Hackaquantum - Theoretical - Genessi e seus Capangas

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Hermite-Gaussian and Laguerre-Gaussian Modes

Exercise 1

(a) Show that $\mathrm{HG}_\theta(x,y) = \cos\theta\,\mathrm{HG}_{10}(x,y) + \sin\theta\,\mathrm{HG}_{01}(x,y)$

Solution:

1. **Understand the HG Modes:**
   The first-order HG modes are:
   - $\mathrm{HG}_{10}(x,y) \propto x\,e^{-\frac{x^2+y^2}{w^2}}$
   - $\mathrm{HG}_{01}(x,y) \propto y\,e^{-\frac{x^2+y^2}{w^2}}$
     These modes represent beam profiles with one node along the x-axis or y-axis, respectively.
2. **Define the Rotated Coordinates:**
   When we rotate the coordinate system counterclockwise by an angle θ, the new coordinates (x', y') are:
   - $x' = x\cos\theta + y\sin\theta$
   - $y' = -x\sin\theta + y\cos\theta$
3. **Express the Rotated HG Mode:**
   The rotated HG mode $\mathrm{HG}_\theta(x,y)$ corresponds to HG_{10} in the rotated frame:
   - $\mathrm{HG}_\theta(x,y) \propto x'\,e^{-\frac{x^2+y^2}{w^2}}$
   - Substitute x' in terms of x and y:

   $$\mathrm{HG}_\theta(x,y) \propto (x\cos\theta + y\sin\theta)\,e^{-\frac{x^2+y^2}{w^2}}$$

4. **Relate to Original HG Modes:**
   Recognize that $\mathrm{HG}_{10}(x,y) \propto x\,e^{-\frac{x^2+y^2}{w^2}}$ and $\mathrm{HG}_{01}(x,y) \propto y\,e^{-\frac{x^2+y^2}{w^2}}$.
   Therefore:

$$\mathrm{HG}_\theta(x,y) = \cos\theta\,\mathrm{HG}_{10}(x,y) + \sin\theta\,\mathrm{HG}_{01}(x,y)$$

(b)

The integral of the multiplication of $HG_\theta(x,y)$ and its complex conjugate is given by:

$$\int (\cos\theta\, HG_{10}(x,y) + \sin\theta\, HG_{01}(x,y))(\cos\theta\, HG_{10}(x,y) + \sin\theta\, HG_{01}(x,y))^*\, dx\, dy \tag{1}$$

Expanding the integrand gives:

$$\int \Big(\cos^2\theta\,|HG_{10}(x,y)|^2 + \sin^2\theta\,|HG_{01}(x,y)|^2 + \cos\theta\sin\theta\,(HG_{10}(x,y)HG^*_{01}(x,y) + HG_{01}(x,y)$$

We have:

$$I = \int (\cos\theta\, HG_{10}(x,y) + \sin\theta\, HG_{01}(x,y))(\cos\theta\, HG_{10}(x,y) + \sin\theta\, HG_{01}(x,y))^* dx\, dy \tag{3}$$

Expand the product inside the integral:

$$I = \int \left[\cos^2\theta\, HG^2_{10}(x,y) + \sin^2\theta\, HG^2_{01}(x,y) + 2\cos\theta\sin\theta\, HG_{10}(x,y)\, HG_{01}(x,y)\right] dx\, dy \tag{4}$$

Hermite-Gaussian modes are **orthonormal**, so

1. **First Term:**

$$I_1 = \cos^2\theta \int HG^2_{10}(x,y)\, dx\, dy = \cos^2\theta \times 1 = \cos^2\theta \tag{5}$$

2. **Second Term:**

$$I_2 = \sin^2\theta \int HG^2_{01}(x,y)\, dx\, dy = \sin^2\theta \times 1 = \sin^2\theta \tag{6}$$

3. **Cross Terms:**

$$I_{\text{cross}} = 2\cos\theta\sin\theta \int HG_{10}(x,y)\, HG_{01}(x,y)\, dx\, dy = 2\cos\theta\sin\theta \times 0 = 0 \tag{7}$$

The cross terms are zero because $HG_{10}(x,y)$ and $HG_{01}(x,y)$ are orthogonal:

$$\int HG_{10}(x,y)\, HG_{01}(x,y)\, dx\, dy = 0 \tag{8}$$

Adding all the terms together:

$$I = I_1 + I_2 + I_{\text{cross}} = \cos^2\theta + \sin^2\theta + 0 = 1 \tag{9}$$

