Bose-Einstein Condensates and the Involvement in Advances for New Technologies.

(1) [Exercise] Interactions between atoms and the low-energy limit.

a

Considering l the momentum of a particle, it's impact parameter is $\frac{l}{\hbar k}$. For the low limit energy every particle with positive angular momemtum misses the potencial, which means $\frac{l}{\hbar k} > R$ for every $l > 0$, so $1 > R\hbar k$. The $l = 0$ is the most important contribution since it always is affected by the potential.

b

The s-wave scattering length is the dominant parameter in low-energy two-body interactions, defining an effective range over which over which two particles influence each other.

c

To solve the exercise, we need to find the constant U_0 for the effective potential $U_{\text{eff}}(r, r') = U_0\delta(r - r')$ such that the scattering length a matches the one calculated using the Born approximation.
The scattering length a is given by:

$$a = \frac{m_r}{2\pi\hbar^2}\int d^3r\, U(r)$$

The effective potential is given by:

$$U_{\text{eff}}(r, r') = U_0\delta(r - r')$$

This potential acts as a contact interaction, meaning it only has a non-zero value when $r = r'$.
For the effective potential to produce the same scattering length a, we can substitute U_{eff} into the scattering length formula. Since U_{eff} is a delta function, we have:

$$a = \frac{m_r}{2\pi\hbar^2}\int d^3r\, U_{\text{eff}}(r, r) = \frac{m_r}{2\pi\hbar^2}U_0$$

Now, we can solve for U_0 by equating this expression to the given scattering length a:

$$U_0 = \frac{2\pi\hbar^2 a}{m_r}$$

(2) [Exercise] The Gross-Pitaevskii equation.

a

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + V(\vec{r}_i)\right] + U_0\sum_{i<j}\delta(\vec{r}_i - \vec{r}_j)$$

and

$$\Psi(\vec{r}_i, ..., \vec{r}_n) = \prod_{i=1}^{N} \phi(\vec{r}_i)$$

then

$$\langle H \rangle = \int d\vec{r}_i......d\vec{r}_N \prod_{i-1}^{N} \phi(\vec{r}_i) \left\{ \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + V(\vec{r}_i) \right] + U_0 \sum_{i<j} \delta(\vec{r}_i - \vec{r}_j) \right\} \prod_{l=1}^{N} \phi(\vec{r}_l)$$

$$\langle H \rangle = \sum_{i=1}^{N} (\mathcal{I}_i + \mathcal{I}'_j) + \sum_{j<k} \mathcal{I}_{i,k}$$

Now we can consider each term separately

$$\begin{aligned}
\mathcal{I}_j &= \int d\vec{r}_1...d\vec{r}_N \prod_{i-1}^{N} \phi(\vec{r}_i) p_j^2/2m\phi(\vec{r}_l) \\
&= \int d\vec{r}_1 |\phi(\vec{r}_1)|^2 \cdot ... \cdot \int d\vec{r}_j \phi^*(\vec{r}_j) \frac{p_j^2}{2m} \phi(\vec{r}_j) \cdot ... \cdot \int d\vec{r}_N |\phi(\vec{r}_N)|^2 \\
&= \int d\vec{r}_j \phi^*(\vec{r}_j) \frac{p_j^2}{2m} \phi(\vec{r}_j) \\
&= \int d\vec{r}_j \phi^*(\vec{r}_j) \frac{\hbar^2 \nabla_j^2}{2m} \phi(\vec{r}_j)
\end{aligned}$$

Analogously we have that

$$\begin{aligned}
\mathcal{I}'_j &= \int_N d\vec{r}_1...d\vec{r}_n \prod_{i}^{N} \phi^*(\vec{r}_i) V(\vec{r}_j) \phi(\vec{r}_i) \\
&= \int_N d\vec{r}_j V(\vec{r}_j) |\phi(\vec{r}_j)|^2
\end{aligned}$$

