Functional Analysis of Stock and Cryptocurrency Market Dynamics

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Abstract

This report analyzes the price behaviors of stocks and cryptocurrencies using functional data analysis (FDA) methods to uncover their distinct dynamics. Fourier analysis reveals that stock prices are driven by a combination of long-term trends and short-term fluctuations, while cryptocurrency prices are dominated by high-frequency components, indicating rapid, volatile changes. Band Depth (BD2) analysis highlights distinct clusters for stocks and cryptocurrencies, and fANOVA confirms a significant difference in their price behaviors.

Mean function comparisons show that stocks have higher and more variable average values over time, whereas cryptocurrencies exhibit a flatter trend with occasional high volatility. Dynamic Time Warping (DTW) further demonstrates the differences by showing largely distinct price movement patterns across the two groups.

These findings suggest that stocks and cryptocurrencies follow unique trajectories, underscoring the need to view them as separate asset classes with different risk and behavior profiles in investment contexts.

Introduction

In recent years, financial markets have been witnessing an increasing interest in cryptocurrency assets alongside traditional stocks, as investors seek diversification and alternative investment opportunities. This study aims to analyze and compare the behavior of selected stocks and cryptocurrencies using Functional Data Analysis (FDA) techniques. By representing time series data as continuous functions, FDA enables us to explore trends, volatility, and underlying correlations in a novel way that considers the entire trajectory of asset price movements.

The key objectives of this project are:

- Volatility Analysis: To examine and interpret volatility trends in stocks versus cryptocurrencies.
- Correlation Analysis: To assess rolling correlations over time, which provides insights into periods when certain assets might have exhibited stronger or weaker associations.
- Functional ANOVA Application: To determine whether significant differences exist in the average behavior of stocks and cryptocurrencies over the observed period.
- Fourier Analysis of Periodicity: Analyze cyclical patterns within the asset prices to identify underlying periodic behaviors that may distinguish stocks from cryptocurrencies.
- Depth Analysis: Utilize depth measures to understand the centrality and outliers in price behavior, comparing the central trends and variability within each asset group.

Through this study, we aim to provide insights into whether the unique properties of cryptocurrency markets affect their alignment with more traditional stock markets, contributing to a better understanding of their role within the larger financial landscape. The results may also shed light on potential diversification benefits, risk factors, and market behavior trends that could be useful for portfolio management and strategic investment decisions.

1 Data Collection and Preprocessing

1.1 Data Selection

For this study, we selected 10 stocks and 10 cryptocurrencies to represent a balanced comparison between traditional and digital assets. The chosen stocks—AAPL, JNJ, KO, WMT, XOM, PG, JPM, UNH, MCD, and NKE—were selected to cover a range of sectors, while the cryptocurrencies—ETH, DSH, XLM, PPC, DOGE, XEM, XRP, XMR, BTC, and LTC—represent a variety of popular and widely traded digital assets. This selection was carefully made to ensure a degree of independence among the different assets, which is essential for accurate comparative analysis.
Daily closing prices were collected for each asset from August 7, 2015, to December 29, 2017. Since stock prices are not available on weekends, the corresponding weekend data points for cryptocurrencies were also removed to maintain homogeneity in the time series and enable a more direct comparison across both asset types

1.2 Data Normalization and Scaling

To facilitate comparison across assets with differing price scales and volatilities, a *min-max normalization* approach was applied. For each time series X, the formula used was:

$$X_{scaled} = \frac{X - X_{\min}}{X_{\max} - X_{\min}}$$

where:

- X_{scaled} is the normalized value,
- $X_{\min}$ and $X_{\max}$ are the minimum and maximum values of X over the time period.

This approach scaled each asset's price series to a [0,1] range, preserving the shape and trends within each series while standardizing the range. Min-max normalization was chosen for its simplicity and interpretability, enabling us to visualize relative price movements more intuitively across assets with different absolute price levels. This was particularly useful in functional data analysis to reveal trends, volatility patterns, and correlations across assets on a comparable scale.

