Cerezo and Zelaquett exercises - Hackaton

Adonai Hilário da Silva, Arthur Dutra de Oliveira, Claudio Alves Pessoa Junior, Gustavo Kerdole, Ygor de Castro Lourenço

QuaCK - Quantum Computing Knights

Prospects and Challenges for Quantum Machine Learning (October 14-16)

Exercises proposed by: Marco Cerezo (Los Alamos National Laboratory, USA)

(1) [**Exercise**] Let $V = \mathbb{C}_2$ be the Hilbert space of a single qubit. Then, consider the set of objects $\{\mathbb{1}, X\}$, where $\mathbb{1}$ is the 2×2 identity matrix and X the Pauli-x matrix. Show that these objects, which represent bit-flips, form a group.

Solution

It suffices to show that this set satisfies the properties of a group. They are:

- *Closure*:
  $\mathbb{1}\mathbb{1} = \mathbb{1}$; $\mathbb{1}X = X\mathbb{1} = X$; and $XX = \mathbb{1}$, so this is a closed set.
- *Associativity*:
  This is a property naturally satisfied by matrix multiplication.
- *Identity*:
  $\mathbb{1}$ is the identity element since $\mathbb{1}\mathbb{1} = \mathbb{1}$ and $\mathbb{1}X = X\mathbb{1} = X$.
- *Inverses*:
  $\mathbb{1} = \mathbb{1}^{-1}$ since $\mathbb{1}\mathbb{1} = \mathbb{1}$, and $X = X^{-1}$ since $XX = \mathbb{1}$.

Therefore this set forms a group.

(2) [**Exercise**] Prove that the set of all unitaries of the form $U = e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}$ constitutes a representation of the unitary Lie group SU(2).

Solution

In the fundamental representation of SU(2), any generic element $U \in \mathrm{SU}(2)$ can be written as

$$U = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix}, \text{ with } \det(U) = |A|^2 + |B|^2 = 1,$$

where star denotes complex conjugation. So if we show that the product of exponentials results in matrices with the exact same form, we show that it is a representation of the group. Let us then explicitly open the quantity $e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y}$. We get

$$
\begin{aligned}
e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y} &= (\cos\phi_3 \mathbb{1} - i\sin\phi_3 Y)(\cos\phi_2 \mathbb{1} - i\sin\phi_2 X)(\cos\phi_1 \mathbb{1} - i\sin\phi_1 Y) = \\
(\cos\phi_3 \mathbb{1} - i\sin\phi_3 Y)&(\cos\phi_2\cos\phi_1 \mathbb{1} - i\cos\phi_2\sin\phi_1 Y - i\sin\phi_2\cos\phi_1 X - i\sin\phi_2\sin\phi_1 Z) = \\
\cos\phi_3\cos\phi_2\cos\phi_1\mathbb{1} &- i\cos\phi_3\cos\phi_2\sin\phi_1 Y - i\cos\phi_3\sin\phi_2\cos\phi_1 X - i\cos\phi_3\sin\phi_2\sin\phi_1 Z \\
-i\sin\phi_3\cos\phi_2\cos\phi_1 Y &- \sin\phi_3\cos\phi_2\sin\phi_1\mathbb{1} + i\sin\phi_3\sin\phi_2\cos\phi_1 Z - i\sin\phi_3\sin\phi_2\sin\phi_1 X.
\end{aligned}
$$

Joining terms appropriately and using the identities for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ we finaly obtain

$$
\cos\phi_2\cos(\phi_1+\phi_3)\mathbb{1} - i\sin\phi_2\cos(\phi_1-\phi_3)X - i\cos\phi_2\sin(\phi_1+\phi_3)Y - i\sin\phi_2\sin(\phi_1-\phi_3)Z.
$$

Writing the matrices explicitly we get

$$
\begin{pmatrix}
\cos\phi_2\cos(\phi_1+\phi_3) - i\sin\phi_2\sin(\phi_1-\phi_3) & -\cos\phi_2\sin(\phi_1+\phi_3) - i\sin\phi_2\cos(\phi_1-\phi_3) \\
\cos\phi_2\sin(\phi_1+\phi_3) - i\sin\phi_2\cos(\phi_1-\phi_3) & \cos\phi_2\cos(\phi_1+\phi_3) + i\sin\phi_2\sin(\phi_1-\phi_3)
\end{pmatrix}.
$$

Now, identifying the complex numbers

$$
\begin{aligned}
A &\equiv \cos\phi_2\cos(\phi_1+\phi_3) - i\sin\phi_2\sin(\phi_1-\phi_3), \\
B &\equiv \cos\phi_2\sin(\phi_1+\phi_3) - i\sin\phi_2\cos(\phi_1-\phi_3),
\end{aligned}
$$

we conclude that

$$
e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y} = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix},
$$

which is precisely the form of a generic element of SU(2). Now it only remains to verify if $|A|^2+|B|^2=1$. We have that

$$
\begin{aligned}
|A|^2 &= \cos^2\phi_2\cos^2(\phi_1+\phi_3) + \sin^2\phi_2\sin^2(\phi_1-\phi_3), \\
|B|^2 &= \cos^2\phi_2\sin^2(\phi_1+\phi_3) + \sin^2\phi_2\cos^2(\phi_1-\phi_3),
\end{aligned}
$$

hence

$$
\begin{aligned}
|A|^2+|B|^2 &= \cos^2\phi_2\left[\cos^2(\phi_1+\phi_3)+\sin^2(\phi_1+\phi_3)\right] + \sin^2\phi_2\left[\sin^2(\phi_1-\phi_3)+\cos^2(\phi_1-\phi_3)\right] \\
&= \cos^2\phi_2 + \sin^2\phi_2 \\
&= 1.
\end{aligned}
$$

Therefore, $e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y}$ is a representation of the SU(2) group. This is analogous to the Euler angles representation for the SO(3) group.

