Higher Topos Theory

Taught by Marc Hoyois

Typed by Ou Liu

December 26, 2025

Contents

Change log

The following is an overview of the major updates of these notes:

-

- **November 2025**:
  - Started a change log (adopting this good habit from Bastiaan).
  - Added Chapter 3.
  - Moved the section "Modality" from Chapter 2 to Chapter 3.

Literature and notation

Remark • *These notes are based on lectures and standard references.*

- *Please note that the organization of the material may not strictly follow the chronological order of Marc's lectures.*
- *Furthermore, the update pace of these notes will be significantly slower than that of Bastiaan's, with completion expected by April 2026. However, I do not intend to discontinue the updates.*
- *Any mistakes or inaccuracies are the sole responsibility of the author.*

These notes draw on several modern sources in higher category theory and higher topos theory, most notably:

1. [Cnossen, 2025] — for many foundational results in higher category theory and higher algebra that we will rely on. The notation in these notes is consistent with this reference.
2. [Haugseng, 2025] — for the theory of presentable categories and model-independent categorical tools.
3. [Lurie, 2009], [Lurie, 2017], and [Lurie, 2018b] — for the foundations of higher topos theory and higher algebra.
4. Anel–Biedermann–Finster–Joyal:
   - [Anel et al., 2022], [Anel et al., 2023a], [Anel et al., 2023b] — on left-exact localizations of topoi.
   - [Anel et al., 2020] — for the Blakers–Massey theorem.
5. (Historical reference) [Rezk, 2010].
6. [Scholze, 2025] — for its geometric motivation and perspective on topoi as ambient geometric worlds.

These notes are written entirely in the setting of higher category theory. For readability, we systematically omit the "∞" symbol from the notation—thus, "category" always means "∞-category", and "topos" means "∞-topos".

- $\mathbb{\Delta}$ denotes the simplex category.
- An denotes the category of animae.
- For a category $\mathcal{C}$, we write
$$\mathsf{PShv}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{An})$$
for the category of presheaves on $\mathcal{C}$.

We adopt the classical convention that a morphism is *n-connected* if its n-truncation vanishes, that is, if $\tau_n(f) \simeq *$. This differs by a shift from Lurie's terminology, where the term *n-connective* is used.

Unless otherwise stated, all categories are assumed to be presentable, and all limits and colimits are taken in the ∞-categorical sense. References to "localizations," "accessibility," or "adjunctions" always refer to their ∞-categorical versions.

Acknowledgements. — My heartfelt thanks go to the course professor, Marc, and to Bastiaan for carefully reading these notes, pointing out many mistakes, offering insightful suggestions, and for many helpful discussions. I am also grateful to Yuchen Wu and Longke Tang for many helpful discussions. Their feedback and insights greatly improved the work.

In these notes, I borrowed content from Bastiaan's notes. For instance, in § 2.1.1, I borrowed his discussion of descent. In § 2.4, I also adopted the notion 'Logos', which was used in Bastiaan's notes. This notion, originally introduced by Anel and Joyal in "Topo-logie", was not covered in Marc's course.

Chapter 1

Introduction

Classical topos theory was introduced by Grothendieck as a conceptual framework for algebraic geometry. Originally conceived to formalize descent and sheaf-theoretic constructions, it soon revealed deep categorical and logical structures. However, classical topos theory is built upon *ordinary* category theory—a setting that treats all morphisms strictly, with no notion of higher homotopy. Consequently, it lacks the expressive power needed to capture inherently homotopical or higher-cohomological phenomena.

Modern mathematics, however, has made homotopy an indispensable part of geometry. To include this higher information, one must refine the classical theory into a *homotopical* or *higher-categorical* framework. Heuristically, one might visualize a (higher) topos via the following schematic:

$$\text{Topos} \simeq \text{classical topos} + \text{animae} + \text{Goodwillie calculus} + \cdots$$

This is not a literal formula, but rather an indication that a topos refines a classical one by allowing objects to carry animated (homotopy-theoretic) information.

1.1 From affine geometry to topos theory

The familiar duality between geometry and algebra—for instance, between affine schemes and commutative rings—is not a perfect equivalence. Geometric information is lost when one passes to global functions. For example,

$$\Gamma(\mathbb{P}^1_k, \mathcal{O}) = k,$$

so the projective line cannot be distinguished from a point solely by its ring of functions. Geometry reappears only once we understand *how affine pieces glue together.*

This process—gluing affine pieces to obtain global spaces—is universal in modern geometry. Grothendieck abstracted it into the notions of *site* and *topos.* Given a category C of "affine" objects and a Grothendieck topology τ (such as Zariski, étale, fppf, or Nisnevich), one forms the category of sheaves

$$\mathsf{Shv}_\tau(C).$$

Objects of this topos can be viewed as spaces obtained by gluing affine pieces along coverings. The resulting category is complete, cocomplete, and satisfies descent.

Idea *Any good category of geometric objects—one that allows affine gluing and all small colimits—should form a* classical topos.

However, when one studies moduli problems, sheaves of sets are too rigid. A moduli problem, such as that of elliptic curves, involves automorphisms; to capture these symmetries, one must use *sheaves of groupoids* or, more generally, *sheaves of anima.* These encode not only objects but their higher symmetries as well. Lurie's higher topos theory generalizes Grothendieck's work in precisely this direction: a topos of anima behaves like a category of "animated geometric spaces", where descent and gluing hold in a homotopical sense.

Idea *A* topos *is the natural setting for geometry that includes all colimits and allows gluing in the homotopical sense.*

1.2 The big picture—two towers

Every topos $\mathcal{T}$ admits two canonical filtrations that expose complementary aspects of its structure:

$$* \simeq \overbrace{\mathcal{T}_{\leq -2} \subseteq \mathcal{T}_{\leq -1} \subseteq \underbrace{\mathcal{T}_{\leq 0}}_{\text{classical topos}} \subseteq \mathcal{T}_{\leq 1} \subseteq \cdots \subseteq \mathcal{T}_{\leq \infty}}^{\text{Postnikov tower}} = \mathcal{T}^{\text{hyp}} \subseteq \underbrace{\mathcal{T}^{(1)} \subseteq \mathcal{T}^{(2)} \subseteq \cdots \subseteq \mathcal{T}^{(\infty)}}_{\text{Goodwillie tower}} \subseteq \mathcal{T}.$$

1.2.1 Homotopy theory

A fundamental difference between higher and classical category theory lies in the presence of higher coherence data. In a topos, such coherence manifests as genuine homotopical information, measured by two complementary notions:

- *Truncation*, which limits how much higher homotopy an object or morphism carries.
- *Connectivity*, which measures how far an object or morphism is from being trivial.

Truncation. A morphism $f\colon X \to Y$ is said to be (-2)-*truncated* if it is an isomorphism. Inductively, f is called n-*truncated* if its diagonal

$$\Delta_f\colon X \longrightarrow X \times_Y X$$

is $(n-1)$-truncated. An object X is n-truncated if the terminal map $X \to *$ is n-truncated. In particular, (-1)-truncated morphisms are precisely the monomorphisms—maps whose fibers contain no nontrivial homotopical information. A 0-truncated object is often called *static*, in the sense that it carries no higher or "animated" structure.

Connectivity. Conversely, an object X of a topos $\mathcal{T}$ is said to be n-*connected* if its n-truncation vanishes:

$$\tau_n X \simeq *.$$

Equivalently, all its homotopy groups $\pi_k(X)$ vanish for $k \leq n$. A morphism $f\colon X \to Y$ is n-connected if it is so in the slice topos $\mathcal{T}_{/Y}$. For instance, a morphism is (-1)-connected if and only if it is an effective epimorphism, and "connected" means 1-connected.

Idea *Truncation and connectivity provide dual measures of homotopical complexity. Truncation limits the* height *of higher structure, while connectivity measures its* depth.

These notions are not independent: together they organize the homotopical geometry internal to a topos. For each integer $n \geq -2$, the interplay between n-connected and n-truncated morphisms defines a canonical *factorization system* on $\mathcal{T}$,

$$(\,n\text{-connected},\ n\text{-truncated}\,).$$

This generalizes the classical effective epi–mono factorization that appears when $n = -1$. Every morphism f admits a factorization

$$f \;=\; (n\text{-truncated}) \;\circ\; (n\text{-connected}),$$

and the two classes are orthogonal in the sense of Definition 2.2.16. These factorization systems fit coherently into the Postnikov tower

$$* \simeq \mathcal{T}_{\leq -2} \subseteq \mathcal{T}_{\leq -1} \subseteq \mathcal{T}_{\leq 0} \subseteq \mathcal{T}_{\leq 1} \subseteq \cdots \subseteq \mathcal{T}_{\leq \infty} = \mathcal{T}^{\text{hyp}} \subseteq \mathcal{T}.$$

This tower expresses that every object can be reconstructed from its successive truncations; each stage records the next layer of nontrivial homotopy. For $n \geq 0$, one defines the *n-th homotopy group* of X as

$$\pi_n(X) \coloneqq \tau_0\big(X^{S^n} \to X\big) \in \mathcal{T}_{/X},$$

which records how spheres of dimension n map into X in a pointed sense.

The hierarchy of factorization systems thus encodes a topos's internal homotopy theory. At the lowest level, (-1)-connected morphisms are effective epimorphisms, and (-1)-truncated morphisms are monomorphisms. At higher levels, the same pattern persists: every morphism factors uniquely through layers of increasing connectivity and bounded truncation, reflecting a universal balance between surjectivity and injectivity in the homotopical sense.

An *n-gerbe* in $\mathcal{T}$ is an object that is both $(n-1)$-connected and n-truncated. Equivalently, it is a geometric object whose only nontrivial homotopy sheaf occurs in degree n. When such a gerbe is pointed, it is called an *Eilenberg–MacLane object of degree n*. These pointed n-gerbes correspond to commutative group objects in the discrete topos $\mathcal{T}_{\leq 0}$. If A is such a group object, its associated Eilenberg–MacLane object $\mathrm{B}^n A$ represents cohomology:

$$H^n(Y;A) \;\simeq\; \pi_0\,\mathrm{Hom}_{\mathcal{T}}(Y,\mathrm{B}^n A).$$

Thus, classical sheaf cohomology appears as the degree-0 case of this higher-gerbe hierarchy.

1.2.2 Goodwillie calculus

The *calculus of functors* introduced by Goodwillie mirrors classical differential calculus: it studies functors between higher categories via polynomial and linear approximations. Just as a smooth map $f\colon M \to N$ is locally approximated by its derivative, a homotopy functor $F\colon \mathcal{C} \to \mathcal{D}$ is locally approximated by its first derivative ∂F, a stable functor between the stabilizations of $\mathcal{C}$ and $\mathcal{D}$.

Classical Calculus	**Goodwillie Calculus**
Riemannian manifold M	Category $\mathcal{C}$
Point $x \in M$	Object $X \in \mathcal{C}$
Tangent space $T_x M$	Stabilization $\mathrm{Sp}(\mathcal{C}_{/X})$
Exponential map $\exp_x\colon T_x M \to M$	Destabilization $\Omega^\infty\colon \mathrm{Sp}(\mathcal{C}_{/X}) \to \mathcal{C}$
Smooth map $f\colon M \to N$	Functor $F\colon \mathcal{C} \to \mathcal{D}$
Derivative $\mathrm{d}_x f$	Linearization $\partial_X F$
Higher derivatives $\mathrm{d}_x^n f$	Higher excisive approximations $\partial^n F$

Remark 1.2.1. In the Goodwillie calculus seminar, Marc suggested a Riemannian manifold as the appropriate analogy, noting the parallel between the exponential map and the Ω^∞ functor.

Goodwillie's theory reveals that homotopy functors admit Taylor-like towers

$$F \longrightarrow P_1 F \longrightarrow P_2 F \longrightarrow \cdots,$$

where each $P_n F$ is the universal n-excisive approximation to F. In this sense, Goodwillie calculus stands as the "differential geometry of higher categories": it linearizes and analyzes how homotopical constructions behave infinitesimally.

While the full machinery belongs to higher algebra, its conceptual role in these notes is parallel to that of the Postnikov tower: the Postnikov tower decomposes objects by connectivity, and the Goodwillie tower decomposes functors by excision. The two hierarchies interact through stabilization and descent, offering a unified framework for understanding how local homotopy data assemble into global geometric phenomena.

Chapter 2

Foundations

The aim of this chapter is to develop the foundational homotopy theory of a topos, building upon the ideas sketched in § 1.2.1. We will make precise the notions of truncation and connectivity, introduce the associated factorization systems, and explain how these structures govern the homotopical geometry internal to a topos. A more detailed introduction is still in progress.

2.1 The definition of topos

In [Lurie, 2009], Lurie defines a topos $\mathcal{T}$ as an accessible left-exact localization of a presheaf category $\mathsf{PShv}(\mathcal{C})$. This description is elegant and technically powerful, but it obscures the essential geometric intuition behind topoi—namely, the *descent principle* that governs how objects glue along coverings. Following Rezk's approach, we will instead define a topos as a presentable category that satisfies the descent principle. This perspective emphasizes the homotopy-theoretic nature of geometry: the defining feature of a topos is not its presentation as a localization, but its ability to reconstruct objects from their local data in a homotopically coherent way.

2.1.1 Topoi and descent

Consider the evaluation functor

$$\mathrm{ev}_1 \colon \mathsf{Fun}([1], \mathcal{C}) \longrightarrow \mathcal{C}.$$

This is a cartesian fibration. By straightening, we obtain a functor

$$\mathcal{C}_{/-} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat},$$

which sends an object X to the slice category $\mathcal{C}_{/X}$ and a map $X \to Y$ to the pullback functor

$$(Z \xrightarrow{f} Y) \longmapsto (Z \times_Y X \to X).$$

Definition 2.1.1. Let $\mathcal{C}$ be a category. We say that a colimit in $\mathcal{C}$ is *van Kampen*, or *satisfies descent*, if it is preserved by the functor

$$\mathcal{C}_{/-} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}.$$

Definition 2.1.2. (Rezk) A *topos* is a presentable category $\mathcal{T}$ satisfying descent for all colimits.

To understand this condition, we first describe an alternative presentation of the limit $\lim_i \mathcal{T}_{/X_i}$ in terms of cartesian natural transformations.

Definition 2.1.3. Given two functors $F, G \colon \mathcal{I} \to \mathcal{T}$, a natural transformation $\alpha \colon F \Rightarrow G$ is called *cartesian* if for every map $i \to j$ in $\mathcal{I}$, the naturality square

$$\begin{tikzcd} F(i) \arrow[r, "\alpha(i)"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G(i) \arrow[d] \\ F(j) \arrow[r, "\alpha(j)"] & G(j) \end{tikzcd}$$

is a pullback square. We denote by $\mathsf{Fun}^{\mathrm{cart}}(\mathcal{I}, \mathcal{T}) \subseteq \mathsf{Fun}(\mathcal{I}, \mathcal{T})$ the wide subcategory spanned by the cartesian natural transformations.

Given a diagram $X_\bullet \colon \mathcal{I} \to \mathcal{T}$, evaluation at $i \in \mathcal{I}$ determines a functor

$$\mathrm{ev}_i \colon \mathsf{Fun}^{\mathrm{cart}}(\mathcal{I}, \mathcal{T})_{/X_\bullet} \longrightarrow \mathcal{T}_{/X_i}.$$

By the definition of cartesianness, these evaluation functors are compatible with base change, inducing a comparison functor

$$\mathsf{Fun}^{\mathrm{cart}}(\mathcal{I}, \mathcal{T})_{/X_\bullet} \longrightarrow \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{T}_{/X_i}.$$

Lemma 2.1.4. *Let $\mathcal{C}$ be a category with pullbacks and let $X_\bullet \colon \mathcal{I} \to \mathcal{C}$ be a functor. Then the comparison functor above is an equivalence.*

Intuitively, this implies that objects of the limit can be identified with $\mathcal{I}$-indexed diagrams $Y_\bullet$ in $\mathcal{C}$ equipped with a cartesian natural transformation $Y_\bullet \Rightarrow X_\bullet$.

Proof. By the general description of limits in Cat, the limit of a diagram $\mathcal{I}^{\mathrm{op}} \to \mathsf{Cat}$ can be computed as the category of cartesian sections of the cartesian unstraightening of the diagram. Since $\mathcal{C}_{/-}$ is the straightening of ev_1, one can form $\mathcal{C}_{/X_\bullet}$ as the pullback

$$\begin{tikzcd} \mathcal{C}_{/X_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}([1], \mathcal{C}) \arrow[d] \\ \mathcal{I} \arrow[r, "X_\bullet"] & \mathcal{C}. \end{tikzcd}$$

Hence $\lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{C}_{/X_i}$ is equivalent to the category of cartesian sections $\Gamma^{\mathrm{ct}}_{\mathcal{I}}(\mathcal{C}_{/X_\bullet})$. From the pullback diagram, a cartesian section $s \colon \mathcal{I} \to \mathcal{C}_{/X_\bullet}$ corresponds to a functor $F \colon \mathcal{I} \to \mathsf{Fun}([1], \mathcal{C})$ such that the composition $\mathrm{ev}_1 \circ F$ is $X_\bullet$. Under currying/uncurrying, we observe that F corresponds to a functor $\tilde{F} \colon [1] \to \mathsf{Fun}(\mathcal{I}, \mathcal{C})$. Since ev_1-cartesian maps in $\mathsf{Fun}([1], \mathcal{C})$ correspond precisely to pullback squares in $\mathcal{C}$, $\tilde{F}$ represents a cartesian natural transformation. This proves the claim. $\square$

Now consider the colimit functor

$$\mathrm{colim} \colon \mathsf{Fun}^{\mathrm{cart}}(\mathcal{I}, \mathcal{T}) \longrightarrow \mathcal{T}.$$

Lemma 2.1.5. *Let $\mathcal{T}$ be a presentable category and let $X = \mathrm{colim}_i X_i$ be a colimit in $\mathcal{T}$. Then the functor $\mathcal{T}_{/X} \to \lim_i \mathcal{T}_{/X_i}$ admits a left adjoint*

$$\mathrm{colim} \colon \lim_i \mathcal{T}_{/X_i} \to \mathcal{T}_{/X}$$

sending a natural transformation $Y_\bullet \Rightarrow X_\bullet$ to its colimit $\mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$.

