Algebraic Geometry

Based on lectures by Marc Hoyois

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Preface

These notes record and complete the *Algebraic Geometry* course taught by Professor Marc Hoyois during the Winter 2025/2026 semester at the University of Regensburg. While the course script has been publicly released, it omits proofs for most theorems. These notes aim to supplement and refine the core content by providing complete proofs based on lecture notes and additional research.

Course Structure

The defining feature of this course is its systematic development of algebraic geometry through the *functor of points* perspective. As is well known, the fundamental objects of study in algebraic geometry can be described as functors of the form:

$$X\colon \mathsf{CAlg} \to \mathsf{Set}.$$

***Remark* 0.0.1** (Connection to Derived Algebraic Geometry)**.** When studying *derived algebraic geometry*, one simply replaces the source category CAlg with the corresponding category of derived rings (such as animated rings or $\mathbb{E}_n$-algebras), and the target category Set with the ∞-category of animae An.

Specifically, we will study several special classes of algebraic functors following this outline:

(i) *Affine Geometry:* Study *affine schemes*, i.e., functors $\operatorname{Spec}(R)$ represented by rings R, as well as their open subfunctors—*quasi-affine schemes*.

(ii) *Projective Geometry:* Study *projective schemes*, i.e., closed subfunctors of projective space, and their open subfunctors—*quasi-projective schemes*.

(iii) *Scheme Theory:* Study *schemes*. These are algebraic functors with good properties: they are sheaves for the Zariski topology and are locally isomorphic to affine schemes (i.e., admit affine open covers).

(iv) *Algebraic Functors:* This is our ambient category. We will see that many constructions (such as quasi-coherent modules) may leave the category of schemes, but remain well-defined algebraic functors.

The hierarchy among these objects is summarized in the following diagram (note that affine and projective are distinct geometric objects, both contained in the category of schemes):

$$\begin{tikzcd}
\{\text{Affine schemes}\} \arrow[r, hook] & \{\text{Quasi-affine schemes}\} \arrow[d, hook] & & \\
\{\text{Projective schemes}\} \arrow[r, hook] & \{\text{Quasi-projective schemes}\} \arrow[r, hook] & \{\text{Schemes}\} \arrow[r, hook] & \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})
\end{tikzcd}$$

The course proceeds in the following order:

- *Chapter 1:* We introduce the concept of algebraic functors and define *affine schemes*. We classify open and closed subfunctors of affine schemes and show that affine schemes satisfy Zariski descent.

- *Chapter 2:* We study projective spaces, introduce graded rings, define projective schemes, and show that the Proj construction satisfies Zariski descent. Finally, we classify open and closed subfunctors of $\operatorname{Proj}(A)$.

- *Chapter 3:* For a functor $F\colon \mathsf{CAlg} \to \mathsf{Cat}$, we extend F from CAlg to $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}}$. Objects of F over an algebraic functor X are called quasi-coherent F-objects.

- *Chapter 4:* We introduce the concept of locales and discover that, from the functor of points perspective, locales (rather than topological spaces) are fundamental. We describe how to construct topological spaces from algebraic functors.

- *Chapter 5:* We introduce sites and sheaves.
- *Chapter 6:* We introduce schemes.

Acknowledgments

The exposition of the theory of functorial algebraic geometry presented in these notes is entirely based on the lectures and insights of Professor Marc Hoyois. All credit for the mathematical content belongs to him, while any errors, misconceptions, or inaccuracies that remain are solely my own responsibility.

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Chapter 1.

Affine Geometry

1.1. Algebraic Functors and Affine Schemes

Definition 1.1.1 (Algebraic Functor). Let k be a ring.

- An *algebraic k-functor* is a functor $\mathsf{CAlg}_k \to \mathsf{Set}$. When $k = \mathbb{Z}$, we simply call it an *algebraic functor*.
- Given an algebraic k-functor X and a k-algebra R, we call elements of $X(R)$ the *R-points* of X.

The most fundamental algebraic functors are *affine spaces*.

Definition 1.1.2 (Affine Space). Let I be a set and k a ring. The *affine I-space over k* is the algebraic k-functor

$$\mathbb{A}_k^I : \mathsf{CAlg}_k \to \mathsf{Set}, \quad R \mapsto R^I.$$

When $k = \mathbb{Z}$, we simply write $\mathbb{A}^I$. For $n \geq 0$, we also write $\mathbb{A}_k^n = \mathbb{A}_k^{\{1,\dots,n\}}$ for the affine n-space over k. When $n = 1$, we call it the *affine line*; when $n = 2$, the *affine plane*.

***Remark* 1.1.3.** It is easy to see that:

- $\mathbb{A}_k^0$ is the terminal object of $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.
- $\mathbb{A}_k^1$ is the forgetful functor $\mathsf{CAlg}_k \to \mathsf{Set}$.
- $\mathbb{A}_k^{(-)} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is functorial.
- By the universal property of polynomial rings, $\mathbb{A}_k^I$ is represented by $k[x_i \mid i \in I]$, i.e., there is an isomorphism

$$\mathbb{A}_k^I \simeq \mathrm{Hom}_{\mathsf{CAlg}_k}(k[x_i \mid i \in I], -).$$

1.1.1. Presheaves and Yoneda Embedding

We begin by recalling the concept of presheaves.

Definition 1.1.4 (Presheaf). Let C be a category. A *presheaf* on C is a functor $C^{\mathrm{op}} \to \mathsf{Set}$. We denote

$$\mathsf{PShv}(C) := \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Set})$$

for the category of presheaves on C.

Note that algebraic k-functors defined earlier are precisely presheaves on $\mathsf{CAlg}_k^{\mathrm{op}}$.

***Remark* 1.1.5.** In functor categories, limits and colimits are computed pointwise. Thus $\mathsf{PShv}(C)$ inherits many properties of the category of sets, such as: filtered colimits commute with finite limits, epimorphisms are automatically effective, etc.

We now study the canonical functor from C to $\mathsf{PShv}(C)$, called the *Yoneda embedding*.

Definition 1.1.6 (Yoneda Embedding). Let C be a category. The *Yoneda embedding* of C is the functor

$$y : C \to \mathsf{PShv}(C), \quad y(X) := \mathrm{Hom}_C(-, X).$$

A functor in C is called *representable* if it lies in the essential image of the Yoneda embedding.

For $\mathcal{C} = \mathsf{CAlg}^{\mathrm{op}}$, the Yoneda embedding is

$$\mathsf{CAlg}^{\mathrm{op}} \to \mathsf{PShv}(\mathsf{CAlg}^{\mathrm{op}}) = \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}),$$

sending a ring R to $\mathrm{Hom}_{\mathsf{CAlg}^{\mathrm{op}}}(-, R) = \mathrm{Hom}_{\mathsf{CAlg}}(R, -)$. For this case, we use the special notation

$$\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$$

to denote this embedding. Thus Remark 1.1.3 shows that $\mathbb{A}^I_k \simeq \mathrm{Spec}(k[x_i \mid i \in I])$.

1.1.2. Category of Elements

For a functor $F\colon \mathcal{C} \to \mathsf{Set}$, we can encode its information as a category $\int_{\mathcal{C}} F$. Let Set_* denote the category of pointed sets:

- Objects are pairs (X, x) where X is a set and $x \in X$ is a chosen point.
- Morphisms $(X, x) \to (Y, y)$ are maps $f\colon X \to Y$ satisfying $f(x) = y$.

Definition 1.1.7 (Category of Elements). Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathsf{Set}$ a functor. The *category of elements* $\int_{\mathcal{C}} F$ is defined as the pullback:

$$\begin{tikzcd} \int_{\mathcal{C}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Set}_* \arrow[d] \\ \mathcal{C} \arrow[r] & \mathsf{Set} \end{tikzcd}$$

Concretely, $\int_{\mathcal{C}} F$ is the category where:

- Objects are pairs (C, x) with $C \in \mathcal{C}$ and $x \in F(C)$.
- A morphism $(C, x) \to (C', x')$ is a morphism $f\colon C \to C'$ in $\mathcal{C}$ such that $F(f)\colon F(C) \to F(C')$ satisfies $F(f)(x) = x'$.

For presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, we define the category of elements similarly:

Definition 1.1.8 (Category of Elements for Presheaves). Let $\mathcal{C}$ be a category and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a functor. The *category of elements* $\int^{\mathcal{C}} F$ is defined as the pullback:

$$\begin{tikzcd} \int^{\mathcal{C}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathsf{Set}_*)^{\mathrm{op}} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathsf{Set}^{\mathrm{op}} \end{tikzcd}$$

Concretely, $\int^{\mathcal{C}} F$ is the category where:

- Objects are pairs (C, x) with $C \in \mathcal{C}$ and $x \in F(C)$.
- A morphism $(C, x) \to (C', x')$ is a morphism $f\colon C \to C'$ in $\mathcal{C}$ such that $F(f)\colon F(C') \to F(C)$ satisfies $F(f)(x') = x$.

Note that there is a category equivalence

$$\left(\int_{\mathcal{C}} F\right)^{\mathrm{op}} \simeq \int^{\mathcal{C}^{\mathrm{op}}} F.$$

In Marc's script, only $\int^{\mathcal{C}} F$ is given, and we also denote $\mathrm{El}(F) = \int^{\mathcal{C}} F$.

For $\mathcal{C} = \mathsf{CAlg}^{\mathrm{op}}$ and an algebraic functor $X\colon \mathsf{CAlg} \to \mathsf{Set}$, we can consider its category of elements $\mathrm{El}(X)$. However, note that the more natural construction for our purposes, $\int_{\mathsf{CAlg}} X$, is actually $\mathrm{El}(X)^{\mathrm{op}}$.

***Remark* 1.1.9** (Grothendieck Construction). This construction generalizes to category-valued functors $F\colon \mathcal{C} \to \mathsf{Cat}$, yielding the *Grothendieck construction*. In the setting of ∞-categories, this extends to $F\colon \mathcal{C} \to \mathsf{An}$ and $F\colon \mathcal{C} \to \mathsf{Cat}_\infty$, giving left fibrations and coCartesian fibrations respectively.

1.1.3. Yoneda Lemma

Theorem 1.1.10 (Yoneda Lemma). *Let C be a category.*

(i) For $X \in C$ and $F \in \mathsf{PShv}(C)$, there is a bijection

$$\mathrm{Hom}_{\mathsf{PShv}(C)}(y(X), F) \simeq F(X), \quad f \mapsto f_X(\mathrm{id}_X).$$

The inverse is given by $x \mapsto ((f : Y \to X) \mapsto f^*(x))$.

(ii) The Yoneda embedding $y : C \to \mathsf{PShv}(C)$ is fully faithful.

(iii) The Yoneda embedding preserves all limits that exist in C.

(iv) (Density of Yoneda embedding) Every presheaf $F \in \mathsf{PShv}(C)$ can be written as a colimit of representable functors:

$$\operatorname{colim}\left(\mathrm{El}(F) \xrightarrow{\text{forget}} C \xrightarrow{y} \mathsf{PShv}(C)\right) \simeq F.$$

(v) (Universal property of presheaves) For any cocomplete category $\mathcal{E}$, there is a category equivalence

$$y^* \colon \mathsf{Fun}^{\mathrm{colim}}(\mathsf{PShv}(C), \mathcal{E}) \xrightarrow{\sim} \mathsf{Fun}(C, \mathcal{E}),$$

where $\mathsf{Fun}^{\mathrm{colim}}$ denotes colimit-preserving functors. The inverse is constructed via left Kan extension: for $F : C \to \mathcal{E}$, consider

$$\begin{tikzcd} C \arrow[d, hook, "y"] \arrow[dr, "F"] & \\ \mathsf{PShv}(C) \arrow[r, "\mathrm{Lan}_y F"'] & \mathcal{E} \end{tikzcd}$$

Since $\mathcal{E}$ has all colimits, this extension exists automatically.

(vi) For a category $\mathcal{E}$ and a colimit-preserving functor $K \colon \mathsf{PShv}(C) \to \mathcal{E}$, it has a right adjoint $N \colon \mathcal{E} \to \mathsf{PShv}(C)$ given by $e \mapsto \mathrm{Hom}_{\mathcal{E}}(K(y(-)), e)$. This is the realization-nerve adjunction.

Proof. We prove each part:

(1) Define $\Phi \colon \mathrm{Hom}_{\mathsf{PShv}(C)}(y(X), F) \to F(X)$ by $\Phi(\alpha) = \alpha_X(\mathrm{id}_X)$.
For the inverse, given $x \in F(X)$, define $\Psi(x) \colon y(X) \to F$ as follows: for each $Y \in C$,

$$\Psi(x)_Y \colon y(X)(Y) = \mathrm{Hom}_C(Y, X) \to F(Y), \quad (f : Y \to X) \mapsto F(f)(x).$$

To verify $\Psi(x)$ is natural: for $g \colon Z \to Y$, we need

$$\begin{tikzcd} \mathrm{Hom}(Y, X) \arrow[r, "\Psi(x)_Y"] \arrow[d, "g^*"'] & F(Y) \arrow[d, "F(g)"] \\ \mathrm{Hom}(Z, X) \arrow[r, "\Psi(x)_Z"'] & F(Z) \end{tikzcd}$$

to commute. For $f \in \mathrm{Hom}(Y, X)$: $(F(g) \circ \Psi(x)_Y)(f) = F(g)(F(f)(x)) = F(f \circ g)(x) = \Psi(x)_Z(f \circ g) = (\Psi(x)_Z \circ g^*)(f)$.

Check $\Phi \circ \Psi = \mathrm{id}$: $\Phi(\Psi(x)) = \Psi(x)_X(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = x$.

Check $\Psi \circ \Phi = \mathrm{id}$: For $\alpha \colon y(X) \to F$, we have $\Psi(\Phi(\alpha)) = \Psi(\alpha_X(\mathrm{id}_X))$. For any Y and $f : Y \to X$:

$$\Psi(\alpha_X(\mathrm{id}_X))_Y(f) = F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Y(y(X)(f)(\mathrm{id}_X)) = \alpha_Y(f),$$

where the second equality uses naturality of α.

(2) Apply (1) with $F = y(X')$ to get $\mathrm{Hom}(y(X), y(X')) \simeq y(X')(X) = \mathrm{Hom}(X, X')$.

(3) Limits in $\mathsf{PShv}(C)$ are computed pointwise. If $\lim_i X_i$ exists in C, then for any Y:

$$y(\lim_i X_i)(Y) = \mathrm{Hom}(Y, \lim_i X_i) \simeq \lim_i \mathrm{Hom}(Y, X_i) = \lim_i y(X_i)(Y).$$

Thus $y(\lim_i X_i) \simeq \lim_i y(X_i)$ in $\mathsf{PShv}(\mathcal{C})$.

(4) The colimit is computed as

$$\operatorname{colim}_{(C,x)\in \mathrm{El}(F)} y(C).$$

For any $Y \in \mathcal{C}$, evaluate at Y:

$$\left(\operatorname{colim}_{(C,x)} y(C)\right)(Y) = \operatorname{colim}_{(C,x)} y(C)(Y) = \operatorname{colim}_{(C,x)} \operatorname{Hom}(Y, C).$$

We claim this equals $F(Y)$. Indeed, an element of the colimit is represented by (C, x, f) where $x \in F(C)$ and $f : Y \to C$, modulo the equivalence: $(C, x, f) \sim (C', x', f')$ if there exists $g \colon C \to C'$ with $F(g)(x) = x'$ and $g \circ f = f'$. But this is precisely the data of an element of $F(Y)$: given $y \in F(Y)$, take (Y, y, id_Y).

(5) and (6) These follow from general category theory (Kan extensions and adjoint functor theorems). ■

1.2. Affine Schemes

1.2.1. Polynomial Equations

Definition 1.2.1 (System of Polynomial Equations)**.** Let k be a ring and let I and J be sets. A *system of J polynomial equations in I variables over k* is a J-tuple $\Sigma = (f_j)_{j\in J}$ in the polynomial ring $k[x_i \mid i \in I]$. We denote by (Σ) the ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j\in J}$ and by $k[\Sigma]$ the k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

***Remark* 1.2.2.** Every k-algebra R is isomorphic to $k[\Sigma]$ for some system of polynomial equations Σ. A choice of isomorphism $R \simeq k[\Sigma]$ is exactly a presentation of R by generators and relations.

Given a system of polynomial equations over k, we can consider its solutions in any k-algebra. To that end, recall that there is, for any k-algebra R, an *evaluation map*

$$k[x_i \mid i \in I] \times R^I \to R, \quad (f, a) \mapsto f(a),$$

which is defined as follows: for each $a \in R^I$, $f \mapsto f(a)$ is the unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to a_i.

Definition 1.2.3 (Vanishing Locus)**.** Let $F \subset k[x_i \mid i \in I]$ be a subset. The *vanishing locus of F in $\mathbb{A}^I_k$* is the subfunctor $V(F) \subset \mathbb{A}^I_k$ given by

$$V(F)(R) = \{a \in R^I \mid f(a) = 0 \text{ for all } f \in F\} \subset R^I.$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $R^I \to S^I$ sends $V(F)(R)$ to $V(F)(S)$.

***Remark* 1.2.4.** It is clear that the vanishing locus of F depends only on the ideal generated by F: if $(F) = (F')$, then $V(F) = V(F')$. We will see below that the converse also holds (Corollary 1.2.11).

Definition 1.2.5 (Solution Functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be a system of J polynomial equations in I variables over k. Its *solution functor* $\mathrm{Sol}_\Sigma \colon \mathsf{CAlg}_k \to \mathsf{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{A}^I_k$:

$$\mathrm{Sol}_\Sigma = V(\{f_j \mid j \in J\}) \subset \mathbb{A}^I_k.$$

By the universal property of polynomial rings, there is a one-to-one correspondence between systems of J polynomial equations in I variables and k-algebra maps

$$k[x_j \mid j \in J] \to k[x_i \mid i \in I].$$

By the Yoneda lemma, these are in turn equivalent to natural transformations

$$\mathbb{A}^I_k \to \mathbb{A}^J_k \colon \mathsf{CAlg}_k \to \mathsf{Set}.$$

Unraveling these equivalences, the map $\mathbb{A}^I_k \to \mathbb{A}^J_k$ corresponding to a system $\Sigma = (f_j)_{j\in J}$ is given on a k-algebra R by

$$R^I \to R^J, \quad a \mapsto (f_j(a))_{j\in J}.$$

By definition, the solution functor Sol_Σ is the kernel of this map, i.e., there is a pullback square

$$\begin{tikzcd}
\mathrm{Sol}_\Sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^I_k \arrow[d] \\
0 \arrow[r] & \mathbb{A}^J_k
\end{tikzcd} \tag{1.1}$$

where 0 is the subfunctor of $\mathbb{A}^J_k$ given by $0(R) = \{0\} \subset R^J$.

Definition 1.2.6 (Affine Scheme). A functor $\mathsf{CAlg}_k \to \mathsf{Set}$ is called an *affine k-scheme* if it is isomorphic to Sol_Σ for some system of polynomial equations Σ over k. We denote by $\mathsf{Aff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ the full subcategory spanned by the affine k-schemes. An *affine scheme* is an affine $\mathbb{Z}$-scheme.

Example 1.2.7. The affine I-space $\mathbb{A}^I_k$ is an affine k-scheme, as it is the solution functor of the empty system of equations in I variables.

Lemma 1.2.8. *Let Σ be a system of polynomial equations over k. Then the solution functor Sol_Σ is represented by the k-algebra $k[\Sigma]$, i.e., there is an isomorphism*

$$\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])\colon \mathsf{CAlg}_k \to \mathsf{Set}.$$

Proof. By definition, $k[\Sigma] = k[x_i \mid i \in I]/(\Sigma)$. For any k-algebra R, a k-algebra map $\varphi\colon k[\Sigma] \to R$ corresponds to a k-algebra map $\tilde{\varphi}\colon k[x_i \mid i \in I] \to R$ that sends each $f_j \in \Sigma$ to zero.

By the universal property of the polynomial ring, $\tilde{\varphi}$ is determined by the tuple $a = (\tilde{\varphi}(x_i))_{i\in I} \in R^I$. The condition that $\tilde{\varphi}(f_j) = 0$ for all $j \in J$ is exactly the condition that $f_j(a) = 0$ for all $j \in J$, i.e., $a \in \mathrm{Sol}_\Sigma(R)$.

Thus we have a bijection

$$\mathrm{Hom}_{\mathsf{CAlg}_k}(k[\Sigma], R) \xrightarrow{\sim} \mathrm{Sol}_\Sigma(R), \quad \varphi \mapsto (\varphi(x_i))_{i\in I},$$

which is natural in R. Therefore $\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])$. ∎

Theorem 1.2.9 (Characterization of Affine Schemes). *Let k be a ring. The following conditions are equivalent for an algebraic k-functor $X\colon \mathsf{CAlg}_k \to \mathsf{Set}$:*

(i) X is an affine k-scheme.

(ii) X is representable, i.e., isomorphic to $\mathrm{Spec}(A)$ for some k-algebra A.

(iii) X preserves limits and is accessible.

Proof. *(i) ⇒ (ii):* If X is an affine k-scheme, then $X \simeq \mathrm{Sol}_\Sigma$ for some system of polynomial equations Σ. By Lemma 1.2.8, $X \simeq \mathrm{Spec}(k[\Sigma])$, so X is representable.

(ii) ⇒ (iii): If $X \simeq \mathrm{Spec}(A)$ for some k-algebra A, then $X(R) = \mathrm{Hom}_{\mathsf{CAlg}_k}(A, R)$. Since $\mathrm{Hom}_{\mathsf{CAlg}_k}(A, -)$ is a representable functor, it preserves all limits (by the Yoneda lemma).

For accessibility: A functor $F\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is accessible if there exists a regular cardinal κ such that F preserves κ-filtered colimits. For a representable functor $\mathrm{Hom}(A, -)$, we can choose κ to be larger than the cardinality of A. Then $\mathrm{Hom}(A, -)$ preserves κ-filtered colimits because such colimits are computed pointwise, and elements of A are finitely presented.

(iii) ⇒ (i): Suppose X preserves limits and is accessible.

Since X is accessible, there exists a small diagram of representable functors whose colimit is X. More precisely, there exist k-algebras A_α and a diagram

$$X \simeq \mathrm{colim}_\alpha \, \mathrm{Spec}(A_\alpha).$$

Since colimits in $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ are computed pointwise and Spec is contravariant, this corresponds to a limit diagram in $\mathsf{CAlg}_k^{\mathrm{op}}$:

$$X \simeq \mathrm{Spec}\left(\lim_\alpha A_\alpha\right) = \mathrm{Spec}(A)$$

for some k-algebra A that is the limit of the A_α.

By Remark 1.2.2, we can write $A \simeq k[\Sigma]$ for some system of polynomial equations Σ. Thus $X \simeq \mathrm{Spec}(k[\Sigma]) \simeq \mathrm{Sol}_\Sigma$. ∎

Corollary 1.2.10. *The Yoneda embedding of* $\mathsf{CAlg}_k^{\mathrm{op}}$ *induces an equivalence of categories*

$$\mathrm{Spec}\colon \mathsf{CAlg}_k^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set}).$$

Under this equivalence, the affine k*-scheme* Sol_Σ *corresponds to the* k*-algebra* $k[\Sigma]$*.*

Proof. This follows immediately from Theorem 1.2.9. The Yoneda embedding is fully faithful and essentially surjective onto the representable functors, which by (i) $\Leftrightarrow$ (ii) are exactly the affine k-schemes. ■

Under the equivalence of Corollary 1.2.10, the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ of affine k-schemes corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[\Sigma]$. This implies the following result:

Corollary 1.2.11 (Functorial Nullstellensatz)**.** *Sending a subset* $F \subset k[x_i \mid i \in I]$ *to its vanishing locus* $V(F) \subset \mathbb{A}^I_k$ *induces an order-reversing bijection*

$$V\colon \{\textit{ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{A}^I_k\}.$$

Proof. By Remark 1.2.4, $V(F)$ depends only on the ideal (F), so we can view V as a map from ideals.

Injectivity: Suppose $V(F) = V(F')$ for ideals (F) and (F'). The embedding $V(F) \hookrightarrow \mathbb{A}^I_k$ corresponds under the equivalence Spec to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[x_i \mid i \in I]/(F)$. Similarly for $V(F')$.

Since $V(F) = V(F')$ as subfunctors, their corresponding quotient maps define the same subfunctor, hence the same ideal. Thus $(F) = (F')$.

Surjectivity: Let $Z \subset \mathbb{A}^I_k$ be a vanishing locus. By definition, there exists a system of polynomial equations Σ such that $Z = \mathrm{Sol}_\Sigma = V(\Sigma)$. Thus $Z = V((\Sigma))$ for the ideal (Σ).

The order-reversing property is clear: if $(F) \subset (F')$, then $V(F') \subset V(F)$. ■

Example 1.2.12. Consider the following systems of polynomial equations over $\mathbb{R}$ in one variable:

$$\Sigma_1 = (x^2+1), \quad \Sigma_2 = ((x^2+1)^2), \quad \Sigma_3 = (x^2+x+1), \quad \Sigma_4 = (x^4+1).$$

Then:

$$\begin{array}{llll} \mathrm{Sol}_{\Sigma_1}(\mathbb{R}) = \varnothing, & \mathrm{Sol}_{\Sigma_2}(\mathbb{R}) = \varnothing, & \mathrm{Sol}_{\Sigma_3}(\mathbb{R}) = \varnothing, & \mathrm{Sol}_{\Sigma_4}(\mathbb{R}) = \varnothing, \\ \mathrm{Sol}_{\Sigma_1}(\mathbb{C}) = \{\pm i\}, & \mathrm{Sol}_{\Sigma_2}(\mathbb{C}) = \{\pm i\}, & \mathrm{Sol}_{\Sigma_3}(\mathbb{C}) = \{\zeta_3, \bar{\zeta}_3\}, & \mathrm{Sol}_{\Sigma_4}(\mathbb{C}) = \{\pm\zeta_8, \pm\bar{\zeta}_8\}, \end{array}$$

where $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$. All four equations have the same solutions in $\mathbb{R}$. However, as the four ideals (Σ_i) in $\mathbb{R}[x]$ are pairwise distinct, they define four different subfunctors of $\mathbb{A}^1_{\mathbb{R}}$ by Corollary 1.2.11. The solutions in $\mathbb{C}$ distinguish them, except for Sol_{Σ_1} and Sol_{Σ_2}. To see that $\mathrm{Sol}_{\Sigma_1} \neq \mathrm{Sol}_{\Sigma_2}$ as subfunctors of $\mathbb{A}^1_{\mathbb{R}}$, we can compute the solutions in the $\mathbb{R}$-algebra $\mathbb{C}[\varepsilon]$ of dual complex numbers (where $\varepsilon^2 = 0$):

$$\mathrm{Sol}_{\Sigma_1}(\mathbb{C}[\varepsilon]) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbb{C}[\varepsilon]) = \{\pm i + a\varepsilon \mid a \in \mathbb{C}\}.$$

On the other hand, the associated $\mathbb{R}$-algebras are

$$\mathbb{R}[\Sigma_1] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_2] \simeq \mathbb{C}[\varepsilon], \quad \mathbb{R}[\Sigma_3] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_4] \simeq \mathbb{C} \times \mathbb{C}.$$

By Lemma 1.2.8, Sol_{Σ_1} and Sol_{Σ_3} are both isomorphic to $\mathrm{Spec}(\mathbb{C})$. The different ideals (Σ_1) and (Σ_3) correspond to two different embeddings of the affine $\mathbb{R}$-scheme $\mathrm{Spec}(\mathbb{C})$ into $\mathbb{A}^1_{\mathbb{R}}$, and the systems Σ_1 and Σ_3 themselves are two different presentations of the $\mathbb{R}$-algebra $\mathbb{C}$.

***Remark* 1.2.13.** In summary, given a system of polynomial equations Σ over k, we have the following relations between Σ and Sol_Σ:

(i) The data of the pullback square (1.1) is equivalent to the data of Σ itself.

(ii) The data of the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ is equivalent to the data of the ideal (Σ) in the polynomial ring $k[x_i \mid i \in I]$.

(iii) The data of the affine k-scheme Sol_Σ alone is equivalent to the data of the k-algebra $k[\Sigma]$.

This can be compared with the following types of data in differential geometry:

(i) A smooth manifold M given as the vanishing locus of a smooth function $\mathbb{R}^n \to \mathbb{R}^m$.

(ii) A smooth manifold M given as a closed submanifold of $\mathbb{R}^n$.

(iii) A smooth manifold M.

Smooth manifolds are the basic objects of interest in differential geometry. Embedding a manifold M into a Euclidean space or realizing it as the vanishing locus of a function are often useful ways to understand M, but we do not consider this additional data to be part of the manifold M itself. The situation in algebraic geometry is entirely similar: the basic objects of interest are affine schemes. Any affine scheme X can be embedded into an affine space ($X \hookrightarrow \mathbb{A}^I$) or realized as the solution functor of a system of polynomial equations ($X \simeq \mathrm{Sol}_\Sigma$), but this data is not part of the affine scheme X itself.

1.2.2. Examples

Example 1.2.14 (Terminal Scheme). The algebraic functor $\mathsf{CAlg} \to \mathsf{Set}, R \mapsto \{*\}$ is isomorphic to $\mathrm{Spec}(\mathbb{Z})$, hence is an affine scheme. In fact, it is the terminal object of $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.

Example 1.2.15 (Initial Scheme). This example shows that the Yoneda embedding $\mathrm{Spec}\colon \mathsf{Aff} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ does not preserve colimits.

- In Aff, the initial object is the functor

$$\mathsf{CAlg} \to \mathsf{Set}, \quad R \mapsto \begin{cases} \varnothing, & R \neq 0 \\ \{*\}, & R = 0, \end{cases}$$

  which is isomorphic to $\mathrm{Spec}(0)$.

- However, in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, the initial object is the constant functor

$$\mathsf{CAlg} \to \mathsf{Set}, \quad R \mapsto \varnothing,$$

  which is not representable, hence not an affine scheme.

Example 1.2.16 (Idempotent Classifier). Let $\mathrm{Idem}\colon \mathsf{CAlg} \to \mathsf{Set}$ be the functor sending R to its set of idempotent elements $\mathrm{Idem}(R) = \{e \in R \mid e^2 = e\}$. There is a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z} \times \mathbb{Z}, R) &\xrightarrow{\sim} \mathrm{Idem}(R) \\ \varphi &\mapsto \varphi(1,0). \end{aligned}$$

Since this is natural in R, we have $\mathrm{Idem} \simeq \mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})$, hence affine.

Example 1.2.17 (Multiplicative Group $\mathbb{G}_m$). Let $\mathbb{G}_m\colon \mathsf{CAlg} \to \mathsf{Ab}$ be the functor sending a ring R to its group of units $R^\times$. We claim this functor is represented by the *monoid ring* $\mathbb{Z}[\mathbb{Z}]$.

First, recall that the monoid ring functor $\mathbb{Z}[-]\colon \mathsf{CMon} \to \mathsf{CAlg}$ is left adjoint to the forgetful functor. For any ring R and commutative monoid M:

$$\mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z}[M], R) \simeq \mathrm{Hom}_{\mathsf{CMon}}(M, (R, \cdot)),$$

where $(R, \cdot)$ denotes the multiplicative monoid structure on R.

Taking $M = \mathbb{Z}$ (the additive group, viewed as a monoid): since $\mathbb{Z}$ is a group, any monoid homomorphism $\mathbb{Z} \to (R, \cdot)$ factors through $R^\times$. Moreover, $\mathbb{Z}$ is the free group on one generator, so group homomorphisms $\phi\colon \mathbb{Z} \to R^\times$ are determined by $\phi(1) \in R^\times$.

Therefore:

$$\mathbb{G}_m(R) = R^\times \simeq \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, R^\times) \simeq \mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z}[\mathbb{Z}], R).$$

This proves $\mathbb{G}_m \simeq \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}])$. Concretely, $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ (Laurent polynomials).

***Remark* 1.2.18** ($\mathbb{G}_m$ as Tate Circle). In higher category theory, the classifying space of $\mathbb{Z}$ is homotopy equivalent to the circle: $B\mathbb{Z} \simeq S^1$. For a presentable monoidal category $\mathcal{C}$, we have

$$\mathsf{Fun}(B\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{Z}}(\mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{1}[\mathbb{Z}]}(\mathcal{C}),$$

where $\mathbb{1}[\mathbb{Z}]$ is the monoid ring object in $\mathcal{C}$.

Since $\mathbb{Z}[\mathbb{Z}]$ is the coordinate ring $\mathcal{O}(\mathbb{G}_m)$, this suggests a deep duality between the topological circle S^1 (corresponding to $\mathbb{Z}$) and the algebraic $\mathbb{G}_m$ in algebraic geometry. Therefore, $\mathbb{G}_m$ is often called the *Tate circle*.

In motivic homotopy theory $\mathsf{H}(S)$, we have two types of "circles":

- *Simplicial circle:* $S^{1,0} := S^1$ (constant sheaf).
- *Tate circle:* $S^{1,1} := \mathbb{G}_m$.

The motivic (i,j)-sphere is defined as:

$$S^{i,j} := (S^1)^{\wedge(i-j)} \wedge \mathbb{G}_m^{\wedge j}.$$

The projective line satisfies $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \simeq S^{2,1}$.

The stable motivic homotopy category $\mathsf{SH}(S)$ is constructed by formally inverting $(\mathbb{P}^1, \infty) \simeq S^{2,1}$ in the pointed motivic homotopy category $\mathsf{H}(S)_*$.

1.2.3. Functions on Algebraic Functors

Recall that $\mathbb{A}^1 \colon \mathsf{CAlg} \to \mathsf{Set}$ is the forgetful functor $R \mapsto R$. Thus $\mathbb{A}^1$ is a ring object in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$:

$$\begin{tikzcd} & \mathsf{CAlg} \arrow[d, "\text{forget}"] \\ \mathsf{CAlg} \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "\mathbb{A}^1"'] & \mathsf{Set} \end{tikzcd}$$

For any algebraic functor X, we have $\mathrm{Hom}(X, \mathbb{A}^1) \in \mathsf{CAlg}$. Pointwise, for a ring R:

$$\mathrm{Hom}(X(R), \mathbb{A}^1(R)) = \mathrm{Hom}_{\mathsf{Set}}(X(R), R).$$

Each $f \in \mathrm{Hom}(X, \mathbb{A}^1)$ functorially maps an R-point x to an element $f(x) \in R$. Thus we can view $\mathrm{Hom}(X, \mathbb{A}^1)$ as "functions" on X.

Similarly, $\mathbb{G}_m \colon \mathsf{CAlg} \to \mathsf{Set}$ factors as

$$\mathsf{CAlg} \xrightarrow{R \mapsto R^\times} \mathsf{Ab} \xrightarrow{\text{forget}} \mathsf{Set}.$$

For $f \in \mathrm{Hom}(X, \mathbb{G}_m)$, we have $f(x) \in R^\times$ for any $x \colon \mathrm{Spec}(R) \to X$ (with $R \neq 0$), so $f(x) \neq 0$. Thus f represents a "nowhere-vanishing function" on X.

Definition 1.2.19 (Functions)**.** Let X be an algebraic functor.

(i) A *function* on X is a morphism $X \to \mathbb{A}^1$. We denote

$$\mathcal{O}(X) := \mathrm{Hom}(X, \mathbb{A}^1)$$

for the ring of all functions on X, called the *coordinate ring* of X.

(ii) A *nowhere-vanishing function* on X is a morphism $X \to \mathbb{G}_m$. We denote

$$\mathcal{O}^\times(X) := \mathrm{Hom}(X, \mathbb{G}_m)$$

for the group of all nowhere-vanishing functions on X.

***Remark* 1.2.20.** It is easy to see:

- $\mathbb{G}_m \subset \mathbb{A}^1$ is a monomorphism, hence $\mathcal{O}^\times(X) \subset \mathcal{O}(X)$.
- $\mathcal{O}(\mathrm{Spec}(R)) = \mathrm{Hom}(\mathrm{Spec}(R), \mathbb{A}^1) \simeq \mathbb{A}^1(R) \simeq R$.

- Since every presheaf can be written as a colimit of representables:
$$\mathcal{O}(X) = \mathrm{Hom}\left(\mathrm{colim}_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Spec}(R), \mathbb{A}^1\right) \simeq \lim_{x\colon \mathrm{Spec}(R)\to X} R.$$
Thus $f \in \mathcal{O}(X)$ corresponds to $(f \circ x)_x$. Similarly:
$$\mathcal{O}^\times(X) \simeq \lim_{x\colon \mathrm{Spec}(R)\to X} R^\times.$$
Since $(-)^\times$ commutes with limits (as $R^\times \simeq \mathrm{Hom}(\mathbb{Z}[u^{\pm 1}], R)$ and $\mathrm{Hom}(\mathbb{Z}[u^{\pm 1}], -)$ preserves limits), we have $\mathcal{O}^\times(X) \simeq \mathcal{O}(X)^\times$.
- The functor
$$\mathcal{O}\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} \longrightarrow \mathsf{CAlg}, \quad X \mapsto \mathcal{O}(X)$$
preserves limits. By the universal property of presheaves:
$$\mathsf{Fun}^{\lim}(\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}}, \mathsf{CAlg}) \simeq \mathsf{Fun}(\mathsf{CAlg}, \mathsf{CAlg}),$$
where $\mathsf{Fun}^{\lim}$ denotes functors preserving small limits. The functor $\mathcal{O}$ corresponds to $\mathrm{id}_{\mathsf{CAlg}}$ under this equivalence.

***Warning* 1.2.21.** In Remark 1.2.20, since $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is a large category, the indexing category for $\mathcal{O}(X)$ may become large, so $\mathcal{O}(X)$ might not be an object of CAlg but rather a large ring (an object of $\widehat{\mathsf{CAlg}}$).

There are two approaches to resolve this issue:

(i) Introduce $\widehat{\mathsf{Set}}$ and $\widehat{\mathsf{CAlg}}$, allowing us to describe $\mathcal{O}$ as a functor
$$\mathcal{O}\colon \mathsf{Fun}(\mathsf{CAlg}, \widehat{\mathsf{Set}})^{\mathrm{op}} \longrightarrow \widehat{\mathsf{CAlg}}.$$
This would be the unique large-limit-preserving extension of $\mathsf{CAlg} \hookrightarrow \widehat{\mathsf{CAlg}}$.

(ii) Consider the full subcategory $\mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})$ of accessible functors. Recall that for a presentable category $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *accessible* if there exists κ such that F preserves κ-filtered colimits. For $\mathcal{D} = \mathsf{Set}$ (which is a topos), κ-filtered colimits commute with κ-small limits, so F can be written as a κ-small colimit of representables. For $X \in \mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})$, it can be written as a small colimit of $\mathrm{Spec}(R_i)$, giving a functor
$$\mathcal{O}\colon \mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} \longrightarrow \mathsf{CAlg}.$$
This approach is more common in modern derived algebraic geometry.

For a ring A, we have:
$$\begin{aligned}\mathrm{Hom}(X, \mathrm{Spec}(A)) &\simeq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Hom}(\mathrm{Spec}(R), \mathrm{Spec}(A)) \\ &\simeq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Hom}(A, R) \\ &\simeq \mathrm{Hom}(A, \lim_x R) \\ &\simeq \mathrm{Hom}(A, \mathcal{O}(X)).\end{aligned}$$

Thus we have an adjoint pair:
$$\begin{tikzcd} \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathsf{CAlg} \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

1.2.4. Loci

We have the diagram:
$$0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m$$

For an algebraic functor X and a function $f\colon X \to \mathbb{A}^1$, we can consider $f^{-1}(0)$ and $f^{-1}(\mathbb{G}_m)$. We view $f^{-1}(0)$ as the locus where f vanishes, and $f^{-1}(\mathbb{G}_m)$ as the locus where f is nowhere-vanishing.

Definition 1.2.22 (Loci). Let X be an algebraic functor and $F \subset \mathcal{O}(X)$ a subset of functions on X.

(i) The *vanishing locus* of F is the subfunctor $V(F) \subset X$ defined by

$$V(F)(R) := \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in F\}.$$

(ii) The *non-vanishing locus* of F is the subfunctor $D(F) \subset X$ defined by

$$D(F)(R) := \{x \in X(R) \mid (f(x))_{f \in F} \text{ generates the unit ideal in } R\}.$$

***Remark* 1.2.23.** Note that $V(F)$ depends only on the ideal $(F) \subset \mathcal{O}(X)$. If $\sqrt{(G)} = \sqrt{(F)}$, then $D(F) = D(G)$, so $D(F)$ depends only on $\sqrt{(F)}$.

Example 1.2.24 (Punctured Affine Space). We have $\mathbb{A}^I = \operatorname{Spec}(\mathbb{Z}[x_i \mid i \in I])$. Define $\mathbb{A}^I - 0$ as the algebraic functor

$$(\mathbb{A}^I - 0)(R) = \{a \in R^I \mid (a_i)_{i \in I} \text{ generates } R\}.$$

Here (a) denotes the ideal generated by all components of $a \in R^I$.

The coordinate functions $x_i \colon \mathbb{A}^I \to \mathbb{A}^1$ send $a \in \mathbb{A}^I(R)$ to its i-th component a_i. Therefore:

$$\mathbb{A}^I - 0 = D(\{x_i \mid i \in I\}) = \{a \in R^I \mid (x_i(a))_{i \in I} \text{ generates } R\}.$$

Note that when $I = \{1\}$, we have $\mathbb{A}^1 - 0 \simeq \mathbb{G}_m$.

For an algebraic functor X and $F \subset \mathcal{O}(X)$, we can define a morphism

$$X \xrightarrow{f} \mathbb{A}^F, \quad (x \colon \operatorname{Spec}(R) \to X) \mapsto (g(x))_{g \in F},$$

such that

$$V(F) = f^{-1}(0) = \{x \in X(R) \mid f(x) = (g(x))_{g \in F} = 0\},$$
$$f^{-1}(\mathbb{A}^F - 0)(R) = \{x \in X(R) \mid (g(x))_{g \in F} \text{ generates } R\}.$$

***Warning* 1.2.25.** It is tempting to think that $D(F)$ is somehow the "complement" of $V(F)$ in X, but this is false. While this holds over fields, it fails over general rings. For example, consider $R = 0$: then $D(F)(0) = V(F)(0) = X(0)$, so they are not even disjoint.

However, in certain subcategories of Fun(CAlg, Set), such as Sch (schemes), this does hold.

Similarly, $V(F)$ is not the complement of $D(F)$, because $D(F) = D(F')$ can hold while $V(F) \neq V(F')$ (since V depends on ideals while D depends on radical ideals).

Proposition 1.2.26 (Properties of Loci). *Let X be an algebraic functor.*

(i) For any family $(F_i)_{i \in I}$ of subsets of $\mathcal{O}(X)$:

$$\bigcap_{i \in I} V(F_i) = V\left(\bigcup_{i \in I} F_i\right)$$

and

$$\bigcup_{i \in I} D(F_i) \subset D\left(\bigcup_{i \in I} F_i\right).$$

The inclusion becomes equality when evaluated on local rings.

(ii) For finitely many subsets $F_1, \ldots, F_n \subset \mathcal{O}(X)$:

$$D(F_1) \cap \cdots \cap D(F_n) = D(F_1 \cdots F_n),$$

and

$$V(F_1) \cup \cdots \cup V(F_n) \subset V(F_1 \cdots F_n).$$

The inclusion becomes equality when evaluated on integral domains.

Proof. We verify these pointwise on all rings R.

(1) For the vanishing loci:

$$\begin{aligned}\bigcap_{i\in I} V(F_i)(R) &= \bigcap_{i\in I}\{x \in X(R) \mid f_i(x) = 0 \text{ for all } f_i \in F_i\} \\ &= \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in \bigcup_{i\in I} F_i\} \\ &= V\left(\bigcup_{i\in I} F_i\right)(R).\end{aligned}$$

For the non-vanishing loci:

$$\begin{aligned}\bigcup_{i\in I} D(F_i)(R) &= \bigcup_{i\in I}\{x \in X(R) \mid (f_i(x))_{f_i\in F_i} = R\} \\ &\subset \{x \in X(R) \mid (f(x))_{f\in\bigcup_{i\in I} F_i} = R\} \\ &= D\left(\bigcup_{i\in I} F_i\right)(R).\end{aligned}$$

The inclusion holds because if $(f_i(x))_{f_i\in F_i} = R$ for some i, then certainly $\bigcup_i F_i$ generates R.

When R is a local ring with unique maximal ideal $\mathfrak{m}$: if $(f(x))_{f\in\bigcup F_i} = R$, then at least one $f(x)$ is a unit. This f belongs to some F_i, so $(f_i(x))_{f_i\in F_i} = R$, hence $x \in D(F_i)(R)$. Thus equality holds.

(2) The equation $D(F_1) \cap \cdots \cap D(F_n) = D(F_1 \cdots F_n)$ is straightforward: x satisfies $(f_i(x))_{f_i\in F_i} = R$ for all i if and only if $(f_1(x) \cdots f_n(x))_{f_i\in F_i} = R$.

For the vanishing loci, the inclusion $V(F_1) \cup \cdots \cup V(F_n) \subset V(F_1 \cdots F_n)$ is clear.

When R is an integral domain: if $f(x) = 0$ for all $f \in F_1 \cdots F_n$, write $f = f_1 \cdots f_n$ with $f_i \in F_i$. Since R is a domain and $f_1(x) \cdots f_n(x) = 0$, at least one $f_i(x) = 0$. Thus $x \in V(F_i)(R)$ for some i. ■

Example 1.2.27. We have $\mathbb{Z} = (2, 3)$, but $V(2) \cap V(3) = V(1)$, while $D(2) \cup D(3) \neq D(1)$. This confirms that $D(F)$ is not the complement of $V(F)$.

Now we characterize open and closed subfunctors of affine schemes.

Proposition 1.2.28 (Subfunctors of $\operatorname{Spec}(A)$). *Let A be a ring.*

(i) For any subset $F \subset A$, the quotient map $A \to A/(F)$ induces an isomorphism

$$\operatorname{Spec}(A/(F)) \xrightarrow{\sim} V(F) \subset \operatorname{Spec}(A).$$

Therefore $V(F)$ is an affine scheme.

(ii) For any $f \in A$, the localization map $A \to A_f$ induces an isomorphism

$$\operatorname{Spec}(A_f) \xrightarrow{\sim} D(f) \subset \operatorname{Spec}(A).$$

Therefore $D(f)$ is an affine scheme.

Proof. *(1)* For $F \subset A$:

$$V(F)(R) = \{\varphi \in \operatorname{Hom}(A, R) \mid \varphi(f) = 0 \text{ for all } f \in F\}.$$

This condition is equivalent to φ factoring through $A/(F)$, i.e., $\varphi \in \operatorname{Hom}(A/(F), R)$.

(2) For a ring R, a map $\psi\colon A_f \to R$ corresponds to a map $\tilde{\psi}\colon A \to R$ sending f to a unit in R. Thus $\operatorname{Hom}(A_f, R)$ is in bijection with $D(f)(R)$. ■

***Remark* 1.2.29.** For a general subset $F \subset A$, $D(F)$ is typically not affine. For instance, Example 1.2.24 shows $\mathbb{A}^I - 0 \simeq D(\{x_i \mid i \in I\})$ is not affine when $|I| > 1$.

1.2.5. Open and Closed Subfunctors

Definition 1.2.30 (Open and Closed Subfunctors). Let X be an algebraic functor and $Z \subset X$ a subfunctor.

(i) Z is a *closed subfunctor* of X if for every R-point $x\colon \operatorname{Spec}(R) \to X$, there exists $F \subset R$ such that $x^{-1}(Z) \simeq V(F)$.

(ii) Z is an *open subfunctor* of X if for every R-point $x\colon \operatorname{Spec}(R) \to X$, there exists $F \subset R$ such that $x^{-1}(Z) \simeq D(F)$.

Proposition 1.2.31 (Classification of Subfunctors of $\operatorname{Spec}(A)$). *Let A be a ring.*

(i) $F \mapsto V(F)$ induces an order-reversing bijection

$$\{\textit{ideals in } A\} \xrightarrow{\sim} \{\textit{closed subfunctors of } \operatorname{Spec}(A)\}.$$

(ii) $F \mapsto D(F)$ induces an order-preserving bijection

$$\{\textit{radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \operatorname{Spec}(A)\}.$$

Proof. *(1)* First, $V(F) \subset \operatorname{Spec}(A)$ is indeed a closed subfunctor: for any $x\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ (corresponding to $\varphi\colon A \to R$):

$$x^{-1}(V(F)) \simeq V(\varphi(F)) \subset \operatorname{Spec}(R).$$

Injectivity: Note that $f \in (F)$ if and only if $V(F) \subset V(f)$. If $V(F) = V(F')$, then for any $f \in F$:

$$V(F') = \bigcap_{f' \in F'} V(f') = V(F) \subset V(f),$$

so $f \in (F')$. Thus $(F) \subset (F')$. By symmetry, $(F') \subset (F)$, so $(F) = (F')$.

