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Dynamique jointe action/option et application aux stratégies de trading sur options

Stock/Option joint dynamics and application to option trading strategies

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A mes parents pour leur dévouement,
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Abstract

0.1 Résumé

Cette thèse explore théoriquement et empiriquement les implications de la dynamique jointe action/option sur divers problématiques liées au trading d'options. Dans un premier temps, nous commençons par l'étude de la dynamique jointe entre une option sur un stock et une option sur l'indice de marché. Le modèle CAPM fournit un cadre mathématique adéquat pour cette étude car il permet de modéliser la dynamique jointe d'un stock et son indice de marché. En passant aux prix d'options, nous montrons que le beta et la volatilité idiosyncratique, paramètres du modèle, permettent de caractériser la relation entre les surfaces de volatilité implicite du stock et de l'indice. Nous nous penchons alors sur l'estimation du paramètre beta sous la probabilité risque-neutre en utilisant les prix d'options. Cette mesure, appelée beta implicite, représente l'information contenue dans les prix d'options sur la réalisation du paramètre beta dans le futur. Pour cette raison, nous essayons de voir, si le beta implicite a un pouvoir prédictif du beta futur. En menant une étude empirique, nous concluons que le beta implicite n'améliore pas la capacité de prédiction en comparaison avec le beta historique qui est calculé à travers la régression linéaire des rendements du stock sur ceux de l'indice. Mieux encore, nous remarquons que l'oscillation du beta implicite autour du beta futur peut entraîner des opportunités d'arbitrage, et nous proposons une stratégie d'arbitrage qui permet de monétiser cet écart. D'un autre côté, nous montrons que l'estimateur du beta implicite pourrait être utilisé pour la couverture d'options sur le stock en utilisant des instruments sur l'indice, cette couverture concerne notamment le risque de volatilité et aussi le risque de delta. Dans la deuxième partie de notre travail, nous nous intéressons au problème de market making sur options. Dans cette étude, nous supposons que le modèle de dynamique du sous-jacent sous la probabilité risque-neutre pourrait être mal spécifié ce qui traduit un décalage entre la distribution implicite du sous-jacent et sa distribution historique. Dans un premier temps, nous considérons le cas d'un market maker risque neutre qui vise à maximiser l'espérance de sa richesse future. A travers l'utilisation d'une approche de contrôle optimal stochastique, nous déterminons les prix optimaux d'achat et de vente sur l'option et nous interprétons l'effet de présence d'inefficience de prix sur la stratégie optimale. Dans un deuxième temps, nous considérons que le market maker est averse au risque et essaie donc de réduire l'incertitude liée à son inventaire. En résolvant un problème d'optimisation basé sur un critère moyenne-variance, nous obtenons des approximations analytiques des prix optimaux d'achat et de vente. Nous montrons aussi les effets de l'inventaire et de l'inefficience du prix sur la stratégie optimale. Nous nous intéressons par la suite au market making d'options dans une dimension plus élevée. Ainsi, en suivant le même raisonnement, nous présentons un cadre pour le market making de deux options ayant des sous-jacents différents avec comme contrainte la réduction de variance liée au risque d'inventaire détenu par le market-maker. Nous déterminons dans ce cas la stratégie optimale et nous appuyons les résultats théoriques par des simulations numériques. Dans la dernière partie de notre travail, nous étudions la dynamique jointe entre la volatilité implicite à la monnaie et le sous jacent, et nous essayons d'établir le lien entre cette dynamique jointe et le skew implicite. Nous nous intéressons à un indicateur appelé "Skew Stickiness Ratio" qui a été introduit dans la littérature récente. Cet indicateur mesure la sensibilité de la volatilité implicite à la monnaie face aux mouvements du sous-jacent. Nous proposons une méthode qui permet d'estimer la valeur de cet indicateur sous la probabilité risque-neutre sans avoir besoin d'admettre des hypothèses sur la dynamique du sous-jacent. Ensuite, en menant une étude empirique, nous montrons que le Skew Stickiness ratio peut avoir des valeurs différentes sous la probabilité risque-neutre et la probabilité historique, ce qui entraîne une inefficience de prix d'options. Nous proposons une stratégie d'arbitrage qui permet de monétiser cet écart et nous testons la validité de cette stratégie empiriquement. Nous donnons par la suite, un exemple de modèle de dynamique du sous-jacent qui justifie que le Skew Stickiness Ratio peut atteindre des valeurs supérieures à 2 dans la limite des maturités courtes. On montre alors, qu'au delà de l'utilité du Skew Stickiness Ratio pour la couverture d'options, cet indicateur peut être utilisé pour des fins d'arbitrage quand la relation entre la dynamique jointe spot/volatilité implicite et le skew implicite est mal spécifiée sous la probabilité risque-neutre.