Using the trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1 \tag{10}$$

Conclusion:

The integral evaluates to:

$$I = 1 \tag{11}$$

(c)

To show that:

$$\int \mathrm{HG}_\theta^*(x,y)\,\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y)\,dx\,dy = 0 \tag{12}$$

We use the following trigonometric identities

- $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$
- $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

Express $\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y)$:

Since:

$$\mathrm{HG}_\theta(x,y) = \cos\theta\,\mathrm{HG}_{10}(x,y) + \sin\theta\,\mathrm{HG}_{01}(x,y) \tag{13}$$

Then:

$$\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y) = \cos\left(\theta + \frac{\pi}{2}\right)\mathrm{HG}_{10}(x,y) + \sin\left(\theta + \frac{\pi}{2}\right)\mathrm{HG}_{01}(x,y) \tag{14}$$

Substitute the trigonometric identities:

$$\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y) = (-\sin\theta)\mathrm{HG}_{10}(x,y) + \cos\theta\,\mathrm{HG}_{01}(x,y) \tag{15}$$

Compute the integral:

$$I = \int \mathrm{HG}_\theta^*(x,y)\,\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y)\,dx\,dy \tag{16}$$

Since Hermite-Gaussian functions are real:

$$\mathrm{HG}_\theta^*(x,y) = \mathrm{HG}_\theta(x,y) = \cos\theta\,\mathrm{HG}_{10}(x,y) + \sin\theta\,\mathrm{HG}_{01}(x,y) \tag{17}$$

Multiply the expressions:

$$I = \int \left[\cos\theta\,\mathrm{HG}_{10}(x,y) + \sin\theta\,\mathrm{HG}_{01}(x,y)\right]\left[-\sin\theta\,\mathrm{HG}_{10}(x,y) + \cos\theta\,\mathrm{HG}_{01}(x,y)\right] dx\,dy \tag{18}$$

Expand the integrand:

1. **First term:**

$$(\cos\theta)\,(-\sin\theta)\mathrm{HG}_{10}^2 = -\cos\theta\sin\theta\,\mathrm{HG}_{10}^2$$

2. **Second term:**

$$(\cos\theta)\,(\cos\theta)\mathrm{HG}_{10}\mathrm{HG}_{01} = \cos^2\theta\,\mathrm{HG}_{10}\mathrm{HG}_{01}$$

3. **Third term:**

$$(\sin\theta)\,(-\sin\theta)\mathrm{HG}_{01}\mathrm{HG}_{10} = -\sin^2\theta\,\mathrm{HG}_{01}\mathrm{HG}_{10}$$

4. **Fourth term:**

$$(\sin\theta)\,(\cos\theta)\mathrm{HG}_{01}^2 = \sin\theta\cos\theta\,\mathrm{HG}_{01}^2$$

Combine the terms:

$$I = \int \left[-\cos\theta\sin\theta\,\mathrm{HG}_{10}^2 + \cos^2\theta\,\mathrm{HG}_{10}\mathrm{HG}_{01} - \sin^2\theta\,\mathrm{HG}_{01}\mathrm{HG}_{10} + \sin\theta\cos\theta\,\mathrm{HG}_{01}^2\right] dx\,dy \quad (19)$$

Simplify using orthogonality:

- **Orthogonality of Hermite-Gaussian modes:**

$$\int \mathrm{HG}_{n,m}(x,y)\,\mathrm{HG}_{n',m'}(x,y)\,dx\,dy = 0 \quad \text{if} \quad (n,m) \neq (n',m')$$

- **Normalization:**

$$\int \mathrm{HG}_{10}^2(x,y)\,dx\,dy = \int \mathrm{HG}_{01}^2(x,y)\,dx\,dy = 1$$

- **Cross terms vanish:**

$$\int \mathrm{HG}_{10}(x,y)\,\mathrm{HG}_{01}(x,y)\,dx\,dy = 0$$

Evaluate each term:

1. **First term:**

$$I_1 = -\cos\theta\sin\theta \int \mathrm{HG}_{10}^2(x,y)\,dx\,dy = -\cos\theta\sin\theta \times 1 = -\cos\theta\sin\theta$$