Proof. The functor $\mathrm{colim} \colon \mathsf{Fun}(\mathcal{I}, \mathcal{T}) \to \mathcal{T}$ is left adjoint to the constant functor $\mathrm{const} \colon \mathcal{T} \to \mathsf{Fun}(\mathcal{I}, \mathcal{T})$. By [Lurie, 2009, Proposition 5.2.5.1], given a diagram $X_\bullet \in \mathsf{Fun}(\mathcal{I}, \mathcal{T})$, the adjunction above induces an adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{I}, \mathcal{T})_{/X_\bullet} \arrow[r, shift left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathcal{T}_{/X} \arrow[l, shift left] \end{tikzcd}$$

where the right adjoint is given by the composite $\mathcal{I}_{/X} \xrightarrow{\mathrm{const}} \mathsf{Fun}(\mathcal{I}, \mathcal{T})_{\mathrm{const}\, X} \to \mathsf{Fun}(\mathcal{I}, \mathcal{T})_{/X_\bullet}$, in which the second map is given by base change along the colimit cocone $X_\bullet \to \mathrm{const}_X$. We now make the following two observations:

- The category $\mathsf{Fun}(\mathcal{I}, \mathcal{T})_{/X_\bullet}$ contains the limit $\lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{C}_{/X_i} \simeq \mathsf{Fun}^{\mathrm{cart}}(\mathcal{I}, \mathcal{T})_{/X_\bullet}$ as a full subcategory: given natural transformations $Y_\bullet \Rightarrow Z_\bullet \Rightarrow X_\bullet$, if both $Z_\bullet \Rightarrow X_\bullet$ and $Y_\bullet \Rightarrow X_\bullet$ are cartesian, then so is $Y_\bullet \Rightarrow Z_\bullet$ by the pasting law for pullback squares.
- Since any transformation $\mathrm{const}\, Y \Rightarrow \mathrm{const}\, X$ between constant diagrams is cartesian, and cartesian transformations are closed under base change, the right adjoint $\mathcal{T}_{/X} \to \mathsf{Fun}(\mathcal{I}, \mathcal{T})_{/X_\bullet}$ lands in the full subcategory $\lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{C}_{/X_i}$.

$\square$

Observation 2.1.6. (Characterization of van Kampen colimits) Let $X \coloneqq \mathrm{colim}_{i \in \mathcal{I}} X_i$. Then X is a van Kampen colimit if and only if the adjunction above is an equivalence of categories. Equivalently, the unit and counit of the adjunction are pointwise isomorphisms, described explicitly as follows:

- The *counit* at $Y \in \mathcal{T}_{/X}$ is the canonical map
$$\operatorname{colim}_i(Y \times_X X_i) \longrightarrow Y.$$
This map is an isomorphism precisely when the colimit $X = \operatorname{colim}_i X_i$ is *universal*.

- The *unit* at $(Y_i)_i \in \lim_i \mathcal{T}_{/X_i}$ is the natural map
$$(Y_i)_i \longrightarrow \big((\operatorname{colim}_i Y_i) \times_X X_i\big)_i.$$
Thus, this map being an isomorphism is equivalent to the condition that the extended natural transformation $\overline{Y}_\bullet \Rightarrow \overline{X}_\bullet$ of colimit cocones $\mathcal{I}^\triangleright \to \mathcal{T}$ is cartesian:
$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y := \operatorname{colim}_i Y_i \arrow[d] \\ X_i \arrow[r] & X := \operatorname{colim}_i X_i. \end{tikzcd}$$

Example 2.1.7. By [Lurie, 2018a, Tag 05W0], the category of anima An is a topos.

Exercise 2.1.8. Determine which colimits are van Kampen in the following cases:

- Set and $\mathsf{An}_{\leq n}$;
- a stable cocomplete category;
- $\mathsf{An}^{\mathrm{op}}$.

Example 2.1.9. (van Kampen property) Let $\mathcal{T}$ be a topos.

- Let G be a group in $\mathcal{T}$, and let $\mathrm{B}G := |G|$ be its classifying object. Then, by Observation 2.1.6, we have
$$\mathcal{T}_{/\mathrm{B}G} \simeq \lim_{[n] \in \Delta^{\mathrm{op}}} \mathcal{T}_{/G^n}.$$
The right-hand side is equivalent to the category of left G-modules in $\mathcal{T}$ (where "module" is taken with respect to the cartesian monoidal structure). Therefore,
$$\mathcal{T}_{/\mathrm{B}G} \simeq \mathsf{LMod}_G(\mathcal{T}).$$

- (Grothendieck construction) Let $A \in \mathsf{An} \subseteq \mathsf{Cat}$ be an anima. As an anima, A can be expressed as the colimit of its points: $A \simeq \operatorname{colim}_{a \in A} *$. Applying the van Kampen property (which states that $\mathcal{T}_{/(-)}$ sends colimits to limits) yields the following chain of equivalences:
$$\mathsf{Fun}(A, \mathcal{T}) \simeq \lim_{a \in A} \mathcal{T}_{/*} \simeq \mathcal{T}_{/\operatorname{colim}_{a \in A} *} \simeq \mathcal{T}_{/A}.$$
This equivalence identifies functors $A \to \mathcal{T}$ with objects of $\mathcal{T}$ parameterized by A, i.e., this is the Grothendieck construction.

There is a neat way to formulate descent using colimits in the category
$$\mathsf{Fun}^{\mathrm{pb}}([1], \mathcal{T})$$
of pullback squares in $\mathcal{T}$, defined as the *wide* subcategory of $\mathsf{Fun}([1], \mathcal{T})$ with the same objects (arrows of $\mathcal{T}$) and whose maps are precisely the pullback squares in $\mathcal{T}$.

Proposition 2.1.10. (Rezk) *A presentable category $\mathcal{T}$ is a topos if and only if $\mathsf{Fun}^{\mathrm{pb}}([1], \mathcal{T})$ admits colimits and the inclusion*
$$\mathsf{Fun}^{\mathrm{pb}}([1], \mathcal{T}) \hookrightarrow \mathsf{Fun}([1], \mathcal{T})$$
preserves colimits.

Proof. Assume $\mathcal{T}$ is a topos. We claim that the inclusion $\mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T}) \hookrightarrow \mathsf{Fun}([1],\mathcal{T})$ is closed under colimits and preserves them. A diagram $\mathcal{I} \to \mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ corresponds (by currying) to a cartesian natural transformation $Y_\bullet \Rightarrow X_\bullet$ in $\mathsf{Fun}(\mathcal{I},\mathcal{T})$. Form its colimit cone in $\mathcal{T}$:

$$\overline{Y}_\bullet \Rightarrow \overline{X}_\bullet.$$

By the *unit* description in Observation 2.1.6, for a van Kampen colimit the comparison maps remain pullbacks; hence $\overline{Y}_\bullet \Rightarrow \overline{X}_\bullet$ is again cartesian. Equivalently, the induced cocone $\mathcal{I}^{\triangleright} \to \mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ exists, and its image in $\mathsf{Fun}([1],\mathcal{T})$ is a colimit cocone. Concretely, write the induced square in $\mathcal{T}$ as

$$\begin{tikzcd} \operatorname{colim}_i Y_i \arrow[r] \arrow[d] & A \arrow[d] \\ \operatorname{colim}_i X_i \arrow[r] & B \end{tikzcd}$$

and note that it is a pullback if and only if for each i the composite squares

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim}_i Y_i \arrow[r] \arrow[d] & A \arrow[d] \\ X_i \arrow[r] & \operatorname{colim}_i X_i \arrow[r] & B \end{tikzcd}$$

are pullbacks. This is an immediate consequence of the universality of colimits, specifically the counit described in Observation 2.1.6. Therefore $\mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ admits colimits and the inclusion preserves them.

Conversely, assume $\mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ admits colimits and the inclusion preserves them. Let $X = \operatorname{colim}_i X_i$ be any colimit in $\mathcal{T}$. Consider the adjunction from Observation 2.1.6:

$$\begin{tikzcd} \lim_i \mathcal{T}_{/X_i} \arrow[r, shift left, "(Y_i)\mapsto \operatorname{colim}_i Y_i"] \arrow[r, phantom, "\perp"] & \mathcal{T}_{/X} \arrow[l, shift left, "Y \mapsto (Y \times_X X_i)"] \end{tikzcd}$$

along with the associated *unit* and *counit*. The hypothesis implies that colimit cocones in $\mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ and in $\mathsf{Fun}([1],\mathcal{T})$ agree; therefore the unit comparison

$$(Y_i)_i \longrightarrow \big((\operatorname{colim}_i Y_i) \times_X X_i\big)_i$$

is pointwise an isomorphism (the "unit isomorphism" in Observation 2.1.6). For the counit, observe that the square

$$\begin{tikzcd} \operatorname{colim}_i (X_i \times_X Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{colim}_i X_i \arrow[r, equal] & X \end{tikzcd}$$

lies in $\mathsf{Fun}^{\mathrm{pb}}([1],\mathcal{T})$ and its image is a colimit square in $\mathsf{Fun}([1],\mathcal{T})$; hence it is a pullback, implying that the counit

$$\operatorname{colim}_i (Y \times_X X_i) \longrightarrow Y$$

is an isomorphism. Thus, both unit and counit are pointwise isomorphisms, so the adjunction is an equivalence. Equivalently, X is a van Kampen colimit. As X was arbitrary, all colimits are van Kampen and $\mathcal{T}$ is a topos. ☐

Our goal in this section is to study several equivalent characterizations of a topos.

Theorem 2.1.11. *Let $\mathcal{T}$ be a presentable category. The following conditions are equivalent:*

1. *$\mathcal{T}$ is a topos.*
2. *Colimits in $\mathcal{T}$ are universal, and $\mathcal{T}$ admits an object classifier.*
3. *$\mathcal{T}$ satisfies* Giraud's axioms*:*
   a) *Colimits in $\mathcal{T}$ are universal.*
   b) *Coproducts in $\mathcal{T}$ are disjoint.*
   c) *Every groupoid object in $\mathcal{T}$ is effective.*
4. *There exists a small category $\mathcal{C}$ such that $\mathcal{T}$ is a left–exact localization of* $\mathsf{PShv}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{An})$.

In what follows, we introduce each of the concepts appearing in Theorem 2.1.11, and then proceed to prove it.

2.1.2 Some terminology

Let $\mathcal{T}$ be a presentable category. Let $\widehat{\mathsf{An}}$ denote the category of large animae in a very large universe. We define the *free cocompletion* of $\mathcal{T}$ in the very large universe, denoted by $\widehat{\mathcal{T}}$, to be the full subcategory

$$\widehat{\mathcal{T}} \subseteq \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \widehat{\mathsf{An}})$$

spanned by those presheaves that preserve small limits.

Definition 2.1.12. (Object classifier) An *object classifier* for $\mathcal{T}$ is an object $\Omega \in \widehat{\mathcal{T}}$ such that, for every $X \in \mathcal{T}$, there is a natural equivalence

$$\mathrm{Hom}_{\widehat{\mathcal{T}}}(X, \Omega) \simeq \left(\mathcal{T}_{/X}\right)^{\simeq},$$

where $\left(\mathcal{T}_{/X}\right)^{\simeq}$ denotes the groupoid core of the slice category $\mathcal{T}_{/X}$.

Definition 2.1.13. (Disjoint coproducts) Let $\mathcal{C}$ be a category that admits finite coproducts. We say that coproducts in $\mathcal{C}$ are *disjoint* if, for every pair of objects $X, Y \in \mathcal{C}$, the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd}$$

is a pullback square.

Next, we introduce groupoid objects and recall their relation with Segal objects.

Definition 2.1.14. (Groupoid object) Let $\mathcal{C}$ be a category that admits finite limits, and let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. We say that $X_\bullet$ is a *groupoid object* if for $n \geq 2$ and every (not necessarily order-preserving) partition

$$[n] = S \cup T, \qquad S = \{i_0, \cdots, i_k\}, \quad T = \{i_k, \cdots, i_n\},$$

the induced diagram

$$\begin{tikzcd} X_n = X([n]) \arrow[r, "{(i_0,\cdots,i_k)^*}"] \arrow[d, "{(i_k,\cdots,i_n)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X(S) = X_k \arrow[d, "i_k^*"] \\ X_{n-k} = X(T) \arrow[r, "i_k^*"] & X(\{i_k\}) = X_1 \end{tikzcd}$$

is a pullback square in $\mathcal{C}$. Let $\mathsf{Gpd}(\mathcal{C}) \subseteq \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory spanned by groupoid objects.

Definition 2.1.15. (Segal object) Let $\mathcal{C}$ be a category that admits finite limits. A *Segal object* in $\mathcal{C}$ is a simplicial object $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that, for every $n \geq 2$, the natural map

$$X_n \longrightarrow \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ factors}}$$

is an equivalence. Intuitively, $X_\bullet$ encodes an internal category in $\mathcal{C}$.

A groupoid object is precisely a Segal object in which all maps are invertible. Concretely, for every partition $[n] = S \cup T$ with $S \cap T = \{i_k\}$, the natural map

$$X([n]) \longrightarrow X(S) \times_{X(\{i_k\})} X(T)$$

must be an equivalence; this is exactly the condition in Definition 2.1.14.

Lemma 2.1.16. ([Lurie, 2009, Proposition 6.1.2.11]) *Let $\mathcal{C}$ be a category and consider an augmented simplicial object $U_\bullet^+ \colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$. The following are equivalent:*

1. *The diagram $U_\bullet^+$ is right Kan extended from $(\Delta_+^{\leq 0})^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$;*
2. *The simplicial object $U_\bullet := U_\bullet^+ |_{\Delta^{\mathrm{op}}}$ is a groupoid object and the square*

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$

is a pullback square.

Definition 2.1.17. (Čech nerve) Let $\mathcal{C}$ admit finite products. The functor

$$\mathrm{ev}_{[0]} \colon \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathcal{C}, \qquad X_\bullet \longmapsto X_0,$$

admits a right adjoint

$$\check{C}_\bullet \colon \mathcal{C} \longrightarrow \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

given by right Kan extension along $i \colon [0] \hookrightarrow \Delta^{\mathrm{op}}$. We refer to $\check{C}_\bullet(X)$ as the *Čech nerve* of X.

Remark 2.1.18. Using the pointwise formula for right Kan extensions, for each $0 \leq i \leq n$, the maps $\rho_i \colon \check{C}_n(X) \to \check{C}_0(X)$ induced by $[0] \simeq \{i\} \subseteq [n]$ exhibit

$$\check{C}_n(X) \simeq X^{\times(n+1)}.$$

Definition 2.1.19. (Čech nerve, relative version) Let $\mathcal{C}$ admit pullbacks and let $f \colon X \to Y$ be a map. Regard f as an object of $\mathcal{C}_{/Y}$. The *Čech nerve* $\check{C}_\bullet(f)$ is the Čech nerve of f computed in $\mathcal{C}_{/Y}$.

Remark 2.1.20. Concretely, $\check{C}_\bullet(f)$ can be written as $\check{C}_n(f)$ satisfying

$$\check{C}_n(f) \simeq \underbrace{X \times_Y \cdots \times_Y X}_{n+1 \text{ factors}}.$$

Viewing this as an augmented simplicial object $\check{C}^+_\bullet(f)$ in $\mathcal{C}$ with $\check{C}^+_{-1}(f) := Y$, one checks that it is right Kan extended from $(\Delta^{\leq 0}_+)^{\mathrm{op}} \subseteq \Delta^{\mathrm{op}}_+$, hence $\check{C}_\bullet(f)$ is a groupoid object by Lemma 2.1.16.

Definition 2.1.21. (Effectivity of groupoid objects) Let $\mathcal{C}$ admit finite limits and let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a groupoid object. We say that $X_\bullet$ is *effective* if its geometric realization $|X_\bullet|$ exists and $X_\bullet$ is equivalent to the Čech nerve of the map $X_0 \to |X_\bullet|$.

Hence, condition 3c asserts that *every groupoid object in $\mathcal{T}$ arises as the Čech nerve of some map.*

2.1.3 Proof of Theorem 2.1.11

We now prove Theorem 2.1.11. The relationships among the four equivalent characterizations are summarized schematically by the following diagram:

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (3) \arrow[d, Rightarrow, "\text{Proposition 2.1.22}"] \\
(2) & (4) \arrow[ul, Rightarrow]
\end{tikzcd}$$

1. $\Leftrightarrow$ 2. The existence of an object classifier is equivalent to the condition that the functor

$$(\mathcal{T}_{/-})^{\simeq} \colon \mathcal{T} \longrightarrow \widehat{\mathsf{An}}$$

preserves all limits. By the adjoint functor theorem, this holds precisely when the functor admits a left adjoint, which gives rise to the classifier object. Suppose condition 1 holds. Then, by Observation 2.1.6, for every colimit $X = \mathrm{colim}_i X_i$ in $\mathcal{T}$, there is an equivalence of categories

$$\mathcal{T}_{/X} \simeq \lim_i \mathcal{T}_{/X_i}.$$

Since the groupoid-core functor $(-)^{\simeq}$ is a right adjoint, it preserves limits, and thus $(\mathcal{T}_{/-})^{\simeq}$ preserves limits. Hence, an object classifier exists. Furthermore, by Observation 2.1.6, all colimits in $\mathcal{T}$ are universal.

Conversely, assume condition 2. By the characterization of the object classifier, the functor $\mathcal{T}_{/X} \to \lim_i \mathcal{T}_{/X_i}$ is essentially surjective. Using the universality of colimits, one verifies that this functor is fully faithful: the counit in Observation 2.1.6 is an isomorphism. Therefore, all colimits satisfy descent, and $\mathcal{T}$ is a topos.

1. $\Rightarrow$ 3. Suppose $\mathcal{T}$ is a topos.

3a). Universality of colimits follows directly from Observation 2.1.6.

3b). A coproduct in $\mathcal{T}$ is van Kampen if and only if it is universal and disjoint. Indeed, applying Observation 2.1.6 to a coproduct diagram shows that $\mathcal{T}$ is *extensive*, hence coproducts are disjoint.

3c). It remains to verify that every groupoid object is effective. Let $X_\bullet$ be a groupoid object of $\mathcal{T}$. Extend it to an augmented simplicial object $X_\bullet^+$ by setting

$$X_{-1}^+ := \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} X_n.$$

The diagram $X_\bullet^+$ is left Kan extended from $\Delta^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$. By Remark 2.1.20, it suffices to show that $X_\bullet^+$ is also right Kan extended from $(\Delta_+^{\leq 0})^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$. The face map induced by $[-1] \to [0]$ defines a natural transformation $X_{\bullet+1}^+ \Rightarrow X_\bullet^+$ of functors $\Delta_+^{\mathrm{op}} \to \mathcal{T}$. By Definition 2.1.14, the induced transformation $X_{\bullet+1} \Rightarrow X_\bullet$ of simplicial objects is cartesian. Using Observation 2.1.6, we obtain pullback squares of the form

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{colim}_{[m]\in\Delta^{\mathrm{op}}} X_{m+1} \arrow[d] \\
X_n \arrow[r] & X.
\end{tikzcd}$$

Since $X_{\bullet+1}^+$ has extra degeneracies, [Cnossen, 2025, Lemma 4.2.4] implies

$$\operatorname{colim}_{[m]\in\Delta^{\mathrm{op}}} X_{m+1} \simeq X_0.$$

In particular, when $n = 0$, we obtain the pullback square

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d] \\
X_0 \arrow[r] & X_{-1}.
\end{tikzcd}$$

By Lemma 2.1.16, $X_\bullet^+$ is right Kan extended from $(\Delta_+^{\leq 0})^{\mathrm{op}}$, hence $X_\bullet$ is effective.