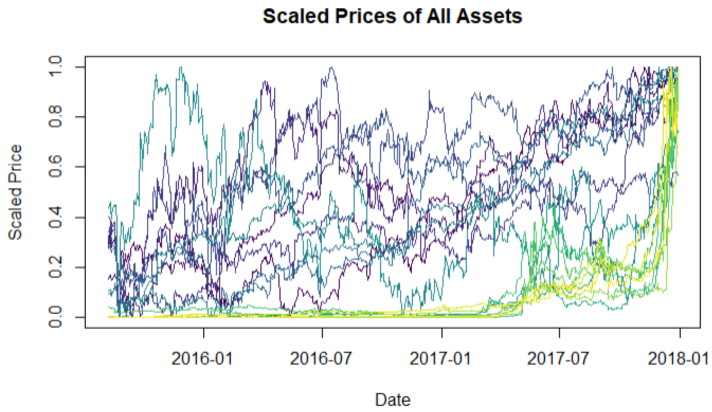

Figure 1: Scaled prices

1.3 B-Spline Smoothing

In this study, we applied B-spline smoothing to transform the discrete daily price data into continuous functional representations. A B-spline function is constructed from a series of basis functions $B_i(t)$, each defined over specific intervals (or "knots") in the data. The smoothed price function $f(t)$ for each asset can be expressed as a weighted sum of these basis functions:

$$f(t) = \sum_{i=1}^{K} c_i B_i(t)$$

where c_i are coefficients determined to best fit the observed data, and K is the number of basis functions (or *nbasis*) selected. By adjusting the number of basis functions, we control the smoothness of the curve: more basis functions allow for a closer fit to the data, capturing finer details, while fewer basis functions result in a smoother representation. For this project, the optimal *nbasis* for each asset was chosen based on visual inspection, ensuring the smoothed curves accurately reflect the unique characteristics of each price trajectory.

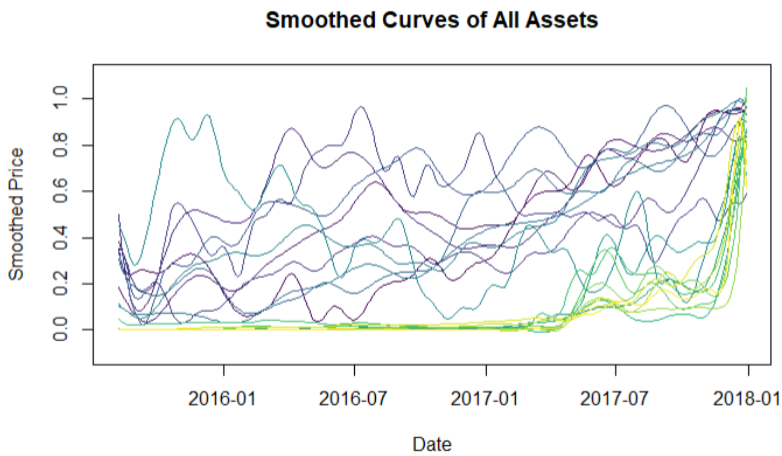

Figure 2: Smoothed prices

1.4 Creating Functional Objects

After applying B-spline smoothing to the daily price data, the next step was to create functional objects that represent the smoothed price trajectories of each asset. This process involved converting the smoothed B-spline curves into a functional data format, which allows for the analysis of the entire trajectory rather than discrete data points. Each functional object $f_i(t)$ for asset i is defined over a continuous interval, capturing the variation in price over time as a function of t. By representing the price data in this way, we enable the application of various Functional Data Analysis (FDA) techniques to explore temporal dynamics, volatility, and correlation patterns among the selected stocks and cryptocurrencies. This functional representation enhances our ability to analyze the underlying trends and behaviors of the assets in a comprehensive manner, facilitating deeper insights into their performance over the study period.