Mots-clés: Skew, corrélation spot-volatilité, volatilité stochastique, dynamique jointe, modèle CAPM, calibration, market making, misspécification de paramètres, contrôle optimal stochastique

0.2 Abstract (English)

This thesis explores theoretically and empirically the implications of the stock/option joint dynamics on applications related to option trading. In the first part of the thesis, we look into the relations between stock options and index options under the risk-neutral measure. The Capital Asset Pricing Model offers an adequate mathematical framework for this study as it provides a modeling approach for the joint dynamics between the stock and the index. As we compute option prices according to this model, we find out that the beta and the idiosyncratic volatility of the stock, which are parameters of the model, characterize the relation between the implied volatility surface of the stock and the one of the index. For this reason, we focus on the estimation of the parameter beta under the risk-neutral measure through the use of option prices. This measure, that we call implied beta, is the information contained in option prices concerning the realization of the parameter beta in the future. Trying to use this additional information, we carry out an empirical study in order to investigate whether the implied beta has a predictive power of the forward realized beta. We conclude that the implied beta doesn't perform better than the historical beta which is estimated using the linear regression of the stock's returns on the index returns. We conclude also that the oscillation of the implied beta around the forward realized beta can engender arbitrage opportunities, and we propose an arbitrage strategy which enables to monetize this difference. In addition, we show that the implied beta is useful to hedge stock options using instruments on the index. In the second part of our work, we consider the problem of option market making. We suppose that the model used to describe the dynamics of the underlying under the risk-neutral probability measure can be misspecified which means that the implied distribution of the underlying may be different from its historical one. We consider first the case of a risk neutral market maker who aims to maximize the expectation of her final wealth. Using a stochastic control approach, we determine the optimal bid and ask prices on the option and we interpret the effect of price inefficiency on the optimal strategy. Next to that, we suppose that the market maker is risk averse as she tries to minimize the variance of her final wealth. We solve a mean-variance optimization problem and we provide analytic approximations for the optimal bid and ask prices. We show the effects of option inventory and price inefficiency on the optimal strategy. We try then to extrapolate the study to a higher dimension in order to see the effect of joint dynamics of the different underlyings on the optimal strategy. Thus, we study market making strategies on a pair of options having different underlyings with the aim to reduce the risk due to accumulated inventories in these two options. Through the resolution of the HJB equation associated to the new optimization problem, we determine the optimal strategy and we support our theoretical finding with numerical simulations. In the final part of the thesis, we study the joint dynamics of the at-the-money implied volatility and the spot process. We try to establish a relation between this joint dynamics and the implied skew through the use of a quantity called the Skew Stickiness Ratio which was introduced in the recent literature. The Skew Stickiness Ratio quantifies the effect of the log-return of the spot on the increment of the at-the-money volatility. We suggest a model-free approach for the estimation of the SSR (Skew Stickiness Ratio) under the risk-neutral measure, this approach doesn't depend on hypothesis on the dynamics of the underlying. Next to that, we carry out an empirical study in order to show that the Skew Stickiness Ratio may have different values under the real-world probability measure and the risk-neutral one. Since this discrepancy engenders a price inefficiency, we propose an arbitrage strategy which aims to monetize the difference between the implied SSR and the realized SSR, and we test the strategy on real data. We also propose a model for the dynamics of the underlying which enables the Skew Stickiness Ratio to have a value larger than 2 in the limit of short maturities.

