A Consistent Pricing Model for Index Options and Volatility Derivatives *

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Abstract

We propose a flexible modeling framework for the joint dynamics of an index and a set of forward variance swap rates written on this index. Our model reproduces various empirically observed properties of variance swap dynamics and enables volatility derivatives and options on the underlying index to be priced consistently, while allowing for jumps in volatility and returns.

An affine specification using Lévy processes as building blocks leads to analytically tractable pricing formulas for volatility derivatives, such as VIX options, as well as efficient numerical methods for pricing of European options on the underlying asset. The model has the convenient feature of decoupling the vanilla skews from spot/volatility correlations and allowing for different conditional correlations in large and small spot/volatility moves. We show that our model can simultaneously fit prices of European options on S&P 500 across strikes and maturities as well as options on the VIX volatility index.

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1 Introduction

Volatility indices –such as the VIX index [15]– and derivatives written on such indices have gained popularity in markets as tools for hedging volatility risk and as market-based indicators of volatility. Variance swap contracts [18] are increasingly used by market operators to take a pure exposure to volatility or hedge the volatility exposure of options portfolios.

The existence of a liquid market for volatility derivatives such as VIX options, VIX futures and a well developed over-the-counter market for options on realized variance, and the use of variance swaps and volatility index futures as hedging instruments for other derivatives have led to the need for a pricing framework in which volatility derivatives and derivatives on the underlying asset can be priced in a consistent manner. In order to yield derivative prices in line with their hedging costs, such models should be based on a realistic representation of the joint dynamics of the underlying asset and variance swaps written on this, while also being able to match the observed prices of the liquid derivatives –futures, calls, puts and variance swaps– used as hedging instruments.

In principle, any continuous-time model with stochastic volatility and/or jumps implies some joint dynamics for variance swaps and the underlying asset price. Broadie and Jain [9] studied the valuation of volatility derivatives in the Heston model; Carr et al. [12] studied the pricing of volatility derivatives in models based on Lévy processes. However, many existing models imply unrealistic dynamics for variance swaps[11, 5]: for example, exponential Lévy models imply a constant variance swap term structure while one-factor stochastic volatility models predict perfect correlation among movements in variance swaps at all maturities.

Also, as pointed out by Bergomi [5, 7], the joint dynamics of forward volatilities and the underlying asset is neither explicit nor tractable in most commonly used models, which makes parameter calibration difficult and does not enable the user to choose parameter values to take a view on forward volatility. As a result, classical models such as the Heston model or (time changed) exponential-Lévy models are unable to match empirical properties of variance swaps and VIX options [5, 7, 6, 11, 21]. Some of these issues can be tackled using multi-factor stochastic volatility models [11, 21] but these models are not capable of reproducing finer features of the data such as the magnitude of the VIX option skew or the level of conditional correlations in large and small spot/volatility moves (see Table 1).

An interesting modeling approach, recently developed by Bergomi [5, 7, 6] (see also related work of Bühler [11] and Gatheral [21]) is one in which, instead of modeling "instantaneous" volatility, one models directly the (forward) variance swaps for a discrete tenor of maturities. This approach, which can be seen as the analog of the LIBOR market model for volatility modeling, turns out to be quite flexible and allows to deal with the issues raised above while retaining some tractability.

These models are based on diffusion dynamics where market variables are driven by a multidimensional Brownian motion. Price history across most asset classes has pointed to the importance of discontinuities in the evolution of prices; this anecdotal evidence is supplemented by an increasing body of statistical evidence for jumps in price dynamics [1, 2, 3, 16]. Volatility indices such as the VIX have exhibited, especially during the recent crisis, large fluctuations which strongly point to the existence of jumps, or spikes, in volatility. Figure 1, which depicts the daily closing levels of S&P 500 and the VIX volatility index from September 22nd, 2003 to February 27th, 2009, reveals to the simultaneity of large drops in the S&P 500 with spikes in volatility, which corresponds to the well-known "leverage effect". Comparing the relative changes in the two series (Figure 2) reveals that, while there is a negative correlation of -45% in small changes in the series, large changes –which represent jumps– exhibit an even stronger negative correlation: as shown in Table 1, the correlation between S&P 500 and VIX returns, conditional on the index return being less than $x < 0$ approaches -1 as x becomes more negative, an indication of significant negative dependence in the tails. For a Gaussian vector, or more generally a vector with a Gaussian copula dependence structure, the same conditional correlation would tend to zero as x decreases [26]. Further statistical evidence that the jumps in the level of the S&P 500 index are accompanied by jumps of opposite sign in its volatility is provided by nonparametric tests for simultaneous jumps [23, 29]. These empirical facts need to be accounted for in a realistic model for variance swap dynamics.

Jumps are also important from a pricing perspective: as noted by Neuberger [27], if the underlying asset follows continuous dynamics, its quadratic variation and hence also the payoff from a variance swap can be replicated by continuous rebalancing of a position in the underlying asset and a static position in a log contract, which in turn can be replicated by a static portfolio of calls and puts [8, 20]. The fact that the tracking error associated with this hedging strategy is found to be small in historical data does not entail that the contribution of jumps to the pricing of variance swaps is negligible. This common fallacy is akin to saying that since crashes occur infrequently, deep out-of-the-money puts should be priced at zero: recent experience shows that such mistaken assertions can be costly. In fact adding jumps with negative mean value to the log-price dynamics should increase the value of the variance swap rate, since being long a variance swap can be seen as an insurance against downward movements in the underlying asset and therefore have a risk premium attached to it [14]. Broadie and Jain [10] find that, for a wide range of models and parameter specifications, the effect of discrete sampling on the valuation of volatility derivatives is typically small while the effect of jumps can be significant. Moreover, jumps in volatility are also important in order to produce the positive "skew" of implied volatilities of VIX options [6, 21].

1.1 Contribution

Following the approach proposed by Bergomi [5, 7, 6], the present work proposes an arbitrage-free modeling framework for the joint dynamics of forward variance swap rates along with the underlying index, which

1. captures the information in index option prices by matching the index implied volatility smiles.
2. is capable of reproducing any observed term structure of variance swap rates.
3. captures the information in options on VIX futures by matching their prices/ implied volatility smiles.
4. implies a realistic joint dynamics of spot and forward implied volatilities, allowing in particular for jumps in volatility and returns.
5. allows for the spot/volatility correlation and the implied volatility skews (of vanilla options) to be parameterized independently.
6. is able to handle the term structure of vanilla skew separately from the term structure of volatility of volatility.
7. is tractable and enables efficient pricing of vanilla options, which is a key point for calibration and implementation.

We address these different issues, while retaining tractability, by introducing a common jump factor which affects variance swaps and the underlying index with opposite signs. Describing these jumps in terms of a Poisson random measure leads to an analytically tractable framework, where VIX options *and* calls/puts on the underlying index can be simultaneously priced using Fourier-based methods. Our modeling framework allows for various specifications of the jump size distribution. We give two examples of flexible parameterizations and detail their implementation.

The difference between our modeling framework and the Bergomi model [7] is mainly the ability to meet points 1), 5), 6) and 7) above. In particular, thanks to a semi-analytic representation of call option prices, our model also satisfies the tractability condition 1), allowing efficient calibration to the whole implied volatility surface. This is an advantage over the Bergomi model where only the short-term implied volatility smile can be matched [5, 7].

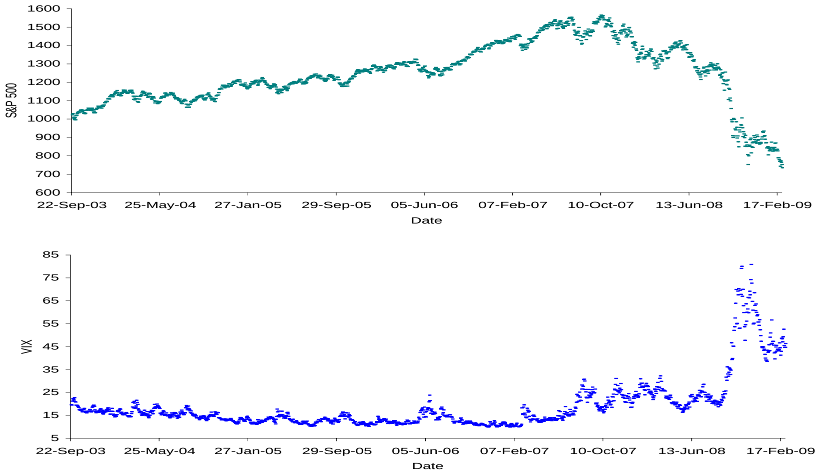

Figure 1: Time series of the VIX index (bottom) depicted together with the S&P 500 (top) covering the period from September 22nd, 2003 to February 27th, 2009.

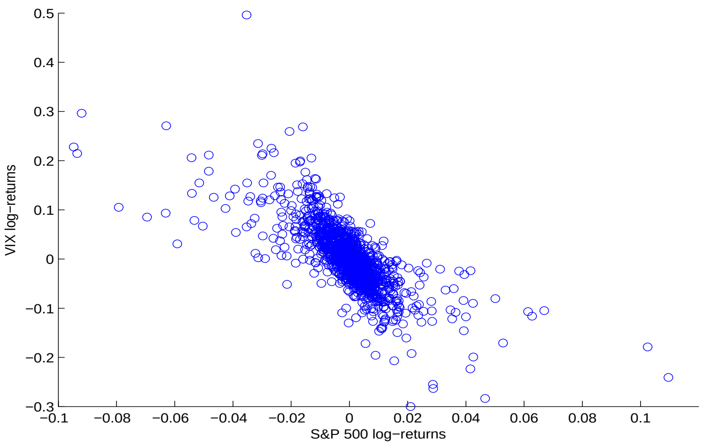

Figure 2: Daily log-returns in the VIX (vertical axis) vs daily log-returns in the S&P 500 index (horizontal axis), September 22nd, 2003 to February 27th, 2009.

Table 1: Conditional correlation between the daily log-returns on S&P 500 and the VIX from September 22nd, 2003 to February 27th, 2009, given the index log-return r_t is below a threshold.