Now, for the last term

$$\begin{aligned}
\mathcal{I}''_{j,k} &= \int d\vec{r}_1...d\vec{r}_n \prod_{i,l} \phi^*(\vec{r}_i) U_0 \delta(\vec{r}_i - -\vec{r}_k) \phi(\vec{r}_l) \\
&= U_0 \int d\vec{r}_j d\vec{r}_k \phi^*(\vec{r}_j) \phi^*(\vec{r}_k) \delta(\vec{r}_i - -\vec{r}_k) \phi(\vec{r}_j) \phi(\vec{r}_k) \\
&= U_0 \int d\vec{r}_j \phi^*(\vec{r}_j) \phi^*(\vec{r}_j) \phi(\vec{r}_j) \phi(\vec{r}_j) \\
&= U_0 \int d\vec{r}_j |\phi(\vec{r}_j)|^4
\end{aligned}$$

than

$$\begin{aligned}\langle H\rangle &= \sum_{j=1}^{N} d\vec{r}_j \phi^*(\vec{r}_j)\left[\frac{p_i^2}{2m}+V(\vec{r}_i)\right]+\sum_{k<j}\int \vec{r}_j|\phi(\vec{r}_j)|^4 \\ &= \sum_{j=1}^{N} d\vec{r}_j \phi^*(\vec{r}_j)\left[\frac{p_i^2}{2m}+V(\vec{r}_i)+(j-1)|\phi(\vec{r}_j)|^2\right]\phi(\vec{r}_j)\end{aligned}$$

b

We have that

$$E[\Psi]=\int\int\int dV\left[\frac{\hbar^2}{2m}|\nabla\Psi(\vec{r})|^2+V(\vec{r})|\Psi(\vec{r})|^2+\frac{1}{2}u_0|\Psi(\vec{r})|^4-\right]$$

and also:

$$\mathcal{N}[\Psi]=\int\int\int dV|\Psi(\vec{r})|^2$$

let μ being de chemical potential, hence:

$$\delta E-\mu\delta\mathcal{N}=0\leftrightarrow\delta\int\int\int d^3V\left[\frac{\hbar^2}{2m}|\nabla\Psi(\vec{r})|^2+V(\vec{r})|\Psi(\vec{r})|^2+\frac{1}{2}u_0|\Psi(\vec{r})|^4-\mu|\Psi(\vec{r})|^2\right]=0$$

let's take:

$$\overline{\Psi_\alpha}(\vec{r})=\overline{\Psi}(\vec{r})+\alpha\eta(\vec{r}),$$

where $\overline{\Psi}(\vec{r})$ stands for the optimal function and $\eta(\vec{r})$ is smoth and vanishes in the boundary of the integrating region, therefore; defining $\mathcal{I}(\Psi,\partial_x\Psi,\partial_y\Psi,\partial_z\Psi,x,y,z)$ as the integrating function:

$$0=\frac{d}{d\alpha}(E-\mu\mathcal{N})=\int\int\int dV[\partial_\alpha\overline{\Psi_\alpha}\ \ \partial\overline{\Psi_\alpha}+\sum_{u=x,y,z}\partial_\alpha(\partial_u\overline{\Psi}_\alpha)\partial_{\partial_u\overline{\Psi}_\alpha}]\mathcal{I},$$

since

$$\int du\ \ \partial_{\partial_u\overline{\Psi}_\alpha}\ \ \mathcal{I}\ \ \partial_\alpha(\partial_u\overline{\Psi}_\alpha)=-\int du\ \ \partial_u(\partial_{\partial_u\overline{\Psi}_\alpha}\ \ \mathcal{I})\ \ \partial_\alpha\overline{\Psi}_\alpha,$$

which is obtained by integration by parts, we could write the minization criteria as:

$$\begin{aligned}0&=\int\int\int dV[\partial_\alpha\overline{\Psi}_\alpha\partial_{\overline{\Psi_\alpha}}\ \ \mathcal{I}\ \ -\sum_{u=x,y,z}\partial_u(\partial_{\partial_u\overline{\Psi}_\alpha}\mathcal{I})\partial_\alpha\overline{\Psi}_\alpha] \\ &=\int\int\int dV[\eta(\vec{r})\partial_{\overline{\Psi_\alpha}}\ \ \mathcal{I}\ \ -\sum_{u=x,y,z}\partial_u(\partial_{\partial_u\overline{\Psi}_\alpha}\mathcal{I})\eta(\vec{r})] \\ &=\int\int\int dV[\partial_{\overline{\Psi_\alpha}}\ \ \mathcal{I}\ \ -\sum_{u=x,y,z}\partial_u(\partial_{\partial_u\overline{\Psi}_\alpha}\mathcal{I})]\eta(\vec{r})\end{aligned}$$

by the arbitraryness of $\eta(\vec{r})$, we could ensure the minimization criteria as:

$$[\partial_{\overline{\Psi}} - \partial_x(\partial_{\partial_x \overline{\Psi}^*}) - \partial_y(\partial_{\partial_y \overline{\Psi}^*}) - \partial_y(\partial_{\partial_y \overline{\Psi}^*})]\mathcal{I} = 0$$

evaluating:

$$\begin{aligned}
\partial_{\overline{\Psi}}\mathcal{I} &= \partial_{\overline{\Psi}}[\frac{\hbar^2}{2m}|\nabla\Psi(\vec{r})|^2 + V((\vec{r})\Psi\overline{\Psi} + \frac{1}{2}u_0\Psi\overline{\Psi}\Psi\overline{\Psi} - \mu\Psi\overline{\Psi}] \\
&= V(\vec{r})\Psi + u_0\Psi\overline{\Psi}\Psi - \mu\Psi \\
&= V(\vec{r})\Psi + u_0|\Psi|^2\Psi - \mu\Psi \\
\partial_u\partial_{\partial_u\overline{\Psi}}\mathcal{I} &= \partial_u\left\{\partial_{\partial_u\overline{\Psi}}\left[\frac{\hbar^2}{2m}(\partial_x\Psi\partial_x\overline{\Psi} + \partial_y\Psi\partial_y\overline{\Psi} + \partial_z\Psi\partial_z\overline{\Psi} + ...)\right]\right\} \\
&= \partial_u\left(\frac{\hbar^2}{2m}\partial_u\Psi\right) \\
&= \frac{\hbar^2}{2m}\partial_u^2\Psi.
\end{aligned}$$

Finally, resulting in the time-independent GPE:

$$(V(\vec{r} + u_0|\Psi|^2 - \mu)\Psi - \frac{\hbar^2}{2m}\partial_x^2 - \frac{\hbar^2}{2m}\partial_y^2 - \frac{\hbar^2}{2m}\partial_z^2 = 0$$

$$(V(\vec{r} + u_0|\Psi|^2 - \mu)\Psi - \frac{\hbar^2}{2m}\nabla^2\Psi = 0$$

c

By Thomas-Fermi approximation, $\frac{-\hbar^2\nabla^2}{2m} \approx 0$ we have

$$\begin{aligned}
\mu\Psi(\vec{r}) &= \left[\left(\frac{-\hbar^2\nabla^2}{2m}\right)V(\vec{r}) + u_0|\Psi|^2\right]\Psi((\vec{r})), \\
\mu &= V(\vec{r}) + u_0|\Psi|^2
\end{aligned}$$

So $V(\vec{r}) = \mu - u_0|\Psi|^2$, so in the region where $|\Psi|^2$ is uniform $|\Psi|^2 = \mathcal{P}_0$, so:

$$V(\vec{r}) = \mu - u_0\mathcal{P}_0,$$

in such way that:

$$i\hbar\partial_t\Psi = \left[\frac{-\hbar^2\nabla^2}{20} + (\mu - u_0\mathcal{P}_0) + u_0|\Psi|^2\right]\Psi(\vec{r})$$

Prospects and Challenges for Quantum Machine Learning.

(1) [Exercise] Let $V = \mathbb{C}^2$ be the Hilbert space of a single qubit. Then, consider the set of objects $\{\mathbb{1}, X\}$, where $\mathbb{1}$ is the 2×2 identity matrix and X the Pauli-X matrix. Show that these objects, which represent bit-flips, form a group. .

Step 1: Definition of the matrices

- The 2×2 identity matrix 1 is:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- The Pauli-X matrix X is:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Step 2: Proving Closure
We need to check if the product of any two elements in $\{1, X\}$ is still in $\{1, X\}$:

$$\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$$

$$\mathbb{1} \cdot X = X$$

$$X \cdot \mathbb{1} = X$$

$$X \cdot X = \mathbb{1}$$

Since the result of any multiplication is either 1 or X, the set is closed under matrix multiplication.
Step 3: Proving Associativity
Matrix multiplication is known to be associative for all matrices of the same dimensions. Since both 1 and X are 2×2 matrices, it follows that for any $A, B, C \in \{1, X\}$, we have:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

Therefore, the associativity axiom is satisfied for all elements in $\{\mathbb{1}, X\}$ without needing to check individual cases.
Step 4: Identity element

$$\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$$

$$\mathbb{1} \cdot X = X$$

$$X \cdot \mathbb{1} = X$$

Thus, $\mathbb{1}$ is the identity element.
Step 5: Inverse
An inverse of an element A in a group is an element A^{-1} such that
$A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{1}$.

$$\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$$

$$X \cdot X = \mathbb{1}$$

Therefore, $\mathbb{1}$ is its own inverse, and X is also its own inverse.
Conclusion
Since the set $\{\mathbb{1}, X\}$ satisfies the closure, associativity, identity, and inverse axioms, as seen in steps 1, 2, 3, 4 and 5; it forms a group.