2. **Second term:**

$$I_2 = \cos^2\theta \int \mathrm{HG}_{10}(x,y)\,\mathrm{HG}_{01}(x,y)\,dx\,dy = \cos^2\theta \times 0 = 0$$

3. **Third term:**

$$I_3 = -\sin^2\theta \int \mathrm{HG}_{01}(x,y)\,\mathrm{HG}_{10}(x,y)\,dx\,dy = -\sin^2\theta \times 0 = 0$$

4. **Fourth term:**

$$I_4 = \sin\theta\cos\theta \int \mathrm{HG}_{01}^2(x,y)\,dx\,dy = \sin\theta\cos\theta \times 1 = \sin\theta\cos\theta$$

Sum the terms:

$$I = I_1 + I_2 + I_3 + I_4 = (-\cos\theta\sin\theta) + 0 + 0 + (\sin\theta\cos\theta) \tag{20}$$

Simplify:

$$I = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0 \tag{21}$$

Conclusion:

$$\int \mathrm{HG}_\theta^*(x,y)\,\mathrm{HG}_{\theta+\frac{\pi}{2}}(x,y)\,dx\,dy = 0 \tag{22}$$

(d) Show that $\mathrm{LG}_\pm(r,\phi) = \frac{1}{\sqrt{2}}[\mathrm{HG}_{10}(x,y) \pm i\,\mathrm{HG}_{01}(x,y)]$

Expressing the HG modes in polar coordinates
Recall that $x = r\cos\phi$ and $y = r\sin\phi$.
Therefore:

- $\mathrm{HG}_{10}(x,y) \propto r\cos\phi\, e^{-\frac{r^2}{w^2}}$
- $\mathrm{HG}_{01}(x,y) \propto r\sin\phi\, e^{-\frac{r^2}{w^2}}$

Combining the two HG modes
Form $\mathrm{HG}_{10} \pm i\,\mathrm{HG}_{01}$:

$$\mathrm{HG}_{10} \pm i\,\mathrm{HG}_{01} \propto r(\cos\phi \pm i\sin\phi)\,e^{-\frac{r^2}{w^2}} = re^{\pm i\phi}e^{-\frac{r^2}{w^2}}$$

Recognize that $\cos\phi \pm i\sin\phi = e^{\pm i\phi}$.

The first-order LG modes with azimuthal index $l = \pm 1$ are:

$$\mathrm{LG}_\pm(r,\phi) \propto r\,e^{\pm i\phi}\,e^{-\frac{r^2}{w^2}}$$

Thus:

$$\mathrm{LG}_\pm(r,\phi) = \frac{1}{\sqrt{2}}[\mathrm{HG}_{10}(x,y) \pm i\,\mathrm{HG}_{01}(x,y)]$$

Exercise 2

Show that:

$$\sum_{n,m=0} u_{nm}(x,y)u^*(x',y') = \delta(x-x')\delta(y-y') \tag{23}$$

Solution:

1. Following the H_{int}:

$$\sum_{n,m=0}^{\infty} \langle x, y||u_{nm}\rangle\langle u_{nm}||x', y'\rangle$$

2. As the summation term is only included inside of the $\langle x, y|$ and $|x', y'\rangle$, we can put:

$$\langle x, y| \sum_{n,m}^{\infty} |u_{nm}\rangle\langle u_{nm}||x', y'\rangle$$

3. As u_{nm} forms a basis set, it must obey the **Completeness** relation, which implies that:

$$\sum_{n,m} P_{n,m} = \mathbb{1} \tag{24}$$

Where $P_{n,m}$ are the projections in said basis.

In this case, we have

$$P_{n,m} = |u_{nm}\rangle\langle u_{nm}| = \mathbb{1} \tag{25}$$

Plugging this back we obtain:

$$\langle x, y|\mathbb{1}|x', y'\rangle = \langle x||x'\rangle\langle y||y'\rangle \tag{26}$$

4. As the position basis is **Orthonormal**, by definition $\langle x||x'\rangle = \delta(x - x')$, which results in:

$$\langle x||x'\rangle\langle y||y'\rangle = \delta(x - x')\delta(y - y') \tag{27}$$

Which in term completes the proof.

