A Review of SU(N) Group and Application in Particle Physics

Zixi Ouyang*
Sun Yat-sun University, Haizhu District, Guangzhou City, Guangdong Province, China
(Dated: September 19, 2022)

CONTENTS

* Email: ouyzx6@mail2.sysu.edu.cn

I. INTRODUCTION

Group theory is the study of groups. Groups are sets equipped with an operation (like multiplication, addition, or composition) that satisfies certain basic properties. As the building blocks of abstract algebra, groups are so general and fundamental that they arise in nearly every branch of mathematics and the sciences.[1][2]

Group theory is widely used in physics since it reveal the symmetry of a physics system. For instance, in condensed matter physcis, the lattice, resulting from atoms arrangement in a 3-dimensional periodic fashion, has certain symmetry which can be categorized into 230 kinds of space groups. Another example is the symmetry of hydrogen system which indicates 4 invariant mechanical quantities without figuring out the Newton's Law or Schrodinger equation.

II. GROUP THEORY FOR PHYSICISTS

II.1. Definition

A finite group with finite number of elements, which is called the order of the group $[G]$, obeys some operation rules as below:

1. **Closure**: $\forall g_1, g_2 \in G, g_1 g_2 \in G$.
2. **Associativity**: $g_1(g_2 g_3) = (g_1 g_2)g_3$.
3. **Inverse element**:$\forall g \in G, \exists g^{-1} \in G$, with $g^{-1}g = gg^{-1} = e$.
4. **Identity Element**: every groups contains $e \in G$, with $eg = ge = g$.

According to the definition, the product of $gG, \forall g \in G$ gives all the elements in G without repetition.

In **abelian group**, commutative law of multiplication works, $g_1 g_2 = g_2 g_1, g_1, g_2 \in G$.

II.2. Properties

II.2.1. Subgourp

A subgroup H of a group G is a subset of elements of G that ensure **closure**. that is $\forall g_1, g_2 \in H, g1g2 \in H, G$. For example, D_3 is a subgroup of S_3. The trivial subgroups are the identity element e and the group G.

The right(or left) **coset** of the subgroup H in G is a set of elements formed by the multiplication gH(or Hg) of

H on the right(orleft) of a element of $g \in G$ but $g \notin H$. But now many consider $hH = H, \forall h \in H$ is part of the series of cosets formed by Lagrange's Theorem, which is shown as below.

Cosets do not overlap, otherwise they are the same.

Lagrange's Theorem: The order of the coset $[H]$, is a divisor of $[G]$, $[G] = [H]n_{cosets}$, where n_{cosets} is the number of cosets on G.

With Lagrange's Theorem, we can divide the group G into a series of cosets with subgroup H.

II.2.2. Class

The element $h \in G$ has its conjugating element f, if $gfg^{-1} = h, \exists g \in G$, which is denoted as $h\ f(\Leftrightarrow f\ h)$. And if $f\ h, h\ k$, then $f\ k$.

A Class of a group is its subset in which elements conjugate with each other. Different classes do not overlap.

We can divide group G by classes. $\forall g \in G$, there is a subgroup $H^h = \{h|hg = gh\}$. The number of elements of class h, $n_g = [G]/[H^h]$ equals the number of the elements in the series of cosets. Then each element in class h correlates a coset.

Invariant subgroup, also self conjugated group, is defined as $gHg^{-1} = H, \forall g \in G$. Its left cosets and right cosets are the same, since $gH = Hg$.

Quotient subgroup, denoted as G/H, where H is the invariant subgroup of group G, yields a series of cosets which can be considered as new elements in quotient subgroup, where H is the new identity element.

II.2.3. Homomorphism and Isomorphism

Isomorphism is the bijection relationship between two groups that preserve the group operations. For example, if $g \in G \Leftrightarrow f \in F$, then $g_1 g_2 \Leftrightarrow f_1 f_2$. Groups of isomorphism are the same in essence.

Homomorphism is injection relationship between two groups that preserve the group operations.

II.2.4. Representation and Characteristic

Representations of a group can be considered as the homorphism mapping from a group to matrices. In group theory, we merely concerned irreducible representations, which cannot be changed as a diagonal matrices by similarity transformation.

