

## Chapter 04.11

### Cholesky and $LDL^T$ Decomposition

*After reading this chapter, you should be able to:*

- 1. understand why the  $LDL^T$  algorithm is more general than the Cholesky algorithm,*
- 2. understand the differences between the factorization phase and forward solution phase in the Cholesky and  $LDL^T$  algorithms,*
- 3. find the factorized  $[L]$  and  $[D]$  matrices,*
- 4. obtain the forward solution phase,*
- 5. obtain the diagonal scaling phase,*
- 6. obtain the backward solution phase,*
- 7. solve a set of simultaneous linear equations using  $LDL^T$  algorithm.*

#### Introduction

Solving large (and sparse) system of simultaneous linear equations (SLE) has been (and continues to be) a major challenging problem for many real-world engineering/science applications [1-2]. In matrix notation, a set of SLE can be represented as:

$$[A][x] = [b] \quad (1)$$

where

$[A]$  = known coefficient matrix, with dimension  $n \times n$

$[b]$  = known right-hand-side (RHS)  $n \times 1$  vector

$[x]$  = unknown  $n \times 1$  vector.

#### Symmetrical Positive Definite (SPD) SLE

For many practical SLE, the coefficient matrix  $[A]$  (see Equation (1)) is Symmetric Positive Definite (SPD). In this case, the efficient 3-step Cholesky algorithm [1-2] can be used. A symmetric matrix  $[A]_{n \times n}$  is SPD if either of the following conditions is satisfied:

- If each and every determinant of sub-matrix  $A_{ii}$  ( $i = 1, 2, \dots, n$ ) is positive, or..
- If  $y^T A y > 0$  for any given vector  $[y]_{n \times 1} \neq \vec{0}$

**Example 1**

Find if

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

is SPD?

**Solution**

*Criterion a:* If each and every determinant of sub-matrix  $A_{ii}$  ( $i = 1, 2, \dots, n$ ) is positive.

The given  $3 \times 3$  matrix  $[A]$  is symmetrical, because  $a_{ij} = a_{ji}$ . Furthermore, one has

$$\det[A]_{1 \times 1} = |2| = 2 > 0$$

$$\begin{aligned} \det[A]_{2 \times 2} &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 3 > 0 \end{aligned}$$

$$\begin{aligned} \det[A]_{3 \times 3} &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} \\ &= 1 > 0 \end{aligned}$$

Hence  $[A]$  is SPD.

*Criterion (b):* If  $y^T A y > 0$  for any given vector  $[y]_{n \times 1} \neq \vec{0}$

For any given vector

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq \vec{0},$$

one computes

$$\begin{aligned} y^T A y &= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= (2y_1^2 - 2y_1y_2 + 2y_2^2) + \{y_3^2 - 2y_2y_3\} \\ &= (y_1 - y_2)^2 + y_1^2 + y_2^2 + \{y_3^2 - 2y_2y_3\} \\ &= (y_1 - y_2)^2 + y_1^2 + (y_2 - y_3)^2 > 0 \end{aligned}$$

Since the above scalar is always positive, hence matrix  $[A]$  is SPD.

**Step 1: Matrix Factorization phase**

In this step, the coefficient matrix  $[A]$  that is SPD can be decomposed (or factorized) into

$$[A] = [U]^T [U] \quad (2)$$

where,

$[U]$  is a  $n \times n$  upper triangular matrix.

The following simple  $3 \times 3$  matrix example will illustrate how to find the matrix  $[U]$ .

Various terms of the factorized matrix  $[U]$  can be computed/derived as follows (see Equation (2)):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (3)$$

Multiplying two matrices on the right-hand-side (RHS) of Equation (3), and then equating each upper-triangular RHS terms to the corresponding ones on the upper-triangular left-hand-side (LHS), one gets the following 6 equations for the 6 unknowns in the factorized matrix  $[U]$ .

$$u_{11} = \sqrt{a_{11}}; u_{12} = \frac{a_{12}}{u_{11}}; u_{13} = \frac{a_{13}}{u_{11}} \quad (4)$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{\frac{1}{2}}; u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}}; u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{\frac{1}{2}} \quad (5)$$