2 Exploratory Data Analysis

2.1 Functional Cross-Correlation

The **correlation** between two functions $f(t)$ and $g(t)$ is defined as the **normalized inner product** of the two functions over a specified interval $[a, b]$. It measures the degree of similarity or alignment between the functions, with values ranging from -1 (perfect negative correlation) to 1 (perfect positive correlation). A value of 0 indicates no correlation. The correlation is computed as:

$$Corr(f, g) = \frac{\langle f, g \rangle}{\|f\|\|g\|}$$

Where:

$$\langle f, g \rangle = \int_a^b f(t) \cdot g(t)\, dt$$

And $\|f\|$ and $\|g\|$ are the norms (magnitudes) of the functions.
The result:

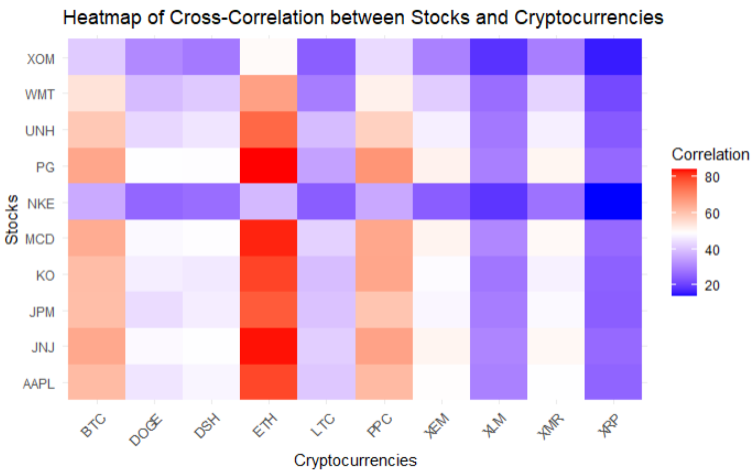

Figure 3: Functional Cross Correlation

The heatmap reveals a limited correlation between the 10 stocks and 10 cryptocurrencies examined. While a few pairs show strong positive correlations, most of the heatmap is dominated by weak to no correlations, indicating a lack of significant relationship between the two asset classes.

2.2 Rolling Volatility Analysis

In Functional Data Analysis (FDA), rolling volatility measures the variability of a function over a moving time window. This approach provides insights into the function's dynamic fluctuations, helping to identify periods of high and low volatility. By calculating the standard deviation within each time window, we can observe how the function's variability evolves over time.

The rolling volatility $V(t)$ at a given time t is calculated as:

$$V(t) = \sqrt{\frac{1}{n} \sum_{i=t-n+1}^{t} \left(f(i) - \bar{f}_t\right)^2}$$

where:

- n is the window size,
- $f(i)$ is the function value at time i,
- $\bar{f}_t$ is the mean of f over the n-point window $\{t-n+1, \dots, t\}$.

For our analysis, we chose a window of 10 days, so $n = 10$. This means that each volatility measure reflects the standard deviation of the function over the previous 10-day period, highlighting trends in variability as time progresses.

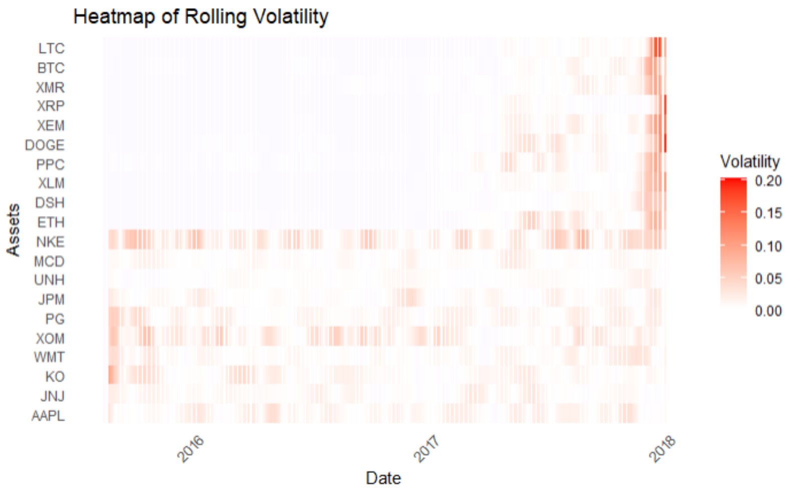

Figure 4: Rolling Volatility

Cryptocurrencies exhibit significantly higher volatility especially at the end of the series (late 2017, early 2018). Stocks, in general, show stable volatility levels.