Mots-clés: Skew, leverage effect, stochastic volatility, joint dynamics, CAPM model, parameters calibration, parameters misspecification, optimal stochastic control.

Contents

Introduction (In French)

Le travail de Black et Scholes sur la valorisation des options européennes, publié en 1973, représente l'une des avancées majeures que l'industrie financière a connues. Ce travail a mis en place les premières briques pour construire des fondements théoriques de la valorisation des options et leur couverture dynamique. Il a été remarqué depuis que l'hypothèse de la volatilité constante, sur laquelle repose ce modèle, est erronée. En effet, cette hypothèse contredit le caractère stochastique, vérifié empiriquement, du processus de la volatilité. De plus, le modèle Black et Scholes ne permet pas de reproduire les faits empiriques qui sont observés sur le marché.

En effet, si on note $(C_{Market}(K_i,T_j))_{\{1\leq i\leq n,1\leq j\leq p\}}$ les prix du marché des call européens de strikes $(K_i)_{1\leq i\leq n}$ et de maturités $(T_j)_{1\leq j\leq p}$, et puis on inverse la formule de valorisation d'option du modèle de Black et Scholes, on obtient les volatilités implicites $(\sigma(K_i,T_j))_{\{1\leq i\leq n,1\leq j\leq p\}}$ vérifiant:

$$C_{Market}(K_i,T_j) = C_{Black-Scholes}\left(K_i,T_j,\sigma(K_i,T_j)\right).$$

On peut alors définir les notations suivantes:

- $(\sigma(K_i,T_j))_{\{1\leq i\leq n\}}$ représente le brin ou le "smile" de volatilité implicite de maturité T_j observé sur le marché.
- $(\sigma(K_i,T_j))_{\{1\leq i\leq n,1\leq j\leq p\}}$ représente la surface de volatilité implicite observée sur le marché.

L'hypothèse de volatilité constante dans le modèle de Black et Scholes entraîne que la volatilité implicite $\sigma(K,T)$ soit constante pour tous les strikes K et maturités T. Ce résultat est en contradiction avec la réalité du marché où la surface de volatilité observée peut exhiber des comportements spécifiques en fonction du strike K et de la maturité T. Pour cette raison, il a été considéré, depuis longtemps, que le modèle de Black et Scholes a des limites en terme de valorisation et couverture d'options. Plusieurs chercheurs se sont penchés sur ce sujet et ont proposé par la suite des modèles plus sophistiqués pour décrire la dynamique du sous-jacent et justifier la forme de la surface de volatilité implicite observée sur le marché. A titre d'exemple, Merton a proposé un modèle à sauts dans lequel le sous-jacent S a la dynamique suivante:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (J-1)\,dN_t$$

où N est un processus de Poisson d'intensité λ, et J est une variable aléatoire positive. Ce modèle a l'avantage de pouvoir reproduire la forme du brin de volatilité implicite pour une maturité T donnée. D'autres chercheurs se sont tournés vers une modélisation plus fine du processus de volatilité instantanée du sous jacent, et ont proposé des modèles de la forme:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma(y_t)dW_t^{(1)},\\ dy_t &= a(y_t)dt + b(y_t)dW_t^{(2)},\end{aligned}$$

où $d\left\langle W^{(1)},W^{(2)}\right\rangle_t = \rho dt$. Les exemples plus connus dans cette catégorie de modélisation sont Heston (1993) et Hagan et al. (2002). Le premier objectif d'une telle modélisation est d'avoir

une surface de volatilité implicite qui est en adéquation avec celle observée sur le marché. Le deuxième objectif du modèle est de reproduire la dynamique jointe de la volatilité implicite et du sous-jacent afin de pouvoir mener des stratégies de couverture dynamique.
Plus récemment, d'autres chercheurs ont abordé le problème différemment en essayant de modéliser directement la dynamique de la volatilité implicite au lieu de la volatilité instantanée. Une telle approche a l'avantage de modéliser une quantité observable sur le marché (la volatilité implicite) à la place de la volatilité instantanée qui devrait être estimée soit à travers l'utilisation de techniques de filtrage ou bien à travers l'emploi de données haute fréquence. Les travaux développés dans Schönbucher (1999) et Cont et al. (2002) représentent de bons exemples de cette approche.