In general, for a  $n \times n$  matrix, the diagonal and off-diagonal terms of the factorized matrix  $[U]$  can be computed from the following formulas:

$$u_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} (u_{ki})^2\right)^{\frac{1}{2}} \quad (6)$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki}u_{kj}}{u_{ii}} \quad (7)$$

It is noted that if  $i = j$ , then the numerator of Equation (7) becomes identical to the terms under the square root in Equation (6). In other words, to factorize a general term  $u_{ij}$ , one simply needs to do the following steps:

Step 1.1: Compute the numerator of Equation (7), such as

$$Sum = a_{ij} - \sum_{k=1}^{i-1} u_{ki}u_{kj}$$

Step 1.2 If  $u_{ij}$  is an off-diagonal term (say,  $i < j$ ) then from Equation (7)

$$u_{ij} = \frac{Sum}{u_{ii}}$$

else, if  $u_{ij}$  is a diagonal term (that is,  $i = j$ ), then from Equation (6)

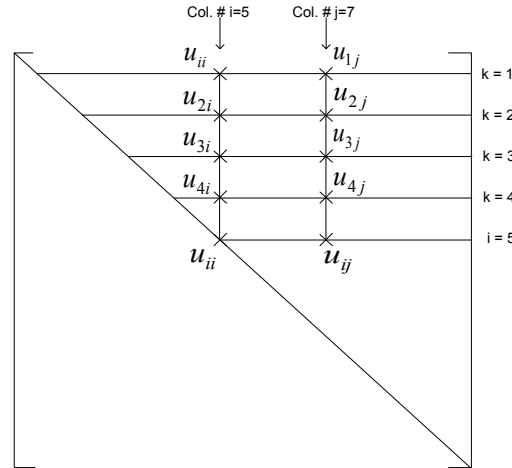
$$u_{ii} = \sqrt{\text{Sum}}$$

As a quick example, one computes:

$$u_{57} = \frac{a_{57} - u_{15}u_{17} - u_{25}u_{27} - u_{35}u_{37} - u_{45}u_{47}}{u_{55}} \quad (8)$$

Thus, for computing  $u(i=5, j=7)$ , one only needs to use the (already factorized) data in columns  $\#i(=5)$ , and  $\#j(=7)$  of  $[U]$ , respectively.

In general, to find the (off-diagonal) factorized term  $u_{ij}$ , one only needs to utilize the “already factorized” columns  $\#i$ , and  $\#j$  information (see Figure 1). For example, if  $i=5$ , and  $j=7$ , then Figure 1 will lead to the same formula as shown earlier in Equation (7), or in Equation (8). Similarly, to find the (diagonal) factorized term  $u_{ii}$ , one simply needs to utilize columns  $\#i$ , and  $\#i$  (again!) information (see Figure 1). In this case, Figure 1 will lead to the same formula as shown earlier in Equation (6).



**Figure 1** Cholesky Factorization for the term  $u_{ij}$

Since the square root operation involved during the Cholesky factorization phase (see Equation (6)), one must make sure the term under the square root is non-negative. This requirement is satisfied by  $[A]$  being SPD.

### Step 2: Forward Solution phase

Substituting Equation (2) into Equation (1), one gets

$$[U]^T [U] [x] = [b] \quad (9)$$

Let us define

$$[U] [x] \equiv [y] \quad (10)$$

Then, Equation (9) becomes

$$[U]^T [y] = [b] \quad (11)$$

Since  $[U]^T$  is a lower triangular matrix, Equation (11) can be efficiently solved for the intermediate unknown vector  $[y]$ , according to the order

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

hence the name “forward solution”.

As a quick example, one has from Equation (11)

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (12)$$

From the 1st row of Equation (12), one gets

$$\begin{aligned} u_{11}y_1 &= b_1 \\ y_1 &= \frac{b_1}{u_{11}} \end{aligned} \quad (13)$$

From the 2<sup>nd</sup> row of Equation (12), one gets

$$\begin{aligned} u_{12}y_1 + u_{22}y_2 &= b_2 \\ y_2 &= b_2 - \frac{u_{12}y_1}{u_{22}} \end{aligned} \quad (14)$$

Similarly

$$y_3 = \frac{b_3 - u_{13}y_1 - u_{23}y_2}{u_{33}} \quad (15)$$

In general, from the  $j^{th}$  row of Equation (12), one has

$$y_j = \frac{b_j - \sum_{i=1}^{j-1} u_{ij}y_i}{u_{jj}} \quad (16)$$

### Step 3: Backward Solution phase

Since  $[U]$  is an upper triangular matrix, Equation (10) can be efficiently solved for the original unknown vector  $[x]$ , according to the order

$$\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \cdot \\ x_1 \end{bmatrix}$$

and hence the name “backward solution”.