3 Statistical Analysis

3.1 Functional ANOVA (fANOVA)

The functional analysis of variance (fANOVA) is a method used to test for differences between groups when the data consist of functions or curves. Similar to traditional ANOVA, which tests for mean differences across groups, fANOVA assesses whether there are significant differences in the average functions across multiple groups.

In fANOVA, the goal is to test whether the functional data from different groups differ significantly across the entire domain (e.g., time). The key idea is to compare the overall mean functions of each group and see if there is a statistically significant difference between them.

The general model for fANOVA can be expressed as:

$$f_i(t) = \mu(t) + \epsilon_i(t)$$

Where:

- $f_i(t)$ is the observed function for the i-th observation at time t,
- $\mu(t)$ is the mean function for the entire population or the group mean at time t,
- $\epsilon_i(t)$ is the error term or the deviation of the i-th function from the mean function at time t.

In the case of multiple groups, the fANOVA model is extended to include group-specific mean functions, where we test if the mean functions for the groups are different:

$$f_i(t) = \mu_{g_i}(t) + \epsilon_i(t)$$

Here, $\mu_{g_i}(t)$ represents the mean function for group g_i, and we test whether the group-specific mean functions, $\mu_{g_1}(t), \mu_{g_2}(t), \ldots$, differ significantly across the domain.

The null hypothesis in fANOVA is that the group mean functions are equal, i.e., $H_0 : \mu_{g_1}(t) = \mu_{g_2}(t) = \ldots$, while the alternative hypothesis is that at least one of the group mean functions differs from the others.

This approach allows for the detection of differences not just in the mean values, but in the entire functional trajectories, providing a richer understanding of group differences over time or space.

Analysis Result:

The results of the fANOVA test show a highly significant difference in the price behaviors of stocks and cryptocurrencies. The F-statistic was calculated to be **29.52**, with 1 degree of freedom between groups and 18 degrees of freedom within groups. The corresponding **p-value** is extremely small (3.669766×10^{-5}), well below the commonly used significance level of 0.05. This strong statistical evidence allows us to reject the null hypothesis, which posits that there is no difference in the price behavior between stocks and cryptocurrencies. Consequently, we can conclude that the price behaviors of the two asset classes are significantly different. This finding highlights the distinct dynamics of stocks and cryptocurrencies, suggesting that their price movements exhibit unique patterns, volatility, and trends over the period analyzed.

The mean function represents the average behavior of a set of functions over time or another continuous domain. For each point in time, the mean function calculates the average value across all observations in the dataset, providing a single, smooth curve that summarizes the central tendency or typical behavior of the functions.

The plot shows the mean functions for stocks and cryptocurrencies over time. The turquoise line (stocks) has a consistently higher mean value and exhibits notable fluctuations, indicating more dynamic price behavior. In contrast, the red line (cryptocurrencies) is flatter and remains close to zero, suggesting lower average volatility and a more stable trend. This visual difference aligns with the fANOVA results, which indicated a significant difference between the two groups, with stocks showing more variable and higher average price behavior than cryptocurrencies.

3.2 Depth Analysis

Depth analysis is a technique used to measure how "central" or "typical" a function is within a set of functions. Depth measures provide a way to rank functions from the most central (most

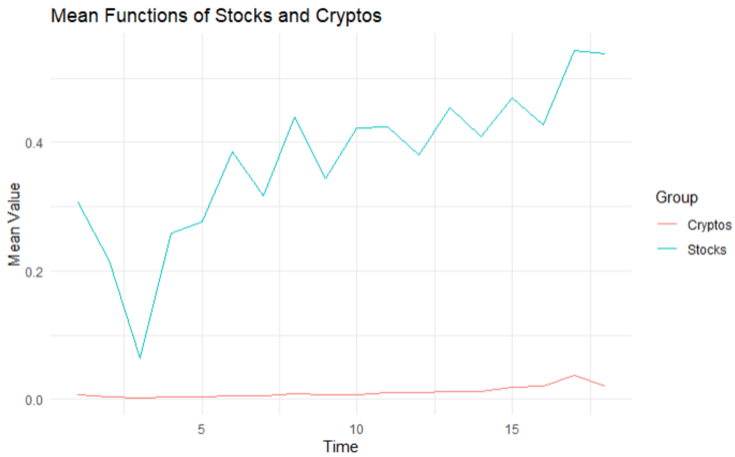

Figure 5: Mean Functions

representative) to the most outlying (least representative). A function with high depth is closer to the central tendency of the dataset, while a function with low depth is considered an outlier.

