Exemples et Exercices

AFC

Exemple 1:

- Soit la matrice des probabilités suivante de type (3,2)

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{12} \end{pmatrix}$$

- Que l'on peut écrire sous forme d'un tableau

			Total
	1/6	1/6	1/3
	1/12	1/4	1/3
	1/4	1/12	1/3
Total	1/2	1/2	1

Nuage $\mathcal{N}(I)$:

$$L_1 = \left(\frac{p_{11}}{p_{1\bullet}}; \frac{p_{12}}{p_{1\bullet}}\right) = \left(\frac{1/6}{1/3}; \frac{1/6}{1/3}\right) = \left(\frac{1}{2}; \frac{1}{2}\right)$$

$$L_2 = \left(\frac{p_{21}}{p_{2\bullet}}; \frac{p_{22}}{p_{2\bullet}}\right) = \left(\frac{1/12}{1/3}; \frac{1/4}{1/3}\right) = \left(\frac{1}{4}; \frac{3}{4}\right)$$

$$L_3 = \left(\frac{p_{31}}{p_{3\bullet}}; \frac{p_{32}}{p_{3\bullet}}\right) = \left(\frac{1/4}{1/3}; \frac{1/12}{1/3}\right) = \left(\frac{3}{4}; \frac{1}{4}\right)$$

Nuage $\mathcal{B}(I)$:

$$M_1 = (\beta_{11};\beta_{12});\quad M_2 = (\beta_{21};\beta_{22});\quad M_3 = (\beta_{31};\beta_{32})$$

$$\beta_{ij} = \frac{p_{ij}}{p_{i\bullet}\sqrt{p_{\bullet j}}}$$

$$M_1 = \left(\frac{p_{11}}{p_{1\bullet}\sqrt{p_{\bullet 1}}};\frac{p_{12}}{p_{1\bullet}\sqrt{p_{\bullet 2}}}\right) = \left(\frac{1/2}{\sqrt{1/2}};\frac{1/2}{\sqrt{1/2}}\right)$$

$$M_1 = \left(\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right)$$

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$$M_2 = \left(\frac{p_{21}}{p_{2\bullet}\sqrt{p_{\bullet 1}}};\frac{p_{22}}{p_{2\bullet}\sqrt{p_{\bullet 2}}}\right) = \left(\frac{1/4}{\sqrt{1/2}};\frac{3/4}{\sqrt{1/2}}\right)$$

$$M_2 = \left(\frac{\sqrt{2}}{4};\frac{3\sqrt{2}}{4}\right)$$

$$M_3 = \left(\frac{p_{31}}{p_{3\bullet}\sqrt{p_{\bullet 1}}};\frac{p_{32}}{p_{3\bullet}\sqrt{p_{\bullet 2}}}\right) = \left(\frac{3/4}{\sqrt{1/2}};\frac{1/4}{\sqrt{1/2}}\right)$$

$$M_3 = \left(\frac{3\sqrt{2}}{4};\frac{\sqrt{2}}{4}\right)$$

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Matrice des variances-covariances:

- Carrée d'ordre 2

$$W = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$v_{11} = \sum_i p_{i\bullet}\left(\beta_{i1} - \sqrt{p_{\bullet 1}}\right)^2 = p_{1\bullet}\left(\beta_{11} - \sqrt{p_{\bullet 1}}\right)^2 + p_{2\bullet}\left(\beta_{21} - \sqrt{p_{\bullet 1}}\right)^2 + p_{3\bullet}\left(\beta_{31} - \sqrt{p_{\bullet 1}}\right)^2$$

$$v_{11} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2$$

$$v_{11} = \frac{1}{3}\left(\frac{2}{16}\right) + \frac{1}{3}\left(\frac{2}{16}\right) = \frac{1}{12}$$

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$$v_{22} = p_{1\bullet}\left(\beta_{12} - \sqrt{p_{\bullet 2}}\right)^2 + p_{2\bullet}\left(\beta_{22} - \sqrt{p_{\bullet 2}}\right)^2 + p_{3\bullet}\left(\beta_{32} - \sqrt{p_{\bullet 2}}\right)^2$$

$$v_{22} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2$$

$$v_{22} = \frac{1}{3}(0)^2 + \frac{1}{3}\left(\frac{2}{16}\right) + \frac{1}{3}\left(\frac{2}{16}\right) = \frac{1}{12}$$