Unconditional	$r_t < -6.5\%$	$r_t < -5\%$	$r_t < -4\%$	$r_t < -3\%$	$\lvert r_t \rvert < 0.5\%$
-0.74	-0.88	-0.55	-0.45	-0.24	-0.45

1.2 Outline

Section 2 describes variance swap contracts, forward variance swap rates and options on variance swaps. The model is described in Section 3 and two model specifications are presented. Section 4 discusses the VIX index and the connection between forward variance swap rates and VIX index futures. Section 5 implements the two specifications of the model and examines their performance in jointly matching VIX options and options on the S&P 500. Section 6 concludes.

2 Variance Swaps and Forward Variances

Consider an underlying asset whose price S is modeled as a stochastic process $(S_t)_{t\geq 0}$ on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ represents the history of the market. We assume the market is arbitrage-free and prices of traded instruments are represented as conditional expectations with respect to a pricing measure $\mathbb{Q}$. We shall neglect in the sequel corrections due to stochastic interest rates.

2.1 Variance Swaps

The annualized *realized variance* of a (price) process S over a time grid $t = t_0 < ... < t_k = T$ is given by

$$RV_t^T = \frac{1}{T-t} \sum_{i=1}^{k} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 . \tag{1}$$

A variance swap (VS) with maturity T initiated at $t < T$ pays the notional of the contract multiplied by the difference between the annualized realized variance of the log-returns RV_t^T less a strike called the variance swap rate, determined such that the contract has zero value at the time of initiation t.

For any semimartingale S, as $\sup_{i=1,\dots,k} |t_i - t_{i-1}| \to 0$ the realized variance converges to the quadratic variation of the log price:

$$\sum_{i=1}^{k} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \xrightarrow{\mathbb{Q}} [\log S]_T - [\log S]_t \ .$$

Approximating the realized variance by the quadratic variation of the log returns is justified when the sampling frequency is daily [10]; we shall use it in the sequel. The fixed leg paid in a variance swap is then approximated by

$$V_t^T = \frac{1}{T-t}\mathbb{E}\left([\log S]_T - [\log S]_t \mid \mathcal{F}_t\right) \ , \tag{2}$$

and we refer to this quantity as the (spot) variance swap (VS) rate prevailing at date t for the maturity T.

2.2 Forward Variance Swap Rates

The forward variance swap rate, quoted at date t for the period $[T_1, T_2]$ is the strike that sets the value of a forward variance swap running from T_1 to T_2 to zero at time t; it is given by

$$V_t^{T_1,T_2} = \frac{1}{T_2 - T_1}\mathbb{E}\left([\log S]_{T_2} - [\log S]_{T_1} \mid \mathcal{F}_t\right) \tag{3}$$

$$= \frac{(T_2 - t)\, V_t^{T_2} - (T_1 - t)\, V_t^{T_1}}{T_2 - T_1} \ , \tag{4}$$

where $t < T_1 < T_2$. The last equality follows easily by substitution of (2). In particular, as noted in [5, 6] forward variance swap rates have the martingale property under the pricing measure: Choosing $t < s < T_1 < T_2$ it follows by substitution of (3) and the use of the law of iterated expectations that

$$\mathbb{E}\left(V_s^{T_1,T_2} \mid \mathcal{F}_t\right) = V_t^{T_1,T_2} \tag{5}$$

which shows that forward variance swap rates are martingales.

Assume that a set of settlement dates is given

$$T_0 < T_1 < ... < T_n$$

known as the tenor structure and that the interval between two tenor dates is fixed, $\tau_i = T_{i+1} - T_i = \tau$ (equal to 30 days if we use the tenor structure of the VIX futures). We define the *forward variance swap rate* over the time interval $[T_i, T_{i+1}]$ as

$$V_t^i = V_t^{T_i,T_{i+1}}. \tag{6}$$

An example of a forward variance swap term structure is given in Figure 3.

2.3 Options on Forward Variance Swaps

A call option with strike K and maturity T_1 on a forward variance swap for the period $[T_1, T_2]$ gives the holder the option to enter at date T_1 into

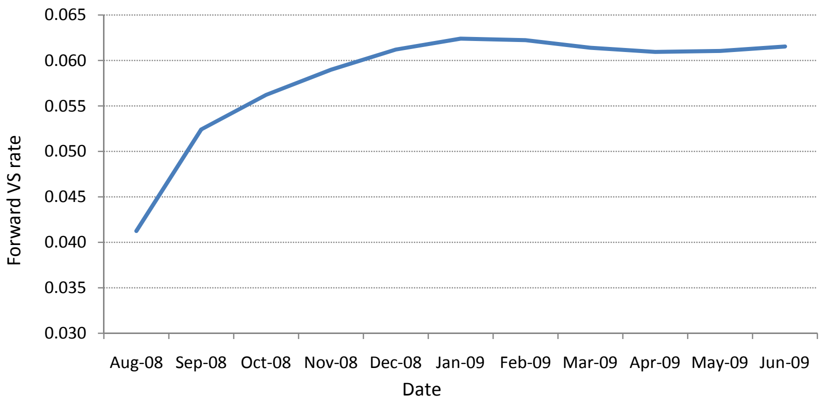

Figure 3: The term structure of 1 month forward variance swaps for the S&P 500 on August 20th, 2008.

a variance swap running from T_1 to T_2 with some predetermined strike K. Hence, the value at time T_1 is

$$\begin{aligned}\pi(T_1, T_1, T_2, K) = &\left(e^{-\int_{T_1}^{T_2} r_s ds}\mathbb{E}\left(RV_{T_1}^{T_2} - K \mid \mathcal{F}_{T_1}\right)\right)^+ \\ =&e^{-\int_{T_1}^{T_2} r_s ds}\left(V_{T_1}^{T_1,T_2} - K\right)^+, \end{aligned} \tag{7}$$

which gives us the time t value

$$\pi(t, T_1, T_2, K) = e^{-\int_t^{T_2} r_s ds}\mathbb{E}\left(\left(V_{T_1}^{T_1,T_2} - K\right)^+ \mid \mathcal{F}_t\right). \tag{8}$$

From this expression it is clear that having tractable dynamics for the forward variance swap rates enables the pricing of options on variance swaps. The market for options on variance swaps is currently illiquid, so in fact the main input for the calibration of the model will be options on VIX futures, which are more liquid. The VIX index is discussed in Section 4.

3 A Model for the Joint Dynamics of Variance Swaps and the Underlying Index

Our goal is to construct a model which

- allows for an arbitrary initial variance swap term structure.
- allows to specify directly the dynamics of the observable forward variance swap rates V_t^i on a discrete tenor of maturities $(T_i, i = 1..n)$.

- allows for flexible modeling of the variance swap curve: for example, it should be able to accommodate the fact that variance swap rates at the long end of the maturity curve have a lower variability than those at the short end.

- allows for jumps in volatility [29] and in the underlying asset [1, 2, 3, 16].

- allows for a flexible specification of the spot price dynamics.

- is analytically tractable i.e. leads to efficient numerical methods for pricing/calibration of calls and puts on the underlying and volatility derivatives such as options on forward variance swaps and options on forward volatility.

In order to achieve these goals, we first specify the dynamics of (a discrete tenor of) forward variance swap rates (Section 3.1) using an affine specification which allows Fourier-based pricing of European-type volatility derivatives. Once the dynamics of forward variance swap rates has been specified, we specify a jump-diffusion dynamics for the underlying asset which is compatible with the variance swap rate dynamics (Section 3.2). Presence of a jump component as well as a Brownian component in the underlying asset allows us to satisfy this compatibility condition while simultaneously matching values/implied volatilities of options on the underlying asset (Section 3.3).

3.1 Variance Swap Dynamics

Given that the forward variance swap rate is a (positive) martingale under the pricing measure, we model it as

$$\begin{aligned} V_t^i &= V_0^i e^{X_t^i} \\ &= V_0^i \exp\left\{ \int_0^t \mu_s^i ds + \int_0^t \omega e^{-k_1(T_i - s)} dZ_s + \int_0^t \int_{\mathbb{R}} e^{-k_2(T_i - s)} x J\left(dx\ ds\right) \right\}, \end{aligned} \tag{9}$$

where $J\left(dxdt\right)$ is a Poisson random measure with compensator $\nu\left(dx\right)dt$ and Z a Wiener process independent of J. The martingale condition imposes

$$\mu_t^i = -\frac{1}{2}\omega^2 e^{-2k_1(T_i - t)} - \int_{\mathbb{R}} \left(\exp\left\{ e^{-k_2(T_i - t)} x \right\} - 1 \right) \nu\left(dx\right) \ . \tag{10}$$

For $t > T_i$ we let $V_t^i = V_{T_i}^i$. In the case of finite jump intensity (9) reduces to

$$V_t^i = V_0^i \exp\left\{ \int_0^t \mu_s^i ds + \int_0^t \omega e^{-k_1(T_i - s)} dZ_s + \sum_{j=0}^{N_t} e^{-k_2(T_i - \tau_j)} Y_j \right\}, \tag{11}$$

where N is a Poisson process with intensity λ, τ_j its jump times, and Y_j i.i.d. random variables with distribution F independent of Z. In this case the Lévy measure has the form $\nu(dx) = \lambda F(dx)$.

The specification (9) allows for jumps and the exponential functions inside the integrals allow to control the term structure of volatility of volatility. For example, if $k_1 > 0$ is large and $k_2 > 0$ small, the diffusion Z results mostly in fluctuations at the short end of the variance swap curve, while the jumps impact the entire curve. Correlated Brownian factors can be added if more flexibility in the variance swap curve dynamics is desired. Likewise, functional forms other than exponentials can be used in (9), although the exponential specification is quite flexible as we will observe in the examples.