(2) [Exercise] Prove that the set of all unitaries of the form

$$U = e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}$$

constitutes a representation of the unitary Lie group $SU(2)$.

For any Pauli matrix $\sigma_j^2 = \mathbb{1}$, then $\sigma_j^{2k} = \mathbb{1}$, with $k \in \mathbb{N}$, $\sigma_j^{2k+1} = \sigma_j$ hence:

$$e^{i\alpha\sigma_j} = \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_j)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_j)^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_j)^{2n+1}}{(2n+1)!}$$

$$e^{i\alpha\sigma_j} = \sum_{n=0}^{\infty} \frac{i^{2n}\alpha^{2n}}{2n!} + i\sigma_j \sum_{n=0}^{\infty} \frac{i^{2n}\alpha^{2n+1}}{(2n+1)!}$$

$$e^{i\alpha\sigma_j} = \sum_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n}}{2n!} + i\sigma_j \sum_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n+1}}{(2n+1)!}$$

$$e^{i\alpha\sigma_j} = \cos\alpha + i\sigma_j \sin\alpha,$$

hence

$$U = e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y} = (\cos\phi_3 - iY\sin\phi_3)(\cos\phi_2 - iX\sin\phi_2)(\cos\phi_1 - iY\sin\phi_1)$$

$$\begin{aligned} U = {} & \cos\phi_1\cos\phi_2\cos\phi_3 \cdot \mathbb{1} \\ & - iY(\sin\phi_3\cos\phi_2\cos\phi_1 + \cos\phi_3\cos\phi_2\sin\phi_1) \\ & - iX(\cos\phi_3\sin\phi_2\cos\phi_1) \\ & + iYXY(\sin\phi_3\sin\phi_2\sin\phi_1) \\ & - Y^2(\sin\phi_3\cos\phi_2\sin\phi_1) \\ & - YX(\sin\phi_3\sin\phi_2\cos\phi_1) \\ & - XY(\cos\phi_3\sin\phi_2\sin\phi_1) \end{aligned}$$

now using that $YXY = -X$, $XY = iZ = -YX$ we have

$$\begin{aligned} U = {} & \mathbb{1}\cos\phi_2(\cos\phi_3\cos\phi_1 - \sin\phi_3\sin\phi_1) \\ & - iY\cos\phi_2(\sin\phi_3\cos\phi_1 + \cos\phi_3\sin\phi_1) \\ & - iX\sin\phi_2(\cos\phi_3\cos\phi_1 + \sin\phi_3\sin\phi_1) \\ & - iZ\sin\phi_2(\cos\phi_3\sin\phi_1 - \sin\phi_3\cos\phi_1) \end{aligned}$$

Now, using the expression of $\sin a \pm b$ anda $\cos a \pm b$:

$$
\begin{aligned}
U = &\mathbb{1} \cos\phi_2 \cos(\phi_3 + \phi_1) \\
&- iY \cos\phi_2 \sin(\phi_3 + \phi_1) \\
&- iX \sin\phi_2 \cos(\phi_1 - \phi_3) \\
&- iZ \sin\phi_2 \sin(\phi_1 - \phi_3)
\end{aligned}
$$

Which proves that for the right choice of $\phi_1, \phi_2, \phi 3$, U generate any element of $SU(2)$ since if $A \in SU(2)$ then exist (a, b, c, d) real such that $a^2 + b^2 + c^2 + d^2 = 1$and
$A = a\mathbb{I} + i(bX + cY + dZ)$.
For the unitary matrix U, $a = \cos\phi_2 \cos(\phi_1 + \phi_3)$, $b = \cos\phi_2 \sin(\phi_1 + \phi_3)$,
$c = \sin\phi_2 \cos(\phi_1 - \phi_3)$, $d = \sin\phi_2 \sin(\phi_1 - \phi_3)$
so, it's easy to see that

$$
\begin{aligned}
a^2 + b^2 + c^2 + d^2 = &\cos^2\phi_2 \cos^2(\phi_1 + \phi_3) + \\
&\cos^2\phi_2 \sin^2(\phi_1 + \phi_3) + \\
&\sin^2\phi_2 \cos^2(\phi_1 + \phi_3) + \\
&\sin^2\phi_2 \sin^2(\phi_1 + \phi_3) = 1
\end{aligned}
$$

solving that the four parameters satisfies the constraint, representating 3 free parameters

