dxdy \\ &= 0, \end{aligned}$$

since both integrals are equal by relabeling the coordinates. Now let us denote $\mathbb{RE}(z)$ the real part of z, it follows that

$$\begin{aligned} S_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{RE}(HG_{10}HG_{01}^*)dxdy \\ &= 0, \end{aligned}$$

because of the orthogonality of the different Hermite-Gauss function. Finally denoting $\mathbb{IM}(z)$ the imaginary part of z.

$$\begin{aligned} S_3 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{IM}(HG_{10}HG_{01}^*)dxdy \\ &= 0, \end{aligned}$$

because the phases cancel out when we multiply the Hermite-Gauss functions, and the product has no imaginary part.

Analogously, let us consider

$$\Phi_\theta(x, y) = HG_{01}\hat{\mathbf{e}}_H - HG_{10}\hat{\mathbf{e}}_V,$$

so

$$\begin{cases} E_x = HG_{01} \\ E_y = -HG_{10} \end{cases}$$

The intensity measured in the horizontal direction I_H and in the vertical direction I_V can be expressed as:

$$\begin{aligned} S_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|HG_{01}|^2 - |-HG_{10}|^2)dxdy \\ &= 0, \end{aligned}$$

because again both integrals are equal by relabeling the coordinates. Now

$$\begin{aligned} S_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{RE}(-HG_{01}HG_{10}^*)dxdy \\ &= 0, \end{aligned}$$

because of the orthogonality of the different Hermite-Gauss function. Finally denoting $\mathbb{IM}(z)$ the imaginary part of z.

$$\begin{aligned} S_3 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{IM}(-HG_{01}HG_{10}^*)dxdy \\ &= 0, \end{aligned}$$

because the phases cancel out when we multiply the Hermite-Gauss functions, and the product has no imaginary part.