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Modified Band Depths (MBD)

The Modified Band Depth (MBD) is a specific depth measure that identifies how central or outlying each function is relative to the others. Unlike simple measures, MBD accounts for the functional nature of the data, taking into consideration the structure and continuity of the functions. It is particularly useful for visualizing central and extreme behaviors in functional datasets, as it provides a clear picture of which functions lie within the typical range and which are more unusual.

The purple-shaded region represents the central functions with high depth, indicating the "typical" behavior of the data. The black line within this region is the mean function, which summarizes the central trend. The light blue lines on the outer boundaries represent more "outlying" functions with low depth values, showing behaviors that differ significantly from the central trend.

Band Depth of Order 2 (BD2)

Band Depth of Order 2 (BD2) is a statistical measure used to assess the centrality of a data point within a functional dataset. It calculates the proportion of time a given function lies completely within the bands formed by pairs of other functions in the dataset. Unlike Modified Band Depth, BD2 requires a function to be entirely contained within the band to contribute to its depth. This makes BD2 more sensitive to outliers and can be useful for identifying extreme values or anomalies in functional data.

The plot shows a general upward trend, indicating that the overall relationship between the assets has become stronger over time. Also, there are two primary clusters of assets visible: one

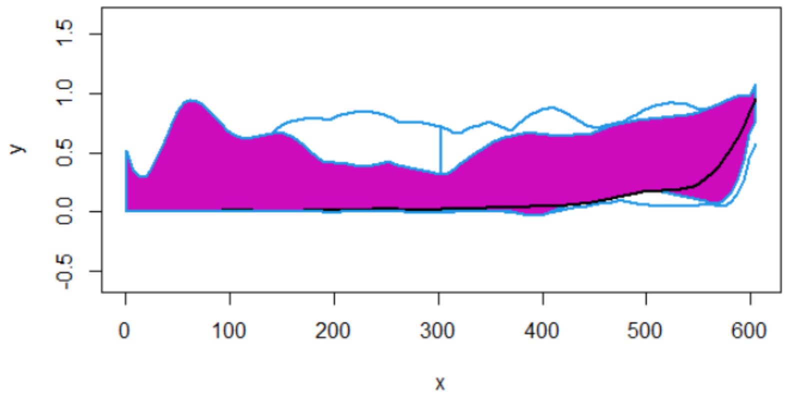

Figure 6: Modified Band Depths (MBD)

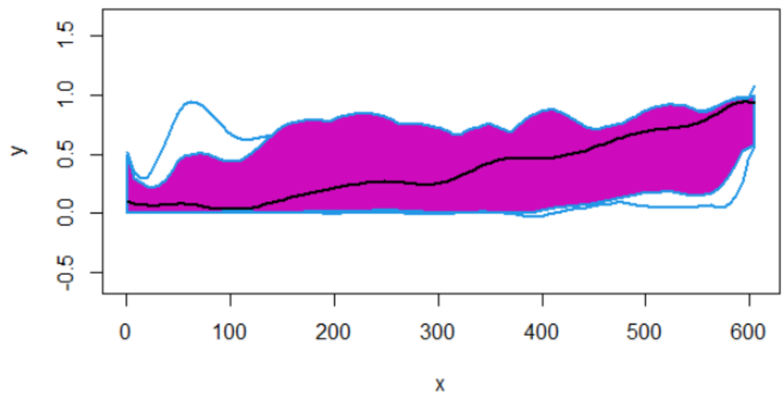

Figure 7: Band Depth of Order 2 (BD2)

at the bottom (cryptocurrencies) and one at the top (stocks). These clusters suggest that certain assets tend to move together more closely than others.