Dans cette thèse, on s'intéresse à la dynamique jointe sous-jacent/option, et on essaie de comprendre les implications de cette dynamique sur des applications concrètes de trading d'options. Cette thèse contient tois chapitres indépendants qui pourraient être lus séparément. Chaque chapitre s'intéresse à l'effet de la dynamique jointe spot/volatilité sur une application concrète de trading d'options.

Dans le premier chapitre de la thèse, on essaie de comprendre les implications de la dynamique jointe entre deux sous-jacents sur la relation entre leurs volatilités implicites. On se place dans le cadre du modèle CAPM pour modéliser la dynamique jointe d'un stock et de l'indice du marché. On suppose que les volatilités instantanées du stock et de l'indice sont stochastiques ce qui permet d'avoir un cadre adapté pour la valorisation des options. Ensuite, on mène une étude théorique afin de donner des approximations analytiques des prix d'options européennes sur le stock et l'indice respectivement. A travers la réalisation d'une étude empirique, on teste la validité de nos résultats sur des données réelles et on essaie de comprendre s'il y a une incohérence sur le marché entre la dynamique jointe stock/indice d'une part et la relation entre la volatilité implicite sur stock et la volatilité implicite sur indice d'autre part.

Dans le deuxième chapitre de la thèse, on s'intéresse au problème de market making sur options. La difficulté de cette tâche provient de la multitude de risques supportés par le market maker, cela inclue le risque du mouvement du sous-jacent, le risque de volatilité, le risque de liquidité de l'option, le risque de l'inventaire,... Pour étudier ce problème, on construit un cadre théorique qui tient compte de la dynamique jointe spot/volatilité instantanée, la fréquence d'arrivée des ordres d'achat et de vente au carnet d'ordre,.... En spécifiant la fonction d'utilité du market maker en terme de gain et de gestion de risques, on formule le problème d'optimisation de ce dernier, et on utilise une approche de contrôle optimal stochastique afin de déterminer sa stratégie optimale de quotation. L'organisation de ce chapitre est comme suit. Dans la première partie, on s'intéresse au market making d'une seule option. On traite d'abord le cas où le market maker est risque neutre, et on mène une étude analytique afin de déterminer sa stratégie optimale. Ensuite, on considère le cas où le market maker est averse au risque et on donne, à travers l'étude analytique de ce problème, une approximation des ses prix limites à l'achat et à la vente. Dans la deuxième partie, on réplique le même raisonnement au market making sur deux options et on pointe l'effet de la dynamique jointe des deux sous-jacents sur la stratégie optimale de market making.

Dans le troisième chapitre de la thèse, on étudie la relation entre le rendement du sous-jacent et l'incrément de la volatilité implicite. En introduisant la fonction f suivante:

$$\begin{aligned} f : \mathbb{R}^+ \times \mathbb{R}^+ &\rightarrow \mathbb{R}^+, \\ (K,T) &\rightarrow \sigma(K,T) \end{aligned}$$

on s'intéresse à la relation entre la dynamique jointe spot/volatilité implicite à la monnaie ($K = S_t$) de maturité T d'une part et la quantité $\frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}$. Pour explorer cette relation, on étudie la quantité suivante:

$$\mathcal{R}_T = \frac{d \langle \sigma(S,T), \log(S) \rangle_t}{d \langle \log(S) \rangle_t \frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}}.$$

En utilisant la quantité $\mathcal{R}_T$, on charactérise la relation de dépendance entre la dynamique jointe spot/volatilité implicite d'une part et le "skew" implicite de maturité T définie par la quantité $\frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}$ d'autre part. On étudie aussi l'éventuelle présence d'opportunités d'arbitrage qui découle dans le cas où cette relation n'est pas respectée.

