Perturbation theory applied to options prices in the two factor PDV

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Recall the dynamics of the two factor PDV model, for all $t \in [0, T]$:

$$\begin{aligned}
\mathrm{d}S_t &= S_t \sigma\left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{1,t} &= -\lambda_1 R_{1,t}\, \mathrm{d}t + \lambda_1 \sigma\left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{2,t} &= \lambda_2 \left(\sigma\left(R_{1,t}, R_{2,t}\right)^2 - R_{2,t}\right) \mathrm{d}t
\end{aligned}$$

where $\sigma : (R_1, R_2) \mapsto \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2}$.

Denote the log price $X := \log(S)$ and suppose that the vol is driven by a very fast and very slow mean reverting factors as follows:

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\sigma\left(R^{\varepsilon}_{1,t}, R^{\delta}_{2,t}\right)^2 \mathrm{d}t + \sigma\left(R_{1,t}\varepsilon, R^{\delta}_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R^{\varepsilon}_{1,t} &= -\frac{1}{\varepsilon}\lambda_1 R^{\varepsilon}_{1,t}\, \mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\lambda_1 \sigma\left(R^{\varepsilon}_{1,t}, R^{\delta}_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R^{\delta}_{2,t} &= \sqrt{\delta}\lambda_2 \left(\sigma\left(R^{\varepsilon}_{1,t}, R^{\delta}_{2,t}\right)^2 - R^{\delta}_{2,t}\right) \mathrm{d}t
\end{aligned}$$

where $\varepsilon, \delta > 0$. The reader must understand the latter variables to be very small, i.e. in the neighbourhood of 0.

Let $(t, x, R_1, R_2) \mapsto v_{\varepsilon,\delta}(t, x, R_1, R_2)$ be the pricing function of a European option on S with payoff $x \mapsto h(x)$ (typically of exponential sub-growth). We know that $v_{\varepsilon,\delta}$ is a classical solution of the boundary problem :

$$\begin{cases}
\left(\partial_t + \mathcal{L}_{\varepsilon,\delta}\right) v = 0 & ;\ (t, (x, R_1, R_2)) \in [0, T) \times \mathbb{R}^2 \times \mathbb{R}_+ \\
v(T, x, R_1, R_2) = h(x) & ;\ (x, R_1, R_2) \in \mathbb{R}^2 \times \mathbb{R}_+
\end{cases} \tag{0.1}$$

such that :

$$\begin{aligned}
\partial_t + \mathcal{L}_{\varepsilon,\delta} &:= \partial_t + \frac{1}{2}\sigma^2\left(\partial^2_{x,x} - \partial_x\right) - \frac{1}{\varepsilon}\lambda_1 R_1 \partial_{R_1} + \sqrt{\delta}\lambda_2\left(\sigma^2 - R_2\right)\partial_{R_2} + \frac{1}{2\varepsilon}\lambda_1^2 \sigma^2 \partial^2_{R_1,R_1} \\
&= L_{\mathrm{BS}} + \frac{1}{\varepsilon}\left(\frac{1}{2}\lambda_1^2\sigma^2\partial^2_{R_1,R_1} - \lambda_1 R_1 \partial_{R_1}\right) + \sqrt{\delta}\lambda_2\left(\sigma^2 - R_2\right)\partial_{R_2} \\
&= L_{\mathrm{BS}} + \frac{1}{\varepsilon}L_1 + \sqrt{\delta}L_2
\end{aligned}$$

Our goal is to devise an expansion of $v_{\varepsilon,\delta}$ in $(\sqrt{\varepsilon}, \sqrt{\delta})$ of the form :

$$v_{\varepsilon,\delta} = \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} \varepsilon^{\frac{k}{2}}\delta^{\frac{j}{2}} v_{k,j}$$

where $(v_{k,j})$ are functions to be determined.

Let's begin with an expansion at first order of $\sqrt{\delta}$:

$$v_{\varepsilon,\delta} = v_0^\varepsilon + \sqrt{\delta} v_1^\varepsilon$$

Inserting this expression in probelem 0.1, we get :

$$\left(L_{\mathrm{BS}} + \frac{1}{\varepsilon} L_1 + \sqrt{\delta} L_2\right)(v_0^\varepsilon + \sqrt{\delta} v_1^\varepsilon) = 0$$

Now, regrouping the terms in powers of $\sqrt{\delta}$ at first order, one has :

$$\left(L_{\mathrm{BS}} + \frac{1}{\varepsilon} L_1\right) v_0^\varepsilon + \sqrt{\delta}\left(\left(L_{\mathrm{BS}} + \frac{1}{\varepsilon} L_1\right) v_1^\varepsilon + L_2 v_0^\varepsilon\right) = 0$$