Expressions such as (8) can be evaluated using Fourier-based methods [13, 25, 17] given the characteristic function of $X^i_{T_i}$, which in this case has a simple form:

$$\mathbb{E}\left[e^{\mathrm{i}uX^i_{T_i}}\right] = \exp\left\{-\frac{1}{2}u^2\int_0^{T_i}\omega^2 e^{-2k_1(T_i-s)}ds + \mathrm{i}u\int_0^{T_i}\mu^i_s ds\right.$$
$$\left.+\int_0^{T_i}\int_{\mathbb{R}}\left(\exp\left\{\mathrm{i}ue^{-k_2(T_i-s)}x\right\}-1\right)\nu(dx)\,dt\right\}. \tag{12}$$

3.2 Dynamics of the Underlying Asset

Once the dynamics of forward variance swap rates V^i_t for a discrete set of maturities $T_i, i = 1, \ldots, n$ has been specified, it imposes some constraints on the (risk neutral) dynamics of the underlying asset $(S_t)_{t\geq 0}$. S should be such that

1. the discounted spot price $\hat{S}_t = \exp(-\int_0^t (r_s - q_s)\,ds)S_t$ is a (positive) martingale, where q_t is the dividend yield.

2. the dynamics of the spot price is compatible with the specification of V^i_t, which puts the following constraint on the quadratic variation process $[\log S]$ of the log-price:

$$\frac{1}{T_{i+1}-T_i}\mathbb{E}[\,[\log S]_{T_{i+1}} - [\log S]_{T_i}\,|\mathcal{F}_t] = V^i_t \tag{13}$$

   To these constraints we add a third requirement, namely that:

3. the model matches the observed prices of calls/puts on S across strikes and maturities.

Typically we need at least two distinct parameters/degrees of freedom in the dynamics of the underlying asset in order to accommodate points 2) and 3) above.

Bergomi [5, 7] proposes to achieve this by introducing a random "local volatility" function which is reset at each tenor date and chosen to match at time T_i the observed value of $V^i_{T_i}$. This procedure guarantees coherence between the variance swaps and the underlying asset dynamics but leads to a loss of tractability: even vanilla call options need to be priced by Monte Carlo simulation for maturities $T > T_1$.

We adopt here a different approach which allows a greater tractability while simultaneously allowing for jumps in the volatility and the price. In fact, as we will see, the presence of jumps *is* the key to tractability.

The underlying asset is driven by

- a Brownian motion W, correlated with the diffusion component Z driving the variance swaps: $< W, Z >_t = \rho\, t$.
- a jump component, which is driven by the *same* Poisson random measure J which drives jumps in the variance. However, we allow for different jump amplitudes in the underlying and the forward variance swap rates.

The presence of a jump component as well as a diffusion component in the underlying asset allow us to satisfy the compatibility condition 2) while simultaneously matching prices/implied volatilities of options on the underlying asset (Section 3.3).

For each interval $[T_i, T_{i+1})$ we introduce a random variable σ_i and a function $u_i(x, V^i_{T_i})$ which expresses the size of the jump in the underlying asset in terms of the jump size x in forward variance rates V^is and $V^i_{T_i}$ itself. We will give in section 3.4 some simple and flexible specifications for this function u_i but most developments below hold for arbitrary choice of u_i.

We will present here the detailed computations in the (typically useful) case where $T = T_m,\ m = 1, ..., n$, but the analysis can be easily generalized to any maturity. The dynamics of the underlying asset is then specified as

$$S_{T_m} = S_0 \exp\left\{ \int_0^{T_m} (r_s - q_s)\, ds + \sum_{i=0}^{m-1} \left[\mu_i \left(T_{i+1} - T_i\right) + \sigma_i \left(W_{T_{i+1}} - W_{T_i}\right) \right.\right.$$
$$\left.\left. + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i\left(x, V^i_{T_i}\right) J\left(dx\ ds\right) \right] \right\}, \tag{14}$$

where σ_is are $\mathcal{F}_{T_i}$-measurable random variables chosen to match the realized variance swap rate $V^i_{T_i}$ via (18) and $\mu_i = -\frac{1}{2}\sigma_i^2 - \int_{\mathbb{R}} \left(e^{u_i\left(x, V^i_{T_i}\right)} - 1 \right) \nu\left(dx\right)$.

When the jump intensity is finite, (14) reduces to

$$S_{T_m} = S_0 \exp\left\{ \int_0^{T_m} (r_s - q_s)\, ds + \sum_{i=0}^{m-1} \left[\mu_i (T_{i+1} - T_i) + \sigma_i \left(W_{T_{i+1}} - W_{T_i} \right) \right.\right.$$
$$\left.\left. + \sum_{T_i \le \tau_j < T_{i+1}} u_i \left(Y_j, V_{T_i}^i \right) \right] \right\}. \tag{15}$$

As far as pricing of vanilla instruments is concerned, the model can be viewed simply as a flexible (and analytically tractable) parametrization of the joint distribution of forward variance swap rates and the underlying asset on a set of tenor dates. At this level the only assumption we are making is that this joint distribution is infinitely divisible [28] and our model is just a flexible parametrization of the Lévy triplet of the distribution. This makes it particularly easy to price any payoff which involves these variables only at tenor dates.

The assumption of a common jump factor which affects the variance swap rates and the underlying in opposite directions is not only analytically convenient but supported by empirical evidence. As revealed in Table 1, while the unconditional correlation of daily returns of the VIX index and the S&P 500 from September 22nd, 2003 to February 27th, 2009 is -74%, the *conditional* correlation drops to -0.45 on days when the return of S&P 500 is within $\pm 0.5\%$, while the conditional correlation for drops in the index greater than -6.5%, which can be interpreted as jumps, is -88%, close to -100%. This suggests that a large portion of the overall negative correlation is due to simultaneous jumps in opposite directions. This observation is also in agreement with econometric evidence for common jumps in price and volatility based on non-parametric tests [29, 23]. This feature, which has no equivalent in diffusion-based stochastic volatility models such as [7, 21], is a generic property of our framework.

Given the dynamics (14) of the underlying asset, the quadratic variation over the time interval $[T_0, T_m]$ is given by

$$[\log S]_{T_m} - [\log S]_{T_0} = \sum_{i=0}^{m-1} \left\{ \sigma_i^2 (T_{i+1} - T_i) + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i^2 \left(x, V_{T_i}^i \right) J(ds\ dx) \right\}. \tag{16}$$

By taking expectation on (16) the variance swap rate in (2) equals

$$V_{T_0}^{T_m} = \sum_{i=0}^{m-1} \frac{T_{i+1} - T_i}{T_m - T_0} \left(\mathbb{E}\left[\sigma_i^2 \mid \mathcal{F}_{T_0} \right] + \mathbb{E}\left[\int_{\mathbb{R}} u_i^2 \left(x, V_{T_i}^i \right) \nu(dx) \mid \mathcal{F}_{T_0} \right] \right),$$

and the forward variance swap rate at time t for the interval $[T_i, T_{i+1}]$ equals

$$V_t^i = \mathbb{E}\left[\sigma_i^2 \mid \mathcal{F}_t \right] + \mathbb{E}\left[\int_{\mathbb{R}} u_i^2 \left(x, V_{T_i}^i \right) \nu(dx) \mid \mathcal{F}_t \right]. \tag{17}$$

Equation (17) has to hold at all times, but since V_t^i is a martingale we just need to ensure that at time T_i

$$V_{T_i}^i = \sigma_i^2 + \int_{\mathbb{R}} u_i^2\left(x, V_{T_i}^i\right)\nu\left(dx\right). \tag{18}$$

Given a choice for ν, the variance swap rate realized at date T_i may be matched by choosing the parameter σ_i which leaves the parameters in u_i free to calibrate to option prices.

3.3 Pricing of Vanilla Options

Let us now explain the procedure for pricing European calls and puts on S in this framework. The aim is to compute in an efficient manner the value of call options of various strikes and maturities at, some initial date $t = 0$:

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds}\mathbb{E}[(S_{T_m} - K)^+|\mathcal{F}_0]. \tag{19}$$

Denote by $\mathcal{F}_t^{(Z,J)}$ the filtration generated by the Wiener process Z and the Poisson random measure J. By first conditioning on the factors driving the variance swap curve and using the iterated expectation property

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds}\mathbb{E}[\mathbb{E}[(S_{T_m} - K)^+|\mathcal{F}_{T_m}^{(Z,J)}]\ |\mathcal{F}_0] \tag{20}$$

we obtain a *mixing formula* for valuing call options:

Proposition 1 *The value $C(0, S_0, K, T_m)$ of a European call option with maturity T_m and strike K is given by*

$$C(0, S_0, K, T_m) = \mathbb{E}[C^{BS}\left(U_{T_m}, K, T_m; \sigma_*\right)], \tag{21}$$

where $C^{BS}(S, K, T; \sigma)$ denotes the Black-Scholes price for a call option with strike K and maturity T:

$$\sigma_*^2 = \frac{1}{T_m}\sum_{i=0}^{m-1}\sigma_i^2\left(1-\rho^2\right)\left(T_{i+1}-T_i\right), \tag{22}$$

and U_{T_m} is a $\mathcal{F}_{T_m}^{(Z,J)}$-measurable random variable given by

$$\begin{aligned} U_{T_m} = S_0 \exp\Bigg\{\sum_{i=0}^{m-1}\Bigg[&-\left(\frac{1}{2}\sigma_i^2\rho^2 + \int_{\mathbb{R}}\left(e^{u_i\left(x, V_{T_i}^i\right)} - 1\right)\nu\left(dx\right)\right)\left(T_{i+1}-T_i\right) \\ &+\rho\left(Z_{T_{i+1}} - Z_{T_i}\right)\sigma_i + \int_{T_i}^{T_{i+1}}\int_{\mathbb{R}} u_i(x, V_{T_i}^i)J(dx\ ds)\Bigg]\Bigg\} \end{aligned} \tag{23}$$

and the expectation in (21) *is taken with respect to the law of* (Z, J).