Dans le reste de l'introduction, on présente les notations suivantes:

- $(\Omega, \mathcal{F}, \mathbb{P})$ représente l'espace de probabilité équippé de la filtration $\mathcal{F}$ et de la mesure de probabilité objective $\mathbb{P}$.
- $(\Omega, \mathcal{F}, \mathbb{Q})$ représente l'espace de probabilité équippé de la filtration $\mathcal{F}$ et de la mesure de probabilité risque-neutre $\mathbb{Q}$.

Les résultats des trois chapitres de la thèse sont résumés ci-dessous.

0.3 Effet de la dynamique jointe stock/indice sur la relation entre leurs volatilités implicites respectives

Dans ce chapitre, on s'intéresse à l'étude de l'effet qu'a la dynamique jointe entre deux sous-jacents sur la relation entre leurs surfaces de volatilité implicite respectives. Pour répondre à ce problème, on considère le cas d'un stock et de son indice de marché et on utilise le modèle CAPM, connu sous le nom "Capital asset pricing model", afin de décrire la dynamique jointe stock/indice. A travers ce modèle, le rendement $R_S(t)$ du stock S à la date t s'écrit en fonction du rendement $R_I(t)$ de l'indice I à la date t de la façon suivante:

$$R_S(t) = \alpha + \beta R_I(t) + \epsilon(t),$$

où ϵ représente la partie idiosyncratique indépendante du rendement de l'indice du marché. Ce modèle est couramment utilisé dans des applications liées à la construction de portefeuille et la gestion du risque. Dans notre étude, on va utiliser ce modèle afin d'investiguer les relations pouvant exister entre la surface de volatilité implicite du stock et celle de l'indice.

0.3.1 Estimation du paramètre beta en utilisant des prix d'options

Le paramètre beta dans le modèle CAPM caractérise la dépendance entre le rendement de l'actif et celui du marché et permet ainsi de quantifier l'effet du risque systématique sur le risque de l'actif. Ce paramètre peut être estimé à travers la régression linéaire des rendements de l'actif S sur ceux de l'indice I, et peut donc être déterminé comme suit:

$$\beta = \frac{cov(R_S, R_I)}{Var(R_I)}.$$

L'estimation de ce paramètre est un problème statistique largement étudié. La complexité de cette tâche vient du fait que ce paramètre est constant dans le cadre du modèle CAPM alors que son estimation par une régression linéaire sur fenêtre glissante montre qu'il varie dans le temps. Ainsi, une régression linéaire des données historiques permettrait de fournir une estimation de ce paramètre sur une fenêtre passée mais ne garantit pas d'avoir cette même valeur dans le futur. Dans notre étude, on a pensé à utiliser les prix d'options afin d'estimer ce paramètre sous la probabilité risque-neutre. Cette réflexion vient du fait que les prix d'option reflètent souvent l'attente des acteurs de marché sur la réalisation de certains paramètres dans le futur. Ainsi, la volatilité implicite $\sigma_I(K,T)$ de strike K et de maturité T représente l'attente du marché de la valeur future de la volatilité "Break-even" $\sigma_{BEVL}(K,T)$, définie par Dupire comme suit:

$$\sigma^2_{BEVL}(K,T) = \frac{E\left(\int_0^T \Gamma_{BS,t} S_t^2 \sigma_t^2 dt\right)}{E\left(\int_0^T \Gamma_{BS,t} S_t^2 dt\right)},$$

Par analogie, on s'attendait à ce que le paramètre beta, estimée à partir des prix des options sous la probabilité risque neutre, pourrait contenir de l'information sur la réalisation du paramètre beta dans le futur sous la probabilité historique. L'idée d'estimation du paramètre beta à travers l'utilisation des prix d'options a été d'abord étudiée dans Fouque and Kollman (2011), où les auteurs ont considéré un modèle CAPM en temps continu sous $\mathcal{P}$, où la volatilité de l'indice I est stochastique:

$$\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + \sigma dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)},
\end{aligned}$$

où $W_t^{(3)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(4)}$ et $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(4)} \end{pmatrix}$ est un mouvement brownien sous $\mathbb{P}$.

