Second Quantum Computing School
HACKAQUANTUM

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1 Introduction to Applications of Quantum Computing to Quantum Chemistry

1.1 [Exercise] Obtaining expectation value of a Hamiltonian.

Given a Hamiltonian K for a 3-qubit system, we want to prepare a superposition state $|\psi\rangle$ and compute the expectation value $\langle\psi|K|\psi\rangle$.

For $n = 3$ qubits, the Hamiltonian K is defined as:

$$K = \frac{1}{3}\sum_{i<j} X_i X_j - \sum_{i=0}^{2} Z_i$$

where X_i and Z_i are the Pauli-X and Pauli-Z operators, respectively, acting on the i-th qubit. Expanding the terms for $n = 3$:

1. The first term, $\frac{1}{3}\sum_{i<j} X_i X_j$, sums over all unique pairs of qubits:

$$\frac{1}{3}(X_0X_1 + X_0X_2 + X_1X_2)$$

2. The second term, $-\sum_{i=0}^{2} Z_i$, includes the Pauli-Z operators on each qubit:

$$-(Z_0 + Z_1 + Z_2)$$

Thus, the Hamiltonian K for $n = 3$ qubits can be written as:

$$K = \frac{1}{3}(X_0X_1 + X_0X_2 + X_1X_2) - (Z_0 + Z_1 + Z_2)$$

Step 1: Prepare the State $|\psi\rangle$

Starting with all qubits in the $|0\rangle$ state, we apply a Hadamard gate H to each qubit, creating an equal superposition state:

1. **Single Qubit Transformation:** The Hadamard gate H transforms $|0\rangle$ into a superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

2. **Three-Qubit Superposition:** After applying a Hadamard gate to each qubit, the three-qubit system evolves to the state $|\psi\rangle$, a superposition of all 8 possible states:

$$|\psi\rangle = H|0\rangle \otimes H|0\rangle \otimes H|0\rangle = \frac{1}{\sqrt{8}} \sum_{x=0}^{7} |x\rangle$$

Expanding, we have:

$$|\psi\rangle = \frac{1}{\sqrt{8}} \left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

Step 2: Calculate the Expectation Value $\langle\psi|K|\psi\rangle$

To compute the expectation value $\langle\psi|K|\psi\rangle$, we separately evaluate each term in K for the state $|\psi\rangle$.

Calculation of $\langle\psi|\frac{1}{3}(X_0X_1 + X_0X_2 + X_1X_2)|\psi\rangle$

1. **Expectation of X_iX_j in $|\psi\rangle$:** In the uniform superposition state $|\psi\rangle$, each pair X_iX_j has an expectation value of 1 because the superposition is symmetric, leading to an equal probability of finding qubits in $|0\rangle$ or $|1\rangle$.

2. **Sum of Expectation Values:**

$$\langle\psi|(X_0X_1 + X_0X_2 + X_1X_2)|\psi\rangle = 3$$

Multiplying by $\frac{1}{3}$, we get:

$$\frac{1}{3} \cdot 3 = 1$$

Calculation of $\langle\psi| - (Z_0 + Z_1 + Z_2)|\psi\rangle$

1. **Expectation of Z_i in $|\psi\rangle$:** In a uniform superposition, the expectation value of each Z_i is 0 because there is an equal probability of measuring $|0\rangle$ (where Z gives $+1$) or $|1\rangle$ (where Z gives -1).

2. **Sum of Expectation Values:**

$$\langle\psi|(Z_0 + Z_1 + Z_2)|\psi\rangle = 0$$

Thus,

$$-\langle\psi|(Z_0 + Z_1 + Z_2)|\psi\rangle = 0$$

Final Result

Combining both parts, we find:

$$\langle\psi|K|\psi\rangle = 1 - 0 = 1$$

therefore, the analytical solution to this problem is

$$\boxed{\langle K\rangle = \frac{N(N-1)}{6}}$$

1.2 [Challenge] Ground state energy for molecule and spin system with Variational Quantum Algorithms and Trotterization.

1.2.1 a)

The Jordan-Wigner transformation is a method used to map fermionic operators to spin operators. Fermions follow strict anti-commutation relations:

$$\{a_i, a_j^\dagger\} = \delta_{ij},$$

where a_i and $a_j^\dagger$ are the fermionic annihilation and creation operators, respectively. When transforming these fermionic operators to spin operators (i.e. qubit mapping), we need to ensure that the anti-commutation property of fermions is preserved. To achieve this, the Jordan-Wigner transformation introduces strings of Pauli-Z matrices, so that when we act on different qubits, the operators correctly anti-commute.

1.2.2 b)

To compare the ground state energies obtained through VQE with the Hardware Efficient Ansatz, we can compute the exact ground energy value. The Hamiltonian in this case is:

$$H = -\sum_{i=1}^{N} Z_i \otimes Z_{i+1} - h \sum_{i=1}^{N} X_i.$$

This Hamiltonian can be treated by applying a Jordan-Wigner transformation, followed by both Fourier and Bogoliubov transformations, to obtain:

$$H = \sum_k \epsilon_k \left(a_k^\dagger a_k - \frac{1}{2} \right),$$

where ϵ_k is the energy of the quasi-particles and a_k and $a_k^\dagger$ are, again, the fermionic annihilation and creation operators, respectively. The dispersion relation of these particles is:

$$\epsilon_k = 2\sqrt{(h - cosk)^2 + sin^2 k},$$

thus, the ground state energy for a finite particle system is obtained by summing over all the quasi-particle energies in the Brillouin zone:

$$\boxed{E = -\frac{1}{2} \sum_k \epsilon_k}$$

where $k = 2\pi n/N$, with $n = \{0, 1, 2, \dots\}$.

1.2.3 c)

Found in Colab notebook.

2 Bose-Einstein Condensates and the Involvement in Advances for New Technologies

2.1 (1) [Exercise] Interactions between atoms and the low-energy limit

2.1.1 a)

The low energy limit is defined by the condition $kR << 1$, where k is the wave number associated with the particle's energy and R is the (finite) range of the potential $U(r)$. This means that the de Broglie wavelength $\lambda = 2\pi/k$ of the particle is much larger than the range of the potential, so the particles interact over very small distances compared to their wavelength.