Exercise 3

Problem:

Consider the linear polarization unit vectors rotated counter-clockwise by angle θ:

$$\hat{e}_\theta = \cos\theta\,\hat{e}_H + \sin\theta\,\hat{e}_V \tag{28}$$

Show that the vector structures used for alignment-free quantum communication:

$$\Psi_\theta(x, y) = HG_\theta(x, y)\,\hat{e}_\theta + HG_{\theta+\frac{\pi}{2}}(x, y)\,\hat{e}_{\theta+\frac{\pi}{2}} \tag{29}$$

are rotation invariant, and that the polarization Stokes parameters for these vector structures are all equal to zero when measured with large area detectors.

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1. Showing Rotation Invariance of $\Psi_\theta(x, y)$

Hermite-Gaussian Modes:

The Hermite-Gaussian mode rotated by angle θ is defined as:

$$HG_\theta(x,y) = \cos\theta\, HG_{10}(x,y) + \sin\theta\, HG_{01}(x,y) \tag{30}$$

For the angle $\theta + \frac{\pi}{2}$, using trigonometric identities:

- $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$
- $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

Thus,

$$HG_{\theta+\frac{\pi}{2}}(x,y) = -\sin\theta\, HG_{10}(x,y) + \cos\theta\, HG_{01}(x,y) \tag{31}$$

Polarization Unit Vectors:

Similarly, the polarization unit vector rotated by $\theta + \frac{\pi}{2}$ is:

$$\hat{e}_{\theta+\frac{\pi}{2}} = -\sin\theta\, \hat{e}_H + \cos\theta\, \hat{e}_V \tag{32}$$

Substituting into $\Psi_\theta(x,y)$:

Substitute $HG_\theta(x,y)$, $HG_{\theta+\frac{\pi}{2}}(x,y)$, $\hat{e}_\theta$, and $\hat{e}_{\theta+\frac{\pi}{2}}$ into the expression for $\Psi_\theta(x,y)$:

$$\begin{aligned}\Psi_\theta(x,y) = &\left(\cos\theta\, HG_{10} + \sin\theta\, HG_{01}\right)\left(\cos\theta\, \hat{e}_H + \sin\theta\, \hat{e}_V\right) \\ &+ \left(-\sin\theta\, HG_{10} + \cos\theta\, HG_{01}\right)\left(-\sin\theta\, \hat{e}_H + \cos\theta\, \hat{e}_V\right)\end{aligned} \tag{33}$$

Expanding the Terms:

Multiply each term carefully:

1. **First Pair:**

$$\left(\cos\theta\, HG_{10}\right)\left(\cos\theta\, \hat{e}_H\right) = \cos^2\theta\, HG_{10}\, \hat{e}_H$$

$$\left(\cos\theta\, HG_{10}\right)\left(\sin\theta\, \hat{e}_V\right) = \cos\theta\sin\theta\, HG_{10}\, \hat{e}_V$$

2. **Second Pair:**

$$\left(\sin\theta\, HG_{01}\right)\left(\cos\theta\, \hat{e}_H\right) = \cos\theta\sin\theta\, HG_{01}\, \hat{e}_H$$

$$\left(\sin\theta\, HG_{01}\right)\left(\sin\theta\, \hat{e}_V\right) = \sin^2\theta\, HG_{01}\, \hat{e}_V$$

3. **Third Pair:**

$$\left(-\sin\theta\, HG_{10}\right)\left(-\sin\theta\, \hat{e}_H\right) = \sin^2\theta\, HG_{10}\, \hat{e}_H$$

$$\left(-\sin\theta\, HG_{10}\right)\left(\cos\theta\, \hat{e}_V\right) = -\sin\theta\cos\theta\, HG_{10}\, \hat{e}_V$$

4. **Fourth Pair:**

$$\left(\cos\theta\, HG_{01}\right)\left(-\sin\theta\, \hat{e}_H\right) = -\sin\theta\cos\theta\, HG_{01}\, \hat{e}_H$$

$$\left(\cos\theta\, HG_{01}\right)\left(\cos\theta\,\hat{e}_V\right) = \cos^2\theta\, HG_{01}\,\hat{e}_V$$

Combining Like Terms:

- **For $HG_{10}\,\hat{e}_H$:**

$$\cos^2\theta\, HG_{10}\,\hat{e}_H + \sin^2\theta\, HG_{10}\,\hat{e}_H = \left(\cos^2\theta + \sin^2\theta\right) HG_{10}\,\hat{e}_H = HG_{10}\,\hat{e}_H$$