(3) [Exercise] Show that the dimension D of the commutant $C(k)(G)$, outlined in Definition 8 of the lecture notes, is determined by $D = \sum_\lambda m_\lambda^2$.

Suppose the representation associated with the commutant (R(g)) is completely reducible, then we may write:

$$
R(\mathbf{g}) \simeq \bigoplus_\lambda I_{m_\lambda} \otimes R_\lambda(\mathbf{g})
$$

for (R_λ), an irreducible representation, with (m_λ) its multiplicity, and (I_{m_λ}) being the identity matrix of size ($m_\lambda \times m_\lambda$).
Now, for $(d_\lambda = \dim(R_\lambda(\mathbf{g}))$), the elements in the commutant must be of the form:

$$
A = \bigoplus_\lambda A_\lambda \otimes I_{d_\lambda}
$$

for an arbitrary ($m_\lambda \times m_\lambda$) matrix (A_λ) (which commutes with (I_{d_λ})). Then we have (m_λ^2) degrees of freedom to choose each (A_λ), so the degrees of freedom in (A) are the sum of all of them. Thus,

$$
D = \sum_\lambda m_\lambda^2.
$$

High Dimensional Quantum Communication with Structured Light.

(1) [Exercise] Let $HG_\theta(x, y)$ be the first order Hermite-Gauss mode rotated counter- clockwise by θ. Show that.

By definition

$$HG_{l,m}(x,y) = \sqrt{\frac{2}{\pi w_0^2 2^{l+m} l! m!}} \mathrm{H}_l(\frac{\sqrt{2}}{w_0}x)\mathrm{H}_m(\frac{\sqrt{2}}{w_0}y)\exp\left[-(r/w_0)^2\right]$$

which is an orthonormal basis, due to the orthogonality of hermite polynomials. Therefore

$$HG_{1,0}(x,y) = \sqrt{\frac{1}{\pi w_0^2}} \mathrm{H}_1(\frac{\sqrt{2}}{w_0}x)\mathrm{H}_0(\frac{\sqrt{2}}{w_0}y)\exp\left[-(r/w_0)^2\right]$$

$$HG_{0,1}(x,y) = \sqrt{\frac{1}{\pi w_0^2}} \mathrm{H}_0(\frac{\sqrt{2}}{w_0}x)\mathrm{H}_1(\frac{\sqrt{2}}{w_0}y)\exp\left[-(r/w_0)^2\right]$$

where $\mathrm{H}_0(u) = 1$ and $\mathrm{H}_1(u) = 2u$.

a. $HG_\theta(x, y) = cos\theta HG_{1,0}(x, y) + sin\theta HG_{0,1}(x, y)$

by the definition

$$HG_\theta(x,y) = HG_{1,0}(xcos\theta + ysin\theta, ycos\theta - xsin\theta)$$

thus

$$\begin{aligned}
HG_\theta(x,y) &= \sqrt{\frac{1}{\pi w_0^2}} \mathrm{H}_1(\frac{\sqrt{2}}{w_0}xcos\theta + ysin\theta)\mathrm{H}_0(\frac{\sqrt{2}}{w_0}ycos\theta - xsin\theta)\exp\left[-(r/w_0)^2\right] \\
&= \sqrt{\frac{1}{\pi w_0^2}} 2(\frac{\sqrt{2}}{w_0}(xcos\theta + ysin\theta))\exp\left[-(r/w_0)^2\right] \\
&= cos\theta\sqrt{\frac{1}{\pi w_0^2}} 2(\frac{\sqrt{2}}{w_0}x)\exp\left[-(r/w_0)^2\right] + sin\theta\sqrt{\frac{1}{\pi w_0^2}} 2(\frac{\sqrt{2}}{w_0}y)\exp\left[-(r/w_0)^2\right] \\
&= cos\theta HG_{1,0}(x,y) + sin\theta HG_{0,1}(x,y)
\end{aligned}$$