For this reasons, the $l = 0$ component, related to the angular momentum, is the only relevant term, because higher values of angular momentum are suppressed at low energies due to the centrifugal barrier, which effectively prevents low-energy particles from interacting via higher partial waves.

Mathematically, solving Schrodinger's equation for the scattering problem between two particles by means of the Green's function method, we obtain the stationary solution:

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{e^{ikr}}{r} f_{\boldsymbol{k}}(\theta, \varphi)$$

where the interpretation becomes clear that after the incident wave interacts with the other wave, one part of the wave continues as a plane wave (i.e., it does not interact) and another part interacts producing a spherical wave mediated by the interaction, where

$$f_{\boldsymbol{k}}(\theta, \varphi) = -\frac{m_\mu}{2\pi\hbar^2} \int d^3x e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'} U(\boldsymbol{x}')\psi_{\boldsymbol{k}}(\boldsymbol{r}') \tag{1}$$

is the scattering amplitude. Now, we can expand the scattering amplitude in

spherical harmonics to define the partial wave expansion, taking into account that for this case the scattering is cylindrically symmetric:

$$f_{\boldsymbol{k}}(\theta) = \sum_{l=0}^{\infty}(2l+1)f_l P_l(cos\theta),$$

where

$$f_l = \frac{e^{i\delta l}sin(\delta_l)}{k}$$

is defined by comparing the formal scattering solution:

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \sum_{l=0}^{\infty}\frac{(2l+1)}{kr}P_l(cos\theta)\left[\frac{i^l}{2i}\left(e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}\right) + kf_l e^{ikr}\right]$$

with the asymptotic expression for the partial wave expansion:

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \sum_{l=0}^{\infty}\frac{i^l(2l+1)}{kr2i}\left(e^{i(kr-l\pi/2+2\delta_l)} - e^{-i(kr-l\pi/2)}\right)$$

and δ_l is the phase shift produced by the interaction potential. Taking into account that only $l = 0$ contributes, we can write

$$f_{\boldsymbol{k}}(\theta) \approx \frac{1}{k}e^{i\delta_0}sin\delta_0. \tag{2}$$

For small k, the radial wave function (for the s-wave) behaves asymptotically at large r as: $R_0(r) \approx sin(kr + \delta_0)$. At very low energies, the wave number k is small, and the scattering is dominated by long-wavelength behavior. The potential $U(r)$ only affects the wave function at short distances. Outside this range, the solution can be matched to a free particle solution, where the scattering length a emerges as a parameter.

2.1.2 b)

For small k, the phase shift δ_0 can be related to the scattering length a. The scattering length a is defined as the value of a where the asymptotic form of the

wave function behaves as if the potential were a hard-wall boundary at $r = a$. This means that at distances much larger than the range of the potential, the wave function can be written as:

$$R_0(r) \approx sin[k(r-a)]$$

This expression indicates that the scattering length a shifts the wave function outwardly proportional to a. Comparing this to the asymptotic form with a phase shift $sin(kr+\delta_0)$ (point a), we see that the two are equivalent if $kr+\delta_0 \approx k(r-1)$, which implies $\delta_0 \approx -ka$.

2.1.3 c)

The result from b) ($\delta_0 \approx -ka$), can be introduced into the scattering amplitude (Eq 2.). For the forward scattering direction $\theta = 0$, the Legendre polynomial $P_0(cos0) = 1$, therefore

$$f_{\boldsymbol{k}}(0) \approx \frac{e^{i\delta_0} sin\delta_0}{k} = \frac{e^{-ika}(-ka)}{k} = -ae^{ika}.$$

Now we can apply Eq. 1, taking into account that the influence of the potential is only on the first partial wave and is small due to the low energy limit, therefore, one can replace the wave function $\psi_{\boldsymbol{k}}(\boldsymbol{r})$ in the integral representation (Eq. 1):

$$f_{\boldsymbol{k}}(\theta) = -\frac{m_\mu}{2\pi\hbar^2}\int d^3x\ e^{i\boldsymbol{k'}\cdot\boldsymbol{r}} U(\boldsymbol{r'})$$

such that in the low energy limit $k \to 0$

$$f_{\boldsymbol{k}\to\boldsymbol{0}}(\theta) = -\frac{m_\mu}{2\pi\hbar^2}\int d^3x\ U(\boldsymbol{x'}) \qquad \& \qquad f_{\boldsymbol{k}\to 0}(0) = -a$$

finally, combining this last results we arrive to

$$f_{\boldsymbol{k}\to 0}(0) = -\frac{m_\mu}{2\pi\hbar^2}\int d^3x\ U(\boldsymbol{x'}) = -a, \quad \to \quad a = \frac{m_\mu}{2\pi\hbar^2}\int d^3x\ U(\boldsymbol{x'}).$$

Introducing the potential $U_{eff} = U_0\delta(\boldsymbol{r}-\boldsymbol{r'})$ into the expression we get

$$a = \frac{m_\mu}{2\pi\hbar^2} \int d^3x \; U_{eff} \delta(\boldsymbol{r} - \boldsymbol{r}') = \frac{m_\mu}{2\pi\hbar^2} U_0$$

with this,

$$U_0 = \frac{2\pi\hbar^2}{(m/2)} a = a \frac{4\pi\hbar^2}{m}$$

and finally,

$$\boxed{U_{eff} = a \frac{4\pi\hbar^2}{m} \delta(\boldsymbol{r} - \boldsymbol{r}')}$$

2.2 [Exercise] The Gross-Pitaevskii equation

2.2.1 a)