- **For $HG_{01}\,\hat{e}_H$:**

$$\cos\theta\sin\theta\, HG_{01}\,\hat{e}_H - \sin\theta\cos\theta\, HG_{01}\,\hat{e}_H = 0$$

- **For $HG_{10}\,\hat{e}_V$:**

$$\cos\theta\sin\theta\, HG_{10}\,\hat{e}_V - \sin\theta\cos\theta\, HG_{10}\,\hat{e}_V = 0$$

- **For $HG_{01}\,\hat{e}_V$:**

$$\sin^2\theta\, HG_{01}\,\hat{e}_V + \cos^2\theta\, HG_{01}\,\hat{e}_V = \left(\sin^2\theta + \cos^2\theta\right) HG_{01}\,\hat{e}_V = HG_{01}\,\hat{e}_V$$

Resulting Expression:

After combining, we have:

$$\Psi_\theta(x,y) = HG_{10}(x,y)\,\hat{e}_H + HG_{01}(x,y)\,\hat{e}_V \tag{34}$$

This final expression is independent of θ, demonstrating that $\Psi_\theta(x,y)$ is rotation invariant.

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2. Calculating the Polarization Stokes Parameters

Definition of Stokes Parameters:

The Stokes parameters are given by:

- S_0 **(Total Intensity):**

$$S_0 = \int |\Psi(x,y)|^2\, dx\, dy$$

- S_1, S_2, S_3 **(Polarization Components):**

$$S_i = \int \Psi^\dagger(x,y)\,\sigma_i\,\Psi(x,y)\,dx\,dy \quad \text{for} \quad i = 1,2,3$$

Here, σ_i are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{35}$$

Expressing $\Psi(x,y)$ in Vector Form:

$$\Psi(x,y) = \begin{pmatrix} HG_{10}(x,y) \\ HG_{01}(x,y) \end{pmatrix} \tag{36}$$

Computing S_0:

Calculate the total intensity:

$$S_0 = \int \left(|HG_{10}(x,y)|^2 + |HG_{01}(x,y)|^2 \right) dx\, dy \tag{37}$$

Since the Hermite-Gaussian modes are orthonormal:

$$\int |HG_{10}(x,y)|^2 \, dx\, dy = 1, \quad \int |HG_{01}(x,y)|^2 \, dx\, dy = 1 \tag{38}$$

Therefore:

$$S_0 = 1 + 1 = 2 \tag{39}$$

Computing S_1:

Compute the integrand:

$$\Psi^\dagger \sigma_1 \Psi = \begin{pmatrix} HG_{10}^* & HG_{01}^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} HG_{10} \\ HG_{01} \end{pmatrix} = HG_{10}^* \, HG_{01} + HG_{01}^* \, HG_{10} \tag{40}$$

Since the functions are real:

$$HG_{10}^* = HG_{10}, \quad HG_{01}^* = HG_{01} \tag{41}$$

So:

$$\Psi^\dagger \sigma_1 \Psi = 2HG_{10}(x,y)\, HG_{01}(x,y) \tag{42}$$

Integrate:

$$S_1 = 2 \int HG_{10}(x,y)\, HG_{01}(x,y)\, dx\, dy \tag{43}$$

Due to orthogonality:

$$\int HG_{10}(x,y)\, HG_{01}(x,y)\, dx\, dy = 0 \tag{44}$$

Therefore:

$$S_1 = 0 \tag{45}$$

Computing S_2:

Compute the integrand:

$$\Psi^\dagger \sigma_2 \Psi = \begin{pmatrix} HG_{10} & HG_{01} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} HG_{10} \\ HG_{01} \end{pmatrix} = -iHG_{10}\, HG_{01} + iHG_{01}\, HG_{10} = 0 \tag{46}$$

Since the imaginary parts cancel out.

