13.7 Reduction of Filtration and Hazard Intensity Pre-Default Credit Risk Modeling

This section gives mathematical tools underlying the so-called reduced-form intensity approach in credit risk modeling, based on reduction of filtration (see Crépey (2013)). Given a $[0,T] \cup \{+\infty\}$-valued stopping time τ without atom on $[0,T]$, let $J_t = \mathbb{1}_{\{\tau > t\}}$ denote the related survival indicator process and let $\bar{\tau} = \tau \wedge T$. We assume further that $\mathbb{G} = \mathbb{G}^\tau := \mathbb{F} \vee \mathbb{H}$, where the filtration $\mathbb{H}$ is generated by the process J and where $\mathbb{F}$ is some reference filtration. The Azéma supermartingale associated with τ is the process A defined, for $t \in [0,T]$, by:

$$A_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \mathbb{E}(J_t \,|\, \mathcal{F}_t). \tag{13.62}$$

Assuming a positive $A_t =: e^{-\Gamma_t}$, where Γ is called the hazard process, we have the following "key lemma" of single-name credit risk (see e.g. page 143 of Bielecki and Rutkowski (2001)).

Lemma 13.7.1 *If ξ is an (integrable) random variable, then*

$$J_t \mathbb{E}[\xi | \mathcal{G}_t] = J_t \frac{\mathbb{E}(\xi J_t \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)} = J_t e^{\Gamma_t} \mathbb{E}(\xi J_t \,|\, \mathcal{F}_t). \tag{13.63}$$

For ξ of the form $J_s \chi$, for some $\mathcal{F}_s$-measurable χ with $s \geq t$, we have:

$$\mathbb{E}[J_s \chi | \mathcal{G}_t] = J_t e^{\Gamma_t} \mathbb{E}(\chi J_s \,|\, \mathcal{F}_t) = J_t \mathbb{E}(\chi e^{-(\Gamma_s - \Gamma_t)} \,|\, \mathcal{F}_t). \tag{13.64}$$

Proof. The left-hand side in (13.63) (where the right-hand side is only notational) results from the fact that, on $\{\tau > t\}$, the σ-field $\mathcal{G}_t$ is generated by $\mathcal{F}_t$ and the random variable $\{\tau > t\}$. In (13.64), the left-hand side follows by an application of (13.63) to $\xi = J_s \chi$; the right-hand side then results from the tower law by taking an inner conditional expectation with respect to $\mathcal{F}_s$. □

In particular, (13.64) with $\chi = e^{\Gamma_s}$ proves that the process $\mathfrak{E}_t = J_t e^{\Gamma_t}$ is a $\mathbb{G}$-martingale, since for $s \geq t$:

$$\mathbb{E}[J_s e^{\Gamma_s} | \mathcal{G}_t] = J_t \mathbb{E}(e^{\Gamma_s} e^{-(\Gamma_s - \Gamma_t)} | \mathcal{F}_t] = J_t e^{\Gamma_t}.$$

Lemma 13.7.2 *For any $\mathbb{G}$-adapted, respectively $\mathbb{G}$-predictable, process Y, there exists a unique $\mathbb{F}$-adapted, respectively $\mathbb{F}$-predictable, process $\widetilde{Y}$, called the pre-default value process of Y, such that $JY = J\widetilde{Y}$, respectively $J_- Y = J_- \widetilde{Y}$.*

Proof. In view of (13.63), we can take, in the adapted case, $\widetilde{Y}_t = e^{\Gamma_t} \mathbb{E}(Y_t J_t \,|\, \mathcal{F}_t)$. For the predictable case see §75, page 186 in Dellacherie, Maisonneuve, and Meyer (1992) and Proposition 9.12 in (Nikeghbali 2006). □

Additionally, assuming a continuous and nonincreasing process A_t and letting $\mathfrak{M}_t = -(J_t + \Gamma_{t \wedge \tau})$, we have that $d\mathfrak{E}_t = -\mathfrak{E}_{t-} d\mathfrak{M}_t$ and therefore $d\mathfrak{M}_t = -e^{-\Gamma_t} d\mathfrak{E}_t$, so that $\mathfrak{M}_t$ also is a $\mathbb{G}$-martingale. Moreover:

Lemma 13.7.3 **(i)** *An $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.*
(ii) *An $\mathbb{F}$-adapted càdlàg process cannot jump at τ. We thus have that $\Delta X_\tau = 0$, almost surely, for any $\mathbb{F}$-adapted càdlàg process X.*

***Proof*.** **(i)** Since τ has a positive, continuous and nonincreasing Azéma supermartingale, it is known from Elliot, Jeanblanc, and Yor (2000) that an $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.
(ii) As A is continuous, τ avoids $\mathbb{F}$-stopping times, i.e. $\mathbb{Q}(\tau = \sigma) = 0$ for any $\mathbb{F}$-stopping time σ (see for instance Coculescu and Nikeghbali (2012)). Now, by Theorem 4.1, page 120 in He, Wang, and Yan (1992), there exists a sequence of $\mathbb{F}$-stopping times exhausting the jump times of an $\mathbb{F}$-adapted càdlàg process. □

Remark 13.7.4 Our assumptions on A exclude that τ could be an $\mathbb{F}$-stopping time (otherwise one would have $A = J$, which jumps at τ). However, they imply that τ is an $\mathbb{F}$-pseudo stopping time, meaning that an $\mathbb{F}$-local martingale stopped at τ is a $\mathbb{G}$-local martingale. This is a slight relaxation of the immersion or $(\mathcal{H})$-hypothesis which would mean that an $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale (see the remark 5.2.2). Note that the "key lemma" 13.7.1 is true regardless whether immersion holds, so from this perspective immersion or not is irrelevant.