We are applying a many-body wave function given by

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \prod_{i=1}^{N} \phi(\boldsymbol{r}_i),$$

where,

$$\int d^3r \phi(\boldsymbol{r})^* \phi(\boldsymbol{r}) = 1$$

is the product of identical single-particle wave functions $\phi(\boldsymbol{r})$. Hence, the expectation value of the sum of one-particle operators simplifies as

$$\langle H\rangle = \left\langle \psi \left| \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + V(r_i) \right] + U_0 \sum_{i<j} \delta(\boldsymbol{r}_i - \boldsymbol{r}_j) \right| \psi \right\rangle$$
$$= N \int d^3r \ \phi^*(\boldsymbol{r}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) \right] \phi(\boldsymbol{r})$$
$$+ U_0 \sum_{i<j} \int d^3r_k \prod_{k=1}^{N} |\psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k)|^2 \delta(\boldsymbol{r}_i - \boldsymbol{r}_j).$$

The interaction term can be simplified, taking into account that when we integrate over $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ the symmetry property of the wave function makes it such that each pair contributes equally due to their indistinguishable nature. Therefore:

$$\langle \psi | \delta(\boldsymbol{r_i} - \boldsymbol{r}_j) | \psi \rangle = \int d^3r_k \prod_{k=1}^{N} |\psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)|^2 \delta(\boldsymbol{r}_i - \boldsymbol{r}_j)$$
$$= \int \left(\prod_{k=1}^{N} d^3r_k \ |\phi(\boldsymbol{r}_k)|^2 \right) \delta(\boldsymbol{r}_i - \boldsymbol{r}_j).$$

The Dirac delta function collapses the integrator over $\boldsymbol{r}_j$. Then, setting $\boldsymbol{r}_j = \boldsymbol{r}_i$:

$$\langle \psi | \delta(\boldsymbol{r_i} - \boldsymbol{r}_j) | \psi \rangle = \int d^3r_i d^3r_j |\phi(\boldsymbol{r}_i)|^2 |\phi(\boldsymbol{r}_j)|^2 \delta(\boldsymbol{r}_i - \boldsymbol{r}_j) = \int d^3r_i |\phi(\boldsymbol{r}_i)|^4,$$

the integrals over the remaining variables just contribute factors of 1 due to the normalization condition. There are $N(N-1)/2$ unique pairs (i, j) with $i < j$. Each pair contributes the same integral because of symmetry. Therefore,

$$\langle \psi | \delta(\boldsymbol{r_i} - \boldsymbol{r}_j) | \psi \rangle = U_0 \frac{N(N-1)}{2} \int d^3r |\phi(\boldsymbol{r})|^2.$$

Finally,

$$\langle H\rangle = N \int d^3r \ \phi^*(\boldsymbol{r}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) \right] \phi(\boldsymbol{r}) + U_0 \frac{N(N-1)}{2} \int d^3r |\phi(\boldsymbol{r})|^2.$$

For large N we can take $N(N-1) \approx N$:

$$\boxed{\langle H\rangle \approx N\int d^3r\ \phi^*(\boldsymbol{r})\left[-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{r})\right]\phi(\boldsymbol{r})+U_0\frac{N^2}{2}\int d^3r|\phi(\boldsymbol{r})|^2}$$

2.2.2 b)

We can introduce the condensate state wave function into the expression for the expectation value of the energy:

$$\begin{aligned}E(\psi) &= N\int d^3r\frac{\psi^*(\boldsymbol{r})}{N^{1/2}}\left[-\frac{\hbar^2}{2m}\nabla^2+V(r)\right]\frac{\psi(\boldsymbol{r})}{N^{1/2}}+\frac{1}{2}U_0N^2\int d^3r\left|\frac{\psi(\boldsymbol{r})}{N^{1/2}}\right|^2\\ &= \int d^3r\left[\frac{\hbar^2}{2m}|\nabla\psi(\boldsymbol{r})|^2+V(\boldsymbol{r})|\psi(\boldsymbol{r})|^2+\frac{1}{2}U_0|\psi(\boldsymbol{r})|^4\right].\end{aligned}$$

To find the optimal $\psi(\boldsymbol{r})$ we can minimize the functional,

$$\delta[E(\psi)-\mu N]=0.$$

Considering small variations $\delta\psi^*(\boldsymbol{r})$, and treating $\psi(\boldsymbol{r})$ and $\psi^*(\boldsymbol{r})$ as independent variables:

$$\delta E=\int d^3r\left[\frac{\hbar^2}{2m}\nabla\delta\psi^*(\boldsymbol{r})\cdot\nabla\psi(\boldsymbol{r})+V(\boldsymbol{r})\delta\psi^*(\boldsymbol{r})\psi(\boldsymbol{r})+U_0|\psi(\boldsymbol{r})|^2\delta\psi^*(\boldsymbol{r})\psi(\boldsymbol{r})\right],$$

$$\delta N=\int d^3r\delta\psi^*(\boldsymbol{r})\psi(\boldsymbol{r}).$$

With these results, the variational condition becomes,

$$\delta[E-\mu N]=\int d^3r\delta\psi^*(\boldsymbol{r})\left[-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r})+V(\boldsymbol{r})\psi(\boldsymbol{r})+U_0|\psi(\boldsymbol{r})|^2\psi(\boldsymbol{r})-\mu\psi(\boldsymbol{r})\right]=0,$$

where we apply integration by parts to perform this integral, using the identity

$$\int_V (\nabla f)\cdot(\nabla g)dV = -\int_V \nabla^2 g dV + \int_{\partial V} f(\nabla g)\cdot d\boldsymbol{S}.$$

Since $\delta\psi^*(\boldsymbol{r})$ is arbitrary, the integrand must vanish, and,

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r}) + V(\boldsymbol{r})\psi(\boldsymbol{r}) + U_0|\psi(\boldsymbol{r})|^2\psi(\boldsymbol{r}) = \mu\psi(\boldsymbol{r})}$$

2.2.3 c)

For a uniform Bose gas, first we can assume that the external potential $V(\boldsymbol{r}) = 0$, such that,

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U_0|\psi(\boldsymbol{r})|^2\right)\psi(\boldsymbol{r}) = \mu\psi(\boldsymbol{r}).$$

For a uniform gas, the condensate wave function $\psi(\boldsymbol{r}) = \psi_0$, therefore,

$$U_0|\psi_0|^2\psi_0 = \mu\psi_0, \quad \rightarrow \quad U_0|\psi_0|^2 = \mu.$$

Finally, taking into account that $\psi_0^2 = n$

$$\boxed{\mu = U_0 n}$$

For a constant potential $V(\boldsymbol{r}) = V_0$, we have,

$$(V_0 + U_0|\psi_0|^2)\psi_0 = \mu\psi_0, \quad \rightarrow \quad \boxed{\mu = V_0 + U_0 n}$$

For a position dependent potential, the assumption of a uniform $\psi(\boldsymbol{r})$ is generally invalid because the particles experience different potentials at different positions.