Characteristic is the trace of a representation matrix of a group element. Under irreducible representations, the following equation reveals the relationship between the dimension of group and irreducible representation.

$$[G]^2 = \sum_{i=1}^{all} D_i^2 \tag{1}$$

II.3. Example

Group S_n: the set of bijective functions $[n] \to [n]$, where $[n] = 1, 2, ..., n,$, with the group operation of function composition.

Group A_n: even permutation of a set $X_1, X_2, ...$, which is the subgroup of S_n.

Group Z_n: the set of integers $0, 1, ..., n-1$, with group operation of addition modulo n.

Group D_n: rotational groups that are they contain only rotational axis of symmetry of regular polygons.

1-dimensional Lie group: group $\mathbb{R}$: group of all reals with addition as operation. group $U(1)$: circle group consisting of all complex numbers with absolute value 1 under multiplication.

III. SU(N) GROUP

SU(N), also special unitary group, is the group of $\mathrm{N}\times\mathrm{N}$ unitary matrices U with $\mathrm{U}^\dagger{=}\mathrm{U}^{-1}$ and $|\mathrm{U}| = 1$. The rank of SU(N) is $\mathrm{N}-1$ and the number of generators is N^2-1. Each matrix element can be written as exponential form $\mathrm{U} = \mathrm{e}^{\mathrm{i}\alpha_\mathrm{n}\mathrm{T}_\mathrm{n}}$, where Einstein's summation convention is applied. The traceless constraint $\mathrm{tr}(\alpha_\mathrm{n}\mathrm{T}_\mathrm{n}){=}0$ provides a unitary value of $|\mathrm{U}|$, since $|\mathrm{U}| = \left|\mathrm{e}^{\mathrm{i}\alpha_\mathrm{n}\mathrm{T}_\mathrm{n}}\right| = \mathrm{e}^{\mathrm{tr}(\alpha_\mathrm{n}\mathrm{T}_\mathrm{n})} = 1$.

If unitary determinant is not required but $|\mathrm{U}| = \pm 1$, we have U(N) group or unitary group with $\mathrm{U}^\dagger{=}\mathrm{U}^{-1}$. The rank of U(N) is N and the number of generators is N^2. U(1) group is one dimensional abelian lie group, the other one is real number group $\mathbb{R}^1$.

III.1. Commutators of SU(N)'s Generators

Since unitary matrices U near identity $\mathrm{I}_{\mathrm{N}\times\mathrm{N}}$ can be expanded as $\mathrm{U_A} = \mathbb{1} + \mathrm{A}, \mathrm{U_B} = \mathbb{1} + \mathrm{B}$, where A and B must be traceless due to the unitary constraint of U. Therefore the expansions $\mathrm{A} = \mathrm{ia_m T_m}, \mathrm{B} = \mathrm{ib_n T_n}$ work, where $\mathrm{T_m}, \mathrm{T_n}$ is generators of SU(N). Therefore the commutator of A and B is $[\mathrm{A}, \mathrm{B}] = -\mathrm{a_m b_n}[\mathrm{T_m}, \mathrm{T_n}]$.

Consider the similarity transformation

$$\mathrm{U_B U_A U_B^{-1}} \approx \mathrm{U_A} + [\mathrm{A}, \mathrm{B}]$$

Therefore, the commutator $[\mathrm{A}, \mathrm{B}]$ is traceless, which means $[\mathrm{A}, \mathrm{B}] = \mathrm{c_k T_k}$. All of above indicate that there must be some linear combinations of the generators that can be used to write commutators.

$$[\mathrm{T_m}, \mathrm{T_n}] = \mathrm{i} f_{\mathrm{mnk}} \mathrm{T_k}$$

where f_{mnk} is called structure constant.

III.2. Young Diagram

On the Young tableaux theory, each tableau represents a specific process of symmetrization and anti-symmetrization of a tensor $A^{\alpha_1,\ldots,\alpha_n}$ in which index α_n

can take any integer value 1 to N. For instance, one Young diagram tableau of A^α represents N-dimension vector $|A\rangle$ on the fundamental representation of SU(N).[3]

A Young diagram, whose height is limited to not more than N tableaux, is a irreducible representation(short for irreps) of SU(N).