Surjectivity: For any closed subfunctor $Z \subset \operatorname{Spec}(A)$, consider $x = \mathrm{id}_{\operatorname{Spec}(A)}$. By definition, $\mathrm{id}^{-1}(Z) \simeq V(F)$ for some $F \subset A$. Thus $Z \simeq V(F)$.

(2) For F with $\sqrt{(F)} = F$, the well-definedness and surjectivity are clear from the definition.

Injectivity: We need to show $f \in F$ if and only if $D(f) \subset D(F)$.

($\Leftarrow$) Assume $D(f) \subset D(F)$. Evaluating at $R = A_f$, we have $D(f)(A_f)$ contains the canonical map $\varphi\colon A \to A_f$. Since $D(f) \simeq \operatorname{Spec}(A_f)$, this is the unique element of $D(f)(A_f)$.

The inclusion $D(f) \subset D(F)$ means $\varphi \in D(F)(A_f)$, i.e., $(\varphi(F)) = A_f$. Therefore, there exist $a \in F$ and $n \in \mathbb{N}$ such that

$$\frac{a}{f^n} = 1 \in A_f.$$

Thus there exists $k \in \mathbb{N}$ with $f^k a = f^{n+k}$, so $f^{n+k} \in (F)$. Since F is radical, $f \in F$.

($\Rightarrow$) is immediate from the definition of D. ∎

Definition 1.2.32 (Embeddings). Let $Y \to X$ be a morphism of algebraic functors.

(i) $Y \to X$ is a *closed embedding* if it is a monomorphism with closed image.

(ii) $Y \to X$ is an *open embedding* if it is a monomorphism with open image.

Definition 1.2.33 (Locally Closed Subfunctors). Let X be an algebraic functor.

(i) A subfunctor $Y \subset X$ is *locally closed* if there exists an open subfunctor $U \subset X$ such that $Y \subset U$ is a closed subfunctor of U.

(ii) A morphism $Y \to X$ is an *embedding* if it is a monomorphism with locally closed image.

Proposition 1.2.34 (Stability of Embeddings). *(i) (Base change) Consider a pullback square:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If f is a closed embedding, then so is f'. The same holds for open embeddings and embeddings.

(ii) (Composition) Consider a commutative diagram:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

If f and g are closed embeddings, then so is h. If h is a closed embedding and f is a monomorphism, then g is a closed embedding. The same holds for open embeddings and embeddings.

Proof. (1) We prove for closed embeddings; the other cases are similar.

If f is a closed embedding, then f is a monomorphism and has closed image. Monomorphisms are stable under pullback, so f' is a monomorphism.

For the image, let $x'\colon \operatorname{Spec}(R) \to X'$ be an R-point, and let $x\colon \operatorname{Spec}(R) \to X$ be the composition. By the pasting lemma:

$$\begin{tikzcd} x'^{-1}(f'(Y')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R) \arrow[d, "x'"] \\ Y' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

the outer rectangle is a pullback, so $x^{-1}(f(Y)) \simeq x'^{-1}(f'(Y'))$. Since $f(Y)$ is closed in X, we have $x^{-1}(f(Y)) \simeq V(I)$ for some ideal $I \subset R$. Thus $f'(Y')$ is closed in X'.

(2 - Composition)

Closed case: If f and g are closed embeddings, they are both monomorphisms, so $h = f \circ g$ is a monomorphism (monomorphisms are closed under composition).

For the image: for $x\colon \operatorname{Spec}(R) \to X$, we have

$$x^{-1}(h(Z)) = x^{-1}(f(g(Z))).$$

Since f has closed image, $x^{-1}(f(Y)) \simeq \operatorname{Spec}(R/I_1)$ for some ideal $I_1 \subset R$. This gives $y\colon \operatorname{Spec}(R/I_1) \to Y$.

Since g has closed image in Y:

$$y^{-1}(g(Z)) \simeq \operatorname{Spec}(R/(I_1 + I_2))$$

for some ideal I_2. Thus $x^{-1}(f(g(Z))) \simeq \operatorname{Spec}(R/(I_1 + I_2))$, which is closed.

Open case: Assume $X \simeq \operatorname{Spec}(A)$. We need to show: for any subset $F \subset A$ and open subfunctor $U \subset D(F)$, U is an open subfunctor of $\operatorname{Spec}(A)$.

For each $f \in F$, we have $D(f) \subset \operatorname{Spec}(A)$ with $D(f) \simeq \operatorname{Spec}(A_f)$. Consider $U \cap D(f)$. Since $U \subset D(F)$ is open, $U \cap D(f)$ is an open subfunctor of $\operatorname{Spec}(A_f)$, so we can write

$$U \cap D(f) = D(\bar{H}_f) \subset \operatorname{Spec}(A_f)$$

for some $\bar{H}_f \subset A_f$. Choose $H_f \subset A$ whose image in A_f equals $\bar{H}_f$ (essentially clearing denominators). Then

$$D(fH_f) = D(H_f) \cap D(f) = D(\bar{H}_f) = U \cap D(f).$$

Let $H = \bigcup_{f \in F} fH_f \subset A$. We claim $U = D(H)$.

For $\varphi\colon A \to R$ (i.e., $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$):

($U \subset D(H)$): If $\operatorname{Spec}(\varphi) \in U$, we need $(\varphi(H)) = R$. Since $U \subset D(F)$, we have $(\varphi(F)) = R$. For any $f \in F$, the map $\operatorname{Spec}(R_{\varphi(f)}) \to \operatorname{Spec}(A_f)$ lies in $U \cap D(f) = D(fH_f)$. Thus $(\varphi(fH_f)) = R_{\varphi(f)}$, so $\varphi(fH_f)$ contains some power of $\varphi(f)$, i.e., $\varphi(f) \in \sqrt{(\varphi(fH_f))}$.

Taking the union over all $f \in F$:

$$R = \sqrt{(\varphi(F))} \subset \sqrt{(\varphi(H))}.$$

($D(H) \subset U$): If $(\varphi(H)) = R$, then since $H \subset (F)$, we have $D(H) \subset D(F)$. Since $U \subset D(F)$ is open, $\operatorname{Spec}(\varphi)^{-1}(U) = D(I)$ for some radical ideal $I \subset R$.

We need $I = R$. Consider the pullback with $D(f)$:

$$D(\varphi(f)I) = \mathrm{Spec}(\varphi)^{-1}(U \cap D(f)) = D(\varphi(fH_f)).$$

Thus $\sqrt{(\varphi(f)I)} = \sqrt{(\varphi(fH_f))}$. Taking the union over $f \in F$:

$$I \supset \sqrt{(\varphi(F)I)} = \sqrt{(\varphi(H))} = R.$$

Therefore $I = R$.

Embedding case: Let $X = \mathrm{Spec}(A)$, $Z \subset X$ closed, and $U \subset Z$ open. We show $U \subset X$ is locally closed.

Write $Z = \mathrm{Spec}(A/I)$ and $U = D(F)$ where $F \subset A/I$. Let $\hat{F}$ be the preimage of F in A and $\hat{U} = D(\hat{F})$. Then $\hat{U} \cap Z = U$:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ \hat{U} \arrow[r] & X \end{tikzcd}$$

Since $Z \to X$ is a closed embedding, so is $U \to \hat{U}$. Thus $U \to X$ is locally closed.

(2 - Decomposition) If h is a closed embedding and f is a monomorphism, consider the pullback:

$$\begin{tikzcd} Z \times_X Y \arrow[r, hook, "p_Y"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "f"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

Factor g as $Z \xrightarrow{(\mathrm{id},g)} Z \times_X Y \xrightarrow{p_Y} Y$. Since closed embeddings are stable under base change, $Z \times_X Y \to Y$ is a closed embedding.

It suffices to show (id, g) is a closed embedding. Consider:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \times_X Y \arrow[d] \\ Y \arrow[r, "\Delta_f"] & Y \times_X Y \end{tikzcd}$$

Since f is a monomorphism, Δ_f is an isomorphism, so $Z \simeq Z \times_X Y$, automatically a closed embedding. ∎

1.2.6. Quasi-Affine Schemes

Definition 1.2.35 (Quasi-Affine Scheme). Let k be a ring. An algebraic k-functor X is a *quasi-affine k-scheme* if there exist a k-algebra A and a finite subset $F \subset A$ such that $X \simeq D(F) \subset \mathrm{Spec}(A)$.

Denote $\mathsf{QAff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ for the full subcategory spanned by quasi-affine k-schemes.

Proposition 1.2.36 (Closed Subfunctors of Quasi-Affine Schemes). *Let X be a quasi-affine k-scheme and $Z \hookrightarrow X$ a closed embedding. Then Z is also a quasi-affine k-scheme.*

Proof. We prove for $k = \mathbb{Z}$; the general case is identical. Since X is quasi-affine, there exist a ring A and finite $F \subset A$ with $X \simeq D(F)$.

Let Y be the image of $Z \hookrightarrow X$, so Y is a closed subfunctor of $D(F)$. For any $f \in F$, $Y \cap D(f)$ is a closed subfunctor of $D(f)$. Let $J(f) \subset A_f$ be the corresponding ideal, and let $I(f) \subset A$ be its preimage under $A \to A_f$. Then $I(f)_f = J(f)$, so

$$V(I(f)) \cap D(f) = Y \cap D(f).$$

Define $I = \bigcap_{f \in F} I(f)$.

Claim: $Y \simeq D(\bar{F}) \subset \mathrm{Spec}(A/I) = V(I)$, where $\bar{F}$ is the image of F in A/I.

Let $\bar{Y}$ be the closure of Y in $\mathrm{Spec}(A)$ (the smallest closed subfunctor containing Y).

Step 1: $\bar{Y} = V(I)$.

First, $Y \subset V(I)$: Let $\varphi \colon A \to R$ be a point in $Y(R)$. Since $Y \subset D(F)$, we have $(\varphi(F)) = R$. We need $(\varphi(I)) = 0$.

For any $f \in F$, consider the composition $A \xrightarrow{\varphi} R \to R_{\varphi(f)}$. This corresponds to a point in $Y \cap D(f) = V(I(f)) \cap D(f)$ evaluated at $R_{\varphi(f)}$. Thus $\varphi(I(f)) = 0$ in $R_{\varphi(f)}$ for each f, which implies $\varphi(I(f)) = 0$ in R. Taking the intersection over $f \in F$ gives $\varphi(I) = 0$.

Conversely, $V(I) \subset \bar{Y}$: By Proposition 1.2.31, closed subfunctors of $\mathrm{Spec}(A)$ correspond to ideals. If $Y \subset V(K)$, we need $K \subset I$. By definition of I, this reduces to showing $K_f \subset J(f)$ in A_f.

From $V(J(f)) = Y \cap D(f) \subset V(K_f)$, we get $K_f \subset J(f)$.

Step 2: $Y = \bar{Y} \cap D(F)$.

First, we show $I_g = J(g)$ for all $g \in F$. One inclusion is clear. For $J(g) \subset I_g$, note that for any $f \in F$:

$$J(g)_f = J(f)_g \quad \text{in } A_{fg}$$

(both equal the ideal cutting out $Y \cap D(fg)$).

Since localization is exact, $I(f)_g$ is the preimage of $J(f)_g$ under $A_g \to A_{fg}$. Thus $J(g) \subset I(f)_g$ for all f, so $J(g) \subset I_g$.

A point $x\colon \mathrm{Spec}(R) \to \bar{Y} \cap D(F)$ corresponds to $\varphi\colon A \to R$ with $\varphi(I) = 0$ and $(\varphi(F)) = R$.

Since $Y \subset D(F)$ is closed in $D(F)$, we have $Y \subset \bar{Y} \cap D(F)$ is also closed. To show equality, consider the pullback $x^{-1}(Y) \simeq V(G)$ for some $G \subset R$. We need $G = 0$.

By Zariski descent (proved in the next section), it suffices to show $G_{\varphi(f)} = 0$ for all $f \in F$. But

$$V(G_{\varphi(f)}) = x^{-1}(Y) \cap D(\varphi(f)) = x^{-1}(Y \cap D(f)) = x^{-1}(V(I_f)) = V(\varphi(I_f)) = V(0).$$

Thus $G_{\varphi(f)} = 0$. ∎

1.2.7. Zariski Descent

Let $(f_i)_{i\in I}$ be a family of elements in a ring R generating the unit ideal. In this case,

$$\bigcap_{i\in I} V(f_i) = V\left(\bigcup_{i\in I}\{f_i\}\right) = \emptyset.$$

However, as mentioned in Warning 1.2.25, $D(f_i)$ is not the complement of $V(f_i)$. Thus, in general, the equality $\bigcup_{i\in I} D(f_i) = \mathrm{Spec}(R)$ does not hold in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.

In this section, we explain that if we restrict to the category of affine schemes $\mathsf{Aff} \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, then $\mathrm{Spec}(R)$ is indeed the union (i.e., colimit) of the $D(f_i)$. More precisely, a morphism $\varphi\colon \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is uniquely determined by a family $\{\varphi_i\colon D(f_i) \to \mathrm{Spec}(S)\}_{i\in I}$ satisfying the gluing condition: for all $i, j \in I$,

$$\varphi_i|_{D(f_i)\cap D(f_j)} = \varphi_j|_{D(f_i)\cap D(f_j)}.$$

For two elements $f, g \in R$, the statement "$D(f) \cup D(g) = \mathrm{Spec}(R)$ in Aff" is equivalent to saying that

$$\begin{tikzcd} D(f)\cap D(g) \arrow[r, hook] \arrow[d, hook] & D(f) \arrow[d] \\ D(g) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Spec}(R) \end{tikzcd}$$

is a pushout diagram in Aff. Equivalently, for any affine scheme X,

$$\begin{tikzcd} X(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X(R_f) \arrow[d] \\ X(R_g) \arrow[r] & X(R_{fg}) \end{tikzcd}$$

is a pullback (i.e., limit) diagram in Set.

More generally, the condition "$\bigcup_{i\in I} D(f_i) = \mathrm{Spec}(R)$ in Aff" is equivalent to the statement that for any affine scheme X,

$$X(R) \longrightarrow \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j})$$

is an equalizer diagram. We begin by establishing this for modules.

Theorem 1.2.37 (Zariski Descent for Modules). *Let R be a ring and let $(f_i)_{i\in I}$ be a family of elements in R generating the unit ideal.*

(i) (Descent for morphisms) *For R-modules M and N, the diagram*

$$\mathrm{Hom}_R(N,M) \longrightarrow \prod_{i\in I}\mathrm{Hom}_{R_{f_i}}(N_{f_i},M_{f_i}) \rightrightarrows \prod_{i,j\in I}\mathrm{Hom}_{R_{f_if_j}}(N_{f_if_j},M_{f_if_j})$$

is an equalizer in Set.

(ii) (Descent for objects) *Given the following data:*

- *For each $i\in I$, an R_{f_i}-module M_i;*
- *For each pair $(i,j)\in I\times I$, an $R_{f_if_j}$-linear isomorphism $\alpha_{ij}\colon (M_i)_{f_j}\xrightarrow{\sim}(M_j)_{f_i}$;*
- (Cocycle condition) *For each triple $(i,j,k)\in I\times I\times I$, we have*

$$\alpha_{jk}\circ\alpha_{ij}=\alpha_{ik}\quad \text{in } \mathrm{Hom}_{R_{f_if_jf_k}}((M_i)_{f_jf_k},(M_k)_{f_if_j}).$$

Then there exists an R-module M and, for each $i\in I$, an R_{f_i}-linear isomorphism $\beta_i\colon M_{f_i}\xrightarrow{\sim}M_i$ such that $\alpha_{ij}\circ\beta_i=\beta_j$ for all i,j. Moreover, M is unique up to unique isomorphism.

***Remark* 1.2.38.** For the higher categorical version, one uses faithfully flat descent, since the flat topology is finer than the Zariski topology. This reduces Zariski descent to faithfully flat descent via a standard argument.

Proof. (1) Descent for morphisms:

We write the equalizer explicitly as

$$E=\left\{(\varphi_i\colon N_{f_i}\to M_{f_i})_{i\in I}\;\middle|\;\forall i,j\in I,\ (\varphi_i)_{f_j}=(\varphi_j)_{f_i}\text{ in }\mathrm{Hom}_{R_{f_if_j}}(N_{f_if_j},M_{f_if_j})\right\}.$$

We must show that the natural map $\mathrm{Hom}_R(N,M)\to E$ is a bijection.

Setting $N=R$, the problem reduces to proving descent for elements of M. We establish this first.

Descent for elements (Existence): Given $(x_i)_{i\in I}\in\prod_{i\in I}M_{f_i}$ satisfying the compatibility condition $(x_i)_{f_j}=(x_j)_{f_i}$ for all $i,j\in I$.

Since $(f_i)_{i\in I}$ generates the unit ideal, we can choose a finite subset $J\subset I$ such that $(f_j)_{j\in J}$ still generates the unit ideal. For each $j\in J$, write $x_j=a_j/f_j^{n_j}$ where $a_j\in M$ and $n_j\geq 0$.

Choose a sufficiently large uniform integer $N\geq\max_j n_j$ such that for all $i,j\in J$:

- Each x_i can be written as $x_i=a_i'/f_i^N$ for some $a_i'\in M$ (by multiplying numerator and denominator by $f_i^{N-n_i}$);
- The compatibility condition $(x_i)_{f_j}=(x_j)_{f_i}$ translates to: there exists $m\geq 0$ such that

$$(f_if_j)^m(a_j'f_i^N-a_i'f_j^N)=0\quad\text{in } M.$$

After possibly increasing N, we may assume this holds with $m=0$, i.e.,

$$a_j'f_i^N=a_i'f_j^N\quad\text{for all } i,j\in J.$$

- There exist $b_i\in R$ such that $\sum_{i\in J}b_if_i^N=1$.

Define $x=\sum_{i\in J}b_ia_i'\in M$. For any $k\in J$, we verify that the image of x in M_{f_k} equals x_k:

$$\begin{aligned}x&=\frac{\sum_{i\in J}b_ia_i'}{1}=\frac{\sum_{i\in J}b_ia_i'f_k^N}{f_k^N}\\&=\frac{\sum_{i\in J}b_i(a_k'f_i^N)}{f_k^N}\quad(\text{using } a_i'f_k^N=a_k'f_i^N)\\&=\frac{a_k'\sum_{i\in J}b_if_i^N}{f_k^N}=\frac{a_k'\cdot 1}{f_k^N}=\frac{a_k'}{f_k^N}=x_k.\end{aligned}$$

Thus $x|_{f_k} = x_k$ for all $k \in J$. For any $i \in I$, since $(f_j)_{j\in J}$ generates the unit ideal, by repeating the above argument on the covering $\{D(f_j) \cap D(f_i)\}_{j\in J}$ of $D(f_i)$, we conclude $x|_{f_i} = x_i$.

Descent for elements (Uniqueness): Suppose $x \in M$ satisfies $x|_{f_i} = 0$ for all $i \in I$. This means that for each i, there exists $n_i \geq 0$ such that $f_i^{n_i} x = 0$ in M.

Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates the unit ideal. Choose N large enough so that $f_j^N x = 0$ for all $j \in J$ and $\sum_{j\in J} c_j f_j^N = 1$ for some $c_j \in R$. Then:

$$x = 1 \cdot x = \left(\sum_{j\in J} c_j f_j^N \right) x = \sum_{j\in J} c_j (f_j^N x) = \sum_{j\in J} c_j \cdot 0 = 0.$$

General case for morphisms: For general N, given compatible morphisms $\varphi_i : N_{f_i} \to M_{f_i}$, define $\varphi : N \to M$ by the following: for $n \in N$, the element $\varphi(n) \in M$ is the unique element (by descent for elements) whose image in each M_{f_i} is $\varphi_i(n|_{f_i})$. One checks that φ is R-linear using uniqueness.

(2) Descent for objects:

Define M as the equalizer of the following diagram:

$$M \coloneqq \mathrm{eq}\left(\prod_{i\in I} M_i \underset{b}{\overset{a}{\rightrightarrows}} \prod_{(i,j)\in I\times I} (M_i)_{f_j} \right),$$

where the morphisms are defined componentwise:

- The (i, j)-component of a is the localization $M_i \to (M_i)_{f_j}$;
- The (i, j)-component of b is the composition $M_j \to (M_j)_{f_i} \xrightarrow{\alpha_{ij}^{-1}} (M_i)_{f_j}$.

By construction, there are canonical morphisms $\beta_i : M_{f_i} \to M_i$ induced by the projection $M \to M_i$ followed by localization.

Case 1: I is finite.

Fix $0 \in I$. Since I is finite and localization commutes with finite products, we have $(-)_{f_0}(\prod_{i\in I} M_i) \cong \prod_{i\in I}(M_i)_{f_0}$.

Consider the following commutative diagram:

$$\begin{tikzcd}
M_{f_0} \arrow[r] \arrow[d, dashed, "\beta_0"] \arrow[dr, phantom, "(1)"] & \prod_{i\in I}(M_i)_{f_0} \arrow[r, shift left, "a_{f_0}"] \arrow[r, shift right, "b_{f_0}"'] \arrow[d, "\alpha_{i0}", "\simeq"'] \arrow[dr, phantom, "(2)"] & \prod_{i,j\in I}(M_i)_{f_j f_0} \arrow[d, "(\alpha_{i0})_{f_j}", "\simeq"'] \\
M_0 \arrow[r] & \prod_{i\in I}(M_0)_{f_i} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j\in I}(M_0)_{f_i f_j}
\end{tikzcd}$$

We verify that all parts of this diagram commute:

(1) The left square commutes by the compatibility $\alpha_{ij} \circ \beta_i = \beta_j$ and the definition of M as an equalizer.

(2) The right square commutes. This follows from the cocycle condition: the commutativity of the upper path (via a) is immediate, while the lower path (via b) uses $\alpha_{i0} \circ \alpha_{ji}^{-1} = \alpha_{j0}^{-1}$, which is the cocycle condition.

The bottom row is the trivial descent diagram for M_0 (any module descends to itself). Since the vertical maps are isomorphisms and the diagram commutes, the induced map $\beta_0 : M_{f_0} \to M_0$ is an isomorphism.

Uniqueness: Given two solutions $(M, \{\beta_i\})$ and $(N, \{\gamma_i\})$, define for each $i \in I$:

$$\varphi_i : M_{f_i} \xrightarrow{\beta_i} M_i \xrightarrow{\gamma_i^{-1}} N_{f_i}.$$

The compatibility $\alpha_{ij} \circ \beta_i = \beta_j$ and $\alpha_{ij} \circ \gamma_i = \gamma_j$ ensures that $(\varphi_i)_{f_j} = (\varphi_j)_{f_i}$.

By descent for morphisms (part 1), there exists a unique morphism $\varphi : M \to N$ with $\varphi_{f_i} = \varphi_i$ for all i. Since $\ker(\varphi)_{f_i} = \ker(\varphi_i) = 0$ and $\mathrm{coker}(\varphi)_{f_i} = \mathrm{coker}(\varphi_i) = 0$ for all i, and both kernel and cokernel satisfy descent (as they are determined by exactness of sequences), we have $\ker(\varphi) = 0$ and $\mathrm{coker}(\varphi) = 0$. Thus φ is an isomorphism.

Case 2: I is arbitrary.

Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates the unit ideal. For any finite $J \subset J' \subset I$, define $M_{J'}$ as the equalizer constructed from the data restricted to J'. The inclusion $J \subset J'$ induces a map $M_{J'} \to M_J$, and by the finite case, both are isomorphic to some M_k (where $k \in J$). Thus $M_{J'} \cong M_J$ canonically.

Since limits commute with limits, we have

$$M = \mathrm{eq}\left(\prod_{i\in I} M_i \rightrightarrows \prod_{i,j\in I} (M_i)_{f_j}\right) = \lim_{\substack{J\subset I\\ J \text{ finite}}} \mathrm{eq}\left(\prod_{i\in J} M_i \rightrightarrows \prod_{i,j\in J} (M_i)_{f_j}\right) \cong M_J.$$

This completes the proof. ∎

Corollary 1.2.39 (Descent for Module Elements). *Let R be a ring, and $(f_i)_{i\in I}$ a family of elements generating the unit ideal of R. For an R-module M, the diagram*

$$M \longrightarrow \prod_{i\in I} M_{f_i} \rightrightarrows \prod_{i,j\in I} M_{f_i f_j}$$

is an equalizer in Mod_R.

Proof. This is the special case of Theorem 1.2.37(1) with $N = R$. ∎

Taking $M = R$ in Corollary 1.2.39, we obtain:

Corollary 1.2.40 (Zariski Descent for Rings). *Let R be a ring, and let $(f_i)_{i\in I}$ be a family of elements generating the unit ideal of R. Then the diagram*

$$R \longrightarrow \prod_{i\in I} R_{f_i} \rightrightarrows \prod_{i,j\in I} R_{f_i f_j}$$

is an equalizer in CAlg.

This is precisely Theorem 1.2.37 from the previous section, now proved in full detail.

Corollary 1.2.41 (Zariski Descent for Affine Schemes). *Let R be a ring, and let $(f_i)_{i\in I}$ be a family of elements generating the unit ideal of R. Then for any affine scheme X, the diagram*

$$X(R) \longrightarrow \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j})$$

is an equalizer in Set.

Proof. Write $X \simeq \mathrm{Spec}(A)$ for some ring A. Since $\mathrm{Spec}(A)(R) = \mathrm{Hom}_{\mathsf{CAlg}}(A, R)$ and the functor $\mathrm{Hom}_{\mathsf{CAlg}}(A, -)$ preserves limits (being representable), the result follows from Corollary 1.2.40. ∎

Corollary 1.2.42 (Descent for Non-vanishing Loci). *Let R be a ring and $F \subset R$ a subset. Then for any affine scheme X, the diagram*

$$\mathrm{Hom}(D(F), X) \longrightarrow \prod_{f\in F} X(R_f) \rightrightarrows \prod_{f,g\in F} X(R_{fg})$$

is an equalizer in Set.

Proof. Recall from Theorem 1.1.10 that $D(F)$ can be written as a colimit:

$$D(F) \simeq \operatorname{colim}_{\substack{R\to S\\ F\mapsto S^\times}} \mathrm{Spec}(S).$$

Let C_F denote the indexing category of ring maps $R \to S$ that send all elements of F to units. Since $\mathrm{Hom}(-, X)$ converts colimits to limits:

$$\mathrm{Hom}(D(F), X) \simeq \mathrm{Hom}\left(\operatorname{colim}_{S\in C_F} \mathrm{Spec}(S), X\right) \simeq \lim_{S\in C_F} X(S).$$

By Corollary 1.2.41, for each S with $R \to S$ sending F to units, the diagram

$$X(S) \longrightarrow \prod_{f \in F} X(S_f) \rightrightarrows \prod_{f,g \in F} X(S_{fg})$$

is an equalizer. Taking limits over $S \in C_F$ and using the fact that limits commute with limits:

$$\lim_{S \in C_F} X(S) \longrightarrow \prod_{f \in F} \lim_{S \in C_F} X(S_f) \rightrightarrows \prod_{f,g \in F} \lim_{S \in C_F} X(S_{fg})$$

is an equalizer.

It remains to show $\lim_{S \in C_F} X(S_g) \simeq X(R_g)$ for each $g \in F$.

For $g \in F$, consider the localization $S \to S_g$. In the category C_F, the map $R \to S_g$ sends F to units (since g is already a unit in S and other elements of F are sent to units in S, hence also in S_g). Moreover, for $R \to S \to S_g$, we have $X((S_g)_g) \simeq X(S_g)$ since g is already a unit in S_g.

This shows that the subcategory of C_F consisting of ring maps $R \to S$ where g is sent to a unit is cofinal. Therefore:

$$\lim_{\substack{R \to S \\ F \mapsto S^\times}} X(S_g) \simeq \lim_{\substack{R \to S \\ g \mapsto S^\times}} X(S_g).$$

The latter category has an initial object R_g, so the limit is simply $X(R_g)$. ■

Example 1.2.43. In Corollary 1.2.42, let $X = \mathbb{A}^1$. For $|I| \geq 2$, we have an isomorphism induced by the inclusion $\mathbb{A}^I - 0 \hookrightarrow \mathbb{A}^I$:

$$\mathcal{O}(\mathbb{A}^I) \xrightarrow{\sim} \mathcal{O}(\mathbb{A}^I - 0).$$

By the equivalence of categories $\mathsf{Aff}^{\mathrm{op}} \simeq \mathsf{CAlg}$ (Corollary 1.2.10), this shows that $\mathbb{A}^I - 0$ is not an affine scheme when $|I| \geq 2$, even though it is a quasi-affine scheme. Thus we have a proper inclusion:

$$\mathsf{Aff} \subsetneq \mathsf{QAff}.$$

1.2.8. Zariski Local Properties

A key observation in the proof of Theorem 1.2.37 is that a morphism φ is an isomorphism if and only if all its localizations φ_{f_i} are isomorphisms. This property of being verifiable locally on R_{f_i} is ubiquitous in algebraic geometry. In this section, we systematically study such properties.

Definition 1.2.44 (Zariski Local Property). Let $\mathcal{P}$ be a property of modules (or linear maps, algebras, etc.). We say that $\mathcal{P}$ is *Zariski local* if for any ring R and any family $(f_i)_{i \in I}$ of elements generating the unit ideal, an R-module M (or R-linear map, R-algebra, etc.) satisfies $\mathcal{P}$ if and only if for each $i \in I$, the localized object M_{f_i} (or corresponding localized object) satisfies $\mathcal{P}$.

Proposition 1.2.45 (Examples of Zariski Local Properties). *The following properties are Zariski local:*

Properties of modules:

(a) Being zero.

(b) Finite generation.

(c) Finite presentation.

(d) Projectivity.

(e) Flatness.

Properties of module homomorphisms:

(f) Being zero.

(g) Injectivity.

(h) Surjectivity.

(i) Being an isomorphism.

Properties of algebras:

(j) Finite generation.

(k) *Finite presentation.*

Properties of sequences:

(l) *Exactness.*

***Remark* 1.2.46.** Proving that projectivity satisfies Zariski descent is more subtle and beyond our current scope. For a classical reference (in French), see Raynaud-Gruson [RG71, p.81]. Modern approaches typically deduce Zariski descent from faithfully flat descent; see Stacks Project [Sta18, Tag 058B, Tag 05A5].

Proof. We prove several representative cases. Throughout, let $(f_i)_{i\in I}$ generate the unit ideal of R.

(a) Being zero: A module $M = 0$ if and only if $M_{f_i} = 0$ for all $i \in I$. This follows from the injectivity of $M \hookrightarrow \prod_{i\in I} M_{f_i}$ (Corollary 1.2.39).

(g), (h), (i) Properties of maps: For an R-linear map $\varphi \colon M \to N$:

- φ is injective $\Leftrightarrow \ker(\varphi) = 0 \Leftrightarrow \ker(\varphi)_{f_i} = 0$ for all i (by exactness of localization) $\Leftrightarrow \ker(\varphi_{f_i}) = 0$ for all $i \Leftrightarrow \varphi_{f_i}$ is injective for all i.
- Similarly for surjectivity using $\operatorname{coker}(\varphi)$.
- Being an isomorphism is equivalent to being both injective and surjective.

(e) Flatness: Assume M_{f_i} is a flat R_{f_i}-module for all $i \in I$. To show M is flat over R, consider any injection $\varphi \colon N \hookrightarrow N'$ of R-modules. We must show $\varphi \otimes \mathrm{id}_M \colon N \otimes_R M \to N' \otimes_R M$ is injective.

By Zariski locality of injectivity, it suffices to verify injectivity after localizing at each f_i. Since tensor product commutes with localization:

$$(\varphi \otimes \mathrm{id}_M)_{f_i} \cong \varphi_{f_i} \otimes \mathrm{id}_{M_{f_i}} \colon N_{f_i} \otimes_{R_{f_i}} M_{f_i} \to N'_{f_i} \otimes_{R_{f_i}} M_{f_i}.$$

Since localization preserves injectivity and M_{f_i} is flat over R_{f_i}, this map is injective. Thus M is flat.

(b) Finite generation: Assume M_{f_i} is finitely generated over R_{f_i} for all i. Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates R.

For each $j \in J$, choose finitely many elements $m_{j,1}, \ldots, m_{j,n_j} \in M$ whose images generate M_{f_j} (clearing denominators if necessary). Define

$$\phi_j \colon R^{n_j} \to M, \quad (r_1, \ldots, r_{n_j}) \mapsto \sum_k r_k m_{j,k}.$$

Consider $\Phi = \sum_{j\in J} \phi_j \colon \bigoplus_{j\in J} R^{n_j} \to M$. To show Φ is surjective, it suffices to show Φ_{f_k} is surjective for each $k \in J$ (by Zariski locality of surjectivity). But $(\phi_k)_{f_k}$ is surjective by construction, so Φ_{f_k} is surjective. Thus M is finitely generated.

(c) Finite presentation: Assume M_{f_i} is finitely presented over R_{f_i} for all i. By (b), M is finitely generated. Choose a surjection $\pi \colon R^n \twoheadrightarrow M$ with kernel K, giving an exact sequence

$$0 \to K \to R^n \xrightarrow{\pi} M \to 0.$$

Localizing (exactness is preserved):

$$0 \to K_{f_i} \to R^n_{f_i} \to M_{f_i} \to 0.$$

Since M_{f_i} is finitely presented and $R^n_{f_i}$ is finitely generated free, the kernel K_{f_i} is finitely generated over R_{f_i} (this is the definition of finite presentation). By (b), K is finitely generated over R. Therefore M is finitely presented.

(l) Exactness: A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is exact at M if and only if $\ker(g) = \operatorname{im}(f)$, which holds if and only if $\ker(g)_{f_i} = \operatorname{im}(f)_{f_i}$ for all i (by part (a)), if and only if the localized sequence is exact at M_{f_i} for all i. ∎

Corollary 1.2.47. *Let X be a quasi-affine k-scheme and $Z \hookrightarrow X$ a closed embedding. Then Z is also a quasi-affine k-scheme.*

Proof. We prove the case $k = \mathbb{Z}$; the general case is identical. Since X is quasi-affine, there exist a ring A and a finite subset $F \subset A$ such that $X \simeq D(F) \subset \operatorname{Spec}(A)$.

Let Y be the image of Z in X, so $Y \subset D(F)$ is a closed subfunctor. For each $f \in F$, $Y \cap D(f)$ is a closed subfunctor of $D(f) \simeq \operatorname{Spec}(A_f)$. Let $J(f) \subset A_f$ be the corresponding ideal, and let $I(f) \subset A$ be its preimage under $A \to A_f$. Then $I(f)_f = J(f)$, so

$$V(I(f)) \cap D(f) = Y \cap D(f).$$

Define $I = \bigcap_{f \in F} I(f)$.

Claim 1: The closure $\bar{Y}$ of Y in $\operatorname{Spec}(A)$ *equals $V(I)$.*

First, $Y \subset V(I)$: Let $\varphi\colon A \to R$ be a point in $Y(R)$. Since $Y \subset D(F)$, we have $(\varphi(F)) = R$. For each $f \in F$, the composition $A \xrightarrow{\varphi} R \to R_{\varphi(f)}$ corresponds to a point in $Y \cap D(f) = V(I(f)) \cap D(f)$ evaluated at $R_{\varphi(f)}$. Thus $\varphi(I(f)) = 0$ in $R_{\varphi(f)}$ for each f, which implies $\varphi(I(f)) = 0$ in R. Taking the intersection over $f \in F$ gives $\varphi(I) = 0$.

Conversely, $V(I) \subset \bar{Y}$: By Proposition 1.2.31, closed subfunctors of $\operatorname{Spec}(A)$ correspond to ideals. If $Y \subset V(K)$, we need $K \subset I$. By definition of I, this reduces to showing $K_f \subset J(f)$ in A_f. From $V(J(f)) = Y \cap D(f) \subset V(K) \cap D(f) = V(K_f)$, we get $K_f \subset J(f)$.

Claim 2: $Y = \bar{Y} \cap D(F) = V(I) \cap D(F)$.

First, we show $I_g = J(g)$ for all $g \in F$. One inclusion is clear. For $J(g) \subset I_g$, note that for any $f \in F$:

$$J(g)_f = J(f)_g \quad \text{in } A_{fg}$$

(both equal the ideal cutting out $Y \cap D(fg)$). Since localization is exact, $I(f)_g$ is the preimage of $J(f)_g$ under $A_g \to A_{fg}$. Thus $J(g) \subset I(f)_g$ for all f, so $J(g) \subset I_g$.

A point $x\colon \operatorname{Spec}(R) \to \bar{Y} \cap D(F)$ corresponds to $\varphi\colon A \to R$ with $\varphi(I) = 0$ and $(\varphi(F)) = R$. Since $Y \subset D(F)$ is closed, $Y \subset \bar{Y} \cap D(F)$ is also closed. To show equality, consider the pullback $x^{-1}(Y) \simeq V(G)$ for some ideal $G \subset R$. We need $G = 0$.

By Zariski descent, it suffices to show $G_{\varphi(f)} = 0$ for all $f \in F$. But

$$V(G_{\varphi(f)}) = x^{-1}(Y) \cap D(\varphi(f)) = x^{-1}(Y \cap D(f)) = x^{-1}(V(I_f)) = V(\varphi(I_f)) = V(0)$$

(using $I_f = J(f)$ and $\varphi(I) = 0$). Thus $G_{\varphi(f)} = 0$ for all $f \in F$, so $G = 0$.

Therefore $Y = V(I) \cap D(F) = D(\bar{F})$ where $\bar{F}$ is the image of F in A/I. This shows $Z \simeq Y$ is quasi-affine. ∎

Chapter 2.

Projective Geometry

The goal of this chapter is to introduce projective geometry and transfer certain results from affine geometry to the projective setting. Specifically, we establish the following analogy:

Affine Geometry	*Projective Geometry*
Affine n-space $\mathbb{A}^n$	Projective n-space $\mathbb{P}^n$
Affine schemes	Projective schemes
Rings	Graded rings
$\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$	$\mathrm{Proj}\colon (\mathsf{CAlg}^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$
Classify open/closed subfunctors of $\mathrm{Spec}(A)$ via ideals in A	Classify open/closed subfunctors of $\mathrm{Proj}(A)$ via homogeneous ideals in A

***Remark* 2.0.1.** It is worth noting that Spec induces an anti-equivalence of categories between rings and affine schemes, but Proj does not, as it is not fully faithful.

2.1. Projective Spaces

2.1.1. Projective Spaces over Fields

First, we study projective spaces over fields. Let k be a field and $n \geq -1$. The *projective n-space over k* is the set of all lines through the origin in k^{n+1}:

$$\mathbb{P}^n(k) = \{\text{one-dimensional subspaces of } k^{n+1}\}.$$

Given a nonzero $(n+1)$-tuple $(a_0, \dots, a_n) \in k^{n+1} - \{0\}$, we can consider the one-dimensional subspace it generates in k^{n+1}, denoted $[a_0 : \cdots : a_n]$. That is,

$$[a_0 : \cdots : a_n] = [b_0 : \cdots : b_n] \Leftrightarrow \exists \lambda \in k^\times \text{ such that } \lambda(a_0, \dots, a_n) = (b_0, \dots, b_n).$$

In this way, we can write $\mathbb{P}^n(k)$ as the set of orbits of the free $k^\times$-action on $k^{n+1} - \{0\}$:

$$(k^{n+1} - \{0\})/k^\times \xrightarrow{\sim} \mathbb{P}^n(k), \quad (a_0, \dots, a_n) \mapsto [a_0 : \cdots : a_n].$$

In fact, we can view $\mathbb{P}^n(k)$ as the union of $n+1$ subsets $U_0, \dots, U_n$, where

$$U_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(k) \mid a_i \text{ is invertible}\}.$$

By the definition of $[a_0 : \cdots : a_n]$, we can assume $a_i = 1$. Therefore, we can identify U_i via:

$$\begin{gathered} U_i \simeq \mathbb{A}^n(k), \\ [a_0 : \cdots : a_n] \mapsto \left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right), \end{gathered}$$

with inverse given by $(a_0, \dots, \widehat{a_i}, \dots, a_n) \mapsto [a_0 : \cdots : 1 : \cdots : a_n]$.

The complement $H_i = \mathbb{P}^n(k) - U_i$ is given by

$$H_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(k) \mid a_i = 0\}.$$

According to the above characterization, we have

$$H_i \xrightarrow{\sim} \mathbb{P}^{n-1}(k), \quad [a_0 : \cdots : a_n] \mapsto [a_0 : \cdots : \widehat{a_i} : \cdots : a_n].$$

Therefore, we can summarize:

- $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$.
- For $0 \leq i \leq n$, we have $\mathbb{P}^n(k) = U_i \cup H_i$, where $U_i \simeq \mathbb{A}^n(k) = k^n$ and $H_i \simeq \mathbb{P}^{n-1}(k)$. Iterating, we get
$$\mathbb{P}^n(k) = k^n \cup k^{n-1} \cup \cdots \cup k \cup k^0.$$
- Taking $i = 0$, we can view $\mathbb{P}^n(k)$ as the space obtained by compactifying $U_0 = \mathbb{A}^n(k)$ by adding a "point at infinity" for each line through the origin:
$$[0 : a_1 : \cdots : a_n] = \lim_{\lambda \to \infty} (\lambda a_1, \ldots, \lambda a_n),$$
with
$$(\lambda a_1, \ldots, \lambda a_n) = [1 : \lambda a_1 : \cdots : \lambda a_n] = [\lambda^{-1} : a_1 : \cdots : a_n].$$

Some additional observations:

- k^n is a vector space over k, and by choosing a basis, it is self-dual. For each subspace V, we can consider its orthogonal complement $V^\perp \subseteq k^n$. This gives the following correspondence:
$$\begin{tikzcd}
V \arrow[d, mapsto] & \{d\text{-dim subspaces of } k^n\} \arrow[r, leftrightarrow, "\sim"] \arrow[d, leftrightarrow, "\sim"'] & \{(n-d)\text{-dim subspaces of } k^n\} \arrow[d, leftrightarrow, "\sim"'] & V^\perp \arrow[d, mapsto] \\
k^n/V & \{(n-d)\text{-dim subspaces of } k^n\} \arrow[r, leftrightarrow, "\sim"] & \{d\text{-dim subspaces of } k^n\} & k^n/V^\perp
\end{tikzcd}$$
In particular, for lines (1-dimensional subspaces), we have:
$$\begin{tikzcd}
\{1\text{-dim subspaces of } k^{n+1}\} \arrow[r] \arrow[d, mapsto] & \{1\text{-dim quotients of } k^{n+1}\} \arrow[d, mapsto] \\
L \arrow[r, mapsto] & k^{n+1}/L^\perp
\end{tikzcd}$$
with the diagram commuting via $k^{n+1} - \{0\}$ at the bottom, where $(a_0, \ldots, a_n)$ maps to both sides via the standard inclusion and quotient maps.

We will follow this line of thought to generalize the definition of projective space to rings.

2.1.2. Vector Spaces and Lines over Rings

Proposition 2.1.1 (Characterization of Vector Spaces). *Let R be a ring and $M \in \mathrm{Mod}_R$ a module. The following conditions are equivalent:*

(i) M is a finitely generated projective module.

(ii) M is a finitely presented flat module.

(iii) There exists $n \in \mathbb{N}$ such that M is a direct summand of R^n.

(iv) There exist $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and each M_{f_i} is a finitely generated free R_{f_i}-module.

(v) M is a dualizable module, i.e., there exists $M^\vee \in \mathrm{Mod}_R$ and linear maps $\mathrm{ev}\colon M^\vee \otimes M \to R$ and $\mathrm{coev}\colon R \to M \otimes M^\vee$ such that the following diagrams commute:
$$\begin{tikzcd}
& M \otimes M^\vee \otimes M \arrow[dr, "\mathrm{id} \otimes \mathrm{ev}"] & \\
M \arrow[ur, "\mathrm{coev} \otimes \mathrm{id}"] \arrow[rr, equal] & & M
\end{tikzcd}$$
and
$$\begin{tikzcd}
& M^\vee \otimes M \otimes M^\vee \arrow[dr, "\mathrm{ev} \otimes \mathrm{id}"] & \\
M^\vee \arrow[ur, "\mathrm{id} \otimes \mathrm{coev}"] \arrow[rr, equal] & & M^\vee
\end{tikzcd}$$

***Remark* 2.1.2.** In practice, condition (4) is most commonly used, as it is closely related to Zariski descent.

Proof of Proposition 2.1.1.

$(1 \Leftrightarrow 3 \Rightarrow 2)$ Let M be a finitely generated projective module. We need to show that there exists $n \in \mathbb{N}$ such that M is a retract of R^n.

Since M is finitely generated, there exists $n \in \mathbb{N}$ and a surjection $\pi\colon R^n \twoheadrightarrow M$. We now construct a section.

Since M is projective, $\mathrm{Hom}_R(M,-)$ is an exact functor, hence preserves surjections. Therefore:

$$\mathrm{Hom}_R(M,R^n) \twoheadrightarrow \mathrm{Hom}_R(M,M).$$

Taking a preimage of id_M gives the desired section, so M is a retract of R^n.

Conversely, note that projective modules are closed under retracts. It suffices to show that R^n is projective, which is obvious. Moreover, direct summands of R^n are clearly finitely presented and flat, giving (2).

$(2 \Rightarrow 4)$ We give the following claim:

Claim 2.1.3. *If M is a finitely presented flat module, then for any maximal ideal $\mathfrak{m} \subset R$, $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module.*

Proof of claim. Take a maximal ideal $\mathfrak{m} \subset R$. We have $M/\mathfrak{m}M$ is an $R/\mathfrak{m}$-module. Since $\mathfrak{m}$ is maximal, $R/\mathfrak{m}$ is a field. Therefore there exists $n \in \mathbb{N}$ such that $(R/\mathfrak{m})^n \simeq M/\mathfrak{m}M$.

Consider the commutative diagram:

$$\begin{tikzcd}
(R/\mathfrak{m})^n \arrow[r, "\sim"] & M/\mathfrak{m}M \\
R_\mathfrak{m}^n \arrow[r, "f"] \arrow[u, two heads] & M_\mathfrak{m} \arrow[u, two heads]
\end{tikzcd}$$

It is easy to see that f is surjective. We need to show f is injective.

By flatness of M, we have $\mathrm{Tor}_1^R(R/\mathfrak{m},M) = 0$. Therefore, the sequence

$$0 \to \ker(f) \otimes_R R/\mathfrak{m} \to (R/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0$$

is exact.

Since M is finitely presented over R, $M_\mathfrak{m}$ is finitely presented over $R_\mathfrak{m}$. Consider the exact sequence:

$$R^m \to R^n \to M \to 0.$$

Localizing gives:

$$R_\mathfrak{m}^m \to R_\mathfrak{m}^n \to M_\mathfrak{m} \to 0,$$

which is also exact. Thus $\ker(f)$ is a finitely generated $R_\mathfrak{m}$-module.

Therefore:

$$\ker(f) \otimes_{R_\mathfrak{m}} (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \simeq \ker(f) \otimes_R R/\mathfrak{m} = 0.$$

By Nakayama's lemma over $R_\mathfrak{m}$, we have $\ker(f) = 0$. ∎

Now we use the claim to prove (4). Recall that $M_\mathfrak{m} = M[(R-\mathfrak{m})^{-1}]$, so:

$$M_\mathfrak{m} = \mathrm{colim}_{f \notin \mathfrak{m}} M_f.$$

Since this is a filtered colimit, a basis of $M_\mathfrak{m}$ can be lifted to M_f for some $f \notin \mathfrak{m}$. Taking the union over all maximal ideals $\mathfrak{m} \subset R$ gives the result.

$(4 \Rightarrow 1)$ Use that finite generation and projectivity are Zariski local properties (see Proposition 1.2.45), together with the fact that free modules are automatically projective.

(4 ⇒ 5) We know that M is dualizable if and only if $M \otimes_R -$ has a right adjoint $\mathrm{Hom}_R(M,-)$ and for any R-module N, we have $N \otimes M^\vee \simeq \mathrm{Hom}_R(M,N)$.

We can reduce to $D(f_i)$ and use Zariski descent.

(5 ⇒ 1) Use that $\mathrm{Hom}_R(M,-)$ preserves colimits when M is dualizable.

■

Based on the previous observations, we view the collection of all dualizable R-modules as finite-dimensional vector spaces.

Definition 2.1.4 (Vector Spaces over Rings)**.** Let R be a ring. We say that an R-module M is a *vector space* if M satisfies the equivalent conditions in Proposition 2.1.1. We denote by

$$\mathsf{Vect}_R \subset \mathrm{Mod}_R$$

the full subcategory spanned by all R-vector spaces.