2.3 [Challenge] Computational project

Found in Colab notebook.

Prospects and Challenges for Quantum Machine Learning

(1) [Exercise]

Let V = $\mathbb{C}_2$ be the Hilbert space of a single qubit. Then, consider the set of objects $\{1, X\}$, where $\mathbb{1}$ is the 2×2 identity matrix and X the Pauli-X matrix. Show that these objects, which represent bit-flips, form a group.

The two matrices that we need to form a group are:

1. The identity matrix $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. The Pauli-x matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

To show that $\{\mathbb{1}, \mathbf{x}\}$ forms a group under matrix multiplication, we need to verify four properties: closure, associativity, identity, and inverses.

Here we show all the base calculations for the fallowing analysis:

$$\mathbb{1} \times \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\mathbb{1} \times X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$X \times \mathbb{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$X \times X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Closure:

We need to check that multiplying any two elements from the set results in an element within the set.

- $\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$
- $\mathbb{1} \cdot \mathbf{x} = \mathbf{x}$
- $\mathbf{x} \cdot \mathbb{1} = \mathbf{x}$
- $\sigma_x \cdot \sigma_x = \mathbb{1}$

Since all results are within $\{\mathbb{1}, \mathbf{x}\}$, the set is closed under multiplication.

Associativity:

Matrix multiplication is associative. Therefore, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ holds for all $A, B, C \in \{\mathbb{1}, \mathbf{x}\}$.

Identity Element:

The identity matrix $\mathbb{1}$ acts as the identity element since $\mathbb{1} \cdot A = A \cdot \mathbb{1} = A$ for all $A \in \{\mathbb{1}, \mathbf{x}\}$.

Inverses:

Each element must have an inverse within the set.

- The inverse of $\mathbb{1}$ is $\mathbb{1}$ itself.
- The inverse of $\mathbf{x}$ is $\mathbf{x}$, since $\mathbf{x} \cdot \mathbf{x} = \mathbb{1}$.

All group properties are satisfied; thus, $\{\mathbb{1}, \mathbf{x}\}$ forms a group under matrix multiplication. This group is isomorphic to $\mathbb{C}_2$, the cyclic group of order 2.

(2) [Exercise]

Prove that the set of all unitaries of the form $U = e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_3 Y}$ constitutes a representation of the unitary Lie group $SU(2)$.

The generators of the Lie algebra are the Pauli matrices X, Y and Z, which are given by:

$$X = \sigma_x, \quad Y = \sigma_y, \quad Z = \sigma_z.$$

The Pauli matrices have the following properties:

$$\sigma_i^2 = \mathbb{1}, \quad \sigma_i^\dagger = \sigma_i, \quad i \in \{x, y, z\},$$

and the following commutation relations:

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k,$$

where ε_{ijk} is the Levi-Civita symbol. The Pauli matrices are Hermitian matrices, which implies:

$$(e^{-i\sigma_i})^\dagger = e^{i\sigma_i}.$$

Unitary:

Starting with the set of all unitaries of the form $U = e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_3 Y}$, we see that

$$U^\dagger = (e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y})^\dagger = e^{i\phi_3 Y}e^{i\phi_2 X}e^{i\phi_1 Y},$$

and, using the commutation relations of the Pauli matrices,

$$U^\dagger U = (e^{i\phi_3 Y}e^{i\phi_2 X}e^{i\phi_1 Y})(e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y}) = \mathbb{1}$$

Therefore,

$$\boxed{U^\dagger U = \mathbb{1}}.$$

Unitary determinant:

For a matrix A, the determinant can be approximated as $\det(e^A) = e^{\mathrm{tr}(A)}$. Given the fact that the trace of the Pauli matrices is $\mathrm{tr}(\sigma_i) = 0$ for $i \in \{x, y, z\}$, the determinant of U is given by:

$$\begin{aligned}\det(U) &= \det(e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y}) \\ &= \det(e^{-i\phi_3 Y})\det(e^{-i\phi_2 X})\det(e^{-i\phi_1 Y}) \\ &= e^{\mathrm{tr}(-i\phi_3 Y)}e^{\mathrm{tr}(-i\phi_2 X)}e^{\mathrm{tr}(-i\phi_1 Y)} \\ &= e^0 e^0 e^0 \\ &= 1.\end{aligned}$$

Therefore,

$$\boxed{det(U) = 1}$$

The set of all matrices of the form $U = e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_3 Y}$ satisfies all the properties required to form a representation of the unitary Lie group $SU(2)$ since all are unitary, have determinant 1, are closed under multiplication, contain the identity, and every element has an inverse.

(3) [Challenge]

Show that the dimension D of the commutant $C^{(k)}(G)$ is determined by $D = \sum_\lambda m_\lambda^2$.

First, we start by considering a representation $R : G \to GL(A)$, where G is a finite group and A is a finite-dimensional vector space. Invoking representation theory, the vector space A can be decomposed into a direct sum or irreducible representations (irreps). If we label the irreps of G by λ, the vector space A decomposes as:

$$A = \bigoplus_\lambda m_\lambda A_\lambda,$$

where m_λ is the multiplicity of λ in the decomposition of A and A_λ is the irrep corresponding to λ.

Now, the commutant $C(k)(G)$ is the set of all linear operators on A that commute with the action of the group G. Formally, the commutant is defined as:

$$C^{(k)}(G) = \{A \in \mathcal{B}(\mathcal{H}^{\otimes k}) | [A, R(g)^{\otimes k}] = 0, \forall g \in G\}.$$

We invoke Schur's Lemma, which says that the commutant of an irreducible representation is 1-dimensional and consists only of scalar multiples of the identity. However, when the representation is not irreducible, we need to consider the multiplicity of each irreducible component. Therefore the commutant $C^{(k)}(G)$ acts as block-diagonal matrices, where each block corresponds to an irreducible component A_λ. These blocks are $m_\lambda \times m_\lambda$ matrices that represent how the multiplicities of A_λ can act independently.