$$v_{12} = v_{21} = p_{1\bullet}\left(\beta_{11} - \sqrt{p_{\bullet 1}}\right)\left(\beta_{12} - \sqrt{p_{\bullet 2}}\right) + p_{2\bullet}\left(\beta_{21} - \sqrt{p_{\bullet 1}}\right)\left(\beta_{22} - \sqrt{p_{\bullet 2}}\right) + p_{3\bullet}\left(\beta_{31} - \sqrt{p_{\bullet 1}}\right)\left(\beta_{32} - \sqrt{p_{\bullet 2}}\right)$$

$$v_{12} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right) + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)$$

$$v_{12} = \frac{-1}{12} = v_{21}$$

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- D'où

$$W = \begin{pmatrix} 1/12 & -1/12 \\ -1/12 & 1/12 \end{pmatrix}$$

- Variabilité de B(I):

$$V_B = tr(W) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

- Valeurs propres de W:

$$\det(W - \lambda I) = \begin{vmatrix} \frac{1}{12} - \lambda & \frac{-1}{12} \\ \frac{-1}{12} & \frac{1}{12} - \lambda \end{vmatrix} = 0$$

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$$\Rightarrow \left(\frac{1}{12} - \lambda\right)^2 - \left(\frac{1}{12}\right)^2 = 0$$

$$\Rightarrow \left(\frac{1}{12}\right)^2 - \frac{\lambda}{6} + \lambda^2 - \left(\frac{1}{12}\right)^2 = 0$$

$$\Rightarrow \lambda\left(\lambda - \frac{1}{6}\right) = 0 \Rightarrow \lambda = 0;\ \lambda = \frac{1}{6}$$

$$\Rightarrow \lambda_{\max} = \frac{1}{6}$$

$$\Rightarrow V_C = \lambda_{\max} = \frac{1}{6}$$

Variabilité expliquée est $\delta = \frac{V_C}{V_B} = \frac{\lambda_{\max}}{tr(W)} = 1$

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Distance χ² dans $\mathcal{N}(I)$:

$$d^2(i,i')=\sum_j \frac{1}{p_{\bullet j}}\left(\frac{p_{ij}}{p_{i\bullet}}-\frac{p_{i'j}}{p_{i'\bullet}}\right)^2$$

$$i=1;\ i'=2$$

$$d^2(1,2)=\frac{1}{2}\left(\frac{p_{11}}{p_{1\bullet}}-\frac{p_{21}}{p_{2\bullet}}\right)^2+\frac{1}{2}\left(\frac{p_{12}}{p_{1\bullet}}-\frac{p_{22}}{p_{2\bullet}}\right)^2$$

$$d^2(1,2)=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)^2+\frac{1}{2}\left(\frac{1}{2}-\frac{3}{4}\right)^2=\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{2}\left(\frac{-1}{4}\right)^2=\frac{1}{16}$$

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$$i=1;\ i'=3$$

$$d^2(1,3)=\frac{1}{2}\left(\frac{p_{11}}{p_{1\bullet}}-\frac{p_{31}}{p_{3\bullet}}\right)^2+\frac{1}{2}\left(\frac{p_{12}}{p_{1\bullet}}-\frac{p_{32}}{p_{3\bullet}}\right)^2$$

$$=\frac{1}{2}\left(\frac{1}{2}-\frac{3}{4}\right)^2+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)^2=\frac{1}{2}\left(\frac{-1}{4}\right)^2+\frac{1}{2}\left(\frac{1}{4}\right)^2=\frac{1}{16}$$

$$i=2;\ i'=3$$

$$d^2(2,3)=\frac{1}{p_{\bullet 1}}\left(\frac{p_{21}}{p_{2\bullet}}-\frac{p_{31}}{p_{3\bullet}}\right)^2+\frac{1}{p_{\bullet 2}}\left(\frac{p_{22}}{p_{2\bullet}}-\frac{p_{32}}{p_{3\bullet}}\right)^2$$

$$=\frac{1}{2}\left(\frac{1}{4}-\frac{3}{4}\right)^2+\frac{1}{2}\left(\frac{3}{4}-\frac{1}{4}\right)^2=\frac{1}{2}\left(\frac{-1}{2}\right)^2+\frac{1}{2}\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

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Distances χ² dans $\mathcal{B}(I)$:

- Entre M_i et $M_{i'}$ $\Rightarrow d^2(M_i;M_{i'}) = \sum_j (\beta_{ij} - \beta_{i'j})^2$

$$d^2(M_1,M_2) = (\beta_{11}-\beta_{21})^2 + (\beta_{12}-\beta_{22})^2$$

$$= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)^2 = \left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{-\sqrt{2}}{4}\right)^2 = \frac{1}{4}$$

$$d^2(M_1,M_3) = (\beta_{11}-\beta_{31})^2 + (\beta_{12}-\beta_{32})^2$$

$$= \left(\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right)^2 = \left(\frac{-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