Proof. Conditional on $\mathcal{F}_{T_m}^{(Z,J)}$, the increments of Z and the paths of the variance swap rates, and in particular $V_{T_i}^i$, thus the corresponding σ_i are known for $i = 0, 1, ..., m$ from equation (18). Moreover

$$W_{T_{i+1}} - W_{T_i} \stackrel{d}{=} \rho\left(Z_{T_{i+1}} - Z_{T_i}\right) + \sqrt{(1-\rho^2)}\left(\hat{W}_{T_{i+1}} - \hat{W}_{T_i}\right) \ ,$$

where the Wiener processes Z and $\hat{W}$ are independent. The process S_t can therefore be decomposed as

$$\begin{aligned} S_{T_m} = U_{T_m} \exp\Bigg\{ \int_0^{T_m} (r_s - q_s)\, ds + \sum_{i=0}^{m-1} \Bigg[& -\frac{1}{2}\sigma_i^2 \left(1-\rho^2\right)(T_{i+1} - T_i) \\ & + \sigma_i \sqrt{1-\rho^2}\left(\hat{W}_{T_{i+1}} - \hat{W}_{T_i}\right) \Bigg] \Bigg\}, \end{aligned}$$

where $\sigma_i^2 = V_{T_i}^i - \int_{\mathbb{R}} u_i^2\left(x, V_{T_i}^i\right)\nu\left(dx\right)$ and

$$\begin{aligned} U_{T_m} = S_0 \exp\Bigg\{ \sum_{i=0}^{m-1} \Bigg[& -\left(\frac{1}{2}\sigma_i^2\rho^2 + \int_{\mathbb{R}}\left(e^{u_i\left(x, V_{T_i}^i\right)} - 1\right)\nu\left(dx\right)\right)(T_{i+1} - T_i) \\ & + \rho\left(Z_{T_{i+1}} - Z_{T_i}\right)\sigma_i + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx\ ds) \Bigg] \Bigg\} \end{aligned}$$

is $\mathcal{F}_{T_m}^{(Z,J)}$-measurable. In distribution we have

$$S_{T_m} \stackrel{d}{=} U_{T_m} e^{\int_0^{T_m}(r_s - q_s)ds + \mu_* T_m + \sigma_* \hat{W}_{T_m}},$$

where σ_* is given in (22) and

$$\mu_* = \frac{1}{T_m}\sum_{i=0}^{m-1} -\frac{1}{2}\sigma_i^2\left(1-\rho^2\right)(T_{i+1} - T_i)\,, \tag{24}$$

where we notice that $\mu_* = -\frac{1}{2}\left(\sigma_*\right)^2$. Hence, given $\mathcal{F}_{T_m}^{(Z,J)}$, the inner conditional expectation in (20) reduces to the evaluation of the Black-Scholes formula

$$\mathbb{E}[(S_{T_m} - K)_+ | \mathcal{F}_{T_m}^{(Z,J)}] = C^{BS}\left(U_{T_m}, K, T_m; \sigma_*\right),$$

where σ_* depends on all the u_is through (18) and (22). ■

This result has interesting consequences for pricing and calibration of vanilla contracts. Note that the outer expectation can be computed by simulating Z and J: with N simulated sample paths for Z and J we obtain the following approximation

$$\hat{C}_N = \frac{1}{N}\sum_{k=1}^{N} C^{BS}\left(U_{T_m}^{(k)}, K, T_m; \sigma_*\left(k\right)\right) \stackrel{N\to\infty}{\to} C\left(0, S_0, K, T_m\right). \tag{25}$$

Since the averaging is done over the variance swap factors Z and J, the call option price in equation (25) is a deterministic function of the parameters in the u_is. This will prove useful when calibrating the model using option data, since we *do not* have to resimulate N independent samples for each calibration trial.

Equation (25) thus allows to calibrate the model to the *entire* implied volatility surface in an efficient manner, in contrast to the Bergomi model where it is only possible to calibrate to at-the-money slopes of the implied volatilities (ATM skews).

3.4 Parametric examples

Different classes of models in the above framework can be obtained by considering various parameterizations of the Lévy measure $\nu(dx)$ describing the intensity of jumps and different functions u_i appearing in (14). We consider as examples Gaussian jumps and double exponential jumps in the forward variance swap rates and the index.

3.4.1 Gaussian Jumps

In the first example, we specify the Lévy measure as

$$\nu(dx) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}} dx, \tag{26}$$

where m and δ^2 is the mean and variance in a normal distribution, respectively and λ the intensity of the jumps. With $\nu(dx)$ specified the characteristic function in (12) can be calculated (see (45) in Appendix A) and prices of options on the variance swaps can be computed by Fourier transform methods.

We now consider the following specification for u_i, which relates the jumps in the variance swap rates to the jumps in the underlying asset

$$u_i\left(x, V_{T_i}^i\right) = \left(\frac{V_{T_i}^i}{V_0^i}\right)^{\frac{1}{2}} b_i x \ , \tag{27}$$

but any functional form can be used as long as (18) leads to positive values for σ_i. This gives us

$$\begin{aligned}\sigma_i^2 &= V_{T_i}^i - \frac{V_{T_i}^i}{V_0^i}\int_{\mathbb{R}} (b_i x)^2 \nu(dx) \\ &= V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i}\left(b_i^2 m^2 + b_i^2\delta^2\right).\end{aligned}$$

In order to achieve non-negative values for σ_i^2 we require

$$\lambda\left(b_i^2 m^2 + b_i^2 \delta^2\right) \leq V_0^i. \tag{28}$$

3.4.2 Exponentially Distributed Jumps

To introduce an asymmetry in the tail of the jump size distribution we can use a two-sided exponential distribution [24]:

$$\nu(dx) = \lambda \left(p\alpha_+ e^{-\alpha_+ x} \mathbf{1}_{x \geq 0} + (1-p)\,\alpha_- e^{-\alpha_- |x|} \mathbf{1}_{x<0} \right) dx, \tag{29}$$

where p denotes the probability of a positive jump, $\alpha_+ > 1$ and $\alpha_- > 0$ and $1/\alpha_+$ and $1/\alpha_-$ the expected positive and negative jump sizes, respectively. In this specification the characteristic function for the forward variance swap rate is given by (47) in Appendix A.

As above, we specify the u_i function as in (27), which gives rise to the following constraint on the σ_is

$$\sigma_i^2 = V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i} \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)\,b_i^2}{\alpha_-^2} \right).$$

To ensure positive σ_is we constrain the calibration by

$$\lambda \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)\,b_i^2}{\alpha_-^2} \right) \leq V_0^i. \tag{30}$$

The parameters b_i in the above specifications can take any value as long as (28) respectively (30) are satisfied. We will see later, how the calibration of b_i entails that jumps in the underlying have opposite direction to the jumps in the VIX futures. Together with negative correlation between Z and W, this feature enables the model to generate positive skews for the implied volatility of VIX options and negative skews for the implied volatility of calls and puts on the underlying index, as observed in empirical data. This property is achieved without the need to add extra volatility factors, as proposed recently in e.g. [11, 19, 21] where multi-factor mean-reverting dynamics on volatility and volatility of volatility are imposed to accommodate this behavior, resulting in a loss of tractability.

4 VIX futures

4.1 The VIX index

The CBOE VIX index provides investors with a quote on the expected volatility of the S&P500 index over the next 30 calendar days. The VIX is constructed by first synthesizing the variance swap from call/put prices on the index using the log-contract formula:

$$VS_t^T = \frac{2}{T-t} \sum_i \frac{\Delta K_i}{K_i^2} e^{r_t^T (T-t)} Q(K_i, T; t) - \frac{1}{T-t} \left(\frac{F_t}{K_0} - 1 \right)^2, \tag{31}$$

where K_i are strikes of a set of out-of-the money options $\Delta K_i = \frac{K_{i+1}-K_{i-1}}{2}$, r_t^T is the risk-free interest rate to expiration, $Q(K_i, T; t)$ the midprice of a call (if $K_i > F_t$) or a put (if $K_i < F_t$) option with strike K_i, and maturity T, and F_t is the forward value. K_0 is the first strike below the forward index level F_t. The VIX is then computed by linearly interpolating the forward variances $VS_t^{T_1}$ and $VS_t^{T_2}$ for the two quoted option maturities closest to 30 days:

$$VIX_t = 100\sqrt{\frac{365}{30}\left((T_1-t)\,VS_t^{T_1}\frac{N_{T_2}-30}{N_{T_2}-N_{T_1}} + (T_2-t)\,VS_t^{T_2}\frac{30-N_{T_1}}{N_{T_2}-N_{T_1}}\right)}\,, \tag{32}$$

where N_{T_1} and N_{T_2} denote the actual number of days to expiration for the maturities, $N_{T_1} < 30 < N_{T_2}$.

The VIX is often presented as an indicator of the expected volatility over the next 30 days. It is not immediately clear from either (31) or (32) that this is in fact the case. For notational simplicity assume that the jumps exhibit finite variation. Then it can be shown that the realized variance of returns can be decomposed into three components (see [14])

$$\begin{aligned}
[\log S]_T - [\log S]_t =& \frac{2}{T-t}\left(\int_0^{F_t}\frac{1}{K^2}(K-S_T)^+\,dK\right.\\
&+\int_{F_t}^{\infty}\frac{1}{K^2}(S_T-K)^+\,dK + \int_t^T\left(\frac{1}{F_{s-}}-\frac{1}{F_t}\right)dF_s\\
&\left.-\int_t^T\int_{\mathbb{R}}\left(e^y-1-y-\frac{y^2}{2}\right)J\,(dy\;ds)\right)\,,
\end{aligned} \tag{33}$$

where y is the jump size in the index (in our framework $y = u_i\left(x, V_{T_i}^i\right)$) and J is a Poisson random measure with Lévy measure $\nu(dx)$, driving the jumps in the underlying asset. Taking expectations with respect to the pricing measure in (33) we obtain

$$\begin{aligned}
V_t^T =& \frac{2}{T-t}e^{\int_t^T r_s ds}\int_0^{\infty}\frac{Q(K,T;t)}{K^2}dK\\
&-\frac{2}{T-t}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}}\left(e^y-1-y-\frac{y^2}{2}\right)\nu(dy)\,ds \mid \mathcal{F}_t\right]\,.
\end{aligned} \tag{34}$$

Equation (31) is a discretization of the first term in (34) [15]. The square of the VIX index is thus a model free estimate of the realized variance expected over the next 30 days, up to a correction term given by

$$VIX_t^2 = V_t^T + \frac{2}{T-t}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}}\left(e^y-1-y-\frac{y^2}{2}\right)\nu(dy)\,ds \mid \mathcal{F}_t\right], \tag{35}$$

where $T = t + 30$ days.