Therefore, the dimension of the commutant $C^{(k)}(G)$ is determined by the structure of these block-diagonal matrices. Since the space of $m_\lambda \times m_\lambda$ has dimension m_λ^2, the total dimension D of the commutant $C^{(k)}(G)$ is the sum of the dimensions of all of these blocks:

$$\boxed{D = \sum_\lambda m_\lambda^2}.$$

This solution reflects the fact that the commutant corresponds to the degree of freedom for each irrep that appears in the decomposition of the original representation A. Furthermore, each component A_λ can act independently on its multiplicity space, and the square of the multiplicity m_λ^2 accounts for the dimension of the space of possible transformations on these components.

High Dimensional Quantum Communication with Structured Light

(1)[Exercise]

Let $HG_\theta(x, y)$ be the first order Hermite-Gauss mode rotated counter clockwise by θ. Show that:

•(a) $HG_\theta(x, y) = cos\theta HG_{10}(x, y) + sin\theta HG_{01}(x, y)$

To find the relation of the rotated Hermite-Gaussian mode $HG_{1,0}(x', y')$, we need to apply a coordinate transformation to the original mode $HG_{1,0}(x, y)$.

First, let's show explicitly the coordinate transformation we used. The rotated coordinates (x', y') are related to the original coordinates (x, y) by a rotation angle θ, as follows:

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

Now, we place this in the definition of the Hermite-Gaussian mode that we will apply the rotated coordinates to. In the rotated coordinate system, the mode $HG_{1,0}(x', y')$ becomes:

$$HG_{1,0}(x', y') = x' e^{-\frac{x'^2+y'^2}{w_0^2}}$$

Substituting the expressions for x' and y' into this equation:

$$HG_{1,0}(x', y') = (x\cos\theta + y\sin\theta) exp\left(\frac{-\left((x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2\right)}{w_0^2}\right)$$

Then, by simplifying the exponent, we obtain:

$$(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = x^2 + y^2$$

Thus, the Gaussian envelope remains unchanged, and the expression for $HG_{1,0}(x', y')$ simplifies to:

$$HG_{1,0}(x', y') = (x\cos\theta + y\sin\theta)\, e^{-\frac{x^2+y^2}{w_0^2}}$$

We recognize that:

$$xe^{-\frac{x^2+y^2}{w_0^2}} = HG_{1,0}(x, y), \quad ye^{-\frac{x^2+y^2}{w_0^2}} = HG_{0,1}(x, y)$$

Thus, we can express the rotated mode as a linear combination of $HG_{1,0}(x, y)$ and $HG_{0,1}(x, y)$:

$$HG_{1,0}(x', y') = \cos\theta\, HG_{1,0}(x, y) + \sin\theta\, HG_{0,1}(x, y)$$

This proves that the rotated Hermite-Gaussian mode $HG_\theta(x, y)$ is a linear combination of the original modes:

$$\boxed{HG_\theta(x,y) = \cos\theta\, HG_{1,0}(x,y) + \sin\theta\, HG_{0,1}(x,y)}$$

•(b) $\int HG^*_\theta(x,y)HG_\theta(x,y)dxdy = 1$

Here the conjugate of HG_θ is given by the equation:

$$HG^*_\theta = cos\theta HG^*_{10}(x,y) + sin\theta HG^*_{01}(x,y)$$

Replacing in the integral for normalization:

$$\begin{aligned}\int HG^*_\theta(x,y)HG_\theta(x,y)dxdy &= \int (cos\theta HG^*_{10}(x,y) \\ &+ sin\theta HG^*_{01}(x,y))(cos\theta HG_{10}(x,y) + sin\theta HG_{01}(x,y))dxdy \\ &= \int cos^2\theta HG^*_{10}(x,y)HG_{10}(x,y)dxdy \\ &+ \int cos\theta HG^*_{10}(x,y)sin\theta HG_{01}(x,y))dxdy \\ &+ \int sin\theta HG^*_{01}(x,y)cos\theta HG_{10}(x,y)dxdy \\ &+ \int sin^2\theta HG^*_{01}(x,y)HG_{01}(x,y)dxdy\end{aligned}$$

The orthogonality relations of the Hermite-Gauss tell us:

$$\int HG^*_{nm}HG_{kl} = \delta_{nk}\delta_{ml}$$

From this condition, we see that the second and third integrals become zero, and based on the aforementioned orthogonality of the Hermite-Gauss modes, the first and last integrals simply become $sin^2\theta + cos^2\theta = 1$. Thus:

$$\boxed{\int HG^*_\theta(x,y)HG_\theta(x,y)dxdy = 1}$$

• (c) $\int HG^*_\theta(x,y)HG_{\theta+\pi/2}(x,y)dxdy = 0$

We can express $HG_{\theta+\frac{\pi}{2}}$ in terms of HG_{10} and HG_{01}.

$$
\begin{aligned}
HG_{\theta+\frac{\pi}{2}}(x,y) &= \cos\left(\theta+\frac{\pi}{2}\right) HG_{10}(x,y) + \sin\left(\theta+\frac{\pi}{2}\right) HG_{01}(x,y) \\
&= -\sin\theta\, HG_{10}(x,y) + \cos\theta\, HG_{01}(x,y)
\end{aligned}
$$

where we have used the trigonometric identities $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ and $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, resulting in $\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta$ and $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$.

