$\boxed{8.16}$ Consider the following first-order analog filter.

$$H_a(s) = \frac{s}{s + 4\pi}$$

(a) What type of frequency-selective filter is this (lowpass, highpass, bandpass, bandstop)?
(b) What is the 3 dB cutoff frequency f_0 of this filter?
(c) Suppose $f_s = 10$ Hz. Find the prewarped cutoff frequency F_0.
(d) Design a digital equivalent filter $H(z)$ using the bilinear transformation method.

Solution

(a) The frequency response is

$$\begin{aligned} H(f) &= H(s)|_{s=j2\pi f} \\ &= \frac{j2\pi f}{j2\pi f + 4\pi} \\ &= \frac{jf}{jf + 2} \\ &= A(f)\exp[j\phi(f)] \end{aligned}$$

Here the magnitude and phase responses are

$$\begin{aligned} A(f) &= \frac{f}{\sqrt{4 + f^2}} \\ \phi(f) &= \pi/2 - \tan^{-1}(f/2) \end{aligned}$$

Since $A(0) = 0$ and $A(\infty) = 1$, this is a *highpass* filter.

(b) Setting $A^2(f) = 0.5$ and solving for f yields

$$\begin{aligned} f^2 &= (4 + f^2)0.5 \\ &= 2 + f^2/2 \end{aligned}$$

Thus $f^2/2 = 2$ or

$$f_0 = 2 \text{ Hz}$$

(c) Using (8.5.10) with $f_s = 10$ Hz yields

$$
\begin{aligned}
F_0 &= \frac{\tan(\pi f_0 T}{\pi T} \\
&= \frac{\tan(\pi 2/10)}{\pi/10} \\
&= 2.3127 \text{ Hz}
\end{aligned}
$$

(d) The prototype analog highpass filter is

$$
\begin{aligned}
H_a(s) &= \frac{s/F_0}{s/F_0 + 1} \\
&= \frac{s}{s + F_0} \\
&= \frac{s}{s + 2.3127}
\end{aligned}
$$

Using (8.5.5), the digital equivalent filter using the bilinear transformation is

$$
\begin{aligned}
H(z) &= H_a(s)|_{s=g(z)} \\
&= \frac{\dfrac{2(z-1)}{T(z+1)}}{\dfrac{2(z-1)}{T(z+1)} + F_0} \\
&= \frac{2(z-1)}{2(z-1) + F_0 T(z+1)} \\
&= \frac{2(z-1)}{(2 + F_0 T)z + F_0 T - 2} \\
&= \frac{2(z-1)}{2.2313z - 1.7687} \\
&= \frac{0.8964(z-1)}{z - 0.7927} \\
&= \frac{0.8964(1 - z^{-1})}{1 - 0.7927z^{-1}}
\end{aligned}
$$