***Remark* 2.1.5.** Note that when $R = k$ is a field, the resulting vector spaces are precisely the *finite-dimensional* vector spaces over k.

Recall that over fields, we can discuss bases of vector spaces, and different bases have the same dimension, or rank. To generalize the notion of rank to our vector spaces, we need the concept of an R-field, i.e., a field that is also an R-algebra.

Definition 2.1.6 (Rank)**.** Let R be a ring, M an R-module, and κ an R-field. The *rank* $\mathrm{rk}_\kappa(M)$ of M is the rank of $M \otimes_R \kappa$.

We say that M has *constant rank* if there exists r such that for every R-field κ, we have $\mathrm{rk}_\kappa(M) = r$.

***Remark* 2.1.7.** We have the following relations:

- $\mathrm{rk}_\kappa(M \oplus N) = \mathrm{rk}_\kappa(M) + \mathrm{rk}_\kappa(N)$.
- $\mathrm{rk}_\kappa(M \otimes N) = \mathrm{rk}_\kappa(M) \cdot \mathrm{rk}_\kappa(N)$.
- $\mathrm{rk}_\kappa(R^n) = n$.

Proposition 2.1.8 (Properties of Rank)**.** *Let R be a ring.*

(i) For an R-vector space V, we have $V = 0$ if and only if V has constant rank 0.

(ii) For a short exact sequence in Vect_R*:*

$$0 \to U \to V \to W \to 0,$$

and any R-field κ, we have $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(U) + \mathrm{rk}_\kappa(W)$.

Proof. (i) The direction $V = 0 \Rightarrow \mathrm{rk}_\kappa(V) = 0$ is obvious. Now take V with constant rank 0. By Proposition 2.1.1, choose $f_1, \dots, f_n$ such that $(f_1, \dots, f_n) = R$. Then for any κ:

$$\mathrm{rk}_\kappa(V_{f_i}) = 0,$$

which means $V_{f_i} = 0$. By Zariski descent, $V = 0$.

(ii) By Proposition 2.1.1, vector spaces are projective modules. Therefore the short exact sequence

$$0 \to U \to V \to W \to 0$$

splits, reducing to Remark 2.1.7.

■

***Remark* 2.1.9.** Note that having constant rank 0 implies $V = 0$ only for R-vector spaces. For general modules, there are counterexamples. For instance, take $R = \mathbb{Z}$; the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ has rank 0.

For R, recall that the *residue field* at a prime ideal $\mathfrak{p} \subset R$ is the R-field:

$$\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \simeq \mathrm{Frac}(R/\mathfrak{p}).$$

In fact, {residue fields of R} forms the (coproduct) initial object of Field_R. More precisely, for any R-field κ, there exists a unique prime ideal $\mathfrak{p}$ (namely $\ker(R \to \kappa)$) such that $R \to \kappa$ factors as:

$$R \longrightarrow \kappa(\mathfrak{p}) \longrightarrow \kappa,$$

and this factorization is unique. In other words, the category of R-fields decomposes as:

$$\mathsf{Field}_R = \coprod_{\mathfrak{p} \subseteq R} \mathsf{Field}_{\kappa(\mathfrak{p})}.$$

Using the fact that scalar extension between fields preserves dimension, we deduce that for κ lying in $\mathsf{Field}_{\kappa(\mathfrak{p})}$, we have $\mathrm{rk}_\kappa(M) = \mathrm{rk}_{\kappa(\mathfrak{p})}(M)$.

Next, we discuss what "one-dimensional" vector spaces are. Rather than thinking of "one-dimensional" in terms of dimension, we prefer to think of it as something invertible.

Proposition 2.1.10 (Characterization of Lines). *Let R be a ring. For any R-module L, the following conditions are equivalent:*

(i) L is a vector space with constant rank 1.

(ii) L is an invertible module, i.e., there exists an R-module $L^\vee$ such that $L \otimes_R L^\vee \simeq R$.

***Remark* 2.1.11.** It is easy to see that invertible modules are automatically vector spaces (including their inverses).

Proof of Proposition 2.1.10. (1 $\Rightarrow$ 2) Let L be an R-vector space. For $f \in R$, we show that $(L^\vee)_f = (L_f)^\vee$.

By Proposition 2.1.1, L is finitely presented. Therefore we have a right exact sequence:

$$R^m \longrightarrow R^n \longrightarrow L \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(-, R)$ gives a left exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(L, R) \longrightarrow \mathrm{Hom}_R(R^n, R) \longrightarrow \mathrm{Hom}_R(R^m, R).$$

Since R_f is flat over R, we obtain a left exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(L, R)_f \longrightarrow \mathrm{Hom}_R(R^n, R)_f \longrightarrow \mathrm{Hom}_R(R^m, R)_f.$$

Localizing the right exact sequence first and then applying $\mathrm{Hom}_{R_f}(-, R_f)$ gives:

$$0 \longrightarrow \mathrm{Hom}_{R_f}(L_f, R_f) \longrightarrow \mathrm{Hom}_{R_f}(R_f^n, R_f) \longrightarrow \mathrm{Hom}_{R_f}(R_f^m, R_f).$$

Since $\mathrm{Hom}_R(R^n, R) \simeq R^n$ for any ring R, using the five lemma gives the isomorphism:

$$\mathrm{Hom}_R(L, R)_f \simeq \mathrm{Hom}_{R_f}(L_f, R_f).$$

Since $L^\vee = \mathrm{Hom}_R(L, R)$, we have proved our claim. Now use that being free is a Zariski local property and apply descent.

(2 $\Rightarrow$ 1) Since invertible modules are automatically dualizable, we only need to show the rank is 1. This follows directly from:

$$\mathrm{rk}_\kappa(M \otimes_R M^{-1}) = \mathrm{rk}_\kappa(R) = 1.$$

∎

Therefore, we establish the principle: in R-vector spaces, "one-dimensional" = invertible.

Definition 2.1.12 (Lines). Let R be a ring. A *line over* R is an invertible R-module M. We denote by

$$\mathsf{Line}_R \subset \mathsf{Vect}_R$$

the full subcategory spanned by all R-lines.

R-lines need not be trivial.

Example 2.1.13 (Non-trivial Line). Let $R = \mathbb{Z}[\sqrt{-5}] \subseteq \mathbb{Q}(\sqrt{-5}) = \mathbb{Q} \oplus \mathbb{Q}\sqrt{-5} \subseteq \mathbb{C}$. This is a classical example of a non-UFD, since:

$$(1+\sqrt{-5})(1-\sqrt{-5}) = 6 = 2 \cdot 3.$$

Consider the ideal:

$$I = (2, 1+\sqrt{-5}) = (2, 1-\sqrt{-5}) \subseteq R.$$

We can construct an R-linear map:

$$I \otimes_R I \longrightarrow (2) \subseteq R, \quad (i,j) \mapsto i \cdot j.$$

Since (2) is principal, $(2) \simeq R$. We claim the above map is an isomorphism. Consider the map:

$$(2) \longrightarrow I \otimes_R I, \quad 2 \mapsto (1+\sqrt{-5}) \otimes (1-\sqrt{-5}) - 2 \otimes 2.$$

First, the composition $(2) \longrightarrow (2)$ is the identity. For the composition $I \otimes_R I \longrightarrow I \otimes_R I$, write $I \otimes_R I = (2, 1+\sqrt{-5}) \otimes (2, 1-\sqrt{-5})$ and check on the four generators:

$$2 \otimes 2, \quad 2 \otimes (1-\sqrt{-5}), \quad (1+\sqrt{-5}) \otimes 2, \quad (1+\sqrt{-5}) \otimes (1-\sqrt{-5}).$$

To show I is non-trivial, we show it is not principal. If $a + b\sqrt{-5}$ divides 2, then so does its conjugate $a - b\sqrt{-5}$. Thus $a^2 + 5b^2$ divides 4. The only possibilities are $b = 0$ and $a = \pm 2$ or ± 1. None of these combinations generate I.

2.1.3. Maps Between Vector Spaces

Next, we discuss morphisms between vector spaces. We want to understand what injectivity and surjectivity mean for maps between vector spaces.

In fact, for vector spaces, surjectivity remains surjectivity. Let M and N be R-vector spaces, and $f : M \twoheadrightarrow N$ a surjection. We have an exact sequence:

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since N is a vector space, hence projective, the sequence splits. Thus $\ker(f)$ is a retract of the projective module M, hence also projective, thus a vector space.

However, if f is injective, $\operatorname{coker}(f)$ need not be a vector space. For example, $\mathbb{Z} \xhookrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ is not a vector space.

In other words, we need "injective"$^{\mathrm{op}}$ = surjective maps.

Therefore, we need to extend the notion of injectivity to match the concept of injectivity between vector spaces.

Definition 2.1.14 (Universal Injectivity). Let R be a ring, and M, N be R-modules. We say that an R-linear map $\varphi : M \longrightarrow N$ is *universally injective* if for any ring homomorphism $R \longrightarrow S$, the map $\varphi \otimes_R S$ is injective.

***Remark* 2.1.15.** The property "universal XX" generally means being preserved by base change.

We believe that universal injectivity is the correct notion of injectivity for vector spaces.

***Remark* 2.1.16.**
- If $f : M \hookrightarrow N$ is universally injective, it is automatically injective (apply $\otimes_R R$). In fact, for any R-module P, $f \otimes_R P$ is injective. This follows by considering the symmetric algebra $\mathrm{Sym}_R(P)$ and the canonical map $R \longrightarrow \mathrm{Sym}_R(P)$.
- Due to right exactness of tensor products, there is no need to consider "universal surjectivity."

Example 2.1.17 (Universal Injectivity Examples).

(i) Any map with a retraction is universally injective.

(ii) Maps without retractions can also be universally injective, e.g., $\bigoplus_{\mathbb{N}} \mathbb{Z} \hookrightarrow \prod_{\mathbb{N}} \mathbb{Z}$.

(iii) Let $f \in R$. The map $R \xrightarrow{\cdot f} R$ is injective if and only if f is not a zero divisor. Universal injectivity requires f to be a unit (invertible element), since you can always base change to $R/(f)$ to make f a zero divisor.

Since finite presentation, finite generation, and projectivity are all Zariski local properties, both R-vector spaces and R-lines are Zariski-local properties. Since exactness of module sequences is also a Zariski local property, universal injectivity is also Zariski local.

Definition 2.1.18 (Subspaces and Quotient Spaces). Let R be a ring and M an R-module.

- We say that a module N is a *subspace* of M if $N \subset M$ is a submodule, N is a vector space, and the embedding $N \hookrightarrow M$ is universally injective.
- We say that a module N is a *quotient space* of M if N is a quotient module of M and N is a vector space.

The following proposition characterizes subspaces and quotient spaces.

Proposition 2.1.19 (Characterization of Subspaces). *Let R be a ring, and U, V be R-vector spaces. For an R-linear map $f : U \to V$, the following are equivalent:*

(i) f is universally injective.

(ii) For any R-field κ, the base change $f \otimes_R \kappa$ is injective.

(iii) f is injective and $\operatorname{coker}(f)$ is a vector space.

(iv) f has a retraction.

(v) The dual map $f^\vee : V^\vee \to U^\vee$ is surjective.

Proof.

- First, by Example 2.1.17, we immediately get $(4 \Rightarrow 1)$.
- By definition, $(1 \Rightarrow 2)$.
- Since $\operatorname{coker}(f)$ is a vector space, the exact sequence
$$0 \to U \to V \to \operatorname{coker}(f) \to 0$$
splits, so f has a retraction, giving $(3 \Rightarrow 4)$.
- If f has a retraction, this means the exact sequence
$$0 \to U \to V \to \operatorname{coker}(f) \to 0$$
splits, so $\operatorname{coker}(f)$ is also a retract of V, hence $\operatorname{coker}(f)$ is a vector space, proving $(4 \Rightarrow 3)$.
- If the dual map $f^\vee : V^\vee \to U^\vee$ is surjective, then for any R-module M, $-\otimes_R M$ preserves surjections, so $\operatorname{Hom}_R(V, N) \twoheadrightarrow \operatorname{Hom}_R(U, N)$ is surjective. Taking $N = U$ and a preimage of $\mathrm{id}_U \in \operatorname{Hom}_R(U, U)$ gives the retraction, so $(5 \Rightarrow 4)$.
- For any R-field κ, $f \otimes_R \kappa$ is automatically a κ-linear map, so it is injective if and only if
$$(f \otimes_R \kappa)^\vee = f^\vee \otimes_R \kappa$$
is surjective. For a maximal ideal $\mathfrak{m}$, take $\kappa(\mathfrak{m}) = R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Then:
$$f^\vee \otimes_R \kappa(\mathfrak{m}) = f_\mathfrak{m}^\vee \otimes_{R_\mathfrak{m}} (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = V_\mathfrak{m}^\vee/\mathfrak{m}V_\mathfrak{m}^\vee \to U_\mathfrak{m}^\vee/\mathfrak{m}U_\mathfrak{m}^\vee$$
is surjective. Therefore $\operatorname{coker}(f_\mathfrak{m}^\vee) \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} = 0$. Since $V_\mathfrak{m}$ is finitely generated over $R_\mathfrak{m}$, by Nakayama's lemma, $f_\mathfrak{m}^\vee$ is surjective.

  Since localizing at $\mathfrak{m}$ is the filtered colimit of localizations at $g \notin \mathfrak{m}$, there exists $g \notin \mathfrak{m}$ such that $f^\vee \otimes_R R_g$ is surjective (more precisely, one should take stalks in the module sheaf). Using that surjectivity is a Zariski local property gives $(2 \Rightarrow 5)$.

∎

Dually, we can give the quotient space version.

Proposition 2.1.20 (Characterization of Quotient Spaces). *Let R be a ring, and U, V be R-vector spaces. For an R-linear map $f : U \to V$, the following are equivalent:*

(i) *f is surjective.*

(ii) *For any R-field κ, the base change $f \otimes_R \kappa$ is surjective.*

(iii) *f is surjective and $\ker(f)$ is a vector space.*

(iv) *f has a section.*

(v) *The dual map $f^\vee : V^\vee \to U^\vee$ is universally injective.*

We immediately obtain the following corollaries.

Corollary 2.1.21 (Isomorphisms Detected on Fields). *Let R be a ring, and V, W be R-vector spaces. Then $f : V \to W$ is an isomorphism if and only if for any R-field κ, $f \otimes_R \kappa$ is an isomorphism.*

Proof. Note that universal injectivity + surjectivity = isomorphism. ■

Corollary 2.1.22 (Pigeonhole for Vector Spaces). *Let R be a ring, and V, W be R-vector spaces with the same rank. For an R-linear map $f : V \to W$, the following are equivalent:*

(i) *f is an isomorphism.*

(ii) *f is surjective.*

(iii) *f is universally injective.*

Proof. Since V and W have the same rank, for all R-fields κ, we have $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(W)$. If $f \otimes_R \kappa$ is injective/surjective, by the pigeonhole principle, f is an isomorphism. ■

Corollary 2.1.23 (Duality Correspondence). *Let R be a ring, and V, W be R-vector spaces. Then dualization induces bijections:*

$$\begin{gathered}\{\textit{subspaces of } V\} \simeq \{\textit{quotient spaces of } V^\vee\} \\ (U \hookrightarrow V) \mapsto (V^\vee \twoheadrightarrow U^\vee)\end{gathered}$$

and

$$\begin{gathered}\{\textit{quotient spaces of } V\} \simeq \{\textit{subspaces of } V^\vee\} \\ (V \twoheadrightarrow W) \mapsto (W^\vee \hookrightarrow V^\vee).\end{gathered}$$

2.1.4. Projective Space

Now we can define projective space.

Definition 2.1.24 (Projective Space). Let k be a ring and I a set. The *projective I-space over k* is the algebraic k-functor:

$$\mathbb{P}^I_k : \mathsf{CAlg}_k \to \mathsf{Set}, \quad R \mapsto \{\text{quotient lines of } R^{(I)}\}.$$

When $k = \mathbb{Z}$, we simply write $\mathbb{P}^I$. For $n \geq -1$, the *projective n-space over k* is $\mathbb{P}^n_k = \mathbb{P}^{\{0,\dots,n\}}_k$. When $n = 1$, we also call it the *projective line*; when $n = 2$, the *projective plane*.

***Remark* 2.1.25.** (i) $\mathbb{P}^{-1}_k = \mathbb{P}^{\emptyset}_k$ is the empty k-scheme $\emptyset$ (note that it is not the initial object of $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$).

(ii) $\mathbb{P}^0_k = \mathrm{Spec}(k)$ is the terminal k-scheme.

(iii) If I is finite, by Corollary 2.1.23, we have:

$$\mathbb{P}^I_k(R) \simeq \{\text{subspace lines of } R^I\}.$$

However, when I is infinite, $R^{(I)}$ is no longer a vector space, so the definition must use quotient lines.

Notation 2.1.26 (Projective Coordinates). Let R be a ring.

- Let L be a line over R.
- Let $(a_0, \ldots, a_n) \in L^{n+1}$ be a family of elements in L that generate the module.

This is equivalent to saying that the R-linear map induced by the a_i:

$$R^{n+1} \longrightarrow L, \quad e_i \longmapsto a_i$$

is surjective. The *coimage* defines a *quotient line* of R^{n+1}, which we denote:

$$[a_0 : \cdots : a_n] \in \mathbb{P}^n(R).$$

For another line M and generating family $(b_0, \ldots, b_n) \in M^{n+1}$, we define:

$$[a_0 : \cdots : a_n] = [b_0 : \cdots : b_n]$$

if and only if there exists a unique isomorphism $\varphi\colon L \xrightarrow{\sim} M$ such that $\varphi(a_i) = b_i$ for all $0 \leq i \leq n$.

2.1.5. Relationship Between Affine and Projective Spaces

Next, we discuss the relationship between affine I-space and projective I-space. They are connected through the punctured affine space $\mathbb{A}^I_k - 0$.

Recall that the *punctured affine I-space* is the functor defined by:

$$(\mathbb{A}^I_k - 0)\colon R \mapsto \{a \in R^I \mid (a) = R\}.$$

Here (a) denotes the ideal in R generated by all components of $a = (a_i)_i \in R^I$.

It is easy to see that $\mathbb{A}^I_k - 0$ is a subfunctor of $\mathbb{A}^I_k$. By duality, giving a tuple generating the unit ideal is equivalent to giving a surjection $R^{(I)} \twoheadrightarrow R$. This induces a canonical morphism $\mathbb{A}^I_k - 0 \longrightarrow \mathbb{P}^I_k$, mapping an I-tuple valued in R to the corresponding quotient line of $R^{(I)}$ (i.e., the quotient object determined by the kernel of the surjection).

In summary, we have the following relationship diagram:

$$\mathbb{A}^I_k \hookleftarrow (\mathbb{A}^I_k - 0) \longrightarrow \mathbb{P}^I_k.$$

In particular, for $n \geq -1$, we have:

$$\mathbb{A}^{n+1}_k \hookleftarrow (\mathbb{A}^{n+1}_k - 0) \longrightarrow \mathbb{P}^n_k.$$

Proposition 2.1.27 (Trivial Quotient Lines). *Let I be a set and R a ring. Then the morphism $\mathbb{A}^I - 0 \longrightarrow \mathbb{P}^I$ induces an injection:*

$$(\mathbb{A}^I - 0)(R)/R^\times \hookrightarrow \mathbb{P}^I(R).$$

Here $R^\times$ acts on $(\mathbb{A}^I - 0)(R)$ by scalar multiplication. The image of this map is precisely the trivial quotient lines *of $R^{(I)}$ (i.e., quotient lines isomorphic to R).*

In particular, if all lines over R are trivial (e.g., R is a local ring or a principal ideal domain), then this map is a bijection.

Proof. Recall that elements of $(\mathbb{A}^I - 0)(R)$ correspond to surjections $u\colon R^{(I)} \twoheadrightarrow R$. The map $\mathbb{A}^I - 0 \longrightarrow \mathbb{P}^I$ sends u to its corresponding quotient line $[u] \in \mathbb{P}^I(R)$, which is R, hence obviously trivial.

Injectivity: We need to characterize when two surjections $a, b\colon R^{(I)} \twoheadrightarrow R$ define the same point in $\mathbb{P}^I(R)$. By definition, $[a] = [b]$ if and only if there exists an isomorphism $\varphi\colon R \xrightarrow{\sim} R$ making the following diagram commute:

$$\begin{tikzcd}
R^{(I)} \arrow[rr, two heads, "a"] \arrow[dr, two heads, "b"'] & & R \arrow[dl, "\simeq"', "\varphi"] \\
& R &
\end{tikzcd}$$

Note that $\mathrm{Aut}_R(R) \cong R^\times$, so φ must be multiplication by some unit $\lambda \in R^\times$. Therefore, $[a] = [b]$ if and only if there exists $\lambda \in R^\times$ such that $b = \lambda a$ in $(\mathbb{A}^I - 0)(R)$. This is precisely the equivalence relation in $(\mathbb{A}^I - 0)(R)/R^\times$.

Image: Clearly, any element from $(\mathbb{A}^I - 0)(R)$ gives a quotient line with target R, i.e., a trivial quotient line. Conversely, if some quotient line L of $R^{(I)}$ is trivial, then there exists an isomorphism $L \simeq R$. Composing $R^{(I)} \twoheadrightarrow L \xrightarrow{\sim} R$ gives an element of $(\mathbb{A}^I - 0)(R)$. ∎

***Remark* 2.1.28** ($\mathbb{G}_m$-Action and Quotient Functors). Note that the scalar action of $R^\times$ on $(\mathbb{A}^I - 0)(R)$ is functorial in R. In other words, this action gives an action of the affine group scheme $\mathbb{G}_m$ on the algebraic functor $\mathbb{A}^I - 0$. According to the previous proposition, this observation means there exists an injection of algebraic functors:

$$(\mathbb{A}^I - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^I.$$

(This also means that $\mathbb{P}^I$ contains the quotient functor of $\mathbb{A}^I - 0$, and when all lines over R are trivial, the two coincide.)

According to Proposition 2.1.1, for an R-quotient line L, we can always find $f_1, \dots, f_n \in R$ such that $(f_1, \dots, f_n) = R$ and each L_{f_i} is a free R_{f_i}-module. Therefore, Zariski locally, $(\mathbb{A}^I - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^I$ is surjective. This means that as Zariski sheaves, we have:

$$(\mathbb{A}^I - 0)/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^I.$$

***Remark* 2.1.29** (Functoriality and Topological Property Differences). (i) *Functoriality difference:* Unlike $I \mapsto \mathbb{A}^I$, which upgrades to a (contravariant) functor $\mathbb{A}^- : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ on the entire category of sets, the constructions $I \mapsto (\mathbb{A}^I - 0)$ and $I \mapsto \mathbb{P}^I$ do not have this global functoriality. They are only functorial *locally*, i.e., only for $\mathsf{Set}^{\mathrm{surj}} \subset \mathsf{Set}$ (the subcategory of all sets and surjections) as contravariant functors. This is because to pull back a quotient line of $R^{(I)}$ to a quotient line of $R^{(J)}$, we generally need a surjection $J \twoheadrightarrow I$.

(ii) *Topological property difference (quasi-compactness):* If we want to distinguish $\mathbb{A}^I$ from $\mathbb{P}^I$ from a topological perspective, we can observe their *quasi-compactness*.

- For $\mathbb{A}^I$, it is quasi-compact for *any* set I (even infinite).
- But $\mathbb{P}^I$ (and $\mathbb{A}^I - 0$) is quasi-compact if and only if $|I|$ is *finite*.

In algebraic geometry, quasi-compactness (and quasi-separatedness) are very central finiteness conditions. For example, when discussing the six-functor formalism for étale cohomology, we typically assume schemes are quasi-compact and quasi-separated (qcqs). Therefore, many classical results in projective geometry only hold when I is finite.

2.2. Graded Rings

Definition 2.2.1 (Homogeneous Polynomials). Let k be a ring, I a set, and $d \in \mathbb{N}$. We say that a polynomial $f \in k[x_i \mid i \in I]$ is a *homogeneous polynomial of degree d* if it is a k-linear combination of monomials of the form $\prod_{i \in I} x_i^{n_i}$ with $\sum_{i \in I} n_i = d$. We denote by $k[x_i \mid i \in I]_d \subset k[x_i \mid i \in I]$ the k-submodule spanned by all homogeneous polynomials of degree d.

Note that according to this concept, we have a direct sum decomposition:

$$k[x_i \mid i \in I] = \bigoplus_{d \in \mathbb{N}} k[x_i \mid i \in I]_d,$$

and it is compatible with multiplication in the polynomial ring in the following way: a homogeneous polynomial of degree d multiplied by a homogeneous polynomial of degree e gives a homogeneous polynomial of degree $(d + e)$.

This property motivates the concept of graded rings.

Definition 2.2.2 (Graded Rings). Let $(\Gamma, +, 0)$ be a commutative monoid (e.g., $\mathbb{N}$ or $\mathbb{Z}$). A Γ-*graded ring* is a ring A together with, for each $\gamma \in \Gamma$, a subgroup $A_\gamma \subset A$ such that:

- A decomposes as a direct sum: $\bigoplus_{\gamma\in\Gamma} A_\gamma \xrightarrow{\sim} A$.
- $1 \in A_0$ and for $\gamma, \delta \in \Gamma$, we have $A_\gamma A_\delta \subset A_{\gamma+\delta}$.

For $\gamma \in \Gamma$, we call elements of A_γ *homogeneous elements of degree* γ. For a ring k, a Γ-*graded* k-*algebra* is a Γ-graded ring A with a ring homomorphism $k \to A$ whose image lies in A_0.

Definition 2.2.3 (Graded Modules). Let A be a Γ-graded ring. A Γ-*graded* A-*module* is an A-module M together with, for each $\gamma \in \Gamma$, a subgroup $M_\gamma \subset M$ such that:

- M decomposes as a direct sum: $\bigoplus_{\gamma\in\Gamma} M_\gamma \xrightarrow{\sim} M$.
- For all $\gamma, \delta \in \Gamma$, we have $A_\gamma M_\delta \subset M_{\gamma+\delta}$.

We say that an ideal $I \subset A$ is a *homogeneous ideal* if it is a Γ-graded A-module satisfying $I_\gamma = I \cap A_\gamma$.

***Remark* 2.2.4** (Properties of Graded Rings and Modules). Let A be a Γ-graded ring and M a Γ-graded A-module.

(i) A_0 is a subring of A, each A_γ is an A_0-module, and each M_γ is also an A_0-module.

(ii) Let $\Gamma \to \Delta$ be a homomorphism of commutative monoids. Then A can be made into a Δ-graded ring by: for $\delta \in \Delta$, defining

$$A_\delta \coloneqq \bigoplus_{\gamma\mapsto\delta} A_\gamma.$$

Similarly, M can be made into a Δ-graded A-module in the same way. For example, an $\mathbb{N}$-graded ring A can be viewed as a $\mathbb{Z}$-graded ring by taking $A_n = 0$ for $n < 0$.

(iii) An ideal $I \subset A$ is homogeneous if and only if it is generated by homogeneous elements.

(iv) For a homogeneous ideal $I \subset A$, the quotient ring A/I inherits a unique Γ-graded ring structure making $A \twoheadrightarrow A/I$ a graded map. Moreover, the A/I-module M/IM has a unique Γ-graded structure making $M \to M/IM$ a graded map.

(v) Let $S \subset A$ be the set of homogeneous elements whose degrees are invertible in Γ. Then the localization $A[S^{-1}]$ inherits a unique Γ-graded structure making $A \to A[S^{-1}]$ a graded map. Moreover, the $A[S^{-1}]$-module $M[S^{-1}]$ inherits a unique Γ-graded structure making $M \to M[S^{-1}]$ a graded map.

Here are some examples:

Example 2.2.5 (Examples of Graded Rings). Let k be a ring.

- As mentioned earlier, for any set I, the polynomial ring $k[x_i \mid i \in I]$ is a $\mathbb{Z}$-graded ring.
- The Laurent polynomial ring $k[x^{\pm 1}]$ is a $\mathbb{Z}$-graded ring.
- Let M be a k-module. Then the symmetric algebra $\mathrm{Sym}_k(M)$ has a canonical $\mathbb{N}$-graded structure:

$$\mathrm{Sym}_k(M) = \bigoplus_{d\in\mathbb{N}} \mathrm{Sym}_k^d(M),$$

where $\mathrm{Sym}_k^d(M)$ is the d-th symmetric power of M.

- Let L be a line over k. Then $\bigoplus_{d\in\mathbb{Z}} L^{\otimes d}$ is automatically a $\mathbb{Z}$-graded ring. In fact, we have $L^{\otimes d} = \mathrm{Sym}_k^d(L)$ and $L^{\otimes -1} = L^\vee$.

We introduce the following notation:

Notation 2.2.6 (Notation for Graded Rings and Modules). Let A be a Γ-graded ring, M a Γ-graded A-module, and $\gamma \in \Gamma$.

(i) We denote by $M(\gamma)$ the Γ-graded A-module defined as follows: its underlying A-module is M, and for $\delta \in \Gamma$, its graded component is:

$$(M(\gamma))_\delta = M_{\gamma+\delta}.$$

The above construction gives an endofunctor $M \mapsto M(\gamma)$. When $\gamma \in \Gamma$ is invertible, this endofunctor is a categorical equivalence (i.e., a shift operator).

(ii) We denote by $A^{(\gamma)}$ the $\mathbb{N}$-graded ring $\bigoplus_{n\in\mathbb{N}} A_{n\gamma}$, where $(A^{(\gamma)})_n = A_{n\gamma}$. Similarly, we define $M^{(\gamma)}$ to be the $\mathbb{N}$-graded $A^{(\gamma)}$-module.

(iii) If $\gamma \in \Gamma$ is invertible, and given $f \in A_\gamma$, by the previous remark, A_f is a Γ-graded ring. We denote by $A_{(f)} = (A_f)_0$ its degree 0 part (i.e., the *homogeneous localization*). Similarly, we denote by $M_{(f)} = (M_f)_0$ the $A_{(f)}$-module. Note that there is an isomorphism of $\mathbb{Z}$-graded rings:

$$A_{(f)}[x^{\pm 1}] \xrightarrow{\sim} A_f^{(\gamma)}, \quad x \mapsto f.$$

2.2.1. Projective Schemes

Now we can define projective schemes.

Construction 2.2.7 (Evaluation Map for Homogeneous Polynomials). Let R be a k-algebra, $L \in \mathrm{Line}_R$ an R-line, I a set, and $d \in \mathbb{N}$ a natural number. We can define the following *evaluation map*:

$$k[x_i \mid i \in I]_d \times L^I \to L^{\otimes d}, \quad (f, a) \mapsto f(a).$$

Construction details: An element $a \in L^I$ is equivalent to an R-linear map $a\colon R^{(I)} \to L$ (uniquely determined by the images of the basis vectors). Therefore $(f, a) \mapsto f(a)$ actually means the following composite map:

$$k[x_i \mid i \in I]_d = \mathrm{Sym}_k^d(k^{(I)}) \to \mathrm{Sym}_R^d(R^{(I)}) \xrightarrow{\mathrm{Sym}_R^d(a)} \mathrm{Sym}_R^d(L) \cong L^{\otimes d}.$$

(Note: for a rank 1 module L, there is a canonical isomorphism $\mathrm{Sym}^d(L) \cong L^{\otimes d}$).

Concretely, this is a k-linear map uniquely determined by the images of monomials:

$$\prod_i x_i^{n_i} \mapsto \bigotimes_i a_i^{\otimes n_i} \in L^{\otimes d}.$$

In particular, when $L = R$ (i.e., L is a trivial R-line), we have $L^{\otimes d} \cong R$, which recovers the usual polynomial evaluation.

Recall that we defined affine schemes as loci of polynomial systems. For projective schemes, we naturally need to use homogeneous polynomials.

Definition 2.2.8 (System of Homogeneous Polynomials). Let k be a ring, and I, J be sets. A *system of J homogeneous polynomials in I variables over k* is a J-tuple $\Sigma = (f_j)_{j\in J}$, where each f_j is a *homogeneous polynomial* in the polynomial ring $k[x_i \mid i \in I]$.

We denote by (Σ) the homogeneous ideal generated by Σ in $k[x_i \mid i \in I]$, and write

$$k[\Sigma] = k[x_i \mid i \in I]/(\Sigma)$$

for the corresponding $\mathbb{N}$-graded k-algebra.

Definition 2.2.9 (Projective Vanishing Locus). Let $F \subset k[x_i \mid i \in I]$ be a family of homogeneous polynomials. The *vanishing locus of F in $\mathbb{P}_k^I$* is the subfunctor $V(F) \subset \mathbb{P}_k^I$ defined by:

$$V(F)(R) = \left\{ a\colon R^{(I)} \twoheadrightarrow L \,\middle|\, \text{for all } f \in F \cap k[x_i]_d, \text{ we have } f(a) = 0 \in L^{\otimes d} \right\}.$$

Verification as Subfunctor. We need to verify that for any k-algebra homomorphism $\varphi\colon R \to S$, the induced map $\mathbb{P}^I(\varphi)$ maps $V(F)(R)$ into $V(F)(S)$.

Let $a \in V(F)(R)$ be a quotient line $R^{(I)} \twoheadrightarrow L$ satisfying $f(a) = 0$. Let a_S be the base change of a along φ, i.e., the quotient line:

$$S^{(I)} \cong R^{(I)} \otimes_R S \xrightarrow{a\otimes 1} L \otimes_R S =: L_S.$$

We need to show $f(a_S) = 0 \in L_S^{\otimes d}$.

This follows from the *commutativity of evaluation with base change*. Consider the following commutative diagram (for $d = \deg f$):

$$\begin{tikzcd} \mathrm{Sym}^d_k(k^{(I)}) \arrow[r, "f(a)"] \arrow[dr, "f(a_S)"'] & L^{\otimes d} \arrow[d, "\varphi_{L^{\otimes d}}"] \\ & L^{\otimes d} \otimes_R S \cong (L \otimes_R S)^{\otimes d} \end{tikzcd}$$

where $\varphi_{L^{\otimes d}}$ is the canonical map sending x to $x \otimes 1$. Since $a \in V(F)(R)$, we have $f(a) = 0_R$ (the zero element in $L^{\otimes d}$). Since $\varphi_{L^{\otimes d}}$ is an additive group homomorphism, it maps 0_R to 0_S (the zero element in $L_S^{\otimes d}$). By commutativity of the diagram, $f(a_S) = 0_S$. Therefore $a_S \in V(F)(S)$. ∎

It is easy to see that the subfunctor $V(F) \subset \mathbb{P}^I_k$ is determined only by the homogeneous ideal (F). However, unlike the affine case, different homogeneous ideals can correspond to the same vanishing locus. We will discuss this in detail later.

Now we can define the solution set functor.

Definition 2.2.10 (Homogeneous Solution Functor). Let $\Sigma = (f_j)_{j \in J}$ be a system of J homogeneous polynomials in I variables over k. Its *solution set functor* $\mathrm{hSol}_\Sigma \colon \mathsf{CAlg}_k \to \mathsf{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{P}^I_k$, i.e.,

$$\mathrm{hSol}_\Sigma = V(\{f_j \mid j \in J\}) \subset \mathbb{P}^I_k.$$

***Remark* 2.2.11.** Of course, the "h" in hSol stands for homogeneous.

As in the affine case, we define projective schemes as algebraic functors isomorphic to solution set functors.

Definition 2.2.12 (Projective Schemes). Let k be a ring. We say that a k-algebraic functor $X \colon \mathsf{CAlg}_k \to \mathsf{Set}$ is a *projective k-scheme* if there exists a system of homogeneous polynomials in *finitely many variables* Σ such that $X \simeq \mathrm{hSol}_\Sigma$.

We denote by $\mathrm{Proj}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ the full subcategory spanned by all projective k-schemes.

***Remark* 2.2.13.** The finiteness condition here is just for convenience. It is easy to see that with the finiteness condition, projective k-schemes are automatically of finite type and automatically proper (compact).

Next, we want to ask whether in the projective case there is an analogue of the phenomenon $\mathrm{Spec}(k[\Sigma]) = \mathrm{Sol}_\Sigma$ from the affine case. But we need to note that not every $\mathbb{N}$-graded k-algebra A can be written in the form $k[\Sigma]$: this is possible if and only if A is generated as a k-algebra by A_1, i.e., $\mathrm{Sym}_k(A_1) \twoheadrightarrow A$ is surjective. Therefore, in the latter half of this section, we temporarily only consider the case where A is generated as a k-algebra by A_1 (this is also the most commonly used case). Of course, this can be extended to more general situations.

Construction 2.2.14 (The Proj Functor - Preliminary Version). Let k be a ring and A an $\mathbb{N}$-graded k-algebra generated by A_1. Define the algebraic k-functor $\mathrm{Proj}(A) \colon \mathsf{CAlg}_k \to \mathsf{Set}$ by:

$$\mathrm{Proj}(A)(R) = \left\{\text{quotient line } A_1 \otimes_k R \twoheadrightarrow L \,\middle|\, \mathrm{Sym}_k(A_1) \to \mathrm{Sym}_R(L) \text{ factors through } A\right\}.$$

Proposition 2.2.15 (Comparison with Solution Functors). *Let Σ be a system of homogeneous polynomials over k. Then there is a canonical isomorphism:*

$$\mathrm{hSol}_\Sigma \simeq \mathrm{Proj}(k[\Sigma]) \colon \mathsf{CAlg}_k \to \mathsf{Set}.$$

Proof. We construct the bijection for any k-algebra R.

$\Rightarrow$: Let $a \in \mathrm{hSol}_\Sigma(R)$. By definition, a is a quotient line $R^{(I)} \twoheadrightarrow L$ satisfying $f(a) = 0$ for all $f \in \Sigma$. According to Construction 2.2.7 and the isomorphism $\mathrm{Sym}_R(R^{(I)}) \cong R[x_i \mid i \in I]$, giving a is equivalent to giving a graded surjection of $\mathbb{N}$-graded R-algebras:

$$\varphi_a \colon \mathrm{Sym}_R(R^{(I)}) \twoheadrightarrow \mathrm{Sym}_R(L).$$

The condition "$f(a) = 0$ for all $f \in \Sigma$" is equivalent to saying that the kernel of φ_a contains the ideal $(\Sigma) \otimes_k R$. This means φ_a factors through the quotient algebra $k[\Sigma] \otimes_k R$, i.e., there exists a unique factorization:

$$\begin{tikzcd}
\mathrm{Sym}_R(R^{(I)}) \arrow[r, two heads] \arrow[dr, two heads, "\mathrm{Sym}_R(a)"'] & \mathrm{Sym}_R(k[\Sigma]_1 \otimes_k R) \arrow[r, two heads] \arrow[d] & k[\Sigma] \otimes_k R \arrow[dl, dashed, two heads] \\
 & \mathrm{Sym}_R(L) &
\end{tikzcd}$$

This corresponds precisely to an element of $\mathrm{Proj}(k[\Sigma])(R)$.

$\Leftarrow$: Conversely, given an element of $\mathrm{Proj}(k[\Sigma])(R)$, i.e., given $k[\Sigma]_1 \otimes_k R \twoheadrightarrow L$ such that the induced Sym map factors through $k[\Sigma] \otimes_k R$. Since $k[\Sigma]$ is generated by its degree 1 part (as a quotient of $k[x_i]$), we can pull this back to $R^{(I)} \twoheadrightarrow k[\Sigma]_1 \otimes_k R \twoheadrightarrow L$. Clearly this composition satisfies the equations in Σ. ■

Recall that for a k-module M, we can consider its symmetric algebra $\mathrm{Sym}_k(M)$.

Example 2.2.16 (Projective Space as $\mathbb{P}(M)$). Let M be a k-module. Define $\mathbb{P}(M)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ by:

$$\mathbb{P}(M)(R) = \{\text{quotient line } M \otimes_k R \twoheadrightarrow L\}.$$

It is easy to see that $\mathbb{P}(M) \simeq \mathrm{Proj}(\mathrm{Sym}_k(M))$. If M is also finitely generated, then $\mathbb{P}(M)$ is a projective k-scheme. Of course, if we consider the free module $k^{(I)}$, we have $\mathbb{P}^I_k \simeq \mathbb{P}(k^{(I)})$.

Similarly, we can consider the dual functor $\mathbb{P}^\vee(M)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ defined by:

$$\mathbb{P}^\vee(M)(R) = \{\text{subspace line } L \hookrightarrow M \otimes_k R\}.$$

If M is a vector space, we have $\mathbb{P}^\vee(M^\vee) \simeq \mathbb{P}(M)$. However, in general, $\mathbb{P}^\vee(M)$ is not a scheme.

***Remark* 2.2.17.** If A is an $\mathbb{N}$-graded k-algebra generated by A_1, then $\mathrm{Proj}(A)$ is a subfunctor of $\mathbb{P}(A_1)$.

2.2.2. Loci of Linear Maps

Now consider the projective n-space $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$. According to the previous content, its R-points are nothing but quotient lines $a\colon R^{\{0,\dots,n\}} \twoheadrightarrow L$.

Given a homogeneous polynomial $f \in \mathbb{Z}[x_0, \dots, x_n]_d = \mathrm{Sym}^d(\mathbb{Z}^{\{0,\dots,n\}})$, we can consider the subfunctor $D(f)$ of $\mathbb{P}^n$, whose R-points are quotient lines a satisfying the following condition: $L^{\otimes d}$ is generated by the image of f. Equivalently, the composite map:

$$s_f\colon R \xrightarrow{f} \mathrm{Sym}^d_R(R^{\{0,\dots,n\}}) \xrightarrow{\mathrm{Sym}^d(a)} L^{\otimes d}$$

is an R-module *isomorphism* (this is precisely the functorial expression that f is non-zero/non-degenerate in $L^{\otimes d}$). Of course, this is also equivalent to saying that for every R-field κ, we have $f(a) \in L^{\otimes d} \otimes_R \kappa$ is non-zero.

Recall that in the affine case, for a k-algebra A and $f \in A$, we defined $D(f) \subset \mathrm{Spec}(A)$ by: for a commutative k-algebra R, $D(f)(R) = \{x \in \mathrm{Hom}(A, R) \mid (x(f)) \text{ generates all of } R\}$.

We can rewrite this condition in a form completely parallel to the projective case. For the affine case, the "function" f evaluated at point x gives an R-linear map:

$$R \xrightarrow{x(f)} R, \quad 1 \mapsto x(f).$$

The condition "$(x(f)) = R$" (i.e., $x(f)$ is invertible) is equivalent to saying the above linear map is an *isomorphism*.

Summary (Unified Description of Non-vanishing Loci). Therefore, whether in the affine or projective case, the non-vanishing locus $D(f)$ can be uniformly described as: the collection of points making the evaluation map of the section f:

$$R \xrightarrow{f(\text{point})} \mathcal{L}$$

an *isomorphism*.

- In the affine case, the target line is the trivial line $\mathcal{L} = R$, and the evaluation map is $R \xrightarrow{f} R$.
- In the projective case, the target line is $\mathcal{L} = L^{\otimes d}$, and the evaluation map is $R \xrightarrow{f} L^{\otimes d}$.

But at this point, we encounter a problem: $f \in \mathbb{Z}[x_0, \ldots, x_n]_d$ is not a function on $\mathbb{P}^n$; it is actually a section in the *line bundle* $\mathcal{O}(d)$ on $\mathbb{P}^n$. Therefore, we cannot immediately conclude that $D(f) \subset \mathbb{P}^n$ is an open subfunctor. The goal of this section is to show that it is indeed an open subfunctor, and in fact, it is a *relatively affine* embedding (i.e., any pullback of it is affine).

To do this, consider any A-point $x\colon \operatorname{Spec}(A) \to \mathbb{P}^n$ (i.e., a quotient line $A^{n+1} \twoheadrightarrow L$ over A). Consider the pullback diagram:

$$\begin{tikzcd} x^{-1}(D(f)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(A) \arrow[d, "x"] \\ D(f) \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

We need to show that $x^{-1}(D(f))$ is an open subfunctor of $\operatorname{Spec}(A)$.

First, note that we can actually characterize $x^{-1}(D(f))$ in the following way: for an A-algebra R, we have:

$$x^{-1}(D(f))(R) = \{\varphi\colon A \to R \mid \varphi^*(s)\colon R \to L^{\otimes d} \otimes_A R \text{ is an isomorphism}\},$$

where s is the linear map over A:

$$s\colon A \xrightarrow{f} \operatorname{Sym}^d_A(A^{\{0,\ldots,n\}}) \xrightarrow{x} L^{\otimes d}.$$

This means that $x^{-1}(D(f))$ is the locus in $\operatorname{Spec}(A)$ where the linear map "s is an isomorphism".

Next, we will approach this from a broader perspective. Let P be some property of modules that is preserved by base change (e.g., being zero, finitely generated, flat, etc.). For a ring A and an A-module M, we can talk about "M having property P". This can actually be encoded as the following subfunctor of $\operatorname{Spec}(A)$:

$$R \mapsto \{\varphi\colon A \to R \mid \text{the } R\text{-module } \varphi^*(M) \text{ has property } P\}.$$

Similarly, for an A-linear map $u\colon M \to N$, we can also define the locus where it has a specific property:

Definition 2.2.18 (Loci of Linear Maps). Let $u\colon M \to N$ be an A-module homomorphism. We define:

- *Vanishing locus* $V(u)$: the locus where u is zero.
- *Surjection locus* $\operatorname{Epi}(u)$: the locus where u is surjective.
- *Injection locus* $\operatorname{Mono}(u)$: the locus where u is universally injective.
- *Isomorphism locus* $\operatorname{Iso}(u)$: the locus where u is an isomorphism.

Returning to the problem of $D(f)$, we find that $x^{-1}(D(f))$ is precisely the *isomorphism locus* $\operatorname{Iso}(s)$ of the linear map $s\colon A \to L^{\otimes d}$.

Proposition 2.2.19 (Loci as Open/Closed Subfunctors). *Let A be a ring and $f\colon M \to N$ an A-linear map. Then:*

(i) If N is a vector space, then $V(f) \subset \operatorname{Spec}(A)$ is a closed subfunctor.

(ii) If N is a vector space, then $\operatorname{Epi}(f) \subset \operatorname{Spec}(A)$ is an open subfunctor.

(iii) If M and N are both vector spaces, then $\operatorname{Mono}(f) \subset \operatorname{Spec}(A)$ is an open subfunctor.

(iv) If M and N are both vector spaces, then $\operatorname{Iso}(f) \subset \operatorname{Spec}(A)$ is an open subfunctor.

***Remark* 2.2.20.** It is easy to see that (4) implies that $D(f)$ is an open subfunctor (since $D(f)$ is essentially the isomorphism locus of the linear map $s\colon A \to L^{\otimes d}$).

Proof. (i) *Vanishing locus:* f is zero if and only if for any $x \in M$, the composite map $f(x)\colon R \xrightarrow{x} M \xrightarrow{f} N$ is zero. That is, $V(f) = \bigcap_{x\in M} V(f(x))$. Since arbitrary intersections of closed subfunctors remain closed subfunctors, we may assume $M = R$. In this case, $f\colon R \to N$ corresponds to an element $n = f(1) \in N$.

Since N is a vector space, N is reflexive ($N \cong N^{\vee\vee}$), so $n = 0$ if and only if for all $\lambda \in N^\vee$, we have $\lambda(n) = 0$. Therefore:

$$V(R \xrightarrow{f} N) = V(N^\vee \xrightarrow{f^\vee} R) = V(\mathrm{im}(f^\vee)),$$

where $f^\vee\colon N^\vee \to R$ is the dual map induced by f. For any R-algebra homomorphism $\varphi\colon R \to S$, we have $\varphi \in V(f)$ if and only if $\varphi^*(\mathrm{im}(f^\vee)) = 0$. This is precisely the closed subfunctor $V(\mathrm{im}(f^\vee))$ defined by the ideal $\mathrm{im}(f^\vee) \subset R$.

(ii) *Surjection locus:* Since N is a vector space, it is a direct summand of R^n. There exists an R-module N' such that $N \oplus N' \simeq R^n$. Clearly, f is surjective if and only if $f \oplus \mathrm{id}_{N'}$ is surjective. Thus:

$$\mathrm{Epi}(f) = \mathrm{Epi}(f \oplus \mathrm{id}_{N'}\colon M \oplus N' \to R^n).$$

Therefore, we *may assume* $N = R^n$ (i.e., $N' = 0$), so $f\colon M \to R^n$.

Claim 2.2.21. *The surjection locus can be detected by the exterior power functor, i.e.,*

$$\mathrm{Epi}(f) = \mathrm{Epi}(\Lambda^n(f)).$$

Proof of Claim. To prove this, we examine the action of f on n elements. For any $m_1, \dots, m_n \in M$, their images $(f(m_1), \dots, f(m_n))$ form n column vectors in R^n, i.e., a matrix in $R^{n\times n}$. We can detect the linear independence of these vectors through the determinant map $\det\colon R^{n\times n} \to R$.