b. $\int dxdyHG^*_\theta(x,y)HG_\theta(x,y) = 1$

$$\begin{aligned}\int dxdyHG^*_\theta(x,y)HG_\theta(x,y) &= \int dxdy\left[cos\theta HG^*_{1,0}(x,y) + sin\theta HG^*_{0,1}(x,y)\right]\times\\ &\qquad [cos\theta HG_{1,0}(x,y) + sin\theta HG_{0,1}(x,y)]\\ &= \int dxdycos\theta cos\theta HG^*_{1,0}(x,y)HG_{1,0}(x,y)+\\ &\qquad sin\theta cos\theta HG^*_{0,1}(x,y)HG_{1,0}(x,y)+\\ &\qquad cos\theta sin\theta HG^*_{1,0}(x,y)HG_{0,1}(x,y)+\\ &\qquad sin\theta sin\theta HG^*_{0,1}(x,y)HG_{0,1}(x,y)\end{aligned}$$

by the orthonormality of HG modes

$$\int dxdyHG^*_\theta(x,y)HG_\theta(x,y) = cos^2\theta + sin^2\theta = 1$$

c. $\int dxdyHG^*_\theta(x,y)HG_{\theta+\pi/2}(x,y) = 0$

$$\begin{aligned}\int dxdyHG^*_\theta(x,y)HG_{\theta+\pi/2}(x,y) &= \int dxdy\left[cos\theta HG^*_{1,0}(x,y) + sin\theta HG^*_{0,1}(x,y)\right]\times\\ &\qquad [-sin(\theta)HG_{1,0}(x,y) + cos(\theta)HG_{0,1}(x,y)]\\ &= \int dxdy - cos\theta sin(\theta)HG^*_{1,0}(x,y)HG_{1,0}(x,y)+\\ &\qquad - sin\theta sin(\theta)HG^*_{0,1}(x,y)HG_{1,0}(x,y)+\\ &\qquad cos\theta cos(\theta)HG^*_{1,0}(x,y)HG_{0,1}(x,y)+\\ &\qquad sin\theta cos(\theta)HG^*_{0,1}(x,y)HG_{0,1}(x,y)\end{aligned}$$

by the orthonormality of HG modes

$$\int dxdyHG^*_\theta(x,y)HG_\theta(x,y) = -cos\theta sin(\theta) + sin\theta cos(\theta) = 0$$

d. $LG_\pm(x,y) = \frac{1}{\sqrt{2}}(HG_{1,0}(x,y) \pm \jmath HG_{0,1}(x,y))$

by the definition

$$LG_\pm(r,\phi) = \sqrt{\frac{2}{\pi w_0^2}}\frac{\sqrt{2}}{w_0}r\exp\left[-(r/w_0)^2\right]\exp(\pm\jmath\phi)$$

since $r\exp(\pm\jmath\phi) = rcos\phi \pm \jmath rsin\phi = x \pm \jmath y$

$$
\begin{aligned}
LG_{\pm}(r,\phi) &= \sqrt{\frac{1}{\pi w_0^2}}\frac{2}{w_0}\exp\left[-(r/w_0)^2\right](x \pm \jmath y) \\
&= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{\pi w_0^2}}2\frac{\sqrt{2}}{w_0}x\exp\left[-(r/w_0)^2\right] \pm \jmath\sqrt{\frac{1}{\pi w_0^2}}2\frac{\sqrt{2}}{w_0}y\exp\left[-(r/w_0)^2\right])\right] \\
&= \frac{1}{\sqrt{2}}\left[HG_{1,0}(x,y) \pm \jmath HG_{0,1}(x,y))\right]
\end{aligned}
$$

(2) [Exercise] Let $u_{nm}(x,y)$ be a basis set of the square integrable functions in $\mathbb{R}^2$. Show that

$$\sum_{n,m=0}^{\infty} u_{nm}(x,y)u_{nm}^*(x',y') = \delta(x-x')\delta(y-y'). \tag{19}$$

H_{int}: Use bra-ket notation to write $u_{n,m}(x,y) = \langle x,y|u_{n,m}\rangle$.