3. $\Rightarrow$ 4. Suppose $\mathcal{T}$ satisfies Giraud's axioms. Since $\mathcal{T}$ is presentable, there exists a regular cardinal κ and a small category $\mathcal{C}$ that admits finite limits and κ-small colimits such that

$$\mathcal{T} \simeq \operatorname{Ind}_\kappa(\mathcal{C}).$$

Denote by $i\colon \mathcal{C} \hookrightarrow \mathcal{T}$ the canonical inclusion. One has an adjunction

$$\begin{tikzcd}
\mathsf{PShv}(\mathcal{C}) \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{T}, \arrow[l, shift left, "i"]
\end{tikzcd}$$

where F is the left Kan extension $\operatorname{Lan}_{よ} i$. Hence F is a left Bousfield localization. By Proposition 2.1.22, F is moreover left exact, showing that $\mathcal{T}$ is a left exact localization of a presheaf topos.

4. $\Rightarrow$ 1. Finally, if $\mathcal{T}$ is a left exact localization of $\mathsf{PShv}(\mathcal{C})$, then $\mathcal{T}$ inherits the descent property because An is a topos (Example 2.1.7), and left exact localizations preserve finite limits. Thus, $\mathcal{T}$ is a topos.

Proposition 2.1.22. *Let $\mathcal{C}$ be a category that admits finite limits, and let $\mathcal{T}$ be a presentable category satisfying condition 3 in Theorem 2.1.11. Let $f\colon \mathcal{C} \to \mathcal{T}$ be a left exact functor. Then the left Kan extension*

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "よ"'] \arrow[dr, "f"] & \\
\mathsf{PShv}(\mathcal{C}) \arrow[r, dashed, "F"'] & \mathcal{T}
\end{tikzcd}$$

is left exact.

Proof. By [Lurie, 2009, Corollary 4.4.2.4], to prove that F is left exact it suffices to show that F preserves pullbacks and final objects. Let $*_{\mathcal{C}}$ denote the final object of $\mathcal{C}$. Since the Yoneda embedding preserves limits, $よ(*_{\mathcal{C}})$ is the final object of $\mathsf{PShv}(\mathcal{C})$, and since f is left exact, $f(*_{\mathcal{C}})$ is also the final object of $\mathcal{T}$. Hence F preserves final objects.

We now show that F preserves pullbacks. We say that an object $X \in \mathsf{PShv}(\mathcal{C})$ is *good* if, for every pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

in $\mathsf{PShv}(\mathcal{C})$, the induced square

$$\begin{tikzcd} F(W) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F(Z) \arrow[r] & F(X) \end{tikzcd}$$

is a pullback in $\mathcal{T}$. To prove that F is left exact, it therefore suffices to show that every object of $\mathsf{PShv}(\mathcal{C})$ is good.

- By the universality of colimits, every representable $よ(X)$ for some $X \in \mathcal{C}$ is good.
- Since coproducts in $\mathcal{T}$ are disjoint, the coproduct of good objects is good.
- Since $よ\colon \mathcal{C} \to \mathsf{PShv}(\mathcal{C})$ generates $\mathsf{PShv}(\mathcal{C})$ under colimits, it remains to check that coequalizers of good objects are good.

We call a map $f\colon X \to Y$ in $\mathsf{PShv}(\mathcal{C})$ *surjective* if it induces a surjection

$$\pi_0(X(c)) \twoheadrightarrow \pi_0(Y(c))$$

for every $c \in \mathcal{C}$. The canonical map defining a coequalizer is always surjective, so it suffices to show that the property "good" descends along surjective maps in $\mathsf{PShv}(\mathcal{C})$. Suppose $X_0 \to X$ is a surjection in $\mathsf{PShv}(\mathcal{C})$ and that X_0 is good; we show that X is good. Since F is a left Kan extension, it preserves colimits, and both $\mathsf{PShv}(\mathcal{C})$ and $\mathcal{T}$ have universal colimits. Therefore, to verify that X is good, it suffices to test the condition on diagrams generated by representable presheaves. Let Y be a representable presheaf, say $Y = よ(C)$ for some $C \in \mathcal{C}$. Because $\pi_0(X_0) \to \pi_0(X)$ is surjective, every map $Y \to X$ factors (up to connected components) through $X_0 \to X$. Hence it suffices to consider diagrams of the form

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \times_X X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X_0 \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times_X X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d] \\ Y \arrow[r] & X_0 \arrow[r] & X, \end{tikzcd}$$

in which every square is a pullback and both Y, Z are representable presheaves. We must show that the outer rectangle remains a pullback after applying F. Since X_0 is good, this reduces to proving that

$$F(X_0 \times_X X_0) \simeq F(X_0) \times_{F(X)} F(X_0).$$

Let $X_\bullet$ denote the Čech nerve of $X_0 \to X$. By Lemma 2.1.16, $X_\bullet$ is automatically a groupoid object. Since X_0 is good, $F(X_\bullet)$ is again a groupoid object in $\mathcal{T}$. As every groupoid object in $\mathcal{T}$ is effective, we obtain

$$|F(X_\bullet)| \simeq F(X).$$

Therefore, X is good. Thus F preserves pullbacks and final objects, and hence is left exact. ☐

2.2 Truncation and connectivity

2.2.1 Truncation and connectivity

A key difference between ordinary and higher category theory is that, in a higher category, maps and objects carry genuine homotopical information. This allows one to speak of how much "higher structure" a map contains (*truncation*), and how far an object is from being trivial (*connectivity*). Both notions play a fundamental role in the homotopy theory internal to a topos.

Truncation

Definition 2.2.1. Let $\mathcal{C}$ be a category that admits pullbacks.

1. A map $f\colon X \to Y$ is (-2)-*truncated* if it is an isomorphism.
2. For $n \geq -2$, a map f is n-*truncated* if its diagonal

$$\Delta_f\colon X \longrightarrow X \times_Y X$$

   is $(n-1)$-truncated. Write $\mathsf{Trun}_n \subseteq \mathsf{Ar}(\mathcal{C})$ for the full subcategory spanned by n-truncated maps.
3. An object $X \in \mathcal{C}$ is n-*truncated* if its terminal map $X \to *$ is n-truncated. Let $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ denote the full subcategory spanned by n-truncated objects.
4. If the inclusion $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits a left adjoint, we denote it by

$$\tau_n\colon \mathcal{C} \to \mathcal{C}_{\leq n},$$

   and call τ_n the n-*truncation functor*.
5. A map (or object) is *truncated* if it is n-truncated for some $n \geq -2$.

In An, truncation can be tested on homotopy groups.

Lemma 2.2.2. *Let $X \in \mathsf{An}$ be an anima. For $n \geq -1$, X is n-truncated if and only if for every $i > n$ and every point $x \in X$,*

$$\pi_i(X, x) \simeq *.$$

When $n = -2$, X is n-truncated precisely when it is contractible. Moreover, for a map $f\colon X \to Y$ of animae, f is n-truncated if and only if for every point $y \in Y$, the fiber

$$f^{-1}(y) \coloneqq \{y\} \times_Y X$$

is n-truncated.

Proof. Left as an exercise for the reader. □

Remark 2.2.3. A map $f\colon X \to Y$ is n-truncated if and only if, viewed as an object of the slice category $\mathcal{C}_{/Y}$, it is n-truncated there. This local viewpoint is often convenient when reasoning about fibers.

Truncation generalizes the classical idea of "injectivity".

Remark 2.2.4. (-1)-truncated maps are exactly the monomorphisms.

Remark 2.2.5. A 0-truncated object is sometimes called *static*: it carries no higher, or "animated", information. The full subcategory $\mathcal{C}_{\leq 0}$ corresponds to the classical topos.

Lemma 2.2.6. ([Lurie, 2009, Lemma 5.5.6.15]) *Let $\mathcal{C}$ be a category admitting finite limits, $f\colon X \to Y$ a map, and $n \geq -1$. Then the following are equivalent:*

1. *f is an n-truncated map;*
2. *For every $C \in \mathcal{C}$, the induced map*

$$\mathrm{Hom}_{\mathcal{C}}(C, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{C}}(C, Y)$$

   is n-truncated in An.

The n-truncated objects form a reflective subcategory.

Proposition 2.2.7. ([Lurie, 2009, Proposition 5.5.6.18]) *Let $\mathcal{C}$ be a presentable category and $n \geq -2$. Then the inclusion $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits a left adjoint*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\tau_n"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq n} \arrow[l, shift left, hook] \end{tikzcd}$$

exhibiting τ_n as an accessible left Bousfield localization.

Sketch of proof. Since $\mathcal{C}$ is presentable, it is tensored and cotensored over An (cf. [Lurie, 2009, Remark 5.5.1.7 and Remark 5.5.2.6]); that is, there exist functors

$$- \otimes - \colon \mathsf{An} \times \mathcal{C} \to \mathcal{C}, \qquad (-)^{(-)} \colon \mathsf{An}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C},$$

satisfying $* \otimes X \simeq X \simeq X^*$ for all $X \in \mathcal{C}$. We also have natural identifications

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes C, X) \simeq \mathrm{Hom}_{\mathcal{C}}(C, X^A) \simeq \mathrm{Hom}_{\mathcal{C}}(C, X)^A$$

for all $C \in \mathcal{C}$ and anima $A \in \mathsf{An}$. Let S^n denote the n-sphere in An. By a simple induction, one observes that an anima A is n-truncated if and only if the map $A \to A^{S^{n+1}}$ induced by the canonical map $S^{n+1} \to *$ is an isomorphism. Therefore, an object $X \in \mathcal{C}$ is n-truncated if and only if the map

$$X \longrightarrow X^{S^{n+1}}$$

induced by $S^{n+1} \to *$ is an isomorphism. Since $\mathcal{C}$ is presentable, there exists a regular cardinal κ such that the κ-compact objects detect isomorphisms. Hence, the n-truncated objects are precisely the local objects with respect to the class of maps

$$\{\, S^{n+1} \otimes Y \longrightarrow Y \simeq * \otimes Y \mid Y \text{ is } \kappa\text{-compact} \,\}.$$

Consequently, $\mathcal{C}_{\leq n}$ is an accessible left Bousfield localization of $\mathcal{C}$, with localization functor τ_n. □

Lemma 2.2.8. *Let $\mathcal{C}$ be a category. Then the class of n-truncated maps in $\mathcal{C}$ is closed under base change.*

Proof. Consider a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

Since the Yoneda embedding $よ \colon \mathcal{C} \to \mathsf{PShv}(\mathcal{C})$ preserves limits, for every $C \in \mathcal{C}$ we obtain a pullback square in An:

$$\begin{tikzcd} \mathrm{Hom}(C, X') \arrow[r] \arrow[d, "f' \circ -"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}(C, X) \arrow[d, "f \circ -"] \\ \mathrm{Hom}(C, Y') \arrow[r] & \mathrm{Hom}(C, Y). \end{tikzcd}$$

In An, a commutative square is a pullback if and only if it induces isomorphisms on all its fibers. By Lemma 2.2.2, $f' \circ -$ is n-truncated in An whenever $f \circ -$ is. The claim follows from Lemma 2.2.6. □

Lemma 2.2.9. *Let $\mathcal{C}$ be a category, and let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps.*

1. *If g is n-truncated, then f is n-truncated if and only if $g \circ f$ is n-truncated.*
2. *If gf is n-truncated and g is $(n+1)$-truncated, then f is n-truncated.*

Proof.
1. We prove the claim by induction on n. For $n = -2$, the fact follows from the 2-out-of-3 property for isomorphisms. For $n \geq -1$, consider the following commutative triangle:

$$\begin{tikzcd} & X \times_Y X \arrow[dr] & \\ X \arrow[ur, "\Delta_f"] \arrow[rr, "\Delta_{gf}"] & & X \times_Z X. \end{tikzcd}$$

   The right diagonal map is the base change of $\Delta_g \colon Y \to Y \times_Z Y$, hence is $(n-1)$-truncated. Thus, by the induction hypothesis, Δ_f is $(n-1)$-truncated if and only if Δ_{gf} is $(n-1)$-truncated.

2. We may factor f as the composite

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

   The first map is the base change of the n-truncated map $\Delta_g \colon Y \to Y \times_Z Y$, while the second map is a base change of $gf \colon X \to Z$. Thus f is n-truncated.

□

Lemma 2.2.10. *Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$.*

1. *If F preserves pullbacks, then it preserves n-truncated objects for $n \geq -2$.*
2. *If F is furthermore a left adjoint, and $\mathcal{C}$, $\mathcal{D}$ admit n-truncation functors, then F commutes with n-truncation: for any $X \in \mathcal{C}$, the induced map*

$$\tau_n(F(X)) \to F(\tau_n X)$$

is an isomorphism.

Proof. 1. Since for $n \geq -1$, n-truncatedness is formulated in terms of pullbacks, and a (-2)-truncated map is an isomorphism, we obtain the result.

2. If F is a left adjoint, then its right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ preserves limits, and hence pullbacks. Thus G preserves n-truncated objects.

□

Corollary 2.2.11. *Let $\mathcal{T}$ be a topos, and let $f\colon X \to Y$ be a map in $\mathcal{T}$. Then the pullback functor $f^*\colon \mathcal{T}_{/Y} \to \mathcal{T}_{/X}$ preserves n-truncated objects and commutes with n-truncation: for any $Z \in \mathcal{T}_{/Y}$, we have $\tau_n(f^*Z/X) \simeq f^*\tau_n(Z/Y)$.*

Connectivity

Definition 2.2.12. Let $\mathcal{T}$ be a topos and $n \geq -2$.

- An object $X \in \mathcal{T}$ is *n-connected* if $\tau_n X \simeq *$, or equivalently, if $\pi_k(X) \simeq *$ for all $k \leq n$.
- A map $f\colon X \to Y$ is *n-connected* if it is n-connected in the slice $\mathcal{T}_{/Y}$.
- Denote by $\mathsf{Conn}_n(\mathcal{T}) \subseteq \mathsf{Ar}(\mathcal{T})$ the full subcategory spanned by n-connected maps.
- Let $\mathcal{T}^{\geq n+1} \subseteq \mathcal{T}$ be the full subcategory spanned by n-connected objects.

Example 2.2.13.
- (-2)-connected maps are isomorphisms.
- (-1)-connected maps are *effective epimorphisms* (see Section 2.2.2 for the definition and Lemma 2.2.36 for the proof).
- Every object is (-2)-connected.

Factorization system

Connectivity and truncation are orthogonal notions: an n-connected map is precisely one that has the left lifting property against all n-truncated ones. This observation leads to the general theory of factorization systems.

Definition 2.2.14. (Orthogonality) Let $\mathcal{C}$ be a category. A map $l\colon A \to B$ is *left orthogonal* to $r\colon X \to Y$, written $l \perp r$, if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the anima of dashed fillers is contractible.

Remark 2.2.15. It is equivalent to saying that the following square is a pullback square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[r, "- \circ l"] \arrow[d, "r \circ -"] & \mathrm{Hom}_{\mathcal{C}}(A, X) \arrow[d, "r \circ -"] \\ \mathrm{Hom}_{\mathcal{C}}(B, Y) \arrow[r, "- \circ l"] & \mathrm{Hom}_{\mathcal{C}}(A, Y). \end{tikzcd}$$

For any class S of maps, write $S^\perp$ for the right-orthogonal class and ${}^\perp S$ for the left-orthogonal one.

Definition 2.2.16. (Factorization system) Let $\mathcal{C}$ be a category. A pair of classes of maps $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ is a *factorization system* if:

1. Every map $f\colon X \to Y$ factors as
$$X \xrightarrow{l} Z \xrightarrow{r} Y,$$
with $l \in \mathcal{L}$ and $r \in \mathcal{R}$;

2. $\mathcal{L} \perp \mathcal{R}$, i.e., each map in $\mathcal{L}$ is left orthogonal to each map in $\mathcal{R}$.

Remark 2.2.17. For $n \geq -2$, a map l is n-connected if and only if $l \perp r$ for all n-truncated r.

For later purposes, we will record some properties of factorization systems in a presentable category.

Proposition 2.2.18. ([Anel et al., 2022, Proposition 3.1.11]) *Let $(\mathcal{L}, \mathcal{R})$ be a factorization system on $\mathcal{C}$. Then:*

1. *$\mathcal{L}^{\perp} = \mathcal{R}$ and ${}^{\perp}\mathcal{R} = \mathcal{L}$;*

2. *$\mathcal{L}$ is right cancellable (i.e., $g \circ f \in \mathcal{L}$ and $f \in \mathcal{L} \implies g \in \mathcal{L}$), and $\mathcal{R}$ is left cancellable (i.e., $g \circ f \in \mathcal{R}$ and $g \in \mathcal{R} \implies f \in \mathcal{R}$);*

3. *$\mathcal{L}$ is closed under colimits, while $\mathcal{R}$ is closed under limits;*

4. *$\mathcal{L} \cap \mathcal{R}$ is the class of isomorphisms.*

Definition 2.2.19. Let $\mathcal{C}$ be a presentable category, and let $\mathcal{L}$ be a class of maps in $\mathcal{C}$. We say that $\mathcal{L}$ is *saturated* if it satisfies the following properties:

1. $\mathcal{L}$ contains all isomorphisms.

2. $\mathcal{L}$ is closed under compositions.

3. The full subcategory of $\mathsf{Fun}([1], \mathcal{C})$ spanned by $\mathcal{L}$ is closed under colimits.

Remark 2.2.20. Sometimes, $\mathcal{L}$ is additionally asked to be closed under pushouts. However, this is automatic: given maps $f\colon X \to Y$ and $g\colon X \to X'$, the pushout map $f'\colon X' \to Y \sqcup_X X'$ can be regarded as a pushout in $\mathsf{Fun}([1], \mathcal{C})$ of f, id_X and $\mathrm{id}_{X'}$.

Observation 2.2.21. Note that the intersection of a collection of saturated classes is again saturated. In particular, if S denotes a class of maps in $\mathcal{C}$, there exists a smallest saturated class $\overline{S}$ containing S, called the *saturation* of S. We say that a saturated class is *of small generation* if it is of the form $\overline{S}$ for a small collection of maps S.

Proposition 2.2.22. ([Lurie, 2009, Proposition 5.5.5.7]) *Let $\mathcal{C}$ be a presentable category and let $\mathcal{L}$ be a saturated subcategory. Then the pair $(\mathcal{L}, \mathcal{L}^{\perp})$ is a factorization system on $\mathcal{C}$.*

In a topos, one expects the canonical hierarchy of factorization systems
$$\big(\mathsf{Conn}_n,\ \mathsf{Trun}_n\big), \qquad n \geq -2,$$
where every map factors uniquely as an n-connected map followed by an n-truncated one. The base case $n = -1$ already yields the classical effective–epi/mono factorization:
$$\big(\mathsf{Conn}_{-1},\ \mathsf{Trun}_{-1}\big) = (\text{Effective epimorphisms}, \text{Monomorphisms}).$$
We will establish this in detail in Section 2.2.2, before extending it to all $n \geq -2$ levels.

2.2.2 Effective epimorphism

In higher category theory, the classical notion of an *epimorphism* (Definition 3.1.5) is generally not very useful. It cannot be used to describe the quotient. Fortunately, the situation is not as hopeless as it may appear: one can still define quotients in higher category theory using the more robust notion of an *effective epimorphism*.