This can be expressed through the following commutative diagram:

$$\begin{tikzcd}
M^n \arrow[r, "f^n"] \arrow[d, two heads, "\mathrm{can}"'] & (R^n)^n \cong R^{n\times n} \arrow[r, "\det"] \arrow[d, two heads, "\mathrm{can}"'] & R \arrow[d, equal] \\
\Lambda^n M \arrow[r, "\Lambda^n(f)"] & \Lambda^n(R^n) \arrow[r, "\simeq"] & R
\end{tikzcd}$$

- If f is surjective, then the standard basis $e_1, \dots, e_n$ of R^n all lie in the image of f. There exist m_i such that $f(m_i) = e_i$. Then in the bottom row of the diagram, $\Lambda^n(f)(m_1 \wedge \cdots \wedge m_n) = 1$, so the image of $\Lambda^n(f)$ contains a unit, hence is surjective.
- If $\Lambda^n(f)$ is surjective, there exists $\omega \in \Lambda^n M$ such that $\Lambda^n(f)(\omega) = 1$. Since ω is a finite sum of elements of the form $m_{i_1} \wedge \cdots \wedge m_{i_n}$, there exists a set $m_1, \dots, m_n \in M$ whose image on the right side (i.e., the determinant) is a unit. In other words, the matrix $(f(m_1), \dots, f(m_n))$ is an invertible matrix in $R^{n\times n}$.

  This means that the column vectors $f(m_1), \dots, f(m_n)$ form a basis of R^n. Since these vectors all lie in $\mathrm{im}(f)$, we have $\mathrm{im}(f) = R^n$, i.e., f is surjective.

■

Through the claim, we reduce the problem from $N = R^n$ to the rank 1 case (since $\Lambda^n R^n \simeq R$). In this case, $\Lambda^n(f)\colon \Lambda^n M \to R$. For maps to R, surjectivity is equivalent to the image generating the unit ideal. That is:

$$\mathrm{Epi}(f) = \mathrm{Epi}(\Lambda^n(f)) = D(\mathrm{im}(\Lambda^n(f))).$$

This is clearly an open subfunctor of $\mathrm{Spec}(A)$.

(iii) *Injection locus:* When M, N are both vector spaces, $f\colon M \to N$ is universally injective if and only if $f^\vee\colon N^\vee \to M^\vee$ is surjective. Since $N^\vee, M^\vee$ are also vector spaces, by (2), $\mathrm{Epi}(f^\vee)$ is an open subfunctor.

(iv) *Isomorphism locus:* f is an isomorphism if and only if it is surjective and universally injective. That is, $\mathrm{Iso}(f) = \mathrm{Epi}(f) \cap \mathrm{Mono}(f)$. The intersection of two open subfunctors is still an open subfunctor.

■

2.2.3. Localization at a Section

Next, we show that for $s\colon R \to L$, we have that $D(s)$ is affine.

Now consider a map between R-lines $s\colon L' \to L$. Since L' is an R-line, hence invertible, we can consider $s \otimes_R (L')^\vee$. Then we have $D(s) = D(s \otimes_R (L')^\vee)$. Therefore, we may assume $s\colon R \to L$. We know this is equivalent to giving an element $s \in L$. When $L = R$, this means $D(s) = \mathrm{Spec}(R_s)$. Now we consider this for L being a non-trivial R-line.

Definition 2.2.22 (Periodic Modules). Let R be a ring, L an R-line, and $s \in L$. We say that an R-module M is *s-periodic* if the morphism:

$$\mathrm{id}_M \otimes s\colon M \otimes_R R \to M \otimes_R L$$

is an isomorphism.

Proposition 2.2.23 (Localization at a Line Section). *Let R be a ring, L an R-line (i.e., an invertible module over R), and $s \in L$ (i.e., a section $s\colon R \to L$). Then:*

(i) There exists an s-periodic R-algebra R_s that is initial among all s-periodic R-algebras.

(ii) The forgetful functor $\mathrm{Mod}_{R_s} \to \mathrm{Mod}_R$ identifies Mod_{R_s} with the full subcategory spanned by s-periodic modules. Therefore, it has a left adjoint $\mathrm{Mod}_R \to \mathrm{Mod}_{R_s}$ sending M to $M_s := M \otimes_R R_s$.

(iii) Concretely, M_s can be computed as the following colimit in Mod_R:

$$M \xrightarrow{\cdot s} M \otimes_R L \xrightarrow{\cdot s} M \otimes_R L^{\otimes 2} \xrightarrow{\cdot s} \cdots .$$

Here the map $M \otimes L^{\otimes n} \xrightarrow{\cdot s} M \otimes L^{\otimes n+1}$ is defined by $x \mapsto x \otimes s$ (using the isomorphism $M \otimes L^{\otimes n} \cong M \otimes L^{\otimes n} \otimes R \xrightarrow{1 \otimes s} M \otimes L^{\otimes n} \otimes L$).

Proof. The proof is left to the reader. ∎

***Remark* 2.2.24** (Classical Localization as Special Case). When $M = R$ and $L = R$ (trivial line), s corresponds to an element $f \in R$. In this case, the above colimit is $R \xrightarrow{f} R \xrightarrow{f} R \cdots$, whose colimit is precisely the classical localization $R_f = R[1/f]$.

For a general line bundle L, we cannot directly write "$1/s$" since s is not an element of the ring. But R_s precisely captures the algebraic property of "making s invertible" (i.e., making $\cdot s\colon M \to M \otimes L$ an isomorphism) through the above construction. This works because L is an R-line, so tensoring with L behaves like "multiplication" in terms of categorical properties.

Therefore, we immediately obtain the following corollary:

Corollary 2.2.25 (Non-vanishing Locus is Affine). *Let R be a ring, L an R-line, and $s \in L$. Then $D(s) = \mathrm{Epi}(R \xrightarrow{s} L) \simeq \mathrm{Spec}(R_s)$.*

Summary (Relative Affineness of $D(f)$). Therefore, for $f \in \mathbb{Z}[x_0, \dots, x_n]_d$, we have that $D(f) \subset \mathbb{P}^n$ is an open subfunctor. Moreover, for any $x\colon \mathrm{Spec}(R) \to \mathbb{P}^n$ corresponding to a quotient line $a\colon R^{n+1} \twoheadrightarrow L$, we have $x^{-1}D(f) = D(s)$, where $s = f(a) \in L^{\otimes d}$. Therefore, $D(f) \hookrightarrow \mathbb{P}^n$ is relatively affine.

2.2.4. Generalized Zariski Descent

We can apply the previous generalization to Zariski descent.

Proposition 2.2.26 (Generalized Zariski Descent). *Let R be a ring, L an R-line, and $(s_i)_{i \in I}$ a generating set for L. Then for any R-module M, the diagram:*

$$M \longrightarrow \prod_{i \in I} M_{s_i} \rightrightarrows \prod_{i,j \in I} M_{s_i s_j}$$

is an equalizer, where $s_i s_j = s_i \otimes s_j \in L^{\otimes 2}$.

Proof. We know that for $s: R \to L$, since L is an R-line, we can consider its dual:

$$s^\vee : L^\vee = R \otimes_R L^\vee \to L \otimes_R L^\vee = R.$$

Let $F \subset L^\vee$ be a generating set for $L^\vee$. We can consider $F_s := s^\vee(F) \subset R$. Consider the ideal $(F_s) \subset R$, viewing it as an R-module. Then consider $(F_s) \otimes_R R_s$. Since R_s is an s-periodic R-algebra, we have:

$$R_s \otimes_R R \xrightarrow{\mathrm{id}_{R_s} \otimes s} R_s \otimes_R L$$

is an isomorphism, i.e., $R_s \xrightarrow{\sim} L_s$. By duality, $s^\vee = s \otimes_R L^\vee$, and since the isomorphism property is preserved by base change, $s^\vee \otimes_R R_s$ is also an isomorphism. Thus (F_s) generates all of R_s. Similarly, for $t: R \to L$, we also have that $F_s F_t$ generates R_{st}.

Now consider the generating set $(s_i)_{i \in I}$ of L. It is easy to see that $(s_i^\vee)_{i \in I}$ also generates $L^\vee$. Now consider $M_{s_i} \in \mathrm{Mod}_{R_{s_i}}$. According to the previous description, we know that F_{s_i} generates all of R_{s_i}. Therefore, by local Zariski descent, we have:

$$M_{s_i} \longrightarrow \prod_{f \in F_{s_i}} (M_{s_i})_f \rightrightarrows \prod_{f,g \in F_{s_i}} (M_{s_i})_{fg}$$

is an equalizer. Similarly, we can write $M_{s_i s_j}$ as an equalizer over $F_{s_i} F_{s_j}$.

Since $(s_i)_{i \in I}$ generates all of L, the set $\mathcal{F} := \bigcup_{i \in I} F_{s_i}$ generates all of R. By standard Zariski descent over R with respect to $\mathcal{F}$, we know that M is the equalizer of its corresponding descent diagram. To relate M with (M_{s_i}), we construct the following double diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \prod_{i \in I} M_{s_i} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j \in I} M_{s_i s_j} \arrow[d] \\
M \arrow[r] \arrow[d] & \prod_{i \in I} \prod_{f_i \in F_i} M_{f_i} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,j \in I} \prod_{f_i f_j \in F_i F_j} M_{f_i f_j} \arrow[d, shift left] \arrow[d, shift right] \\
M \arrow[r] & \prod_{i \in I} \prod_{f_i, g_i \in F_i} M_{f_i g_i} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j \in I} \prod_{f_i f_j, f_i' f_j' \in F_i F_j} M_{f_i f_j f_i' f_j'}
\end{tikzcd}$$

(Note: in the diagram, we abbreviate F_{s_i} as F_i).

In this diagram, each column is a local Zariski descent diagram, so the columns are exact. Using the commutativity of limits, the global limit (the equalizer of the first row) is equivalent to first computing the equalizer of rows and then the equalizer of columns.

We need to show that the top row (i.e., $M \to \cdots$) is an equalizer diagram. Since the columns are known to be exact, everything reduces to showing that the descent conditions formed by the second and third rows are equivalent to the standard descent condition with respect to $\bigcup_{i \in I} F_{s_i}$.

The key is to show that the double products in the diagram cover all compatibility conditions:

$$\prod_{i,j \in I} \prod_{f_i f_j \in F_i F_j} M_{f_i f_j} \quad \text{and} \quad \prod_{f,g \in \bigcup_{i \in I} F_{s_i}} M_{fg} \quad \text{define consistent restriction conditions.}$$

Note that for any two elements $f, g \in \bigcup_{i \in I} F_{s_i} = \bigcup_{k \in I} F_k$, there always exist $i, j \in I$ such that $f \in F_i$ and $g \in F_j$. Therefore, any compatibility condition $x_f|_{fg} = x_g|_{fg}$ that must be satisfied over $\bigcup_{i \in I} F_{s_i}$ can be encoded as the compatibility of the component corresponding to $f_i f_j \in F_i F_j$ in the double product.

This shows that the equalizer of the row direction is precisely the elements satisfying all pairwise compatibility conditions, i.e., M. Therefore, the first row is also an equalizer diagram. ∎

2.3. The Proj Construction

Notation 2.3.1 (Graded Ring Notation). Let A be an $\mathbb{N}$-graded ring and $d \in \mathbb{N}$ a natural number.

- $A^{(d)}$ denotes the $\mathbb{N}$-graded ring $\bigoplus_{n \in \mathbb{N}} A_{nd}$ (Veronese subring), where $(A^{(d)})_n = A_{nd}$.

- $A_+ = \bigoplus_{d\geq 1} A_d$ is the irrelevant ideal.
- For a homogeneous element $f \in A$, A_f is a $\mathbb{Z}$-graded ring, and $A_{(f)} := (A_f)_0$ is its degree 0 part.

Next, we construct the general Proj functor.

Definition 2.3.2 (Eventually Surjective). Let A, B be $\mathbb{N}$-graded rings. We say that a graded map $A \to B$ is *eventually surjective* if for all $d \geq 1$ and every homogeneous element $b \in B_d$, there exists $n \geq 1$ such that b^n lies in the image of the map $A_{nd} \otimes_{A_0} B_0 \to B_{nd}$.

We denote by $\mathsf{CAlg}^{\mathbb{N},\mathrm{es}}$ the category of $\mathbb{N}$-graded rings and eventually surjective maps between them.

Construction 2.3.3 (The Proj Functor). We construct the functor:

$$\mathrm{Proj}\colon (\mathsf{CAlg}^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}).$$

In fact, it comes with a natural transformation $\mathrm{Proj}(-) \Rightarrow \mathrm{Spec}((-)_0)$.

Step 1: Define $\mathrm{Proj}_1(A)$. We define $\mathrm{Proj}_1(A)$ as an object in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{Spec}(A_0)} \simeq \mathsf{Fun}(\mathsf{CAlg}_{A_0}, \mathsf{Set})$. Given an A_0-algebra R, the set $\mathrm{Proj}_1(A)(R)$ consists of pairs (L, φ) where:

- L is a quotient line of $A_1 \otimes_{A_0} R$, i.e., a surjective R-module homomorphism $A_1 \otimes_{A_0} R \twoheadrightarrow L$;
- $\varphi\colon A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ is a graded R-algebra morphism that extends the above quotient map (i.e., φ restricted to degree 1 is precisely $A_1 \otimes R \to L$).

Note that φ is automatically surjective. Therefore, $\mathrm{Proj}_1(A)(R)$ can be viewed as the collection of all graded quotient algebras of $A \otimes_{A_0} R$ that are isomorphic to the symmetric algebra of a line.

Step 2: Define $\mathrm{Proj}_d(A)$ *and the colimit.* For $d \in \mathbb{N}$, define $\mathrm{Proj}_d(A) = \mathrm{Proj}_1(A^{(d)})$. For $n \in \mathbb{N}$, there exists a canonical morphism:

$$\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A), \quad (L, \varphi) \mapsto (L^{\otimes n}, \varphi^{(n)}),$$

where $\varphi^{(n)}$ is the restriction of φ to the Veronese subring (note that $\mathrm{Sym}_R(L)^{(n)} \cong \mathrm{Sym}_R(L^{\otimes n})$).

This defines a functor:

$$\mathbb{N}^{\mathrm{div}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}), \quad d \mapsto \mathrm{Proj}_d(A),$$

where $\mathbb{N}^{\mathrm{div}}$ is the poset ordered by divisibility. We define $\mathrm{Proj}(A)$ as its colimit:

$$\mathrm{Proj}(A) = \mathrm{colim}_{d\in\mathbb{N}^{\mathrm{div}}_{>0}} \mathrm{Proj}_d(A).$$

(Since 1 divides all d, we also have $\mathrm{Proj}(A) \cong \mathrm{colim}_{d\to\infty} \mathrm{Proj}_1(A^{(d)})$).

Since 0 is the terminal object of $\mathbb{N}^{\mathrm{div}}$, there exists a canonical morphism:

$$\mathrm{Proj}(A) \to \mathrm{Proj}_0(A) = \mathrm{Proj}_1(A^{(0)}) = \mathrm{Proj}_1(A_0[x]) \cong \mathrm{Spec}(A_0).$$

Step 3: Functoriality. Let $\alpha\colon A \to B$ be a map between $\mathbb{N}$-graded rings. Let (L, φ) be an R-point of $\mathrm{Proj}_d(B)$. Then there exist finitely many $b_1, \dots, b_r \in B_d$ whose images under φ_d generate L. If there exists $n \geq 1$ such that all b_i^n lie in the image of $\alpha_{nd}\colon A_{nd} \otimes_{A_0} B_0 \to B_{nd}$, then we can pull back $L^{\otimes n}$ to a quotient line of $A^{(nd)}$. Specifically, $(L^{\otimes n}, \varphi^{(n)} \circ \alpha^{(nd)})$ forms an R-point of $\mathrm{Proj}_{nd}(A)$.

Therefore, if α is *eventually surjective*, this process is well-defined for any R-point, inducing a morphism:

$$\mathrm{Proj}(\alpha)\colon \mathrm{Proj}(B) \to \mathrm{Proj}(A).$$

***Remark* 2.3.4** (Veronese Isomorphism). For $d \geq 1$, since $d\mathbb{N}^{\mathrm{div}} \subset \mathbb{N}^{\mathrm{div}}$ is cofinal, we have an isomorphism:

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

***Remark* 2.3.5** (Degree d Maps). We can extend the domain of Proj in the following way.

For $\mathbb{N}$-graded rings A and B and $d \in \mathbb{N}$, a *degree d map* from A to B is a graded ring map $A \to B^{(d)}$.

If it is *eventually surjective* and $d \geq 1$, it induces:

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}(B^{(d)}) \to \mathrm{Proj}(A).$$

This also holds for $d = 0$:

$$\mathrm{Proj}(B) \longrightarrow \mathrm{Spec}(B_0) = \mathrm{Proj}(B^{(0)}) \longrightarrow \mathrm{Proj}(A).$$

We can compose a degree d map with a degree e map to get a *degree $d \cdot e$ map*. Therefore, we can define an extended category (an extension of $\mathsf{CAlg}^{\mathbb{N},\mathrm{es}}$):

- Objects are the same: $\mathbb{N}$-graded rings.
- Morphisms from A to B: $\coprod_{d\in\mathbb{N}} \mathrm{Hom}_{\mathsf{CAlg}^{\mathbb{N},\mathrm{es}}}(A, B^{(d)})$.

The functor Proj extends to this category.

One of our main goals in this section is to prove the following theorem:

Theorem 2.3.6 (Proj Structure Theorem). *Let A be an $\mathbb{N}$-graded ring and $d \geq 1$.*

(i) The canonical map $\mathrm{Proj}_d(A) \longrightarrow \mathrm{Proj}(A)$ is an open immersion.

(ii) If A is generated as an A_0-algebra by homogeneous elements whose degrees are divisible by d, *then the canonical map $\mathrm{Proj}_d(A) \longrightarrow \mathrm{Proj}(A)$ is an isomorphism.*

In particular, if A is generated by $A_{\leq 1}$ (i.e., A_0 and A_1), then $\mathrm{Proj}_1(A) \simeq \mathrm{Proj}(A)$.

***Remark* 2.3.7.** At this point, we can roughly imagine $\mathrm{Proj}(A)$ as the union of some open things.

This structure theorem immediately gives the following corollary, establishing the relationship between weighted projective spaces and standard projective spaces:

Corollary 2.3.8 (Projective Schemes are Projective). *If A is finitely generated as an A_0-algebra (i.e., A is a finitely generated $\mathbb{N}$-graded A_0-algebra), then $\mathrm{Proj}(A)$ is a* projective A_0-scheme *(i.e., there exists some N such that $\mathrm{Proj}(A)$ is isomorphic to a closed subscheme of $\mathbb{P}^N_{A_0}$).*

Proof. Since A is finitely generated as an A_0-algebra and A is an $\mathbb{N}$-graded ring, we can choose finitely many homogeneous generators $x_1, \ldots, x_n$. Let $\gamma_i = \deg(x_i) \in \mathbb{N}_{\geq 1}$.

Choose d to be a *common multiple* of all degrees (e.g., $d = \mathrm{lcm}(\gamma_1, \ldots, \gamma_n)$ or $d = \prod \gamma_i$).

Now consider the Veronese subring $A^{(d)} = \bigoplus_{k\geq 0} A_{kd}$. For any generator x_i, since $\gamma_i \mid d$, we have $x_i^{d/\gamma_i} \in A_d$. Although $A^{(d)}$ may not be generated by A_d, by a classical result, for sufficiently large multiples of d (or in this simple setting, as long as d contains enough powers of all factors), $A^{(d)}$ will be generated by $A_d = (A^{(d)})_1$.

More precisely, the inclusion map $A^{(d)} \hookrightarrow A$ is eventually surjective. Therefore, it induces an isomorphism $\mathrm{Proj}(A) \xrightarrow{\sim} \mathrm{Proj}(A^{(d)})$.

Since $A^{(d)}$ (after appropriate adjustment) is generated by its degree 1 part, by Theorem 2.3.6(2), we have:

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}) \simeq \mathrm{Proj}_1(A^{(d)}).$$

Finally, $\mathrm{Proj}_1(A^{(d)})$ by definition is a closed subscheme of $\mathbb{P}(A_d^\vee)$ (defined by the relations in $A^{(d)}$). Therefore $\mathrm{Proj}(A)$ is a projective A_0-scheme. ∎

Example 2.3.9 (Veronese Embedding). Let k be a ring, M a k-module, and $d \in \mathbb{N}_{\geq 1}$ a natural number. The *d-fold Veronese embedding* is a natural transformation between k-algebraic functors:

$$\rho_d\colon \mathbb{P}(M) \longrightarrow \mathbb{P}(\mathrm{Sym}^d_k(M)),$$

which sends a quotient line $M \otimes_k R \twoheadrightarrow L$ to its d-th symmetric power:

$$\mathrm{Sym}^d_k(M) \otimes_k R \twoheadrightarrow \mathrm{Sym}^d_R(L) \cong L^{\otimes d}.$$

(Note: since L is an R-line, its symmetric power is isomorphic to its tensor power).

Coordinate description: If $M = k^{(I)}$ is a free module, then the d-fold Veronese embedding is the map:

$$\rho_d\colon \mathbb{P}(k^{(I)}) \longrightarrow \mathbb{P}(\mathrm{Sym}^d(k^{(I)})).$$

In particular, when $M = k^{n+1}$ (i.e., $I = \{0, \dots, n\}$), $\mathrm{Sym}^d(M)$ has rank $\binom{n+d}{n}$. The embedding map is:

$$\rho_d : \mathbb{P}^n_k \to \mathbb{P}^N_k, \quad \text{where } N = \binom{n+d}{n} - 1.$$

Its action on coordinates is given by:

$$[a_0 : \cdots : a_n] \mapsto [\text{all degree } d \text{ monomials } M_J(a_0, \dots, a_n)].$$

Geometric property: We claim that for $d \geq 1$, ρ_d is a *closed immersion*.

We can prove this using the functoriality of Proj. Let $S = \mathrm{Sym}_k(M)$. The map ρ_d decomposes as the composition of the following two maps:

$$\mathrm{Proj}_1(S) \xrightarrow{\sim} \mathrm{Proj}_d(S) \xrightarrow{i} \mathrm{Proj}_1(\mathrm{Sym}_k(S_d)).$$

Here $\mathrm{Proj}_1(\mathrm{Sym}_k(S_d))$ is precisely the target space $\mathbb{P}(\mathrm{Sym}^d(M))$.

(i) *First map:* $(L, \varphi) \mapsto (L^{\otimes d}, \varphi^{(d)})$ is an isomorphism. This is guaranteed by *Theorem* 2.3.6 (Proj Structure Theorem), since S is generated by $S_1 = M$, so $S^{(d)}$ is generated by S_d.

(ii) *Second map:* i is a closed immersion induced by a graded ring surjection. Consider the canonical k-algebra homomorphism:

$$\Phi\colon\ \mathrm{Sym}_k(S_d) \twoheadrightarrow S^{(d)} = \bigoplus_{n \geq 0} S_{nd}.$$

This map sends the "formal generators" in $\mathrm{Sym}_k(S_d)$ to actual products of degree d polynomials in $S^{(d)}$. Since S is generated by S_1, $S^{(d)}$ must be generated by S_d. This means the ring homomorphism Φ is *surjective*.

By the contravariant functoriality of Proj, the surjective ring homomorphism Φ induces a closed immersion:

$$\mathrm{Proj}(\Phi)\colon\ \mathrm{Proj}(S^{(d)}) \hookrightarrow \mathrm{Proj}(\mathrm{Sym}_k(S_d)).$$

That is, $\mathrm{Proj}_d(S) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d(M))$.

In summary, ρ_d is the composition of an isomorphism and a closed immersion, hence is a closed immersion.

Next, we discuss whether the Proj functor satisfies Zariski descent.

Proposition 2.3.10 (Zariski Descent for Proj). *Let A be an $\mathbb{N}$-graded ring, $d \in \mathbb{N}$ a natural number, and X be $\mathrm{Proj}_d(A)$ (or $\mathrm{Proj}(A)$). Then for any ring R, R-line L, and generating set $(s_i)_{i\in I}$ of L, the diagram:*

$$X(R) \longrightarrow \prod_{i\in I} X(R_{s_i}) \rightrightarrows \prod_{i,j\in I} X(R_{s_is_j})$$

is an equalizer.

Proof. Without loss of generality, assume $X = \mathrm{Proj}_1(B)$ (if $X = \mathrm{Proj}_d(A)$, take $B = A^{(d)}$). Let $M = B_1 \otimes_{B_0} R$ be the generating module over R.

Consider a family of local data $x_i = (L_i, \varphi_i) \in X(R_{s_i})$, where L_i is an R_{s_i}-line and $\varphi_i : B \otimes R_{s_i} \twoheadrightarrow \mathrm{Sym}(L_i)$. This particularly induces a degree 1 surjection $\pi_i : M_{s_i} \twoheadrightarrow L_i$.

Assume these data are compatible on intersections, i.e., there exist isomorphisms $\theta_{ij} : (L_i)_{s_j} \xrightarrow{\sim} (L_j)_{s_i}$ such that $(L_i, \varphi_i)|_{ij} \cong (L_j, \varphi_j)|_{ij}$. This means the following diagram commutes:

$$\begin{tikzcd} M_{s_is_j} \arrow[rr, two heads, "\pi_j"] \arrow[dr, two heads, "\pi_i"'] & & (L_j)_{s_i} \\ & (L_i)_{s_j} \arrow[ur, "\theta_{ij}"', "\sim"] & \end{tikzcd}$$

Note: since π_i is surjective, the isomorphism θ_{ij} making the above diagram commute, if it exists, must be *unique*.

1. *Descent of the line L:* We need to prove there exists a global R-line L with its quotient map $\pi\colon M \twoheadrightarrow L$. For this, we only need to verify that the gluing maps θ_{ij} satisfy the cocycle condition: $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$ on the triple intersection $R_{s_i s_j s_k}$.

Consider the following diagram (all modules are localized at $R_{s_i s_j s_k}$):

$$\begin{tikzcd}
& (L_i) \arrow[rr, two heads, "\theta_{ij}"] \arrow[dr] & & (L_j) \arrow[dl, "\theta_{jk}"] \\
M \arrow[ur, two heads, "\pi_i"] \arrow[urrr, two heads] \arrow[rr, two heads, "\pi_k"'] & & (L_k) &
\end{tikzcd}$$

- The upper left triangle commutes (by definition of θ_{ij}).
- The right triangle commutes (by definition of θ_{jk}).
- The outer large triangle commutes (by definition of θ_{ik}).

Since $M \twoheadrightarrow (L_i)$ is surjective, we can verify by tracing the images of elements:

$$\theta_{jk}(\theta_{ij}(\pi_i(m))) = \theta_{jk}(\pi_j(m)) = \pi_k(m) = \theta_{ik}(\pi_i(m)).$$

Since π_i is surjective, $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$.

By generalized Zariski descent for R-modules (Proposition 2.2.26), there exists a unique R-module L with isomorphisms $L_{s_i} \cong L_i$. Moreover, since "being a line bundle (invertible module)" is a local property, and L is locally isomorphic to the line bundles L_i, L is an R-line.

2. *Descent of the morphism φ:* Now we have the global line L and local isomorphisms. The local morphisms φ_i can be viewed as $\psi_i\colon B \otimes R_{s_i} \to \mathrm{Sym}(L)_{s_i}$. The compatibility condition ensures that ψ_i and ψ_j agree on intersections.

By Zariski descent for morphisms, this family ψ_i uniquely glues to a global algebra morphism:

$$\varphi\colon B \otimes R \to \mathrm{Sym}(L).$$

Finally, surjectivity is also a local property. Since φ is locally isomorphic to the surjections φ_i, φ is also surjective (or more precisely, the induced map $B \to \mathrm{Sym}(L)$ is eventually surjective, which in the case $X = \mathrm{Proj}_1$ means surjective).

Thus we have found a unique $(L, \varphi) \in X(R)$ restricting to the given local data. ■

Next, we prove Theorem 2.3.6. First, we need to establish the following lemma.

Lemma 2.3.11 (Properties of $D(f)$). *Let A be an $\mathbb{N}$-graded ring, $d \geq 1$, and $f \in A_d$. Let $D(f) \subset \mathrm{Proj}_d(A)$ be the subfunctor defined by: for a ring R, $D(f)(R) = \{(L, \varphi) \mid \varphi(f) \text{ generates } L\}$.*

(i) There is a canonical isomorphism $D(f) \simeq \mathrm{Spec}(A_{(f)})$.

(ii) For each $n \geq 1$, the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$ induces an isomorphism:

$$D(f) \xrightarrow{\sim} D(f^n).$$

In fact, we have a pullback diagram:

$$\begin{tikzcd}
D(f) \arrow[r, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & D(f^n) \arrow[d, hook] \\
\mathrm{Proj}_d(A) \arrow[r] & \mathrm{Proj}_{nd}(A)
\end{tikzcd}$$

Proof. (i) We need to construct mutually inverse maps $D(f) \to \mathrm{Spec}(A_{(f)})$ and $\mathrm{Spec}(A_{(f)}) \to D(f)$.

(a) Constructing $D(f) \to \mathrm{Spec}(A_{(f)})$:

Let $(L, \varphi) \in D(f)(R)$. By definition, $\varphi(f)$ generates L. This means $\varphi(f)$ is invertible in $\mathrm{Sym}_R(L)$ (geometrically corresponding to introducing the dual $L^\vee$). Localizing at $\varphi(f)$ gives the $\mathbb{Z}$-graded algebra $\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$. This induces the following commutative diagram:

$$\begin{tikzcd}
A^{(d)} \otimes_{A_0} R \arrow[r, "\varphi"] \arrow[d, "f^{-1}"'] & \mathrm{Sym}_R(L) \arrow[d, "\varphi(f)^{-1}"] \\
A_f^{(d)} \otimes_{A_0} R \arrow[r, dashed, "\tilde{\varphi}"] & \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}
\end{tikzcd}$$

Take the *degree 0 part* of the bottom row. The left side is $A_{(f)} \otimes R$; since L is trivialized by $\varphi(f)$, the right side gives $L^{\otimes 0} \cong R$. Therefore, $\tilde{\varphi}$ restricted to degree 0 gives a homomorphism $\psi\colon A_{(f)} \to R$.

(b) Constructing $\mathrm{Spec}(A_{(f)}) \to D(f)$*:*

Let $\psi\colon A_{(f)} \to R$. We take the trivial line $L = R$ with generator T, so $\mathrm{Sym}_R(L) = R[T]$. Note that we have a graded ring isomorphism $A_f^{(d)} \cong A_{(f)}[u, u^{-1}]$ (where u is the degree 1 variable corresponding to f). We can extend and restrict the map using ψ:

$$\begin{tikzcd}
A^{(d)} \arrow[r, dashed, "\varphi"] \arrow[d, hook] & R[T] \arrow[d, hook] \\
A_{(f)}[u, u^{-1}] \arrow[r, "\tilde{\varphi}"] & R[T, T^{-1}]
\end{tikzcd}$$

Here $\tilde{\varphi}$ is uniquely determined by ψ (on coefficients) and $u \mapsto T$ (on variables). Clearly, $\varphi(f) = T$ generates L, so $(R, \varphi) \in D(f)(R)$.

It is easy to see that the two constructions above are mutually inverse.

(ii) Since localization of A at f and at f^n as graded rings are the same ($A_f = A_{f^n}$), their degree 0 parts are completely identical ($A_{(f)} = A_{(f^n)}$). Therefore, we have the following commutative diagram (where the vertical arrows are the isomorphisms established in part 1):

$$\begin{tikzcd}
D(f) \arrow[r] \arrow[d, "\wr"'] & D(f^n) \arrow[d, "\wr"] \\
\mathrm{Spec}(A_{(f)}) \arrow[r, equal] & \mathrm{Spec}(A_{(f^n)})
\end{tikzcd}$$

This proves that $D(f) \xrightarrow{\sim} D(f^n)$ is an isomorphism.

Finally, we show that the diagram:

$$\begin{tikzcd}
D(f) \arrow[r, "\sim"] \arrow[d, hook] & D(f^n) \arrow[d, hook] \\
\mathrm{Proj}_d(A) \arrow[r] & \mathrm{Proj}_{nd}(A)
\end{tikzcd}$$

is a pullback. Since we are considering algebraic functors, we only need to check the fiber product property on sets for any ring R.

The map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$ sends (L, φ) to $(L^{\otimes n}, \varphi^{(n)})$. We need to verify:

$$(L, \varphi) \in D(f)(R) \iff (L^{\otimes n}, \varphi^{(n)}) \in D(f^n)(R).$$

By definition, the left side is equivalent to $\varphi(f)$ generating L. The right side is equivalent to $\varphi^{(n)}(f^n) = \varphi(f)^{\otimes n}$ generating $L^{\otimes n}$. For a line bundle L and section s, we have: s generates L if and only if $s^{\otimes n}$ generates $L^{\otimes n}$ (locally, this is $u \in R^\times \iff u^n \in R^\times$). Thus, the diagram is a pullback.

∎

Now we give the proof of Theorem 2.3.6.

Proof of Theorem 2.3.6. From the conceptual viewpoint, we can think of $\mathrm{Proj}_d(A) = \bigcup_{f\in A_d} D(f)$. This is not quite right, but we have pullback diagrams:

$$\begin{tikzcd} \mathrm{Spec}(R_{\varphi(f)}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & D(f) \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "x"] & \mathrm{Proj}_d(A) \end{tikzcd} \tag{2.1}$$

From the construction of $\mathrm{Proj}_d(A)$, we know that x corresponds to $A^{(d)} \otimes_{A_0} R \xrightarrow{\varphi} \mathrm{Sym}_R(L)$. Therefore, if we can show that $(\varphi(f))_{f\in A_d}$ generates L, then we can use Zariski descent for Proj_d. By Remark 2.3.4, we only need to prove this for all $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$. We now proceed with the proof based on this idea:

Injectivity We first show that $i\colon \mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is a subfunctor. Consider $x_1, x_2\colon \mathrm{Spec}(R) \to \mathrm{Proj}_d(A)$ satisfying $i \circ x_1 = i \circ x_2$. By Lemma 2.3.11, for any $f \in A_d$ and $n \geq 1$, we have the commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(R_{\varphi_1(f)}) \arrow[r, shift left, "y_1"] \arrow[r, shift right, "y_2"'] \arrow[d, hook, "j_f"] \arrow[dr, phantom, "\lrcorner", very near start] & D(f) \arrow[r, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & D(f^n) \arrow[d, hook] \\ \mathrm{Spec}(R) \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & \mathrm{Proj}_d(A) \arrow[r] & \mathrm{Proj}_{nd}(A) \end{tikzcd}$$

We take the pullback along x_1 (taking the pullback along x_2 is the same; they are isomorphic) to get $\mathrm{Spec}(R_{\varphi_1(f)})$. Then y_1 corresponds to $(L_1, \varphi_1\colon A^{(d)} \otimes_{A_0} R_{\varphi_1(f)} \to L)$ satisfying that $\varphi_1(f)$ generates L_1. Therefore, by Zariski descent, we have an embedding:

$$\mathrm{Proj}_d(A)(R) \hookrightarrow \prod_{f\in A_d} \mathrm{Proj}_d(A)(R_{\varphi_1(f)}), \quad x \mapsto (x \circ j_f)_{f\in A_d}.$$

Since $D(f) \simeq D(f^n)$, we naturally have $y_1 = y_2$, i.e., $x_1 \circ j_f = x_2 \circ j_f$, hence $x_1 = x_2$. Thus, we have injectivity.

Openness Next, we show that $i\colon \mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is open. We need to show that for any R-point $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$, we can explicitly write the pullback as:

$$\begin{tikzcd} \{(\varphi(f^n))_{f\in A_d} \text{ generates } L\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(R) \arrow[d, "x"] \\ \mathrm{Proj}_d(A) \arrow[r, hook] & \mathrm{Proj}_{nd}(A) \end{tikzcd}$$

We know that $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}_{nd}(A)$ actually corresponds to $\varphi\colon A^{(nd)} \otimes_{A_0} R \to \mathrm{Sym}_R(L)$. The pullback actually refers to the preimage, which means:

$$x \text{ lies in } \mathrm{Proj}_d(A) \iff L = \langle \varphi(f^n) \mid f \in A_d \rangle.$$

We still use the Zariski descent idea to reduce to $D(f)$:

$\Rightarrow$ If x lies in $\mathrm{Proj}_d(A)$, then x corresponds to $\psi\colon A^{(d)} \otimes_{A_0} R \to \mathrm{Sym}_R(M)$ such that $M^{\otimes n} = L$, $\psi^{(n)} = \varphi$, and $M = \langle \psi(f) \mid f \in A_d \rangle$. This is equivalent to saying $L = M^{\otimes n} = \langle \psi(f^n) \mid f \in A_d \rangle$.

$\Leftarrow$ Assume $x \in \mathrm{Proj}_{nd}(A)(R)$ satisfies the generation condition, i.e., L is generated by $\{\varphi(f^n)\}_{f\in A_d}$. We need to prove there exists a unique $x' \in \mathrm{Proj}_d(A)(R)$ such that $i(x') = x$.

Using Zariski descent, we construct x' by tracing through the following diagram:

$$\begin{tikzcd}
\exists ! x' \arrow[r, dashed, mapsto] & (x'_f)_f & (x'_{fg})_{f,g} \\
\mathrm{Proj}_d(A)(R) \arrow[r] \arrow[d, hook] & \prod_f \underbrace{D(f)(R_{\varphi(f^n)})}_{\text{local lift}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, squiggly] & \prod_{f,g} D(fg)(R_{\varphi(f^n g^n)}) \arrow[d, "\wr"] \\
\mathrm{Proj}_{nd}(A)(R) \arrow[r] & \prod_f D(f^n)(R_{\varphi(f^n)}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{f,g} D(f^n g^n)(R_{\varphi(f^n g^n)}) \\
x \arrow[r, mapsto] & (x_f)_f & (x_{fg})_{f,g}
\end{tikzcd}$$

Construction logic:

(i) *Restriction:* Restrict x to $(D(f))_{f \in A_d}$ to get (x_f). Since $\varphi(f^n)$ generates L on $D(f)$, x_f actually lies in $D(f^n) \subset \mathrm{Proj}_{nd}(A)$.

(ii) *Local lifting:* Using the local isomorphism $D(f^n) \simeq D(f)$, each x_f uniquely corresponds to $x'_f \in D(f)(R_{\varphi(f^n)})$.

(iii) *Gluing:* Clearly (x_f) satisfies the descent condition. By the canonicity of the vertical isomorphisms, the lifted (x'_f) also satisfies the descent condition.

(iv) *Global existence:* $\mathrm{Proj}_d(A)$ is a sheaf, so (x'_f) uniquely glues to a global object x'.

This proves that $\mathrm{Proj}_d(A)$ is indeed isomorphic to the open subfunctor of $\mathrm{Proj}_{nd}(A)$ satisfying the generation condition.

Now consider $\Psi \colon R^{(A_d)} \to L$ sending e_f to $\varphi(f^n) \in L$. It is easy to see that $\varphi(f^n)_{f \in A_d}$ generates L is equivalent to saying Ψ is surjective. Therefore, $\{(\varphi(f^n))_{f \in A_d}$ generates $L\}$ is actually $\mathrm{Epi}(\Psi)$, hence we have openness.

Surjectivity Finally, we show surjectivity. Now assume A is generated as an A_0-algebra by homogeneous elements whose degrees are divisible by d. For any $f \in A_{nd}$, it is an A_0-linear combination of $f_1 \cdots f_k$ where $\deg(f_i)$ is divisible by d. Therefore, we may assume $f = f_1 \cdots f_k$. Consider an R-point $x \colon \mathrm{Spec}(R) \to \mathrm{Proj}_{nd}(A)$ in $\mathrm{Proj}_{nd}(A)$. We want to show there exists $x' \colon \mathrm{Spec}(R) \to \mathrm{Proj}_d(A)$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathrm{Proj}_d(A) \arrow[d, hook] \\
\mathrm{Spec}(R) \arrow[ur, dashed, "x'"] \arrow[r, "x"] & \mathrm{Proj}_{nd}(A)
\end{tikzcd}$$

Here x corresponds to (L, φ) where $L = \langle \varphi(f) \mid f \in A_{nd}$ is a product of generators$\rangle$. Let $\deg(f) \cdot m = d$. We have:

$$\begin{tikzcd}
 & D(f_1^m) \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{Proj}_d(A) \arrow[d, hook] \\
D(f) \quad \subset & D(f_1^{mn}) \arrow[r, hook] & \mathrm{Proj}_{nd}(A)
\end{tikzcd}$$

Here $D(f) \subset D(f_1^{mn})$ because we have $\varphi(f) = \varphi(f_1) \otimes \cdots \otimes \varphi(f_k)$, and for $s_i \in L$ with $\langle s_i \rangle = L$, this is equivalent to $\langle s_1 \otimes \cdots \otimes s_r \rangle = L_1 \otimes \cdots \otimes L_r$. Thus, $D(f) \subset D(f_1^{mn})$, so $D(f) \subset D(f_1^m) \to \mathrm{Proj}_d(A)$ gives the lift. Finally, use Zariski descent to get the result.

■

2.3.1. Comparison Between Spec and Proj

Now we can compare Spec and Proj more systematically.

***Remark* 2.3.12** (Relationship Between Spec and Proj)**.** For an $\mathbb{N}$-graded ring A, we have the following canonical diagram over $\mathrm{Spec}(A_0)$:

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[d] & D(A_+) \arrow[l, hook'] \arrow[r] \arrow[d] & \mathrm{Proj}(A) \arrow[d] \\
\mathrm{Spec}(A_0) \arrow[r, equal] & \mathrm{Spec}(A_0) \arrow[r, equal] & \mathrm{Spec}(A_0)
\end{tikzcd}$$

Here:

(i) $D(A_+) = \mathrm{colim}_{d\geq 1} D(A_d)$ is the *non-vanishing locus of the irrelevant ideal* $A_+ = \bigoplus_{d\geq 1} A_d$ in $\mathrm{Spec}(A)$.

(ii) The map $D(A_+) \to \mathrm{Proj}(A)$ is constructed as follows: For $f \in A_d$ with $d \geq 1$, we have $D(f) \subset \mathrm{Spec}(A)$ corresponding to the localization $\mathrm{Spec}(A_f)$. On the other hand, from Lemma 2.3.11, $D(f) \subset \mathrm{Proj}_d(A)$ corresponds to $\mathrm{Spec}(A_{(f)})$. The relationship between these two is:

$$A_f \cong A_{(f)}[u, u^{-1}],$$

where u is the degree 1 variable corresponding to f. This gives a canonical map $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A_{(f)})$, hence $D(f) \to D(f) \subset \mathrm{Proj}_d(A)$.

(iii) The map $D(f) \to \mathrm{Proj}_d(A)$ sends an A-algebra homomorphism $\varphi\colon A \to R$ to the graded map:

$$\varphi^{(d)}\colon A^{(d)} \otimes_{A_0} R \to \bigoplus_{n\in\mathbb{N}} R \cdot \varphi(f)^n \cong \mathrm{Sym}_R(R),$$

where the right side is identified with the symmetric algebra of the trivial line R with generator $\varphi(f)$. This is precisely an element of $\mathrm{Proj}_d(A)(R)$.

(iv) $\mathbb{G}_m$*-equivariance:* Note that $D(A_+) \subset \mathrm{Spec}(A)$ carries a natural $\mathbb{G}_m$-action (coming from the grading). The map $D(A_+) \to \mathrm{Proj}(A)$ is $\mathbb{G}_m$-equivariant. When $d = 1$ (i.e., A is generated by A_1), the map $D(A_1) \to \mathrm{Proj}_1(A)$ has image precisely the *trivial quotient lines*, and induces an injection:

$$D(A_1)/\mathbb{G}_m \hookrightarrow \mathrm{Proj}_1(A).$$

***Remark* 2.3.13** (Extension to Spec)**.** We can extend the domain of the Proj functor to include Spec as follows: For any ring R (viewed as a trivially graded ring concentrated in degree 0), we can define $\mathrm{Proj}(R[x])$ where x has degree 1. Then:

$$\mathrm{Proj}(R[x]) \simeq \mathrm{Proj}_1(R[x]) = \mathbb{P}^0_R = \mathrm{Spec}(R).$$

This shows that Proj can be viewed as a generalization of Spec.

***Remark* 2.3.14** (Comparison with Previous Definition)**.** Earlier, we defined projective schemes using homogeneous polynomial systems. How does this relate to the current definition?

For an $\mathbb{N}$-graded k-algebra A generated by A_1, we have a canonical surjection:

$$\mathrm{Sym}_{A_0}(A_1) \twoheadrightarrow A.$$

According to the previous definition, hSol_Σ for a homogeneous system Σ is defined as the set of quotient lines $A_1 \otimes_k R \twoheadrightarrow L$ through which the composition $\mathrm{Sym}_k(A_1) \to A \to \mathrm{Sym}_R(L)$ factors. By the universal property of the symmetric algebra, this is precisely the definition of $\mathrm{Proj}_1(A)(R)$.

Therefore, the two definitions are completely consistent.

2.3.2. Vanishing Loci and Non-vanishing Loci in Proj

Now we define vanishing loci and non-vanishing loci for Proj, analogous to the affine case.

Definition 2.3.15 (Vanishing Loci and Non-vanishing Loci). Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a set of homogeneous elements. We define:

(i) The *vanishing locus* $V(F) \subset \mathrm{Proj}(A)$ is defined as the colimit:

$$V(F) = \mathrm{colim}_{d \geq 1} V_d(F),$$

where for each $d \geq 1$:

$$V_d(F)(R) = \left\{ \varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \,\middle|\, \text{for all } f \in F^{(d)}, \varphi(f) = 0 \right\}.$$

(ii) The *non-vanishing locus* $D(F) \subset \mathrm{Proj}(A)$ is defined as the colimit:

$$D(F) = \mathrm{colim}_{d \geq 1} D_d(F),$$

where for each $d \geq 1$:

$$D_d(F)(R) = \left\{ \varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \,\middle|\, \exists n \geq 1 : \varphi((F)_{nd}) \text{ generates } L^{\otimes n} \right\}.$$

Here $(F)_{nd}$ denotes the degree nd part of the ideal generated by F in A.

***Remark* 2.3.16** (Properties of $V(F)$ and $D(F)$). These definitions have properties similar to the affine case:

- $\bigcap_i V(F_i) = V(\bigcup_i F_i)$.
- $\bigcup_i D(F_i) \subset D(\bigcup_i F_i)$ (not necessarily equality).
- $V(F)$ is determined by the ideal (F) generated by F.
- $D(F)$ is determined by the radical $\sqrt{(F)}$ of the ideal generated by F.
- When A is generated by $A_{\leq 1}$, we have simpler descriptions. For instance, for a single homogeneous element $f \in A_d$:

$$V(f)(R) = \{\varphi \mid \varphi(f) = 0 \in L^{\otimes d}\},$$
$$D(f)(R) = \{\varphi \mid \varphi(f) \text{ generates } L^{\otimes d}\}.$$

***Remark* 2.3.17** (Further Properties). (i) For a homogeneous ideal $I \subset A$, we have $\mathrm{Proj}(A/I) \simeq V(I)$ canonically. This follows from the surjection $A \twoheadrightarrow A/I$ inducing a closed immersion $\mathrm{Proj}(A/I) \hookrightarrow \mathrm{Proj}(A)$.

(ii) For $f \in A_d$ with $d \geq 1$, we have $D(f^n) \simeq D(f) \simeq \mathrm{Spec}(A_{(f)})$ for all $n \geq 1$. This was proved in Lemma 2.3.11.

(iii) For $f \in A_0$ (i.e., a degree 0 element), we have $\mathrm{Proj}(A_{(f)}) \simeq D(f)$. Here $A_{(f)}$ means the localization of A at f viewed as an element of the degree 0 ring A_0.

Proposition 2.3.18 (Closedness and Openness). *Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a set of homogeneous elements.*

(i) $V(F)$ is a closed subfunctor *of* $\mathrm{Proj}(A)$.

(ii) $D(F)$ is an open subfunctor *of* $\mathrm{Proj}(A)$.

Proof. We may assume A is generated by $A_{\leq 1}$ (the general case follows from passing to Veronese subrings and using cofinality).

(i) *Closedness of* $V(F)$*:* We can write $V(F) = \bigcap_{f\in F} V(f)$, so it suffices to show $V(f)$ is closed for a single homogeneous element $f \in A_d$.