Since $U_{n,m}(X,Y) = \langle X,Y|U_{n,m}\rangle$ form a basis set for the square-integrable functions, $|U_{n,m}\rangle$ is also a basis in an abstract Hilbert space where $\langle X,Y|$ represents functionals acting on such space, and we have that:

$$\sum_{n,m=0}^{\infty} |U_{n,m}\rangle\langle U_{n,m}| = 1$$

is the identity of the Hilbert space.
Thus,

$$
\begin{aligned}
\sum_{n,m=0}^{\infty} U_{n,m}(X,Y)U_{n,m}^*(X',Y') &= \sum_{n,m=0}^{\infty}\langle X,Y|U_{n,m}\rangle\langle U_{n,m}|X',Y'\rangle \\
&= \langle X,Y|\left(\sum_{n,m=0}^{\infty}|U_{n,m}\rangle\langle U_{n,m}|\right)|X',Y'\rangle \\
&= \langle X,Y|X',Y'\rangle = \langle X|X'\rangle\langle Y|Y'\rangle \\
&= \delta(X-X')\delta(Y-Y')
\end{aligned}
$$

(3) [Exercise]Consider the linear polarization unit vectors rotated counter-clockwise by θ.

a.

Applying trigonometric identities

$$\hat{e}_{\theta+\pi/2} = -sin\theta\hat{e}_H + cos\theta\hat{e}_V$$

$$HG_{\theta+\pi/2}(x,y) = -sin\theta HG_{10}(x,y) + cos\theta HG_{01}(x,y)$$

First relation.
Evaluating

$$\begin{aligned}HG_\theta(x,y)\hat{e}_\theta = &\\ &\cos(\theta)^2 HG_{10}(x,y)\hat{e}_H + \\ &\sin\theta\cos\theta(HG_{10}(x,y)\hat{e}_V(x,y) + HG_{01}(x,y)\hat{e}_H) + \\ &\sin(\theta)^2 HG_{01}(x,y)\hat{e}_V\end{aligned}$$

$$\begin{aligned}HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta+\pi/2} = &\\ &\sin(\theta)^2 HG_{10}(x,y)\hat{e}_H - \\ &\sin\theta\cos\theta(HG_{10}(x,y)\hat{e}_V(x,y) + HG_{01}(x,y)\hat{e}_H) + \\ &\cos(\theta)^2 HG_{01}(x,y)\hat{e}_V\end{aligned}$$

summing both equations we have that

$$\begin{aligned}HG_\theta(x,y)\hat{e}_\theta + HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta+\pi/2} &= (\cos(\theta)^2 + \sin(\theta)^2)(HG_{10}\hat{e}_H + HG_{01}(x,y)\hat{e}_V) \\ &= HG_{10}\hat{e}_H + HG_{01}(x,y)\hat{e}_V\end{aligned}$$

Conclusion. both fields are independent of θ.

b.

We are given two vector fields involving Hermite-Gaussian (HG) modes and their corresponding polarization vectors. We will compute the Stokes parameters S_1, S_2, S_3 and demonstrate that they are all zero when integrated over a large area detector.
First Field:
The first field is given by:

$$\psi(x,y) = HG_\theta(x,y)\hat{e}\theta + HG\theta + \pi/2(x,y)\hat{e}\theta + \pi/2 = HG10(x,y)\hat{e}H + HG01(x,y)\hat{e}_V$$

The Hermite-Gaussian modes $HG_{10}(x,y)$ and $HG_{01}(x,y)$ correspond to the horizontal and vertical polarization directions, respectively. We proceed to compute the Stokes parameters:
1. First Stokes Parameter S_1:
The first Stokes parameter measures the difference in intensity between the horizontal and vertical components of the field:

$$S_1 = \langle|\psi_x|^2\rangle - \langle|\psi_y|^2\rangle$$

where $\psi_x = HG_{10}(x,y)$ and $\psi_y = HG_{01}(x,y)$. The corresponding integrals are:

$$\begin{aligned}S_1 &= \int HG_{10}^{(}x,y)HG_{10}(x,y)\,dxdy - \int HG_{01}^{(}x,y)HG_{01}(x,y)\,dxdy \\ &= 1-1 \\ &= 0\end{aligned} \tag{1}$$

Since the modes HG_{10} and HG_{01} are normalized, their integrals over the entire space both equal 1, leading to $S_1 = 0$.