Definition 2.2.23. Let $\mathcal{C}$ be a category that admits pullbacks. A map $f\colon X \to Y$ is called an *effective epimorphism* if its Čech nerve $\check{C}(f) \to Y$ is a colimit diagram. We denote by $\mathsf{EffEpi}(\mathcal{C}) \subseteq \mathsf{Ar}(\mathcal{C})$ the full subcategory spanned by the effective epimorphisms.

Remark 2.2.24. One observes the following:

- If groupoid objects in $\mathcal{C}$ are effective, then there is an equivalence of categories

$$\begin{tikzcd} \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) & & \mathsf{Ar}(\mathcal{C}) \\ \cup\!\! \mid & & \cup\!\! \mid \\ \mathsf{Gpd}(\mathcal{C}) \arrow[dr, "\mathrm{ev}_{[0]}"'] & & \mathsf{EffEpi}(\mathcal{C}) \arrow[ll, "\check{C}", "\sim"'] \arrow[dl, "s"] \\ & \mathcal{C} & \end{tikzcd}$$

- If colimits of groupoid objects are universal, then effective epimorphisms are closed under pullback.

Lemma 2.2.25. *Let $\mathcal{C}$ be a category admitting pullbacks such that colimits of groupoid objects are universal. Then:*

1. *For every effective epimorphism $f\colon X \to Y$, the pullback functor*

$$f^*\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$$

is conservative.

2. EffEpi *is closed under base change in $\mathcal{C}$.*

Proof. 1. By Observation 2.1.6, it suffices to show that the unit and counit of the relevant adjunction are pointwise isomorphisms. It is straightforward to show that the units are pointwise isomorphisms. The fact that the counit is also an isomorphism follows from our assumption on the universality of colimits. Therefore, we have the equivalence:

$$\mathcal{C}_{/Y} \xrightarrow{\sim} \lim_n \mathcal{C}_{/\check{C}_n(f)}.$$

In particular, the functors $\mathcal{C}_{/Y} \to \mathcal{C}_{/\check{C}_n(f)}$ are jointly conservative. By the construction of the Čech nerve, each of these functors factors through the pullback functor $f^*\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$, which proves the claim.

2. Let $g\colon Y' \to Y$ be any map, and consider the pullback diagram:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

We need to show that f' is also an effective epimorphism. First, we note that the Čech nerve of f', $\check{C}(f')$, is the base change of the Čech nerve of f along g: $\check{C}(f') \simeq \check{C}(f) \times_Y Y'$. We can then compute the colimit of $\check{C}(f')$:

$$\operatorname{colim} \check{C}(f') \simeq \operatorname{colim} \left(\check{C}(f) \times_Y Y'\right) \simeq \operatorname{colim} \check{C}(f) \times_Y Y' \simeq X \times_Y Y' \simeq X'.$$

Here, the second isomorphism holds because colimits of groupoid objects are universal, and the third isomorphism holds because f is an effective epimorphism by assumption. Since $\operatorname{colim} \check{C}(f') \simeq X'$, f' is by definition an effective epimorphism.

□

Next, we show that under suitable hypotheses, effective epimorphisms and monomorphisms together form a factorization system on $\mathcal{C}$.

Proposition 2.2.26. *Let $\mathcal{C}$ be a category that admits pullbacks and satisfies the following:*

- *Every groupoid object in $\mathcal{C}$ is effective.*

- *Colimits of groupoid objects in $\mathcal{C}$ are universal.*

Then the pair $(\mathsf{EffEpi}, \mathsf{Mono})$ *forms a factorization system on $\mathcal{C}$, where* $\mathsf{Mono} = \mathsf{Trun}_{-1}$ *denotes the full subcategory spanned by the monomorphisms.*

Proof. Let $f\colon X \to Y$ be a map, and set $V = \operatorname{colim} \check{C}(f)$, the colimit of the Čech nerve of f. We obtain a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & V \arrow[ur, "m"'] & \end{tikzcd}$$

Since groupoid objects in $\mathcal{C}$ are effective, we have $\check{C}(f) = \check{C}(e)$. Hence e is an effective epimorphism.

We now show that m is a monomorphism. By Definition 2.2.1, this is equivalent to the diagonal map $\Delta_m\colon V \to V \times_Y V$ being an isomorphism. We may identify V with $V \times_Y V$. Since colimits of groupoid objects are universal, it suffices to prove that $\check{C}(e) \times_V \check{C}(e) \to \check{C}(f) \times_Y \check{C}(f)$ is an isomorphism. Concretely, for each $m, n \geq 0$, the induced map

$$p_{m,n}\colon \check{C}_m(e) \times_V \check{C}_n(e) \longrightarrow \check{C}_m(f) \times_Y \check{C}_n(f)$$

is a pullback of

$$p_{0,0}\colon X \times_V X \to X \times_Y X.$$

The left and right sides can be identified with $\check{C}_1(e)$ and $\check{C}_1(f)$, respectively, so $p_{0,0}$ is an isomorphism by the equality $\check{C}_1(e) \simeq \check{C}_1(f)$.

Next, let $l \in \mathsf{EffEpi}$ and $r \in \mathsf{Mono}$, and consider a commutative square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, two heads, "l"'] & E \arrow[d, hook, "r"] \\ X \arrow[r] & Y. \end{tikzcd}$$

We must show that there exists an essentially unique map $X \to E$ making both triangles commute. Factor $X \to Y$ as above:

$$X \xrightarrow{e} V \xrightarrow{m} Y$$

with $e \in \mathsf{EffEpi}$ and $m \in \mathsf{Mono}$. We may enlarge the diagram to

$$\begin{tikzcd} & & & E \arrow[d, hook] \\ Z \arrow[r, two heads] \arrow[urrr] & X \arrow[r, two heads] & V \arrow[r, hook] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since e is effective, the object V is the colimit of the Čech nerve of $X \to V$. The map $Z \to E$ induces compatible maps from the Čech nerve of $X \to V$ into E. By the universal property of this colimit, we obtain a map $V \to E$ making the entire diagram commute. Finally, since r (and m) is a monomorphism, the space of such lifts is contractible. Therefore, every map in EffEpi is left orthogonal to every map in Mono, and together with the factorization constructed above, this shows that $(\mathsf{EffEpi}, \mathsf{Mono})$ forms a factorization system on $\mathcal{C}$. $\square$

This factorization system yields several immediate consequences.

Corollary 2.2.27. *Let $\mathcal{C}$ be a category satisfying the hypotheses of Proposition 2.2.26. If $f\colon X \to Y$ admits a section, then f is an effective epimorphism.*

Proof. If f admits a section s, then the Čech nerve $\check{C}(f)$ carries extra degeneracies. Hence $\operatorname{colim} \check{C}(f) = Y$, so f is an effective epimorphism. $\square$

Corollary 2.2.28. *Let $\mathcal{C}$ be a category satisfying the hypotheses of Proposition 2.2.26. Given a commutative triangle*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z, \end{tikzcd}$$

if $gf \in \mathsf{EffEpi}$, then $g \in \mathsf{EffEpi}$.

Proof. We may factor the map g as
$$Y \xhookrightarrow{i_Y} Y \sqcup_X Z \xrightarrow{\langle g, \mathrm{id}_Z\rangle} Z,$$
where i_Y is the inclusion of the Y-component and $\langle g, \mathrm{id}_Z\rangle$ is induced by $g\colon Y \to Z$ and $\mathrm{id}_Z\colon Z \to Z$. The first map i_Y is an effective epimorphism since it can be described as a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "gf"'] & Y \arrow[d] \\ Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Z \sqcup_X Y. \end{tikzcd}$$

The second map $\langle g, \mathrm{id}_Z\rangle$ is an isomorphism, since it can be written as the pushout

$$\begin{tikzcd} Y \sqcup_X Y \arrow[r, "Y \sqcup_X f"] \arrow[d, "\nabla_f"'] & Y \sqcup_X Z \arrow[d, "\langle g, \mathrm{id}_Z\rangle"] \\ Y \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Z. \end{tikzcd}$$

For any map f, the fold map ∇_f admits a section; hence, by Corollary 2.2.27, it is an effective epimorphism, and by Proposition 2.2.18, an isomorphism. Consequently, g is an effective epimorphism. $\square$

We now turn to the *delooping principle*, which expresses the equivalence between connected pointed objects and group objects in a topos. Intuitively, it asserts that the passage between a group G and its classifying object $\mathrm{B}G$ behaves internally to any topos exactly as it does in spaces.

Definition 2.2.29. Let $\mathcal{T}$ be a topos. A *pointed connected object* is an effective epimorphism
$$* \twoheadrightarrow X.$$
We denote by $\mathcal{T}_*^{\geq 1}$ the full subcategory of $\mathcal{T}_*$ spanned by such pointed connected objects.

Proposition 2.2.30. (Delooping Principle) *For any topos $\mathcal{T}$, there is an equivalence*

$$\mathsf{Grp}(\mathcal{T}) \underset{\Omega}{\overset{\mathrm{B}}{\underset{\leftarrow}{\overset{\rightarrow}{\simeq}}}} \mathcal{T}_*^{\geq 1}$$

between group objects and pointed connected objects, given by the usual bar and loop constructions.

Proof. The equivalence $\mathsf{Gpd}(\mathcal{T}) \simeq \mathsf{EffEpi}(\mathcal{T})$ from Remark 2.2.24 fits into a commutative triangle

$$\begin{tikzcd} \mathsf{Gpd}(\mathcal{T}) \arrow[rr, "\check{C}", "\sim"'] \arrow[dr, "X_\bullet \mapsto X_0"'] & & \mathsf{EffEpi}(\mathcal{T}) \arrow[dl, "(U \to X) \mapsto U"] \\ & \mathcal{T}. & \end{tikzcd}$$

Taking the fiber over the terminal object $* \in \mathcal{T}$ yields the desired equivalence $\mathsf{Grp}(\mathcal{T}) \simeq \mathcal{T}_*^{\geq 1}$. $\square$

Remark 2.2.31. In other words, every pointed connected object in $\mathcal{T}$ arises as the classifying object $\mathrm{B}G$ of a unique group object $G \in \mathsf{Grp}(\mathcal{T})$, and conversely, taking the loop object ΩX recovers the corresponding group.

We conclude this discussion with a useful lemma that connects the notion of effective epimorphism in a topos to that in its 0-truncation.

Lemma 2.2.32. *Let $\mathcal{T}$ be a topos. A map $X \to Y$ is an effective epimorphism if and only if $\tau_0 X \to \tau_0 Y$ is an effective epimorphism in the classical topos $\tau_0\mathcal{T}$.*

Proof.
- For the "only if" direction, it suffices to show that the canonical map $p\colon X \to \tau_0 X$ is an effective epimorphism. By Proposition 2.2.26, we may factor p as
$$X \twoheadrightarrow V \hookrightarrow \tau_0 X,$$
where $V \in \tau_0\mathcal{T}$ since it is a subobject of a 0-truncated object. The inclusion $V \hookrightarrow \tau_0 X$ is a monomorphism of 0-truncated objects, and τ_0 is left adjoint to the inclusion, so there exists a section $X \to V$. By Corollary 2.2.27, this shows that $V \hookrightarrow \tau_0 X$ is an effective epimorphism, hence an isomorphism. Thus, p is an effective epimorphism.

- Conversely, consider the diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d] & V \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ \tau_0 X \arrow[r] & \tau_0 V \arrow[r] & \tau_0 Y. \end{tikzcd}$$

  The right square is cartesian and $\tau_0 V \to \tau_0 Y$ is a monomorphism. Indeed, in An, a monomorphism induces an injection on path components, and since $\mathcal{T}$ is a left exact localization of $\mathsf{PShv}(\mathcal{C})$, this property is preserved in $\mathcal{T}$. Since $\tau_0 X \to \tau_0 Y$ is an effective epimorphism, the monomorphism $\tau_0 V \hookrightarrow \tau_0 Y$ must be an isomorphism, hence $V \hookrightarrow Y$ is an isomorphism. Therefore $X \to Y$ is an effective epimorphism.

□

Corollary 2.2.33. *Let $\mathcal{T}$ be a topos and $*$ its terminal object. Then the map $* \to X$ is an effective epimorphism if and only if $\tau_0 X \simeq *$.*

Using Lemma 2.2.32, we can establish several fundamental exactness properties of a topos.

Proposition 2.2.34. (Exactness Properties) *Let $\mathcal{T}$ be a topos. Then in $\mathcal{T}$:*

1. *Filtered colimits commute with finite limits.*
2. *Sifted colimits commute with finite products.*
3. *Let $A \in \mathsf{An}$ be an anima. Then A-indexed colimits commute with weakly contractible limits. For example,*
$$(X \times_Z Y)/G \;\simeq\; X/G \times_{Z/G} Y/G.$$
4. *Consider a diagram of simplicial objects*
$$\begin{tikzcd} & Y_\bullet \arrow[d] \\ X_\bullet \arrow[r] & Z_\bullet, \end{tikzcd}$$
where $X_\bullet$, $Y_\bullet$, and $Z_\bullet$ are simplicial objects in $\mathcal{T}$. If $\tau_0 Z_\bullet$ is constant, then
$$\operatorname{colim}(X_n \times_{Z_n} Y_n) \;\simeq\; \operatorname{colim} X_n \times_{\operatorname{colim} Z_n} \operatorname{colim} Y_n.$$

Before proving Proposition 2.2.34, we recall a basic structural fact.

Remark 2.2.35. Let $\mathcal{T}$ be a topos and let $X \in \mathcal{T}$ be an object. Then the slice category $\mathcal{T}_{/X}$ is again a topos. This follows from the fact that the projection functor $\mathcal{T}_{/X} \to \mathcal{T}$ preserves colimits and pullbacks and is conservative.

Proof of Proposition 2.2.34. 1. The first statement follows from the corresponding fact in An.[1]

2. Follows from 1, the siftedness of Δ^{op}, and the fact that sifted colimits are generated by filtered colimits and geometric realizations.
3. By Example 2.1.9, we have a commutative diagram
$$\begin{tikzcd} \mathsf{Fun}(A, \mathcal{T}) \arrow[rr, phantom, "\simeq"] \arrow[dr, "\operatorname{colim}_A"'] & & \mathcal{T}_{/\operatorname{colim}_A *} \arrow[dl, "F"] \\ & \mathcal{T} & \end{tikzcd}$$
   where F denotes the forgetful functor. The forgetful functor from a slice category always preserves weakly contractible limits, hence the desired commutation property follows.

[1]See [Haugseng, 2025, § 9.9] for a model-independent proof of this fact.

4. Consider the pullback functor
$$p^*\colon \mathcal{T}_{/\tau_0 Z_0} \longrightarrow \mathcal{T}_{/Z_0}.$$
By Lemma 2.2.32, the map $p\colon Z_0 \to \tau_0 Z_0$ is an effective epimorphism. Therefore, by Lemma 2.2.25, p^* is conservative. In the slice $\mathcal{T}_{/\tau_0 Z_0}$, we may regard $Z_\bullet$ as a pointed simplicial object
$$Z_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{T}_*^{\geq 1}.$$
By the equivalence $\mathcal{T}_*^{\geq 1} \simeq \mathsf{Grp}(\mathcal{T})$ (Proposition 2.2.30), we may identify $Z_\bullet$ with a simplicial group object $\mathrm{B}G_\bullet$. Accordingly, we can write $X_\bullet \simeq X'_\bullet/G_\bullet$ and $Y_\bullet \simeq Y'_\bullet/G_\bullet$, so that
$$X \times_{\mathrm{B}G} Y \;\simeq\; (X' \times Y')/G$$
by statement 3.

□

2.2.3 The (n-connected, n-truncated) factorization system

We now show that effective epimorphisms are precisely the (-1)-connected maps.

Lemma 2.2.36. *Let $\mathcal{T}$ be a topos. A map $f\colon X \to Y$ is an effective epimorphism if and only if it is (-1)-connected.*

Proof. Let $f\colon X \twoheadrightarrow Y$ be an effective epimorphism in $\mathcal{T}$, and regard it as an object T of the slice topos $\mathcal{T}_{/Y}$. By Proposition 2.2.26, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[dr, two heads] & \tau_{-1}T \arrow[r, hook] \arrow[d, hook] & * \\ & V \arrow[ur, hook] & \end{tikzcd}$$

where, by Remark 2.2.3, V is (-1)-truncated. By Proposition 2.2.7, there is a canonical monomorphism
$$\tau_{-1}T \hookrightarrow V.$$
Since $T \twoheadrightarrow V$ is an effective epimorphism, Corollary 2.2.28 implies that the monomorphism $\tau_{-1}T \hookrightarrow V$ is also an effective epimorphism. Hence $\tau_{-1}T \simeq V$, so that $T \twoheadrightarrow \tau_{-1}T$ is an effective epimorphism. As T corresponds to the map $f\colon U \to X$, we obtain $\tau_{-1}T \simeq *$ in $\mathcal{T}_{/X}$, and therefore f is (-1)-connected.

Conversely, assume that f is (-1)-connected. Then in the slice topos $\mathcal{T}_{/Y}$ we have $\tau_{-1}T \simeq *$. By Proposition 2.2.26, f admits a factorization
$$X \xrightarrow{h} V \xhookrightarrow{k} Y$$
where h is an effective epimorphism and k a monomorphism. Applying the argument above to this factorization, we find that
$$\tau_{-1}X \simeq V.$$
Since $\tau_{-1}X \simeq Y$ by assumption, we conclude that $V \simeq Y$. Hence k is an isomorphism, and therefore f is an effective epimorphism. □

Thus Proposition 2.2.26 can be restated as saying that
$$(\mathsf{Conn}_{-1},\, \mathsf{Trun}_{-1})$$
forms a factorization system on $\mathcal{T}$. We will use this base case to deduce the general result.

Proposition 2.2.37. *Let $\mathcal{T}$ be a topos and $n \geq -2$. Then the pair*
$$(\mathsf{Conn}_n,\, \mathsf{Trun}_n)$$
forms a factorization system on $\mathcal{T}$.