4.2 VIX Futures

Let VIX_t^i denote the VIX futures price for the interval $[T_i, T_{i+1}]$ seen at time t. For $t = T_i$ we have

$$\left(VIX_{T_i}^i\right)^2 = V_{T_i}^i + 2\int_{\mathbb{R}} (e^{u_i\left(x, V_{T_i}^i\right)} - 1 - u_i\left(x, V_{T_i}^i\right) - \frac{u_i^2\left(x, V_{T_i}^i\right)}{2})\nu\left(dx\right).$$

We also have the martingale property for $t < T_i$

$$VIX_t^i = \mathbb{E}\left[VIX_{T_i}^i | \mathcal{F}_t\right]$$

By Jensen's inequality for convex functions

$$\begin{aligned}\left(VIX_t^i\right)^2 &\leq \mathbb{E}\left[\left(VIX_{T_i}^i\right)^2 | \mathcal{F}_t\right] \\ &= V_t^i + 2\mathbb{E}\left[\int_{\mathbb{R}} e^{u_i\left(x, V_{T_i}^i\right)} - 1 - u_i\left(x, V_{T_i}^i\right) - \frac{u_i^2\left(x, V_{T_i}^i\right)}{2}\nu\left(dx\right) \mid \mathcal{F}_t\right],\end{aligned} \tag{36}$$

since V_t^i is a martingale. Equation (36) shows that there is a "convexity correction" connecting VIX futures to forward variance swap rates.

To obtain a tractable framework, we first characterize the dynamics of $\sqrt{V_t^i}$ then propose an approximation which has a closed-form characteristic function.

Proposition 2 *The process $\sqrt{V_t^i}$ has the multiplicative decomposition*

$$\sqrt{V_t^i} = \sqrt{V_0^i} M_t^i A_t^i$$

where M_t^i is an exponential martingale given by

$$M_t^i = \exp\left\{\int_0^t \eta_s^i ds + \frac{1}{2}\int_0^t \omega e^{-k_1(T_i - s)} dZ_s + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i - s)} x J\left(dx\ ds\right)\right\}, \tag{37}$$

where

$$\eta_t^i = -\frac{1}{8}\omega^2 e^{-2k_1(T_i - t)} - \int_{\mathbb{R}} \left(\exp\left\{\frac{1}{2}e^{-k_2(T_i - t)}x\right\} - 1\right)\nu\left(dx\right) \tag{38}$$

and A_t^i is a (deterministic) finite variation process given by

$$\begin{aligned}A_t^i = \exp\Bigg\{\int_0^t \Bigg[&-\frac{1}{8}\omega^2 e^{-2k_1(T_i - s)} - \frac{1}{2}\int_{\mathbb{R}} \left(\exp\left\{e^{-k_2(T_i - s)}x\right\} - 1\right)\nu\left(dx\right) \\ &+ \int_{\mathbb{R}} \left(\exp\left\{\frac{1}{2}e^{-k_2(T_i - s)}x\right\} - 1\right)\nu\left(dx\right)\Bigg] ds\Bigg\}.\end{aligned} \tag{39}$$

Proof. Taking the square root on both sides of (9) we arrive at

$$\sqrt{V_t^i} = \sqrt{V_0^i} \exp\left\{ \int_0^t \widehat{\mu}_s^i ds + \frac{1}{2} \int_0^t \omega e^{-k_1(T_i - s)} dZ_s \right. \\ \left. + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i - s)} x J\left(dx\ ds\right) \right\}, \tag{40}$$

where

$$\widehat{\mu}_t^i = -\frac{1}{4}\omega^2 e^{-2k_1(T_i - t)} - \frac{1}{2}\int_{\mathbb{R}} \left(\exp\left\{ e^{-k_2(T_i - t)} x \right\} - 1 \right) \nu\left(dx\right).$$

The stochastic integrals in (40) being processes with independent increments, a straightforward application of the exponential formula for Poisson random measures [17, Prop. 3.6.] yields the multiplicative decomposition in Proposition 2 where M is given by (37)-(38) and A is given by (39). ■

Using this result we approximate VIX_t^i with

$$VIX_t^i \simeq VIX_0^i M_t^i = VIX_0^i \exp(Y_t^i)\ . \tag{41}$$

where $Y_t^i = \log M_t^i$. This approximation leaves the initial level of the VIX^i and the volatility of VIX^i untouched and thus is relevant for pricing VIX derivatives. As in the case of the forward variance swap rates, once the Lévy measure $\nu\left(dx\right)$ is specified, the characteristic function of Y^i in (41) can be easily computed, so options on the VIX index can be priced by Fourier transform methods.[1]

5 Implementation

In this section the model specifications described in Section 3.4 are implemented on data on the VIX and the S&P 500 implied volatility smiles.

5.1 Data Description

We assess the performance of the model using prices from August 20th, 2008 on a range of VIX put and call options for five maturities, VIX futures for the same maturities, the dividend yield on S&P 500, call and put options on S&P 500 for six maturities and for the various maturities we also have the corresponding prices of the futures on S&P 500, from which we also derive a discount curve. Options for which the bid price is zero were removed. All the options data are extracted from the Chicago Board Options Exchange homepage. The variance swap rates are extracted from Bloomberg and equation (4) is used to find the term structure of the 1 month forward variance swap rates depicted in Figure 3.

[1]The characteristic functions for the examples in Section 3.4 are given in Appendix A.

5.2 Calibration and Performance

The calibration of the model consists of three steps:

1. Determine the parameters controlling the variance swap rate dynamics by calibration to VIX options using the characteristic function in (46) or (48) and Fourier transform methods.

2. Use the parameters from the first step to simulate N paths of the forward variance swap rates and store the increments of Z, the jump times and jump sizes along with the $V^i_{T_i}$s.

3. Calibrate to options on the index recursively by the use of (25).

In the first step we compute model prices by the modified Fourier transform method described in [17]. The calibration is performed by minimizing the sum of squared errors weighted by the inverse bid-ask spread across all maturities and strikes on out-of-the-money options on the VIX futures

$$\mathrm{SE} = \sum_{\text{options}} \frac{1}{Q_{Ask} - Q_{Bid}} \left(Q_{Market,Mid} - Q_{Model}\right)^2 \ , \tag{42}$$

using a gradient-based minimization algorithm.

The resulting parameters for the two specifications are shown in Table 2 and it also includes the relative calibration error defined as

$$\frac{1}{\#\left\{\text{options}\right\}} \sum_{\text{options}} \frac{\max\left\{\left(Q_{Model} - Q_{Ask}\right)^+, \left(Q_{Bid} - Q_{Model}\right)^+\right\}}{Q_{Market,Mid}} \ . \tag{43}$$

We see how both model specifications are able to achieve very low calibration error. Note the high value of k_1 compared to k_2, which implies (in the risk-neutral dynamics) that the diffusion component Z is mainly causing fluctuations at the short end of the variance swap curve, while the jumps impact the entire curve.

Figures 4 and 5 depict the (Black) implied volatility for the bid, ask, mid and model prices of VIX options as a function of moneyness. We observe

Table 2: Parameters calibrated to VIX options on August 20th, 2008 and calibration errors.

Gaussian jumps							
λ	ω	k_1	k_2	m	δ		Calibration error (%)
3.52	2.04	21.9	2.07	0.54	0.25		0.64
Double exponential jumps							
λ	ω	k_1	k_2	p	α_+	α_-	Calibration error (%)
13.6	1.98	22.3	2.20	0.86	4.25	19.9	0.85

that model prices fall within the bid-ask spread for almost all observations in both specifications. This performance should be compared with flat model implied volatilities in the Bergomi model [5] and downward-sloping volatilities in the Heston model [22]. The Bergomi model [7] is able to generate positive skews on volatility of volatility via the use of a Markov functional mapping for the forward variances, but it should be noted that the model can only price vanilla options on the underlying index by Monte Carlo simulation.

Step 2 is done by discretization of (9). In this example we have used $2 \cdot 10^5$ simulated sample paths. Simulation of the jumps is straightforward, since they arrive at constant intensity and the jump sizes are computed by draws from a normal/exponential distribution times the scaling $e^{-k_2(T_i-\tau)}$, where τ is the jump time. For details on how to simulate the increments in the forward variance swap rates due to the diffusion the reader is referred to [5].

We estimate the correlation parameter between the continuous components to be $\rho = -0.45$. This may be done either using a threshold method [16] or by using the values in Table 1. The last step is implemented by matching the model to market prices of out-of-the-money options: starting from the shortest maturity, we minimize the calibration error (42) to compute, b_0, then b_1, etc. Parameter estimates for the two different specifi-

Table 3: Model parameters calibrated to S&P 500 volatility smiles on August 20th, 2008. The second and third row in each panel correspond to the parameters determining mean and variance of the jumps in the index.

i	0	1	2	3	4	5
Gaussian jumps						
b_i	-0.140	-0.161	-0.162	-0.187	-0.198	-0.199
$b_i m$	-0.075	-0.087	-0.088	-0.101	-0.107	-0.107
$\lvert b_i \delta \rvert$	0.034	0.040	0.040	0.046	0.049	0.049
Calibration error (%)	3.9	0.6	0.6	1.5	1.2	1.3
Double exponential jumps						
b_i	-0.141	-0.159	-0.158	-0.187	-0.195	-0.192
$\left(\frac{b_i p}{\alpha_+} - \frac{b_i(1-p)}{\alpha_-}\right)$	-0.028	-0.031	-0.031	-0.037	-0.039	-0.038
$\left(\frac{b_i^2 p}{\alpha_+^2} + \frac{b_i^2(1-p)}{\alpha_-^2}\right)^{\frac{1}{2}}$	0.031	0.035	0.035	0.041	0.043	0.042
Calibration error (%)	2.7	0.7	1.1	1.8	1.3	1.8

cations of the jump size distributions are shown in Table 3 along with the pricing error in (43) for each maturity. The mean and standard deviations of the jumps before scaling with $\left(V_{T_i}^i / V_0^i\right)^{\frac{1}{2}}$ are also included. The values of b_i for $i = 0, ..., 3$ relate to the monthly distributions of the index up to the maturity of the options expiring on December 19th, 2008. For the last two option maturities the interval between the expirations are three months and hence b_i for $i = 4, 5$ relate to the three month distributions of the index. Double exponential jumps perform slightly better on the first interval, but model performance is not very sensitive to the choice of the jump size distribution.