(3) [**Challenge**] Show that the dimension D of the commutant $\mathcal{C}^{(k)}(G)$, outlined in Definition 8 of the lecture notes, is determined by $D = \sum_\lambda m_\lambda^2$.

Solution

In general we will be dealing with a reducible representation. So given an appropriate choice of basis, we can write the vector space V as a direct sum o subspaces

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

where n is the number o distinct irreps. All elements of the representation can then be written as

$$R(g) = R_1^{(m_1)}(g) \oplus R_2^{(m_2)}(g) \oplus \cdots \oplus R_n^{(m_n)}(g),$$

where each $R_\lambda^{(m_\lambda)}(g)$ is a block-diagonal matrix containing the $\lambda-$th irrep with its respective multiplicity m_λ, that is,

$$R_\lambda^{(m_\lambda)}(g) \equiv \mathbb{1}_{m_\lambda} \otimes R_\lambda(g),$$

where $R_\lambda(g)$ is he actual $\lambda-$th irrep, and $\mathbb{1}_{m_\lambda}$ is an $m_\lambda \times m_\lambda$ identity matrix. Any bounded operator A acting on V, with the appropriate basis, can be written as

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n.$$

The commutant is defined to be the set of such bounded operators on V that commutes with all elements $R(g)$ of the representation. So the condition is that

$$\left[A_\lambda, R_\lambda^{(m_\lambda)}(g)\right] \equiv [A_\lambda, \mathbb{1}_{m_\lambda} \otimes R_\lambda(g)] = 0 \quad \forall\ \lambda.$$

Since the identity commutes with any matrix, and elements of an irrep only commute with the identity, any generic element A_λ must have the form

$$A_\lambda = A_{m_\lambda} \otimes \mathbb{1}_{d_\lambda},$$

where A_{m_λ} is an arbitrary $m_\lambda \times m_\lambda$ matrix, and d_λ is the dimension o the respective irrep $R_\lambda(g)$. Therefore, for each A_λ we need m_λ^2 linearly independent matrices. So for obtaining the dimension D of the entire commutant, we must only sum over all A_λ, that is,

$$D = \sum_{\lambda=1}^{n} m_\lambda^2.$$

If the representation is irreducible then the block-diagonal form is really just $R(g)$, meaning $n = 1$, and also $m_n = 1$, so the commutant is trivially just $\mathbb{1}_{d_\lambda}$.

High Dimensional Quantum Communication with Structured Light (October 14, 16-17)

Exercises proposed by: Prof. Antonio Zelaquett Khoury (UFF)

(1) [**Exercise**] Let $HG_\theta(x,y)$ be the first order Hermite-Gauss mode rotated counter-clockwise by θ. Show that

a) $HG_\theta(x,y) = \cos\theta HG_{10}(x,y) + \sin\theta HG_{01}(x,y)$;

b) $\int HG_\theta^*(x,y)HG_\theta(x,y)\,\mathrm{d}x\mathrm{d}y = 1$;

c) $\int HG_\theta^*(x,y)HG_{\theta+\pi/2}(x,y)\,\mathrm{d}x\mathrm{d}y = 0$;

d) $LG_\pm(r,\phi) = \frac{1}{\sqrt{2}}\left[HG_{10}(x,y) + iHG_{01}(x,y)\right]$.

Solution

A ser feito...

(2) [**Exercise**] Let $u_{nm}(x,y)$ be a basis set of the square integrable functions in $\mathbb{R}^2$. Show that

$$\sum_{n,m=0}^{\infty} u_{nm}(x,y)u_{nm}^*(x',y') = \delta(x-x')\delta(y-y').$$

Solution

$$\begin{aligned}\sum_{n,m=0}^{\infty} u_{nm}(x,y)u_{nm}^*(x',y') &= \sum_{n,m=0}^{\infty} \langle x,y|u_{nm}\rangle \langle u_{nm}|x',y'\rangle \\ &= \langle x,y| \left(\sum_{n,m=0}^{\infty} |u_{nm}\rangle\langle u_{nm}|\right) |x',y'\rangle\,.\end{aligned}$$

Since the $|u_{nm}\rangle$ form a basis, by summing over all indices we get the identity operator in $\mathbb{R}^2$, that is,

$$\sum_{n,m=0}^{\infty} |u_{nm}\rangle\langle u_{nm}| = \mathbb{1}.$$

Therefore

$$\sum_{n,m=0}^{\infty} u_{nm}(x,y)u_{nm}^*(x',y') = \langle x,y|x',y'\rangle = \delta(x-x')\delta(y-y').$$

(3) [**Challenge**] Consider the linear polarization unit vectors rotated counter-clockwise by θ:

$$\hat{\boldsymbol{e}}_\theta = \cos\theta\hat{\boldsymbol{e}}_H + \sin\theta\hat{\boldsymbol{e}}_V\,.$$

- Show that the vector structures used for alignment-free quantum communication,

$$\Psi_\theta(x,y) = HG_\theta(x,y)\hat{\boldsymbol{e}}_\theta + HG_{\theta+\pi/2}(x,y)\hat{\boldsymbol{e}}_{\theta+\pi/2},$$
$$\Psi_\theta(x,y) = HG_\theta(x,y)\hat{\boldsymbol{e}}_{\theta+\pi/2} + HG_{\theta+\pi/2}(x,y)\hat{\boldsymbol{e}}_\theta,$$

  are rotation invariant.

- Show that the polarization Stokes parameters for these vector structures are all equal to zero, if measured with large area detectors.

Solution

A ser feito...