III.3. Tensor Product on the Fundamental representatoin

Consider the tensor product(FIG 1)

$$\begin{aligned} A^\alpha \otimes B^\beta &= C^{\alpha\beta} \\ &= \frac{1}{2}(A^\alpha B^\beta + A^\beta B^\alpha) \oplus \frac{1}{2}(A^\alpha B^\beta - A^\beta B^\alpha) \\ &= C^{(\alpha\beta)} \oplus C^{[\alpha\beta]} \end{aligned}$$

where the first and second term are symmetric(parenthesis) and anti-symmetric(brackets) part. The Young tableaux are then the box-tensors with N indices, where the first one symmetrizes indices in each row, and the second one anti-symmetrizes all indices in the collums.

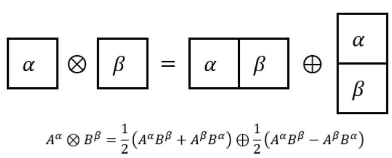

FIG. 1: $A^\alpha \otimes B^\beta$

In general, tensor product is shown as below(FIG 2).

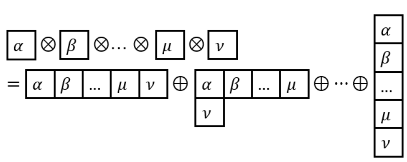

FIG. 2: Tensor Product of Fundamental Representatoins

$$\begin{aligned} &A^\alpha \otimes B^\beta \otimes ... \otimes M^\mu N^\nu = \Omega^{\alpha\beta...\mu\nu} \\ &\quad = \Omega^{(\alpha\beta...\mu\nu)} + \frac{1}{2}(\omega^{(\alpha\beta...\mu)}\kappa^\nu - \omega^{(\nu\beta...\mu)}\kappa^\alpha) \\ &\quad + ... + \Omega^{[\alpha\beta...\mu\nu]} \end{aligned}$$

The decomposition of the fundamental representation specifies how a subgroup H is embedded in the general group G, and since all representations may be built up as products of the fundamental representation, once we know the fundamental representation decomposition, we know all the representation decompositions.

III.4. Young Diagram

III.4.1. Dimension of Young Diagram

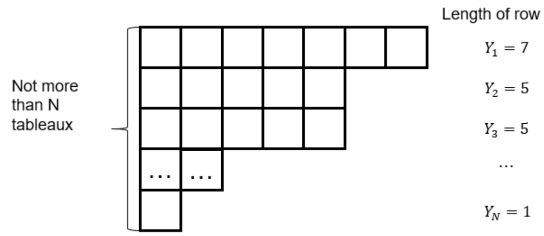

FIG. 3: Young Diagram

Each Young Diagram of SU(N)'s irreps shown as below(FIG 3) can be marked as $\mathrm{R}_Y = \{Y_1, Y_2, ..., Y_{N-1}, Y_N\}$. The dimension of it, which is often use to label $\mathrm{R}_Y = \{Y_1, Y_2, ..., Y_{N-1}, Y_N\}$, can be obtained by equation as below

$$\mathrm{dimR}_Y = \frac{\prod_{r,s=1,r<s}^{N}(Y_r - Y_s + s - r)}{\prod_{r,s=1,r<s}^{N}(s-r)}$$

or Hook length as shown in figure 4.

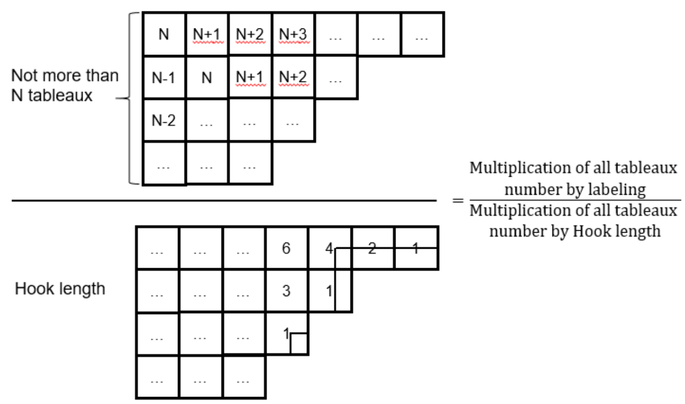

FIG. 4: Dimension of Young Diagram

III.4.2. Rules of Young Diagram

Young diagram is limited to be not more than N tableaux height with $Y_n \leq Y_{n-1}$.