Figures 6 and 7 show the result of the calibration to S&P 500 options. The fit of the two specifications are practically indistinguishable and overall the calibrations perform well for both with slightly less success on the shortest maturity.

To assess parameter stability, we repeated the calibration for July 16th, 2008. The result of the calibration to VIX options is reported in Table 4. Table 5 reports the result of the calibration to S&P 500 options.

5.3 Impact of Jumps on Variance Swap Rates

To study the impact of jumps on the valuation of variance swap contracts, we first consider the contribution of jumps to the forward variance swap rates V^i:

$$\varepsilon_i = -2\mathbb{E}\left[\int_{\mathbb{R}}\left(e^{u_i\left(x,V^i_{T_i}\right)} - 1 - u_i\left(x, V^i_{T_i}\right) - \frac{u_i^2\left(x, V^i_{T_i}\right)}{2}\right)\nu\left(dx\right) \mid \mathcal{F}_0\right]. \tag{44}$$

In Table 6 we compute this term using model parameters calibrated as explained above. For the first interval this contribution represents 1.9% of the value of the contract and increases with maturity.

Finally, Table 7 shows the decomposition of the forward variance swap rate into a diffusion term and a jump term according to (17), where the expectations are computed based on $2 \cdot 10^6$ simulated paths of the variance swap rates. In all cases, we observe that the results are not sensitive with respect to the model specification (i.e. exponential vs Gaussian jumps).

Table 4: Parameters calibrated to VIX options on July 16th, 2008 together with the resulting calibration error.

Gaussian jumps							
λ	ω	k_1	k_2	m	δ		Calibration error (%)
3.52*	2.04*	19.9	1.22	0.45	0.21		0.43
Double exponential jumps							
λ	ω	k_1	k_2	p	α_+	α_-	Calibration error (%)
13.6*	1.98*	19.8	1.36	0.86*	4.90	15.8	0.38

* Fixed parameter from the calibration on August 20th 2008.

Table 5: Model parameters calibrated to S&P 500 volatility smiles on July 16th, 2008. The correlation between the two Brownian components is set to -0.45.

i	0	1	2	3	4	5
Gaussian jumps						
b_i	-0.201	-0.233	-0.237	-0.237	-0.259	-0.234
Calibration error (%)	2.9	1.5	0.4	1.4	0.6	1.2
Double exponential jumps						
b_i	-0.203	-0.232	-0.234	-0.235	-0.250	-0.226
Calibration error (%)	2.2	1.7	0.8	1.9	0.7	2.0

Table 6: The contribution of jumps to the forward variance swap rates, as implied by model parameters calibrated to VIX and SP500 options on August 20th, 2008 (Eq. 44), expressed as a percentage of the forward variance swap rate.

Start (months)	0	1	2	3	4	7
End	1	2	3	4	7	10
Gaussian jumps						
$\frac{\varepsilon_i}{V_0^i}$ (%)	1.9	2.3	2.9	3.4	4.3	4.5
Double exponential jumps						
$\frac{\varepsilon_i}{V_0^i}$ (%)	1.9	2.4	2.8	3.6	4.3	4.5

Table 7: Decomposition of the VS rates into diffusion component and jump component with the specifications calibrated to the August 20th, 2008 data.

Start (months)	0	1	2	3	4	7
End	1	2	3	4	7	10
Gaussian jumps						
V_0^i	0.041	0.052	0.056	0.059	0.062	0.061
$\mathbb{E}\left[\sigma_i^2\right]$	0.017	0.020	0.024	0.015	0.013	0.012
$\mathbb{E}\left[\int_{\mathbb{R}} u_i^2\left(x, V_{T_i}\right) \nu(dx)\right]$	0.024	0.032	0.032	0.044	0.049	0.049
Double exponential jumps						
V_0^i	0.041	0.052	0.056	0.059	0.062	0.061
$\mathbb{E}\left[\sigma_i^2\right]$	0.015	0.019	0.023	0.013	0.012	0.013
$\mathbb{E}\left[\int_{\mathbb{R}} u_i^2\left(x, V_{T_i}\right) \nu(dx)\right]$	0.026	0.033	0.033	0.046	0.050	0.048

5.4 Exotic Derivatives

The two specifications considered above for the Lévy measure/ jump size distribution produce comparable fits to VIX and S&P 500 options and, as observed above, lead to similar values for the correction due to jumps in Variance Swap rates. Where they differ is regarding the valuation of exotic –forward starting or path-dependent– options, which are sensitive to the details of the dynamics of the underlying asset.

Consider a forward straddle whose payoff at T_2 is

$$|S_{T_2} - S_{T_1}|,$$

with $T_1 = 5$ months and $T_2 = 10$ months, and a reverse cliquet with payoff at T_n given by

$$\max\left\{0, C + \sum_{i=1}^{n} \min\left\{\frac{S_{T_i} - S_{T_{i-1}}}{S_{T_{i-1}}}, 0\right\}\right\},$$

where the returns are computed on a monthly basis, $T_n = 10$ months and $C = 30\%$.

In Table 8 the 95%-confidence intervals for the prices of the derivatives are estimated by Monte Carlo using $2 \cdot 10^6$ simulated paths.

Table 8: 95%-confidence intervals of Monte Carlo estimators for exotic option values with the specifications calibrated to the August 20th, 2008 data.

	Gaussian jumps	Double exponential jumps
Forward Straddle	$[139.51, 139.83]$	$[139.70, 140.01]$
Reverse Cliquet	$[0.1065, 0.1068]$	$[0.1033, 0.1036]$

6 Conclusion

We have presented a model for the joint dynamics of a set of forward variance swap rates along with the underlying index. Using Lévy processes as building blocks, this model leads to a tractable pricing framework for variance swaps, VIX futures and vanilla call/put options, which makes calibration of the model to such instruments feasible. This tractability feature distinguishes our model from previous attempts [5, 4, 21] which only allow for full Monte Carlo pricing of vanilla options.

Our model reproduces salient empirical features of variance swap rate dynamics, in particular the strong negative correlation of large index moves with large moves in the VIX and the positive skew observed in implied volatilities of VIX options, by introducing a common jump component in the variance swap rates and the underlying asset. Using two different specifications for the jump size distribution (Lévy measure) we have illustrated the feasibility of the numerical implementation, as well as the capacity of the model to match a complete set of market prices of vanilla options and options on the VIX. Our model can be used to price and hedge various payoffs sensitive to forward volatility, such as cliquets or forward starting options, as well as volatility derivatives, in a manner consistent with market prices of simpler instruments such as calls, puts or variance swaps which are typically used for hedging them.

References

[1] Y. Ait-Sahalia and J. Jacod, *Estimating the Degree of Activity of Jumps in High Frequency Data*, Annals of Statistics, 37 (2009), pp. 2202–2244.

[2] T. Andersen, L. Benzoni, and J. Lund, *An Empirical Investigation of Continuous-Time Equity Return Models*, Journal of Finance, 57 (2002), pp. 1239–1284.

[3] O. Barndorff-Nielsen and N. Shephard, *Econometrics of Testing for Jumps in Financial Economics using Bipower Variation*, Journal of Financial Econometrics, 4 (2006), pp. 1–30.

[4] L. Bergomi, *Smile Dynamics*, Risk, 17 (2004), pp. 117–123.

[5] ——, *Smile Dynamics II*, Risk, 18 (2005), pp. 67–73.

[6] ——, *Dynamic Properties of Smile Models*, in Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling, R. Cont, ed., Wiley, 2008, ch. 3.

[7] ——, *Smile Dynamics III*, Risk, 21 (2008), pp. 90–96.

[8] D. Breeden and R. Litzenberger, *Prices of State-Contingent Claims Implicit in Option Prices*, Journal of Business, 51 (1978), pp. 621–651.

[9] M. Broadie and A. Jain, *Pricing and Hedging Volatility Derivatives*, Journal of Derivatives, 15 (2008), pp. 7–24.

[10] ——, *The Effect of Jumps and Discrete Sampling on Volatility and Variance Swaps*, International Journal of Theoretical and Applied Finance, 11 (2008), pp. 761–797.

[11] H. Buehler, *Consistent Variance Curve Models*, Finance and Stochastics, 10 (2006), pp. 178–203.

[12] P. Carr, H. Geman, D. Madan, and M. Yor, *Pricing Options on Realized Variance*, Finance and Stochastics, 9 (2005), pp. 453–475.

[13] P. Carr and D. Madan, *Option Valuation using the Fast Fourier Transform*, Journal of Computational Finance, 2 (1999), pp. 61–73.

[14] P. Carr and L. Wu, *Variance Risk Premiums*, Review of Financial Studies, 22 (2009), pp. 1311–1341.

[15] CBOE, *VIX: CBOE Volatility Index*, http://www.cboe.com/micro/vix/vixwhite.pdf, 2003.

[16] R. Cont and C. Mancini, *Nonparametric Tests for Pathwise Properties of Semimartingales*, Bernoulli, 17 (2011).

[17] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, 2004.

[18] K. Demeterfi, E. Derman, M. Kamal, and J. Zou, *A Guide to Volatility and Variance Swaps*, Journal of Derivatives, 6 (1999), pp. 9–32.