Expanding the product $HG_{\theta}(x,y)\, HG_{\theta+\frac{\pi}{2}}(x,y)$:

$$
\begin{aligned}
HG^{*}_{\theta}\, HG_{\theta+\frac{\pi}{2}} &= \left(\cos\theta\, HG^{*}_{10} + \sin\theta\, HG^{*}_{01}\right)\left(-\sin\theta\, HG_{10} + \cos\theta\, HG_{01}\right) \\
&= -\cos\theta\sin\theta\, HG^{2}_{10} + \cos^{2}\theta\, HG^{*}_{10} HG_{01} - \sin^{2}\theta\, HG^{*}_{01} HG_{10} + \sin\theta\cos\theta\, HG^{2}_{01}
\end{aligned}
$$

Now, we solve the integral with the expressions for the Hermite-Gauss modes and the orthogonality relations:

$$
\begin{aligned}
\int HG^{*}_{\theta}\, HG_{\theta+\frac{\pi}{2}}\, dx\, dy &= -\cos\theta\sin\theta \int HG^{2}_{10}\, dx\, dy + \sin\theta\cos\theta \int HG^{2}_{01}\, dx\, dy \\
&\quad + \cos^{2}\theta \int HG^{*}_{10}\, HG_{01} - \sin^{2}\theta \int HG^{*}_{01}\, HG^{10\, dx\, dy} \\
&= -\cos\theta\sin\theta + \sin\theta\cos\theta + 0 \\
&= -\cos\theta\sin\theta + \cos\theta\sin\theta \\
&= 0.
\end{aligned}
$$

Therefore:

$$
\boxed{\int HG^{*}_{\theta}(x,y)\, HG_{\theta+\frac{\pi}{2}}(x,y)\, dx\ dy = 0}
$$

This shows that HG^{*}_{θ} and $HG_{\theta+\frac{\pi}{2}}$ are orthogonal functions.

$\bullet(d)\ LG_{\pm}(r,\phi) = \frac{1}{\sqrt{2}}\left[HG_{10}(x,y) \pm i\, HG_{01}(x,y)\right]$

The expanded form of the Hermite-Gauss modes is given by:

$$
HG_{nm}(x,y,z) = \frac{C_{nm}}{w(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) e^{\left(-\frac{(x^2+y^2)}{w(z)^2}\right)} e^{\left(i\left(k\frac{x^2+y^2}{2R(z)} - (n+m+1)\zeta(z)\right)\right)}
$$

The Hermite-Gauss modes HG_{10} and HG_{01} in polar coordinates are given by:

$$HG_{10}(x,y) = Cx\, e^{-\frac{x^2+y^2}{w(z)^2}} e^{ig(x,y)} = Cr\cos\phi\, e^{-\frac{r^2}{w(z)^2}} e^{if(r,\phi)}$$

$$HG_{01}(x,y) = Cy\, e^{-\frac{x^2+y^2}{w(z)^2}} e^{ig(x,y)} = Cr\sin\phi\, e^{-\frac{r^2}{w(z)^2}} e^{if(r,\phi)}$$

Where $g(x,y)$ and $f(r,\phi)$ are functions that contain a phase in each coordinate basis and C are normalization factors, with $f(r,\phi)$ given by

$$f(r,\phi) = e^{k\frac{r^2}{2R(z)} - 2\zeta(z)}.$$

The general form of the Laguerre-Gaussian modes are:

$$LG_{pl}(r,\phi,z) = \frac{C_p^l}{w(z)} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right) e^{\left(-\frac{r^2}{w(z)^2}\right)} e^{\left(i\left(k\frac{r^2}{2R(z)} - l\phi + \psi_{p,l}(z)\right)\right)}$$

The special cases of $LG_{\pm}$ correspond to $p = 0$ and $|l| = 1$. Furthermore, $\psi_{0,\pm1}(z) = 2\zeta(z)$; this is called Gouy phase. Casting the $LG_{\pm}(r,\phi,z)$ equation with this information, the expression becomes:

$$LG_{\pm}(r,\phi,z) = \frac{1}{w(z)} \left(\frac{\sqrt{2}r}{w(z)}\right) e^{-\frac{r^2}{w(z)^2}} e^{\left(i\left(k\frac{r^2}{2R(z)} \mp \phi + 2\zeta(z)\right)\right)}.$$

For simplicity, we can express this equation as:

$$LG_{\pm}(r,\phi,z) = Cre^{-\frac{r^2}{w(z)^2}} e^{if(r,\phi)} e^{\mp\phi}$$

Finally,

$$LG_{\pm}(r,\phi,z) = Cr\cos\phi\; e^{-\frac{r^2}{w(z)^2}} e^{if(r,\phi)} \mp i\; Cr\sin\phi\; e^{-\frac{r^2}{w(z)^2}} e^{if(r,\phi)}.$$

When comparing this answer with the expressions for HG_{10} and HG_{01}, we see that:

$$\boxed{LG_{\pm}(r,\phi) = \frac{1}{\sqrt{2}}\left[HG_{10}(x,y) \pm i\, HG_{01}(x,y)\right]}$$

(2) [Exercise]

Let $u_{nm}(x, y)$ be a basis set of the square-integrable functions in R^2. Show:

$$\sum_{nm} u_n m(x, y) u^*_{nm}(x', y') = \delta(x - x')\delta(y - y')$$

Let $u_{nm}(xy) = \langle xy|nm\rangle$, with its complex conjugate being $u^*_{nm}(x', y') = \langle nm|x'y'\rangle$. We start by looking at the right-hand side of the equation:

$$\sum_{nm} u_n m(x, y) u^*_{nm}(x', y') = \sum_{nm} \langle xy|nm\rangle\langle nm|x'y'\rangle = \langle xy| \left(\sum_{nm} |nm\rangle\langle nm| \right) |x'y'\rangle.$$

The completeness relation of a basis nm tells us that:

$$\sum_{nm} |nm\rangle\langle nm| = \mathbb{1},$$

and the orthogonality tells us that:

$$\langle nm|nm\rangle = \delta_{nm}.$$

Using these properties, we see that:

$$\langle xy|\mathbb{1}|x'y'\rangle = \langle xy|x'y'\rangle = \delta(x - x')\delta(y - y').$$

Putting it all together:

$$\boxed{\sum_{nm} u_n m(x, y) u^*_{nm}(x', y') = \delta(x - x')\delta(y - y')}$$

(3)[Challenge]

Consider the linear polarization unit vectors rotated counter-clockwise by θ:

$$\hat{e}_\theta = cos\theta\hat{e}_H + sin\theta\hat{e}_V$$

• Show that the vector structures used for alignment-free quantum communication,

$$\psi_\theta(x,y) = HG_\theta(x,y)\hat{e}_\theta + HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta+\pi/2}$$
$$\psi_\theta(x,y) = HG_\theta(x,y)\hat{e}_{\theta+\pi/2} + HG_{\theta+\pi/2(x,y)}\hat{e}_\theta$$

are rotation invariant.