The formal calculation of the direct product of two (or more) irreps is given by the following rules:

1. On the second tensor in the tensorial multiplication, add to all boxes of the first row a indices, add to all boxes of the second row b indices, etc.

2. The process of multiplication is made adding one box of index a each time to the others boxes, in

all allowed ways, until ceasing these boxes. Then adding one box of index b each time to all, etc. Never put more than one a or b in the same column. Align the number of a's $\leq$ number of b's (at any position, number of a above and right must be equal or smaller than of b, which must be equal or smaller than c, etc).

3. If two tableaux of the same shape are produced, they are counted as different only if the labels are different.

4. Cancel columns with N boxes since they are the trivial representation.

5. Check the dimension of the products to the dimension of the initial tensors.

By rules above, decomposition of tensor product is feasible.

III.5. Special Irreducible Representation

For SU(N), there are some useful irreps.

1. Fundamental representation: N-dimension representation with $\mathrm{R}_Y = \{Y_1 = 1, Y_{else} = 0\}$.

2. Anti-fundamental representation: N-dimension representation with $\mathrm{R}_Y = \{Y_1 = 1, Y_2 = 1, ..., Y_{N-1} = 1, Y_N = 0\}$.

3. Adjoint representation: the representation of the generators with $(T_a)_{bc} = -\mathrm{i}f_{abc}$ or $\mathrm{R}_Y = \{Y_1 = 2, Y_2 = 1, ..., Y_{N-1} = 1, Y_N = 0\}$.

III.6. Weight Diagram

The Cartan-Weyl basis for the group SU(N) are the maximally commutative basis of N−1 generators. The number of compact generators less the number of non-compact is the rank of the Lie algebra, which is the maximum number of commuting generators.[4]

For SU(N), the maximum number of commuting generators is (N−1). For SU(2), the rank of the Lie algebra is 1 even though we cannot find one commutator.

In weight diagram of (N−1) dimensions, an SU(N) multiplet is uniquely identified or marked by a series of (N−1) non-negative integers: Dynkin indices $(\alpha_1, \alpha_2, ..., \alpha_{\mathrm{N}-1})$, labeling an irrep of SU(N). Any such set of integers specifies a multiplet. Dynkin indices $(\alpha_1, \alpha_2, ..., \alpha_{\mathrm{N}-1})$ is conjugate to $(\alpha_{\mathrm{N}-1}, ..., \alpha_2, \alpha_1)$.

For SU(4), SU(5), SU(6), figure 4 can be marked as (2,1,1), (2,1,1,3), (2,1,1,3,0).

For an SU(2)(ranked 1) multiplet such as an isospin multiplet, the single integer α is the number of steps from one end of the multiplet to the other.

In SU(3)(ranked 2), the two integers α and β are the numbers of steps across the top and bottom levels of the multiplet diagram. Weight diagram of (1,1) and (3,0) in SU(3) is shown in figure 5, whose Young diagrams of them are $\{2, 1, 0\}$ and $\{3, 0, 0\}$

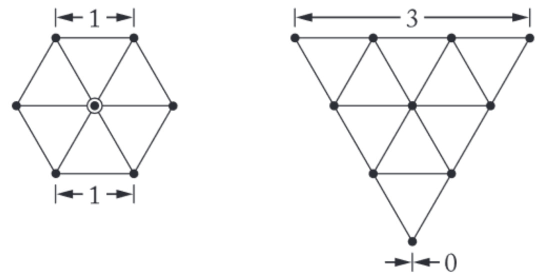

FIG. 5: SU(3): (1,1) and (3,0)

III.6.1. Ways to Draw Weight Diagram of SU(3)

Ways to draw weight diagram of SU(3) are

1. Draw a hexagon with $\alpha_1 + 1$ dots on top and $\alpha_2 + 1$ dots at bottom. When α_1 or α_2 is 0, it degenerates into an upright or inverted triangle;

2. The slope of each hypotenuse is $\frac{\Delta y}{\Delta t_3} = \pm 2$, and the length of three non-adjacent hypotenuse is α_1, and the length of the other three non-adjacent hypotenuse is α_2;

3. Draw a grid of parallel lines between two opposite hypotenuse, and each intersection point of the grid represents a state or degenerate states;

4. place the $t_3 - y$ coordinate frame, and place the origin on the center of gravity of the hexagon.