Thus:

$$S_2 = 0 \tag{47}$$

Computing S_3:

Compute the integrand:

$$\Psi^\dagger \sigma_3 \Psi = \begin{pmatrix} HG_{10} & HG_{01} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} HG_{10} \\ HG_{01} \end{pmatrix} = |HG_{10}(x,y)|^2 - |HG_{01}(x,y)|^2 \tag{48}$$

Integrate:

$$S_3 = \int \left(|HG_{10}(x,y)|^2 - |HG_{01}(x,y)|^2\right) dx\, dy = 1 - 1 = 0 \tag{49}$$

Conclusion:

✓ **All polarization Stokes parameters S_1, S_2, and S_3 are zero when measured with large area detectors.**

Prospects and Challenges for Quantum Machine Learning

(1)

$$V = \mathbb{C}_2 \tag{50}$$

So one possible basis set is given by the **Pauli** matrices and **Identity**:

$$\{\mathbb{1}, X, Y, Z\} \tag{51}$$

We need to prove properties of the base, which are:

1. **Associativity**

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{52}$$

$$\mathbb{1}X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \tag{53}$$

Similarly,

$$X\mathbb{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \tag{54}$$

2. **Existence of Identity**

Applying (53) into (54) we obtain that:

$$X\mathbb{1} = \mathbb{1}X \tag{55}$$

So, as is easily generalized for the other members of the group, $\mathbb{1}$ represents the **Identity** of the group. So, in fact, we do have an Identity for the group in question.

3. **Existence of an Inverse for every member of the group**:

Firstly, every member of the group is Hermitian(as Pauli matrices are defined as observables and the Identity is also hermitian):

$$\sigma_0^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0; \sigma_1^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1; \tag{56}$$

Which means that, in general:

$$\sigma_i = \sigma_i^\dagger \tag{57}$$

Notice that for every member of the set is unitary

$$\begin{aligned} \sigma_0\sigma_0^\dagger &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \\ \sigma_1\sigma_1^\dagger &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{1} \end{aligned} \tag{58}$$

$$\sigma_i\sigma_i^\dagger = \mathbb{1} \ \forall \ \sigma_i \tag{59}$$

Where $\sigma_0 = \mathbb{1}, \sigma_1 = X$.

Plugging (57) into (59) we obtain:

$$\sigma_i\sigma_i = \mathbb{1} \ \forall \ \sigma_i \tag{60}$$

Which shows us that, in fact, every member of the group has an inverse(which is itself).

Cayley Table

The previous properties can be summarized into the Cayley Table:

	X	$\mathbb{1}$
X	$\mathbb{1}$	X
$\mathbb{1}$	X	$\mathbb{1}$

(2)

Proves that the set of all unitaries of the form $U = e^{-i\varphi_3 Y} e^{-i\varphi_2 X} e^{-i\varphi_1 Y}$ constitutes a representation of the unitary Lie group SU(2).

The main objective is prove that the set of all unitaries of the form

$$U = e^{-i\varphi_3 Y} e^{-i\varphi_2 X} e^{-i\varphi_1 Y},$$

where X and Y represent the Pauli matrices, constitutes a representation of the unitary Lie group $SU(2)$.

We introduce a novel set $G = \{X, Y\}$, composed of all rotation matrices within the set U. To ascertain that U represents a Lie group $SU(2)$, it is necessary to demonstrate that the set G is capable of generating a new set $\mathcal{G}$, which satisfies the closure property concerning the commutator operation. Moreover, this new group must adhere to the $SU(2)$ representation relation, specifically, $[\sigma_i, \sigma_j] = 2\varepsilon_{ijk}\sigma_k$ where σ_i represents an element of the group and ε_{ijk} is the Levi-Civita symbol.

The Pauli matrices are represented as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{61}$$

$$[X, Y] = 2iZ, \quad [X, Z] = -2iY, \quad [Y, Z] = 2iX$$

These relations demonstrate that the set $\mathcal{G} = \{X, Y, Z\}$ is closed under the commutator relation and adheres to the $SU(2)$ representation relation. Furthermore, they establish that U constitutes a representation of the unitary Lie group $SU(2)$.

(3)

Our objective is to establish that the dimension of $\mathcal{C}(k)(G)$ is characterized by

$$D = \sum_{\lambda} m_{\lambda}^2.$$

To achieve this, we define the commutant as

Definition

Given some representation R of G, we define its $k-th$ order commutant, denoted as $C(k)(G)$, to be the vector subspace of the space of bounded linear operators on $H \otimes k$ that commutes with $R(g) \otimes k$ for all g in G. That is,

$$C(k)(G) = \{A \in B(H^{\otimes k}) | [A, R(g)^{\otimes k}] = 0, \quad \forall g \in G\}.$$

Then, we denote as $S(k)(G)$ a Hermitian basis for $C(k)(G)$, which consists of D elements.