[19] D. Duffie, J. Pan, and K. Singleton, *Transform Analysis and Asset Pricing for Affine Jump-diffusions*, Econometrica, 68 (2000), pp. 1343–1376.

[20] B. Dupire, *Arbitrage Pricing with Stochastic Volatility*, Paribas Working Paper, (1993).

[21] J. Gatheral, *Consistent Modeling of SPX and VIX Options*, in Bachelier Congress, July 2008.

[22] S. Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, Review of Financial Studies, 6 (1993), pp. 327–343.

[23] J. Jacod and V. Todorov, *Do Price and Volatility Jump Together?*, Annals of Applied Probability, 20 (2010), pp. 1425–1469.

[24] S. Kou, *A Jump-Diffusion Model for Option Pricing*, Management Science, 48 (2002), pp. 1086–1101.

[25] A. Lewis, *A Simple Option Formula For General Jump-Diffusion And Other Exponential Lévy Processes*, tech. report, OptionCity.net, http://optioncity.net/pubs/ExpLevy.pdf, 2001.

[26] Y. Malevergne and D. Sornette, *Testing the Gaussian Copula Hypothesis for Financial Assets Dependences*, Quantitative Finance, 3 (2003), pp. 231–250.

[27] A. Neuberger, *The Log Contract: A New Instrument to Hedge Volatility*, Journal of Portfolio Management, 20 (1994), pp. 74–80.

[28] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[29] V. Todorov and G. Tauchen, *Volatility Jumps*, forthcoming in: Journal of Business and Economic Statistics, (2010).

A Characteristic Functions

A.1 Gaussian Jumps

Variance Swaps

We have from [28] that the characteristic function of $X^i_{T_i}$ is given by

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}uX^i_{T_i}}\right] = \exp\Big\{ & -\frac{1}{2}u^2\int_0^{T_i}\omega^2 e^{-2k_1(T_i-s)}ds + \mathrm{i}u\int_0^{T_i}\mu^i_s ds \\ & + \int_0^{T_i}\int_{\mathbb{R}}\left(\exp\left\{\mathrm{i}ue^{-k_2(T_i-s)}x\right\}-1\right)\nu\left(dx\right)dt\Big\} \ .\end{aligned}$$

Noting that

$$\int_0^{T_i} e^{-2k_1(T_i-s)}ds = \frac{1-e^{-2k_1T_i}}{2k_1} \qquad \text{and} \qquad \mathbb{E}\left[e^Y\right] = e^{m+\frac{\delta^2}{2}} \text{ for } Y\sim N\left(m,\delta^2\right)$$

we arrive at

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}uX^i_{T_i}}\right] = \exp\Big\{ & -\frac{1}{2}\omega^2\mathrm{i}u\frac{1-e^{-2k_1T_i}}{2k_1} - \frac{1}{2}\omega^2u^2\frac{1-e^{-2k_1T_i}}{2k_1} \\ & - \mathrm{i}u\lambda\int_0^{T_i}\left(\exp\left\{e^{-k_2(T_i-s)}m+\frac{1}{2}e^{-2k_2(T_i-s)}\delta^2\right\}-1\right)ds \\ & + \lambda\int_0^{T_i}\left(\exp\left\{\mathrm{i}ue^{-k_2(T_i-s)}m-\frac{1}{2}u^2e^{-2k_2(T_i-s)}\delta^2\right\}-1\right)ds\Big\} .\end{aligned} \tag{45}$$

VIX Futures

Recall from (41) that $VIX^i_t \simeq VIX^i_0 M^i_t$ where $M^i_t = \exp(Y^i_t)$ is a positive exponential martingale. The characteristic function of

$$Y^i_{T_i} = \int_0^{T_i}\eta^i_s ds + \frac{1}{2}\int_0^{T_i}\omega e^{-k_1(T_i-s)}dZ_s + \int_0^{T_i}\int_{\mathbb{R}}\frac{1}{2}e^{-k_2(T_i-s)}xJ\left(dx\ ds\right)$$

can be found in the same way as for X^i. It is given by

$$\mathbb{E}\left[e^{\mathrm{i}uY^i_{T_i}}\right] = \exp\left\{-\frac{1}{8}\omega^2\mathrm{i}u\frac{1-e^{-2k_1T_i}}{2k_1} - \frac{1}{8}\omega^2u^2\frac{1-e^{-2k_1T_i}}{2k_1}\right.$$
$$- \mathrm{i}u\lambda\int_0^{T_i}\left(\exp\left\{\frac{1}{2}e^{-k_2(T_i-s)}m + \frac{1}{8}e^{-2k_2(T_i-s)}\delta^2\right\} - 1\right)ds$$
$$\left.+\lambda\int_0^{T_i}\left(\exp\left\{\frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}m - \frac{1}{8}u^2e^{-2k_2(T_i-s)}\delta^2\right\} - 1\right)ds\right\}. \tag{46}$$

A.2 Exponentially Distributed Jumps

Variance Swaps

The characteristic function in (12) takes the form

$$\mathbb{E}\left[e^{\mathrm{i}uX^i_{T_i}}\right] = \exp\left\{-\frac{1}{2}\omega^2\mathrm{i}u\frac{1-e^{-2k_1T_i}}{2k_1} - \frac{1}{2}\omega^2u^2\frac{1-e^{-2k_1T_i}}{2k_1}\right.$$
$$- \mathrm{i}u\lambda\int_0^{T_i}\left(\frac{p\alpha_+}{\alpha_+ - e^{-k_2(T_i-s)}} + \frac{(1-p)\,\alpha_-}{\alpha_- + e^{-k_2(T_i-s)}} - 1\right)ds$$
$$\left.+\lambda\int_0^{T_i}\left(\frac{p\alpha_+}{\alpha_+ - \mathrm{i}ue^{-k_2(T_i-s)}} + \frac{(1-p)\,\alpha_-}{\alpha_- + \mathrm{i}ue^{-k_2(T_i-s)}} - 1\right)ds\right\}. \tag{47}$$

VIX Futures

Recall from (41) that $VIX^i_t \simeq VIX^i_0M^i_t$ where $M^i_t = \exp(Y^i_t)$ is a positive exponential martingale. In this specification the characteristic function of $Y^i_{T_i}$ is given by

$$\mathbb{E}\left[e^{\mathrm{i}uY^i_{T_i}}\right] = \exp\left\{-\frac{1}{8}\omega^2\mathrm{i}u\frac{1-e^{-2k_1T_i}}{2k_1} - \frac{1}{8}\omega^2u^2\frac{1-e^{-2k_1T_i}}{2k_1}\right.$$
$$- \mathrm{i}u\lambda\int_0^{T_i}\left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}e^{-k_2(T_i-s)}} + \frac{(1-p)\,\alpha_-}{\alpha_- + \frac{1}{2}e^{-k_2(T_i-s)}} - 1\right)ds$$
$$\left.+\lambda\int_0^{T_i}\left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}} + \frac{(1-p)\,\alpha_-}{\alpha_- + \frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}} - 1\right)ds\right\}. \tag{48}$$

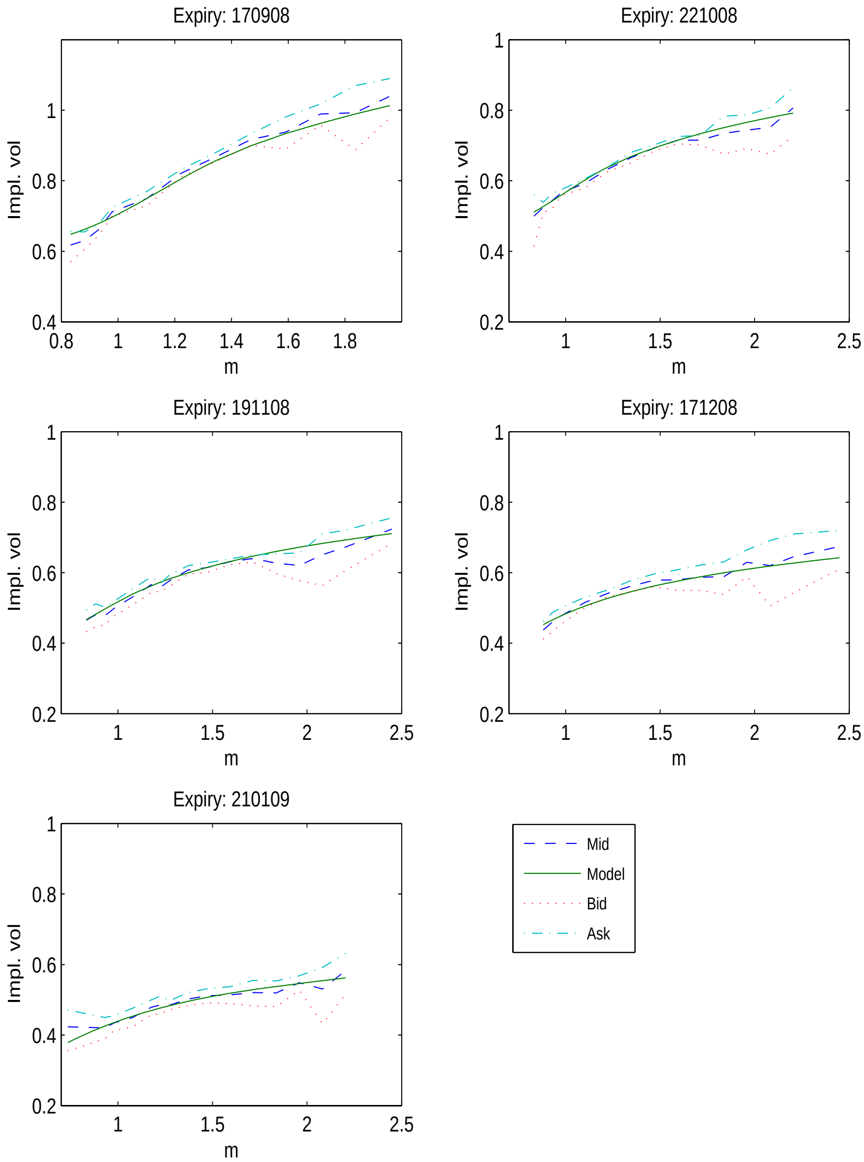

Figure 4: VIX implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the horizontal axis.