   If $(\alpha_1 - \alpha_2)$(mod 3)=0, the origin is on a grid point;

   If $(\alpha_1 - \alpha_2)$(mod 3)=1, the origin is at the center of gravity of an inverted triangle;

   If $(\alpha_1 - \alpha_2)$(mod 3)=-1, the origin is at the center of gravity of a positive triangle;

5. Degeneracy of states: for hexagonal numbers from the outside to the inside, the states of the first layer are singlet states, the second layer are double degenerate states, the third layer is triple degenerate states, and so on until a layer becomes a triangle, at which time the degeneracy stops increasing.

For instance, (4,1) in SU(3), whose number of particle is $35 = 1\times15+2\times9+2\times1$, is shown in figure 6. Although only weight diagrams in SU(3) are discussed here, weight diagrams in SU(N) for N$\geq$ 3 are easy to figure out by SU(3)(figure 6).

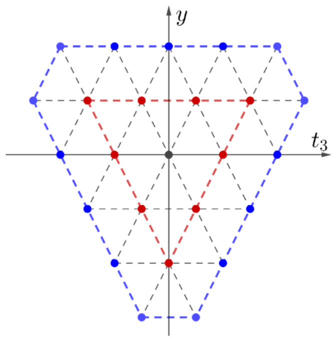

FIG. 6: SU(3): (4,1)

III.6.2. *Number of Particles in Weight Diagram of SU(3)*

The number of particles in a multiplet $\mathrm{N}(\alpha_1, \alpha_2, ...)$ is

$$\mathrm{N}(\alpha_1, \alpha_2, ...) = \left\{ \begin{array}{c} (\alpha_1+1),\ \text{for } (\alpha_1) \text{ in SU(2)} \\ \frac{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+2)}{2},\ \text{for } (\alpha_1, \alpha_2) \text{ in SU(3)} \end{array} \right.$$

III.6.3. *Decomposition of Tensor Product by Weight Diagram*

Decomposition rules of tensor product $(\alpha_1, \alpha_2) \otimes (\beta_1, \beta_2)$ by means of weight diagram are

1. Draw the weight diagram of (α_1, α_2) and (β_1, β_2) and each point in the weight graph represents the state in the representation;
2. Draw the weight diagram of (β_1, β_2) on the center of (α_1, α_2), so that all points obtained are the weights of all states of $(\alpha_1, \alpha_2) \otimes (\beta_1, \beta_2)$. The degree of degeneracy is calculated when repeated points appear, that is, some weights are multiple degeneracy;
3. Find the weight farthest from the coordinate origin, determine its representation and then deduct it from the weight diagram;
4. Repeat the previous step until all points have been deducted. $(\alpha_1, \alpha_2) \otimes (\beta_1, \beta_2)$ is equal to the direct sum of the representations deducted one by one.

$(1,0)$ is self-conjugate fundamental representation of SU(3). $(1,0) \otimes (0,1) = (1,1) \oplus (0,0)$ in SU(3) is a good example for deep understanding of SU(3), as shown in figure 7.

III.7. Structure of SU(2) and SU(3)

SU(2) with generators $\mathrm{T_m} = \frac{1}{2}\sigma_m$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

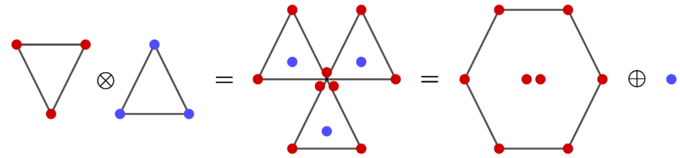

FIG. 7: $(1,0) \otimes (0,1) = (1,1) \oplus (0,0)$ in SU(3)

is 1-dimension in terms of Lie algebra since these generators cannot commute with each others, meaning that the states of SU(2) can be marked ± 1. The commutators of SU(2)'s generators are

$$[\mathrm{T_m}, \mathrm{T_n}] = \mathrm{i}\epsilon_{\mathrm{mnk}}\mathrm{T_k}$$

where Levi-Civita symbol ϵ_{mnk} is structure constant of SU(2).