To begin, we calculate the first HG-mode:

$$HG_\theta(x,y)\hat{e}_\theta = HG_\theta(x,y)\cos(\theta)\hat{e}_H + HG_\theta(x,y)\sin(\theta)\hat{e}_V.$$

With our previous derivation of $HG_\theta(x,y)$ in terms of x, y, and θ:

$$HG_\theta(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}\left(\cos(\theta)x + \sin(\theta)y\right),$$

we substitute this expression into the equation for $HG_\theta(x,y)\hat{e}_\theta$:

$$HG_\theta(x,y)\hat{e}_\theta = e^{-\frac{x^2+y^2}{w_0^2}}\left(\cos(\theta)x + \sin(\theta)y\right)\left(\cos(\theta)\hat{e}_H + \sin(\theta)\hat{e}_V\right).$$

Distributing the terms:

$$HG_\theta(x,y)\hat{e}_\theta = e^{-\frac{x^2+y^2}{w_0^2}}\left(\cos^2(\theta)x\hat{e}_H + \cos(\theta)\sin(\theta)y\hat{e}_H + \cos(\theta)\sin(\theta)x\hat{e}_V + \sin^2(\theta)y\hat{e}_V\right).$$

This expression is the explicit form of $HG_\theta(x,y)\hat{e}_\theta$ in terms of x, y, and θ, along with the polarization vectors $\hat{e}_H$ and $\hat{e}_V$.

Now, for the second Hermite-Gauss term:

$$HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta+\pi/2},$$

we begin with the polarization unit vector:

$$\hat{e}_{\theta+\pi/2} = \cos(\theta+\pi/2)\hat{e}_H + \sin(\theta+\pi/2)\hat{e}_V.$$

Using the trigonometric identities (from before):

$$\cos(\theta + \pi/2) = -\sin(\theta), \quad \sin(\theta + \pi/2) = \cos(\theta),$$

we get:

$$\hat{e}_{\theta+\pi/2} = -\sin(\theta)\hat{e}_H + \cos(\theta)\hat{e}_V.$$

The Hermite-Gauss mode $HG_{\theta+\pi/2}(x, y)$ is:

$$HG_{\theta+\pi/2}(x, y) = e^{-\frac{x^2+y^2}{w_0^2}} \left(\cos(\theta + \pi/2)x + \sin(\theta + \pi/2)y\right).$$

Substituting the trigonometric values:

$$HG_{\theta+\pi/2}(x, y) = e^{-\frac{x^2+y^2}{w_0^2}} \left(-\sin(\theta)x + \cos(\theta)y\right).$$

Now, substituting both the Hermite-Gauss mode and the polarization vector into the original equation:

$$HG_{\theta+\pi/2}(x, y)\hat{e}_{\theta+\pi/2} = e^{-\frac{x^2+y^2}{w_0^2}} \left(-\sin(\theta)x + \cos(\theta)y\right)\left(-\sin(\theta)\hat{e}_H + \cos(\theta)\hat{e}_V\right).$$

Expanding the terms, we have:

$$HG_{\theta+\pi/2}(x, y)\hat{e}_{\theta+\pi/2} = e^{-\frac{x^2+y^2}{w_0^2}} \left(\sin^2(\theta)x\hat{e}_H - \sin(\theta)\cos(\theta)y\hat{e}_H - \sin(\theta)\cos(\theta)x\hat{e}_V + \cos^2(\theta)y\hat{e}_V\right).$$

This is the fully expanded expression of $HG_{\theta+\pi/2}(x, y)\hat{e}_{\theta+\pi/2}$ in terms of x, y, and θ.

From our previous calculations:

$$HG_\theta(x, y)\hat{e}_\theta = e^{-\frac{x^2+y^2}{w_0^2}} \left(\cos^2(\theta)x\hat{e}_H + \cos(\theta)\sin(\theta)y\hat{e}_H + \cos(\theta)\sin(\theta)x\hat{e}_V + \sin^2(\theta)y\hat{e}_V\right),$$

and

$$HG_{\theta+\pi/2}(x, y)\hat{e}_{\theta+\pi/2} = e^{-\frac{x^2+y^2}{w_0^2}} \left(\sin^2(\theta)x\hat{e}_H - \sin(\theta)\cos(\theta)y\hat{e}_H - \sin(\theta)\cos(\theta)x\hat{e}_V + \cos^2(\theta)y\hat{e}_V\right).$$

We add these two expressions to obtain the value of $\psi_\theta(x, y)$:

$$\Psi_\theta(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}\left(\cos^2(\theta)x\hat{e}_H + \cos(\theta)\sin(\theta)y\hat{e}_H + \cos(\theta)\sin(\theta)x\hat{e}_V + \sin^2(\theta)y\hat{e}_V\right)$$
$$+e^{-\frac{x^2+y^2}{w_0^2}}\left(\sin^2(\theta)x\hat{e}_H - \sin(\theta)\cos(\theta)y\hat{e}_H - \sin(\theta)\cos(\theta)x\hat{e}_V + \cos^2(\theta)y\hat{e}_V\right).$$

Now, we simplifying the terms, beginning with $\hat{e}_H$:

$$\left(\cos^2(\theta)x + \sin^2(\theta)x\right)\hat{e}_H = x\hat{e}_H,$$

and

$$\left(\cos(\theta)\sin(\theta)y - \sin(\theta)\cos(\theta)y\right)\hat{e}_H = 0.$$

For the terms with $\hat{e}_V$:

$$\left(\cos(\theta)\sin(\theta)x - \sin(\theta)\cos(\theta)x\right)\hat{e}_V = 0,$$

and

$$\left(\sin^2(\theta)y + \cos^2(\theta)y\right)\hat{e}_V = y\hat{e}_V.$$

The final expression for $\Psi_\theta(x,y)$ is, thus:

$$\boxed{\Psi_\theta(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}\left(x\hat{e}_H + y\hat{e}_V\right).}$$

It is clear from this expression that $\Psi_\theta(x,y)$ doesn't depend on the angle, so it is rotationally invariant.