For any R-point $x\colon \operatorname{Spec}(R) \to \operatorname{Proj}(A)$ corresponding to a quotient line $\varphi\colon A_1 \otimes_k R \twoheadrightarrow L$, we have:

$$x \in V(f) \iff \varphi(f) = 0 \in L^{\otimes d}.$$

Consider the pullback:

$$\begin{tikzcd} x^{-1}(V(f)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V(f) \arrow[d] \\ \operatorname{Spec}(R) \arrow[r, "x"] & \operatorname{Proj}(A) \end{tikzcd}$$

We need to show $x^{-1}(V(f))$ is closed in $\operatorname{Spec}(R)$. By definition:

$$x^{-1}(V(f))(S) = \{\psi\colon R \to S \mid \psi^*(\varphi(f)) = 0 \in L^{\otimes d} \otimes_R S\}.$$

Since L is an R-line, hence projective, $L^{\otimes d}$ is also projective, hence reflexive. Therefore:

$$\varphi(f) = 0 \iff \text{for all } \lambda \in (L^{\otimes d})^\vee, \lambda(\varphi(f)) = 0.$$

The condition $\lambda(\varphi(f)) = 0$ defines a closed subset $V(\lambda(\varphi(f))) \subset \operatorname{Spec}(R)$ (since $\lambda(\varphi(f)) \in R$ generates an ideal). Therefore:

$$x^{-1}(V(f)) = \bigcap_{\lambda\in(L^{\otimes d})^\vee} V(\lambda(\varphi(f)))$$

is closed.

(ii) *Openness of* $D(F)$*:* For an R-point x corresponding to $\varphi\colon A \otimes R \twoheadrightarrow \operatorname{Sym}_R(L)$, we have:

$$x \in D(F) \iff \exists n \geq 1 : \{(\varphi(f))_{f\in F_n}\} \text{ generates } L^{\otimes n},$$

where F_n denotes the homogeneous elements in F of degrees dividing n.

This is equivalent to:

$$x^{-1}(D(F)) = \bigcup_{n\geq 1} \operatorname{Epi}(R^{(F_n)} \to L^{\otimes n}).$$

Each $\operatorname{Epi}(R^{(F_n)} \to L^{\otimes n})$ is an open subfunctor by Proposition 2.2.19. Moreover, we can write:

$$x^{-1}(D(F)) = D\left(\bigcup_{n\geq 1} I_n\right),$$

where $I_n \subset R$ is the radical ideal corresponding to the image of (F_n) under φ. Since each $I_n \subset I_{n+1}$ and their union generates an ideal, this is an open subfunctor.

∎

Example 2.3.19 (Non-uniqueness of Vanishing Loci). Unlike the affine case, in the projective case, $V(F) = V(F')$ does *not* imply $(F) = (F')$ in general.

Example: Consider $A = \mathbb{Z}[x, y]$ with the standard grading. We have:

$$V(x) = V(x^2, xy) \subset \mathbb{P}^1.$$

Verification. Clearly $(x^2, xy) \subset (x)$, so $V(x) \subset V(x^2, xy)$.

For the reverse inclusion, suppose $\varphi\colon \mathbb{Z}[x, y] \to \operatorname{Sym}_R(L)$ satisfies $\varphi(x^2) = 0$ and $\varphi(xy) = 0$ in $L^{\otimes 2}$. Write $\varphi(x) = a$ and $\varphi(y) = b$ where $a, b \in L$ generate L (i.e., $L = \langle a, b\rangle$).

From $\varphi(x^2) = a^{\otimes 2} = 0$ and $\varphi(xy) = a \otimes b = 0$ in $L^{\otimes 2}$, we deduce:

- Since $a^{\otimes 2} = 0$, we have $a \cdot a = 0$ (viewing multiplication in the symmetric algebra).

- Since $a \otimes b = 0$ and $L = \langle a, b \rangle$, we must have $a \cdot L = 0$.

In particular, a acts as zero on all of L. Since L is a line (hence has no nilpotent elements), this implies $a = 0$. Therefore $\varphi(x) = a = 0$, so $\varphi \in V(x)$. ■

This shows that the ideals (x) and (x^2, xy) define the same vanishing locus in $\mathbb{P}^1$, even though they are different ideals.

This phenomenon leads us to introduce the concept of *saturation*.

2.3.3. Saturation of Homogeneous Ideals

Definition 2.3.20 (Saturation). Let A be an $\mathbb{N}$-graded ring and $H \subset A$ a homogeneous ideal. The *saturation* of H is:

$$H^{\mathrm{sat}} = \{x \in A \mid \text{for all } f \in A_+, \exists n \geq 1 : f^n x \in H\}.$$

We say H is *saturated* if $H = H^{\mathrm{sat}}$.

***Remark* 2.3.21** (Properties of Saturation).

(i) The saturation H^{sat} is determined by the "tail" of H: for any infinite subset $D \subset \mathbb{N}$, we have:

$$H^{\mathrm{sat}} = \left(\bigoplus_{d \in D} H_d \right)^{\mathrm{sat}}.$$

(ii) If A is generated by $A_{\leq 1}$, we can replace A_+ with A_1 in the definition of saturation.

(iii) For a radical ideal H (i.e., $\sqrt{H} = H$), the saturation H^{sat} is also radical.

(iv) A homogeneous prime ideal $\mathfrak{p} \subset A$ is saturated if and only if $A_+ \not\subset \mathfrak{p}$.

Example 2.3.22 (Examples of Saturation).

(i) For $A = \mathbb{Z}[x, y]$, we have $(x) = (x^2, xy)^{\mathrm{sat}}$. This can be verified directly: for any $g \in A$ with $y^n g \in (x^2, xy)$ and $x^m g \in (x^2, xy)$, we can show $g \in (x)$.

(ii) For polynomial rings $k[x_i]$, the zero ideal (0) is saturated (since there are no nilpotent elements and no elements killed by all positive degree elements).

Our goal is to show that $V(H) = V(H')$ if and only if $H^{\mathrm{sat}} = H'^{\mathrm{sat}}$.
To prove this, we need to develop the theory of *I-nilpotent modules* and *I-local modules*.

Definition 2.3.23 (I-nilpotent and I-local Modules). Let R be a ring, $I \subset R$ a subset, and M an R-module.

(i) An element $x \in M$ is *I-nilpotent* if for all $f \in I$, there exists $n \geq 1$ such that $f^n x = 0$.

(ii) We denote by $\Gamma_I(M) \subset M$ the *I-nilpotent submodule*, i.e., the submodule consisting of all I-nilpotent elements.

(iii) M is *I-nilpotent* if $\Gamma_I(M) = M$.

(iv) M is *I-local* if for any homomorphism $h \colon P \to Q$ with I-nilpotent kernel and cokernel, the induced map:

$$h^* \colon \mathrm{Hom}_R(Q, M) \xrightarrow{\sim} \mathrm{Hom}_R(P, M)$$

is an isomorphism.

Proposition 2.3.24 (Categories of I-nilpotent and I-local Modules). *Let R be a ring and $I \subset R$ a subset. Then:*

(i) $\mathrm{Mod}_R^{I\text{-nil}}$ *(the full subcategory of I-nilpotent modules) and* $\mathrm{Mod}_R^{I\text{-loc}}$ *(the full subcategory of I-local modules) are Grothendieck abelian categories.*

(ii) There exist exact functors:

$$\mathrm{Mod}_R^{I\text{-nil}} \rightleftarrows \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_R^{I\text{-loc}},$$

where the left adjoints are the inclusion functors, and the right adjoints are Γ_I (for I-nilpotent) and L_I (for I-local).

(iii) M is I-nilpotent if and only if $\mathrm{L}_I(M) = 0$.

(iv) $\Gamma_I(M) = \ker(M \to \mathrm{L}_I(M))$.

(v) For a homomorphism $h\colon M \to N$, $\mathrm{L}_I(h)$ is an isomorphism if and only if $h_f\colon M_f \to N_f$ is an isomorphism for all $f \in I$.

***Remark* 2.3.25** (Explicit Description for Finite I). When $I = \{f_1, \dots, f_n\}$ is finite, $\mathrm{Mod}_R^{I\text{-loc}}$ can be identified with the category of quasi-coherent modules on the open subset $D(I) \subset \mathrm{Spec}(R)$.

In this case, $\mathrm{L}_I(M)$ can be computed as the equalizer:

$$\mathrm{L}_I(M) \to \prod_{i=1}^{n} M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}.$$

In particular, $\mathcal{O}(D(I)) = \mathrm{L}_I(R)$.

Example 2.3.26 (Special Cases). (i) When $I = \{f\}$ consists of a single element, "I-nilpotent" means "f-torsion", and $\mathrm{L}_I(M) = M_f$ (the localization at f).

(ii) For an $\mathbb{N}$-graded ring A and homogeneous ideal $H \subset A$, we have:

$$H^{\mathrm{sat}} = \ker(A \to A/H \to \mathrm{L}_{A_+}(A/H)).$$

(iii) In particular, $(0)^{\mathrm{sat}} = \Gamma_{A_+}(A)$.

Now we can state the key proposition relating Proj to I-local modules.

Proposition 2.3.27 (Characterization of Proj via I-local Modules). *Let A, B be $\mathbb{N}$-graded rings and $\alpha\colon A \to B$ an eventually surjective graded map. Then the following are equivalent:*

(i) $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ is an isomorphism.

(ii) For all $d \geq 1$, the map:

$$\mathrm{L}_{A_d}(A^{(d)}) \to \mathrm{L}_{A_d}(B^{(d)})$$

is an isomorphism.

If α is surjective, these are also equivalent to:

3 For all $d \geq 1$, $\ker(A \to B^{(d)})$ is A_d-nilpotent.

If A is generated by $A_{\leq 1}$, then it suffices to check condition (2) or (3) for $d = 1$ only.

Proof Sketch (for A generated by $A_{\leq 1}$).

$1 \Rightarrow 2$: If $\mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(A)$, then for each $f \in A_1$, the pullback diagram gives $D(\alpha(f)) \simeq D(f)$. By Lemma 2.3.11, this means $B_{\alpha(f)} \simeq A_f$ as graded rings, hence $B_{(\alpha(f))} \simeq A_{(f)}$. Taking the equalizer over all $f \in A_1$ and using Zariski descent gives $\mathrm{L}_{A_1}(A) \xrightarrow{\sim} \mathrm{L}_{A_1}(B)$.

$2 \Rightarrow 1$: Assume $\mathrm{L}_{A_1}(A) \xrightarrow{\sim} \mathrm{L}_{A_1}(B)$. This implies $A_f \xrightarrow{\sim} B_f$ as graded rings for all $f \in A_1$, hence $A_{(f)} \xrightarrow{\sim} B_{(f)}$. For any $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ corresponding to a quotient line $A_1 \otimes R \twoheadrightarrow L$, we can use Zariski descent over the open cover $\{D(f)\}_{f \in A_1}$ to lift x uniquely to $\mathrm{Proj}(B)$. This shows $\mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(A)$.

■

Corollary 2.3.28 (Functorial Projective Nullstellensatz). *For a ring k, the map $F \mapsto V(F)$ induces an order-preserving bijection:*

$$\{\textit{saturated homogeneous ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{P}_k^I\}.$$

Proof. Let $H, H' \subset k[x_i \mid i \in I]$ be homogeneous ideals. We need to show:

$$V(H') \hookrightarrow V(H) \text{ is an isomorphism} \iff H^{\mathrm{sat}} = H'^{\mathrm{sat}}.$$

Note that $V(H') \hookrightarrow V(H)$ being an isomorphism is equivalent to $\mathrm{Proj}(A/H') \hookrightarrow \mathrm{Proj}(A/H)$ being an isomorphism, where $A = k[x_i \mid i \in I]$.

By Proposition 2.3.27, this occurs if and only if the quotient map:

$$A/H \twoheadrightarrow A/H'$$

has kernel that is A_+-nilpotent in A/H, i.e., $H'/H \subset (0)^{\mathrm{sat}}$ in A/H.
This condition is equivalent to $H' \subset H^{\mathrm{sat}}$.
By symmetry (swapping H and H'), we also have $H \subset H'^{\mathrm{sat}}$.
Therefore, $V(H) = V(H')$ if and only if $H^{\mathrm{sat}} = H'^{\mathrm{sat}}$. ∎

Now we can give a complete classification of closed and open subfunctors of $\mathrm{Proj}(A)$.

Proposition 2.3.29 (Classification of Open and Closed Subfunctors). *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$.*

(i) The map $F \mapsto V(F)$ induces an order-preserving injection:

$$\{\textit{saturated homogeneous ideals in } A\} \hookrightarrow \{\textit{closed subfunctors of } \mathrm{Proj}(A)\}.$$

This is a bijection if A is finitely generated over A_0.

(ii) The map $F \mapsto D(F)$ induces an order-preserving bijection:

$$\{\textit{saturated homogeneous radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Proj}(A)\}.$$

Proof. (i) *Injectivity:* This follows directly from Corollary 2.3.28.

Surjectivity (when A finitely generated): Let $Z \subset \mathrm{Proj}(A)$ be a closed subfunctor. We need to construct a saturated homogeneous ideal $H \subset A$ such that $Z = V(H)$.

Let $F \subset A_1$ be a finite generating set (which exists since A is finitely generated over A_0). For each $f \in F$, consider the pullback:

$$\begin{tikzcd} Z_f \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ D(f) \arrow[r, hook] & \mathrm{Proj}(A) \end{tikzcd}$$

Since $D(f) \simeq \mathrm{Spec}(A_{(f)})$ by Lemma 2.3.11, and Z_f is a closed subfunctor of $\mathrm{Spec}(A_{(f)})$, there exists an ideal $I(f) \subset A_{(f)}$ such that $Z_f = V(I(f))$.

For each $f \in F$, define $H(f) \subset A$ as the preimage of $I(f)[x^{\pm 1}] \subset A_f$ under the canonical map $A \to A_f$ (where we use $A_f \cong A_{(f)}[x^{\pm 1}]$).

Define:

$$H = \bigcap_{f \in F} H(f).$$

We claim that $Z = V(H)$.

Step 1: Show $Z \subset V(H)$. For any R-point of Z, its restriction to $D(f)$ lies in $Z_f = V(I(f))$. This means that locally on $D(f)$, the point kills all elements of $H(f)$. Since this holds for all $f \in F$ and $\{D(f)\}$ covers $\mathrm{Proj}(A)$ (by Zariski descent), the point kills H.

Step 2: Show $V(H) \subset Z$. This uses Zariski descent: if a point of $\mathrm{Proj}(A)$ kills H, then its restriction to each $D(f)$ kills $H(f)$, hence lies in $V(I(f)) = Z_f$. By descent, the point lies in Z.

Finally, we need to verify that H is saturated. This follows from the construction: for $h \in H^{\mathrm{sat}}$, we have $f^n h \in H$ for all $f \in A_+$ and sufficiently large n. Since H is defined by its localizations at F, and being killed by high powers of A_+ exactly characterizes saturation, we get $H = H^{\mathrm{sat}}$.

(ii) *Injectivity:* We need to show that if $D(F) = D(F')$, then $\sqrt{(F)}^{\mathrm{sat}} = \sqrt{(F')}^{\mathrm{sat}}$.

Note that $a \in F$ if and only if $D(a) \subset D(F)$. To see this, observe that for any R-point $\varphi\colon A \otimes R \to \mathrm{Sym}_R(L)$:

$$\varphi \in D(a) \iff \varphi(a) \text{ generates } L^{\otimes d} \text{ (where } d = \deg(a)).$$

If $D(a) \subset D(F)$, then whenever $\varphi(a)$ generates $L^{\otimes d}$, there exists some $f \in F$ and $n \geq 1$ such that $\varphi(f^n)$ generates $L^{\otimes dn}$. Taking the pullback to $D(a) \simeq \mathrm{Spec}(A_{(a)})$, this means $1 \in \sqrt{(F_a)}^{(d)}$ (where F_a denotes the image of F in A_a). This implies $a \in \sqrt{(F)}$.

By symmetry, $\sqrt{(F)}^{\mathrm{sat}} = \sqrt{(F')}^{\mathrm{sat}}$.

Surjectivity: Let $U \subset \mathrm{Proj}(A)$ be an open subfunctor. For each $f \in A_1$, the pullback $U_f \subset D(f) \simeq \mathrm{Spec}(A_{(f)})$ is an open subfunctor, hence of the form $D(J(f))$ for some radical ideal $J(f) \subset A_{(f)}$.

Define $H_f \subset A$ as the preimage of $J(f)[x^{\pm 1}]$ under $A \to A_f$. Set:

$$H = \sum_{f \in F} f \cdot H_f,$$

where $F \subset A_1$ generates A_1.

We claim $D(H) = U$. The verification uses Zariski descent over the cover $\{D(f)\}_{f \in F}$, similar to the proof of part (1).

■

***Remark* 2.3.30** (General $\mathbb{N}$-graded Rings). For a general $\mathbb{N}$-graded ring A (not necessarily generated by $A_{\leq 1}$), the notion of saturation needs to be modified. An ideal $H \subset A$ is saturated if it contains all homogeneous elements x such that for all $d \geq 1$:

$$(x \cdot A_d) \subset \ker(A \to \mathrm{L}_{A_d}(A/H)).$$

With this modified definition, the classification theorems above still hold.

2.3.4. Quasi-projective Schemes

Definition 2.3.31 (Quasi-projective Schemes). Let k be a ring. An algebraic k-functor X is a *quasi-projective k-scheme* if there exist:

- An $\mathbb{N}$-graded k-algebra A that is finitely generated over A_0,
- A finite set of homogeneous elements $F \subset A$,

such that $X \simeq D(F) \subset \mathrm{Proj}(A)$.

We denote by $\mathsf{QProj}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ the full subcategory spanned by all quasi-projective k-schemes.

***Remark* 2.3.32** (Relationship with Quasi-affine Schemes). By analogy with the affine case, we have:

$$\mathsf{Aff}_k \subset \mathsf{QAff}_k \quad \text{and} \quad \mathrm{Proj}_k \subset \mathsf{QProj}_k.$$

Moreover, there exists an inclusion $\mathsf{QAff}_k \subset \mathsf{QProj}_k$ (since affine schemes embed into projective space via "cones"), but the two categories are not the same.

2.4. Projective closure

[To be continued...]

Chapter 3.

Quasi-coherent Modules

In this chapter, we develop the theory of modules over general algebraic functors. The fundamental idea is to extend the notion of "module" from rings to arbitrary algebraic functors.

3.1. Quasi-coherence

3.1.1. Category-valued Functors

Before defining modules over algebraic functors, we need to understand limits of category-valued functors.

Definition 3.1.1 (Category-valued Functors). Let $\mathcal{J}$ be a category. A *functor* $F\colon \mathcal{J} \to \mathsf{Cat}$ consists of the following data:

(i) For each object $I \in \mathcal{J}$, a category $F(I)$.

(ii) For each morphism $f : I \to J$ in $\mathcal{J}$, a functor $F(f)\colon F(I) \to F(J)$.

(iii) For each object $I \in \mathcal{J}$, a natural isomorphism $\eta_I\colon \mathrm{id}_{F(I)} \simeq F(\mathrm{id}_I)$.

(iv) For morphisms $f : I \to J$ and $g : J \to K$ in $\mathcal{J}$, a natural transformation $\mu_{f,g}\colon F(g) \circ F(f) \simeq F(g \circ f)$.

These data must satisfy the following compatibility conditions:

5. For any morphism $f : I \to J$ in $\mathcal{J}$, both of the following triangles commute:

$$\begin{tikzcd} F(f) \arrow[r, "\eta_I"] \arrow[dr, "\mathrm{id}"'] & F(f) \circ F(\mathrm{id}_I) \arrow[d, "\mu_{\mathrm{id}_I,f}"] \\ & F(f), \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(f) \arrow[r, "\eta_J"] \arrow[dr, "\mathrm{id}"'] & F(\mathrm{id}_J) \circ F(f) \arrow[d, "\mu_{f,\mathrm{id}_J}"] \\ & F(f). \end{tikzcd}$$

6. For morphisms $f : I \to J$, $g : J \to K$, and $h\colon K \to L$ in $\mathcal{J}$, the following square commutes:

$$\begin{tikzcd} F(h) \circ F(g) \circ F(f) \arrow[r, "\mu_{f,g}"] \arrow[d, "\mu_{g,h}"] & F(h) \circ F(g \circ f) \arrow[d, "\mu_{g\circ f,h}"] \\ F(h \circ g) \circ F(f) \arrow[r, "\mu_{f,h\circ g}"] & F(h \circ g \circ f). \end{tikzcd}$$

Example 3.1.2 (Slice Categories). Let C be a category with pullbacks. We can consider the functor:

$$C_{/-}\colon C^{\mathrm{op}} \to \mathsf{Cat}, \quad X \mapsto C_{/X},$$

called the *self-indexing* of C. For a morphism $f : X \to Y$, it sends f to the pullback functor:

$$f^*\colon C_{/Y} \to C_{/X}, \quad (A \to Y) \mapsto (X \times_Y A \to X).$$

***Remark* 3.1.3** (Topos). If C is a presentable category and the above functor satisfies the property that for any small category $\mathcal{I}$ and functor $F\colon \mathcal{I} \to C$:

$$C_{/\mathrm{colim}_{i\in\mathcal{I}} F(i)} \simeq \lim_{i\in\mathcal{I}} C_{/F(i)},$$

then C is called a *topos*.

Example 3.1.4 (Module Categories). We can construct the functor $R \mapsto \mathsf{Mod}_R$ in three different ways:

(i) *Base change:* There exists a functor:

$$\mathsf{Mod}^* \colon \mathsf{CAlg} \to \mathsf{Cat}, \quad R \mapsto \mathsf{Mod}_R,$$

which sends a ring homomorphism $f: R \to S$ to the base change functor:

$$f^* \colon \mathsf{Mod}_R \to \mathsf{Mod}_S, \quad M \mapsto M \otimes_R S.$$

(ii) *Restriction of scalars:* There exists a functor:

$$\mathsf{Mod}_* \colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Cat}, \quad R \mapsto \mathsf{Mod}_R,$$

which sends a ring homomorphism $f: R \to S$ to the forgetful functor:

$$f_* \colon \mathsf{Mod}_S \to \mathsf{Mod}_R, \quad M \mapsto M,$$

where we view the S-module M as an R-module via composition with f.

(iii) *Internal Hom:* There exists a functor:

$$\mathsf{Mod}^! \colon \mathsf{CAlg} \to \mathsf{Cat}, \quad R \mapsto \mathsf{Mod}_R,$$

which sends a ring homomorphism $f: R \to S$ to the functor:

$$f^! \colon \mathsf{Mod}_R \to \mathsf{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M).$$

Remark **3.1.5.** In the above example, we can replace Mod_R with CAlg_R. Then constructions (1) and (2) work for CAlg_R, but (3) does not.

Example 3.1.6 (Properties Preserved by Base Change). Any property P of modules that is preserved by base change (such as being finitely generated, finitely presented, flat, projective, free, a vector space, or a line) defines a subfunctor of Mod^*.

Example 3.1.7 (Ideal Functors). For a poset P, we can automatically view it as a category. Therefore, for a category C, we can consider functors landing in Pos (the category of posets):

$$C \to \mathsf{Pos}.$$

For example, taking $C = \mathsf{CAlg}$:

- For a ring R, consider the poset Id_R of all ideals of R ordered by inclusion. For $R \to S$, we have $\mathrm{Id}_R \to \mathrm{Id}_S$ sending $I \mapsto IS$.
- For a ring R, consider the poset Rad_R of all radical ideals of R ordered by inclusion. For $R \to S$, we have $\mathrm{Rad}_R \to \mathrm{Rad}_S$ sending $I \mapsto \sqrt{IS}$.

Notation 3.1.8 (Left and Right Cones). Let $\mathcal{J}$ be a category. We denote by $\mathcal{J}^{\triangleleft}$ the category $[0] \star \mathcal{J}$. In other words, it is $\mathcal{J}$ with an initial object $-\infty$ added, such that for any $I \in \mathcal{J}$, we have $\mathrm{Hom}_{\mathcal{J}^{\triangleleft}}(-\infty, I) = *$. We call this the *left cone* over $\mathcal{J}$.

Similarly, we can define the *right cone* $\mathcal{J}^{\triangleright}$.

Now we can introduce the concept of limits of categories.

Definition 3.1.9 (Limits of Categories). Let $F \colon \mathcal{J} \to \mathsf{Cat}$ be a functor. The *limit* $\lim F$ (or $\lim_{I \in \mathcal{J}} F(I)$) is the category defined as follows:

- *Objects:* An object of $\lim F$ is a pair (x, α) where:
  (i) For each $I \in \mathcal{J}$, we specify an object $x_I \in F(I)$.
  (ii) For each morphism $f: I \to J$ in $\mathcal{J}$, we specify an isomorphism $\alpha_f \colon F(f)(x_I) \xrightarrow{\sim} x_J$.

These data must satisfy the following *cocycle condition*:

(i) For each $I \in \mathcal{J}$, the following triangle commutes:

$$\begin{tikzcd} x_I \arrow[r, "\eta_I(x_I)"] \arrow[dr, "\mathrm{id}"'] & F(\mathrm{id}_I)(x_I) \arrow[d, "\alpha_{\mathrm{id}_I}"] \\ & x_I. \end{tikzcd}$$

(ii) For morphisms $f: I \to J$ and $g: J \to K$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{tikzcd} F(g)(F(f)(x_I)) \arrow[r, "F(g)(\alpha_f)"] \arrow[d, "\mu_{f,g}(x_I)"] & F(g)(x_J) \arrow[d, "\alpha_g"] \\ F(g \circ f)(x_I) \arrow[r, "\alpha_{g\circ f}"] & x_K. \end{tikzcd}$$

- *Morphisms:* A morphism φ from (x, α) to (y, β) in $\lim F$ consists of:
  (i) For each $I \in \mathcal{J}$, a morphism $\varphi_I : x_I \to y_I$ in $F(I)$.
  (ii) For each morphism $f : I \to J$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{tikzcd} F(f)(x_I) \arrow[r, "F(f)(\varphi_I)"] \arrow[d, "\alpha_f"] & F(f)(y_I) \arrow[d, "\beta_f"] \\ x_J \arrow[r, "\varphi_J"] & y_J. \end{tikzcd}$$

- *Composition:* Identity morphisms and composition of morphisms are defined pointwise.

Definition 3.1.10 (Limit Diagrams). Let $F: \mathcal{J} \to \mathsf{Cat}$ be a functor. Consider an extension $\bar{F}: \mathcal{J}^{\triangleleft} \to \mathsf{Cat}$. We say $\bar{F}$ is a *limit diagram* if the canonical map $\bar{F}(-\infty) \to \lim F$ is a categorical equivalence.

***Remark* 3.1.11** (Colimits). Similarly, we can define colimits of category-valued functors and colimit diagrams. Moreover, we can verify that given a functor $F: \mathcal{J} \to \mathsf{Cat}$ and an extension $\bar{F}: \mathcal{J}^{\triangleright} \to \mathsf{Cat}$, we have: $\bar{F}$ is a colimit diagram if and only if for any category $\mathcal{E}$:

$$\mathsf{Fun}(\bar{F}(-), \mathcal{E}) : (\mathcal{J}^{\mathrm{op}})^{\triangleleft} \to \mathsf{Cat}$$

is a limit diagram. (Note: in synthetic category theory, we define pushouts in a similar way.)

3.1.2. Quasi-coherent Modules over Algebraic Functors

Now we can define modules over general algebraic functors.

Idea (Modules via Kan Extension). Our core idea is to extend the notion of "module" to general algebraic functors $X : \mathsf{CAlg} \to \mathsf{Set}$. Consider the following commutative diagram:

$$\begin{tikzcd} \mathsf{CAlg} \arrow[r, "\mathsf{Mod}_{(-)}"] \arrow[d, hook] & \mathsf{Cat} \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} \arrow[ur, dashed] & \end{tikzcd}$$

Our goal is to construct the dashed arrow. A natural approach is to use *Kan extension* theory.
We can view a set-valued algebraic functor X as a functor taking values in discrete categories:

$$X : \mathsf{CAlg} \to \mathsf{Cat}.$$

The functor category $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Cat})$ naturally has a Cat-enriched structure (the cartesian closed structure on Cat gives a tensor action and internal Hom on $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Cat})$ via adjunction). In this framework, we can define the category of quasi-coherent modules Mod_X as the following internal Hom object:

$$\mathsf{Mod}_X := \underline{\mathrm{Hom}}_{\mathsf{Cat}}(X, \mathsf{Mod}).$$

By the density of the Yoneda embedding, we have a canonical equivalence:

$$X \simeq \operatorname*{colim}_{x \in \mathrm{El}(X)} \operatorname{Spec} R.$$

Using the property that Hom functors convert colimits to limits, we obtain:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathsf{Cat}}(X, \mathsf{Mod}) &\simeq \underline{\mathrm{Hom}}_{\mathsf{Cat}}\left(\operatorname*{colim}_{x \in \mathrm{El}(X)} \operatorname{Spec} R, \mathsf{Mod}\right) \\ &\simeq \lim_{x \in \mathrm{El}(X)^{\mathrm{op}}} \underline{\mathrm{Hom}}(\operatorname{Spec} R, \mathsf{Mod}) \\ &= \lim_{x \in \mathrm{El}(X)^{\mathrm{op}}} \mathsf{Mod}_R. \end{aligned}$$

Note that the indexing category $x \in \mathrm{El}(X)^{\mathrm{op}}$ is equivalent to the category of morphisms $x \colon \operatorname{Spec} R \to X$ (the opposite of the element category of X). The above limit formula precisely corresponds to the *right Kan extension* of $\mathsf{Mod}_{(-)}$ along the Yoneda embedding.

Definition 3.1.12 (Quasi-coherent Modules). Let X be an algebraic functor. A *quasi-coherent module* over X (or *quasi-coherent O_X-module*) is an element of the limit category:

$$\mathsf{Mod}_X := \lim_{x \colon \operatorname{Spec}(R) \to X} \mathsf{Mod}_R.$$

Explicitly, a quasi-coherent module M over X consists of:

(i) For each ring R and each R-point $x \colon \operatorname{Spec}(R) \to X$, an R-module $M(x) \in \mathsf{Mod}_R$.

(ii) For each ring homomorphism $\varphi \colon R \to S$ and commutative triangle:

$$\begin{tikzcd} \operatorname{Spec}(S) \arrow[rr, "y"] \arrow[dr, "\operatorname{Spec}(\varphi)"'] & & X \\ & \operatorname{Spec}(R) \arrow[ur, "x"'] & \end{tikzcd}$$

an isomorphism $\alpha_\varphi \colon M(x) \otimes_R S \xrightarrow{\sim} M(y)$ in Mod_S.

These data must satisfy:

- *Transitivity:* For composable ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ with compatible points x, y, z, we have:
$$\alpha_{\psi \circ \varphi} = \alpha_\psi \circ (\alpha_\varphi \otimes_S T).$$
- *Identity:* For $\varphi = \mathrm{id}_R$, we have $\alpha_{\mathrm{id}} = \mathrm{id}$.

***Remark* 3.1.13** (Functoriality). The assignment $X \mapsto \mathsf{Mod}_X$ is contravariant functorial. For a morphism $f \colon Y \to X$ of algebraic functors, we obtain a pullback functor:

$$f^* \colon \mathsf{Mod}_X \to \mathsf{Mod}_Y, \quad M \mapsto f^* M,$$

where $(f^*M)(y) = M(f \circ y)$ for any $y \colon \operatorname{Spec}(R) \to Y$.

Similarly, we can define quasi-coherent algebras, ideals, and radical ideals.

Definition 3.1.14 (Quasi-coherent Algebras and Ideals). Let X be an algebraic functor.

(i) A *quasi-coherent O_X-algebra* is an element of:

$$\mathsf{CAlg}_X := \lim_{x \colon \operatorname{Spec}(R) \to X} \mathsf{CAlg}_R.$$

(ii) A *quasi-coherent ideal* is an element of:

$$\mathsf{Id}_X \coloneqq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathsf{Id}_R.$$

(iii) A *quasi-coherent radical ideal* is an element of:

$$\mathsf{Rad}_X \coloneqq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathsf{Rad}_R.$$

3.2. Classification of Open and Closed Subfunctors

In previous sections, we established that for affine and projective schemes, open (closed) subfunctors correspond bijectively to (radical) ideals (with saturation conditions in the projective case). We now show this holds for general algebraic functors.

Proposition 3.2.1 (Classification Theorem). *Let X be an algebraic functor.*

(i) There exists an isomorphism of posets:

$$V\colon \mathsf{Id}_X^{\mathrm{op}} \xrightarrow{\sim} \{\textit{closed subfunctors of } X\}.$$

(ii) There exists an isomorphism of posets:

$$D\colon \mathsf{Rad}_X \xrightarrow{\sim} \{\textit{open subfunctors of } X\}.$$

Proof. Let $\mathrm{Sub}(X)$ denote the poset of all subfunctors (or subobjects) of X. Since $X \simeq \mathrm{colim}_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Spec}(R)$, we have an isomorphism:

$$\mathrm{Sub}(X) \xrightarrow{\sim} \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Sub}(\mathrm{Spec}(R))$$
$$Y \mapsto (x^{-1}(Y) \subset \mathrm{Spec}(R))_x.$$

The inverse is given by $(Y_x \subset \mathrm{Spec}(R))_x \mapsto (Y(R) = \{x\colon \mathrm{Spec}(R) \to X \mid x^{-1}(Y) = Y_x = \mathrm{Spec}(R)\})$.

(i) Consider the diagram:

$$\begin{tikzcd} \mathrm{Sub}(X) \arrow[r, "\sim"] & \lim_x \mathrm{Sub}(\mathrm{Spec}(R)) \\ \mathsf{Closed}(X) \arrow[u, hook] \arrow[r, "\sim"] & \lim_x \mathsf{Closed}(\mathrm{Spec}(R)) \arrow[u, hook] \end{tikzcd}$$

By the definition of closed subfunctors and $X \simeq \mathrm{colim}_x \mathrm{Spec}(R)$, the bottom row is an isomorphism. Therefore, we reduce to showing:

$$\lim_x \mathsf{Closed}(\mathrm{Spec}(R)) \simeq \lim_x \mathsf{Id}_R^{\mathrm{op}}.$$

For this, we need to show that the bijection $V\colon \mathsf{Id}_R^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Closed}(\mathrm{Spec}(R))$ is functorial. For $\varphi\colon R \to S$, consider the diagram:

$$\begin{tikzcd} \mathsf{Id}_R \arrow[r, "V_R"] \arrow[d, "\varphi^*"] & \mathsf{Closed}(\mathrm{Spec}(R)) \arrow[d, "\mathrm{Spec}(\varphi)^{-1}"] \\ \mathsf{Id}_S \arrow[r, "V_S"] & \mathsf{Closed}(\mathrm{Spec}(S)). \end{tikzcd}$$

We verify commutativity by showing $V(IS) = \mathrm{Spec}(\varphi)^{-1}(V(I))$. The right side is:

$$V(I) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \simeq V(S \otimes_R I) \simeq \mathrm{Spec}(S \otimes_R R/I).$$

Using $S/IS \simeq S \otimes_R R/I$, we get:

$$\mathrm{Spec}(\varphi)^{-1}(V(I)) \simeq \mathrm{Spec}(S/IS) = V(IS) = V(\varphi^*(I)).$$

This shows V is a natural transformation from $\mathsf{Id}_{(-)}^{\mathrm{op}}$ to $\mathsf{Closed}(\mathrm{Spec}(-))$.

(ii) Similarly, using $X \simeq \operatorname{colim} \operatorname{Spec}(R)$ and the definition of open subfunctors, we reduce to proving $D\colon \mathsf{Rad}_R \xrightarrow{\sim} \mathsf{Open}(\operatorname{Spec}(R))$ is functorial.

For $\varphi\colon R \to S$ (let $f = \operatorname{Spec}(\varphi)$), we need to show the following diagram commutes:

$$\begin{tikzcd} \mathsf{Rad}_R \arrow[r, "D_R"] \arrow[d, "\varphi^*"] & \mathsf{Open}(\operatorname{Spec}(R)) \arrow[d, "f^{-1}"] \\ \mathsf{Rad}_S \arrow[r, "D_S"] & \mathsf{Open}(\operatorname{Spec}(S)) \end{tikzcd}$$

Note: In the category Rad, for $J \in \mathsf{Rad}_R$, the pullback is defined as $\varphi^*(J) := \sqrt{JS}$.

We verify $f^{-1}(D(J)) = D(\sqrt{JS})$:

- *Left side:* By the definition of open subfunctor pullback:

$$f^{-1}(D(J)) = D(J) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S).$$

For affine schemes, the pullback of $D(J)$ is the open subfunctor defined by the extended ideal JS, so $f^{-1}(D(J)) = D(JS)$.

- *Right side:* By definition, this is $D(\sqrt{JS})$.

Since $D(K)$ depends only on the radical of K (i.e., $D(K) = D(\sqrt{K})$), we have $D(JS) = D(\sqrt{JS})$ identically. This means the diagram commutes, inducing an isomorphism at the limit level:

$$\mathsf{Rad}_X = \lim_x \mathsf{Rad}_R \xrightarrow{\sim} \lim_x \mathsf{Open}(\operatorname{Spec}(R)) \simeq \mathsf{Open}(X).$$

■

Definition 3.2.2 (Open Complement). Let $Z \subset X$ be a closed subfunctor determined by a quasi-coherent ideal I. The *open complement* $X - Z$ is the open subfunctor corresponding to the quasi-coherent radical ideal $\sqrt{I}$.

Example 3.2.3 (Open Complements).

- Let A be a ring and $F \subset A$ a subset. Then in $\operatorname{Spec}(A)$, $D(F)$ is the open complement of $V(F)$. For example, for a set I, take $A = \mathbb{Z}[x_i \mid i \in I]$ and $F = \{x_i\}_{i \in I}$. Then $D(F) = \mathbb{A}^I - 0$ is the open complement of $V(F) = 0$.
- Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a subset of homogeneous elements. Then in $\operatorname{Proj}(A)$, $D(F)$ is the open complement of $V(F)$. For example, for a set I, in $\mathbb{P}^{I \sqcup \{0\}}$, the affine space $D(x_0) \simeq \mathbb{A}^I$ is the open complement of $V(x_0) \simeq \mathbb{P}^I$.

Warning **3.2.4** (No Closed Complement). Different closed subfunctors can have the same open complement, because different ideals can have the same radical. Therefore, there is no notion of "closed complement" of an open subfunctor.

Example 3.2.5 (Scheme-theoretic Image). For any ring homomorphism $\varphi\colon A \to B$, we can factor it as a surjection followed by an injection:

$$A \twoheadrightarrow A/\ker(\varphi) \hookrightarrow B.$$

Therefore, $V(\ker(\varphi)) \subset \operatorname{Spec}(A)$ is the smallest closed subfunctor through which $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors.

This construction generalizes to arbitrary algebraic functors: Let $f\colon Y \to X$ be a morphism of algebraic functors. The *scheme-theoretic image* of f is the smallest closed subfunctor Z through which f factors. By Proposition 3.2.1 (the bijection between closed subfunctors and quasi-coherent ideals), there exists a unique quasi-coherent ideal $I \in \mathsf{Id}_X$ such that $Z = V(I)$.

Explicit characterization: I is the *largest* quasi-coherent ideal such that the algebra homomorphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ factors through the quotient $\mathcal{O}_X/I$. This is because f factoring through the closed subfunctor $V(J)$ is equivalent to the algebra morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ factoring through the quotient $\mathcal{O}_X/J$ (which can be understood intuitively by examining the geometric shape of $V(J(x)) \subset \operatorname{Spec}(\mathcal{O}_X(x))$ pointwise). Since

$V(\cdot)$ is an order-reversing bijection, "smallest closed subfunctor" naturally corresponds to "largest quasi-coherent ideal".

It is important to note that although when X is a scheme, I coincides with the kernel of the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ in the module category Mod_X, for general algebraic functors X, the kernel in the module category may not form a quasi-coherent ideal (i.e., it may not satisfy the naturality requirements in Id_X). Therefore, I should be understood through the above universal property (largest quasi-coherent ideal), rather than simply taking the kernel at the module level.

***Remark* 3.2.6** (Loci of Maps Between Quasi-coherent Modules). Recall that in Section 3.4, we considered loci of linear maps $u: M \to N$. We defined $V(u)$ as:

$$V(u)(R) := \{\varphi: R \to A \mid \varphi^*(u) = 0\}.$$

We can now generalize this to general algebraic functors X. For a morphism $f: M \to N$ of quasi-coherent modules over X, we define $V(f)$ as:

$$V(f)(R) := \{x: \operatorname{Spec}(R) \to X \mid f(x) = 0\}.$$

Note: we have $f(x): M(x) \to N(x)$, so $f(x) = 0$ means the zero homomorphism. We can similarly define $\operatorname{Epi}(f)$, $\operatorname{Mono}(f)$, and $\operatorname{Iso}(f)$.

We can establish the following analogous proposition: If $N \in \mathsf{Vect}_X$ is a vector bundle, then:

- $V(f)$ is a closed subfunctor.
- $\operatorname{Epi}(f)$ is an open subfunctor.

If M is also a vector bundle, then:

- $\operatorname{Mono}(f)$ is an open subfunctor.
- $\operatorname{Iso}(f)$ is an open subfunctor.

3.2.1. Relative Spec and Relative Proj

We now extend the constructions of Spec and Proj to the relative setting over arbitrary algebraic functors.

***Observation* 3.2.7** (Base Change Compatibility). Note that the functor:

$$\operatorname{Spec}: \mathsf{CAlg}_R^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\operatorname{Spec}(R)}$$

is compatible with base change in the following sense: For an R-algebra A and a ring homomorphism $f: R \to S$, we have a pullback diagram:

$$\begin{tikzcd} \operatorname{Spec}(A \otimes_R S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(A) \arrow[d] \\ \operatorname{Spec}(S) \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

Therefore, we can extend the Spec functor to quasi-coherent algebras.

Definition 3.2.8 (Relative Spec). Let X be an algebraic functor and $A \in \mathsf{CAlg}_X$ a quasi-coherent algebra over X. We define $\operatorname{Spec}(A) \in \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ by:

$$\operatorname{Spec}(A)(R) := \{(x, a) \mid x \in X(R) \text{ and } a \text{ is a section of } \operatorname{Spec}(A(x)) \to \operatorname{Spec}(R)\}.$$

This construction is functorial, giving a functor:

$$\operatorname{Spec}: \mathsf{CAlg}_X^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}.$$

Similarly, for an $\mathbb{N}$-graded R-algebra A and a ring homomorphism $f\colon R \to S$, we have a pullback diagram:

$$\begin{tikzcd} \mathrm{Proj}(A \otimes_R S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Proj}(A) \arrow[d] \\ \mathrm{Spec}(S) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

Therefore, we can also extend the Proj functor to quasi-coherent algebras.

Definition 3.2.9 (Relative Proj). Let X be an algebraic functor and $A \in \mathsf{CAlg}_X$ a quasi-coherent $\mathbb{N}$-graded algebra over X. We define $\mathrm{Proj}(A) \in \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ by:

$$\mathrm{Proj}(A)(R) := \{(x, a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Proj}(A(x)) \to \mathrm{Spec}(R)\}.$$

This construction is functorial, giving a functor:

$$\mathrm{Proj}\colon \mathsf{CAlg}_X^{\mathbb{N},\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}.$$

We can also define relative affine and projective spaces.

Definition 3.2.10 (Relative Affine and Projective Spaces). Let X be an algebraic functor and M a quasi-coherent module over X.

(i) The *affine space over* X associated to M is $\mathbb{A}(M) := \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{A}(M)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \to R \text{ is an } R\text{-linear map}\}.$$

(ii) The *punctured affine space over* X is $\mathbb{A}(M) - 0 := D(M) \subset \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$(\mathbb{A}(M) - 0)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadleftarrow R \text{ is surjective}\}.$$

(iii) The *projective space over* X associated to M is $\mathbb{P}(M) := \mathrm{Proj}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{P}(M)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadrightarrow L \text{ is a quotient } R\text{-line}\}.$$

***Remark* 3.2.11** (Functorial Characterization). Let X and Y be algebraic functors.

(i) *Relative Spec:* For any quasi-coherent algebra $A \in \mathsf{CAlg}_X$, a morphism $Y \to \mathrm{Spec}(A)$ consists of:
- A morphism of algebraic functors $f\colon Y \to X$;
- An algebra homomorphism in CAlg_Y: $\varphi\colon f^*(A) \to \mathcal{O}_Y$.

(ii) *Relative Proj:* Let A be a quasi-coherent $\mathbb{N}$-graded algebra over X generated by A_1. A morphism $Y \to \mathrm{Proj}(A)$ consists of:
- A morphism of algebraic functors $f\colon Y \to X$;
- A quotient line bundle on Y, i.e., a surjection $q\colon f^*(A_1) \twoheadrightarrow L$;
- A compatibility condition: the induced map $\mathrm{Sym}(f^*(A_1)) \to \mathrm{Sym}(L)$ must factor through $f^*(A)$. That is, there exists a unique dashed arrow making the following diagram commute:

$$\begin{tikzcd} \mathrm{Sym}(f^*(A_1)) \arrow[r, "\mathrm{Sym}(q)"] \arrow[d, two heads] & \mathrm{Sym}(L) \\ f^*(A) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

(This is equivalent to saying that homogeneous relations in A must vanish when mapped to $L^{\otimes n}$.)

In fact, relative Spec and relative Proj allow us to extend the notions of affine schemes, quasi-affine schemes, projective schemes, and quasi-projective schemes to the relative setting. For an algebraic functor V, let $\mathsf{Rad}_V^{\mathrm{ft}}$ denote the category of finitely generated radical ideals over V.

Definition 3.2.12 (Classification of Relative Morphisms). Let $f: Y \to X$ be a morphism of algebraic functors.

(i) *Affine morphism:* If there exists $A \in \mathsf{CAlg}_X$ such that $Y \simeq \mathrm{Spec}(A)$, then f is called an *affine morphism*, or we say Y is *affine over* X. We denote by $\mathsf{Aff}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory spanned by affine schemes over X.

(ii) *Quasi-affine morphism:* If there exist $V \in \mathsf{Aff}_X$ and $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ such that $Y \simeq D(I)$, then f is called a *quasi-affine morphism*, or we say Y is *quasi-affine over* X. We denote by $\mathsf{QAff}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of quasi-affine schemes over X.

(Note: The condition $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ means I is a radical ideal generated by finitely many sections. This is equivalent to saying $I \subset \mathcal{O}_V$ is a finite subset.)

(iii) *Projective morphism:* If there exists $A \in \mathsf{CAlg}_X^{\mathbb{N}}$ such that A_1 is *finitely generated* (i.e., A_1 is a finitely generated A_0-module and A as an algebra is generated by A_1) and $Y \simeq \mathrm{Proj}(A)$, then f is called a *projective morphism*, or we say Y is *projective over* X. We denote by $\mathsf{Proj}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of projective schemes over X.

(iv) *Quasi-projective morphism:* If there exist $V \in \mathsf{Proj}_X$ and $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ such that $Y \simeq D(I)$, then f is called a *quasi-projective morphism*, or we say Y is *quasi-projective over* X. We denote by $\mathsf{QProj}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of quasi-projective schemes over X.

Now we characterize affine morphisms in terms of fibers.

***Observation* 3.2.13** (Fibers of Affine Morphisms). Let $f: Y \to X$ be an affine morphism. Then there exists $A \in \mathsf{CAlg}_X$ such that $\mathrm{Spec}(A) \simeq Y$. For any R-point $x: \mathrm{Spec}(R) \to X$, consider the pullback:

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "x"] & X. \end{tikzcd}$$

For any ring S, by the universal property of fiber products, we have $Y_x(S) = Y(S) \times_{X(S)} \mathrm{Hom}_{\mathsf{CAlg}}(R, S)$. Concretely, elements of $Y_x(S)$ have the form $((r, a), r')$ where:

- $(r, a) \in Y(S) = \mathrm{Spec}(A)(S)$;
- $r' \in \mathrm{Hom}_{\mathsf{CAlg}}(R, S)$,

satisfying $x \circ r' = r$ and a is a section of $\mathrm{Spec}(A(r)) \to \mathrm{Spec}(S)$.