The generators of SU(3) are $\mathrm{T_m} = \frac{1}{2}\lambda_m$, where

$$\lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

where $i = 1, 2, 3$. The commutators of SU(3)'s generators are

$$[\mathrm{T}_a, \mathrm{T}_b] = \mathrm{i}f_{abc}\mathrm{T}_c$$

where the structure constants are anti-symmetry.

$$f_{123} = 1, f_{485} = f_{678} = \frac{\sqrt{3}}{2}, else = \frac{1}{2}$$

The only zero commutator $[\mathrm{T_3}, \mathrm{T_8}] = 0$ among these 8 generators means 2-rank in terms of Lie algebra, which can be marked as coordinate pair (y, t_3).

IV. APPLICATION IN PARTICLE PHYSICS

IV.1. Decomposition of Angular Momentum Direct Product in SU(2)

Spin j in SU(2), whose dimension is $2j+1$ can be marked as $\mathrm{R}_Y = \{Y_1 = 2j, Y_2 = 0\}$. Then product $R_{j_1} \otimes R_{j_2}$ can be decomposed by means of Young diagram(figure 8).

$$\begin{aligned} R_{j_1} \otimes R_{j_2} &= (2j_1+1) \otimes (2j_2+1) \\ &= R_{|j_1-j_2|} \oplus R_{(|j_1-j_2|+1)} \oplus ... \oplus R_{(j_1+j_2)} \end{aligned}$$

The quantum number j is also used to label a representation besides Young diagram, the dimension of Young diagram $2j+1$ and weight diagram.

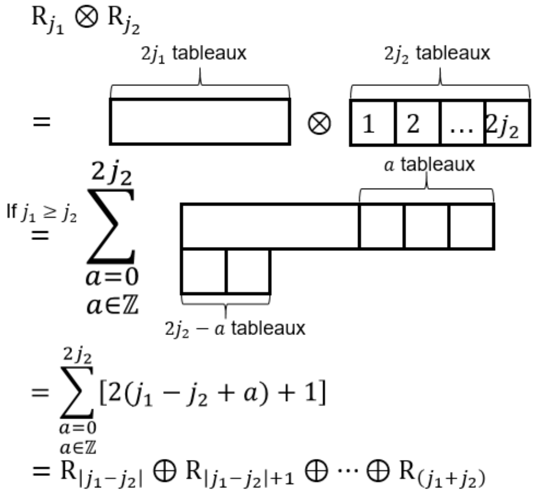

FIG. 8: Decomposition of Angular Momentum Direct Product in SU(2)

IV.2. Hadrons Categorization and Particle Multiplets

IV.2.1. *Internal structure of hadrons*

There are four important internal symmetries among the quarks and anti-quarks inside hadrons, which are associated with:
the color degree of freedom,
the flavor degree of freedom,
the spin degree of freedom,
the orbital degree of freedom.[5]
The total wave function of hardrons are[6][7]

$$\Psi = \Psi(orbital)\Psi(spin)\Psi(color)\Psi(flavor)$$

For a fermion system, the total wave function must be anti-symmetric **under particles exchanging**, while for a boson system, it must be symmetric.

Mesons, belonging to bosons, consist of one quark and one anti-quark. The direct product of spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ manifests that $\Psi(spin)$ can be symmetric or anti-symmetric. The product $\Psi(spin)\Psi(color)\Psi(flavor)$ must be symmetric, if consider the ground state $L = 0$, which means that $\Psi(orbital)$ is symmetric.

Baryons are a kind of fermions with 3 quarks or anti-quarks. Here $L = 0$ is still taken into consideration. But the direct product of spin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$ suggests that the spin part of the total wave function is anti-symmetric. In addition to discussion above, since there exists only 3 kinds of colors and 3 quarks or anti-quarks in baryons, the color part of wave function must be anti-symmetric. In conclusion, the flavor part of wave function must be symmetric.