As a quick example, one has from Equation (10)

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (17)$$

From the last (or  $n^{th} = 3^{rd}$ ) row of Equation (17), one has

$$u_{33}x_3 = y_3.$$

hence

$$x_3 = \frac{y_3}{u_{33}} \quad (18)$$

Similarly

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}} \quad (19)$$

and

$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} \quad (20)$$

In general, one has

$$x_j = \frac{y_j - \sum_{i=j+1}^n u_{ji}x_i}{u_{jj}} \quad (21)$$

Amongst the above 3-step Cholesky algorithms, factorization phase in step 1 consumes about 95% of the total SLE solution time.

If the coefficient matrix  $[A]$  is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following  $LDL^T$  factorized algorithms can be employed

$$[A] = [L][D][L]^T \quad (22)$$

For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \quad (23)$$

Multiplying the three matrices on the RHS of Equation (23), then equating the resulting upper-triangular RHS terms of Equation (23) to the corresponding ones on the LHS, one obtains the following formulas for the diagonal  $[D]$ , and lower-triangular  $[L]$  matrices

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk} \quad (24)$$

$$l_{ij} = \left( a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left( \frac{1}{d_{jj}} \right) \quad (25)$$

Thus, the  $LDL^T$  algorithms can be summarized by the following step-by-step procedures.

**Step1: Factorization phase**

$$[A] = [L][D][L]^T \quad (22, \text{repeated})$$

**Step 2: Forward solution and diagonal scaling phase**

Substituting Equation (22) into Equation (1), one gets

$$[L][D][L]^T [x] = [b] \quad (26)$$

Let us define

$$[L]^T [x] = [y]$$

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (27)$$

$$x_i = y_i - \sum_{k=i+1}^n l_{ki} x_k; \text{ for } i = n, n-1, \dots, 2, 1 \quad (28)$$

Also, define

$$[D][y] = [z]$$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (29)$$

$$y_i = \frac{z_i}{d_{ii}}, \text{ for } i = 1, 2, 3, \dots, n \quad (30)$$

Then Equation (26) becomes

$$[L][z] = [b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (31)$$

$$z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \text{ for } i = 1, 2, 3, \dots, n \quad (32)$$

Equation (31) can be efficiently solved for the vector  $[z]$ , and then Equation (29) can be conveniently (and trivially) solved for the vector  $[y]$ .

**Step 3: Backward solution phase**

In this step, Equation (27) can be efficiently solved for the original unknown vector  $[x]$ .

**Example 2**

Using the Cholesky algorithm, solve the following SLE system for the unknown vector  $[x]$ .

$$[A][x] = [b]$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[b] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

**Solution**

The factorized, upper triangular matrix  $[U]$  can be computed by either referring to Equations (6-7), or looking at Figure 1, as following:

Row 1 of  $[U]$  is given below.

$$\begin{aligned} u_{11} &= \sqrt{a_{11}} \\ &= \sqrt{2} \\ &= 1.414 \end{aligned}$$

$$\begin{aligned} u_{12} &= \frac{a_{12}}{u_{11}} \\ &= \frac{-1}{1.414} \\ &= -0.7071 \end{aligned}$$

$$\begin{aligned} u_{13} &= \frac{a_{13}}{u_{11}} \\ &= \frac{0}{1.414} \\ &= 0 \end{aligned}$$

Row 2 of  $[U]$  is given below

$$\begin{aligned} u_{22} &= \left\{ a_{22} - \sum_{k=1}^{i-1=1} (u_{ki})^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 2 - (u_{12})^2 \right\}^{\frac{1}{2}} \\ &= \sqrt{2 - (-0.7071)^2} \\ &= 1.225 \end{aligned}$$



$$\begin{aligned}
u_{23} &= \frac{a_{23} - \sum_{k=1}^{i-1=1} u_{ki} u_{kj}}{U_{22}} \\
&= \frac{-1 - u_{12} \times u_{13}}{1.225} \\
&= \frac{-1 - (-0.7071)(0)}{1.225} \\
&= -0.8165
\end{aligned}$$