4 Time Series Analysis

4.1 Dynamic Time Warping

Dynamic Time Warping (DTW) is a method used in Functional Data Analysis (FDA) to measure the similarity between two temporal sequences that may vary in speed. In the context of FDA, DTW aligns two functional data curves by optimally warping the time axis, allowing for the comparison of functions that may not be perfectly aligned or may have different temporal lengths. The objective of DTW is to minimize the cumulative distance between the two sequences through an optimal mapping of their time indices. Mathematically, DTW is typically formulated as:

$$D(x, y) = \min_{\pi \in \Pi} \sum_{t=1}^{T} \|x(t) - y(\pi(t))\|^2$$

where $x(t)$ and $y(t)$ are the values of the two curves at time t, and π is a warping path that represents the alignment between the two time series. The goal is to find the path π that minimizes the cumulative squared distance between the two curves. DTW is particularly useful for comparing functions with varying temporal dynamics or different lengths, such as time-series data in medical or economic contexts.

In our analysis, DTW was applied to compare the daily price movements of stocks and cryptocurrencies. By using DTW, we aim to capture the temporal dynamics between these two asset classes, allowing us to identify how similarly or differently they behave over time, even if their price movements are not perfectly aligned. Our objective is to assess the relationships and patterns between stocks and cryptocurrencies based on their temporal dynamics, providing insights into their co-movements and potential correlations.

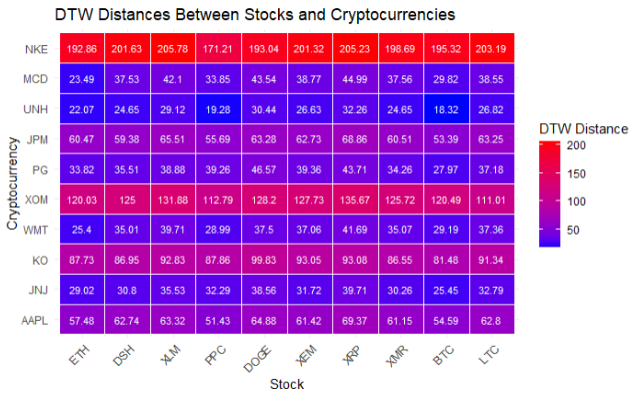

Figure 8: Dynamic Time Warping

The heatmap highlights that certain stocks have closer price movement patterns with specific cryptocurrencies, while others are more distinct. For example, some stocks like McDonald's (MCD) and UnitedHealth (UNH) tend to have lower DTW distances with a range of cryptocurrencies, implying more aligned price movement. In contrast, stocks like Nike (NKE) and JPMorgan (JPM) generally show higher distances with most cryptocurrencies, suggesting they behave differently over time.

4.2 Fourier Analysis

Fourier analysis is a mathematical method used to decompose complex time series data into simpler periodic components, helping to identify underlying patterns and frequencies. In Fourier analysis, a function $f(t)$ can be represented as an infinite sum of sines and cosines, which capture different frequency components. This decomposition, called a Fourier series, is given by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

where a_n and b_n are Fourier coefficients, representing the amplitude of each cosine and sine function at different frequencies n, and ω is the fundamental frequency. These coefficients reveal the contribution of each frequency component to the overall shape of the function.

In this analysis, Fourier smoothing was applied to daily-scaled price data for both stocks and cryptocurrencies. Using a Fourier basis with 12 basis functions (approximately monthly periodicity), the data were smoothed over the observed date range to capture dominant periodic trends while reducing noise. For each asset group, Fourier coefficients were extracted, representing how each frequency contributes to the periodic structure of the data.

To visualize these patterns, we plotted the Fourier coefficients for stocks and cryptocurrencies. The coefficients provide insights into the periodic behavior of the assets, highlighting any dominant frequencies that could indicate cyclical trends, seasonality, or other recurring patterns in the price movements of stocks and cryptos. These visualizations enable a clearer comparison of temporal dynamics between the two asset groups in terms of their frequency characteristics.