Proof. First, we want to show that every map $f\colon X \to Y$ in $\mathcal{T}$ can be factored as an n-connected map followed by an n-truncated map. By Proposition 2.2.7, let

$$X \to \tau_{\le n} f \to Y$$

be a factorization of f such that the second map is n-truncated. It suffices to show that $X \to \tau_{\le n} f$ is an n-connected map. It suffices to check that, for every n-truncated object $U \to \tau_{\le n} f$ in $\mathcal{T}_{/\tau_n f}$, the map

$$\mathrm{Hom}_{\tau_n f}(\tau_n f, U) \to \mathrm{Hom}_{\tau_n f}(X, U)$$

is an isomorphism. Since $\left(\mathcal{T}_{/Y}\right)_{\tau_n f} \simeq \mathcal{T}_{\tau_n f}$, one observes that this map is induced by the following diagram in $\mathcal{T}_{/Y}$:

$$\begin{tikzcd}
\mathrm{Hom}_{/Y}(\tau_n f, U) \arrow[r] \arrow[d] & \mathrm{Hom}_{/Y}(X, U) \arrow[d] \\
\mathrm{Hom}_{/Y}(\tau_n f, \tau_n f) \arrow[r] & \mathrm{Hom}_{/Y}(X, \tau_n f)
\end{tikzcd}$$

Since both U and $\tau_n f$ are n-truncated objects in $\mathcal{T}_{/Y}$, Proposition 2.2.7 ensures that both horizontal maps are isomorphisms, and thus so is the induced map on fibers. Next, we show the orthogonality. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f\in\mathsf{Conn}_n"'] & W \arrow[d, "g\in\mathsf{Trun}_n"] \\
Y \arrow[r] \arrow[ur, dashed] & Z.
\end{tikzcd}$$

We need to show that the anima of dashed fillers is contractible. Since n-truncated maps are closed under base change, we may replace g by the projection $W \times_Z Y \to Y$, reducing the problem to the diagonal fillers in the following commutative square:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f\in\mathsf{Conn}_n"'] & W \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "g\in\mathsf{Trun}_n"] \\
Y \arrow[r, equal] \arrow[ur, dashed] & Y \arrow[r] & Z.
\end{tikzcd}$$

Since f is an n-connected map and the projection $W \times_Z Y \to Y$ is an n-truncated map, we have $\tau_n f \simeq Y$ and we have

$$\mathrm{Hom}_{/Y}(X, W \times_Z Y) \simeq \mathrm{Hom}_{/Y}(\tau_n f, W \times_Z Y) \simeq \mathrm{Hom}_{/Y}(Y, W \times_Z Y).$$

□

We can now record some basic properties of n-connected maps.

Corollary 2.2.38. *Let $\mathcal{T}$ be a topos, and let $n \ge -2$ be an integer. Then:*

1. *If $f\colon X \to Y$ is n-connected, then f is also m-connected for every $m \le n$.*
2. *Suppose we have a cartesian square in $\mathcal{T}$:*

$$\begin{tikzcd}
X' \arrow[r, "q'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y' \arrow[r, "p"'] & Y
\end{tikzcd}$$

   If f is n-connected, then so is f'.

3. *A map $f\colon X \to Y$ is n-connected if and only if the pullback functor*

$$f^*\colon \mathcal{T}_{/Y} \longrightarrow \mathcal{T}_{/X}$$

   is fully faithful when restricted to n-truncated objects.

Proof. 1. Obvious from the definition of n-connected map.

2. By Corollary 2.2.11, we have $\tau_n(X'/Y') \simeq g^*\tau_n(X/Y)$. In particular, if f is n-connected, then we have $\tau_n(X/Y) \simeq Y$, so $\tau_n(X'/Y') \simeq Y'$.

3. We need to show that f is n-connected if and only if for each pair of n-truncated maps $Z \to Y$ and $Z' \to Y$, the induced map

$$f^* \colon \operatorname{Hom}_{/Y}(Z, Z') \to \operatorname{Hom}_{/X}(X \times_Y Z, X \times_Y Z')$$

is an isomorphism. By the universal property of $X \times_Y Z'$, it is equivalent to show that

$$\operatorname{Hom}_{/Y}(Z, Z') \xrightarrow{\sim} \operatorname{Hom}_{/Y}(X \times_Y Z, Z').$$

In other words, for every commutative diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the anima of dashed fillers is contractible. This follows from the argument above and the factorization system in Proposition 2.2.37.

□

2.2.4 Homotopy group

In this section, our goal is to prove the following theorem.

Theorem 2.2.39. *Let $\mathcal{T}$ be a topos and $n \geq -1$ an integer. For a map $f \colon X \to Y$ in $\mathcal{T}$, the following are equivalent:*

1. *f is n-connected;*
2. *f is an effective epimorphism and its diagonal Δ_f is $(n-1)$-connected.*

To prove this theorem, we shall make use of the notion of *homotopy group objects.*

Definition 2.2.40. Let $\mathcal{C}$ be a category with finite products. For $n \geq 0$, we define the notion of an $\mathbb{E}_n$-group in $\mathcal{C}$ iteratively:

- For $n = 0$, we define $\mathsf{Grp}_{\mathbb{E}_0}(\mathcal{C}) \coloneqq \mathcal{C}_*$ to be the category of $\mathbb{E}_0$*-groups* (pointed objects) in $\mathcal{C}$.
- For $n \geq 1$, we define $\mathsf{Grp}_{\mathbb{E}_n}(\mathcal{C}) \coloneqq \mathsf{Grp}(\mathsf{Grp}_{\mathbb{E}_{n-1}}(\mathcal{C}))$ to be the category of $\mathbb{E}_n$*-groups* in $\mathcal{C}$.

If $\mathcal{D}$ admits finite coproducts, an $\mathbb{E}_n$*-cogroup* in $\mathcal{D}$ is defined to be an $\mathbb{E}_n$-group in $\mathcal{D}^{\mathrm{op}}$.

By the proof of Proposition 2.2.7, for any category $\mathcal{C}$, one can consider X^{S^n} where S^n is the n-sphere.

Definition 2.2.41. (Homotopy group objects) Let $\mathcal{T}$ be a topos and let $X \in \mathcal{T}$ be an object. For each integer $n > 0$, the *n-th homotopy group object* of X is defined as

$$\pi_n(X) \coloneqq \tau_0\big(X^{S^n} \to X\big) \in \big(\mathcal{T}_{/X}\big)_{\leq 0} \subseteq \mathcal{T}_{/X},$$

where τ_0 denotes the 0-truncation in the slice topos $\mathcal{T}_{/X}$. Since S^n is an $\mathbb{E}_n$-cogroup in An and the map $* \to S^n$ is a map of $\mathbb{E}_n$-cogroups, $\pi_n(X)$ is an $\mathbb{E}_n$-group. Thus, depending on n, the object $\pi_n(X)$ carries specific algebraic structure:

$$\pi_n(X) \text{ is a } \begin{cases} \text{pointed object}, & n = 0, \\ \text{group object}, & n = 1, \\ \text{abelian group object}, & n \geq 2. \end{cases}$$

From the definition, one can easily deduce the following results.

Lemma 2.2.42. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topoi, and let $F \colon \mathcal{T} \to \mathcal{T}'$ be a functor preserving colimits and finite limits. Then F preserves homotopy groups: for all $X \in \mathcal{T}$ we have a natural isomorphism*

$$F(\pi_n(X)) \simeq \pi_n(F(X)) \in \mathcal{T}_{/F(X)}.$$

Proof. Since $F\colon \mathcal{T}_{/X} \to \mathcal{T}'_{/F(X)}$ preserves finite limits and X^{S^n} is a finite limit, we see that $F(X^{S^n}) \simeq F(X)^{S^n}$. The result then follows from Lemma 2.2.10. $\square$

Lemma 2.2.43. *Let $\mathcal{T}$ be a topos. For every integer $n \geq -2$, the functor*

$$\pi_n\colon \mathcal{T} \longrightarrow \mathcal{T}$$

preserves finite products.

Proof. This reduces to the case $\mathcal{T} = \mathsf{An}$, where it is standard. $\square$

Notation 2.2.44. Let $\mathcal{T}$ be a topos and $f\colon X \to Y$ a map. We denote by

$$\pi_n(f) \in (\mathcal{T}_{/Y})_{/f} \simeq \mathcal{T}_{/X}$$

the n-th homotopy group object of f regarded as an object of the slice topos $\mathcal{T}_{/X}$. (Note: This does **not** refer to the induced map $\pi_n(X) \to \pi_n(Y)$).

Remark 2.2.45.

- Let $f\colon X \to Y$ be a map in $\mathcal{T}$. Viewing f as an object of the slice topos $\mathcal{T}_{/Y}$, we have
$$\pi_0(f) = \tau_0(f).$$
This is an object of $(\mathcal{T}_{/Y})_{/f}$, which under the canonical equivalence $(\mathcal{T}_{/Y})_{/f} \simeq \mathcal{T}_{/X}$ corresponds to
$$\pi_0(f) \simeq X \times_Y \tau_0(f).$$
The natural map $X \to \tau_0(f)$ induces a global section of $\pi_0(f)$, so $\pi_0(f)$ is pointed. In particular, when $f = \mathrm{id}_Y$, we obtain
$$\pi_0(X) \simeq X \times_X \tau_0 X.$$
Equivalently, one may view $\pi_0(X)$ as the fiber product $X \times \tau_0 X$ computed in the slice topos $\mathcal{T}_{/X}$.
- For every integer $n \geq 1$, there is a canonical isomorphism
$$\pi_n(f) \;\simeq\; \pi_{n-1}(\Delta_f),$$
where $\Delta_f\colon X \to X \times_Y X$ denotes the diagonal of f.

Exercise 2.2.46. Show that every subgroup of $\pi_1(X)$ is normal.

Proposition 2.2.47. (Long exact sequence) *Let $\mathcal{T}$ be a topos, and let $f\colon X \to Y$ be a map in $\mathcal{T}$. Then there is a canonical long exact sequence of homotopy group objects in the slice topos $\mathcal{T}_{/Y}$:*

$$\cdots \longrightarrow \pi_n(f) \longrightarrow \pi_n(Y) \longrightarrow f^*\pi_n(X) \longrightarrow \pi_{n-1}(f) \longrightarrow \pi_{n-1}(Y) \longrightarrow \cdots$$

Proof. This follows from the corresponding long exact sequence of homotopy groups in An, using the fact that every topos $\mathcal{T}$ is a left exact localization of An and that the construction of homotopy group objects commutes with base change. $\square$

Now we can use homotopy group objects to characterize the maps appearing in Theorem 2.2.39.

Proposition 2.2.48. *Let $\mathcal{T}$ be a topos and $n \geq -1$ an integer. Let $f\colon X \to Y$ be a map in $\mathcal{T}$.*

1. *If f is n-truncated, then $\pi_k(f) \simeq *$ for all $k > n$. Conversely, if f is m-truncated for some m and $\pi_k(f) \simeq *$ for all $k > n$, then f is n-truncated.*
2. *The map f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) \simeq *$ for all $k < n$.*

Proof.

1. The proof proceeds by induction on n.

   If $n = -1$, a map f is (-1)-truncated if and only if its diagonal Δ_f is an isomorphism. Hence, by the relation $\pi_{k+1}(f) \simeq \pi_k(\Delta_f)$, we deduce that $\pi_{k+1}(f) \simeq *$ for all $k \geq -1$. It remains to show that $\pi_0(f) \simeq *$ when f is (-1)-truncated. Let T denote f regarded as an object of $\mathcal{T}_{/X}$. Then
$$\pi_0(T) \simeq T \times T \in \mathcal{T}_{/X}.$$

Since T is (-1)-truncated, the diagonal $T \to T \times T$ is an isomorphism, so $\pi_0(T) \simeq T$, which means $\pi_0(T) \simeq *$ in the slice $\mathcal{T}_{/T}$, as required.

Assume now that the claim holds for all $(n-1)$-truncated maps with $n \geq 0$. If f is n-truncated, then Δ_f is $(n-1)$-truncated. By the induction hypothesis, $\pi_k(\Delta_f) \simeq *$ for all $k > n-1$. Using $\pi_n(f) \simeq \pi_{n-1}(\Delta_f)$, we deduce that $\pi_k(f) \simeq *$ for all $k > n$.

Conversely, suppose f is m-truncated for some m, and that $\pi_k(f) \simeq *$ for all $k > n$. We proceed by descending induction on n. Assume the statement holds for $(n+1)$; we show it for n. If f is n-truncated and $\pi_n(f) \simeq *$, then

$$\pi_n(f) \simeq \pi_{n-1}(\Delta_f) \simeq *.$$

By the induction hypothesis, this implies that Δ_f is $(n-2)$-truncated, hence f is $(n-1)$-truncated. Finally, for $n = 0$, note that if f is 0-truncated, then $\tau_0(f) \simeq f$, and thus

$$\pi_0(f) \simeq X \times_Y X.$$

Therefore, $\pi_0(f) \simeq *$ means that Δ_f is an isomorphism, which is equivalent to f being (-1)-truncated.

2. By Corollary 2.2.38, for $n \geq -1$, every n-connected map is an effective epimorphism. It remains to show that $\pi_k(f) \simeq *$ for all $k < n$. By Theorem 2.1.11, the topos $\mathcal{T}$ is a left exact localization of a presheaf category:

$$\begin{tikzcd} \mathsf{PShv}(\mathcal{C}) \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{T} \arrow[l, shift left, hook] \end{tikzcd}$$

Let X be an n-connected object in $\mathcal{T}$. Then $\tau_n X \simeq *$. By the adjunction, we can embed X into $\mathsf{PShv}(\mathcal{C})$ and obtain

$$L(\tau_n^{\mathsf{PShv}} X) \simeq \tau_n X \simeq *.$$

Passing to the slice category $\mathsf{PShv}(\mathcal{C})_{/\tau_n^{\mathsf{PShv}} X}$, which is again a presheaf category, we obtain a functor

$$\mathsf{PShv}(\mathcal{C})_{/\tau_n^{\mathsf{PShv}} X} \longrightarrow \mathcal{T}$$

sending an n-connected object X in $\mathsf{PShv}(\mathcal{C})_{/\tau_n^{\mathsf{PShv}} X}$ to X in $\mathcal{T}$. Hence, every n-connected object in $\mathcal{T}$ can be represented by one in the presheaf category, and the statement reduces to the case $\mathcal{T} = \mathsf{An}$, where it is classical.

For the converse, we proceed by induction on n. For $n = -1$, this follows from the fact that effective epimorphisms are (-1)-connected. Assume the claim holds for $n-1$, and let X be such that $\pi_k(X) \simeq *$ for all $k < n$. Consider the truncation map

$$t \colon X \longrightarrow \tau_n X.$$

By Proposition 2.2.37, t is n-connected. By Proposition 2.2.47, we have

$$\pi_k(X) \simeq t^* \pi_k(\tau_n X) \quad \text{for all } k \leq n.$$

By assumption, $t^*\pi_k(\tau_n X) \simeq *$, so by Lemma 2.2.32, t is an effective epimorphism. By part 1, we have $\tau_n X \simeq \tau_{-1} X$. Since $X \to *$ is an effective epimorphism, we get $\tau_{-1} X \simeq *$, hence $\tau_n X \simeq *$. Therefore X is n-connected.

□

Now, we return to the proof of Theorem 2.2.39.

Proof of Theorem 2.2.39. By Proposition 2.2.48, a map f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) \simeq *$ for all $k \leq n$. Similarly, Δ_f is $(n-1)$-connected if and only if it is an effective epimorphism and $\pi_{k-1}(\Delta_f) \simeq \pi_k(f) \simeq *$ for $k \leq n$. Thus, it remains to show that if f is 0-connected, then Δ_f is an effective epimorphism. By Lemma 2.2.32, it suffices to check that the induced map $\tau_0 X \to \tau_0 X \times \tau_0 X$ is an effective epimorphism. But by assumption f is 0-connected, hence $\tau_0 X \simeq *$, and the map $* \to * \times *$ is indeed an effective epimorphism. This completes the proof. □

Next, we provide some consequences.

Corollary 2.2.49. *Let $\mathcal{T}$ be a topos, and let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps in $\mathcal{T}$. If gf is n-connected and g is $(n+1)$-connected, then f is n-connected.*

Proof. This is the dual statement of Lemma 2.2.9 (2). ☐

Corollary 2.2.50. *Let $\mathcal{T}$ be a topos, and let $f\colon X \to Y$ be an n-connected map. Then the pullback functor*

$$f^*\colon \mathcal{T}_{/Y} \longrightarrow \mathcal{T}_{/X}$$

is an equivalence when restricted to $(n-1)$-truncated objects; that is,

$$f^*\colon (\mathcal{T}_{/Y})_{\leq n-1} \xrightarrow{\sim} (\mathcal{T}_{/X})_{\leq n-1}$$

is an equivalence of categories.

Proof. The full faithfulness of f^* was established in point 3 of Corollary 2.2.38. It remains to show that f^* is essentially surjective. Replacing $\mathcal{T}$ by the slice $\mathcal{T}_{/Y}$, we may assume $Y = *$. Let $U \to X$ be an $(n-1)$-truncated map. We must show that it lies in the essential image of the functor

$$(-) \times X\colon \mathcal{T}_{\leq n-1} \longrightarrow \mathcal{T}_{\leq n-1}.$$

We claim that the canonical map

$$U \longrightarrow \tau_{n-1}U \times X$$

is an isomorphism. By the factorization system of Proposition 2.2.37 and the properties in Proposition 2.2.18, it suffices to verify that this map is both $(n-1)$-truncated and $(n-1)$-connected.

- The map is $(n-1)$-truncated because it lies over X, and both structural maps to X are $(n-1)$-truncated; the claim then follows by right cancellation.
- To see that it is $(n-1)$-connected, observe that it factors as

  $$U \longrightarrow U \times X \longrightarrow \tau_{n-1}U \times X.$$

  The first map is the base change of the diagonal $\Delta\colon X \to X \times X$, which is $(n-1)$-connected since X is n-connected by assumption. The second map is the base change of $U \to \tau_{n-1}U$, which is $(n-1)$-connected by the proof of Proposition 2.2.37.

Hence $U \to \tau_{n-1}U \times X$ is an isomorphism, proving essential surjectivity. ☐

Corollary 2.2.51. *For any topos $\mathcal{T}$ and integer $n \geq 0$, there is an equivalence*

$$\mathsf{Grp}_{\mathbb{E}_n}(\mathcal{T}) \underset{\Omega^n}{\overset{\mathrm{B}^n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathcal{T}_*^{\geq n}$$

between $\mathbb{E}_n$-group objects and n-connected pointed objects.

Proof. The proof proceeds by induction on n.

If $n = 0$, the statement is tautological. For $n = 1$, it follows from Proposition 2.2.30. Assume the claim holds for $n - 1$, so that

$$\mathsf{Grp}_{\mathbb{E}_{n-1}}(\mathcal{T}) \simeq \mathcal{T}_*^{\geq n-1}.$$

Then

$$\mathsf{Grp}_{\mathbb{E}_n}(\mathcal{T}) \simeq \mathsf{Grp}\big(\mathsf{Grp}_{\mathbb{E}_{n-1}}(\mathcal{T})\big) \simeq \mathsf{Grp}\big(\mathcal{T}_*^{\geq n-1}\big).$$

By Proposition 2.2.30, we have an equivalence

$$\mathsf{Grp}(\mathcal{T}_*^{\geq n-1}) \underset{\Omega^n}{\overset{\mathrm{B}^n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathcal{T}_*^{\geq n}$$

as required. ☐

2.2.5 ∞-connected maps

One can also consider the notion of ∞-connected maps.

Definition 2.2.52. Let $\mathcal{T}$ be a topos. A map $f\colon X \to Y$ is called ∞-*connected* if, for every n, f is n-connected. It is called ∞-*truncated* (or *hypercomplete*) if it is right orthogonal to the class of ∞-connected maps.

From the definition, one can easily imply the following consequences.