Letting $\beta_t = e^{-\int_0^t r_s ds}$ denote the discount factor at some $\mathbb{F}$-progressively measurable risk-free rate r_t, we model the cumulative discounted future cash flows of a defaultable claim in the form of the following $\mathcal{G}_{\bar{\tau}}$-measurable random variable, assumed to be well-defined for any $t \in [0, \bar{\tau}]$:

$$\beta_t \pi^t = \int_t^{\bar{\tau}} \beta_s f_s ds + \beta_{\bar{\tau}} \Big(\mathbb{1}_{\{t<\tau<T\}} R_\tau + \mathbb{1}_{\tau > T} \xi \Big), \tag{13.65}$$

for some $\mathbb{F}$-progressively measurable dividend rate process f, some $\mathbb{F}$-predictable recovery process R and some $\mathcal{F}_T$-measurable payment at maturity (random variable) ξ. Note that the assumption that the data r_t, f, R and ξ are in $\mathbb{F}$ is not restrictive in view of Lemma 13.7.2.

Now, assuming A_t time-differentiable, we define the hazard intensity γ and the credit-risk-adjusted-discount-factor α by:

$$\gamma_t = -\frac{d \ln A_t}{dt} = \frac{d\Gamma_t}{dt}, \ \alpha_t = \beta_t \exp(-\int_0^t \gamma_s ds) = \exp(-\int_0^t (r_s + \gamma_s) ds).$$

The next result shows that the computation of conditional expectations of cash flows π^t with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of "$\mathbb{F}$-equivalent" cash flows $\widetilde{\pi}^t$ with respect to $\mathcal{F}_t$.

Lemma 13.7.5 *We have*

$$\mathbb{E}\big(\pi^t \,\big|\, \mathcal{G}_t\big) = J_t \, \mathbb{E}\big(\widetilde{\pi}^t \,\big|\, \mathcal{F}_t\big),$$

where $\widetilde{\pi}^t$ is given, with $g = f + \gamma R$, by:

$$\alpha_t \widetilde{\pi}^t = \int_t^T \alpha_s g_s ds + \alpha_T \xi. \tag{13.66}$$

Proof. Since $\mathfrak{M}_t = -(J_t + \Gamma_{t\wedge\tau})$ is a $\mathbb{G}$-martingale,

$$\mathbb{E}[\mathbb{1}_{\{t<\tau<T\}}\beta_\tau R_\tau | \mathcal{G}_t] = -\mathbb{E}[\int_t^T \beta_s R_s dJ_s | \mathcal{G}_t] = \int_t^T \mathbb{E}[J_s \gamma_s \beta_s R_s | \mathcal{G}_t] ds.$$

The proof is concluded by repeated applications of (13.64). □

In order to compute the left-hand side in (13.66), we can apply a pre-default BSDEs/PDEs modeling approach to the $\mathbb{E}(\widetilde{\pi}^t \,|\, \mathcal{F}_t)$-term in the right-hand side, following the lines of Sects. 13.4-13.5 relative to an $\mathbb{F}$-jump-diffusion (pre-default factor process) X. In this approach, the valuation of defaultable claims is handled in essentially the same way as default-free claims, provided the default-free discount factor process β is replaced by a credit risk adjusted discount factor α and a fictitious dividend continuously paid at rate γ is introduced to account for recovery on the claim upon default (note that a "default-free" discount factor β can itself be interpreted in terms of a default risk with "intensity" r_t). This approach can also be applied to the hedging issue by decomposing the $\mathbb{G}$-martingale of the hedging error of a specific hedging scheme as the sum of an $\mathbb{F}$-martingale stopped at τ (hence a $\mathbb{G}$-martingale) and a jump-to-default $\mathbb{G}$-compensated martingale (see Chapter 5 and Crépey (2013)).

13.7.1 Portfolio Credit Risk

We label by $i \in N = \{1, \ldots, n\}$ the n names of a credit pool. We denote by τ_i the default time of name i. Given a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, the full model filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is defined as the progressive enlargement of $\mathbb{F}$ by the τ_i, i.e.

$$\mathcal{G}_t = \mathcal{F}_t \vee \bigvee_{i\in N} \mathcal{H}_t^i,$$

where $(\mathcal{H}_t^i)$ is the natural filtration of the indicator process H^i of τ_i, so that $\mathcal{H}_t^i = (\tau_i \wedge t) \vee (\tau_i > t)$. The filtration $\mathbb{G}$ can be shown to be right-continuous by a combination of the arguments of Amendinger (1999) and of the appendix[6] of Bélanger, Shreve, and Wong (2001).

Moreover, for $I \subseteq N$, we define the filtration $\mathbb{G}^I = (\mathcal{G}_t^I, t \geq 0)$ as the initial enlargement of $\mathbb{F}$ by the τ_i for $i \in I$, i.e.

$$\mathcal{G}_t^I = \mathcal{F}_t \vee \bigvee_{i\in I} \tau_i.$$

[6] Available online, not present in the Mathematical Finance published version of the paper.