By the characterization of quasi-coherent modules (more precisely, quasi-coherent algebras), we have $(r')^*(A(x)) \simeq A(r)$. Therefore, a can be interpreted as an S-algebra homomorphism:

$$\varphi: (r')^*(A(x)) \simeq A(x) \otimes_R S \to S.$$

Using the fact that tensor product is the pushout in the category of rings, the above map corresponds to an R-algebra homomorphism:

$$\psi: A(x) \to S.$$

Therefore, we obtain an isomorphism of sets:

$$Y_x(S) \simeq \mathrm{Hom}_{\mathsf{CAlg}_R}(A(x), S) \simeq \mathrm{Spec}(A(x))(S),$$

yielding an isomorphism:

$$Y_x \simeq \mathrm{Spec}(A(x)).$$

Conversely, if for any ring R and any morphism $x: \mathrm{Spec}(R) \to X$, the pullback Y_x is an affine scheme, we define $A(x) := \mathcal{O}(Y_x)$. Since Y_x satisfies pullback properties with respect to x and the $\mathcal{O}$ functor preserves base change, the assignment $x \mapsto A(x)$ forms a quasi-coherent X-algebra A. Recalling the definition of relative Spec, we have:

$$\mathrm{Spec}(A)(R) = \{(x, \sigma) \mid x \in X(R), \sigma \in \mathrm{Hom}_{\mathsf{CAlg}}(A(x), R)\}.$$

Since Y_x is affine, we have $\mathrm{Hom}_{\mathsf{CAlg}}(A(x), R) \simeq Y_x(R)$ (the fiber of $Y(R)$ projecting to x). Thus we establish $\mathrm{Spec}(A) \simeq Y$.

Proposition 3.2.14 (Affine Morphisms). *Let X be an algebraic functor. We have a categorical equivalence:*

$$\mathrm{Spec}\colon \mathsf{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_X.$$

The inverse sends $(Y \to X)$ to $f_*(\mathcal{O}_Y)$.

Proof. Everything reduces to showing:

$$\mathsf{Aff}_X \xrightarrow{\sim} \lim_x \mathsf{Aff}_R.$$

Note that we have a pullback:

$$\begin{tikzcd} \mathsf{Aff}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_x \mathsf{Aff}_R \arrow[d] \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X} \arrow[r] & \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{colim}_x \mathrm{Spec}(R)}. \end{tikzcd}$$

Since $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is a topos, all colimits in it are van Kampen. This means the bottom row is an isomorphism, hence the top row is also an isomorphism. ■

3.3. Modules over Quasi-projective Schemes

Our next goal is to describe Mod_X for quasi-projective schemes X.

The most important example is when $X = \mathbb{P}^n_k$ is projective n-space. We will discover that there exist line bundles $\mathcal{O}(d)$ on $\mathbb{P}^n_k$ such that $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = k[x_0, \ldots, x_n]_d$.

Generalized Zariski Descent

Recall that for an R-line L and a generating system $(s_i)_{i\in I}$ of L, we have generalized Zariski descent:

$$\mathsf{Mod}_R \longrightarrow \prod_{i\in I} \mathsf{Mod}_{R_{s_i}} \rightrightarrows \prod_{i,j\in I} \mathsf{Mod}_{R_{s_is_j}} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i,j,k\in I} \mathsf{Mod}_{R_{s_is_js_k}}$$

is a limit diagram.

This raises the following question: For $X = \mathbb{P}^n_k$, is there a family of affine open subschemes $\{D(x_i)\}_{i\in I}$ such that for any $\mathrm{Spec}(R) \to X$, we have a pullback diagram:

$$\begin{tikzcd} \mathsf{Mod}_{D(x_i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D(x_i) \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & X? \end{tikzcd}$$

This motivates the following definition:

Definition 3.3.1 (Zariski Locally Surjective). A morphism $Y \to X$ of algebraic functors is called *Zariski locally surjective* if for any affine scheme $\mathrm{Spec}(R)$ and morphism $\mathrm{Spec}(R) \to X$, there exist elements $f_1, \ldots, f_n \in R$ satisfying $(f_1, \ldots, f_n) = R$ (i.e., they generate the unit ideal) such that for each i, the following dashed lift exists:

$$\begin{tikzcd} & & Y \arrow[d] \\ \mathrm{Spec}(R_{f_i}) \arrow[r, hook] \arrow[urr, dashed] & \mathrm{Spec}(R) \arrow[r] & X. \end{tikzcd}$$

Example 3.3.2 (Examples of Zariski Locally Surjective Morphisms).

(i) Every surjection is Zariski locally surjective. In particular, morphisms with sections, such as $\mathbb{A}^n \times X \to X$, are also Zariski locally surjective.

(ii) Let $(F_i)_{i\in I}$ be a family of subsets of $\mathcal{O}(X)$ such that $(\bigcup_{i\in I} F_i)$ generates $\mathcal{O}(X)$. Then $\coprod_{i\in I} D(F_i) \to X$ is Zariski locally surjective.

(iii) Let L be a line bundle over X. We say L is *generated by a family of global sections* $(s_i)_{i\in I}$ if for any $x \in \mathrm{El}(X)$, the elements $(s_i(x))_{i\in I}$ generate $L(x)$. In this case, $\coprod_{i\in I} D(s_i) \to X$ is Zariski locally surjective.

(iv) The morphism $\mathbb{A}^I - 0 \to \mathbb{P}^I$ is Zariski locally surjective. In fact, for a general $\mathbb{N}$-graded algebra A, we have $D(A_1) \to \mathrm{Proj}_1(A)$ is Zariski locally surjective.

Proposition 3.3.3 (Zariski Descent for Quasi-coherent Modules). *Let $(Y_i \to X)_{i\in I}$ be a family of morphisms such that $\coprod_{i\in I} Y_i \to X$ is Zariski locally surjective. Then the following diagram is a limit diagram:*

$$\mathsf{Mod}_X \longrightarrow \prod_{i\in I} \mathsf{Mod}_{Y_i} \rightrightarrows \prod_{i,j\in I} \mathsf{Mod}_{Y_i\times_X Y_j} \rightrightarrows\!\!\!\!\to \prod_{i,j,k\in I} \mathsf{Mod}_{Y_i\times_X Y_j\times_X Y_k}$$

In particular, $\mathsf{Mod}_X \to \prod_{i\in I} \mathsf{Mod}_{Y_i}$ is a conservative functor.

Remark **3.3.4** (Proof Strategy). The proof of this proposition essentially translates descent of morphisms into descent for Grothendieck topologies. We will prove a more general result when we discuss sheaves later.

Corollary 3.3.5 (Zariski Descent for $D(I)$). *Let A be a ring and $I \subset A$ a subset such that $\coprod_{f\in I} D(f) \to D(I)$ is Zariski locally surjective. Then:*

$$\mathsf{Mod}_{D(I)} \longrightarrow \prod_{f\in I} \mathsf{Mod}_{D(f)} \rightrightarrows \prod_{f,g\in I} \mathsf{Mod}_{D(fg)} \rightrightarrows\!\!\!\!\to \prod_{f,g,h\in I} \mathsf{Mod}_{D(fgh)}$$

is a limit diagram.

Proof. Take $X = D(I)$. Then $Y_f = D(f)$. Everything reduces to showing $D(fg) \simeq D(f)\times_{D(I)} D(g)$. We know:

$$D(f)\times_{D(I)} D(g) \simeq D(f)\times_{\mathrm{Spec}(A)} D(g) \simeq D(fg).$$

■

Now we can study modules over Proj.

Example 3.3.6 (Modules over Proj). Let A be an $\mathbb{N}$-graded ring.

(i) Let $I \subset A_+$ be a subset of homogeneous elements satisfying $A_+ \subset \sqrt{(I)}$. Then $\coprod_{f\in I} \mathrm{Spec}(A_{(f)}) \to \mathrm{Proj}(A)$ is Zariski locally surjective. Therefore, Proposition 3.3.3 shows that we have a limit diagram:

$$\mathsf{Mod}_{\mathrm{Proj}(A)} \longrightarrow \prod_{f\in I} \mathsf{Mod}_{A_{(f)}} \rightrightarrows \prod_{f,g\in I} \mathsf{Mod}_{A_{(fg)}} \rightrightarrows\!\!\!\!\to \prod_{f,g,h\in I} \mathsf{Mod}_{A_{(fgh)}}.$$

(ii) Consider $D(A_1) \subset \mathrm{Spec}(A)$. We know that $D(A_1) \to \mathrm{Proj}_1(A)$ is Zariski locally surjective, and it induces an injection $D(A_1)/\mathbb{G}_m \hookrightarrow \mathrm{Proj}_1(A)$. Therefore, considering the n-fold fiber product $D(A_1)\times_{\mathrm{Proj}_1(A)} \cdots \times_{\mathrm{Proj}_1(A)} D(A_1)$, by the injectivity mentioned above, this is equivalent to the n-fold fiber product $D(A_1)\times_{D(A_1)/\mathbb{G}_m} \cdots \times_{D(A_1)/\mathbb{G}_m} D(A_1)$. By the freeness of the $\mathbb{G}_m$-action on $D(A_1)$, this is nothing but $D(A_1)\times \mathbb{G}_m^{n-1}$. Thus we obtain a limit diagram:

$$\mathsf{Mod}_{\mathrm{Proj}_1(A)} \longrightarrow \mathsf{Mod}_{D(A_1)} \rightrightarrows \mathsf{Mod}_{D(A_1)\times\mathbb{G}_m} \rightrightarrows\!\!\!\!\to \mathsf{Mod}_{D(A_1)\times\mathbb{G}_m\times\mathbb{G}_m}.$$

Example 3.3.7 (Modules over $\mathbb{P}^1$). Let k be a ring. Consider the projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$. Let $u = y/x$. We have $k[x,y]_{(x)} = k[u]$. Let $v = x/y$. Then $k[x,y]_{(y)} = k[v]$ and $k[x,y]_{(xy)} = k[u,v] = k[t^{\pm 1}]$. Therefore, the limit diagram from Example 3.3.6 can be rewritten as a pullback diagram:

$$\begin{tikzcd} \mathsf{Mod}_{\mathbb{P}^1_k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Mod}_{k[u]} \arrow[d] \\ \mathsf{Mod}_{k[v]} \arrow[r] & \mathsf{Mod}_{k[t^{\pm 1}]} \end{tikzcd}$$

Thus we can write $\mathsf{Mod}_{\mathbb{P}^1_k}$ as:

$$\mathsf{Mod}_{\mathbb{P}^1_k} = \left\{ (M,N,\alpha) \,\middle|\, \substack{M\in\mathsf{Mod}_{k[u]},\, N\in\mathsf{Mod}_{k[v]},\\ \alpha\colon M[u^{\pm 1}]\xrightarrow{\sim} N[v^{\pm 1}] \text{ is a } k[t^{\pm 1}]\text{-linear isomorphism}} \right\}.$$

Example 3.3.8 (Modules over the Affine Line with Two Origins). Let X be the affine line with two origins. From previous exercises, we know the explicit characterization of X is:

$$X(R) := \{(f, e) \mid f \in R, e \in R/(f), e^2 = e\}.$$

Let $X_0 \subset X$ and $X_1 \subset X$ be the loci where $e = 0$ and $e = 1$, respectively. Then we can verify:

- X_0 and X_1 are both open subfunctors of X, and $X_0 \sqcup X_1 \to X$ is Zariski locally surjective.
- The projection $X \xrightarrow{\sim} \mathbb{A}^1_k$ given by $(f, e) \mapsto f$ induces isomorphisms:

$$X_0 \xrightarrow{\sim} \mathbb{A}^1_k, \quad X_1 \xrightarrow{\sim} \mathbb{A}^1_k, \quad X_0 \cap X_1 \xrightarrow{\sim} \mathbb{G}_{m,k}.$$

Therefore, combining with Proposition 3.3.3, we obtain a pullback diagram:

$$\begin{tikzcd}
\mathsf{Mod}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Mod}_{k[x]} \arrow[d, "x \mapsto x"] \\
\mathsf{Mod}_{k[x]} \arrow[r, "x \mapsto x"] & \mathsf{Mod}_{k[x^{\pm 1}]}
\end{tikzcd}$$

That is:

$$\mathsf{Mod}_X = \left\{ (M, N, \alpha) \,\middle|\, \substack{M \in \mathsf{Mod}_{k[x]}, N \in \mathsf{Mod}_{k[x]}, \\ \alpha : M[x^{-1}] \xrightarrow{\sim} N[x^{-1}] \text{ is a } k[x^{\pm 1}]\text{-linear isomorphism}} \right\}.$$

Recall that in Section 3.4 we introduced the notion of I-local modules. We know that for a set I, there is a functor:

$$\mathrm{L}_I : \mathsf{Mod}_A \to \mathsf{Mod}_A^{I\text{-loc}}.$$

Moreover, for a homomorphism $h : M \to N$, we have $\mathrm{L}_I(h)$ is an isomorphism if and only if for all $f \in I$, the map $h_f : M_f \to N_f$ is an isomorphism.

Theorem 3.3.9 (Modules over $D(I)$). *Let A be a ring and $I \subset A$ a finite subset. Then the base change functor $\mathsf{Mod}_A \to \mathsf{Mod}_{D(I)}$ induced by $D(I) \hookrightarrow \mathrm{Spec}(A)$ gives a categorical equivalence:*

$$\mathsf{Mod}_A^{I\text{-loc}} \xrightarrow{\sim} \mathsf{Mod}_{D(I)}.$$

Moreover, we have:

$$\mathrm{L}_I(M) = \lim \left(\prod_{f \in I} M_f \rightrightarrows \prod_{f, g \in I} M_{fg} \right).$$

***Remark* 3.3.10** (Infinite Sets). Note that when I is an infinite set, since $\prod_I$ is no longer a finite product, we no longer have full faithfulness.

Proof. First, we construct the functor $\mathsf{Mod}_A^{I\text{-loc}} \to \mathsf{Mod}_{D(I)}$. Consider the diagram:

$$\begin{tikzcd}
\mathsf{Mod}_A \arrow[r, "i^*"] \arrow[d, "\mathrm{L}_I"'] & \mathsf{Mod}_{D(I)} \\
\mathsf{Mod}_A^{I\text{-loc}} \arrow[ur, dashed] &
\end{tikzcd}$$

We need to characterize the dashed arrow. By Corollary 3.3.5, we can write $\mathsf{Mod}_{D(I)}$ as a limit diagram. We know that $h \in \mathsf{Mod}_A^{I\text{-loc}}$ is an isomorphism if and only if for all $f \in I$, we have h_f is an isomorphism. Thus the dashed functor, if it exists, is a conservative functor. Its existence is guaranteed by $\mathsf{Mod}_A^{I\text{-loc}} \hookrightarrow \mathsf{Mod}_A$.

In fact, the functor $\mathsf{Mod}_A^{I\text{-loc}} \to \mathsf{Mod}_{D(I)}$ has a right adjoint, given by:

$$(M(f), \alpha_{fg}) \mapsto \lim \left(\prod_{f \in I} M(f) \rightrightarrows \prod_{f, g \in I} M(f)_g \right).$$

By the basic properties of adjoint functors, everything reduces to showing that the right adjoint is fully faithful (left adjoint conservative + right adjoint fully faithful ⇒ categorical equivalence). This further reduces to showing that the counit is a pointwise isomorphism. We need to show that for all $(M(f), \alpha_{fg})$:

$$i^*\left(\lim\left(\prod_{f\in I} M(f) \rightrightarrows \prod_{f,g\in I} M(f)_g \right)\right) \simeq (M(f), \alpha_{fg}).$$

By conservativity, everything reduces to showing that for all $h \in I$:

$$\left(\lim\left(\prod_{f\in I} M(f) \rightrightarrows \prod_{f,g\in I} M(f)_g \right)\right)_h \simeq M(h).$$

This follows by using the exactness of $(-)_h$ and the fact that $\prod_I$ is a finite product, combined with Zariski descent. ∎

Corollary 3.3.11 (Affine Completion of Quasi-affine Schemes). *Let X be a quasi-affine scheme. Then the canonical morphism $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an open embedding induced by some finitely generated radical ideal.*

Proof. Since X is a quasi-affine scheme, there exist a ring A and a finite subset $I \subset A$ such that $X \simeq D(I)$. By Theorem 3.3.9, we have a categorical equivalence:

$$\mathsf{Mod}_A^{I\text{-loc}} \simeq \mathsf{Mod}_{D(I)} \simeq \mathsf{Mod}_X.$$

This actually says that $\mathcal{O}_X \in \mathsf{Mod}_X$ corresponds to $\mathrm{L}_I(A)$. In other words, the global sections ring $\mathcal{O}(X) \in \mathsf{CAlg}$ corresponds to $\mathrm{End}_A(\mathrm{L}_I(A))$. Moreover, we can see that the ring homomorphism $\lambda\colon A \to \mathrm{L}_I A$ induces the canonical morphism:

$$X \to \mathrm{Spec}(\mathcal{O}(X)).$$

Next, consider the pullback:

$$\begin{tikzcd} D(\lambda(I)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(\mathcal{O}(X)) \arrow[d, "\mathrm{Spec}(\lambda)"] \\ D(I) \arrow[r, hook] & \mathrm{Spec}(A). \end{tikzcd}$$

Everything reduces to showing $D(\lambda(I)) \to D(I)$ is an isomorphism. For this, we only need to show that for any $\varphi\colon A \to R$, if φ lies in $D(I)(R)$, then R can be viewed as an I-local A-module via φ. This follows from $D(I)(R) = \{\varphi\colon A \to R \mid (\varphi(I)) = R\}$ and Zariski descent. ∎

Construction 3.3.12 (Quasi-coherent Module from $\mathbb{Z}$-graded Module). Let A be an $\mathbb{N}$-graded ring. We will define a functor:

$$\{\mathbb{Z}\text{-graded } A\text{-modules}\} \to \mathsf{Mod}_{\mathrm{Proj}(A)}, \quad M \mapsto \widetilde{M}.$$

For a $\mathbb{Z}$-graded A-module M, let $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ be an R-point of $\mathrm{Proj}(A)$. Specifically, we place it in $\mathrm{Proj}_d(A)$ corresponding to a quotient $\mathbb{N}$-graded algebra $\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L)$ (of course, when A is generated by $A_{\leq 1}$, by the structure theorem we can take $d = 1$).

Let $\widetilde{\varphi}$ be the induced map between $\mathbb{Z}$-graded rings $A^{(d)} \to \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$. Define:

$$\widetilde{M}(x) = \widetilde{\varphi}^*(M^{(d)})_0 \in \mathsf{Mod}_R.$$

Note that for $d \geq 1$ and $f \in A_d$, the pullback functor $\mathsf{Mod}_{\mathrm{Proj}(A)} \to \mathsf{Mod}_{A_{(f)}}$ sends $\widetilde{M}$ to $M_{(f)}$. Example 3.3.6 shows that $M \mapsto \widetilde{M}$ factors through L_{A_+}.

Theorem 3.3.13 (Quasi-coherent Modules on Quasi-projective Schemes). *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$, and let $I \subset A_1$ be a finite subset such that $D(I) \subset \mathrm{Proj}(A)$. Then the functor $M \mapsto \widetilde{M}$ induces a categorical equivalence:*

$$\{I\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathsf{Mod}_{D(I)}.$$

In particular, if A is also finitely generated, then:

$$\{A_1\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{Proj}(A)}.$$

3.3.1. Serre Twisting

Let A be a $\mathbb{Z}$-graded ring, M a $\mathbb{Z}$-graded A-module, and $d \in \mathbb{Z}$ an integer. Recall that we denote $M(d) := \bigoplus_{n\in\mathbb{N}} M_{n+d}$.

Definition 3.3.14 (Serre Twisting). Let A be an $\mathbb{N}$-graded ring and $d \in \mathbb{Z}$. We denote by $\mathcal{O}(d)$ the quasi-coherent module corresponding to the $\mathbb{Z}$-graded A-module $A(d)$. For a given $M \in \mathsf{Mod}_{\mathrm{Proj}(A)}$, the *d-th Serre twist* of M is:

$$M(d) := M \otimes \mathcal{O}(d).$$

Remark **3.3.15** (Explicit Description). If A is generated by $A_{\leq 1}$, then we know that an R-point $x\colon \operatorname{Spec}(R) \to \operatorname{Proj}(A)$ corresponds to $A_1 \otimes_{A_0} R \twoheadrightarrow L$ satisfying that $A \otimes_{A_0} R \to \operatorname{Sym}_R(L)$ factors through A. In this case, we have $\mathcal{O}(d)(x) = L^{\otimes d}$.

Remark **3.3.16** (Notation Variation). In some references, $\mathcal{O}(-d)$ may be used to denote what we call $\mathcal{O}(d)$.

Definition 3.3.17 (Tautological Line Bundle). Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$. We call $\mathcal{O}(1)$ the *tautological line bundle* on $\operatorname{Proj}(A)$.

Next we study some properties of Serre twisting.

Proposition 3.3.18 (Properties of Serre Twisting). *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$.*

(i) For any $d \in \mathbb{Z}$, $\mathcal{O}(d)$ is a line bundle on $\operatorname{Proj}(A)$.

(ii) For a $\mathbb{Z}$-graded A-module M and $d \in \mathbb{Z}$, there exists an isomorphism $\widetilde{M}(d) \xrightarrow{\sim} \widetilde{M(d)}$. In particular, for all $d, e \in \mathbb{Z}$, there exist isomorphisms:

$$\mathcal{O}(d) \otimes \mathcal{O}(e) \simeq \mathcal{O}(d+e) \quad \text{and} \quad \mathcal{O}(1)^{\otimes d} \xrightarrow{\sim} \mathcal{O}(d).$$

Therefore, $\bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$ can be viewed as a $\mathbb{Z}$-graded quasi-coherent algebra on $\operatorname{Proj}(A)$.

(iii) There exist canonical maps of $\mathbb{Z}$-graded rings:

$$\mathrm{L}_{A_1} A \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\operatorname{Proj}(A), \mathcal{O}(d)),$$

and for any $\mathbb{Z}$-graded A-module M, there exist canonical maps of $\mathbb{Z}$-graded A-modules:

$$\mathrm{L}_{A_1} M \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\operatorname{Proj}(A), \widetilde{M}(d)).$$

When A is finitely generated as an A_0-algebra, these are isomorphisms.

Proof. (i) For $x\colon \operatorname{Spec}(R) \to \operatorname{Proj}(A)$, we know it corresponds to $\varphi\colon A \otimes_{A_0} R \twoheadrightarrow \operatorname{Sym}_R(L)$. Therefore, we can consider the commutative diagram:

$$\begin{tikzcd} A \arrow[r, dashed, "\widetilde{\varphi}"] \arrow[d] & \bigoplus_{n\in\mathbb{Z}} L^{\otimes n} \\ A \otimes_{A_0} R \arrow[r, "\varphi"] & \operatorname{Sym}_R(L). \arrow[u, hook] \end{tikzcd}$$

This gives the dashed $\widetilde{\varphi}$. By definition:

$$\mathcal{O}(d)(x) = \widetilde{A(d)}(x) = \widetilde{\varphi}^*(A(d))_0 = \left(\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}(d)\right)_0 = L^{\otimes d}.$$

(ii) We have:

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}(d))(x) = \widetilde{M}(x) \otimes_R \mathcal{O}(d)(x) = \widetilde{\varphi}^*(M) \otimes_R L^{\otimes d} = \widetilde{\varphi}^*(M(d))_0.$$

(iii) Since $\mathrm{Proj}(A)$ is a quasi-projective scheme, by Theorem 3.3.13, there exists an isomorphism:

$$\mathrm{Hom}_A^{A_1\text{-loc},\mathbb{Z}\text{-graded}}(\mathrm{L}_{A_1}\,A, \mathrm{L}_{A_1}\,A(d)) \xrightarrow[M\mapsto\widetilde{M}]{\sim} \mathrm{Hom}_{\mathsf{Mod}_{\mathrm{Proj}(A)}}(\mathcal{O}, \mathcal{O}(d)) = \Gamma(\mathrm{Proj}(A), \mathcal{O}(d)).$$

The left side is nothing but $\mathrm{Hom}_{\mathsf{Mod}_A^{\mathbb{Z}}}(A, \mathrm{L}_{A_1}\,A(d)) \simeq (\mathrm{L}_{A_1}\,A)_d$.

■

Chapter 4.

Locale

In this chapter, our main goal is to introduce "pointless topology", that is, to discuss the concept of *locales*.

Generally, when understanding topological spaces, we often focus on the relationships between open sets rather than specific points. Therefore, we wish to extract the properties of open sets so that they no longer depend on points. This concept is called a *locale*. Precisely speaking, we will construct the following diagram:

$$\begin{tikzcd} & \mathsf{Loc} \arrow[d, hook] \\ \mathsf{Top} \arrow[r, "X \mapsto \mathrm{Open}(X)"] & \mathsf{Pos}^{\mathrm{op}}. \end{tikzcd}$$

Of course, not all spaces can be recovered from their open sets, but many spaces with nice properties can. We call spaces that can be recovered from their open sets *sober spaces*. We have the following relationship:

$$\mathsf{Top}^{\mathrm{Haus}} \subset \mathsf{Top}^{\mathrm{sober}} \hookrightarrow \mathsf{Loc}.$$

In fact, M. Hochster proved the following theorem in 1969:

Theorem 4.0.1 (Hochster). *Let X be a topological space. The following are equivalent:*

(i) There exists a ring A such that $X \simeq \mathrm{Prim}(A) = |\mathrm{Spec}(A)|$.

(ii) X is quasi-compact and has a basis $\mathcal{B}$ of quasi-compact open sets that is closed under finite intersections, and X is sober.

(iii) X can be written as a limit of finitely many T_0 spaces.

Definition 4.0.2 (Spectral Space). A *spectral space* is a topological space satisfying any of the equivalent conditions in Theorem 4.0.1.

Therefore, at least for algebraic geometry, we will not need spaces worse than sober spaces.

The main goal of this chapter is to show that for any algebraic functor X, the poset $\mathsf{Open}(X)$ of open subfunctors is a locale, and it lies in the essential image of:

$$\mathsf{Top} \xrightarrow{\mathrm{Open}} \mathsf{Loc}.$$

That is, there exists $|X|$ such that $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$. We call $|X|$ the *geometric realization* or *underlying topological space* of X.

4.1. Locales

4.1.1. Review of Partially Ordered Sets

Before we begin, let us briefly review the relevant concepts of *partially ordered sets* (posets).

Definition 4.1.1 (Partially Ordered Set). A *partial order* on a set X is a binary relation $\le$ satisfying:

- *Reflexivity:* For all $x \in X$, we have $x \le x$.
- *Transitivity:* For all $x, y, z \in X$, if $x \le y$ and $y \le z$, then $x \le z$.
- *Antisymmetry:* For all $x, y \in X$, if $x \le y$ and $y \le x$, then $x = y$.

The pair $(X, \leq)$ is called a *partially ordered set* or *poset*.

In these notes, we often view a poset as a *category*:

- *Objects:* Elements of the poset X.
- *Morphisms:* For $x, y \in X$, if $x \leq y$, there exists a unique morphism $x \to y$; otherwise, there is no morphism.

For a map (functor) between posets:

$$f : I \to P, \quad i \mapsto x_i,$$

the limit and colimit (if they exist) correspond precisely to the infimum and supremum in order theory:

- *Colimit:* Corresponds to the *supremum* (join):

$$\mathrm{colim}_{i\in I}\, x_i = \sup_{i\in I} x_i = \bigvee_{i\in I} x_i.$$

- *Limit:* Corresponds to the *infimum* (meet):

$$\lim_{i\in I} x_i = \inf_{i\in I} x_i = \bigwedge_{i\in I} x_i.$$

Example 4.1.2 (Empty Diagram). When $I = \varnothing$ is the empty diagram:

- $\mathrm{colim}_{\varnothing}$ is the initial object in P, which is the *minimum element* (if it exists).
- $\lim_{\varnothing}$ is the terminal object in P, which is the *maximum element* (if it exists).

***Remark* 4.1.3** (Completeness). In fact, for a poset P, "having all small colimits" (i.e., every subset has a supremum) is equivalent to "having all small limits" (i.e., every subset has an infimum). This is because the infimum can be constructed from the supremum:

$$\bigwedge_{i\in I} x_i = \bigvee \{y \in P \mid y \leq x_i, \forall i \in I\}.$$

(That is: the infimum of a family of elements equals the supremum of all their lower bounds.)
A poset satisfying these properties is called a *complete lattice*.

4.1.2. Locales

Definition 4.1.4 (Locale). A poset O is called a *locale* if it satisfies:

(i) *Completeness:* O has all suprema (and hence all infima).

(ii) *Distributivity:* For any $u \in O$ and any family $(v_i)_{i\in I}$:

$$u \wedge \left(\bigvee_{i\in I} v_i \right) = \bigvee_{i\in I} (u \wedge v_i).$$

A morphism $f : O' \to O$ between locales is a map of posets $f^* : O \to O'$ (note the reversed direction) that preserves colimits (corresponding to arbitrary unions of open sets) and finite limits (corresponding to finite intersections of open sets).
We denote by $\mathsf{Loc} \subset \mathsf{Pos}^{\mathrm{op}}$ the full subcategory of all locales.

***Remark* 4.1.5** (Frames). The category $\mathsf{Loc}^{\mathrm{op}}$ is also called the category of *frames* or the category of *complete Heyting algebras*.

The most basic and important example of locales comes from topological spaces.

Example 4.1.6 (Open Sets of Topological Spaces). Let X be a topological space. Then the poset $\mathrm{Open}(X)$ of all open sets of X forms a locale. Let us verify that it satisfies the definition:

- On one hand, for a family $(U_i)_{i\in I}$ of open sets, since arbitrary unions of open sets are open, we have $\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$ exists. We define $\bigwedge_{i\in I} U_i$ to be the *interior* of $\bigcap_{i\in I} U_i$.
- On the other hand, for $U, (V_i)_i \in \mathrm{Open}(X)$, we always have:

$$U \cap \left(\bigcup_{i\in I} V_i\right) = \bigcup_{i\in I}(U \cap V_i).$$

Let $f : T \to S$ be a continuous map of topological spaces. Then $f^{-1} \colon \mathrm{Open}(S) \to \mathrm{Open}(T)$ preserves arbitrary colimits and finite limits, thus giving a morphism of locales $\mathrm{Open}(T) \to \mathrm{Open}(S)$. Therefore, we obtain a functor:

$$\mathrm{Open} \colon \mathsf{Top} \to \mathsf{Loc}.$$

***Remark* 4.1.7** (Direction of Morphisms). From the above example, we can see that the direction of morphisms between locales actually corresponds to the direction of continuous maps between topological spaces. In fact, when studying topoi, a geometric morphism between topoi $\mathcal{T}$ and $\mathcal{S}$ is a functor $\varphi_* \colon \mathcal{T} \to \mathcal{S}$ such that φ_* has a left exact left adjoint $\varphi^* \colon \mathcal{S} \to \mathcal{T}$.

We can illustrate this with topoi on topological spaces. In this case, the topoi are simply sheaf categories $\mathsf{Shv}(S)$ and $\mathsf{Shv}(T)$, and $\varphi_* \colon \mathsf{Shv}(T) \to \mathsf{Shv}(S)$ corresponds to the direction of continuous maps between topological spaces, hence called a *geometric morphism*. The functor φ^* corresponds to the direction $\mathrm{Open}(T) \to \mathrm{Open}(S)$, also called a *logical morphism* (or *algebraic morphism*). When we focus on the logical side, we also call topoi *logoi*.

In fact, we will find that topoi are actually determined by their locales. In other words, in algebraic geometry, locales are the more fundamental objects.

Next, we study some properties of locales.

Proposition 4.1.8 (Properties of Locales).

(i) Loc *has limits and colimits.*

(ii) The embedding functor $\mathsf{Loc} \hookrightarrow \mathsf{Pos}^{\mathrm{op}}$ *preserves colimits.*

(iii) The initial object of Loc *is* $0 = \mathrm{Open}(\varnothing) = \{\varnothing\}$*, and the terminal object is* $1 = \mathrm{Open}(*) = \{\varnothing \to *\}$.

Proof. (i) Let $(P_i)_{i\in I}$ be a diagram in $\mathsf{Loc}^{\mathrm{op}} \subset \mathsf{Pos}$. Let $P = \lim_{i\in I} P_i$ be its limit. We show that P is a locale:

- First, note that limits and colimits in P are computed pointwise:

$$\bigvee_{\alpha} p_\alpha = \left(\bigvee_{\alpha} p_{\alpha,i}\right)_{i\in I}, \quad p \wedge q = (p_i \wedge q_i)_{i\in I}.$$

  Therefore, P is complete.
- Next, we verify the distributive law, which also follows from pointwise computation.

Therefore, Loc has all colimits. A similar argument shows that Loc has all limits.

(ii) Note that for the constructed P, a map $Q \to P$ preserves colimits and finite limits if and only if for each $i \in I$, the map $Q \to P_i$ preserves colimits and finite limits. Therefore, $\mathsf{Loc} \hookrightarrow \mathsf{Pos}^{\mathrm{op}}$ preserves colimits.

(iii) Direct verification.

∎

Next, we discuss points of locales.

Definition 4.1.9 (Points of a Locale). Let O be a locale. We can define its corresponding topological space $\mathrm{Pt}(O)$ as follows:

- *Point set:* A point of $\mathrm{Pt}(O)$ is defined as a morphism $p \colon 1 \to O$ in the category of locales. (Note: In the language of frames, this corresponds to a frame homomorphism $p^* \colon \mathcal{O}(O) \to \mathcal{O}(1) \cong \{0, 1\}$.)

- *Topology:* The open sets of $\mathrm{Pt}(O)$ are determined by elements $u \in \mathcal{O}(O)$. For any u, define the corresponding open set $\mathrm{Pt}(u)$ as:
$$\mathrm{Pt}(u) := \{p \in \mathrm{Pt}(O) \mid p^*(u) = 1\}.$$
(Here 1 denotes the top element in $\mathcal{O}(1)$.)

Given a morphism of locales $f : O' \to O$, since the preimage of $\mathrm{Pt}(u)$ under the map is exactly $\mathrm{Pt}(f^*(u))$, the morphism f naturally induces a continuous map $\mathrm{Pt}(f) \colon \mathrm{Pt}(O') \to \mathrm{Pt}(O)$ of topological spaces. Thus we obtain a functor:
$$\mathrm{Pt} \colon \mathsf{Loc} \to \mathsf{Top}.$$

***Remark* 4.1.10** (Explicit Characterization of Points)**.** Let P be a poset. Viewing P as a set, we find that a set map $f : P \to \{\varnothing \to *\}$ (i.e., a characteristic function $P \to \{0, 1\}$) is completely determined by the subset $F = f^{-1}(*) \subset P$.

Moreover, we find that the above map can be upgraded to morphisms at different levels after satisfying the following conditions:

(i) *Poset morphism:* If F is *upward closed* (i.e., for any $x \in F$, if $x \leq y$, then $y \in F$), then f upgrades to a morphism of posets (monotone map).

(ii) *Finite-limit-preserving morphism:* If additionally P has finite limits (i.e., is a semilattice), and F is non-empty and closed under finite meets (i.e., for any $x, y \in F$, we have $x \wedge y \in F$), then f upgrades to a finite-limit-preserving poset map (corresponding to a filter).

(iii) *Morphism of locales (reversed):* If P is a locale (i.e., its underlying poset forms a frame), and for any $\bigvee_{i \in I} x_i \in F$, there exists $i \in I$ such that $x_i \in F$ (i.e., F is a completely prime filter, analogous to prime ideals), then f upgrades to a morphism of locales *reversed* (i.e., a frame homomorphism).

***Warning* 4.1.11** (Locales Can Be Pointless)**.** A locale can even have no points. For example, consider complete atomless Boolean algebras: First, a complete Boolean algebra is a cocomplete Boolean algebra (since Boolean algebras are posets, they are automatically complete). We call an element of a poset an *atom* if there is no element strictly between it and the minimum element.

Now let $(B, \bot, \top, \vee, \wedge, \neg)$ be a complete Boolean algebra. Since we know that complete Boolean algebras are automatically complete Heyting algebras, B is automatically a locale. Let P be a completely prime filter of B. Then $\bigwedge_{p \in P} p$ is an atom in B. This is because if there exists $x \in \bigwedge_{p \in P} p$, then $x \notin P$. By the complementation law, there exists $\neg x$ such that $x \vee \neg x = \top \in P$ (by upward closure), so $\neg x \in P$. Thus $x < \neg x$, i.e., $x = \bot$.

Finally, we give an example of a complete atomless Boolean algebra to demonstrate existence:

- Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $N \subset \mathcal{A}$ be the ideal of null sets. Then the quotient algebra $\mathcal{A}/N$ is a complete atomless Boolean algebra.

Next, we show that there is an adjoint pair:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "\mathrm{Open}"] \arrow[r, phantom, "\perp"] & \mathsf{Loc} \arrow[l, bend left, "\mathrm{Pt}"] \end{tikzcd}$$

In fact, this adjoint pair is also *idempotent*.

Definition 4.1.12 (Sober and Spatial)**.** Let T be a topological space and O a locale.

(i) We say T is *sober* if it lies in the essential image of $\mathrm{Pt} \colon \mathsf{Loc} \to \mathsf{Top}$. We denote by $\mathsf{Top}^{\mathrm{sob}} \subset \mathsf{Top}$ the full subcategory spanned by sober spaces.

(ii) We say O is *spatial* if it lies in the essential image of $\mathrm{Open} \colon \mathsf{Top} \to \mathsf{Loc}$. We denote by $\mathsf{Loc}^{\mathrm{spa}} \subset \mathsf{Loc}$ the full subcategory spanned by spatial locales.

Proposition 4.1.13 (Adjunction and Idempotence)**.** *In fact,* Open *is left adjoint to* Pt*, and this adjoint pair is idempotent. That is, we have categorical equivalences:*

$$\begin{tikzcd} \mathsf{Top}^{\mathrm{sob}} \arrow[r, bend left, "\mathrm{Open}"] \arrow[r, phantom, "\simeq"] & \mathsf{Loc}^{\mathrm{spa}} \arrow[l, bend left, "\mathrm{Pt}"] \end{tikzcd}$$

Proof. We prove everything from the definitions:

- *Adjunction:* Note that there is a map:

$$\mathrm{Hom}_{\mathsf{Top}}(T,\mathrm{Pt}(L)) \to \mathrm{Hom}_{\mathsf{Loc}}(\mathrm{Open}(T),L),$$

  sending $(f : T \to \mathrm{Pt}(L))$ to the map:

$$\begin{aligned} L &\to \mathrm{Open}(T) \\ u &\mapsto f^{-1}(\mathrm{Pt}(u)) \end{aligned}$$

  The inverse is given by sending $(g\colon \mathrm{Open}(T) \to L)$ to:

$$\begin{aligned} T &\longrightarrow \mathrm{Pt}(L) \\ t &\mapsto g \circ \mathrm{Open}(t) \end{aligned}$$

- *Idempotence:* Consider the unit morphism:

$$T \to \mathrm{Pt}(\mathrm{Open}(T)), \quad t \mapsto (t^{-1}\colon \mathrm{Open}(T) \to \mathrm{Open}(*)).$$

  This induces a homeomorphism:

$$\mathrm{Open}(T) \xrightarrow{\sim} \mathrm{Open}(\mathrm{Pt}(\mathrm{Open}(T))).$$

  Therefore, the unit is an isomorphism. Similarly, we can show that the counit is also an isomorphism.

■

***Remark* 4.1.14** (Soberization and Spatialization). Proposition 4.1.13 also shows that:

$$\mathsf{Top}^{\mathrm{sob}} \hookrightarrow \mathsf{Top}$$

has a left adjoint $\mathrm{Pt} \circ \mathrm{Open}$, called *soberization*.

Additionally, it shows that:

$$\mathsf{Loc}^{\mathrm{spa}} \hookrightarrow \mathsf{Loc}$$

has a right adjoint $\mathrm{Open} \circ \mathrm{Pt}$, called *spatialization*.

Next, we show how to identify sober spaces.

Proposition 4.1.15 (Characterization of Sober Spaces). *Let T be a topological space. Then points of $\mathrm{Open}(T)$ can be viewed as irreducible closed subsets of T. Moreover, the unit $T \to \mathrm{Pt}(\mathrm{Open}(T))$ sends $t \in T$ to its closure $\overline{\{t\}}$.*

Therefore, a topological space T is sober if and only if the map:

$$\begin{aligned} \{\textit{points of } T\} &\to \{\textit{irreducible components of } T\} \\ t &\mapsto \overline{\{t\}} \end{aligned}$$

is a bijection. In other words, every irreducible subset of T has a unique generic point.

Example 4.1.16 (Hausdorff Spaces are Sober). Therefore, Hausdorff spaces are sober, because their irreducible closed subsets are all single points.

Example 4.1.17 (Sober vs T_0 and T_1). Sober spaces are all T_0 spaces, but sober spaces and T_1 spaces do not imply each other. For example, the Sierpiński space is sober but not T_1, while an infinite set with the cofinite topology is T_1 but not sober.

***Remark* 4.1.18** (Sober is a Local Property). Being sober is a local property. That is, if T has an open cover by sober spaces, then T itself is also sober.

4.1.3. Locale of Radical Ideals

Recall that for a ring R, we previously proved a categorical equivalence:

$$\mathsf{Rad}_R \simeq \mathsf{Open}(\mathrm{Spec}(R)).$$

In this section, we show that it is a locale. For a family of radical ideals $(K_i)_{i\in I}$, their supremum $\bigvee_{i\in I} K_i$ is $\sqrt{\sum_{i\in I} K_i}$. The distributive law can then be written as:

$$K \cap \sqrt{\sum_{i\in I} L_i} = \sqrt{\sum_{i\in I}(K \cap L_i)},$$

because for radical ideals K and L, in $\mathsf{Open}(\mathrm{Spec}(R))$ we have $K \cap L = \sqrt{KL}$, and multiplication and sum of ideals satisfy the distributive law. However, Id_R generally cannot form a locale, because intersections and sums of ideals generally do not satisfy the distributive law.

The goal of this section is to show that Rad_R is a spatial locale and to describe its points.

Definition 4.1.19 (Prime Spectrum). Let R be a ring. The *prime spectrum* of R is the topological space $\mathrm{Prim}(R)$ defined as follows:

- The elements of $\mathrm{Prim}(R)$ are prime ideals of R.
- The open sets of $\mathrm{Prim}(R)$ have the form $\mathrm{Prim}(I) = \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$.

Since prime ideals are automatically radical ideals, $\mathrm{Prim}(I)$ depends only on $\sqrt{(I)}$.

Next, we show that $\mathrm{Prim}(R)$ is precisely $\mathrm{Pt}(\mathsf{Rad}_R)$, and use this to show that Rad_R is a spatial locale.

Proposition 4.1.20 (Locale of Rings). *Let R be a ring.*

(i) There exists a homeomorphism:

$$\mathrm{Prim}(R) \xrightarrow{\sim} \mathrm{Pt}(\mathsf{Rad}_R), \quad \mathfrak{p} \mapsto \{I \in \mathsf{Rad}_R \mid I \not\subset \mathfrak{p}\},$$

where the open subset $\mathrm{Pt}(I)$ corresponds to the open subset $\mathrm{Prim}(I)$.

(ii) The locale Rad_R is spatial.

(iii) The topological space $\mathrm{Prim}(R)$ is a spectral space.

Proof. (i) Let $\mathfrak{p}$ be a prime ideal and $F_\mathfrak{p} = \{I \in \mathsf{Rad}_R \mid I \not\subset \mathfrak{p}\}$. We need to show $F_\mathfrak{p} \in \mathrm{Pt}(\mathsf{Rad}_R)$. In other words, we need to show that $F_\mathfrak{p}$ is a completely prime filter.

- *Upward closure:* For $I \in F_\mathfrak{p}$ and $I \le J$, we have $I \subset J$. Since $I \not\subset \mathfrak{p}$ means there exists at least one i such that $i \notin \mathfrak{p}$, we have $J \not\subset \mathfrak{p}$.
- *Filter property:* For $I, J \in F_\mathfrak{p}$, we need to show $I \cap J \in F_\mathfrak{p}$. Note that $I \cap J = \sqrt{IJ}$. For elements $i \in I$ with $i \notin \mathfrak{p}$ and $j \in J$ with $j \notin \mathfrak{p}$, by the definition of prime ideals, we have $ij \notin \mathfrak{p}$ and $ij \in I \cap J$.
- *Completely prime:* For $\bigvee_{i\in I} K_i \in F_\mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal, this is equivalent to saying $\sum_{i\in I} K_i \not\subset \mathfrak{p}$. Therefore, there exists K_i such that $K_i \in F_\mathfrak{p}$.

Therefore, $\mathrm{Prim}(R) \to \mathrm{Pt}(\mathsf{Rad}_R)$ is well-defined. Now we show it is a bijection by constructing its inverse. Consider the map:

$$\mathrm{Pt}(\mathsf{Rad}_R) \to \mathrm{Prim}(R), \quad F \mapsto \bigcup\{J \in \mathsf{Rad}_R \mid J \notin F\},$$

where F is a completely prime filter. Let $\mathfrak{q}_F = \bigcup\{J \in \mathsf{Rad}_R \mid J \notin F\}$. We show it is a prime ideal:

- *Ideal:* Since F is a completely prime filter, it is upward closed, so its complement F^c is downward closed. Since F preserves finite meets, F^c preserves finite joins. Therefore, elements of F^c form an ideal (in the lattice Rad_R). Hence their union $\mathfrak{q}_F$ is still a radical ideal of R.

- *Prime:* Let $a, b \in R$ with $ab \in \mathfrak{q}_F$. This means $\sqrt{(ab)} \subset \mathfrak{q}_F$. By the construction of $\mathfrak{q}_F$, this means $\sqrt{(ab)} \notin F$ (otherwise, if $\sqrt{(ab)} \in F$, since F is upward closed and $\sqrt{(ab)} \subset \mathfrak{q}_F$, we would have $\mathfrak{q}_F \in F$, but $\mathfrak{q}_F$ is generated by elements not in F; by the complete primality of F, we know $\mathfrak{q}_F \notin F$, contradiction).

  Since $\sqrt{(ab)} = \sqrt{(a)} \cap \sqrt{(b)}$ and F is a filter (closed under finite meets), we have:

$$\sqrt{(a)} \cap \sqrt{(b)} \notin F \implies \sqrt{(a)} \notin F \text{ or } \sqrt{(b)} \notin F.$$

  This means $\sqrt{(a)} \subset \mathfrak{q}_F$ or $\sqrt{(b)} \subset \mathfrak{q}_F$, i.e., $a \in \mathfrak{q}_F$ or $b \in \mathfrak{q}_F$.

Now we only need to verify that these are inverses:

- Show $F_{\mathfrak{q}_F} = F$: Note that $F_{\mathfrak{q}_F} = \{I \in \mathsf{Rad}_R \mid I \not\subset \mathfrak{q}_F\} = F$.
- Show $\mathfrak{q}_{F_\mathfrak{p}} = \mathfrak{p}$: Use that radical ideals can be viewed as intersections of all prime ideals containing them.

(ii) Next, we show $I \mapsto \mathrm{Prim}(I)$ is a bijection. Surjectivity is obvious. If $\mathrm{Prim}(I) = \mathrm{Prim}(J)$, then since $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$, we have $I = J$.

(iii) Finally, we show $\mathrm{Prim}(R)$ is a spectral space. That is, we need to show $\mathrm{Prim}(R)$ is sober and quasi-compact, and has a basis $\mathcal{B}$ consisting of quasi-compact open sets that is closed under finite intersections.

Note that $\mathrm{Prim}(R)$ automatically has a basis consisting of $\mathrm{Prim}(f)$ for $f \in R$. For any $I \subset R$, we have $\mathrm{Prim}(I) = \bigcup_{f \in I} \mathrm{Prim}(f)$.

- $\mathrm{Prim}(I) = \mathrm{Prim}(R)$ is equivalent to saying $\sqrt{(I)} = R$. We know there always exist finitely many elements to achieve this, meaning $\mathrm{Prim}(R)$ is quasi-compact.
- $\mathrm{Prim}(f) = \mathrm{Prim}(R_f)$, hence quasi-compact.
- $\mathrm{Prim}(f) \cap \mathrm{Prim}(g) = \mathrm{Prim}(fg)$, hence closed under finite intersections.

■

Graded Ring Version

Now we discuss the case for $\mathbb{N}$-graded rings. Recall that for an $\mathbb{N}$-graded ring A, we have a categorical equivalence:

$$\mathsf{Open}(\mathrm{Proj}(A)) \simeq \mathsf{hRad}_A,$$

where hRad_A denotes the poset (viewed as a category) of saturated homogeneous radical ideals of A. Since Rad_A is a locale and saturation commutes with finite meets, we can deduce that hRad_A is also a locale. Next, we describe the topological space corresponding to hRad_A.

Definition 4.1.21 (Homogeneous Prime Spectrum). Let A be an $\mathbb{N}$-graded ring. The *homogeneous prime spectrum* $\mathrm{hPrim}(A)$ of A is the topological space defined as follows:

- The elements of $\mathrm{hPrim}(A)$ are saturated homogeneous prime ideals of A.
- The open sets of $\mathrm{hPrim}(A)$ have the form $\mathrm{hPrim}(I) = \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$.

Note that $\mathrm{hPrim}(I)$ depends only on $\sqrt{(I)}^{\mathrm{sat}}$.