Row 3 of [U] is given below

$$\begin{aligned}
u_{33} &= \left\{ a_{33} - \sum_{k=1}^{i-1=2} (u_{ki})^2 \right\}^{\frac{1}{2}} \\
&= \left\{ a_{33} - u_{13}^2 - u_{23}^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{1 - (0)^2 - (-0.8165)^2} \\
&= 0.5774
\end{aligned}$$

Thus, the factorized matrix

$$[U] = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

The forward solution phase, shown in Equation (11), becomes

$$[U]^T [y] = [b]$$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, Equation (16) can be used to solve for [y] as

$$\begin{aligned}
y_1 &= \frac{b_1}{u_{11}} \\
&= \frac{1}{1.414} \\
&= 0.7071 \\
y_2 &= \frac{b_2 - \sum_{i=1}^{j-1=1} u_{ij} y_i}{u_{jj}} \\
&= \frac{0 - (u_{12} = -0.7071)(y_1 = 0.7071)}{(u_{22} = 1.225)} \\
&= 0.4082
\end{aligned}$$

$$\begin{aligned}
y_3 &= \frac{b_3 - \sum_{i=1}^{j-1=2} u_{ij} y_i}{u_{jj}} \\
&= \frac{0 - (u_{13} = 0)(y_1 = 0.7071) - (u_{23} = -0.8165)(y_2 = 0.4082)}{(u_{33} = 0.5774)} \\
&= 0.5774
\end{aligned}$$

The backward solution phase, shown in Equation (10), becomes:

$$[U][x] = [y]$$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

Thus, Equation (21) can be used to solve

$$\begin{aligned}
x_3 &= \frac{y_j}{u_{jj}} \\
&= \frac{y_3}{u_{33}} \\
&= \frac{0.5774}{0.5774} \\
&= 1 \\
x_2 &= \frac{y_j - \sum_{i=j+1=3}^{N=3} u_{ji} x_i}{u_{jj}} \\
&= \frac{y_2 - u_{23} x_3}{u_{22}} \\
&= \frac{0.4082 - (-0.8165)(1)}{1.225} \\
&= 1 \\
x_1 &= \frac{y_j - \sum_{i=j+1=2}^{N=3} u_{ji} x_i}{u_{jj}} \\
&= \frac{y_1 - u_{12} x_2 - u_{13} x_3}{u_{11}} \\
&= \frac{0.7071 - (-0.7071)(1) - (0)(1)}{1.414} \\
&= 1
\end{aligned}$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

### Example 3

Using the LDL<sup>T</sup> algorithm, solve the following SLE system for the unknown vector  $[x]$ .

$$[A][x] = [b]$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[b] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

### Solution

The factorized matrices  $[D]$  and  $[L]$  can be computed from Equation (24) and Equation (25), respectively.

$$\left. \begin{aligned} d_{11} &= a_{11} - \sum_{k=1}^{j-1=0} l_{jk}^2 d_{kk} \\ &= a_{11} \\ &= 2 \\ l_{11} &= 1 \text{ (always !)} \\ l_{21} &= \frac{a_{21} - \sum_{k=1}^{j-1=0} l_{ik} d_{kk} l_{jk}}{d_{jj}} \\ &= \frac{a_{21}}{d_{11}} \\ &= \frac{-1}{2} \\ &= -0.5 \\ l_{31} &= \frac{a_{31}}{d_{11}} \\ &= \frac{0}{2} \\ &= 0 \end{aligned} \right\} \text{Column 1 of matrices of } [D] \text{ and } [L]$$

$$\left. \begin{aligned}
 d_{22} &= a_{22} - \sum_{k=1}^{j-1=1} l_{jk}^2 d_{kk} \\
 &= 2 - l_{21}^2 d_{11} \\
 &= 2 - (-0.5)^2 (2) \\
 &= 1.5 \\
 l_{22} &= 1 \text{ (always !)} \\
 l_{32} &= \frac{a_{32} - \sum_{k=1}^{j-1=1} l_{31} d_{11} l_{21}}{d_{22}} \\
 &= \frac{-1 - (0)(2)(-0.5)}{1.5} \\
 &= -0.6667
 \end{aligned} \right\} \text{Column 2 of matrices } [D] \text{ and } [L]$$

$$\left. \begin{aligned}
 d_{33} &= a_{33} - \sum_{k=1}^{j-1=2} l_{jk}^2 d_{kk} \\
 &= 1 - l_{31}^2 d_{11} - l_{32}^2 d_{22} \\
 &= 1 - (0)^2 (2) - (-0.6667)^2 (1.5) \\
 &= 0.3333
 \end{aligned} \right\} \text{Column 3 of matrices } [D] \text{ and } [L]$$