So that :

$$\left(L_{\mathrm{BS}} + \frac{1}{\varepsilon} L_1\right) v_0^\varepsilon = 0 \tag{0.2}$$

$$\left(L_{\mathrm{BS}} + \frac{1}{\varepsilon} L_1\right) v_1^\varepsilon + L_2 v_0^\varepsilon = 0 \tag{0.3}$$

Now suppose that :

$\mathcal{H}$: R_1 has an invariant distribution (possibly parameterized by $R_2 \in \mathbb{R}_+$) denoted by Φ

for $g : \mathbb{R} \to \mathbb{R} \in L^1(\Phi(r)\,\mathrm{d}r)$, introduce the following notation :

$$\langle g \rangle_\Phi := \int_{\mathbb{R}} g(r)\Phi(r)\,\mathrm{d}r$$

Now, suppose that :

$$v_0^\varepsilon = v_0 + \sqrt{\varepsilon} v_{1,0} + \varepsilon v_{2,0} + \varepsilon\sqrt{\varepsilon} v_{3,0}$$

Injecting the latter expression into 0.2 and considering only terms up to order one, yields:

$$\frac{1}{\varepsilon} L_1 v_0 + \frac{1}{\sqrt{\varepsilon}} L_1 v_{1,0} + (L_{\mathrm{BS}} v_0 + L_1 v_{2,0}) + \sqrt{\varepsilon}\,(L_{\mathrm{BS}} v_{1,0} + L_1 v_{3,0}) = 0$$

First we have, from the previous expression, that :

$$L_1 v_0 = L_1 v_{1,0} = 0$$

and noting that L_1 contains only partial derivatives w.r.t R_1, we can choose :

$$v_0(t, x, R_1, R_2) = v_0(t, x, R_2)$$

$$v_{1,0}(t, x, R_1, R_2) = v_{1,0}(t, x, R_2)$$

Next we have :

$$L_{\mathrm{BS}} v_0 + L_1 v_{2,0} = 0$$

This corresponds to a poisson type equation associated to the generator of R_1, with leading terme $-L_{\mathrm{BS}} v_0$, so the latter must satisfy the solvability condition, i.e. :

$$\langle L_{\mathrm{BS}} v_0 \rangle_\Phi = 0$$

knowing that v_0 doesn't depend on R_1, we have then :

$$\langle L_{\mathrm{BS}} \rangle_\Phi v_0 = 0$$

Noting that :

$$L_{\mathrm{BS}}\left(\bar{\sigma}(R_2)\right) := \langle L_{\mathrm{BS}}\rangle_\Phi = \partial_t + \frac{1}{2}\bar{\sigma}(R_2)^2\left(\partial^2_{x,x} - \partial_x\right)$$

where

$$\bar{\sigma}(R_2) = \sqrt{\langle \sigma(.,R_2)^2\rangle_\Phi}$$

We conclude that v_0 solves the B-S PDE with constant volatility parametrized by R_2 as follows :

$$\begin{cases} L_{\mathrm{BS}}\left(\bar{\sigma}(R_2)\right) v_0 = 0 \\ \\ v_0(T,x,R_2) = h(x) \end{cases}$$

Thus:

$$\forall\ (t,x,R_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}_+ : \quad v_0(t,x,R_2) = P_{\mathrm{BS}}\left(t,x,\bar{\sigma}(R_2)\right)$$

Now, cancelling The order one term yields:

$$L_1 v_{3,0} = -L_{\mathrm{BS}} v_{1,0}$$

In the same fashion as v_0, writing the solvability condition of the above problem along with zero terminal condition, yield that :

$$P^\varepsilon_{1,0} := \sqrt{\varepsilon} v_{1,0} = 0$$

Now going back to the problem 0.3, and inserting the follwing form of v_1^ε into it :

$$v_1^\varepsilon = v_{0,1} + \sqrt{\varepsilon} v_{1,1} + \varepsilon v_{2,1} + \varepsilon\sqrt{\varepsilon} v_{3,1},$$

one gets :

$$\frac{1}{\varepsilon}L_1 v_{0,1} + \frac{1}{\sqrt{\varepsilon}}L_1 v_{1,1} + \left(L_{\mathrm{BS}}v_{0,1} + L_1 v_{2,1} + L_2 v_0\right) + \sqrt{\varepsilon}\left(L_{\mathrm{BS}} v_{1,1} + L_1 v_{3,1} + L_2 v_{1,0}\right) = 0$$