Proposition 4.1.22 (Properties of Homogeneous Spectrum). *Let A be an $\mathbb{N}$-graded ring.*

(i) There exists a homeomorphism:

$$\mathrm{hPrim}(A) \xrightarrow{\sim} \mathrm{Pt}(\mathsf{hRad}_A), \quad \mathfrak{p} \mapsto \{I \in \mathsf{hRad}_A \mid I \not\subset \mathfrak{p}\},$$

such that the open subset $\mathrm{hPrim}(I)$ (i.e., $D_+(I)$) corresponds to $\mathrm{Pt}(I)$.

(ii) The locale hRad_A is spatial.

(iii) The topological space $\mathrm{hPrim}(A)$ is sober and has a basis of quasi-compact open sets closed under binary intersections. In particular, when A is a finitely generated A_0-algebra, $\mathrm{hPrim}(A)$ is a spectral space.

Proof. We mainly prove conclusion (3).

For any homogeneous element $f \in A_+$ (positive degree term), consider the open subset $\mathrm{Prim}(f) = \{\mathfrak{p} \in \mathrm{hPrim}(A) \mid f \notin \mathfrak{p}\}$. We claim there exists a homeomorphism:

$$\mathrm{Prim}(f) \simeq \mathrm{Prim}((A_f)_0).$$

Here $(A_f)_0$ denotes the degree-0 component of the localized ring A_f. Proving this amounts to constructing the inverse map Ψ of the above map.

1. Constructing the inverse Ψ:

Let $d = \deg(f)$. For any prime ideal $\mathfrak{q} \in \mathrm{Prim}((A_f)_0)$, we construct a homogeneous ideal $\Psi(\mathfrak{q})$ in A by defining its graded components:

$$\Psi(\mathfrak{q})_n := \left\{ a \in A_n \,\middle|\, \frac{a^d}{f^n} \in \mathfrak{q} \right\}.$$

2. Verifying well-definedness of Ψ:

(i) $\Psi(\mathfrak{q})$ *is a homogeneous ideal:*

- *Additivity:* Let $a, b \in \Psi(\mathfrak{q})_n$, meaning $a^d/f^n \in \mathfrak{q}$ and $b^d/f^n \in \mathfrak{q}$ in $(A_f)_0$. Consider the binomial expansion of $(a+b)^{2d}$. Each term contains a factor $a^k b^{2d-k}$. By the pigeonhole principle, either $k \geq d$ or $2d - k \geq d$, so each term is divisible by a^d or b^d. This means $\frac{(a+b)^{2d}}{f^{2n}}$ is in the ideal $\mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal (hence also a radical ideal), we have $\frac{(a+b)^d}{f^n} \in \mathfrak{q}$. Therefore, $a + b \in \Psi(\mathfrak{q})_n$.
- *Absorption:* Let $a \in \Psi(\mathfrak{q})_n$ and $u \in A_m$. Then:

$$\frac{(ua)^d}{f^{n+m}} = \frac{u^d}{f^m} \cdot \frac{a^d}{f^n}.$$

  Since the second factor is in $\mathfrak{q}$, the product is in $\mathfrak{q}$. Hence $ua \in \Psi(\mathfrak{q})_{n+m}$.

(ii) $\Psi(\mathfrak{q})$ *is a prime ideal:* Let $x \in A_n, y \in A_m$ with $xy \in \Psi(\mathfrak{q})$. This is equivalent to $\frac{(xy)^d}{f^{n+m}} = \frac{x^d}{f^n} \cdot \frac{y^d}{f^m} \in \mathfrak{q}$. By primality of $\mathfrak{q}$, either $\frac{x^d}{f^n} \in \mathfrak{q}$ or $\frac{y^d}{f^m} \in \mathfrak{q}$. That is, $x \in \Psi(\mathfrak{q})$ or $y \in \Psi(\mathfrak{q})$.

(iii) $f \notin \Psi(\mathfrak{q})$*:* Check degree d: $f \in \Psi(\mathfrak{q})_d \iff \frac{f^d}{f^d} = 1 \in \mathfrak{q}$. But this contradicts $\mathfrak{q}$ being a prime ideal. Hence $f \notin \Psi(\mathfrak{q})$.

Therefore, Ψ is a well-defined map. By standard results on graded rings, Ψ and the degree-restriction map $\mathfrak{p} \mapsto (\mathfrak{p}A_f)_0$ are inverses.

3. Verifying homeomorphism:

Consider a standard basic open set $\mathrm{Prim}(g)$ in $\mathrm{Prim}((A_f)_0)$, where $g = a/f^k \in (A_f)_0$ and $a \in A_{kd}$. Its preimage under Ψ is:

$$\Psi^{-1}(\mathrm{Prim}(g)) = \{\mathfrak{q} \mid g \notin \mathfrak{q}\} = \left\{ \mathfrak{q} \,\middle|\, \frac{a^d}{f^{kd}} \notin \mathfrak{q} \right\} \quad \text{(using radical ideal property).}$$

By the definition of Ψ, this is equivalent to:

$$\Psi(\mathfrak{q}) \in \mathrm{Prim}(a) \iff a \notin \Psi(\mathfrak{q}) \iff \frac{a^d}{f^{\deg a}} \notin \mathfrak{q}.$$

This shows that open sets in $\mathrm{Prim}((A_f)_0)$ correspond to open sets in $\mathrm{hPrim}(A)$. Therefore, Ψ is a homeomorphism.

4. Conclusion:

Thus, $\{\mathrm{Prim}(f) \mid f \in A_+\}$ forms a basis for $\mathrm{hPrim}(A)$.

- *Closure under intersections:* $\mathrm{Prim}(f) \cap \mathrm{Prim}(g) = \mathrm{Prim}(fg)$.
- *Quasi-compactness:* Since $\mathrm{Prim}(f) \simeq \mathrm{Prim}((A_f)_0)$, it is an affine scheme, hence a quasi-compact open set.

- If A is a finitely generated A_0-algebra, then A_+ can be generated by finitely many elements $f_1, \dots, f_n$ (i.e., $A_+ \subset \sqrt{(f_1, \dots, f_n)}$). This means the space can be covered by finitely many quasi-compact open sets $\mathrm{Prim}(f_i)$, hence is itself quasi-compact.

This proves it is a spectral space. ∎

4.2. Geometric Realization of Algebraic Functors

Now we consider a general algebraic functor X and the category $\mathsf{Open}(X)$ of its open subfunctors (which is a poset category). In the previous classification theorem, we showed there exists a categorical equivalence:

$$\begin{tikzcd} \mathsf{Rad}_X \arrow[r, "\sim"] \arrow[d, "\simeq"'] & \mathsf{Open}(X) \arrow[d, "\simeq"] \\ \lim\limits_{\mathrm{El}(X)^{\mathrm{op}}} \mathsf{Rad}_R \arrow[r] & \lim\limits_{\mathrm{El}(X)^{\mathrm{op}}} \mathsf{Open}(\mathrm{Spec}(R)). \end{tikzcd}$$

Note that the above limit is taken in the category $\mathsf{Pos} \subset \mathsf{Cat}$. Recall the locale category Loc, where we have $\mathsf{Rad}_R \in \mathsf{Loc}$. Since the embedding $\mathsf{Loc} \hookrightarrow \mathsf{Pos}^{\mathrm{op}}$ preserves colimits (i.e., colimits in Loc correspond to limits in Pos), the above limit process actually corresponds to a colimit in the locale category.

We can extend the open set functor from the affine scheme category to the general algebraic functor category via left Kan extension:

$$\begin{tikzcd} \mathsf{Aff} \arrow[d] \arrow[dr, "\mathsf{Open}"] & \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \arrow[r, "\mathsf{Open}"'] & \mathsf{Loc} \end{tikzcd}$$

Since left Kan extension preserves colimits, the resulting functor $\mathsf{Open}\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Loc}$ preserves colimits.

Thus, we immediately obtain:

Corollary 4.2.1 (Open Subfunctors Form Spatial Locales). *For any algebraic functor X, $\mathsf{Open}(X)$ is a spatial locale.*

***Remark* 4.2.2** (Properties of the Open Set Locale). For an algebraic functor X, $\mathsf{Open}(X)$ as a locale has the following properties:

- *Terminal object:* $D(1) = X$ (the entire functor).
- *Initial object:* $D(0) = \varnothing_X$, where $\varnothing_X$ is defined by: $\varnothing_X(0) = X(0)$ (a singleton over the zero ring), and for non-zero rings $R \neq 0$, we have $\varnothing_X(R) = \varnothing$.
- *Meet operation:* For any $U, V \in \mathsf{Open}(X)$, i.e., two open subfunctors, their "intersection" in the locale is the functorial intersection, i.e., $U \wedge V = U \cap V$.

In this section, our core goal is to find a topological space $|X|$ such that its locale of open sets is isomorphic to the locale of open subfunctors of X, i.e., $\mathrm{Open}(|X|) \simeq \mathsf{Open}(X)$. Precisely, we attempt to construct the dashed functor $|-|$ in the following diagram, making it compatible with the topological space category and the locale category:

$$\begin{tikzcd} \mathsf{Aff} \arrow[d, hook, "\mathrm{Spec}"'] \arrow[dr, "\mathrm{Prim}"] & \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \arrow[r, dashed, "|-|"] \arrow[dr, "\mathsf{Open}"'] & \mathsf{Top} \arrow[d, "\mathrm{Open}"] \\ & \mathsf{Loc}^{\mathrm{spa}} \end{tikzcd}$$

Since Top is cocomplete and the category of algebraic functors (presheaf category) is generated by representable functors, by the universal property, this functor $|-|\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Top}$ is actually the *left Kan extension* of the functor Prim along Spec.

It is easy to verify that for any algebraic functor X, the above construction satisfies:

$$\mathsf{Open}(X) \simeq \mathrm{Open}(|X|),$$

and the soberization of the geometric realization $|X|$ corresponds precisely to the point space of the locale $\mathsf{Open}(X)$:

$$|X|^{\mathrm{sob}} \simeq \mathrm{Pt}(\mathsf{Open}(X)).$$

Next, we attempt to give an explicit characterization of $|X|$. First, recall that for a category C, we say C is *connected* if for any $X, Y \in C$, there exists $n \geq 1$ such that:

$$X = i_0 \leftarrow i_1 \rightarrow i_2 \leftarrow \cdots \rightarrow i_{2n} = Y.$$

Proposition 4.2.3 (Explicit Description of Geometric Realization). *Let $\mathsf{Field}_X \subset \mathrm{El}(X)^{\mathrm{op}}$ be the full subcategory spanned by pairs (k, x), where k is a field and $x\colon \mathrm{Spec}(k) \to X$ is a k-point of X. Then the map $(k, x) \mapsto (|x|\colon |\mathrm{Spec}(k)| \to |X|)$ defines a bijection:*

$$\{\textit{connected components of } \mathsf{Field}_X\} \xrightarrow{\sim} |X|.$$

Proof. First, consider the case where X is an affine scheme. Let $X \simeq \mathrm{Spec}(R)$. We know that:

$$\mathsf{Field}_R \simeq \coprod_{\mathfrak{p} \in \mathrm{Prim}(R)} \mathsf{Field}_{\kappa(\mathfrak{p})}.$$

Therefore, we can deduce there exists a bijection:

$$\begin{aligned} \{\text{connected components of } \mathsf{Field}_R\} &\to \mathrm{Prim}(R) \\ (R \to k) &\mapsto \ker(R \to k), \end{aligned}$$

with inverse given by $\mathfrak{p} \mapsto (R \to \kappa(\mathfrak{p}))$.

Now let X be a general algebraic functor. Since $X \simeq \mathrm{colim}_{\mathrm{El}(X)} \mathrm{Spec}(R)$, we have $|X| \simeq \mathrm{colim}_{\mathrm{El}(X)} \mathrm{Prim}(R)$. Everything reduces to showing there exists a bijection:

$$\mathrm{colim}_{(R,x) \in \mathrm{El}(X)}\{\text{connected components of } \mathsf{Field}_R\} \xrightarrow{\sim} \{\text{connected components of } \mathsf{Field}_X\}.$$

Surjectivity: For any object (k, x) in Field_X, i.e., $x\colon \mathrm{Spec}(k) \to X$, since X is a colimit of affine schemes, this map must factor as $\mathrm{Spec}(k) \to \mathrm{Spec}(R) \xrightarrow{x'} X$. This means the connected component comes from some Field_R, so the map is surjective.

Injectivity: We need to prove that elements identified in the left-side colimit also lie in the same connected component on the right side. This reduces to checking compatibility of morphisms. Consider a morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ in $\mathrm{El}(X)$. This induces a functor $\mathsf{Field}_S \to \mathsf{Field}_R$ (covariant, since Field_X is a subcategory of $\mathrm{El}(X)^{\mathrm{op}}$).

If two points (k_1, x_1) and (k_2, x_2) in Field_X belong to the same connected component, this means there exists a series of morphisms between them. The most crucial step is handling the case of spans: Suppose in CAlg we have a diagram $k_1 \leftarrow k_0 \rightarrow k_2$ (corresponding to $x_1 \rightarrow x_0 \leftarrow x_2$ in Field_X). Consider the pushout (tensor product):

$$\begin{tikzcd} k_0 \arrow[r] \arrow[d] & k_1 \arrow[d] \\ k_2 \arrow[r] & k_1 \otimes_{k_0} k_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Although $A = k_1 \otimes_{k_0} k_2$ may not be a field, since k_i are all fields, this tensor product is non-zero. Therefore, there exists a prime ideal $\mathfrak{p}$ of A and a residue field $K = \kappa(\mathfrak{p})$ such that we have a composite map $A \to K$. This induces the following commutative diagram:

$$\begin{tikzcd} k_0 \arrow[r] \arrow[d] & k_1 \arrow[d] \\ k_2 \arrow[r] & K \end{tikzcd}$$

In the category Field_X, this means there exists an object (K, x_K) and morphisms $(K, x_K) \to (k_1, x_1)$ and $(K, x_K) \to (k_2, x_2)$. This shows that x_1 and x_2 are in the same connected component via x_K.

This construction shows that: any gluing between $\mathrm{Prim}(R)$ given by diagram relations in $\mathrm{El}(X)$ corresponds to connectivity of objects in Field_X. Specifically, the equivalence relation in $\mathrm{colim}_{\mathrm{El}(X)}$ is precisely generated by such zigzag paths. Since tensor products of fields always have non-trivial residue fields, Field_X has enough morphisms to realize these equivalence relations. Therefore, the map is injective. ∎

***Remark* 4.2.4** (Two Meanings of "Point"). In algebraic geometry, "point" has two different meanings:

- For any ring R, an "R-point" is an element of $X(R)$. We know it gives the element category $\mathrm{El}(X)$.
- A point of X generally refers to a point of $|X|$, i.e., an equivalence class of field-valued points of X, which gives the topological space $|X|$.

Therefore, we obtain the following relationship:

$$\mathrm{El}(X)^{\mathrm{op}} \hookleftarrow \mathsf{Field}_X \twoheadrightarrow |X|.$$

It is worth noting that the $|-|$ functor generally does not preserve limits. For example, we have a pullback diagram:

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{C}\times\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(\mathbb{C}) \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{R}) \end{tikzcd}$$

But after applying $|-|$, we get the diagram:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

We know this diagram is clearly not a pullback. Next, we explore when the above diagram is indeed a pullback.

Lemma 4.2.5 (Pullback Preservation). *Consider a diagram in* $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$*:*

$$Y \to X \leftarrow Z.$$

Then there is a canonical map:

$$|Y \times_X Z| \to |Y| \times_{|X|} |Z|.$$

This map is surjective, and is a bijection when $Y \to X$ is a monomorphism.

Proof. Surjectivity: For $y\colon \mathrm{Spec}(k) \to Y$ and $z\colon \mathrm{Spec}(k') \to Z$ such that they map to the same point in $|X|$, i.e., given $(y, z) \in |Y| \times_{|X|} |Z|$, we need to find $\mathrm{Spec}(k'') \to Y \times_X Z$ corresponding to it. We know that y and z mapping to the same point in $|X|$ is equivalent to there existing a field k'' such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{Spec}(k) \arrow[r] & Y \arrow[dr] & \\ \mathrm{Spec}(k'') \arrow[ur] \arrow[dr] & & & X \\ & \mathrm{Spec}(k') \arrow[r] & Z \arrow[ur] & \end{tikzcd}$$

This is equivalent to saying there is a canonical morphism:

$$\mathrm{Spec}(k'') \to Y \times_X Z.$$

Therefore, the map is surjective.

Bijection when $Y \hookrightarrow X$ is a monomorphism: Since monomorphisms are closed under pullback, we have $Y \times_X Z \hookrightarrow Z$ is also a monomorphism. Recall we have a commutative diagram:

$$\begin{tikzcd} {|Y \times_X Z|} \arrow[rr, hook] \arrow[dr] & & {|Z|} \\ & {|Y| \times_{|X|} |Z|} \arrow[ur] & \end{tikzcd}$$

We only need to show $|Y| \times_{|X|} |Z| \to |Z|$ is a monomorphism. Note this is equivalent to saying $|Y| \hookrightarrow |X|$ is injective. This is equivalent to:

$$|Y| \xrightarrow{\sim} |Y| \times_{|X|} |Y|$$

being an isomorphism, which follows immediately from $Y \to X$ being a monomorphism. ■

Proposition 4.2.6 (Geometric Realization of Embeddings). *Let X be an algebraic functor, and $Y \hookrightarrow X$ an open embedding/closed embedding/embedding. Then it induces an open embedding/closed embedding/embedding $|Y| \hookrightarrow |X|$.*

Proof. We only prove the case of closed embeddings. Let $Y \to X$ be a closed embedding. This is equivalent to saying that for any R-point $x\colon \operatorname{Spec}(R) \to X$, considering the pullback:

$$\begin{tikzcd} Y_R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

there exists $I \subset R$ such that $Y_R \simeq \operatorname{Spec}(R/I)$. Therefore, $|Y_R| \simeq \operatorname{Prim}(R/I) \hookrightarrow \operatorname{Prim}(R)$. By Lemma 4.2.5 and the fact that $Y \hookrightarrow X$ is a closed embedding, we can deduce that after geometric realization, it becomes a pullback diagram in Set:

$$\begin{tikzcd} {|Y_R|} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Prim}(R) \arrow[d] \\ {|Y|} \arrow[r, hook] & {|X|} \end{tikzcd}$$

Now everything reduces to showing $|Y| \hookrightarrow |X|$ is a closed map. For a closed subset $Z \subset |Y|$, we have for any ring R, $Z_R \subset |Y_R|$ is a closed subset. Therefore, $Z_R \subset \operatorname{Prim}(R)$ is a closed subset. This is equivalent to saying $Z \subset |X|$ is a closed subset (this is because the topology on $|X|$ is the colimit topology, so being closed when pulled back to each $\operatorname{Prim}(R)$ means being closed in $|X|$). ■

***Remark* 4.2.7** (Closed Subfunctors vs Closed Subspaces). From the above proposition, we can deduce there exists a map of posets:

$$\mathsf{Closed}(X) \to \operatorname{Closed}(|X|), \quad Z \mapsto |Z|.$$

But this map is almost never a bijection. However, if $X = \operatorname{Spec}(A)$ or $X = \operatorname{Proj}(A)$, then it is surjective: a preimage of a closed subset in $|X|$ is given by the vanishing locus of the radical ideal in A corresponding to the open complement of that closed set.

In general, if $Z \subset X$ is a closed subfunctor with open complement U, then $|Z| = |X| \setminus |U|$.

Next, we study some topological properties of geometric realizations of algebraic functors.

Definition 4.2.8 (Topological Properties of Algebraic Functors). Let P be a property of topological spaces (such as connected, locally connected, irreducible, discrete). We say an algebraic functor X has property P if $|X|$ has property P.

Example 4.2.9 (Properties of Affine Schemes). Consider an affine scheme $X \simeq \operatorname{Spec}(R)$. Then we know $|X| \simeq \operatorname{Prim}(R)$. We can immediately see that it satisfies the following properties:

(i) X is connected if and only if R has exactly two idempotents.

(ii) X is irreducible if and only if $\sqrt{(0)} \subset R$ is a prime ideal, i.e., $R/\sqrt{(0)}$ is an integral domain.

(iii) X is discrete if and only if R is an Artinian ring.

Similarly, we can study some topological properties of maps between algebraic functors.

Definition 4.2.10 (Topological Properties of Morphisms)**.** Let P be a property of continuous maps of topological spaces (such as surjective, injective, bijective, dominant, submersive, open, closed, homeomorphism).

- We say a morphism $f: Y \to X$ of algebraic functors is P if $|f|$ has property P.
- We say a morphism $f: Y \to X$ of algebraic functors is *universally* P if for any $X' \to X$, the base change $Y \times_X X' \to X'$ is P.

Proposition 4.2.11 (Universal Properties)**.** *Let $f: Y \to X$ be a morphism of algebraic functors.*

(i) Let $P \in \{$surjective, injective, bijective, dominant, submersive, open, closed, homeomorphism$\}$. Then f is universally P if and only if for any ring R and any R-point $x\colon \operatorname{Spec}(R) \to X$, the map $Y \times_X \operatorname{Spec}(R) \to \operatorname{Spec}(R)$ has property P.

(ii) f is universally surjective if and only if it is surjective.

(iii) If f is a monomorphism, then it is also universally injective.

(iv) If f is Zariski locally surjective, then it is also universally submersive. In particular, it is also universally surjective.

Proof. (i) We only prove the case P = submersive. Recall that a continuous map $\varphi\colon T \to S$ of topological spaces is called *submersive* if φ is surjective and S has the quotient topology.

For each ring R and each R-point $x\colon \operatorname{Spec}(R) \to X$, consider the pullback:

$$\begin{tikzcd} Y_R \arrow[r, "f_R"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R) \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

- If for each R, we have f_R is surjective, we need to show f is surjective. Since we know $X \simeq \operatorname{colim}_{\mathrm{El}(X)} \operatorname{Spec}(R)$, we take the colimit of the above pullback indexed by $\mathrm{El}(X)$ to get:

$$\begin{tikzcd} \operatorname{colim}_{(R,x)\in\mathrm{El}(X)} Y_R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim}_{(R,x)\in\mathrm{El}(X)} \operatorname{Spec}(R) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

  Since the right vertical arrow is an isomorphism, we have an isomorphism $Y \simeq \operatorname{colim}_{(R,x)\in\mathrm{El}(X)} Y_R$. Since the geometric realization functor preserves colimits, we have:

$$|Y| \simeq \operatorname{colim}_{(R,x)\in\mathrm{El}(X)} |Y_R|.$$

  Note that for each point $x \in |X|$, it comes from some $|\operatorname{Spec}(R)|$. Since f_R is surjective, it also comes from some $|Y_R|$. Therefore, $|Y| \to |X|$ is surjective.
- Next, suppose f_R is submersive, meaning we need to show $|X|$ has the quotient topology. Let $U \subset X$ be such that $|f|^{-1}(U) \subset |Y|$ is an open set. We need to show U is open in $|X|$. After writing $|X|$ as a colimit of $\operatorname{Prim}(R)$, we see this is equivalent to saying it is open in each $|\operatorname{Spec}(R)|$. Therefore, it reduces to f_R being submersive.

(ii) Use that the pullback of a surjection is a surjection together with Lemma 4.2.5.

(iii) Use that the pullback of an injection is an injection together with Lemma 4.2.5.

(iv) By (1), we may assume $X = \operatorname{Spec}(R)$. Then f being Zariski locally surjective is equivalent to saying there exist $(f_1, \dots, f_n) = R$ such that the following dashed lifts exist:

$$\begin{tikzcd} & Y \arrow[d] \\ \operatorname{Spec}(R_{f_i}) \arrow[r, hook] \arrow[ur, dashed] & \operatorname{Spec}(R) \end{tikzcd}$$

Note that $\mathrm{Prim}(R) = \bigcup_{i=1}^{n} \mathrm{Prim}(R_{f_i})$, so $|f|$ is surjective.

Next, we show it is submersive. Take $U \subset \mathrm{Prim}(R)$ such that $|f|^{-1}(U) \subset |Y|$ is an open subset. It is easy to see that $U \cap \mathrm{Prim}(R_{f_i})$ is also open, so U is open.

■

Example 4.2.12 (Non-universal Properties). In fact, among the many properties mentioned in Proposition 4.2.11, except for surjectivity, most do not automatically possess universal properties.

(i) *Homeomorphisms are not universal:* Consider $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$. Topologically, this is a map from a single point to a single point, obviously a homeomorphism. But it is not a universal homeomorphism, because it is not even a universal injection. Note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$. If we consider its base change along itself (i.e., pullback), we get:

$$\mathrm{Spec}(\mathbb{C} \times \mathbb{C}) \to \mathrm{Spec}(\mathbb{C}).$$

Topologically, this corresponds to two points mapping to one point $(* \sqcup * \to *)$, obviously no longer a homeomorphism.

(ii) *Closed maps are not universal:* For a field k, the topological space $|\mathrm{Spec}(k)| = *$ contains only one point. Therefore, for any map $X \to \mathrm{Spec}(k)$, it is automatically a closed map. However, consider the affine line $\mathbb{A}^1_k \to \mathrm{Spec}(k)$. It is not a universal closed map. Consider its pullback along itself, i.e., the projection map $p\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$. Take the closed set $Z = V(xy-1) \subset \mathbb{A}^2_k$ (hyperbola). Its image under p is $\mathbb{A}^1_k \setminus \{0\}$, which is an open set, not a closed set.

(iii) *Open maps are not universal:* Consider the normalization map of the nodal cubic:

$$f\colon \mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x,y]/(y^2 - x^2 - x^3)),$$

given by $x \mapsto t^2-1$, $y \mapsto t(t^2-1)$ (where k is a field of characteristic not 2). This is a finite morphism, and topologically it is an open map (since its image is a dense open set and dimensions are equal). But since this map is not flat, it can be shown that it is not a universal open map.

Corollary 4.2.13 (Zariski Codescent). *Let $(Y_i \to X)_{i \in I}$ be a family of morphisms such that $\coprod_{i\in I} Y_i \to X$ is Zariski locally surjective. Then the following diagram is a coequalizer diagram in* Top*:*

$$\coprod_{i,j\in I} |Y_i \times_X Y_j| \rightrightarrows \coprod_{i\in I} |Y_i| \xrightarrow{q} |X|$$

Proof. Let $Y = \coprod_{i\in I} Y_i$. By the conclusion of Proposition 4.2.11, since $Y \to X$ induces a submersion $q\colon |Y| \to |X|$ in the topological space category.

In Top, a submersion $q\colon |Y| \to |X|$ is equivalent to saying $|X|$ is the quotient space of $|Y|$ with respect to the equivalence relation defined by q. In other words, $|X|$ is the colimit of the following coequalizer diagram:

$$|Y| \times_{|X|} |Y| \rightrightarrows |Y| \xrightarrow{q} |X|$$

where $|Y| \times_{|X|} |Y|$ is the fiber product in the topological space category.

On the other hand, consider the fiber product at the algebraic geometry level $Y \times_X Y = \coprod_{i,j} Y_i \times_X Y_j$. By the properties of the functor $|-|$, there exists a canonical continuous map:

$$\phi\colon |Y \times_X Y| \to |Y| \times_{|X|} |Y|.$$

By Lemma 4.2.5, this map ϕ is surjective.

Now we use a general fact from point-set topology: If Q is the quotient space of X modulo an equivalence relation $R \subset X \times X$ (i.e., $R \rightrightarrows X \to Q$ is a coequalizer), and there exists a surjection $R' \twoheadrightarrow R$, then Q is also the coequalizer of $R' \rightrightarrows X \to Q$.

In this case, $R = |Y| \times_{|X|} |Y|$ represents the topological equivalence relation defining $|X|$, while $R' = |Y \times_X Y|$ covers this relation via the surjection ϕ. Therefore, the quotient topology structure is completely determined by $|Y \times_X Y|$, i.e.:

$$\coprod_{i,j\in I} |Y_i \times_X Y_j| \rightrightarrows \coprod_{i\in I} |Y_i| \to |X|$$

is a coequalizer diagram in Top. ■

4.2.1. Open Covers

Note that for a topological space T, the notion of open cover is only related to the locale $\mathrm{Open}(T)$: a family of open subsets $(U_i \subset T)_{i \in I}$ forms an open cover of T if their supremum is T. Now we extend this concept to general algebraic functors.

Proposition 4.2.14 (Characterization of Open Covers). *Let X be an algebraic functor, $(U_i \subset X)_{i \in I}$ a family of open subfunctors, and $K_i \in \mathsf{Rad}_X$ a family of quasi-coherent ideals on U_i. Then the following conditions are equivalent:*

(i) *In the poset $\mathsf{Open}(X)$, we have $X = \bigvee_{i \in I} U_i$.*
(ii) *In the poset Rad_X, we have $\mathcal{O}_X = \bigvee_{i \in I} K_i$.*
(iii) *For each ring R and each $x : \mathrm{Spec}(R) \to X$, we have $\bigcup_{i \in I} K_i(x)$ generates R.*
(iv) *For each local ring R, we have $X(R) = \bigcup_{i \in I} U_i(R)$.*
(v) *For each field k, we have $X(k) = \bigcup_{i \in I} U_i(k)$.*
(vi) *$\coprod_{i \in I} U_i \to X$ is Zariski locally surjective.*
(vii) *$\coprod_{i \in I} U_i \to X$ is surjective.*

Proof. Our proof strategy is:

$$\begin{tikzcd}
& 7 & & \\
5 \arrow[ur, Rightarrow] & 1 \arrow[u, Rightarrow] \arrow[r, Rightarrow] & 2 \arrow[d, Rightarrow] & 6 \arrow[ull, Rightarrow] \\
4 \arrow[u, Rightarrow] & & 3 \arrow[ll, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$1 \iff 2$: Immediately from the categorical equivalence $\mathsf{Open}(X) \simeq \mathsf{Rad}_X$.

$2 \iff 3$: Use $\mathsf{Rad}_X = \lim_{\mathrm{El}(X)^{\mathrm{op}}} \mathsf{Rad}_R$ and pointwise computation of suprema as colimits.

$1 \iff 7$: Use $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$.

$3 \Rightarrow 4$: For a local ring R, consider $x : \mathrm{Spec}(R) \to X$. It is easy to see that $\sum_{i \in I} K_i(x) = R$ implies there exists i such that $1 \in K_i(x)$, i.e., $x \in U_i(R)$.

$4 \Rightarrow 5$: Obvious.

$5 \Rightarrow 7$: Obvious.

$6 \Rightarrow 7$: By Proposition 4.2.11.

$3 \Rightarrow 6$: By (3), we have $\sum_{i \in I} K_i(x) = R$. This means there exist finitely many $i_1, \ldots, i_n \in I$ and elements $f_j \in K_{i_j}(x) \subset R$ such that $1 = \sum_{j=1}^{n} f_j$.

Consider $\{R \to R_{f_j}\}_{j=1}^{n}$. For each j, consider the restriction of x to the open subfunctor $D(f_j) \simeq \mathrm{Spec}(R_{f_j})$, denoted $x|_{D(f_j)}$. By the correspondence between U_{i_j} and K_{i_j} (i.e., U_{i_j} consists of points where K_{i_j} generates the unit ideal), since $f_j \in K_{i_j}(x)$ and its image in R_{f_j} is a unit, we have $K_{i_j}(x)R_{f_j} = R_{f_j}$. This precisely means the restriction morphism $x|_{D(f_j)}$ factors as:

$$\mathrm{Spec}(R_{f_j}) \to U_{i_j} \hookrightarrow X.$$

Therefore, it is Zariski locally surjective.

■

Definition 4.2.15 (Open Cover). For an algebraic functor X, we say a family of open subfunctors $(U_i \subset X)_{i \in I}$ is an *open cover* if it satisfies any of the equivalent conditions in Proposition 4.2.14.

This completes our treatment of topology and locales in algebraic geometry. The key results are:

- For any algebraic functor X, $\mathsf{Open}(X)$ is a spatial locale.
- The geometric realization $|X|$ recovers X as a topological space, with $\mathrm{Open}(|X|) \simeq \mathsf{Open}(X)$.

- Points of $|X|$ correspond to equivalence classes of field-valued points of X.
- Open covers are characterized both algebraically (via radical ideals) and topologically (via Zariski local surjectivity).

Chapter 5.

Sheaves

This chapter aims to establish the theory of *Grothendieck topologies* and *sheaves* on them.

For any category C, we are already familiar with its presheaf category $\mathsf{PShv}(C)$. However, the concept of presheaves is too broad and does not capture the "local-to-global" properties of geometric objects. Therefore, we need to select those presheaves that satisfy local-to-global properties (i.e., *descent* properties). These properties are characterized by a Grothendieck topology τ on C. We call presheaves satisfying the τ-descent condition *τ-sheaves*.

In algebraic geometry, we are primarily concerned with the following scenarios:

(i) *The case of topological spaces:* Let $C = \mathrm{Open}(T)$ be the category of open sets (poset) of a topological space T. Endow it with the *canonical Grothendieck topology*. The resulting sheaves are precisely the classical sheaves on T. This concept was originally introduced by Leray.

(ii) *The case of affine schemes:* Let $C = \mathsf{Aff}_k \simeq \mathsf{CAlg}_k^{\mathrm{op}}$. The main topologies we use include:

- *Zariski topology* (i.e., satisfying Zariski descent);
- *Étale topology*;
- *Nisnevich topology*;
- *fppf topology* (fidèlement plat de présentation finie - faithfully flat of finite presentation);
- *fpqc topology* (fidèlement plat quasi-compact - faithfully flat quasi-compact).

(iii) *Extension and generalization:* We can naturally extend the above topologies from Aff_k to the category of k-schemes Sch_k, and even further generalize them to the functor category $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ (i.e., the presheaf category).

5.1. Introduction: Descent and Sheaves

Before we begin, let us roughly introduce our ideas about sheaves.

We know that colimits in categories generally represent "gluing". For example, in the category of topological spaces Top, we use colimits to glue spaces together.

Now consider a category C and its presheaf category $\mathsf{PShv}(C) = \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Set})$. We know that a presheaf is a functor $F\colon C^{\mathrm{op}} \to \mathsf{Set}$. We want to know whether the presheaf F preserves some gluing information in C. For example, consider a diagram $X\colon \mathcal{I} \to C$. We want to understand the relationship between $F(\mathrm{colim}_{i\in\mathcal{I}} X_i)$ and $\lim_{i\in I} F(X_i)$ (note that colimits in C are limits in C^{op}).

Precisely, we want to know whether we have an isomorphism:

$$\mathrm{Hom}_{\mathsf{PShv}(C)}(\mathrm{colim}_{i\in I}\, y(X_i), F) \simeq \mathrm{Hom}_{\mathsf{PShv}(C)}(y(\mathrm{colim}_{i\in I} X_i), F).$$

This isomorphism is essentially asking: from F's perspective, can X_i be glued together to form $\mathrm{colim}\, X_i$? Of course, this does not hold for general presheaves, but we can consider presheaves for which this does hold. We call presheaves satisfying the above isomorphism as *satisfying descent with respect to X*.

In the category of topological spaces, this local-to-global property is entirely controlled by the concept of open covers. Specifically, for a topological space X, we do not need to concern ourselves with the complex gluing data inside it; we only need to focus on a family of open embeddings $\{U_i \to X\}_{i\in I}$ whose union equals X. This family of maps is called an *open cover* of X.

If we wish to discuss locality and globality on a general category C, we need to generalize the geometric intuition of open covers to C. In a general category, open subsets are replaced by general morphisms $U \to X$. Therefore, a cover should be viewed as a family of morphisms with target X.

To handle these families of morphisms more rigorously in category theory (particularly to handle refinements of covers and base changes), we introduce the concept of sieves. They play the role in category theory that open sets play in topological spaces. This is the starting point of Grothendieck topologies.

5.2. Grothendieck Topology

5.2.1. Sieves and Descent

Definition 5.2.1 (Sieves and Pullbacks). Let C be a category and $X \in C$ an object.

(i) *Sieve:* A *sieve* on X is a full subcategory $\mathcal{R} \subset C_{/X}$ of the slice category satisfying: for any morphism $h\colon V' \to V$ in $C_{/X}$ (i.e., a morphism making the diagram commute), if $V \in \mathcal{R}$, then $V' \in \mathcal{R}$. (In other words, sieves are families of morphisms closed under precomposition.)

(ii) *Pullback:* Let $f : X' \to X$ be a morphism in C, and $\mathcal{R}$ a sieve on X. The *pullback* $f^*\mathcal{R}$ of $\mathcal{R}$ along f is a sieve on X' defined by:

$$f^*\mathcal{R} := \{(g : Y \to X') \in C_{/X'} \mid (f \circ g) \in \mathcal{R}\}.$$

***Remark* 5.2.2** (Right Fibration Composition and Subfunctor Viewpoint). Note that *right fibrations are closed under composition*.

We can view a sieve $\mathcal{R}$ on X as the composite of the following maps:

$$\mathcal{R} \hookrightarrow C_{/X} \to C.$$

Where:

(i) The inclusion $\mathcal{R} \hookrightarrow C_{/X}$ is a right fibration. (This follows directly from the definition of a sieve: a sieve is a full subcategory closed under precomposition, which is precisely the right fibration property for full subcategory inclusions.)

(ii) The forgetful functor (domain projection) $C_{/X} \to C$ is also a right fibration (this is a standard property of slice categories).

Therefore, the composite functor $\mathcal{R} \to C$ is still a *right fibration*.

By straightening, there is an equivalence between right fibrations over C and its presheaf category:

$$\mathrm{St}^R : \mathsf{RFib}(C) \xrightarrow{\sim} \mathsf{PShv}(C) = \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{An}).$$

Under this correspondence:

- The right fibration $C_{/X} \to C$ corresponds to the representable functor $y(X)$.
- The right fibration $\mathcal{R} \to C$ corresponds to a *subfunctor* of $y(X)$.

  Furthermore, if we consider the straightening of the inclusion $\mathcal{R} \hookrightarrow C_{/X}$ itself, we get a functor taking values in (-1)-truncated objects:

  $$\chi_{\mathcal{R}} : (C_{/X})^{\mathrm{op}} \to \mathsf{An}_{\leq -1} \simeq [1].$$

  Here we view $[1]$ as the full subcategory consisting of $\{\varnothing, [0]\}$ (i.e., Boolean values in the homotopy sense: false and true). In this viewpoint:

  - Objects belonging to the sieve $\mathcal{R}$ are mapped to $[0]$ (true/one-point space);
  - Objects not belonging to the sieve $\mathcal{R}$ are mapped to $\varnothing$ (false/empty space).

  Functoriality ensures that: if $V \to X$ is mapped to $[0]$ (i.e., in the sieve), then any factorization $V' \to V \to X$ must also be mapped to $[0]$ (i.e., also in the sieve).

This is the higher categorical interpretation of why we usually define sieves as subfunctors of $y(X)$ (i.e., $\mathcal{R}(V) \subset \mathrm{Hom}(V, X)$).

It is not hard to see that pullbacks of sieves along morphisms are precisely pullbacks in the presheaf category $\mathsf{PShv}(C)$. The viewpoint of $\mathcal{R} \subset y(X)$ as a subfunctor is sometimes more useful.

Next, we introduce sieves generated by morphisms.

Definition 5.2.3 (Generated Sieve). Let C be a category and $X \in C$ an object. The sieve *generated* by a family of morphisms $\{f_i : Y_i \to X\}_{i \in I}$ is defined as the image of the natural transformation:

$$\coprod_{i \in I} y(Y_i) \to y(X).$$

Concretely, this sieve $\mathcal{R} \subset C_{/X}$ contains precisely those morphisms that can be factored through some f_i. That is:

$$\mathcal{R} = \{(g : V \to X) \in C_{/X} \mid \exists i \in I, \exists h : V \to Y_i \text{ such that } g = f_i \circ h\}.$$

Next, we introduce the concept of descent along a sieve.

Definition 5.2.4 (Descent Along a Sieve). Let C be a category, $X \in C$ an object, and $\mathcal{R} \subset C_{/X}$ a sieve on X. Let $F : C^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf.

(i) *Descent data:* The *descent data* of F with respect to the sieve $\mathcal{R}$ is defined as the following limit:

$$\mathrm{Desc}(\mathcal{R}, F) := \lim_{(V \to X) \in \mathcal{R}^{\mathrm{op}}} F(V) \simeq \mathrm{Nat}(\mathcal{R}, F).$$

(Note: If we view the sieve $\mathcal{R}$ as a subfunctor of $y(X)$, we can understand this limit as the set of natural transformations from $\mathcal{R}$ to F.)

(ii) *Descent condition:* We say F *satisfies descent along* $\mathcal{R}$ if the canonical map:

$$F(X) \xrightarrow{\sim} \mathrm{Desc}(\mathcal{R}, F)$$

is an isomorphism.

(iii) *Universal descent:* We say F *universally descends* along $\mathcal{R}$ if for any morphism $f : Y \to X$, F satisfies descent along the pullback sieve $f^*\mathcal{R}$.

***Remark* 5.2.5** (Intuitive Picture). Intuitively, we can explain "descent" through the following correspondence diagram:

$$\begin{array}{ccc} \text{Local data} & \mathrm{Desc}(\mathcal{R}, F) & \{Y_i\} \\ & \downarrow \text{descent} & \downarrow \text{Cover} \\ \text{Global data} & F(X) & X \end{array}$$

- On the right is the *geometric* picture: a family of "pieces" $\{Y_i\}$ (belonging to the sieve $\mathcal{R}$) covers X.
- On the left is the *algebraic* picture: $\mathrm{Desc}(\mathcal{R}, F)$ contains all local sections on the pieces (satisfying compatibility).

So-called "descent" means: given a collection of compatible local data (top layer), can we uniquely "glue/descend" to obtain a global section (bottom layer)? If it is an isomorphism, it means local information completely determines global information.

Here are some examples of sieves:

Example 5.2.6 (Empty Sieve). For any $X \in C$, we can consider the empty presheaf, which automatically forms a sieve on X. It is easy to see that F descending along $\varnothing$ is equivalent to saying $F(X)$ is a singleton.

Example 5.2.7 (Sieve Generated by Two Subobjects). Let $X \in C$ and $U, V \subset X$ be subobjects. Let $\mathcal{R}$ be the sieve on X generated by U and V. If $U \cap V$ exists, then $\mathcal{R}$ is actually the pushout in $\mathsf{PShv}(C)$:

$$\begin{tikzcd} y(U \cap V) \arrow[r] \arrow[d] & y(U) \arrow[d] \\ y(V) \arrow[r] & \mathcal{R} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Therefore, a presheaf F on C satisfying descent along $\mathcal{R}$ is equivalent to the following diagram being a pullback:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \cap V) \end{tikzcd}$$

In fact, (in 1-categories) we can interpret the descent condition as an equalizer diagram.

Lemma 5.2.8 (Sieve as Coequalizer). *Let C be a category and $X \in C$ an object. Let $\mathcal{R}$ be the sieve generated by a family $(Y_i \to X)_{i \in I}$. Then the canonical maps $y(Y_i) \to \mathcal{R}$ induce an isomorphism in the presheaf category:*

$$\operatorname{colim}\left(\coprod_{i,j \in I} y(Y_i) \times_{y(X)} y(Y_j) \rightrightarrows \coprod_{i \in I} y(Y_i) \right) \xrightarrow{\sim} \mathcal{R}.$$

Proof. By Definition 5.2.3, the sieve $\mathcal{R}$ is the image of:

$$\coprod_{i \in I} y(Y_i) \to y(X).$$

Recall from category theory that the *coimage* of a morphism $f : A \to B$ is defined as the coequalizer:

$$\operatorname{coim}(f) := \operatorname{colim}\left(A \times_B A \rightrightarrows A\right).$$

Since all morphisms in the presheaf category $\mathsf{PShv}(C)$ are strict (because this is true in Set), this means the image and coimage of a morphism are isomorphic. Substituting $A = \coprod y(Y_i)$ and $B = y(X)$ gives the conclusion. ∎

***Remark* 5.2.9** (Higher Categorical Perspective: Image and Čech Nerve). In higher categories (∞-categories), we directly use the definition of image from Kerodon Tag 04X1.

Let $p \colon \coprod_{i \in I} y(Y_i) \to y(X)$ be the canonical morphism. The sieve $\mathcal{R}$ generated by $\{Y_i \to X\}$ is the image $\operatorname{im}(p)$. In this case, $\operatorname{im}(p)$ is identified with the *geometric realization* of the *Čech nerve* $\check{C}(p)_\bullet$ of p:

$$\mathcal{R} \simeq \operatorname{im}(p) \simeq |\check{C}(p)_\bullet| := \operatorname{colim}_{\Delta^{\mathrm{op}}}\left(\cdots \rightrightarrows U \times_{y(X)} U \times_{y(X)} U \rightrightarrows U \times_{y(X)} U \to U \right).$$

Where $U = \coprod_{i \in I} y(Y_i)$. Since $\mathsf{Fun}(C, \mathsf{An})$ is a topos, colimits in it are universal, i.e., coproducts commute with fiber products. Therefore, we recover the form of Lemma 5.2.8.

Of course, the reader can actually use Quillen's Theorem A to prove cofinality.

Now we apply Lemma 5.2.8 to descent.

Proposition 5.2.10 (Equivalent Forms of Descent). *Let C be a category, $\mathcal{R}$ a sieve on $X \in C$, and $(Y_i \to X)_{i \in I}$ a family of morphisms generating $\mathcal{R}$. Then for any presheaf $F \in \mathsf{PShv}(C)$, the following conditions are equivalent:*

(i) F satisfies descent along $\mathcal{R}$.

(ii) The canonical morphism:

$$F(X) \to \lim_{Y \in \mathrm{El}(\mathcal{R})} F(Y)$$

is an isomorphism.

(iii) There is an equalizer diagram:

$$F(X) \longrightarrow \prod_{i \in I} F(Y_i) \rightrightarrows \prod_{i,j \in I} F(Y_i \times_X Y_j)$$

Proof. Obvious. ∎

Example 5.2.11 (Monogenic Sieve). Let C be a category and $\mathcal{R}$ a sieve on $X \in C$. We say $\mathcal{R}$ is *monogenic* if it is generated by a single morphism $Y \to X$. Then if the pullback $Y \times_X Y$ exists in C, F satisfies descent along $\mathcal{R}$ in $\mathsf{PShv}(C)$ if and only if:

$$F(X) \longrightarrow F(Y) \rightrightarrows F(Y \times_X Y)$$

is an equalizer diagram.

***Remark* 5.2.12** (Descent for Categories). We can now consider descent for category-valued presheaves $F\colon C^{\mathrm{op}} \to \mathsf{Cat}$. Since we actually view Cat as a $(2,1)$-category, by Remark 5.2.9, the information actually needed is:

$$F(X) \longrightarrow \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j) \Rrightarrow \prod_{i,j,k\in I} F(Y_i \times_X Y_j \times_X Y_k)$$

5.2.2. Grothendieck Topologies

To better simulate the concept of open covers, we need to select sieves satisfying certain conditions.

Definition 5.2.13 (Grothendieck Topology). Let C be a category. A *Grothendieck topology* on C (or simply *topology* when we are being lazy) consists of the following data: for each X, we specify a family of sieves $\tau(X)$ on X, called *τ-covering sieves*, satisfying:

(i) *(Pullback stability)* If $\mathcal{R}$ is a τ-covering sieve on X and $f : Y \to X$ is any morphism, then the pullback sieve $f^*\mathcal{R}$ is also a τ-covering sieve on Y.

(ii) *(Local character)* Let $\mathcal{S}$ be a sieve on X and $\mathcal{R}$ a τ-covering sieve on X. If for every $f \in \mathcal{R}$, we have $f^*\mathcal{S}$ is a τ-covering sieve, then $\mathcal{S}$ is a τ-covering sieve.

(iii) *(Maximality)* For any $X \in C$, $y(X)$ is a τ-covering sieve.

We call a category C equipped with a Grothendieck topology a *site*.

***Remark* 5.2.14** (Additional Properties). In fact, the following two conditions can be derived from the above three:

(4) *(Refinement)* If $\mathcal{R}$ is a τ-covering sieve on X and $\mathcal{S}$ is a sieve containing $\mathcal{R}$, then $\mathcal{S}$ is also a τ-covering sieve.

(5) *(Finite intersections)* Any finite intersection of τ-covering sieves on X is still a τ-covering sieve.

Now we can define the concept of sheaves.

Definition 5.2.15 (Sheaves). Let (C,τ) be a site. A presheaf F is called a *τ-sheaf* if for any τ-covering sieve $\mathcal{R}$, F satisfies descent along $\mathcal{R}$. We denote by:

$$\mathsf{Shv}_\tau(C) \subset \mathsf{PShv}(C)$$

the full subcategory of τ-sheaves.