Hence

$$[D] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

and

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix}$$

The forward solution shown in Equation (31) becomes:

$$[L][z] = [b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.667 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ or}$$

$$z_i = b_i - \sum_{k=1}^{i-1} l_{ik} z_k \quad (32, \text{ repeated})$$

Hence

$$z_1 = b_1 = 1$$

$$\begin{aligned} z_2 &= b_2 - L_{21}z_1 \\ &= 0 - (-0.5)(1) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} z_3 &= b_3 - L_{31}z_1 - L_{32}z_2 \\ &= 0 - (0)(1) - (-0.6667)(0.5) \\ &= 0.3333 \end{aligned}$$

The diagonal scaling phase, shown in Equation (29) becomes

$$[D][y] = [z]$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix}, \text{ or}$$

$$y_i = \frac{z_i}{d_{ii}}$$

Hence

$$\begin{aligned} y_1 &= \frac{z_1}{d_{11}} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} y_2 &= \frac{z_2}{d_{22}} \\ &= \frac{0.5}{1.5} \\ &= 0.3333 \end{aligned}$$

$$\begin{aligned} y_3 &= \frac{z_3}{d_{33}} \\ &= \frac{0.3333}{0.3333} \\ &= 1 \end{aligned}$$

The backward solution phase can be found by referring to Equation (27)

$$[L]^T [x] = [y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^N l_{ki} x_k$$

(28, repeated)

Hence

$$\begin{aligned}
 x_3 &= y_3 \\
 &= 1 \\
 x_2 &= y_2 - l_{32}x_3 \\
 &= 0.3333 - (-0.6667) \times 1 \\
 x_2 &= 1 \\
 x_1 &= y_1 - l_{21}x_2 - l_{31}x_3 \\
 x_1 &= 0.5 - (-0.5)(1) - (0)(1) \\
 &= 1
 \end{aligned}$$

Hence

$$\begin{aligned}
 [x] &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
 \end{aligned}$$

Through this numerical example, one clearly sees that the “square root operations” have NOT been involved during the entire  $LDL^T$  algorithms. Thus, the coefficient matrix  $[A]$ , shown in Equation (1) is NOT required to be SPD.

### Re-ordering Algorithms For Minimizing Fill-in Terms [1,2].

During the factorization phase (of Cholesky, or  $LDL^T$  algorithms), many “zero” terms in the original/given matrix  $[A]$  will become “non-zero” terms in the factored matrix  $[U]$ . These new non-zero terms are often called as “fill-in” terms (indicated by the symbol  $F$ ). It is, therefore, highly desirable to minimize these fill-in terms, so that both computational time/effort and computer memory requirements can be substantially reduced. For example, the following matrix  $[A]$  and vector  $[b]$  are given:

$$[A] = \begin{bmatrix} 112 & 7 & 0 & 0 & 0 & 2 \\ 7 & 110 & 5 & 4 & 3 & 0 \\ 0 & 5 & 88 & 0 & 0 & 1 \\ 0 & 4 & 0 & 66 & 0 & 0 \\ 0 & 3 & 0 & 0 & 44 & 0 \\ 2 & 0 & 1 & 0 & 0 & 11 \end{bmatrix} \quad (33)$$

$$[b] = \begin{bmatrix} 121 \\ 129 \\ 94 \\ 70 \\ 47 \\ 14 \end{bmatrix} \quad (34)$$

The Cholesky factorization matrix  $[U]$ , based on the original matrix  $[A]$  (see Equation 33) and Equations (6-7), or Figure 1, can be symbolically computed as

$$[U] = \begin{bmatrix} \times & \times & 0 & 0 & 0 & \times \\ 0 & \times & \times & \times & \times & F \\ 0 & 0 & \times & F & F & \times \\ 0 & 0 & 0 & \times & F & F \\ 0 & 0 & 0 & 0 & \times & F \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \quad (35)$$

In Equation (35), the symbols  $x$ , and  $F$  represents the “non-zero” and “fill-in” terms, respectively.