Proposition 2.2.53. *Let $\mathcal{T}$ be a topos. Then:*

1. $(\mathsf{Conn}_\infty, \mathsf{Trun}_\infty)$ *is a factorization system on $\mathcal{T}$.*
2. *The ∞-connected maps are closed under base change and diagonals.*
3. *The ∞-connected maps satisfy the 2-out-of-3 property.*

Proof. Left to the reader. $\square$

Definition 2.2.54. Let $\mathcal{T}$ be a topos. The full subcategory

$$\mathcal{T}_{\leq\infty} \subseteq \mathcal{T}$$

spanned by the ∞-truncated objects is called the *hypercompletion* of $\mathcal{T}$, and is also denoted by $\mathcal{T}^{\mathrm{hyp}}$. We say that $\mathcal{T}$ is *hypercomplete* if the canonical functor

$$\mathcal{T} \longrightarrow \mathcal{T}^{\mathrm{hyp}}$$

is an equivalence.

Remark 2.2.55. The term “hypercomplete” is historical but slightly misleading: hypercompleteness is a separation condition rather than a completeness condition. “Postnikov separated” might have been a better term.

2.3 Gerbes

Let $\mathcal{T}$ be a topos, and let $X \in \mathcal{T}$ be an object. One can form the canonical tower of truncations:

$$X \longrightarrow \cdots \longrightarrow \tau_n X \longrightarrow \tau_{n-1}X \longrightarrow \cdots \longrightarrow \tau_0 X \longrightarrow \tau_{-1}X \longrightarrow \tau_{-2}X = *.$$

For each integer $n \geq -2$, the object

$$\tau_n X \in \mathcal{T}_{/\tau_{n-1}X}$$

is both n-truncated and $(n-1)$-connected.

Definition 2.3.1. (Gerbes and Eilenberg–MacLane objects)

- Let $\mathcal{T}$ be a topos and $n \geq -2$ an integer. An n-*gerbe* in $\mathcal{T}$ is an object that is both n-truncated and $(n-1)$-connected. Let

$$\mathsf{Gerb}_n(\mathcal{T}) \subseteq \mathcal{T}$$

denote the full subcategory spanned by the n-gerbes.

- A *pointed n-gerbe* is called an *Eilenberg–MacLane object of degree n*. Denote by

$$\mathsf{EM}_n(\mathcal{T}) \subseteq \mathsf{Gerb}_n(\mathcal{T})$$

the full subcategory spanned by the Eilenberg–MacLane objects of degree n.

Example 2.3.2. Let $K \in \mathsf{Gerb}_{n+1}(\mathcal{T})$. Given two maps $* \overset{a}{\underset{b}{\rightrightarrows}} K$, consider the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "b"] \\ * \arrow[r, "a"] & K. \end{tikzcd}$$

Then P is an n-gerbe. Indeed, the map $b\colon * \to K$ is an n-gerbe by the cancellation properties from Corollary 2.2.49 and Lemma 2.2.9 (2); hence the base change $P \to *$ is also an n-gerbe.

Corollary 2.3.3. *Let $\mathcal{T}$ be a topos and $n \geq 0$. Then the equivalence of Corollary 2.2.51 restricts to the following equivalences:*

- *If $n = 0$, then*
$$\mathsf{EM}_0(\mathcal{T}) \simeq \tau_0(\mathcal{T}_*).$$
- *If $n = 1$, then*
$$\mathsf{EM}_1(\mathcal{T}) \simeq \mathsf{Grp}(\tau_0(\mathcal{T}_*)) \simeq \mathsf{Grp}(\tau_0(\mathcal{T})).$$
- *If $n \geq 2$, then*
$$\mathsf{EM}_n(\mathcal{T}) \simeq \mathsf{Gerb}_n(\mathcal{T})_* \simeq \mathsf{Ab}(\tau_0(\mathcal{T})).$$

Proof. Immediate from Corollary 2.2.51. □

We are interested in n-gerbes because they naturally arise in the problem of lifting along the Postnikov tower. Let X be an object of $\mathcal{T}$, and consider the canonical truncation maps

$$\tau_n X \longrightarrow \tau_{n-1} X.$$

Given an object $Y \in \mathcal{T}$ equipped with a map

$$Y \longrightarrow \tau_{n-1} X,$$

we ask whether it is possible to construct a lift

$$\begin{tikzcd} & \tau_n X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "?"] & \tau_{n-1} X \end{tikzcd}$$

through the projection $\tau_n X \to \tau_{n-1} X$. Since this map is an n-gerbe, the problem of producing such a lift—or describing the obstruction to its existence—is governed precisely by the theory of n-gerbes.

Lemma 2.3.4. *Let $\mathcal{T}$ be a topos. For $n \geq 2$, let X be an n-gerbe of $\mathcal{T}$. Then there exists a unique $A \in \mathsf{Ab}(\mathcal{T}_{\leq 0})$ such that $\pi_n X \simeq X \times A$.*

For $n = 1$, the object $A \in \mathsf{Grp}(\mathcal{T}_{\leq 0})$ is unique if it exists, but it might not exist.

Proof. Since X is $(n-1)$-connected, the functor $X \times -\colon \mathcal{T} \to \mathcal{T}_{/X}$ induces an equivalence $\mathcal{T}_{\leq n-2} \simeq \left(\mathcal{T}_{/X}\right)_{\leq n-2}$ on $(n-2)$-truncated objects, by Corollary 2.2.50. In particular, the abelian group object $\pi_n(X) \in \mathsf{Ab}\left(\left(\mathcal{T}_{/X}\right)_{\leq 0}\right)$ corresponds to an object $A \in \mathsf{Ab}\left(\mathcal{T}_{\leq 0}\right)$ under this equivalence. □

We describe this situation by saying that the n-gerbe X is *banded* by A.

Definition 2.3.5. Let $\mathcal{T}$ be a topos, let $X \in \mathsf{Gerb}_n(\mathcal{T})$ be an n-gerbe, and consider $A \in \mathsf{Ab}\left(\mathcal{T}_{\leq 0}\right)$. We say that X is *banded by A* if it comes equipped with an isomorphism $\pi_n X \simeq X \times A$ in $\mathsf{Ab}\left(\mathcal{T}_{/X}\right)$. We denote by

$$\mathsf{Gerb}_n^A(\mathcal{T})$$

the category of pairs $(X, \pi_n X \simeq X \times A)$ consisting of an n-gerbe X together with an isomorphism $\pi_n X \simeq X \times A$.

For $A \in \mathsf{Ab}\left(\mathcal{T}_{\leq 0}\right)$, the only n-gerbe banded by A which is also an Eilenberg–MacLane object is the *trivial* n-gerbe $\mathrm{B}^n A$.

Theorem 2.3.6. *Let $\mathcal{T}$ be a topos, let $A \in \mathsf{Ab}(\tau_0 \mathcal{T})$, and let $n \geq 1$. Then there is a canonical equivalence of animae*

$$\mathrm{fib}_0\colon\ \mathrm{Hom}_{\mathcal{T}}(X, \mathrm{B}^{n+1} A) \simeq \mathsf{Gerb}_n^A(\mathcal{T}_{/X}), \quad f \mapsto \mathrm{fib}_0(f).$$

Proof. First, note that fib_0 is well defined, since by Example 2.3.2 the map $0\colon * \to \mathrm{B}^{n+1}A$ is an n-gerbe. The claim is that this map is, in fact, the *universal* n-gerbe. To see this, consider an n-gerbe $\widetilde{X} \to X$ banded by A. There is a *unique* cartesian square of the form

$$\begin{tikzcd} \widetilde{X} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ X \arrow[r, dashed] & \mathrm{B}^{n+1}A. \end{tikzcd}$$

More formally, we claim that $\mathrm{Hom}_{\mathsf{Ar}^{\mathrm{pb}}(\mathcal{T})}(\widetilde{X} \to X, * \to \mathrm{B}^{n+1}A)$ is contractible. This statement is local on X: if $U \twoheadrightarrow X$ is an effective epimorphism and the claim holds for $\widetilde{X} \times_X U \to U$, then it also holds for $\widetilde{X} \to X$. We may thus assume that $\widetilde{X} \to X$ admits a section $s\colon X \to \widetilde{X}$, and hence that $\widetilde{X}$ is of the form $X \times \mathrm{B}^n A$. Let us now rewrite the data contained in the pullback square:

- It consists of a map $X \to \mathrm{B}^{n+1}A$ together with an identification of $\widetilde{X} = X \times \mathrm{B}^n A$ with the pullback $P \coloneqq X \times_{\mathrm{B}^{n+1}A} *$.
- An identification $X \times \mathrm{B}^n A \simeq P$ is the same as providing a section $P \to *$, which in turn is equivalent to providing a lift $X \to *$ of the map $X \to \mathrm{B}^{n+1}A$.
- But the anima of pairs of a map $X \to \mathrm{B}^n A$ together with a lift to a map $X \to *$ is just the same as the anima of maps $X \to *$, which is clearly contractible.

This yields the claim. □

Remark 2.3.7. Under this equivalence $\mathcal{T}_{/\mathrm{B}G} \simeq \mathsf{Mod}_G(\mathcal{T})$, passing to fibers over $\mathcal{T}$ (with respect to the forgetful functor $\mathcal{T}_{/\mathrm{B}G} \to \mathcal{T}$ and the quotient functor $\mathsf{Mod}_G(\mathcal{T}) \to \mathcal{T}$) yields

$$\mathrm{Hom}_{\mathcal{T}}(*, \mathrm{B}G) \simeq \{G\text{-torsors}\}.$$

Let us now return to the question of obstruction theory. Given an object X, the map $\tau_n X \to \tau_{n-1} X$ is an n-gerbe banded by $A \in \mathsf{Ab}(\mathcal{T}_{\leq 0})$. We may write it as a pullback:

$$\begin{tikzcd} \tau_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ \tau_{n-1} X \arrow[r] & \mathrm{B}^{n+1}A. \end{tikzcd}$$

In particular, finding a lift $Y \to \tau_n X$ of a map $Y \to \tau_{n-1} X$ is equivalent to lifting

$$\begin{tikzcd} & * \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & \mathrm{B}^{n+1}A. \end{tikzcd}$$

We wish to think of this as the question of whether a certain cohomology class vanishes:

Definition 2.3.8. Let $\mathcal{T}$ be a topos, let $A \in \mathsf{Ab}(\mathcal{T}_{\leq 0})$, and let $Y \in \mathcal{T}$. We define the *n-th cohomology of Y with coefficients in A* as

$$H^n(Y, A) \coloneqq \pi_0 \mathrm{Hom}_{\mathcal{T}}(Y, \mathrm{B}^n A) \in \mathsf{Ab}.$$

2.4 The category of topoi

In this section, we will introduce the category of topoi Top (this notation reflects the idea that the topos is a generalization of a topological space). Since topoi are presentable categories, by the adjoint functor theorem, a geometric morphism $f\colon \mathcal{T} \to \mathcal{S}$ corresponds to a pair of adjoint functors:

- A left exact, colimit-preserving functor $f^*\colon \mathcal{S} \to \mathcal{T}$, which we regard as the *algebraic map* (inverse image).
- Its right adjoint $f_*\colon \mathcal{T} \to \mathcal{S}$, which we regard as the *geometric map* (direct image).

Thus, the geometric map corresponds to the 'topo' aspect, while the algebraic map corresponds to the rest—the 'logie'.

2.4.1 Logoi

Chapter 3

Localization and presentation

3.1 Modality

In this section, we introduce a special type of factorization system, called a *modality*.

In practice, when we study a class of maps (or a property) P in a category, we often expect it to satisfy two natural properties:

- P is closed under composition;
- P is closed under base change.

For a general factorization system $(\mathcal{L}, \mathcal{R})$, Proposition 2.2.18 shows that the right class $\mathcal{R}$ is always closed under base change. However, the same is not true in general for the left class $\mathcal{L}$. When this additional closure property holds, the factorization system enjoys particularly good behavior and is called a *modality*.

Definition 3.1.1. (Modality) Let $\mathcal{C}$ be a category that admits finite limits. A factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ is called a *modality* if the left class $\mathcal{L}$ is closed under base change in $\mathcal{C}$. Equivalently, for every pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

if $f \in \mathcal{L}$, then $f' \in \mathcal{L}$.

Example 3.1.2. Let $\mathcal{T}$ be a topos. Then:

- Both $(\mathsf{Iso}, \mathsf{All})$ and $(\mathsf{All}, \mathsf{Iso})$ are modalities.
- By Corollary 2.2.38, for every $n \geq -2$, the canonical factorization system

  $$(\mathsf{Conn}_n, \mathsf{Trun}_n)$$

  is a modality.

In a topos $\mathcal{T}$, membership in the classes of a modality can be checked after pullback along an effective epimorphism.

Lemma 3.1.3. *Let $\mathcal{T}$ be a topos and let $(\mathcal{L}, \mathcal{R})$ be a modality on $\mathcal{T}$. Then membership in both classes $\mathcal{L}$ and $\mathcal{R}$ can be checked after pullback along an effective epimorphism.*

Proof. We prove the claim for $\mathcal{L}$; the proof for $\mathcal{R}$ is analogous.

Let $f\colon X \to Z$ be a map in $\mathcal{T}$, and let $Z' \to Z$ be an effective epimorphism. Assume that the pullback $f'\colon X \times_Z Z' \to Z'$ is in $\mathcal{L}$. We will show that $f \in \mathcal{L}$.

Since $(\mathcal{L}, \mathcal{R})$ is a factorization system, we can factor f as $X \xrightarrow{l} Y \xrightarrow{r} Z$, where $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Our goal is to show that r is an isomorphism.

We construct the following composite pullback diagram:

$$\begin{tikzcd} X \times_Z Z' \arrow[r] \arrow[d, "l'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "l"] \\ Y \times_Z Z' \arrow[r] \arrow[d, "r'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] \\ Z' \arrow[r] & Z \end{tikzcd}$$

Since $\mathcal{R}$ is closed under base change (as the right class of any factorization system) and $\mathcal{L}$ is closed by the modality assumption, we have $l' \in \mathcal{L}$ and $r' \in \mathcal{R}$.

The composite map $f' \colon X \times_Z Z' \to Z'$ is given by $f' = r' \circ l'$. By our initial assumption, $f' \in \mathcal{L}$. Since $\mathcal{L}$ satisfies the right cancellation property (Definition 2.2.18)—i.e., if $r' \circ l' \in \mathcal{L}$ and $l' \in \mathcal{L}$, then $r' \in \mathcal{L}$—we conclude that $r' \in \mathcal{L}$.

Since $r' \in \mathcal{L}$ and we already know $r' \in \mathcal{R}$, r' must be an isomorphism.

By Lemma 2.2.25, the pullback functor $\mathcal{T}_{/Z} \to \mathcal{T}_{/Z'}$ is conservative because $Z' \to Z$ is an effective epimorphism. Since r' (the pullback of r) is an isomorphism, the conservative functor reflects this property, and thus r must also be an isomorphism.

Finally, since r is an isomorphism and $l \in \mathcal{L}$, the composite $f = r \circ l$ is in $\mathcal{L}$ (as $\mathcal{L}$ is closed under composition with isomorphisms). This completes the proof for $\mathcal{L}$. □

3.1.1 Excursion: Epimorphisms and acyclic maps

We now provide an example of a modality which also serves as an opportunity to discuss the notion of an epimorphism in higher category theory.

Definition 3.1.4. (Cotruncation) Let $\mathcal{C}$ be a category that admits pushouts. We say that a map $f \colon X \to Y$ is *n-cotruncated* if it is n-truncated in $\mathcal{C}^{\mathrm{op}}$.

For $n = -1$, we obtain the dual notion of monomorphisms: the epimorphisms.

Definition 3.1.5. Let $\mathcal{C}$ be a category. A map $f \colon X \to Y$ is called an *epimorphism* if it is (-1)-cotruncated, i.e., if the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a pushout square. We denote by $\mathsf{Epi} \subseteq \mathsf{Ar}(\mathcal{C})$ the full subcategory spanned by epimorphisms.

We denote by $(-)^+ \colon \mathcal{C} \to \mathcal{C}$ the left adjoint to the inclusion of the (-1)-cotruncated objects. This yields a factorization

$$X \to X^+ \to *$$

of the terminal map into an epimorphism followed by a map that is right orthogonal to all epimorphisms.

Warning 3.1.6. The class of epimorphisms is a subclass of effective epimorphisms (in § 3.1.3, we will use the product of modalities to construct Epi from EffEpi); however, the converse does not hold. We will provide a characterization of epimorphisms below.

For instance, let $(X, x) \in \mathsf{An}_*$ be a pointed anima, and consider the inclusion map $i \colon \{x\} \to X$, regarded as a morphism in An. Then:

- i is an epimorphism in An if and only if X is contractible. To see this, observe that the identity map id_X and the constant map $c \colon X \to \{x\} \xrightarrow{i} X$ become homotopic after precomposition with i. Thus, if i is an epimorphism, it follows that id_X is homotopic to c.

- By Proposition 2.2.48, i is an effective epimorphism if and only if X is connected.

Lemma 3.1.7. *Let $\mathcal{T}$ be a topos, and let $\mathcal{L}$ be the collection of n-cotruncated maps in $\mathcal{T}$. Then the pair $(\mathcal{L}, \mathcal{L}^{\perp})$ forms a modality on $\mathcal{T}$.*

Proof. To show that $(\mathcal{L}, \mathcal{L}^{\perp})$ is a modality, we must prove two things:

1. $(\mathcal{L}, \mathcal{L}^\perp)$ is a factorization system.
2. $\mathcal{L}$ is closed under base change.

For (1), by Proposition 2.2.22, it suffices to show that $\mathcal{L}$ is a saturated class of small generation.

- **Saturation:** The class $\mathcal{L}$ contains all isomorphisms and is closed under composition. Since colimits in $\mathsf{Fun}([1], \mathcal{T})$ are computed pointwise, and the class of n-cotruncated maps is closed under colimits in $\mathcal{T}$, $\mathcal{L}$ is also closed under colimits in $\mathsf{Fun}([1], \mathcal{T})$. Thus, $\mathcal{L}$ is saturated.
- **Small Generation:** Fix an object $X \in \mathcal{T}$. Let $\mathcal{P} \subseteq \mathcal{T}_{/X}$ be the full subcategory spanned by the n-cotruncated maps (i.e., maps $Y \to X$ in $\mathcal{L}$). One observes that this category is defined by the following pullback:

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}_{/X} \arrow[d, "- \otimes S^{n+1}"] \\ * \arrow[r, "\mathrm{id}_X"] & \mathcal{T}_{/X}. \end{tikzcd}$$

  By [Lurie, 2009, Proposition 5.4.7.3], the category of accessible categories Acc is closed under limits, so $\mathcal{P}$ is an accessible category. By [Lurie, 2009, Lemma 5.4.4.14], this implies that $\mathcal{L}$ is of small generation.

This establishes that $(\mathcal{L}, \mathcal{L}^\perp)$ is a factorization system.