***Remark* 5.2.16** (Poset of Topologies). Given a category C and topologies τ and ρ on it, we can compare their coarseness. We say τ is *coarser* than ρ, or ρ is *finer* than τ, written $\tau \le \rho$, if every τ-covering sieve is a ρ-covering sieve. It is easy to see that all topologies on C form a poset under the $\le$ relation.

Therefore, we can consider the coarsest topology and the finest topology on C.

Example 5.2.17 (Extreme Topologies). For any category C, consider the following topologies:

(i) The finest topology on C is the *discrete topology*: all sieves are covering sieves. The terminal object $* \in \mathsf{PShv}(C)$ is a sheaf only in the discrete topology.

(ii) The coarsest topology on C is the *indiscrete topology* (or *trivial topology*): only maximal sieves are covering sieves. In this case, all presheaves are sheaves.

Example 5.2.18 (Canonical Topology). Let O be a locale. Viewing it as a category, we can define the following topology on it: a sieve on $u \in O$ is a covering sieve if and only if its supremum is u. The first two conditions in Definition 5.2.13 come from the distributive law (note that pullback is $\wedge$), and the third condition is obvious. We call this topology the *canonical topology* on O.

For a general topological space T, we denote by $\mathsf{Shv}(T)$ the sheaf category $\mathsf{Shv}_{\mathrm{can}}(\mathrm{Open}(T))$, i.e., the category of sheaves on $\mathrm{Open}(T)$ with respect to the canonical topology. We also simply call this the *sheaf category* on T. The resulting sheaves are precisely the classical sheaves on topological spaces.

Example 5.2.19 (κ-Weiss Topology)**.** For a topological space $T \in \mathsf{Top}$ and a cardinal κ, we can consider the *κ-Weiss topology* on the category of open sets $\mathrm{Open}(T)$.

In this topology, a sieve $\mathcal{R}$ on an open set U is a covering sieve if and only if every subset $S \subset U$ of cardinality less than κ (i.e., $|S| < \kappa$) is *contained* in some member of $\mathcal{R}$. Formally:

$$\forall S \subset U, \text{ if } |S| < \kappa, \text{ then } \exists V \in \mathcal{R} \text{ such that } S \subseteq V.$$

(Note: When $\kappa = \aleph_0$, this is the standard *Weiss topology*, which requires that covers can capture all finite point configurations.) Additionally:

- When $\kappa = 0$, we get the discrete topology.
- When $\kappa = 1$, we get the topology consisting of non-empty sieves. The corresponding sheaf category is the category of constant sheaves.
- When $\kappa = 2$, we get the canonical topology.
- When $\kappa \geq |T|$, we get the trivial topology.

***Remark* 5.2.20** (Motivation for Grothendieck Topologies)**.** It is worth noting that Grothendieck topologies are not a direct generalization of classical point-set topology. (Example 5.2.18 illustrates the subtle connection between the two.)

From a historical development perspective, Grothendieck introduced this concept entirely out of considerations for sheaf theory. Recall that in the classical case, sheaves on topological spaces depend only on the properties of open sets and their open covers, not on the specific properties of points in the space.

Grothendieck's insight was: since sheaves only care about "covers", we can abandon the underlying space and directly axiomatize the concept of "covering" on any category.

Next is the core proposition of this section.

Proposition 5.2.21 (Adjunction Between Topologies and Sheaf Categories)**.** *Let C be a category. Then the map between posets:*

$$\{\textit{topologies on } C\}^{\mathrm{op}} \to \{\textit{subcategories of } \mathsf{PShv}(C)\}$$
$$\tau \mapsto \mathsf{Shv}_\tau(C)$$

has a left adjoint $\mathcal{E} \mapsto \tau_{\mathcal{E}}$. That is, for any subcategory $\mathcal{E}$ of the presheaf category, there exists a finest topology *$\tau_{\mathcal{E}}$ making $\mathcal{E}$ into sheaves.*

Specifically, a sieve $\mathcal{R}$ on X is a $\tau_{\mathcal{E}}$-covering sieve if and only if for any presheaf $F \in \mathcal{E}$ and any morphism $f : Y \to X$, F universally descends along $\mathcal{R}$ (i.e., F satisfies the sheaf condition). (Note: The definition can actually be simplified to $F(X) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R}, F)$.)

Proof. We only need to verify that $\tau_{\mathcal{E}}$ actually forms a Grothendieck topology. It is easy to see that conditions (1) and (3) in Definition 5.2.13 are obviously satisfied. The main work is verifying condition (2).

Let $\mathcal{S}, \mathcal{R} \subset y(X)$ be two sieves on X. Suppose $\mathcal{R}$ is a $\tau_{\mathcal{E}}$-covering sieve, and for every $(f : Y \to X) \in \mathcal{R}$, the pullback $f^*\mathcal{S}$ is a $\tau_{\mathcal{E}}$-covering sieve. We need to prove that $\mathcal{S}$ is also a $\tau_{\mathcal{E}}$-covering sieve.

This means for any $F \in \mathcal{E}$, we need to prove the isomorphism:

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{S}, F).$$

We establish this isomorphism through the intermediate term $\mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$ in three steps:

(i) *Using $\mathcal{R}$ as covering sieve:* Since $\mathcal{R}$ is a covering sieve and $F \in \mathcal{E}$, we have an isomorphism:

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{R}, F).$$

(ii) *Using condition (2):* Since $\mathcal{R}$ is a presheaf, we can write $\mathcal{R} = \mathrm{colim}_{Y\in \mathrm{El}(\mathcal{R})}\, y(Y)$. Using the property that Hom pulls out limits, we consider the restriction map from $\mathcal{R}$ to $\mathcal{R} \cap \mathcal{S}$:

$$\mathrm{Hom}(\mathcal{R}, F) \to \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F).$$

Note that $\mathcal{R} \cap \mathcal{S} = \mathcal{R} \times_{y(X)} \mathcal{S} \simeq \mathrm{colim}_{Y\in\mathcal{R}}(y(Y) \times_{y(X)} \mathcal{S})$. The above map can be expanded as a limit:

$$\lim_{Y\in \mathrm{El}(\mathcal{R})^{\mathrm{op}}} \Big(F(Y) \to \mathrm{Hom}_{\mathsf{PShv}(C)}(y(Y) \times_{y(X)} \mathcal{S}, F)\Big).$$

For each $Y \in \mathcal{R}$ (i.e., morphism $f : Y \to X$ in $\mathcal{R}$), the term $y(Y)\times_{y(X)}\mathcal{S}$ is precisely the pullback sieve $f^*\mathcal{S}$. By assumption, $f^*\mathcal{S}$ is a covering sieve, so $F(Y) \xrightarrow{\sim} \mathrm{Hom}(f^*\mathcal{S}, F)$ is an isomorphism. Since limits preserve isomorphisms, we get $\mathrm{Hom}(\mathcal{R}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$.

(iii) *Establishing* $\mathrm{Hom}(\mathcal{S}, F) \simeq \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$*:* Finally, we use the colimit decomposition of $\mathcal{S}$: $\mathcal{S} \simeq \mathrm{colim}_{Z\in \mathrm{El}(\mathcal{S})}\, y(Z)$. Similarly, the map $\mathrm{Hom}(\mathcal{S}, F) \to \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$ can be written as:

$$\lim_{Z\in \mathrm{El}(\mathcal{S})^{\mathrm{op}}} \Big(F(Z) \to \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{R} \times_{y(X)} y(Z), F)\Big).$$

For $Z \in \mathcal{S}$ (i.e., $g : Z \to X$ in $\mathcal{S}$), the term $\mathcal{R} \times_{y(X)} y(Z)$ is precisely the pullback sieve $g^*\mathcal{R}$. Since $\mathcal{R}$ is a covering sieve, $g^*\mathcal{R}$ is also a covering sieve on Z. This means $F(Z) \to \mathrm{Hom}(g^*\mathcal{R}, F)$ is an isomorphism. Again, since limits preserve isomorphisms, we get $\mathrm{Hom}(\mathcal{S}, F) \simeq \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$.

In summary, we have established $F(X) \simeq \mathrm{Hom}(\mathcal{S}, F)$. ■

Corollary 5.2.22 (Supremum of Topologies)**.** *Let C be a category and $(\tau_i)_{i\in I}$ a family of topologies on C. Let $\tau = \bigvee_{i\in I} \tau_i$ be their supremum. Then:*

$$\mathsf{Shv}_\tau(C) = \bigcap_{i\in I} \mathsf{Shv}_{\tau_i}(C).$$

Proof. Simply take $\mathcal{E} = \{F \in \mathsf{PShv}(C) \mid F \text{ descends along } \tau_i \text{ for each } i \in I\}$. ■

Next, we describe a simpler method for describing topologies. However, this method has two drawbacks:

- It requires the existence of corresponding pullbacks in C.
- The relationship with topologies is not injective, meaning there may be multiple pretopologies determining the same topology.

Definition 5.2.23 (Pretopology)**.** Let C be a category. A *pretopology* π on C consists of the following data: for each object X in C, we specify a family of objects in $C_{/X}$, called *π-coverings*, satisfying:

(i) *(Pullback stability)* For a π-covering $(U_i \to X)_{i\in I}$ and any morphism $f : Y \to X$, the pullbacks $U_i \times_X Y$ exist and $(U_i \times_X Y \to Y)_{i\in I}$ is a π-covering.

(ii) *(Transitivity)* For a π-covering $(U_i \to X)_{i\in I}$ and π-coverings $(V_{ij} \to U_i)_{j\in J_i}$, the family $(V_{ij} \to X)_{i\in I, j\in J_i}$ is a π-covering.

(iii) *(Identity)* $(\mathrm{id}_X : X \to X)$ is a π-covering.

The *topology generated by the pretopology* π is the coarsest topology containing all sieves generated by π-coverings.

***Remark* 5.2.24** (From Topology to Pretopology)**.** Let τ be a topology on C. Note that only when C has pullbacks can τ-coverings give rise to a pretopology on C.

Proposition 5.2.25 (Pretopology and Descent)**.** *Let C be a category, π a pretopology on C, and τ the topology generated by π. Then:*

(i) A sieve is a τ-covering sieve if and only if it contains at least one π-covering.

(ii) A presheaf F is a τ-sheaf if and only if for each π-covering $(U_i \to X)_{i\in I}$, there is an equalizer diagram:

$$F(X) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j)$$

Proof. This follows immediately from the fact that τ is the coarsest topology containing all covering sieves generated by π-coverings. ∎

5.2.3. Examples

Next, we introduce some concrete examples.

Example 5.2.26 (Standard Topology on Top). On the category of topological spaces Top, open covers form a pretopology. Let τ be the corresponding topology. Then:

(i) A sieve on X is a τ-covering sieve if and only if it contains an open cover of X. Therefore, $(Y_i \to X)_{i\in I}$ is a τ-covering if and only if it locally admits sections.

(ii) A presheaf $F\colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ is a τ-sheaf if and only if for each topological space T, the restriction $F|_{\mathrm{Open}(T)}$ is a sheaf on T.

Next, we describe the most commonly used topologies in algebraic geometry. (Of course, since we have not discussed étale morphisms and smooth morphisms, we will not define the Nisnevich topology, étale topology, or smooth topology here for now.)

Example 5.2.27 (Zariski Topology). On the category of affine schemes $\mathsf{Aff} \simeq \mathsf{CAlg}^{\mathrm{op}}$, for any commutative ring R, consider a family of elements $(f_i)_{i\in I}$ generating the unit ideal. We specify the covering families to be $\{\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)\}_{i\in I}$. The topology generated by this is called the *Zariski topology*.

We have the following characterizations:

(i) *Covering sieve criterion:* A sieve $\mathcal{S} \subset \mathrm{Spec}(R)$ is a Zariski-covering sieve if and only if the set:

$$\{f \in R \mid (\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)) \in \mathcal{S}\}$$

generates the unit ideal in R.

(ii) *Sheaf condition:* An algebraic functor $X\colon \mathsf{CAlg} \to \mathsf{Set}$ is a Zariski sheaf if and only if for any ring R and elements $(f_i)_{i\in I}$ generating the unit ideal, there is an equalizer diagram:

$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j}).$$

Example 5.2.28 (fpqc and fppf Topologies). The definition of the Zariski topology is actually a special case of a more general construction. Let E be a class of ring homomorphisms that includes isomorphisms and is closed under composition and pushouts. We can define a pretopology on $\mathsf{CAlg}^{\mathrm{op}}$ whose covering families are those families $(R \to R_i)_{i\in I}$ satisfying:

- The index set I is finite;
- The induced map $\coprod_{i\in I} \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ is surjective;
- Each map $R \to R_i$ belongs to the class E.

Depending on the choice of E, we obtain different topologies:

(i) If E is the class of localizations $R \to R_f$, this recovers the Zariski topology.

(ii) If E is the class of flat morphisms, the corresponding topology is called the *fpqc topology*.

(iii) If E is the class of finitely presented flat morphisms, the corresponding topology is called the *fppf topology*.

Since localizations $R \to R_f$ are both flat and finitely presented, we have the following relationship between topologies:

$$\text{Zariski} \le \text{fppf} \le \text{fpqc}.$$

In fact, we have

$$\text{Zariski} \le \text{Nisnevich} \le \text{Étale} \le \text{Smooth} \le \text{fppf} \le \text{fpqc}$$

5.3. Sheafification

Theorem 5.3.1 (Sheafification). *Let C be a small category and τ be a topology on C. Then the inclusion functor $\mathsf{Shv}_\tau(C) \hookrightarrow \mathsf{PShv}(C)$ admits a left adjoint:*

$$a_\tau : \mathsf{PShv}(C) \to \mathsf{Shv}_\tau(C),$$

called τ-sheafification. *Moreover,*

(i) the sheafification functor a_τ is left exact, i.e., it preserves finite limits.

(ii) A sieve $\mathcal{R} \subset y(X)$ is a τ-covering sieve if and only if $a_\tau(y(X)) \to a_\tau(\mathcal{R})$ is an epimorphism.

Proof. See [Sta18, Tag 00ZG] . ∎

Thus, one can obtain the following corollary.

Corollary 5.3.2. *Let (C, τ) be a small site. Then the category $\mathsf{Shv}_\tau(C)$ admits limits and colimits:*

- *Limits are computed as in $\mathsf{PShv}(C)$.*
- *Colimits are computed as in $\mathsf{PShv}(C)$ and then sheafified.*

Combine Theorem 5.3.1 and Proposition 5.2.21, we can obtain the following adjunction between topologies and category of sheaves.

Corollary 5.3.3. *Let C be a small category. Then the map between posets*

$$\begin{aligned} \{\textit{topologies on } C\}^{\mathrm{op}} &\to \{\textit{full subcategories of } \mathsf{PShv}(C) \textit{ admitting left exact left adjoint}\} \\ \tau &\mapsto \mathsf{Shv}_\tau(C) \end{aligned}$$

is an isomorphism.

Proof. The injectivity is ensured by (ii) of Theorem 5.3.1. Now, let's show the surjectivity. Let $\mathcal{E}$ be a full subcategory of $\mathsf{PShv}(C)$ admitting a left exact left adjoint $L : \mathsf{PShv}(C) \to \mathcal{E}$ to the inclusion functor. We need to show that $\mathcal{E}$ is the category of sheaves for some topology τ. Let $\tau = \{\mathcal{R} \subset y(X) \mid L(\mathcal{R}) \xrightarrow{\sim} L(y(X))\}$. We need to verify that τ is a topology and $\mathcal{E} = \mathsf{Shv}_\tau(C)$.

(i) Since L is left exact, it preserves finite limits. Therefore, for any morphism $f : Y \to X$ and sieve $\mathcal{R} \subset y(X)$, we have $L(f^*\mathcal{R}) \simeq L(y(Y) \times_{y(X)} \mathcal{R}) \simeq L(y(Y)) \times_{L(y(X))} L(\mathcal{R})$. This implies that if $\mathcal{R}$ is a τ-covering sieve, then $f^*\mathcal{R}$ is also a τ-covering sieve. Thus, condition (i) in Definition 5.2.13 is satisfied.

(ii) Now, let $\mathcal{S}$ be a sieve on X and $\mathcal{R}$ be a τ-covering sieve on X. If for every $(f : Y \to X) \in \mathcal{R}$, we have $f^*\mathcal{S}$ is a τ-covering sieve, then we have $L(f^*\mathcal{S}) \simeq L(y(Y)) \times_{L(y(X))} L(\mathcal{S}) \simeq L(y(Y))$, thus we have $L(\mathcal{S}) \simeq L(y(X))$.

(iii) Direct from the definition of τ.

Thus, τ is a topology and one can find that we have $\mathcal{E} \subset \mathsf{Shv}_\tau(C)$. So we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[dd, hook] \arrow[dr, hook, shift left] & \\
& \mathsf{PShv}(C) \arrow[ul, "L"', shift left] \arrow[dl, "a_\tau", shift left] \\
\mathsf{Shv}(C) \arrow[ur, hook, shift left] &
\end{tikzcd}$$

Thus, it suffices to show that for all $f : G \to F$ in $\mathsf{PShv}(C)$, if $L(f)$ is an isomorphism, then $a_\tau(f)$ is an isomorphism (which implies that for every $F \in \mathsf{Shv}_\tau(C)$, $F \to L(F)$ is an a_τ-isomorphism).

- If $f : \mathcal{R} \hookrightarrow y(X)$ is the inclusion of a sieve, then $L(f)$ being an isomorphism is equivalent to saying $\mathcal{R}$ is a τ-covering sieve, and hence $a_\tau(f)$ is an isomorphism.

- If f is a monomorphism, since F is a presheaf, we can write F as a colimit $\operatorname{colim}_{c\in \mathrm{El}(F)} y(c)$. Hence we have a canonical map $y(c) \to F$, so we can consider the Pullback

$$\begin{tikzcd} G \arrow[r, hook, "f"] & F \\ \mathcal{R} \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\lrcorner"] & y(c) \arrow[u] \end{tikzcd}$$

  Since f is a monomorphism, we obtain a sieve $\mathcal{R} \subset y(c)$. If $L(f)$ is an isomorphism, we have $\mathcal{R}$ is a τ-covering sieve, and hence by (ii) of Theorem 5.3.1, $a_\tau(\mathcal{R}) \xrightarrow{\sim} a_\tau(y(c))$. Since f can be written as the colimit of such $\mathcal{R} \hookrightarrow y(c)$ and a_τ is a left adjoint functor (preserving colimits), we have $a_\tau(f)$ is an isomorphism.

- Now, for an arbitrary $f : G \to F$, we can consider the epimorphism-monomorphism factorization of f, i.e.,

$$G \xrightarrow{f_{\mathrm{epi}}} \mathrm{im}(f) \xhookrightarrow{f_{\mathrm{mono}}} F.$$

  If $L(f)$ is an isomorphism, then $L(f_{\mathrm{mono}})$ is an isomorphism (as $L(f)$ factors through it, and L preserves monos, so $L(f_{\mathrm{mono}})$ is both mono and split epi). Since f_{mono} is a monomorphism, by the previous case, $a_\tau(f_{\mathrm{mono}})$ is an isomorphism.

  It remains to show $a_\tau(f_{\mathrm{epi}})$ is an isomorphism. Consider the kernel pair of f_{epi}:

$$\begin{tikzcd} G \times_{\mathrm{im}(f)} G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner"] & G \arrow[d, "f_{\mathrm{epi}}"] \\ G \arrow[r, "f_{\mathrm{epi}}"] & \mathrm{im}(f) \end{tikzcd}$$

  The diagonal $\Delta : G \to G \times_{\mathrm{im}(f)} G$ is a monomorphism. Since $L(f_{\mathrm{epi}})$ is an isomorphism (as $L(f)$ and $L(f_{\mathrm{mono}})$ are isos), applying the left exact functor L shows that $L(\Delta)$ is an isomorphism. Since Δ is a monomorphism, by the previous case, $a_\tau(\Delta)$ is an isomorphism.

  Since a_τ is left exact, $a_\tau(\Delta)$ is the diagonal of the kernel pair of $a_\tau(f_{\mathrm{epi}})$. The fact that the diagonal is an isomorphism implies $a_\tau(f_{\mathrm{epi}})$ is a monomorphism. Additionally, a_τ is a left adjoint and thus preserves epimorphisms; hence $a_\tau(f_{\mathrm{epi}})$ is an epimorphism. In a topos (like $\mathsf{Shv}_\tau(C)$), a morphism that is both a monomorphism and an epimorphism is an isomorphism.

  Therefore, $a_\tau(f) = a_\tau(f_{\mathrm{mono}}) \circ a_\tau(f_{\mathrm{epi}})$ is an isomorphism.

This confirms that the essential image of L coincides with $\mathsf{Shv}_\tau(C)$. ■

***Remark* 5.3.4** (Non-topological localizations). The isomorphism established in Corollary 5.3.3 relies essentially on the fact that objects in a 1-category are 0-truncated, which implies that for any morphism f, the diagonal Δ_f is a monomorphism.

In the context of ∞-topoi, the situation is different. As pointed out by Marc (see MO 273085), there are many natural examples of ∞-topoi that arise as left exact localizations but have *a priori* nothing to do with Grothendieck topologies. Examples include:

- The ∞-category of n-excisive functors (in the sense of Goodwillie calculus) with values in an ∞-topos.

- The ∞-category of coalgebras for a left exact comonad in an ∞-topos.

While these ∞-topoi might happen to be equivalent to sheaf ∞-categories "by accident", their defining localizations involve inverting morphisms that are not monomorphisms (e.g., hypercovers), a feature that is invisible in the 1-categorical setting.

We can now characterize the properties of morphisms in the category of sheaves via local properties of morphisms of presheaves.

Definition 5.3.5 (Local epimorphism, monomorphism, isomorphism)**.** Let (C, τ) be a site and let $f : F \to G$ be a morphism in $\mathsf{PShv}(C)$. Let $\Delta_f : F \to F \times_G F$ denote the diagonal morphism.

(i) f is called a *τ-local epimorphism* if for every $X \in C$ and every section $s \in G(X)$, the sieve on X consisting of all morphisms $u : Y \to X$ such that $u^*(s)$ lies in the image of the map $f_Y : F(Y) \to G(Y)$ is a τ-covering sieve.

(ii) f is called a *τ-local monomorphism* if the diagonal Δ_f is a τ-local epimorphism.

(iii) f is called a *τ-local isomorphism* if it is both a τ-local epimorphism and a τ-local monomorphism.

Proposition 5.3.6. *Let (C, τ) be a small site and let f be a morphism in $\mathsf{PShv}(C)$. Then $a_\tau(f)$ is an epimorphism (resp. monomorphism, isomorphism) in $\mathsf{Shv}_\tau(C)$ if and only if f is a τ-local epimorphism (resp. monomorphism, isomorphism).*

Chapter 6.

Scheme

Scheme is the central object of study in algebraic geometry. In this chapter, we will define schemes and explore their properties.

6.1. The Category of Schemes

In this section, we will define what a scheme is and describe how to glue schemes.

Definition 6.1.1 (Scheme). A *scheme* is a functor $X\colon \mathsf{CAlg} \to \mathsf{Set}$ satisfying:

(i) X is a sheaf with respect to the Zariski topology.

(ii) X admits an open cover by affine schemes.

We denote by $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ the full subcategory spanned by schemes. For any scheme S, we denote by Sch_S the slice category $\mathsf{Sch}_{/S}$, and call objects in it S*-schemes*. When $S = \mathrm{Spec}(k)$ is the affine scheme corresponding to a ring k, we also call S-schemes k*-schemes*.

Note that $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ is only a full subcategory. Therefore, many properties that hold in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ may not hold when restricted to Sch. Our goal is to explore which properties do hold in Sch.

Proposition 6.1.2 (Immersions of Schemes). *Let X be a scheme and $Y \hookrightarrow X$ an open immersion/closed immersion/immersion. Then Y is also a scheme.*

Proof. (i) *First, we show the open case.*

Let $Y \subset X$ be an open subfunctor. Since X is a scheme, there exists an open cover $(U_i \to X)_{i\in I}$ where each U_i is an affine scheme. Therefore, $(U_i \cap Y \to Y)_{i\in I}$ is also an open cover (using that open covers are closed under base change).

For each $i \in I$, since $U_i \cap Y \subset U_i$ is an open subfunctor of U_i, we can write $U_i \cap Y$ as $D(J_i)$ for some quasi-coherent ideal J_i. Note that $D(J_i)$ is covered by $(D(f) \to D(J_i))_{f\in J_i}$. Therefore, each $U_i \cap Y$ admits an affine open cover, so $(U_i \cap Y \to Y)_{i\in I}$ can be refined to $(D(f) \to Y)_{i\in I, f\in J_i}$.

Finally, we need to show that Y satisfies Zariski descent. For this, we need to verify that for any ring R and any family of elements $(f_k)_{k\in K}$ generating the unit ideal of R, there is an equalizer diagram:

$$Y(R) \longrightarrow \textstyle\prod_{k\in K} Y(R_{f_k}) \rightrightarrows \textstyle\prod_{k,l\in K} Y(R_{f_k f_l})$$

Consider the diagram:

$$\begin{tikzcd}
Y(R) \arrow[r] \arrow[d, hook] & \prod_{k\in K} Y(R_{f_k}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \prod_{k,l\in K} Y(R_{f_k f_l}) \arrow[d, hook] \\
X(R) \arrow[r] & \prod_{k\in K} X(R_{f_k}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{k,l\in K} X(R_{f_k f_l})
\end{tikzcd}$$

Since X is a scheme, the bottom row is an equalizer diagram. We need to show the top row is also.

For this, take $x \in X(R)$, corresponding to a map $x\colon \operatorname{Spec}(R) \to X$. Consider the pullbacks:

$$\begin{tikzcd} D(I_k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R_{f_k}) \arrow[d] \\ D(I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

where $I_k = I \otimes_R R_{f_k}$. We need to show that if x lies in Y, then $D(I_k) \simeq \operatorname{Spec}(R_{f_k})$. Precisely, we need to show:

$$[\forall k \in K, D(I_k) \simeq \operatorname{Spec}(R_{f_k})] \iff [D(I) \simeq \operatorname{Spec}(R)]$$

The former is equivalent to saying that for all k, we have $I \otimes_R R_{f_k} \twoheadrightarrow R_{f_k}$ (i.e., generates the unit ideal). Therefore, by Zariski descent, we obtain the latter.

(ii) *Next, we show the closed case.*

Again, take an open cover $(U_i \to X)_{i \in I}$. This time, we find that $U_i \cap Y \subset U_i$ is a closed subfunctor, hence can be written as $V(J_i)$ for some quasi-coherent ideal J_i.

We only need to show Zariski descent. Consider the pullback:

$$\begin{tikzcd} V(I_k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R_{f_k}) \arrow[d] \\ V(I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(R) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

It is easy to see that $V(I_k) = \operatorname{Spec}(R_{f_k})$ is equivalent to saying that for all k, the map $I \otimes_R R_{f_k} \to R_{f_k}$ is the zero map. Since the zero map satisfies Zariski descent, we obtain the result.

(iii) *The immersion case is similar.* (Omitted, combine the above two cases.)

■

Next, we give some examples of schemes.

Example 6.1.3 (Examples of Schemes). (i) For any ring A, $\operatorname{Spec}(A)$ is a scheme.

(ii) For an $\mathbb{N}$-graded ring A, $\operatorname{Proj}(A)$ is a scheme, with affine cover $(D(f) \simeq \operatorname{Spec}(A_{(f)}))_{f \in A_+}$.

(iii) The affine line with two origins is also a scheme, where $\mathbb{A}^1$ serves as its open cover.

(iv) For a scheme X and an affine/quasi-affine/projective/quasi-projective morphism $Y \to X$, we have Y is a scheme.

(v) For a ring k, a k-module M, and $n \in \mathbb{N}$, the Grassmannian $\operatorname{Gr}_n(M)$ is a scheme.

Next, we study how to glue schemes. This is somewhat similar to gluing topological spaces. Therefore, we first generalize the notion of equivalence relations to general categories.

Definition 6.1.4 (Equivalence Relation). Let C be a category. An *equivalence relation* in C is a pair of morphisms:

$$R \underset{t}{\overset{s}{\rightrightarrows}} X$$

satisfying: for any $Y \in C$, the induced map:

$$(s_*, t_*)\colon \operatorname{Hom}_C(Y, R) \to \operatorname{Hom}_C(Y, X) \times \operatorname{Hom}_C(Y, X)$$

is *injective*, and its image defines an equivalence relation (in the usual sense) on the set $\operatorname{Hom}_C(Y, X)$.

If the coequalizer of the above diagram exists, it is called the *quotient* of X by the equivalence relation R, denoted X/R.

***Remark* 6.1.5** (Higher Categorical Analogue: Groupoid Objects). The higher categorical analogue of an equivalence relation is a *groupoid object*. It is a simplicial object $X_\bullet$ in C:

$$\cdots \; X_2 \rightrightarrows X_1 \rightrightarrows X_0$$

satisfying the *groupoid condition* (this condition has no standard name, perhaps it should be called the Kan condition): for any $[n] \in \Delta$ and any (not necessarily order-preserving) partition:

$$[n] \simeq \{i_0, \ldots, i_k\} \cup \{i_k, \ldots, i_n\},$$

the following induced diagram is a *pullback square* in C:

$$\begin{tikzcd} X_n \arrow[r, "(i_0,\ldots,i_k)^*"] \arrow[d, "(i_k,\ldots,i_n)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_k \arrow[d, "i_k^*"] \\ X_{n-k} \arrow[r, "i_k^*"'] & X_0 \end{tikzcd}$$

Intuitive explanation: This condition elegantly captures both core properties of groupoids:

(i) *Associativity (Segal condition):* When choosing standard ordered partitions (such as $[2] = \{0,1\} \cup \{1,2\}$), the pullback condition gives $X_2 \simeq X_1 \times_{X_0} X_1$, ensuring composition of morphisms.

(ii) *Invertibility:* When choosing non-order-preserving partitions, this condition forces morphisms to be invertible.

Under this viewpoint, X_0 corresponds to the object set (or space of objects), and X_1 corresponds to the morphism set (or relation). The image of the map $(s,t)\colon X_1 \to X_0 \times X_0$ defines a "higher equivalence relation" on X_0.

Specifically:

- Consider a groupoid object $\mathcal{G}$ in the category of sets Set. Then it is a groupoid in the classical sense. The image of the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ defines an equivalence relation on $\mathcal{G}_0$. Specifically, reflexivity comes from identity maps, transitivity from composition of morphisms, and symmetry from inversion of morphisms.

- Conversely, an equivalence relation $R \subseteq X \times X$ on a set X corresponds bijectively to a groupoid $\mathcal{G}$ in sets satisfying: $\mathcal{G}_0 = X$, and the map $(s,t)\colon \mathcal{G}_1 \hookrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is injective. Under this correspondence, the colimit of the groupoid $|\mathcal{G}|$ corresponds to the quotient set X/R.

***Remark* 6.1.6** (Effective Epimorphisms). For a category with pullbacks, every morphism $f\colon X \to Y$ defines an equivalence relation $X \times_Y X \rightrightarrows X$ (in the higher setting, one should take the groupoid object). We say f is an *effective epimorphism* if its information can be recovered from the quotient by this equivalence relation. Precisely, this means the diagram:

$$X \times_Y X \rightrightarrows X \longrightarrow X/R$$

is a coequalizer.

Note that in 1-categories, epimorphisms are always effective.

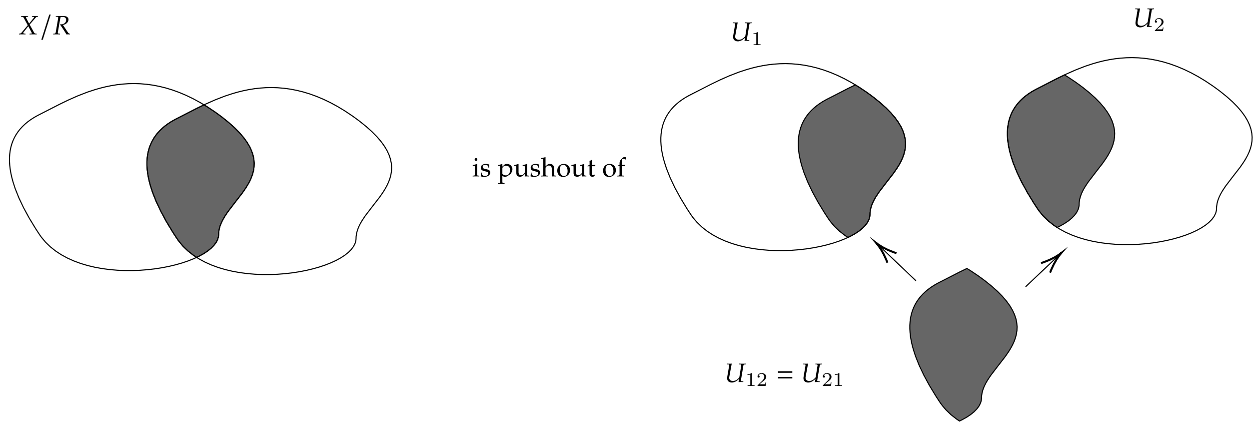
Next, we describe locally trivial relations.

Definition 6.1.7 (Locally Trivial Relation). An equivalence relation $s,t\colon R \rightrightarrows X$ in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ is called *locally trivial* if there exists an open cover $(U_i \subset X)_{i \in I}$ such that:

$$s^{-1}(U_i) \hookrightarrow R \xrightarrow{t} X$$

is an open immersion.

Next, we discuss how to glue schemes. For U_1 and U_2 with $U_{12} = U_{21}$, we can represent the gluing process by the following data:



where $X = U_1 \sqcup U_2$, and $R = \coprod_{i,j\in\{1,2\}} U_{ij} \rightrightarrows X$. Note that when $i = j$, we set $U_{ij} = U_i$. It is easy to see that when $U_{12} \hookrightarrow U_1$ and $U_{21} \hookrightarrow U_2$ are open immersions, R is a locally trivial relation.

This leads to the following definition.

Definition 6.1.8 (Gluing Datum). A *gluing datum* in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ consists of data $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (U_{ij} \to U_i \times U_j)_{i,j\in I})$:

(i) Objects $(U_i)_{i\in I}$ and $(U_{ij})_{i,j\in I}$.

(ii) For $i, j \in I$, morphisms $U_{ij} \to U_i \times U_j$ (note that these need not be isomorphisms).

satisfying the following compatibility conditions:

- For $i, j \in I$, both $U_{ij} \hookrightarrow U_i$ and $U_{ij} \hookrightarrow U_j$ are open immersions.
- The diagram:
$$\coprod_{i,j\in I} U_{ij} \rightrightarrows \coprod_{i\in I} U_i$$
is an equivalence relation in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$.

***Remark* 6.1.9** (Open Covers and Gluing Data). We can convert between open covers and gluing data as follows:

- If X is the quotient by the equivalence relation given by gluing data $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (U_{ij} \to U_i \times U_j)_{i,j\in I})$ in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$, then $(U_i)_{i\in I}$ forms an open cover of X, satisfying $U_{ij} \xrightarrow{\sim} U_i \times_X U_j$.
- Conversely, given a Zariski sheaf $X \in \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ and an open cover $(U_i \to X)_{i\in I}$, the families $(U_i)_{i\in I}$ and $(U_i \cap U_j)$ form gluing data.

Next, we explore which properties are preserved in $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$.

Theorem 6.1.10 (Limits and Colimits of Schemes). *The full subcategory* $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ *is closed under:*

(i) Finite limits.

(ii) Cofiltered diagrams of affine morphisms.

(iii) Arbitrary coproducts.

(iv) Quotients by locally trivial equivalence relations.

Before proving this theorem, we first prove the following lemma.

Lemma 6.1.11 (Coproducts Are Clopen). *Let $(X_i)_{i\in I}$ be a family of Zariski sheaves and $X := \coprod_{i\in I} X_i$. Then $X_i \hookrightarrow X$ is both an open and closed immersion. Therefore, $(X_i \hookrightarrow X)_{i\in I}$ forms an open cover of X.*

Proof. Note that for each $i \in I$, we can decompose $X_i \hookrightarrow X$ as:

$$X_i \hookrightarrow X_i \sqcup \left(\coprod_{j \in I, j \neq i} X_j \right) = X.$$

Therefore, everything reduces to showing: for any Zariski sheaves X, Y, the morphism $X \hookrightarrow X \sqcup Y$ is both an open and closed immersion.

Consider the pullback diagram in the presheaf category:

$$\begin{tikzcd} X \sqcup^{\mathrm{pre}} \varnothing_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}_m \arrow[d, hook] \\ X \sqcup^{\mathrm{pre}} Y \arrow[r, "{(1,0)}"] & \mathbb{A}^1 \end{tikzcd}$$

where $\sqcup^{\mathrm{pre}}$ denotes coproduct in the presheaf category, and the bottom map $(1,0)$ is the algebraic functor morphism that takes constant value 1 on X and constant value 0 on Y.

It is easy to compute that $\varnothing_Y$ in the upper left pullback is concretely:

$$\varnothing_Y(R) = \begin{cases} \varnothing, & R \neq 0 \\ Y(0), & R = 0 \end{cases}$$

Note that there is a unique morphism $\varnothing_Y \to \varnothing = \mathrm{Spec}(0)$, and this morphism is obviously an isomorphism for each $R \neq 0$.

Therefore, we claim that the Zariski sheafification $a_{\mathrm{Zar}}(\varnothing_Y) = \varnothing$. This is equivalent to saying $\varnothing_Y \to \varnothing$ is a Zariski-local isomorphism. That is, we need to show this morphism is both a Zariski-local epimorphism and a Zariski-local monomorphism. Since the zero ring $R = 0$ is an empty cover in the Zariski topology (i.e., $\mathrm{Spec}(0) = \varnothing$), this local isomorphism property obviously holds.

Since the sheafification functor a_{Zar} preserves finite limits (in particular, pullbacks), in the Zariski sheaf category $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$, we obtain the pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}_m \arrow[d, hook] \\ X \sqcup Y \arrow[r, "{(1,0)}"] & \mathbb{A}^1 \end{tikzcd}$$

This shows that $X \hookrightarrow X \sqcup Y$ is the pullback of the open immersion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, hence is itself an open immersion.

More intuitively, the map $(1,0)$ corresponds to defining an idempotent section on the scheme. Therefore, $X \hookrightarrow X \sqcup Y$ can be viewed as corresponding to the standard open set $D(1,0)$ and closed set $V(0,1)$ defined by this section (since the complement of $D(1,0)$ is precisely $V(1,0) = Y$, equivalent to the open set generated by $(0,1)$), making it both open and closed. ∎

By Lemma 6.1.11, we know that condition (2) in Definition 6.1.8 gives a locally trivial equivalence relation. Specifically, condition (2) is equivalent to giving the following information:

(i) *(Reflexivity)* The diagonal map $U_i \to U_i \times U_i$ (its image) lies in U_{ii} (its image).

(ii) *(Symmetry)* The map $U_{ij} \to U_i \times U_j \xrightarrow{\sim} U_j \times U_i$ (its image) lies in U_{ji} (its image).

(iii) *(Transitivity)* The map $U_{ij} \times_{U_j} U_{jk} \to U_i \times U_k$ (its image) lies in U_{ik} (its image).

Now we prove Theorem 6.1.10.

Proof of Theorem 6.1.10. (i) We know that finite limits can be constructed from terminal objects and pullbacks. Therefore, everything reduces to showing that terminal objects and pullbacks are schemes. The terminal object is $\mathrm{Spec}(\mathbb{Z})$, which is obviously a scheme. Therefore, we only need to show that pullbacks are schemes.

Consider a pullback diagram in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $X, Y, Z \in \mathsf{Sch}$ are schemes. Since Z is a scheme, we can take a point $z \in |Z|$ and find an affine open set $W \subset Z$ containing it. For a point $x \in |X|$ with $f(x) = z$, there exists an affine open set $U \subset X$ containing x such that $f(U) \subset W$. Similarly, for $y \in |Y|$ with $g(y) = z$, there exists an affine open set $V \subset Y$ containing y such that $g(V) \subset W$.

This gives an open subfunctor in the pullback:

$$U \times_W V \hookrightarrow X \times_Z Y = P.$$

Since U, V, W are all affine schemes, $U \times_W V$ is also affine. Since $\mathsf{Open}(P) \simeq \mathrm{Open}(|P|)$, we can use such $U \times_W V$ to form an affine open cover of P, thus proving P is a scheme.

(ii) Recall that $f : Y \to X$ is an affine morphism if and only if for any $\mathrm{Spec}(R) \to X$, we have $Y \times_X \mathrm{Spec}(R)$ is an affine scheme. Recall we have a categorical equivalence:

$$\mathrm{Spec}\colon \mathsf{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_X.$$

We also know that $\mathsf{Aff}_X \hookrightarrow \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ preserves limits.

Now let I be a cofiltered diagram and $F : I \to \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ a diagram such that F sends morphisms in I to affine morphisms in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$. By the structure of cofiltered diagrams, for any $x, y \in I$, there always exists z with morphisms $z \to x$ and $z \to y$. Therefore, for any chosen basepoint $x_0 \in I$, the limit actually satisfies $\lim_{i \in I} F \simeq \lim_{i \in I_{/x_0}} F$. Thus we may assume I has a terminal object $*$ (the original x_0). Denote $F(*) = X$. Then the entire diagram F reduces to taking a limit in the slice category Aff_X. Since $\mathsf{Aff}_X \hookrightarrow \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ preserves limits, and filtered colimits exist in the category of algebras, this limit is an affine scheme (relative to X), hence a scheme.

(iii) The existence of arbitrary coproducts is a direct consequence of Lemma 6.1.11. For $X = \coprod_{i \in I} X_i$, we know $X_i \hookrightarrow X$ are open immersions, and $(X_i \to X)$ forms an open cover. Refining using affine open covers on each X_i gives an affine open cover of X, thus X is a scheme.

(iv) Now let R be a locally trivial equivalence relation. Consider its coequalizer:

$$R \underset{t}{\overset{s}{\rightrightarrows}} X \longrightarrow X/R$$

We need to show the quotient X/R is a scheme (given that X and R are schemes). By the definition of locally trivial, there exists an open cover $(U_i \subset X)_{i \in I}$ such that the pullback diagram:

$$\begin{tikzcd} s^{-1}(U_i) \arrow[r, hook, "open"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "s"] \\ U_i \arrow[r, hook, "open"] & X \end{tikzcd}$$

has $s^{-1}(U_i) \to U_i \times (X/R)$ an isomorphism. Since $X \to X/R$ is Zariski-locally surjective, and the property of being an open immersion is Zariski-local on the target (i.e., can descend through covers), the induced map $U_i \to X/R$ is also an open immersion. This means $(U_i \to X/R)_{i \in I}$ forms an open cover of X/R. Therefore, X/R is a scheme.

■

By Theorem 6.1.10, we know that given gluing data $((U_i)_{i \in I}, (U_{ij})_{i,j \in I}, (U_{ij} \to U_i \times U_j)_{i,j \in I})$ where each U_i and U_{ij} is a scheme, the object X constructed from the locally trivial equivalence relation is also a scheme.

6.1.1. Examples

Next, we give some examples.

Example 6.1.12 (Obtaining Proj by Gluing). Let A be an $\mathbb{N}$-graded ring and $I \subset A_+$ a subset of homogeneous elements satisfying $A \subset \sqrt{(I)}$. Then $(D(f) \subset \mathrm{Proj}(A))_{f \in I}$ forms an open cover of $\mathrm{Proj}(A)$. It is easy to see that:

$$D(f) \cap D(g) = \mathrm{Spec}(A_{(f)}) \times_{\mathrm{Proj}(A)} \mathrm{Spec}(A_{(g)}) \simeq D(fg).$$

Therefore, $\mathrm{Proj}(A)$ is the quotient of the equivalence relation:

$$\coprod_{f,g \in I} \mathrm{Spec}(A_{(fg)}) \rightrightarrows \coprod_{f \in I} \mathrm{Spec}(A_{(f)})$$

For $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, we take $I = \{x_0, \dots, x_n\}$. Then:

$$D(x_i) = \mathrm{Spec}(\mathbb{Z}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]) \simeq \mathbb{A}^n,$$

and for $i \neq j$:

$$D(x_i x_j) \simeq \mathbb{A}^{n-1} \times \mathbb{G}_m.$$

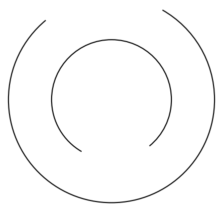
Therefore, we obtain the coequalizer diagram:

$$\coprod_{i<j} \mathbb{A}^{n-1} \times \mathbb{G}_m \rightrightarrows \coprod_{i=0}^{n} \mathbb{A}^n \longrightarrow \mathbb{P}^n$$

When $n = 1$, this collapses to the pushout diagram:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t \mapsto t"] \arrow[d, "t \mapsto t^{-1}"] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

This can be visualized as gluing two copies of $\mathbb{A}^1$ along $\mathbb{G}_m$ with the identification $t \leftrightarrow t^{-1}$



Remark **6.1.13** (Additional Observations).

- If there is a group object G acting on X, then:

$$X/G = \mathrm{colim}\left(X \times G \underset{\mathrm{proj}}{\overset{\mathrm{act}}{\rightrightarrows}} X \right).$$

  When $X = *$, the result is called a *classifying stack*. In the higher categorical setting, by Remark 6.1.5, we need to take the colimit over the entire groupoid object, i.e., take the geometric realization.

- Any scheme can be written as a colimit of affine schemes in the following way: consider an affine cover and the pairwise intersections.

Example 6.1.14 (Obtaining $\mathbb{P}^n$ via Group Action). Note that there is a canonical action:

$$\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1} - 0.$$

We can obtain:

$$(\mathbb{A}^{n+1} - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^n.$$

But we mentioned earlier that this map is Zariski-locally surjective. Therefore, in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$, we have an isomorphism:

$$(\mathbb{A}^{n+1} - 0)/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^n.$$

In fact, for any $\mathbb{N}$-graded ring generated by $A_{\leq 1}$, in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ we have:

$$D(A_+)/\mathbb{G}_m \xrightarrow{\sim} \mathrm{Proj}(A).$$

Example 6.1.15 (Coproducts of Affine Schemes (in the Category of Schemes)).

(i) *Finite coproducts:* First, the initial scheme is $\varnothing = \mathrm{Spec}(0) = a_{\mathrm{Zar}}(\varnothing^{\mathrm{pre}})$.

For $R, S \in$ CAlg, consider their product $R \times S$ in the category of rings. We have the following commutative diagram (pullback in CAlg, pushout in Aff):

$$\begin{tikzcd} R \times S \arrow[r, "{(0,1)}"] \arrow[d, "{(1,0)}"] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d] \\ R \arrow[r] & 0 \end{tikzcd}$$

where $R \times S \to R$ and $R \times S \to S$ are localizations at idempotents $e_1 = (1, 0)$ and $e_2 = (0, 1)$ respectively (note $e_1 + e_2 = 1$).

Let $\mathcal{T}$ be the sieve on $R \times S$ generated by the covering family $\{R \times S \to R, R \times S \to S\}$. Since (e_1, e_2) generate the unit ideal, $\mathcal{T}$ is a Zariski-covering sieve. By the definition of Zariski sheaves, for any $X \in \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$, we have an isomorphism:

$$X(R \times S) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{T}, X) \simeq X(R) \times X(S).$$

This shows that $\mathrm{Spec}(R \times S)$ represents the coproduct of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ in the sheaf category. Therefore, Aff $\subset$ Sch preserves finite coproducts.

(ii) *Counterexample for arbitrary coproducts:* Note that Aff $\subset$ Sch does not preserve arbitrary coproducts. Consider an infinite family $\mathrm{Spec}(R_i)_{i \in I}$. In Sch, its coproduct is the disjoint union $X = \coprod_{i \in I} \mathrm{Spec}(R_i)$ (which is a scheme, but usually not affine).

If it were represented by an affine scheme $\mathrm{Spec}(A)$, we would necessarily have $A \simeq \mathcal{O}(X) = \prod_{i \in I} R_i$. In the ring A, we can take idempotents $e_i = (\delta_{ij})_{j \in I}$ (i.e., 1 in the i-th position, 0 elsewhere). The standard open set $D(e_i) \subset \mathrm{Spec}(A)$ is precisely isomorphic to $\mathrm{Spec}(R_i)$.

However, if the index set I is infinite and infinitely many $R_i \neq 0$, then the ideal generated by all $(e_i)_{i \in I}$ is $\bigoplus_{i \in I} R_i$, which is a proper ideal in $\prod_{i \in I} R_i$ (not containing the unit $1 = (1, 1, \ldots)$). This means:

$$\bigcup_{i \in I} D(e_i) \subsetneq \mathrm{Spec}\left(\prod_{i \in I} R_i\right).$$

That is, the coproduct $\mathrm{Spec}(\prod R_i)$ in Aff contains "more" points than X (these extra points correspond to non-principal ultrafilters on I).

6.2. Separatedness

[To be continued...]

Bibliography

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