Under SU(3) flavor symmetry, fundamental representation $(1, 0)$ or $\{1, 0, 0\}$ of Young diagram is used to represent $u/d/s$ light quarks, whose anti-quarks $\bar{u}/\bar{d}/\bar{s}$ are represented by the conjugate representation $(0, 1)$ or $\{1, 1, 0\}$(figure 9). In figure 9, $y = b + S$ and t_3 are hyper-charge and the third component of isospin, where b and S are baryon number and strangeness. While under SU(4) flavor symmetry, $(1, 0, 0)$ or $\{1, 0, 0, 0\}$ is used to represent $u/d/c/s$ quarks.

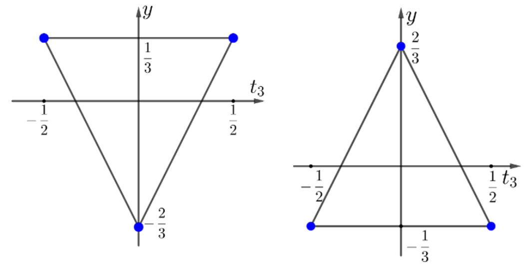

FIG. 9: Flavor Symmetry under SU(3)

IV.3. SU(3) Flavor Symmetry for Mesons and Baryons

Fundamental representation of SU(3) represents $u/d/s$ and its conjugate representation represents $\bar{u}/\bar{d}/\bar{s}$.

Consider mesons consisting of one $u/d/s$ quark and one $\bar{u}/\bar{d}/\bar{s}$ anti-quark, product $(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0)$ or $3 \otimes \bar{3} = 8 \oplus 1$, where 8 and 1 are octet $(1, 1)$ and singlet $(0, 0)$ respectively, can be used to generate all states of mesons, which provides powerful methods to predict new particles.

Octet and singlet mesons states are in figure 10. The left weight diagram represents scalar mesons for $L = 0$, while the right one represents pseudoscalar mesons for $L = 1$.

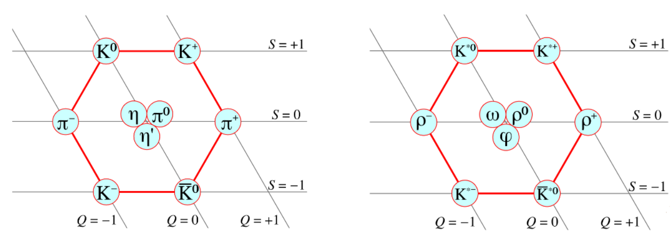

FIG. 10: Octets of mesons consisting of $u/d/s$ and $\bar{u}/\bar{d}/\bar{s}$.

Given baryons consisting of 3 $u/d/s$ quarks, product $(1, 0) \otimes (1, 0) \otimes (1, 0) = (3, 0) \oplus (1, 1) \oplus (1, 1) \oplus (0, 0)$ or $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ generates one completely symmetric decuplet, two partially anti-symmetric octets and one completely anti-symmetric singlet, shown in figure 11.

IV.4. SU(4) Flavor Symmetry for Mesons and Baryons

The weight diagrams of mesons(left mesons, upper: scalar boson, lower: vector boson) and baryons(right:

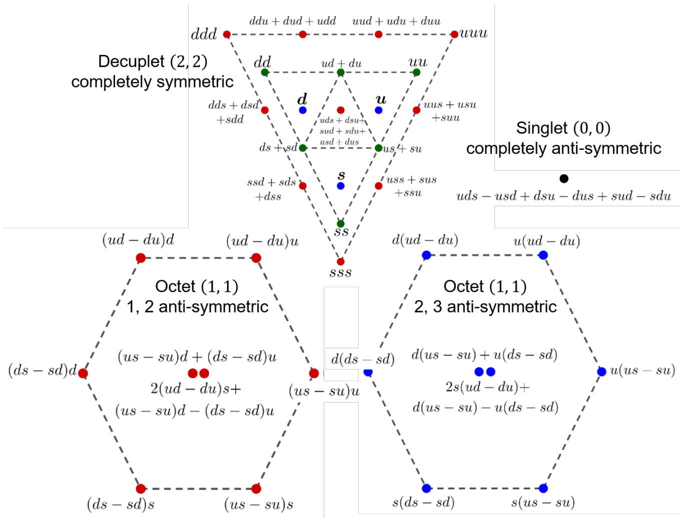

FIG. 11: Octet and decuplet of baryons consisting of $u/d/s$

baryons, upper: spin 1/2, lower: spin 3/2) are shown in figure 12.