Stocks

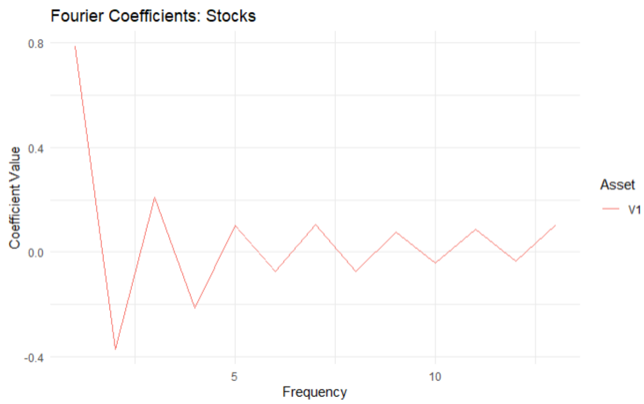

Figure 9: Fourier coefficients - Stocks

The Fourier analysis of the stock price reveals a complex pattern with multiple frequency components. This suggests that the stock's price movement is influenced by a combination of long-term trends, seasonal patterns, and short-term fluctuations. The presence of multiple peaks in the Fourier spectrum indicates that these different components contribute to the overall price dynamics.

Cryptocurrencies

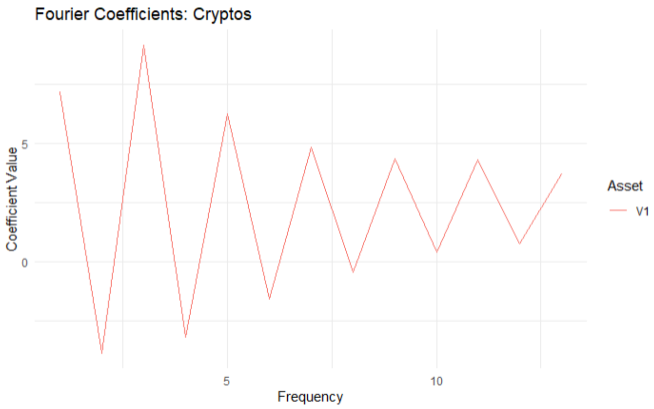

Figure 10: Fourier coefficients - Cryptocurrencies

The Fourier analysis of the cryptocurrency price reveals a pronounced periodic pattern with dominant high-frequency components. This suggests that the cryptocurrency's price movement is largely driven by short-term fluctuations and rapid changes. The presence of sharp peaks at higher frequencies indicates that the market is highly volatile and susceptible to frequent price swings.

Conclusion

This report provides a detailed comparison of the price behaviors of stocks and cryptocurrencies using various methods from functional data analysis. The results show clear differences between the two asset classes, highlighting their unique dynamics and patterns.

The Fourier analysis shows that stocks and cryptocurrencies are driven by different types of frequency components. Stock prices display a complex mix of frequencies, suggesting that they are influenced by a combination of long-term trends, seasonal patterns, and short-term movements. In contrast, cryptocurrency prices are dominated by high-frequency components, indicating that they are driven mostly by rapid, short-term fluctuations. This reveals a more volatile and fast-changing pattern for cryptocurrencies.

The Band Depth of order 2 (BD2) analysis supports these findings by showing two main clusters: one for stocks and one for cryptocurrencies. This separation suggests that stocks and cryptocurrencies have distinct price behaviors, with assets in each group moving more closely together within their own group.

The fANOVA test provides statistical confirmation of these differences, with an F-statistic of 29.52 and a p-value of 3.67×10^{-5}. This highly significant result shows that the price behaviors of stocks and cryptocurrencies are not the same. Specifically, stocks have higher mean values and more fluctuations over time, while cryptocurrencies show a lower, more stable average trend. This is also seen in the mean function plot, where stocks exhibit greater variability compared to the flatter trend of cryptocurrencies.

The Dynamic Time Warping (DTW) analysis adds further insight by measuring the similarity of price movements over time. The high DTW distances between most stocks and cryptocurrencies indicate that they follow different price paths. Although a few stocks show some similarity to certain cryptocurrencies, the overall DTW results confirm that stocks and cryptocurrencies generally move in distinct ways.

In conclusion, the methods used in this report—Fourier analysis, BD2, fANOVA, mean function comparison, and DTW—all indicate significant differences between stocks and cryptocurrencies. Stocks exhibit more complex and varied behaviors, while cryptocurrencies are characterized by rapid, short-term changes. These findings suggest that, from an investment standpoint, stocks and cryptocurrencies should be viewed as separate asset classes with different risk and behavior profiles.