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Distances χ² dans $\mathcal{B}(I)$:

$$d^2(M_2,M_3) = (\beta_{21}-\beta_{31})^2 + (\beta_{22}-\beta_{32})^2$$

$$= \left(\frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right)^2 = \left(\frac{-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

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Matrice R:

- Elle est de type (3,2)

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \\ r_{31} & r_{32} \end{pmatrix}$$

$$r_{ij} = \frac{p_{ij} - p_{i\bullet} p_{\bullet j}}{\sqrt{p_{i\bullet} p_{\bullet j}}}$$

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$$r_{11} = \frac{p_{11} - p_{1\bullet} p_{\bullet 1}}{\sqrt{p_{1\bullet} p_{\bullet 1}}} = \frac{1/6 - (1/3)\cdot(1/2)}{\sqrt{(1/3)\cdot(1/2)}} = 0$$

$$r_{12} = \frac{p_{12} - p_{1\bullet} p_{\bullet 2}}{\sqrt{p_{1\bullet} p_{\bullet 2}}} = \frac{1/6 - (1/3)\cdot(1/2)}{\sqrt{(1/3)\cdot(1/2)}} = 0$$

$$r_{21} = \frac{p_{21} - p_{2\bullet} p_{\bullet 1}}{\sqrt{p_{2\bullet} p_{\bullet 1}}} = \frac{1/12 - (1/3)\cdot(1/2)}{\sqrt{(1/3)\cdot(1/2)}} = \frac{-\sqrt{6}}{12}$$

$$r_{22} = \frac{p_{22} - p_{2\bullet} p_{\bullet 2}}{\sqrt{p_{2\bullet} p_{\bullet 2}}} = \frac{1/4 - (1/3)\cdot(1/2)}{\sqrt{(1/3)\cdot(1/2)}} = \frac{\sqrt{6}}{12}$$

$$r_{31} = \frac{p_{31} - p_{3\bullet} p_{\bullet 1}}{\sqrt{p_{3\bullet} p_{\bullet 1}}} = \frac{1/4 - (1/3)\cdot(1/2)}{\sqrt{(1/3)\cdot(1/2)}} = \frac{\sqrt{6}}{12}$$

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$$r_{32} = \frac{p_{32} - p_{3\bullet} p_{\bullet 2}}{\sqrt{p_{3\bullet} p_{\bullet 2}}} = \frac{1/12 - (1/3)\cdot(1/2)}{1/\sqrt{6}} = \frac{-\sqrt{6}}{12}$$

$$R = \begin{pmatrix} 0 & 0 \\ \frac{-\sqrt{6}}{12} & \frac{\sqrt{6}}{12} \\ \frac{\sqrt{6}}{12} & \frac{-\sqrt{6}}{12} \end{pmatrix} \Rightarrow R' = \begin{pmatrix} 0 & \frac{-\sqrt{6}}{12} & \frac{\sqrt{6}}{12} \\ 0 & \frac{\sqrt{6}}{12} & \frac{-\sqrt{6}}{12} \end{pmatrix}$$

$$R'R = \begin{pmatrix} 0 & \frac{-\sqrt{6}}{12} & \frac{\sqrt{6}}{12} \\ 0 & \frac{\sqrt{6}}{12} & \frac{-\sqrt{6}}{12} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{-\sqrt{6}}{12} & \frac{\sqrt{6}}{12} \\ \frac{\sqrt{6}}{12} & \frac{-\sqrt{6}}{12} \end{pmatrix} = \begin{pmatrix} 1/12 & -1/12 \\ -1/12 & 1/12 \end{pmatrix} = W$$

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Exemple 2:

- Le tableau suivant représente le type d'études poursuivies après le Bac. (université, classes préparatoires, autres) en fonction du parcours suivi au lycée (Littéraire, Sc. Eco., Sc. Ex et Technologique) indiquées par 100 étudiants au cours d'une enquête:

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Etudes post-Bac / Type de Bac	Univ.	Classes Prépa.	IUT-BTS	Total
Lettre	13	2	5	20
Sc. éco.	20	2	8	30
Sc. Ex	10	5	5	20
Tech.	7	1	22	30
Total	50	10	40	100

Questions:

1) Donner le tableau des probabilités.
2) Calculer la distance χ^2 entre les modalités.
3) Déterminer la matrice des variances-covariances W ou la matrice R.
4) Calculer la variabilité de nuage $\mathcal{B}(I)$ et du nuage projeté $\mathcal{C}(I)$. En déduire la variabilité expliquée.