For SU(4), baryons multiplets $4 \otimes 4 \otimes 4 = 20 \oplus 20 \oplus 20 \oplus 4$. Also, its Lie algebra is 3-dimension and charm quantum is the third dimension besides hyper-charge and the third component of isospin.

IV.4.1. SU(4) Mesons

For mesons of 0-baryon-number with gluons otect in $3 \otimes 3 = 1 \oplus 8$, product $4 \otimes \bar{4} = 1 \oplus 15$ or $= (1+3) \otimes (\bar{1} + \bar{3}) = 1 \otimes \bar{1} + 1 \otimes \bar{3} + 3 \otimes \bar{1} + 3 \otimes \bar{3}$ generates singlets, octet and triplets as shown in figure 12. Spins product $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ generates the scalar mesons(spin 0) and vector mesons(spin 1).

IV.4.2. SU(4) Baryons

For baryons, spins product $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$ means spins of baryons can only be 1/2(partially anti-symmetry) and 3/2(completely symmetry). $\Psi(color)$ here must be anti-symmetric as the colorless singlet in $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$.

Given baryons of $L = 0$ and spin 1/2, $\Psi(orbital)$ is symmetric while $\Psi(spin)$ is partially symmetric. Therefore, $\Psi(flavor)$ here must be partially symmetric.

Consider baroyns of $L = 0$ and spin 3/2, which indicate the symmetry of $\Psi(orbital)$ and $\Psi(spin)$. Under the circumstances that wave function Ψ is anti-symmetric, $\Psi(flavor)$ must be symmetric as 3 Young diagram tableaux in a row, which is 20-dimension.

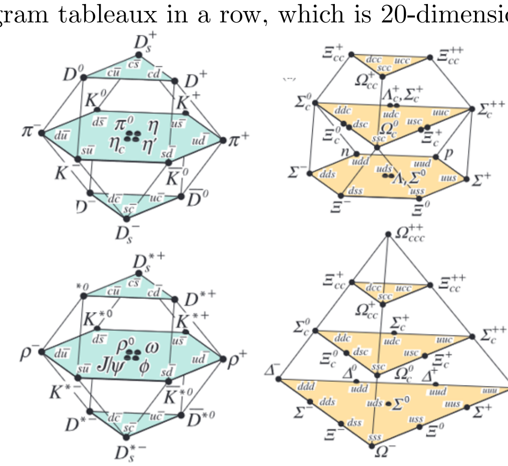

FIG. 12: Hadrons categorization under SU(4), taken from PDG[4]

V. SUMMARY

SU(N) Group theory is useful in particle physics. It can help to decompose reducible tensor product, categories particle states and even predict particle states. In 1960s, SU(3) theory is used to successfully categorizes mesons and baryons multiplets of light quarks for the first time in history, by Gellmann and Ne'eman. Although subtle symmetry breaking of SU(3) for light quarks happens, this theory still provide us an inspiring view of microscopic particles.

VI. ACKNOWLEDGMENTS

Love Sun Yat-sun University from South Campus 180-311.

[1] A. Davidson and K. C. Wali, Multigeneration SU(n) × SU(n) flavor unifying schemes, Phys. Rev. D **23**, 477 (1981).
[2] M. von Steinkirch, Introduction to group theory for physicists (State University of New York at Stony Brook, 2011).
[3] T. Fujita, K. Mukaida, K. Murai, and H. Nakatsuka, SU(n) natural inflation, Phys. Rev. D **105**, 103519 (2022).
[4] P. Zyla *et al.* (Particle Data Group), Review of Particle Physics, PTEP **2020**, 083C01 (2020), and 2021 update.
[5] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, An updated review of the new hadron states, arXiv (2022), arXiv:2204.02649 [hep-ph].
[6] Symmetries, in *Introduction to Elementary Particles* (John Wiley and Sons, Ltd, 1987) Chap. 4, pp. 103–141.
[7] Quantum chromodynamics, in *Introduction to Elementary Particles* (John Wiley and Sons, Ltd, 1987) Chap. 9, pp. 279–299.