It is crucial to acknowledge the property $[A, R(g)^{\otimes k}] = 0$, regardless of A, because of this the dimension of the commuter is the sum of the dimensions of all A_λ. We define A and $R(g)$ as

$$R(g) = \bigoplus_\lambda \mathbb{1}_{m_\lambda} \otimes R_\lambda(g)$$

$$A = \bigoplus_\lambda A_\lambda \otimes \mathbb{1}_{d_\lambda}$$

In the definition of A we can observe that A_λ lives in $\mathbb{C}^{m_\lambda \times m_\lambda}$, this interference of the dimension of A_λ is m_λ^2, so the dimension of $C(k)(G)$ is equal to

$$D = \sum_\lambda m_\lambda^2.$$

Bose-Einstein Condensate

(1)

a)

The low-energy limit is a limit for the Gross-Pitaevskii equation where we consider that the kinetic energy of the particles is very small when compared to the potential energy that is associated with the interaction, something very feasible because the particles are almost all populating the ground state. At this limit, the wavelength is considered very large when compared to the radius (R_{TF}) of the condensate, so higher-energy waves are suppressed. Therefore, the lowest-energy scattering, known as (l = 0) or s-wave scattering, dominates.

When we operate in this regime and consider the interaction potential to be finite for large (r) and bounded, we can express the interaction potential as ($4\pi\hbar^2 a/m$), which simplifies the Gross-Pitaevskii equation significantly.

b)

S-wave scattering refers to the lowest-energy scattering process in quantum mechanics, where the angular momentum quantum number (l = 0). It is characterized by isotropic scattering, meaning the scattering amplitude is uniform in all directions, and it dominates at low energies when the wavelength of the particles is much larger than the range of the interaction potential.

c)

To find the constant (U_0) for the two-body potential ($U_{\text{eff}}(r, r') = U_0 \delta(r - r')$), we can use the expression for the scattering length (a) given by:

$$a = \frac{m}{2\pi\hbar^2} \int d^3r \, U(r). \tag{62}$$

For the contact interaction ($U_{\text{eff}}(r, r') = U_0 \delta(r - r')$), this becomes:

$$a = \frac{m}{2\pi\hbar^2} \int d^3r \, U_0 \delta(r - r'). \tag{63}$$

Since the delta function is only non-zero when (r = r'), the integral simplifies to:

$$a = \frac{m}{2\pi\hbar^2} U_0 \tag{64}$$

Now, solving for (U_0), we get:

$$U_0 = \frac{2\pi\hbar^2 a}{m}. \tag{65}$$

So the constant (U_0) in the contact interaction potential is ($\frac{2\pi\hbar^2 a}{m}$), where (a) is the scattering length and (m) is the mass of the particles.

2.

a)

The many-body Hamiltonian for (N) interacting bosons is given by:

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + V(r_i) \right) + U_0 \sum_{i<j} \delta(r_i - r_j) \tag{66}$$

where (p_i) is the momentum operator for the (i)-th particle, ($V(r_i)$) is the external potential, and ($U_0 \delta(r_i - r_j)$) is the interaction potential between particles.

The wavefunction for a fully condensed Bose-Einstein condensate (BEC) is:

$$\Psi(r_1, r_2, \ldots, r_N) = \prod_{i=1}^{N} \phi(r_i), \tag{67}$$

where ($\phi(r_i)$) is the single-particle wavefunction for each boson.

First, consider the kinetic and potential energy terms. Because the expectation value for all particles is the same, we can calculate the expectation value for one particle and multiply by (N):

$$\langle H \rangle = N \int d^3r \, \phi^*(r) \left(\frac{-\hbar^2 \nabla^2}{2m} + V(r) \right) \phi(r). \tag{68}$$

This represents the total kinetic and potential energy for all (N) particles.