For (2), we must show that $\mathcal{L}$ is closed under base change. Consider the pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

and assume that $f \in \mathcal{L}$. We want to show $f' \in \mathcal{L}$.

By induction (on the definition of n-cotruncated maps), it suffices to show that the following square, which compares the (dual) diagonals of f and f', is a pullback square:

$$\begin{tikzcd} Y' \sqcup_{X'} Y' \arrow[r] \arrow[d, "\nabla_{f'}"] & Y \sqcup_X Y \arrow[d, "\nabla_f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

For this, note that we have the colimit decomposition $Y \sqcup_X Y \simeq \mathrm{colim}_k \left(Y \sqcup X^{\sqcup k} \sqcup Y\right)$, and similarly for $Y' \sqcup_{X'} Y'$.

By Observation 2.1.6 (which implies that pullbacks commute with colimits in $\mathcal{T}$), it suffices to show that each component square

$$\begin{tikzcd} Y' \sqcup X'^{\sqcup k} \sqcup Y' \arrow[r] \arrow[d] & Y \sqcup X^{\sqcup k} \sqcup Y \arrow[d] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a pullback square. This is true because colimits (like $\sqcup$) are universal in a topos:

$$Y' \sqcup X'^{\sqcup k} \sqcup Y' \simeq (Y \times_Y Y') \sqcup (X \times_Y Y')^{\sqcup k} \sqcup (Y \times_Y Y') \simeq (Y \sqcup X^{\sqcup k} \sqcup Y) \times_Y Y'.$$

This latter expression is precisely the definition of the pullback, so each component square is a pullback, and the proof is complete. $\square$

3.1.2 The Blakers–Massey theorem

TO DO

3.1.3 Product of modality

3.2 Congruence

3.2.1 Congruence and left-exact localization

3.2.2 Monogenic and epigenic congruence

3.3 Sieves and sheaves

Definition 3.3.1. Let $\mathcal{C}$ be a category. A *sieve* on $\mathcal{C}$ is a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ such that the inclusion functor is a right fibration.

Remark 3.3.2. Recall that a functor is said to be a right fibration if it is right orthogonal to the map $\{1\} \to [1]$ (Definition 2.2.14). Thus, Definition 3.3.1 is equivalent to the following stronger condition: let $y \in \mathcal{C}_0$ be an object; then for every map $f\colon x \to y$ in $\mathcal{C}$, the map f itself belongs to $\mathcal{C}_0$ (which implies $x \in \mathcal{C}_0$). In other words, a sieve is a *full* subcategory closed under precomposition.

3.4 Sheaves on topos

3.5 More on sheaves

Chapter 4

Goodwillie calculus

4.1 Polynomial functors

4.2 Goodwillie tower

4.3 Goodwillie tower of topos

Chapter 5

Postnikov completeness and boundness

Chapter 6

Coherence

Appendix A

Algebra

In this appendix, we provide a brief introduction to higher algebra via algebraic patterns. We will use this theory to simplify some technical details[1]. However, the reader's intuition from classical algebra will often serve as a reliable guide for the concepts used in the main text.[2] We recommend [Liu, 2025, Chapter 2][3] or [Haugseng, 2023] for details on algebraic patterns.

A.1 Algebraic pattern

An *algebraic pattern* is a blueprint for a notion of functors on a fixed category that satisfy a Segal condition, suitable for formalizing homotopy-coherent algebra in the cartesian setting. Informally, an algebraic pattern generalizes the active and inert morphisms in operads and designates certain objects to control the Segal condition.

Definition A.1.1. An *algebraic pattern* is a category $\mathcal{O}$ equipped with:

1. A pair of subcategories $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ such that any morphism $f\colon X \to Z$ in $\mathcal{O}$ admits a factorization as the composition
$$X \xrightarrow{\in \mathcal{O}^{\mathrm{int}}} Y \xrightarrow{\in \mathcal{O}^{\mathrm{act}}} Z,$$
and the anima of such factorizations is contractible. The morphisms in $\mathcal{O}^{\mathrm{int}}$ are called *inert morphisms*, and those in $\mathcal{O}^{\mathrm{act}}$ are called *active morphisms*.

2. A full subcategory $\mathcal{O}^{\mathrm{el}} \subseteq \mathcal{O}^{\mathrm{int}}$, whose objects are called *elementary objects*.

For any object $X \in \mathcal{O}$, we write
$$\mathcal{O}^{\mathrm{el}}_{X/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{X/}$$

A morphism of algebraic patterns from $\mathcal{O}$ to $\mathcal{P}$ is a functor $f\colon \mathcal{O} \to \mathcal{P}$ that preserves inert and active morphisms and elementary objects.

We will use $\mathsf{AlgPatt}$ to denote the category of algebraic patterns.

Definition A.1.2. (Trivial Pattern) *Trivial Pattern* Triv is the final object of $\mathsf{AlgPatt}$. The underlying category of trivial pattern is the final category $*$.

Definition A.1.3. (Empty Pattern) *Empty Pattern* $\varnothing$ is the initial object of $\mathsf{AlgPatt}$. The underlying category of empty pattern is the initial category $\varnothing$.

Example A.1.4. (Commutative Pattern) Consider the category of pointed finite sets Fin_* with $\langle n \rangle := (\{0, 1, \cdots, n\}, 0)$. We say a morphism $f\colon \langle n \rangle \to \langle m \rangle$ is:

- *Inert*, if f restricts to an isomorphism $\langle n \rangle \setminus f^{-1}(0) \to \langle m \rangle \setminus \{0\}$.
- *Active*, if $f^{-1}(0) = \{0\}$.

This forms an algebraic pattern by taking $\langle 1 \rangle$ as the single elementary object. We denote this pattern by Comm and refer to it as the *commutative pattern*.

[1] The author thinks it is a more natural way to consider higher algebra.
[2] This appendix provides technical background and may be skipped on a first reading.
[3] In fact, this appendix comes from this note.

Remark A.1.5. In Example A.1.4, if we take $\langle 0\rangle$ and $\langle 1\rangle$ to be the elementary objects, we get a new algebraic pattern $\mathsf{Fin}_*^\natural$.

Example A.1.6. (Associative Pattern) Consider the opposite of the simplex category, $\mathbb{\Delta}^{\mathrm{op}}$. A morphism $f\colon [n] \to [m]$ in $\mathbb{\Delta}^{\mathrm{op}}$ is:

- *Inert*, if its corresponding map $g\colon [m] \to [n]$ in $\mathbb{\Delta}$ is an interval inclusion (i.e., its image is a contiguous block of integers).
- *Active*, if its corresponding map $g\colon [m] \to [n]$ preserves endpoints (i.e., $g(0) = 0$ and $g(m) = n$).

We choose the object $[1]$ as the unique elementary object. This algebraic pattern, denoted Assoc, is called the *associative pattern*.

Remark A.1.7. In Example A.1.6, if we take $[0]$ and $[1]$ to be the elementary objects, we get a new algebraic pattern $\mathbb{\Delta}^{\mathrm{op},\natural}$.

Example A.1.8. (Left Module Pattern) The underlying category of the *left module pattern* LM is the category $\mathbb{\Delta}^{\mathrm{op}} \times [1]$. Consider the functor $T\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$ which sends $[n]$ to $[n] \star [0] \simeq [n+1]$. This functor then induces a functor

$$(T \to \mathrm{id})\colon \mathbb{\Delta}^{\mathrm{op}} \times [1] \to \mathbb{\Delta}^{\mathrm{op}}.$$

The algebraic pattern structure of LM is lifted from Assoc along $(T \to \mathrm{id})$. More precisely,

- $\mathbb{\Delta}^{\mathrm{op}} \times \{1\}$ is precisely Assoc.
- For each $[n] \in \mathbb{\Delta}$, the induced morphism $([n], 0) \to ([n], 1)$ is an inert morphism in LM.
- If $f\colon [h] \to [n]$ is an inert morphism in $\mathbb{\Delta}$ such that $f(h) = n$, then the corresponding morphism $([n], 0) \to ([h], 0)$ is an inert morphism in LM.
- If $g\colon [h] \to [n]$ is a morphism in $\mathbb{\Delta}$ such that $g(0) = 0$, then the corresponding morphism $([n], 0) \to ([h], 0)$ is an active morphism in LM.
- The elementary objects of LM are $([0], 0)$ (often denoted by $\mathfrak{m}$) and $([1], 1)$ (often denoted by $\mathfrak{a}$).

Remark A.1.9. Saying 'left' versus 'right' is just a convention; the same algebraic pattern also encodes right actions.

A.2 Segal objects

An algebra represented by an algebraic pattern is called a Segal object.

Definition A.2.1. Let $\mathcal{O}$ be an algebraic pattern. A functor $X\colon \mathcal{O} \to \mathcal{C}$ is called a *Segal $\mathcal{O}$-object* in a category $\mathcal{C}$ if for every $O \in \mathcal{O}$ the induced functor

$$\left(\mathcal{O}^{\mathrm{el}}_{O/}\right)^{\triangleleft} \to \mathcal{O} \xrightarrow{X} \mathcal{C}$$

is a limit diagram. If $\mathcal{C}$ has limits for diagrams indexed by $\mathcal{O}^{\mathrm{el}}_{O/}$ for all $O \in \mathcal{O}$, in which case we say that $\mathcal{C}$ is *$\mathcal{O}$-complete*, then this condition is equivalent to the canonical morphisms

$$X(O) \to \varprojlim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} X(E)$$

being isomorphisms.

Now, we will provide some examples to explain how algebraic patterns work.

Example A.2.2. (Segal Comm-Objects) Let $\mathcal{C}$ be a category with finite products, and let $X\colon \mathsf{Comm} \to \mathcal{C}$ be a functor. The Segal condition on X requires that

$$X(\langle n\rangle) \simeq \varprojlim_{\rho \in \mathsf{Comm}^{\mathrm{el}}_{\langle n\rangle/}} X(\langle 1\rangle).$$

We can identify the category $\mathsf{Comm}^{\mathrm{el}}_{\langle n\rangle/}$ with the discrete set of inert morphisms $\{\rho_i \mid i = 1, \dots, n\}$, where $\rho_i \colon \langle n\rangle \to \langle 1\rangle$ is given by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Then, the Segal condition says that the canonical map

$$(\rho_i^*)_{i=1}^n \colon X(\langle n\rangle) \to \prod_{i=1}^n X(\langle 1\rangle)$$

is an equivalence. This means that for each non-basepoint element $i \in \langle n\rangle$ (where $i \neq 0$), we can specify a corresponding object $x_i \in X(\langle 1\rangle)$. Therefore, we can describe an object in $X(\langle n\rangle)$ as a sequence $(x_1, \dots, x_n)$.

Next, we will show how inert and active morphisms work:

- Let $f \colon \langle n\rangle \to \langle m\rangle$ be an inert morphism in Comm. Then $n = m$ and f corresponds to a permutation. The map $X(f)$ is

$$X(f) \colon X(\langle n\rangle) \simeq X(\langle 1\rangle)^n \to X(\langle 1\rangle)^n \simeq X(\langle m\rangle), \quad (x_1, \dots, x_n) \mapsto (x_{f^{-1}(1)}, \dots, x_{f^{-1}(n)}).$$

- Let $g \colon \langle n\rangle \to \langle m\rangle$ be an active morphism in Comm, and let $I_j \coloneqq g^{-1}(j)$ for $j \in \{1, \dots, m\}$. Then g corresponds to the morphism

$$X(g) \colon X(\langle n\rangle) \simeq X(\langle 1\rangle)^n \to X(\langle 1\rangle)^m \simeq X(\langle m\rangle), \quad (x_1, \dots, x_n) \mapsto \left(\prod_{i_1 \in I_1} x_{i_1}, \dots, \prod_{i_m \in I_m} x_{i_m}\right).$$

  In particular, the active morphism $s \colon \langle 2\rangle \to \langle 2\rangle$ that swaps 1 and 2 corresponds to the map $(x_1, x_2) \mapsto (x_2, x_1)$, which enforces commutativity. Active morphisms, such as the map $\langle 2\rangle \to \langle 1\rangle$, represent the "commutative multiplication".

When we set $\mathcal{C} = \mathsf{Set}$, we find that Comm-Segal objects are precisely commutative monoids.

Example A.2.3. (Segal Assoc-objects) Let $\mathcal{C}$ be a category with finite products, and let $X \colon \mathsf{Assoc} \to \mathcal{C}$ be a simplicial object. The Segal condition on X requires that

$$X([n]) \simeq \varprojlim_{e \in \mathsf{Assoc}^{\mathrm{el}}_{[n]/}} X([1]).$$

Now, let's analyze the limit above. An elementary inert morphism $e_i \colon [n] \to [1]$ in $\mathbb{\Delta}^{\mathrm{op}}$ corresponds to an inclusion $[1] \hookrightarrow [n]$ in $\mathbb{\Delta}$ with image $\{i-1, i\}$. Notice that $[n]$ is a linearly ordered set, and one can think of it as being cut into n pieces:

$$[n] = \left\{0 \xrightarrow{e_1} 1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} n\right\},$$

where the segment $i-1 \to i$ corresponds to the elementary inert map e_i. The Segal condition says that the canonical map

$$(e_i^*)_{i=1}^n \colon X([n]) \to \prod_{i=1}^n X([1])$$

is an equivalence. This means we can associate each arrow $i-1 \to i$ in $[n]$ with a corresponding object $x_i \in X([1])$. Therefore, we can describe an object in $X([n])$ as a sequence $(x_1, \dots, x_n)$.

Next, we will show how inert and active morphisms work:

- Let $f \colon [n] \to [m]$ be an inert morphism in Assoc, and let $f^{\mathrm{op}} \colon [m] \to [n]$ be the corresponding morphism in $\mathbb{\Delta}$. In the Segal object X, the morphism $X(f)$ corresponds to selecting a contiguous subsequence:

$$X(f) \colon X([n]) \simeq X([1])^n \to X([1])^m \simeq X([m]), \quad (x_1, \dots, x_n) \mapsto (x_{f^{\mathrm{op}}(0)+1}, \dots, x_{f^{\mathrm{op}}(m-1)+1}).$$

- Let $g \colon [n] \to [m]$ be an active morphism in Assoc, and let $g^{\mathrm{op}} \colon [m] \to [n]$ be the corresponding map in $\mathbb{\Delta}$. Let $I_j \coloneqq \{g^{\mathrm{op}}(j-1)+1, \cdots, g^{\mathrm{op}}(j)\}$ for $j \in \{1, \dots, m\}$. In the Segal object X, the morphism $X(g)$ corresponds to:

$$X(g) \colon X([n]) \simeq X([1])^n \to X([1])^m \simeq X([m]), \quad (x_1, \dots, x_n) \mapsto \left(\prod_{i_1 \in I_1} x_{i_1}, \dots, \prod_{i_m \in I_m} x_{i_m}\right),$$

  which represents "multiplication".

When we set $\mathcal{C} = \mathsf{Set}$, we find that Assoc-Segal objects are precisely monoids.

Remark A.2.4. We note that Segal $\mathbb{\Delta}^{\mathrm{op},\natural}$-objects are precisely the standard *Segal objects*. That is, they are functors $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ satisfying the condition that for all $n \geq 2$, the canonical *Segal map*

$$X([n]) \to X([1]) \times_{X([0])} \cdots \times_{X([0])} X([1])$$

(where the right-hand side is the $(n-1)$-fold fiber product) is an isomorphism.

Example A.2.5. (Segal LM-objects) Let $\mathcal{C}$ be a category with finite products, and let $X\colon \mathsf{LM} \to \mathcal{C}$ be a functor. The Segal condition on X requires that

$$X([n], i) \simeq \varprojlim_{([h],j) \in \mathsf{LM}^{\mathrm{el}}_{([n],i)/}} X([h], j).$$

Now, let's analyze the limit above. Consider the canonical projection $p\colon ([n], 0) \to ([0], 0)$ in LM, which corresponds to an inclusion $[0] \simeq \{n\} \hookrightarrow [n]$.

When $i = 1$, the Segal condition is equivalent to the map

$$((e_i, 1)^*)_{i=1}^n : X([n], 1) \to \prod_{i=1}^n X([1], 1)$$

being an equivalence.

When $i = 0$, the Segal condition is equivalent to the map

$$((e_i, 0)^*)_{i=1}^n \times p\colon X([n], 0) \to \left(\prod_{i=1}^n X([1], 1)\right) \times X([0], 0)$$

being an equivalence.

We will refer to $X([1], 1)$ as A and $X([0], 0)$ as M.

That is, a Segal LM-object in $\mathcal{C}$ consists of a natural transformation $M_\bullet \Rightarrow A_\bullet$ of simplicial objects $M_\bullet, A_\bullet\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that $A_\bullet$ is a Segal Assoc-object in $\mathcal{C}$ and for all $n \geq 0$, we have an equivalence

$$M_n \simeq A^n \times M.$$

Now we will show how inert and active morphisms work:

- The structure on $\{1\} \times \mathbb{\Delta}^{\mathrm{op}}$ is consistent with Assoc.
- For each $[n] \in \mathbb{\Delta}$, the induced morphism $([n], 0) \to ([n], 1)$ corresponds to the projection

$$\begin{aligned} M_n \simeq A^n \times M &\to A^n \\ (a_1, \cdots, a_n, m) &\mapsto (a_1, \cdots, a_n). \end{aligned}$$

- Let $f\colon ([n], 0) \to ([h], 0)$ be an inert morphism in LM, and denote its image under $(T \to \mathrm{id})$ as $\tilde{f}\colon [n+1] \to [h+1]$. Then $X(f)$ corresponds to the projection

$$\begin{aligned} M_n \simeq A^n \times M &\to A^h \times M \simeq M_h \\ (a_1, \cdots, a_n, m) &\mapsto \left(a_{\tilde{f}^{\mathrm{op}}(0)+1}, \cdots, a_{\tilde{f}^{\mathrm{op}}(h-1)+1}, m\right). \end{aligned}$$

- Let $g\colon ([n], 0) \to ([h], 0)$ be an active morphism in LM, and denote its image under $(T \to \mathrm{id})$ as $\tilde{g}\colon [n+1] \to [h+1]$. Let $I_j := \{\tilde{g}^{\mathrm{op}}(j-1)+1, \cdots, \tilde{g}^{\mathrm{op}}(j)\}$ for $j \in \{1, \ldots, h+1\}$. Then $X(g)$ corresponds to the morphism

$$\begin{aligned} &M_n \simeq A^n \times M \to A^h \times M \simeq M_h \\ &(a_1, \cdots, a_n, m) \mapsto \left(\prod_{i_1 \in I_1} a_{i_1}, \cdots, \prod_{i_h \in I_h} a_{i_h}, \left(\prod_{i_{h+1} \in I_{h+1}} a_{i_{h+1}}\right) \cdot m\right), \end{aligned}$$

  which represents the "left action".

A.3 Operad over algebraic pattern

In this section, we introduce *operads*, which are mathematical structures designed to encode the abstract properties of algebraic operations. Building on the notion of algebraic patterns, operads allow us to describe entire algebraic theories, such as the theory of homotopy-coherent algebras.