In practical applications, however, it is always a necessary step to rearrange the original matrix  $[A]$  through re-ordering algorithms (or subroutines) [Refs 1-2] and produce the following integer mapping array

$$\text{IPERM (new equation \#) = \{old equation \#} \quad (36)$$

such as, for this particular example:

$$\text{IPERM} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad (37)$$

Using the above results (see Equation 37), one will be able to construct the following re-arranged matrices:

$$[A^*] = \begin{bmatrix} 11 & 0 & 0 & 1 & 0 & 2 \\ 0 & 44 & 0 & 0 & 3 & 0 \\ 0 & 0 & 66 & 0 & 4 & 0 \\ 1 & 0 & 0 & 88 & 5 & 0 \\ 0 & 3 & 4 & 5 & 110 & 7 \\ 2 & 0 & 0 & 0 & 7 & 112 \end{bmatrix} \quad (38)$$

and

$$[b^*] = \begin{bmatrix} 14 \\ 47 \\ 70 \\ 94 \\ 129 \\ 121 \end{bmatrix} \quad (39)$$

In the original matrix  $A$  (shown in Equation 33), the nonzero term  $A$  (old row 1, old column 2) = 7 will move to new location of the new matrix  $A^*$  (new row 6, new column 5) = 7, etc.

The non zero term  $A$  (old row 3, old column 3) = 88 will move to  $A^*$  (new row 4, new column 4) = 88, etc.

The value of  $b$  (old row 4) = 70 will be moved to (or located at)  $b^*$  (new row 3) = 70, etc.

Now, one would like to solve the following modified system of linear equations (SLE) for  $[x^*]$ ,

$$[A^*][x^*] = [b^*] \quad (40)$$

rather than to solve the original SLE (see Equation (1)). The original unknown vector  $\{x\}$  can be easily recovered from  $[x^*]$  and  $[IPERM]$ , shown in Equation (37).

The factorized matrix  $[U^*]$  can be “symbolically” computed from  $[A^*]$  as (by referring to either Figure 1 or Equations 6-7):

$$[U^*] = \begin{bmatrix} \times & 0 & 0 & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & \times & 0 \\ 0 & 0 & 0 & \times & \times & F \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \quad (41)$$

You can clearly see the big benefits of solving the SLE shown in Equation (40), instead of solving the original Equation (1), since the factorized matrix  $[U^*]$  has only 1 fill-in term (see the symbol “ $F$ ” in Equation 41), as compared to six fill-in-terms occurred in the factorized matrix  $[U]$  as shown in Equation 35.

#### **On-Line Chess-Like Game For Reordering/Factorized Phase [4].**

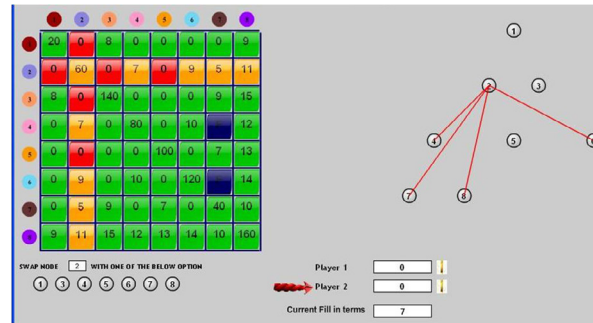
Based on the discussions presented in the previous section 2 (about factorization phase), and section 3 (about reordering phase), one can easily see the similar operations between the symbolic, numerical factorization and reordering (to minimize the number of fill-in terms) phases of sparse SLE.

In practical computer implementation for the solution of SLE, the reordering phase is usually conducted first (to produce the mapping between “old/new” equation numbers, as indicated in the integer array  $IPERM(-)$ , see Equations 36-37).

Then, the sparse *symbolic* factorization phase is followed by using either Cholesky Equations 6-7, or the  $LDL^T$  Equations 24-25 (without requiring the actual/numerical values to be computed). The reason is because during the *symbolic factorization* phase, one only wishes to find the number (and the location) of non-zero *fill-in terms*. This *symbolic* factorization process is necessary for allocating the “computer memory” requirement for the “numerical factorization” phase which will actually compute the exact numerical values of  $[U^*]$ , based on the same Cholesky Equations (6-7) (or the  $LDL^T$  Equations (24-25)).

