Assignment 3

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1 Introduction

This assignment is little bit lighter than the last two. You are to solve the epidemiological SIR model using the 4^{th} order Runge-Kutta method. I put the code I wrote to solve the SIR model using Euler's method on Canvas under the tutorial section (NumericalODE.zip) and you can use this code as guide.

Specific requirements include:

1. Your code must be able to deal with a system of equations of any size, not just 3 equations.
2. Modify the code to implement the 4^{th} order Runge-Kutta method.
3. Use exception handling where necessary and deal with all potential errors.
4. Output the result of the computation to a csv file.
5. Keep your code tidy and readable. Put in a reasonable amount of comments but no essays.
6. Provide code in main which tests your implementation.
7. Submit your work in a single .cs file (combine all files into one file).

The following notes may be of help.

2 The SIR Model

The SIR model comprise three compartments containing subsets of the total population:

1. The susceptible (S) have not been exposed to infection,
2. The infected (I) who are infected and infectious,
3. The recovered (R) who were infected and have acquired immmunity i.e. cannot be reinfected.

For ease lets use proportions of the population in each set and the evolution equations are:

$$\begin{aligned}
\dot{S} &= -\beta IS \\
\dot{I} &= \beta IS - \gamma I \\
\dot{R} &= \gamma I
\end{aligned}$$

At all times $S + I + R = 1$ and $\dot{S} + \dot{I} + \dot{R} = 0$.

The infectious rate, β, represents the probability of transmitting disease between a susceptible and an infectious individual. The recovery rate, $\gamma = \frac{1}{D}$, is determined by the average time a person stays in the infected compartment, D, and is typically measured in days. Usually $D \approx 14$.

Often people in the media talk in terms of the basic reproduction number (R_0)or the R number. An epidemic occurs if $\dot{I} > 0$

$$\begin{aligned}
\dot{I} = \beta IS - \gamma I &> 0 \\
\frac{\beta S}{\gamma} &> 1
\end{aligned}$$

At the beginning of an epidemic then $S \approx 1$ thus the condition is $R_0 = \frac{\beta}{\gamma} > 1$ for a disease to spread. Because we know γ and R_0, i.e. for COVID $R_0 \approx 2.4$, we can estimate values for β.

3 The 4^{th} order Runge-Kutta method

Consider the equation $\frac{dy}{dx} = f(x, y)$ along with some initial condition $y(0) = y_0$, and we wish to numerically solve it so that we know the solution $y_0, y_1, \cdots, y_n$ at times $x_0, x_1, \cdots, x_n$ where the fixed step size $h = x_{i+1} - x_i$. Then the scheme is

$$\begin{aligned}
y_{j+1} &= y_j + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\
x_{j+1} &= x_j + h
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= f\left(x_j, y_j\right) \\
k_2 &= f\left(x_j + \frac{h}{2}, y_j + h\frac{k_1}{2}\right) \\
k_3 &= f\left(x_j + \frac{h}{2}, y_j + h\frac{k_2}{2}\right) \\
k_4 &= f\left(x_j + h, y_j + hk_3\right)
\end{aligned}$$

In order to adapt this scheme to a system of equations let's suppose we have a system of m equations:

$$\begin{aligned} \dot{y_1} &= f_1(x, y_1, y_2, \cdots, y_m) \\ \dot{y_1} &= f_2(x, y_1, y_2, \cdots, y_m) \\ &\vdots \\ \dot{y_n} &= f_m(x, y_1, y_2, \cdots, y_m). \end{aligned}$$

Define the vector of dependent variables $\mathbf{y} = \{y_1, y_2, \cdots, y_m\}^T$ and similarly for their first derivatives $\dot{\mathbf{y}} = \{\dot{y_1}, \dot{y_2}, \cdots, \dot{y_m}\}^T$. Define a vector of the functions on the right hand side to be $\mathbf{f} = \{f_1(x, \mathbf{y}), f_2(x, \mathbf{y}), \cdots, f_m(x, \mathbf{y})\}^T$, then our system of equations is succinctly given as $\dot{\mathbf{y}} = \mathbf{f}$. Notice that if $\mathbf{f}$ is evaluated at a particular value of x and $\mathbf{y}$ the result is a vector of numbers.

Now the scheme can be written as follows (and we need to think about what each part of the scheme represents):

$$\begin{aligned} &\mathbf{y}_{j+1} = \mathbf{y}_j + \frac{1}{6}h(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \\ &x_{j+1} = x_j + h \end{aligned}$$

where

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}\left(x_j, \mathbf{y}_j\right) \\ \mathbf{k}_2 &= \mathbf{f}\left(x_j + \frac{h}{2}, \mathbf{y}_j + h\frac{\mathbf{k}_1}{2}\right) \\ \mathbf{k}_3 &= \mathbf{f}\left(x_j + \frac{h}{2}, \mathbf{y}_j + h\frac{\mathbf{k}_2}{2}\right) \\ \mathbf{k}_4 &= \mathbf{f}\left(x_j + h, \mathbf{y}_j + h\mathbf{k}_3\right) \end{aligned}$$

Each of $\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_3$ and $\mathbf{k}_4$ are vectors because they result from an evaluation of $\mathbf{f}$.