$\boxed{2.32}$ Consider the following second-order digital filter.

$$H(z) = \frac{3(z+1)}{z^2 - 0.81}$$

(a) Find the frequency response $H(f)$.

(b) Find and sketch the magnitude response $A(f)$.

(c) Find and sketch the phase response $\phi(f)$.

(d) Find the steady state response to the following periodic input.

$$x(k) = 10\cos(0.6\pi k)$$

Solution

(a) Let $\theta = 2\pi fT$. Then applying Definition 2.9.1 and using Euler's identity, the frequency response is

$$\begin{aligned}
H(f) &= H(z)|_{z=\exp(j\theta)} \\
&= \frac{3[\exp(j\theta)+1]}{\exp(2j\theta) - 0.81} \\
&= \frac{3[\cos(\theta) + j\sin(\theta) + 1]}{[\cos(2\theta) + j\sin(2\theta) - 0.81]} \\
&= \frac{3[\cos(\theta) + 1 + j\sin(\theta)]}{[\cos(2\theta) - 0.81 + j\sin(2\theta)]} \quad , \quad \theta = 2\pi fT
\end{aligned}$$

(b) The magnitude response is

$$\begin{aligned}
A(f) &= |H(f)| \\
&= \frac{3|\cos(\theta) + 1 + j\sin(\theta)|}{|\cos(2\theta) - 0.81 + j\sin(2\theta)|} \\
&= \frac{3|\sqrt{[\cos(\theta)+1]^2 + \sin^2(\theta)}}{\sqrt{[\cos(2\theta) - 0.81]^2 + \sin^2(2\theta)}} \\
&= \frac{3\sqrt{2[1+\cos(\theta)]}}{\sqrt{[\cos(2\theta) - 0.81]^2 + \sin^2(2\theta)}} \\
&= \frac{3\sqrt{2[1+\cos(2\pi fT)]}}{\sqrt{[\cos(4\pi fT) - 0.81]^2 + \sin^2(4\pi fT)}}
\end{aligned}$$

(c) The phase response is

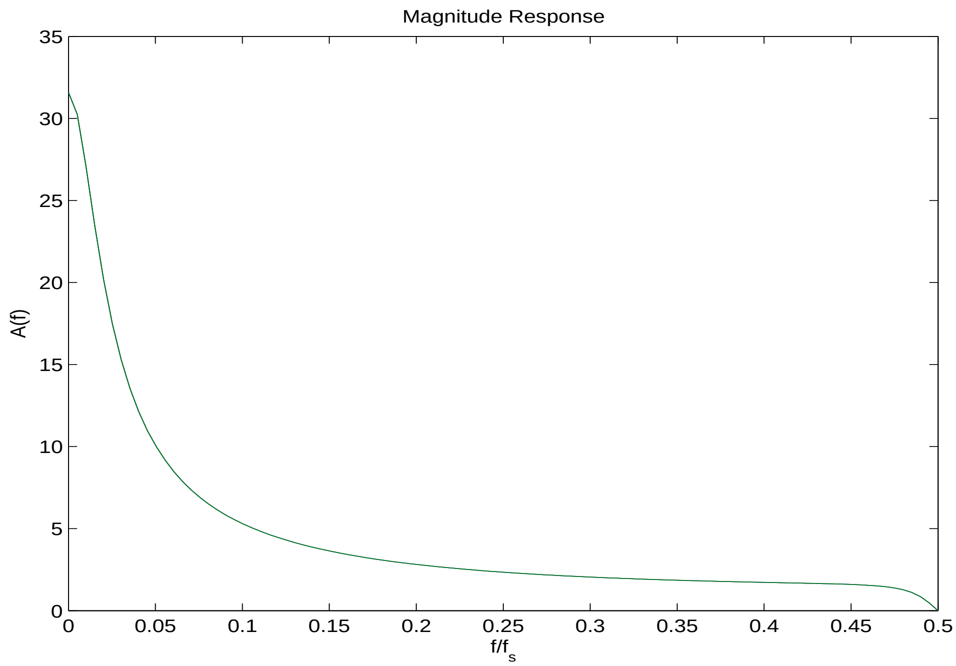
$$
\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle\{3[\cos(\theta) + 1 + j\sin(\theta)]\} - \angle\{[\cos(2\theta) - 0.81 + j\sin(2\theta)]\} \\
&= \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta)+1}\right] - \tan^{-1}\left[\frac{\sin(2\theta)}{\cos(2\theta)-0.81}\right] \\
&= \tan^{-1}\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT)+1}\right] - \tan^{-1}\left[\frac{\sin(4\pi fT)}{\cos(4\pi fT)-0.81}\right]
\end{aligned}
$$

(d) Since $f_s T = 1$, the input can be rewritten as

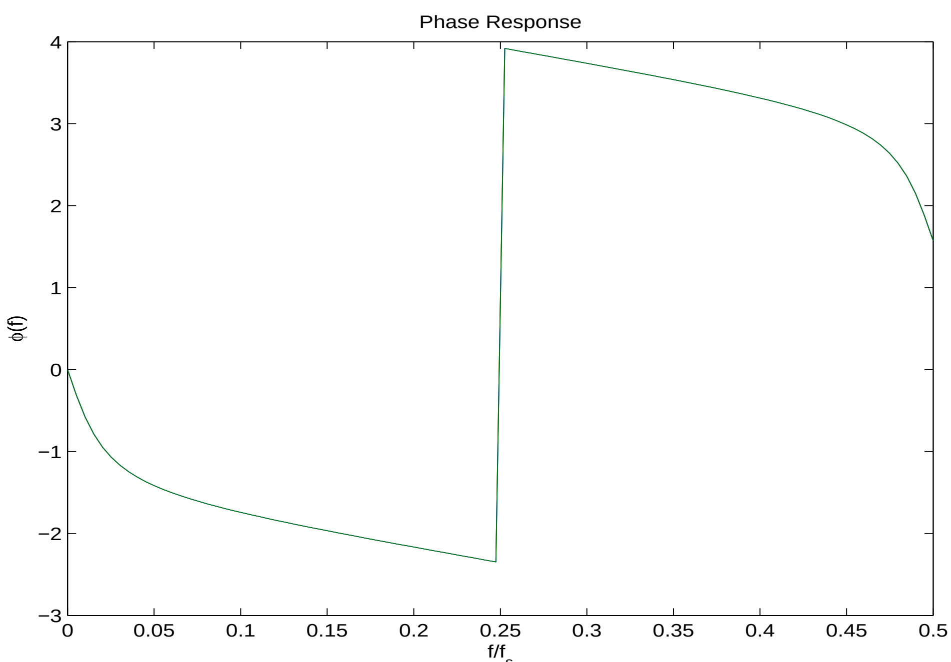
$$
\begin{aligned}
x(k) &= 10\cos(0.6\pi k) \\
&= 10\cos[2\pi(0.3)kf_sT] \\
&= 10\cos(2\pi F_1 kT)
\end{aligned}
$$

Thus the frequency of $x(k)$, expressed a a fraction of f_s, is $F_1 = 0.3f_s$. Since $H(z)$ is BIBO stable, it follows that the steady-state output is

$$
y_{ss}(k) = 2A(0.6\pi)\cos[0.6\pi k + \phi(0.6\pi)]
$$



Magnitude Response



Phase Response