In this work, a chess-like game (shown in Figure 2, Ref. [4]) has been designed with the following objectives:





**Figure 2** A Chess-Like Game For Learning to Solve SLE.

- (A) Teaching undergraduates the process how to use the reordering output  $IPERM(-)$ , see Equations (36-37) for converting the original/given matrix  $[A]$ , see Equation (33), into the new/modified matrix  $[A^*]$ , see Equation (38). This step is reflected in Figure 2, when the “Game Player” decides to swap node (or equation)  $i$  (say  $i = 2$ ) with another node (or equation)  $j$ , and click the *CONFIRM* icon! Since node  $i=2$  is currently connected to nodes  $j=4, 6, 7, 8$ , swapping node  $i=2$  with the above nodes  $j$  will *NOT* change the number/pattern of the *fill-in* terms. However, if node  $i=2$  is swapped with node  $j=1$ , or 3 or 5, then the fill-in terms pattern may change (for better or worse)!
- (B) Helping undergraduates to understand the “symbolic” factorization” phase by symbolically utilizing the Cholesky factorized Equations (6-7). This step is illustrated in Figure 2, for which the “game player” will see (and also hear the computer animated sound, and human voice) the non-zero terms (including fill-in terms) of the original matrix  $[A]$  to move to the new locations in the new/modified matrix  $[A^*]$ .
- (C) Helping undergraduates to understand the *numerical factorization* phase, by numerically utilizing the same Cholesky factorized Equations (6-7).
- (D) Teaching undergraduates to *understand existing reordering concepts*, or to *discover new reordering algorithms*.

### Further Explanation on the Developed Game

1. In the above chess-like game, which is available on-line [4], powerful features of FLASH computer environment [3], such as animated sound, human voice, motions, graphical colors etc... have been incorporated and programmed into the developed game-software for more appeal to game players/learners.
2. In the developed chess-like game, fictitious monetary (or any kind of ‘scoring system’) is rewarded (and broadcasted by computer animated human voice) to game players, based on how he/she swaps the node (or equation) numbers, and consequently based on how many fill-in  $F$  terms occurred. In general, less fill-in terms introduced will result in more rewards.
3. Based on the original/given matrix  $[A]$ , and existing re-ordering algorithms (such as the Reverse Cuthill-McKee, or RCM algorithms [1-2]) the number of fill-in terms  $F$  can be computed using RCM algorithms. This internally generated information will be used to judge how good the players/learners are, and/or broadcast “congratulations

message” to a particular player who discovers a new “chess-like move” (or, swapping node) strategies which are even better than RCM algorithms.

4. Initially, the player(s) will select the matrix size ( $8 \times 8$ , or larger is recommended), and the percentage (50%, or larger is suggested) of zero-terms (or sparsity of the matrix). Then, the *START Game* icon will be clicked by the player.
5. The player will then CLICK one of the selected node  $i$  (or equation) numbers appearing on the computer screen. The player will see those nodes  $j$  which are connected to node  $i$  (based on the given/generated matrix  $[A]$ ). The player then has to decide to swap node  $i$  with one of the possible node  $j$ . After *confirming* the player’s decision, the outcomes/results will be announced by the computer animated human voice, and the monetary-award will (or will not) be given to the players/learners, accordingly. In this software, a maximum of \$1,000,000 can be earned by the player, and the exact dollar amount will be *inversely* proportional to the number of fill-in terms occurred (based on the player’s decision on how to swap node  $i$  with another node  $j$ ).
6. The next player will continue to play, with his/her move (meaning to swap the  $i^{th}$  node with the  $j^{th}$  node) based on the *current best* non-zero terms pattern of the matrix.

## References

<b>CHOLSEKY AND LDL<sup>T</sup> DECOMPOSITION</b>	
Topic	Cholesky and LDL <sup>T</sup> Decomposition
Summary	Textbook chapter of Cholesky and LDL <sup>T</sup> Decomposition
Major	General Engineering
Authors	Duc Nguyen
Date	July 29, 2010
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>