Now, for the interaction term:

$$H_{\text{interaction}} = U_0 \sum_{i<j} \delta(r_i - r_j). \tag{69}$$

The expectation value of this term is:

$$\langle H_{\text{interaction}} \rangle = U_0 \sum_{i<j} \int d^3r \, |\phi(r_i)|^2 |\phi(r_j)|^2. \tag{70}$$

The interaction between two particles is the same for all pairs in the condensate, so we calculate this integral once and multiply by the number of ways to pick two distinct particles, ($\frac{N(N-1)}{2}$):

$$\langle H_{\text{interaction}} \rangle = \frac{U_0 N(N-1)}{2} \int d^3r \, |\phi(r)|^4. \tag{71}$$

Therefore, the total expectation value of the Hamiltonian is:

$$\langle H \rangle = N \int d^3r \, \phi^*(r) \left(\frac{-\hbar^2 \nabla^2}{2m} + V(r) \right) \phi(r) + \frac{U_0 N(N-1)}{2} \int d^3r \, |\phi(r)|^4. \tag{72}$$

b)

We start with the energy functional for a Bose-Einstein condensate:

$$E(\phi, \phi^*) = N \int d^3r \left(\frac{\hbar^2}{2m} |\nabla \phi(r)|^2 + V(r)|\phi(r)|^2 + \frac{1}{2} N U_0 |\phi(r)|^4 \right),$$

where $\phi(r)$ is the normalized condensate wavefunction, $V(r)$ is the external potential, and $U_0 = \frac{4\pi\hbar^2 a}{m}$ represents the interaction term, with a being the scattering length. The number of particles N is given by the normalization constraint:

$$\int d^3r |\phi(r)|^2 = N.$$

To enforce this normalization constraint, we introduce a Lagrange multiplier μ (the chemical potential), and define the Lagrangian:

$$\mathcal{L}(\phi, \phi^*) = E(\phi, \phi^*) - \mu \left(\int d^3r |\phi(r)|^2 - N \right).$$

Variation of the Energy Functional

We calculate the variation of the energy functional $E(\phi, \phi^*)$ with respect to $\phi(r)$:

$$\delta_\phi E = N \int d^3r \left[-\frac{\hbar^2}{2m} \nabla^2 \phi^*(r) + V(r)\phi^*(r) + N U_0 |\phi(r)|^2 \phi^*(r) \right] \delta\phi(r).$$

Similarly, the variation with respect to $\phi^*(r)$ is:

$$\delta_{\phi^*} E = N \int d^3r \left[-\frac{\hbar^2}{2m} \nabla^2 \phi(r) + V(r)\phi(r) + N U_0 |\phi(r)|^2 \phi(r) \right] \delta\phi^*(r).$$

Variation of the Normalization Constraint

The normalization constraint is:

$$\int d^3r|\phi(r)|^2 = N.$$

The variations of this constraint with respect to $\phi(r)$ and
$\phi^*(r)$ are:

$$\delta_\phi \left(\int d^3r|\phi(r)|^2\right) = \int d^3r\, \phi^*(r)\delta\phi(r),$$

and

$$\delta_{\phi^*} \left(\int d^3r|\phi(r)|^2\right) = \int d^3r\, \phi(r)\delta\phi^*(r).$$

Variation of the Lagrange Multiplier Term

The Lagrange multiplier term is:

$$-\mu \int d^3r|\phi(r)|^2.$$

The variations of this term are:

$$\delta_\phi \left(-\mu \int d^3r|\phi(r)|^2\right) = -\mu \int d^3r\, \phi^*(r)\delta\phi(r),$$

and

$$\delta_{\phi^*} \left(-\mu \int d^3r|\phi(r)|^2\right) = -\mu \int d^3r\, \phi(r)\delta\phi^*(r).$$

Final Result

Combining all the terms, we get the variation of the Lagrangian:

$$\delta_\phi \mathcal{L} = N \int d^3r \left[-\frac{\hbar^2}{2m}\nabla^2\phi^*(r) + V(r)\phi^*(r) + NU_0|\phi(r)|^2\phi^*(r) - \mu\phi^*(r)\right]\delta\phi = 0.$$

And similarly for the complex part:

$$\delta_{\phi^*} \mathcal{L} = N \int d^3r \left[-\frac{\hbar^2}{2m}\nabla^2\phi^*(r) + V(r)\phi^*(r) + NU_0|\phi(r)|^2\phi^*(r) - \mu\phi^*(r)\right]\delta\phi = 0 \quad (73)$$

As they vary independently, they must be both $= 0$.

This gives the Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\nabla^2\phi^*(r) + V(r)\phi^*(r) + NU_0|\phi(r)|^2\phi^*(r) = \mu\phi^*(r) \quad (74)$$