2. Second and Third Stokes Parameters S_2 and S_3:

Next, we compute the cross-term $\langle \psi_x^* \psi_y \rangle$, which will determine the values of S_2 and S_3:

$$\langle \psi_x^* \psi_y \rangle = \int HG_{10}^*(x,y) HG_{01}(x,y)\, dxdy$$

By the orthogonality of the Hermite-Gaussian modes, this integral vanishes:

$$\int HG_{10}^*(x,y) HG_{01}(x,y)\, dxdy = 0$$

Thus, both S_2 and S_3 are zero:

$$S_2 = 2\text{Re}(0) = 0 \quad \text{and} \quad S_3 = 2\text{Im}(0) = 0$$

Second Field:

For the second field, the roles of the modes and polarization vectors are swapped:

$$\psi(x,y) = HG_\theta(x,y)\hat{e}\theta + \pi/2 + HG\theta + \pi/2(x,y)\hat{e}\theta = HG10(x,y)\hat{e}V + HG01(x,y)\hat{e}_H$$

1. First Stokes Parameter S_1:

The first Stokes parameter for this field is:

$$S_1 = \langle |\psi_x|^2 \rangle - \langle |\psi_y|^2 \rangle$$

where now $\psi_x = HG_{01}(x,y)$ and $\psi_y = HG_{10}(x,y)$. Performing the integrals:

$$\begin{aligned} S_1 &= \int HG_{01}^(x,y) HG_{01}(x,y)\, dxdy - \int HG_{10}^(x,y) HG_{10}(x,y)\, dxdy \\ &= 1 - 1 \\ &= 0 \end{aligned} \tag{2}$$

Again, the integrals of the normalized modes are equal, so $S_1 = 0$.

2. Second and Third Stokes Parameters S_2 and S_3:

For the cross-term:

$$\langle \psi_x^* \psi_y \rangle = \int HG_{01}^*(x,y) HG_{10}(x,y)\, dxdy$$

Once again, the orthogonality of the modes leads to the vanishing of this term:

$$\int HG_{01}^*(x,y) HG_{10}(x,y)\, dxdy = 0$$

Thus, both S_2 and S_3 are zero:

$$S_2 = 2\text{Re}(0) = 0 \quad \text{and} \quad S_3 = 2\text{Im}(0) = 0$$

now for the second case

Evaluating

$$
\begin{aligned}
HG_{\theta}(x,y)\hat{e}_{\theta+\pi/2} = &\\
&\cos(\theta)^2 HG_{10}(x,y)\hat{e}_V + \\
&\sin\theta\cos\theta(HG_{01}(x,y)\hat{e}_V(x,y) - HG_{01}(x,y)\hat{e}_H) + \\
&-\sin(\theta)^2 HG_{01}(x,y)\hat{e}_H
\end{aligned}
$$

$$
\begin{aligned}
HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta} = &\\
&-\sin(\theta)^2 HG_{10}(x,y)\hat{e}_V + \\
&\sin\theta\cos\theta(HG_{01}(x,y)\hat{e}_V(x,y) - HG_{01}(x,y)\hat{e}_H) + \\
&cos(\theta)^2 HG_{01}(x,y)\hat{e}_H
\end{aligned}
$$

Now we clearly see that the sum of both terms does not lead to a θ independent field

$$
\begin{aligned}
HG_{\theta}(x,y)\hat{e}_{\theta+\pi/2} + HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta} = &\\
&\cos(2\theta)(HG_{10}(x,y)\hat{e}_V - HG_{01}(x,y)\hat{e}_H) + \\
&\sin(2\theta)(HG_{01}(x,y)\hat{e}_V(x,y) - HG_{01}(x,y)\hat{e}_H) +
\end{aligned}
$$

However if we take the difference

$$
\begin{aligned}
\psi_{\theta}(x,y) =& HG_{\theta}(x,y)\hat{e}_{\theta+\pi/2} - HG_{\theta+\pi/2}(x,y)\hat{e}_{\theta} \\
=& (\cos(\theta)^2 + \sin(\theta)^2)(HG_{10}(x,y)\hat{e}_V - HG_{01}(x,y)\hat{e}_H) + \\
=& HG_{10}(x,y)\hat{e}_V - HG_{01}(x,y)\hat{e}_H
\end{aligned}
$$

which independs on θ and give again null stokes parameters due to the orthonormality of HG_{10} and H_{01} (where only making the exchange $E'_x = -E_y$ and E **Conclusion:** For both fields, the Stokes parameters S_1, S_2, S_3 are all zero. This result follows from the normalization of the Hermite-Gaussian modes and their orthogonality, ensuring that the cross-terms vanish.