• Show that the polarization Stokes parameters for these vector structures are all equal to zero, if measured with large area detectors.

In the previous point, we showed that the field $\psi_\theta(x,y)$ simplifies to a rotation-invariant form:

$$\psi(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}\left(x\,\hat{e}_H + y\,\hat{e}_V\right)$$

From there, we can identify the horizontal E_H and vertical E_v components:

$$E_H(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}x$$
$$E_V(x,y) = e^{-\frac{x^2+y^2}{w_0^2}}y$$

Calculating the Stokes parameters S_0, S_1, S_2, and S_3 for this field when measured with large area detectors, these parameters are defined as:

$$\begin{aligned} S_0 &= \int \left(|E_H(x,y)|^2 + |E_V(x,y)|^2\right) dx\, dy \\ S_1 &= \int \left(|E_H(x,y)|^2 - |E_V(x,y)|^2\right) dx\, dy \\ S_2 &= 2\int \mathrm{Re}\left[E_H(x,y)E_V^*(x,y)\right] dx\, dy \\ S_3 &= 2\int \mathrm{Im}\left[E_H(x,y)E_V^*(x,y)\right] dx\, dy \end{aligned}$$

Intensity S_0

$$\begin{aligned} S_0 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|E_H(x,y)|^2 + |E_V(x,y)|^2\right) dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{2(x^2+y^2)}{w_0^2}} \left(x^2+y^2\right) dx\, dy \end{aligned}$$

Switching to polar coordinates:

Let $x = r\cos\phi$, $y = r\sin\phi$, so $x^2 + y^2 = r^2$, and $dx\, dy = r\, dr\, d\phi$,

and substituting into the integral:

$$S_0 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^2 \cdot r\, dr\, d\phi = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^3 dr\, d\phi.$$

This integral is of the form:

$$\int_0^{\infty} r^n e^{-ar^2} dr = \frac{1}{2} a^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Thus, for $n = 3$ and $a = \frac{2}{w_0^2}$:

$$\int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^3 dr = \frac{1}{2}\left(\frac{2}{w_0^2}\right)^{-2} \Gamma(2) = \frac{w_0^4}{8}.$$

This yields:

$$S_0 = \int_0^{2\pi} \left(\frac{w_0^4}{8}\right) d\phi = \frac{w_0^4}{8} \cdot 2\pi = \frac{\pi w_0^4}{4}.$$

$$\boxed{S_0 = \frac{\pi w_0^4}{4}}$$

Linear Polarization. S_1

$$\begin{aligned} S_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|E_H(x,y)|^2 - |E_V(x,y)|^2\right) dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{2(x^2+y^2)}{w_0^2}} \left(x^2 - y^2\right) dx\, dy \end{aligned}$$

In polar coordinates, $x^2 - y^2$:

$$x^2 - y^2 = r^2\cos^2\phi - r^2\sin^2\phi = r^2\cos 2\phi.$$

Substituting again into the integral:

$$S_1 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^2\cos 2\phi \cdot r\, dr\, d\phi = \int_0^{2\pi} \cos 2\phi\, d\phi \int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^3 dr,$$

we see that one of the integrals is zero:

$$\int_0^{2\pi} \cos 2\phi\, d\phi = 0$$

Therefore,

$$S_1 = (0)\left(\frac{w_0^4}{8}\right) = 0$$

$$\boxed{S_1 = 0}.$$

Linear Polarization at $\pm 45°$. S_2

$$\begin{aligned} S_2 &= 2 \int \mathrm{Re}\left[E_H(x,y) E_V^*(x,y)\right] dx\, dy \\ &= 2 \int E_H(x,y) E_V(x,y) dx\, dy \end{aligned}$$

Using the product:

$$E_H(x,y)E_V(x,y) = \left(e^{-\frac{x^2+y^2}{w_0^2}}\, x\right)\left(e^{-\frac{x^2+y^2}{w_0^2}}\, y\right) = e^{-\frac{2(x^2+y^2)}{w_0^2}}\, xy$$

and expressing xy in polar coordinates:

$$xy = r^2 \cos\phi \sin\phi = \frac{r^2}{2} \sin 2\phi.$$

Once again, substituting into the integral:

$$S_2 = 2\int_0^{2\pi} \int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} \frac{r^2}{2} \sin 2\phi \cdot r\, dr\, d\phi = \int_0^{2\pi} \sin 2\phi\, d\phi \int_0^{\infty} e^{-\frac{2r^2}{w_0^2}} r^3 dr$$

Again, the angular integral yields zero:

$$\int_0^{2\pi} \sin 2\phi\, d\phi = 0.$$

Thus, we find that S_2 is equal:

$$S_2 = (0)\left(\frac{w_0^4}{8}\right) = 0$$

$$\boxed{S_2 = 0}$$

Circular Polarization. S_3

$$S_3 = 2 \int \mathrm{Im}\left[E_H(x,y) E_V^*(x,y)\right] dx\, dy$$

Since $E_H(x,y)$ and $E_V(x,y)$ are real functions, their product is real, and thus the imaginary part is zero:

$$\mathrm{Im}\left[E_H(x,y) E_V^*(x,y)\right] = 0$$

Now, computing S_3:

$$S_3 = 2 \int 0\, dx\, dy = 0$$

Which results in:

$$\boxed{S_3 = 0}$$

Based on these results, we arrive at these conclusions:

- No Net Polarization: The parameters S_1, S_2, and S_3 are all zero, meaning there is no preferred polarization direction in the field.
- Isotropic Intensity Distribution: The non-zero S_0 represents the total intensity, which is finite and positive.
- Consistency with Rotation Invariance: The zero values of S_1, S_2, and S_3 are consistent with the rotation invariance shown in Part 1, confirming that the field is not polarized when averaged over the entire plane.