For more on the heuristics of operads, we refer to the excellent overview in [Cnossen, 2025, Chapter 10].

An operad $\mathcal{O}$ over an algebraic pattern $\mathcal{P}$ can be regarded as a category of "$\mathcal{P}$-type" operations, where "$\mathcal{P}$-type" refers to the Segal condition of $\mathcal{P}$.

Definition A.3.1. (Operad) Let $\mathcal{P}$ be an algebraic pattern. An $\mathcal{P}$-*operad* is a functor $p\colon \mathcal{O} \to \mathcal{P}$, where $\mathcal{O}$ has an algebraic pattern structure lifted from $\mathcal{P}$, such that:

1. $\mathcal{O}$ has p-cocartesian lifts of inert morphisms in $\mathcal{P}$.
2. For $P \in \mathcal{P}$, let $\mathcal{O}_P$ denote the fiber of P. For $X \in \mathcal{O}_P$, if $\xi\colon \left(\mathcal{P}^{\mathrm{el}}_{P/}\right)^{\triangleleft} \to \mathcal{C}$ is a diagram of cocartesian morphisms over the objects of $\mathcal{P}^{\mathrm{el}}_{P/}$, then for any $Y \in \mathcal{O}_{P'}$, the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}}(Y, X) & \longrightarrow & \varprojlim_{\alpha\colon P \to O \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathrm{Hom}_{\mathcal{O}}(Y, \xi(\alpha)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\mathcal{P}}(P', P) & \longrightarrow & \varprojlim_{\alpha\colon P \to O \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathrm{Hom}_{\mathcal{P}}(P', O) \end{array}$$

   is a pullback square.
3. The functor $\mathcal{O}_P \to \varprojlim_{O \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}_O$ is an equivalence.

We refer to $\mathsf{Op}(\mathcal{P})$ as the category of $\mathcal{P}$-operads and functors over $\mathcal{P}$ that preserve inert cocartesian morphisms.

Remark A.3.2. When we consider the category (without the algebraic pattern structure) of a $\mathcal{P}$-operad $\mathcal{O}$, we will denote it as $\mathcal{O}^{\otimes}$.

Example A.3.3. (Operad) A Comm-operad is precisely the operad in the sense of [Lurie, 2017, Definition 2.1.1.10]. Unless specified otherwise, we will use the term "operad" to mean a Comm-operad and denote its category by Op.

Example A.3.4. (Generalized operad) By Remark A.1.5, one can consider $\mathsf{Fin}^{\natural}_*$-operads. A $\mathsf{Fin}^{\natural}_*$-operad is precisely the generalized operad in the sense of [Lurie, 2017, Definition 2.3.2.1]. Unless specified otherwise, we will use the term "generalized operad" to mean a $\mathsf{Fin}^{\natural}_*$-operad and denote its category by $\mathsf{Op}^{\mathrm{gen}}$.

Remark A.3.5. Consider the inclusion $\mathsf{Op} \subseteq \mathsf{Op}^{\mathrm{gen}}$. By [Lurie, 2017, Proposition 2.1.4.6 and Remark 2.3.2.4] and [Lurie, 2009, Propositon A.3.7.6], one can find that both Op and $\mathsf{Op}^{\mathrm{gen}}$ are presentable. The inclusion $\mathsf{Op} \hookrightarrow \mathsf{Op}^{\mathrm{gen}}$ preserves limits. Therefore, it admits a left adjoint

$$\mathrm{L}_{\mathrm{gen}}\colon \mathsf{Op}^{\mathrm{gen}} \to \mathsf{Op}.$$

Example A.3.6. (Planar operad) A Assoc-operad is the planar operad in the sense of [Lurie, 2017, Definition 4.1.3.2]. Unless specified otherwise, we will use the term "planar operad" to mean an Assoc-operad and denote its category by $\mathsf{Op}^{\mathrm{ns}}$.

If $f\colon \mathcal{O} \to \mathcal{P}$ is a functor between algebraic patterns that preserves the factorization system and elementary objects, and moreover, the induced functor

$$\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$$

is initial (i.e., for any $F\colon \mathcal{P}^{\mathrm{el}}_{f(X)/} \to \mathcal{C}$, the canonical map $\varprojlim F \to \varprojlim (F \circ f)$ is an isomorphism), then the pullback functor

$$f^*\colon \mathsf{Op}(\mathcal{P}) \to \mathsf{Op}(\mathcal{O})$$

is well-defined. Under mild assumptions, this functor has a left adjoint.

Example A.3.7. Define a functor Cut: $\Delta^{\mathrm{op}} \to \mathsf{Fin}_*$ that takes $[n]$ to $\langle n\rangle$ and a morphism $\phi\colon [n] \to [m]$ in Δ to the map $\mathrm{Cut}(\phi)\colon \langle m\rangle \to \langle n\rangle$ given by

$$\mathrm{Cut}(\phi)(i) = \begin{cases} j, & \text{if } \phi(j-1) < i \leq \phi(j), \\ 0, & \text{otherwise.} \end{cases}$$

Pulling back along this gives a functor $\mathsf{Op} \to \mathsf{Op}^{\mathrm{ns}}$ that informally "forgets symmetric group actions". Its left adjoint is a "symmetrization" functor $\mathsf{Op}^{\mathrm{ns}} \to \mathsf{Op}$. In fact, [?, Theorem 5.1.1] proves a comparison result that gives conditions for certain functors as above to induce equivalences.

Let's examine the structure of a Comm-operad $\mathcal{O}$ in more detail. We refer to the fiber $\mathcal{O}_{\langle 1\rangle} := p^{-1}(\langle 1\rangle)$ over $\langle 1\rangle$ as the *underlying category of the operad* $\mathcal{O}$. We denote its groupoid core by

$$\mathcal{O}^{\simeq} := \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\simeq}$$

and refer to it as the *anima of colors of* $\mathcal{O}$. For $\langle n\rangle \in \mathsf{Fin}_*$, condition 3 guarantees that every object in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ may be uniquely written as a product $\prod_{i=1}^n x_i$ for some colors $x_i \in \mathcal{O}^{\simeq}$. We also denote such a product as an unordered tuple $\{x_i\}_{i\in\{1,\dots,n\}}$.

Given another color $y \in \mathcal{O}^{\simeq}$, we define the *anima of multimorphisms (or anima of operations) in* $\mathcal{O}$ from $\{x_i\}_{i=1}^n$ to y as the anima of morphisms in $\mathcal{O}^{\otimes}$ that map to the active morphism $\langle n\rangle \to \langle 1\rangle$:

$$\begin{tikzcd} \mathcal{O}(\{x_i\}_{i=1}^n; y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\prod_{i=1}^n x_i, y) \arrow[d, "p"] \\ * \arrow[r, "(\langle n\rangle \to \langle 1\rangle)"] & \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle 1\rangle). \end{tikzcd}$$

A multimorphism in an operad $\mathcal{O}$ represents an abstract operation with multiple inputs. When interpreted in a symmetric monoidal category $\mathcal{C}$, this corresponds to a morphism of the form $x_1 \otimes \cdots \otimes x_n \to y$.

At the end of this section, we will give the notion of *weak enrichment*.

Definition A.3.8. Let $\mathcal{C} \to \mathsf{Assoc}$ be a planar operad, and let $\mathcal{M}$ be a category. We say $\mathcal{M}$ *is weakly enriched over* $\mathcal{C}$ if there exists an LM-operad $\mathcal{O}$ such that $\mathcal{O}^{\otimes}_{\mathsf{a}} \simeq \mathcal{C}$ and $\mathcal{O}^{\otimes}_{\mathsf{m}} \simeq \mathcal{M}$.

A.4 $\mathcal{O}$-monoidal category and $\mathcal{O}$-algebra

In this section, we will consider $\mathcal{O}$-monoidal categories and $\mathcal{O}$-algebras for some algebraic patterns $\mathcal{O}$.

By the discussion above, it is natural to consider algebraic objects in the cartesian setting.

Definition A.4.1. (Cartesian pattern) A *cartesian pattern* is an algebraic pattern $\mathcal{O}$ equipped with a morphism of algebraic patterns $|-|\colon \mathcal{O} \to \mathsf{Comm}$ such that for every $O \in \mathcal{O}$, the induced functor

$$\mathcal{O}^{\mathrm{el}}_{O/} \to \mathsf{Comm}^{\mathrm{el}}_{|O|/}$$

is an equivalence.

Remark A.4.2. All the examples we have considered so far are cartesian patterns.

Now, we can define $\mathcal{O}$-monoidal categories and $\mathcal{O}$-algebras in them.

Definition A.4.3. Let $\mathcal{O}$ be a cartesian pattern. An $\mathcal{O}$*-monoidal category* is a cocartesian fibration $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}$ whose associated functor $\mathcal{O} \to \mathsf{Cat}$ is an $\mathcal{O}$-Segal object in Cat.

Remark A.4.4. One can find that every $\mathcal{O}$-monoidal category is an $\mathcal{O}$-operad if we lift the algebraic pattern structure along $p_{\mathcal{C}}$.

Definition A.4.5. Let $\mathcal{O}$ be a cartesian pattern and let $\mathcal{V}^{\otimes}$, $\mathcal{W}^{\otimes}$ be $\mathcal{O}$-monoidal categories. Then a *lax* $\mathcal{O}$*-monoidal functor* between them is a commutative triangle

$$\begin{tikzcd} \mathcal{V}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p_{\mathcal{V}}"'] & & \mathcal{W}^{\otimes} \arrow[dl, "p_{\mathcal{W}}"] \\ & \mathcal{O}. & \end{tikzcd}$$

such that F preserves cocartesian lifts of inert morphisms. Equivalently, a lax $\mathcal{O}$-monoidal functor is a morphism of algebraic patterns over $\mathcal{O}$.

If F preserves *all* cocartesian morphisms over $\mathcal{O}$, we call it an $\mathcal{O}$*-monoidal functor*.

Example A.4.6. A *monoidal category* is an Assoc-monoidal category $\mathcal{C}^{\otimes}$. We will denote the image of $[1]$ by $\mathbb{1}_{\mathcal{C}}$ and refer to it as the unit of the monoidal structure. We also let $\mathcal{C}$ denote the fiber of $[1]$. In this context, $\mathcal{C}^{\otimes}$ will be referred to as the monoidal structure on $\mathcal{C}$. By Example A.2.3, the active morphism $[2] \to [1]$ in Assoc corresponds to a functor $- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we will refer to as the *tensor product functor*.

Example A.4.7. A *symmetric monoidal category* is a Comm-monoidal category $\mathcal{C}^{\otimes}$. We will denote the image of $\langle 1 \rangle$ by $\mathbb{1}_{\mathcal{C}}$ and refer to it as the unit of the monoidal structure. We also let $\mathcal{C}$ denote the fiber of $\langle 1 \rangle$. In this context, $\mathcal{C}^{\otimes}$ will be referred to as the symmetric monoidal structure on $\mathcal{C}$. By Example A.2.2, the active morphism $\langle 2 \rangle \to \langle 1 \rangle$ corresponds to a functor $- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we will refer to as the *tensor product functor*.

Let $X\colon \mathsf{LM} \to \mathsf{Cat}$ be an LM-monoidal category. Using the Grothendieck–Lurie construction, one can obtain a cocartesian fibration $p\colon \mathcal{O} \to \mathsf{LM}$. Let $\mathcal{O}_{\mathfrak{a}}$ and $\mathcal{O}_{\mathfrak{m}}$ be the fibers of $\mathfrak{a}$ and $\mathfrak{m}$, respectively. This implies the existence of the following structures:

- The fiber $\mathcal{O}_{\mathfrak{a}}$ is a monoidal category.
- The fiber $\mathcal{O}_{\mathfrak{m}}$ is a category that is a left module over $\mathcal{O}_{\mathfrak{a}}$, meaning there is an action functor $\otimes\colon \mathcal{O}_{\mathfrak{a}} \times \mathcal{O}_{\mathfrak{m}} \to \mathcal{O}_{\mathfrak{m}}$ which is well-defined up to homotopy.

Definition A.4.8. Let $\mathcal{C}$ be a monoidal category. We say a category $\mathcal{M}$ is $\mathcal{C}$*-linear* if there exists an LM-Segal object in Cat, given by a cocartesian fibration $p\colon \mathcal{O} \to \mathsf{LM}$, satisfying the following two properties:

- $\mathcal{O}_{\mathfrak{a}} \simeq \mathcal{C}$;
- $\mathcal{O}_{\mathfrak{m}} \simeq \mathcal{M}$.

Remark A.4.9. One can find that if $\mathcal{M}$ is linear over $\mathcal{C}$, then $\mathcal{M}$ is weakly enriched over $\mathcal{C}$.

Now, we define the algebra object in a $\mathcal{P}'$-monoidal category $\mathcal{C}$.

Definition A.4.10. Let $\mathcal{P}$ and $\mathcal{P}'$ be cartesian patterns with a morphism $f\colon \mathcal{P} \to \mathcal{P}'$ over Comm, and let $\mathcal{O}$ be a $\mathcal{P}'$-operad. Then a $\mathcal{P}$*-algebra* in $\mathcal{O}$ is a commutative triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "A"] \arrow[dr, "f"'] & & \mathcal{O} \arrow[dl, "p"] \\ & \mathcal{P}' & \end{tikzcd}$$

such that A takes inert morphisms in $\mathcal{P}$ to cocartesian morphisms in $\mathcal{O}$ (relative to p). We write $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$ for the full subcategory of $\mathsf{Fun}_{/\mathcal{P}'}(\mathcal{P}, \mathcal{O})$ spanned by the $\mathcal{P}$-algebras. If $f = \mathrm{id}_{\mathcal{P}'}$, we will denote the category $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$ by $\mathsf{Alg}_{/\mathcal{P}'}(\mathcal{O})$. If $\mathcal{P}' = \mathsf{Comm}$, then we will omit $\mathcal{P}'$ in $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$.

Moreover, if $\mathcal{O} = \mathcal{C}^{\otimes}$ is a $\mathcal{P}'$-monoidal category, then we will omit the notation $\otimes$ in $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{C}^{\otimes})$.

Example A.4.11. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category. Consider the morphism $|-|\colon \mathsf{Trivial} \to \mathsf{Comm}$ that sends $*$ to the elementary object $\langle 1 \rangle$. Then, a $\mathsf{Trivial}$-algebra in $\mathcal{C}^{\otimes}$ is just an object in $\mathcal{C} \simeq \mathcal{C}^{\otimes}_{\langle 1 \rangle}$.

Now, let $\mathcal{P} = \mathsf{Comm}$. We try to describe what an $\mathcal{O}$-algebra is in a symmetric monoidal category $\mathcal{C}$.

Definition A.4.12. Let $\mathcal{O}$ be an operad. Then:

- A Comm-algebra in $\mathcal{O}$ is called a *commutative algebra* in $\mathcal{O}$, and we denote $\mathsf{Alg}_{\mathsf{Comm}}(\mathcal{O})$ by $\mathsf{CAlg}(\mathcal{O})$.
- An Assoc-algebra in $\mathcal{O}$ is called an *associative algebra* in $\mathcal{O}$, and we denote $\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{O})$ by $\mathsf{Alg}(\mathcal{O})$.
- An LM-algebra in $\mathcal{O}$ is called a *left module* in $\mathcal{O}$, and we denote $\mathsf{Alg}_{\mathsf{LM}}(\mathcal{O})$ by $\mathsf{LMod}(\mathcal{O})$.

Remark A.4.13. By Remark A.1.9, one can also use LM to define right modules in $\mathcal{O}$ (in this case, we will use RM to denote LM) and use $\mathsf{RMod}(\mathcal{O}) := \mathsf{Alg}_{\mathsf{RM}}(\mathcal{O})$ to denote the category of right modules.

Definition A.4.14. Let $\mathcal{C}$ be a monoidal category and let $q\colon \mathcal{O} \to \mathsf{LM}$ exhibit $\mathcal{M}$ as weakly enriched over $\mathcal{C}$. We let $\mathsf{LMod}(\mathcal{M})$ denote the category $\mathsf{Alg}_{/\mathsf{LM}}(\mathcal{O})$. We will refer to $\mathsf{LMod}(\mathcal{M})$ as the category of *left module objects of* $\mathcal{M}$. If A is an associative algebra in $\mathcal{C}$, we let $\mathsf{LMod}_A(\mathcal{M})$ denote the pullback

$$\begin{tikzcd} \mathsf{LMod}_A(\mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{LMod}(\mathcal{M}) \arrow[d] \\ * \arrow[r, "\{A\}"] & \mathsf{Alg}(\mathcal{C}). \end{tikzcd}$$

Bibliography

[Anel et al., 2020] Anel, M., Biedermann, G., Finster, E., and Joyal, A. (2020). A generalized blakers–massey theorem. *Journal of Topology*, 13(4):1521–1553.

[Anel et al., 2022] Anel, M., Biedermann, G., Finster, E., and Joyal, A. (2022). Left-exact localizations of ∞-topoi i: Higher sheaves. *Advances in Mathematics*, 400:108268.

[Anel et al., 2023a] Anel, M., Biedermann, G., Finster, E., and Joyal, A. (2023a). Left-exact localizations of ∞-topoi ii: Grothendieck topologies. https://arxiv.org/abs/2201.01236.

[Anel et al., 2023b] Anel, M., Biedermann, G., Finster, E., and Joyal, A. (2023b). Left-exact localizations of ∞-topoi iii: The acyclic product. https://arxiv.org/abs/2308.15573.

[Cnossen, 2025] Cnossen, B. (2025). Stable homotopy theory and higher algebra — an ∞-categorical introduction. https://drive.google.com/file/d/1ivHDIqclbg2hxmUEMTqmj2TnsAHQxVg9/view.

[Haugseng, 2023] Haugseng, R. (2023). An allegedly somewhat friendly introduction to ∞-operads. https://runegha.folk.ntnu.no/iopd.pdf.

[Haugseng, 2025] Haugseng, R. (2025). Yet another introduction to ∞-categories. https://runegha.folk.ntnu.no/naivecat_web.pdf.

[Heyer and Mann, 2024] Heyer, C. and Mann, L. (2024). 6-functor formalisms and smooth representations.

[Liu, 2025] Liu, O. (2025). A breif introduction to monad and higher algebra. https://drive.google.com/file/d/1Quc8ySwaMy3d0pVL1H1myXiO8If9ny2r/view.

[Lurie, 2009] Lurie, J. (2009). *Higher Topos Theory (AM-170)*. Princeton University Press.

[Lurie, 2017] Lurie, J. (2017). Higher algebra. https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Lurie, 2018a] Lurie, J. (2018a). Kerodon. https://kerodon.net.

[Lurie, 2018b] Lurie, J. (2018b). Spectral algebraic geometry. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[Rezk, 2010] Rezk, C. (2010). Toposes and homotopy toposes (version 0.15). https://rezk.web.illinois.edu/homotopy-topos-sketch.pdf.

[Scholze, 2025] Scholze, P. (2025). Geometry and higher category theory. https://people.mpim-bonn.mpg.de/scholze/Gestalten.pdf.