Again the terms in ε^{-1} and $\varepsilon^{-\frac{1}{2}}$, yield:

$$v_{0,1}(t,x,R_1,R_2) = v_{0,1}(t,x,R_2)$$

$$v_{1,1}(t,x,R_1,R_2) = v_{1,1}(t,x,R_2)$$

Now, by cancelling the term of order 0, we get:

$$L_{\mathrm{BS}} v_{0,1} + L_1 v_{2,1} + L_2 v_0 = 0$$

The solvability condition along with the fact that both $v_{0,1}$ and v_0 don't depend on R_1, yield :

$$\langle L_{\mathrm{BS}}\rangle_\Phi v_{0,1} = -\langle L_2\rangle_\Phi v_0$$

Multiplying both equations by $\sqrt{\delta}$, one gets:

$$L_{\mathrm{BS}}\left(\bar{\sigma}(R_2)\right) v^\delta_{0,1} = -2\mathcal{A}_\delta P_{\mathrm{BS}}$$

where :

$$\mathcal{A}^\delta = \frac{1}{2}\sqrt{\delta}\lambda_2\left(\bar{\sigma}(R_2)^2 - R_2\right)\bar{\sigma}(R_2)'\partial_\sigma$$

Noting that $v^\delta_{0,1}(T,.) = 0$, a solution to the last problem is :

$$v^\delta_{0,1}(t,x,R_2) = (T-t)\mathcal{A}^\delta P_{\mathrm{BS}}$$

Hint: one can use the the Vega-Gamma relation:

$$\text{Vega} = (T-t)\sigma e^{2x}\Gamma$$

to conclude that $L_{\text{BS}}\left(\bar{\sigma}(R_2)\right)$ and $\mathcal{A}^{\delta}$ actually commute.

Finally, we get the following expansion, which is independent of ε :

$$v_{\delta}(t,x,R_2) = P_{\text{BS}}(t,x,\bar{\sigma}(R_2)) + \left(\frac{\lambda_2}{2}(T-t)\left(\bar{\sigma}(R_2)^2 - R_2\right)\bar{\sigma}(R_2)'\text{Vega}_{\text{BS}}\right)\sqrt{\delta}$$

Notice that the pricing function doesn't depend on R_1 but only on the volatility factor R_2.

Now, we prove that the hypothesis $\mathcal{H}$ is actually satisfied in our framework. Introduce the following function :

$$\forall\ (R_1,R_2) \in \mathbb{R}\times\mathbb{R}_+ : \quad \psi_{R_2}(R_1) := \frac{1}{\lambda_1^2\sigma(R_1,R_2)^2}\exp\left(-\int_0^{R_1}\frac{2\lambda_1 x}{\sigma(x,R_2)^2}\,\mathrm{d}x\right)$$

Let $R_2 \geq 0$. We have :

$$\forall\ R_1 \in \mathbb{R} : \quad 0 < \psi_{R_2}(R_1) \leq \frac{1}{\lambda_1^2\sigma(R_1,R_2)^2}$$

by noticing that :

$$\frac{1}{\lambda_1^2\sigma(R_1,R_2)^2} \underset{R_1\to+\infty}{\sim} \frac{1}{\lambda^2\beta_1^2R_1^2}$$

so that $\psi_{R_2} \in L^1(\mathrm{d}R_1)$. Thus, The process, R_1 (when the process $R_{2,.}$ is freezed at R_2) has the invariant distribution :

$$\Phi_{R_2} = \frac{\psi_{R_2}}{\displaystyle\int_{-\infty}^{+\infty}\psi_{R_2}(R_1)\,\mathrm{d}R_1}$$

One can even have an explicit expression of ψ_{R_2}, as :

$$\int_0^{R_1}\frac{2\lambda_1 x}{\sigma(x,R_2)^2}\,\mathrm{d}x = \frac{2\lambda_1}{\beta_1^2}\left(\frac{\sigma(0,R_2)}{\sigma(R_1,R_2)} + \log\left(\sigma(R_1,R_2)\right)\right) + \kappa(R_2)$$

Note, however, the interesting tails property :

$$\Phi_{R_2}(R_1) \underset{R_1\to\pm\infty}{\sim} \frac{1}{\lambda_1^2\left(\sigma(0,R_2)+\beta_1R_1\right)^{2(\frac{\lambda_1}{\beta_1^2}+1)}}$$