$\boxed{3.15}$ Consider the following digital filter where $|a| < 1$.

$$H(z) = \frac{1}{1 - az^{-1}}$$

(a) Find the impulse response $h(k)$.
(b) Find the frequency response $H(f)$.
(c) Let $H(i)$ be the N-point DFT of $h(k)$, and let $f_i = if_s/N$. Given an arbitrary $\epsilon > 0$, use (3.7.5) to find a lower bound n such that for $N \geq n$,

$$|H(i) - H(f_i)| \leq \epsilon \quad \text{for} \quad 0 \leq i \leq \frac{N}{2}$$

Solution

(a) The impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\{H(z)\} \\ &= Z^{-1}\left\{\frac{z}{z-a}\right\} \\ &= a^k u(k) \end{aligned}$$

(b) The frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=j2\pi f} \\ &= \left.\frac{z}{z-a}\right|_{z=j2\pi f} \\ &= \frac{\exp(j2\pi f)}{\exp(j2\pi f) - a} \end{aligned}$$

(c) Using (3.7.5) and the geometric series from (2.2.4) we have

$$\begin{aligned} |H(i) - H(f_i)| &\leq \sum_{k=N}^{\infty} |h(k)| \\ &= \sum_{k=N}^{\infty} |a|^k \\ &= \frac{|a|^{N+1}}{1 - |a|} \end{aligned}$$

Setting the upper bound to $\epsilon > 0$ we have

$$\frac{|a|^{N+1}}{1-|a|} = \epsilon$$

Multiplying both sides by $1-|a|$ and taking the logarithm of both sides then yields

$$(N+1)\ln(|a|) = \ln[\epsilon/(1-|a|)]$$

Solving for N, and recalling that N must be an integer, we then have

$$n = \text{ceil}\left\{\frac{\ln[\epsilon/(1-|a|)]}{\ln(|a|)} - 1\right\}$$