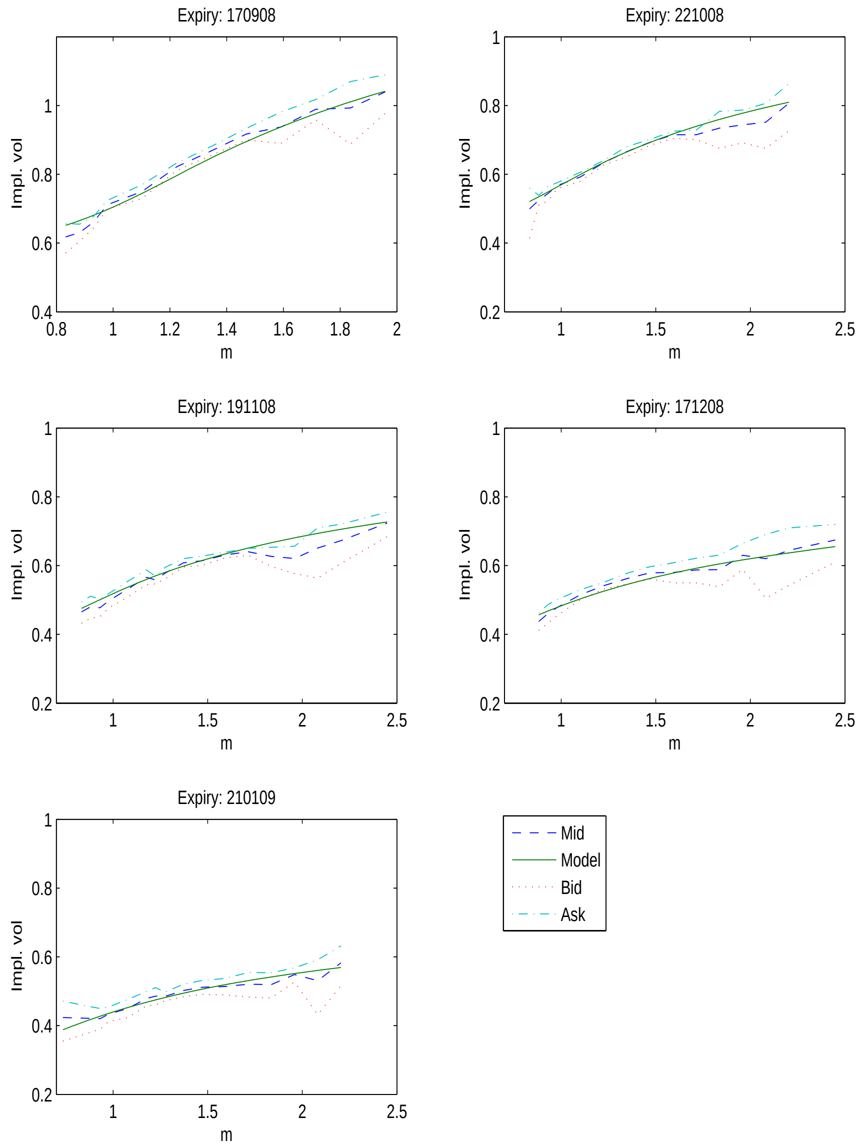

Figure 5: VIX implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the horizontal axis.

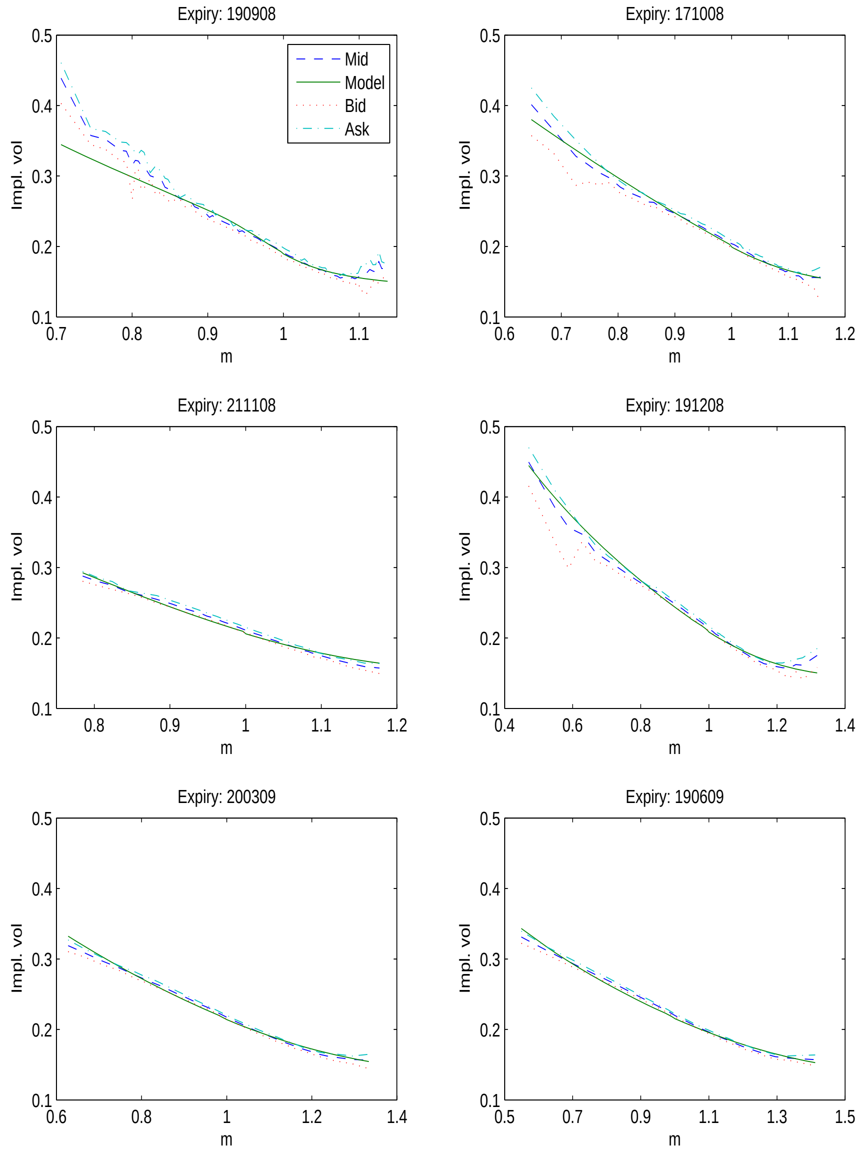

Figure 6: S&P 500 implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/S_t$ on the horizontal axis.

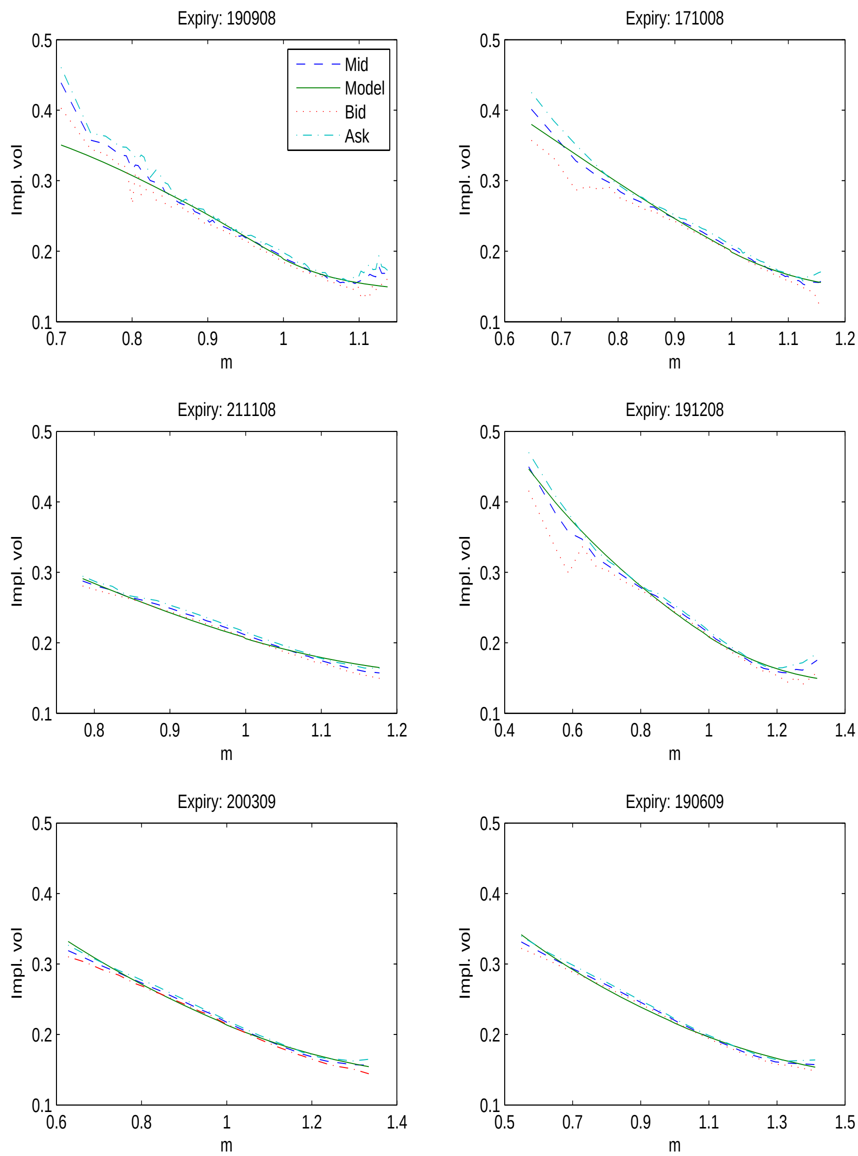

Figure 7: S&P 500 implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against moneyness $m = K/S_t$ on the horizontal axis.