Sous l'hypothèse $0 < \epsilon \ll 1$, une perturbation singulière par rapport au paramètre ϵ peut être faite et les prix d'options européennes sur le stock S et l'indice I peuvent être approximées. Ensuite, les volatilités implicites $\Sigma_S(K,T)$ et $\Sigma_I(K,T)$ du stock et de l'indice sont approximées comme suit:

$$\begin{aligned}
\Sigma_I(K_I,T) &= b_I + a_I \frac{\log(\frac{F_I}{K_I})}{T}, \\
\Sigma_S(K_S,T) &= b_S + a_S \frac{\log(\frac{F_S}{K_S})}{T},
\end{aligned}$$

où les quantités F_I et F_S représentent les prix forward pour la maturité T pour l'indice et le stock respectivement. Le paramètre β peut être approximé par $\hat{\beta}$:

$$\hat{\beta} = \left(\frac{a_S}{a_I}\right)^{\frac{1}{3}} \frac{b_S}{b_I}.$$

L'inconvénient de ce modèle réside dans le caractère constant de la volatilité idiosyncratique du stock S, une propriété qui n'est pas validée empiriquement. Pour remédier à ce problème, on a changé les hypothèses du modèle et on a supposé que la volatilité idiosyncratique du stock est stochastique. Le nouveau modèle est détaillé ci-dessous:

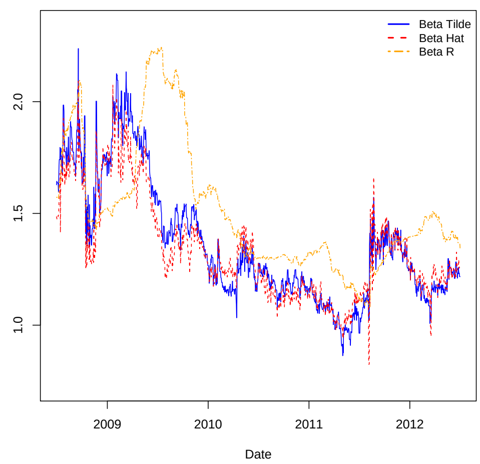
$$\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f_1(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + f_2(Z_t) dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)}, \\
dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt + \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{(4)},
\end{aligned}$$

où $W_t^{(3)} = \rho_Y W_t^{(1)} + \sqrt{1-\rho_Y^2} W_t^{(5)}$, $W_t^{(4)} = \rho_Z W_t^{(2)} + \sqrt{1-\rho_Z^2} W_t^{(6)}$ et $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(5)} \\ W^{(6)} \end{pmatrix}$ est un mouvement brownien sous $\mathbb{P}$.

En utilisant le théorème de Girsanov, on arrive à charactériser la dynamique du stock et de l'indice sous la probabilité risque neutre. Ensuite, par le moyen d'une perturbation singulière par rapport au paramètre ϵ, on fournit des approximations des prix d'option européennes sur le stock et l'indice, et on déduit des approximations analytiques des volatilités implicites du stock et de l'indice. Enfin, on présente un nouvel estimateur du paramètre β:

$$\tilde{\beta} \quad = \quad \sqrt{\frac{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}}{(\bar{\sigma}_I^*)^2}} \left(1 + \frac{V_2^{S,\epsilon}}{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}} - \frac{V_2^{I,\epsilon}}{(\bar{\sigma}_I^*)^2} \right) + o(\epsilon)\,.$$

On a voulu tester le pouvoir prédictif de l'estimateur $\tilde{\beta}$ sur la réalisation du paramètre beta dans le futur sous la probabilité historique. On a ainsi mené une étude comparative entre le beta implicite estimé à partir des prix d'options $\tilde{\beta}$, le beta historique β_H calculé par le moyen d'une régression linéaire sur une fenêtre historique et le beta forward β_F qui va se réaliser effectivement dans le futur. Le graphe ci-dessous donne un exemple pour l'ETF Financial Select Sector (XLF) vs l'indice SP500:



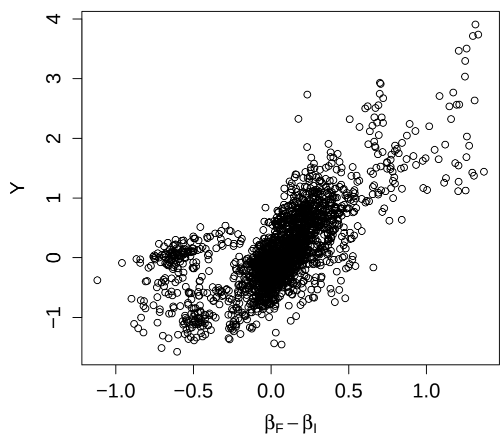
(a) XLF vs SPX

Figure 1: Comparison between $\tilde{\beta}$ (blue line) and $\hat{\beta}$ (red line)

Le but de cette étude est de voir lequel parmi le beta implicite et le beta historique qui contient plus d'informations sur le beta forward. Cela permet ainsi de savoir si le beta implicite, et du coup les prix des options, ont un pouvoir prédictif du beta futur. La conclusion générale de cette étude montrait que c'était plutôt le beta historique qui prédisait mieux le beta forward.

0.3.2 Application pour les stratégies d'arbitrage

On n'a pas pu trouver de pouvoir prédictif supplémentaire dans les prix des options concernant la réalisation future du paramètre beta, mais on a pu tirer un enseignement majeur. En effet, l'écart temporaire entre le beta implicite et le beta forward est un signe de l'existence d'inefficience de prix sur le marché des options, et constitue ainsi une preuve de l'existence d'écart entre la probabilité historique et la probabilité risque-neutre. On peut donc utiliser l'information contenue dans le beta historique, qui est en moyenne le meilleur prédicteur du beta forward, afin de profiter de l'inefficience de prix d'options. On propose dans notré étude une stratégie de trading qui permet de monétiser l'écart entre le beta implicite et le beta réalisé, le $P\&L$ de cette stratégie est bien proportionnel à la différence $(\beta_{Realized} - \beta_{Implied})$. Le graphe suivant représente le $P\&L$ de la stratégie quand elle est testée sur l'ETF XLF (Financial select sector) projeté sur l'indice SPX.



(a) $P\&L$ quotidien de la stratégie en fonction du biais $(\beta_{Realized} - \beta_{Implied})$ dans le cas XLF-SPX

0.3.3 Application pour la couverture des options

Dans le cadre du modèle CAPM en temps continu, on a essayé de définir des stratégies de couverture d'options sur l'actif S en utilisant des instruments sur l'indice I.

Dans un premier temps, on a montré qu'on peut hedger partiellement le risque de volatilité d'une option sur stock en utilisant des options sur l'indice. En effet, si on note $P^{S,\epsilon}$ le prix de l'option sur le stock, $P^{I,\epsilon}$ le prix de l'option sur l'indice, Δ_S le delta de l'option sur le stock et Δ_I le delta de l'option sur l'indice, alors le portefeuille X défini comme suit:

$$X_t = P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} + \Delta_{I,t}\vartheta_t I_t - \Delta_{S,t} S_t,$$

avec:

$$\vartheta_t = \beta^2 \frac{S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2 \frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}} + o(\epsilon)\,,$$

alors le portefeuille X est insensible aux variations du processus Y.

Dans un deuxième temps, on a montré que le paramètre beta peut aussi être utilisé pour la couverture en delta d'options sur stock en utilisant des futures sur l'indice. En effet, si on prend un portefeuille L défini comme suit:

$$L_t = P_t^{S,\epsilon} - \vartheta_t P^{I,\epsilon} - \varphi_{I,t} I_t,$$

où:

$$\varphi_{I,t} = \frac{\beta S_t}{I_t}\frac{\partial P^{S,\epsilon}}{\partial S} - \vartheta_t \frac{\partial P^{I,\epsilon}}{\partial I},$$

alors le portefeuille L serait insensible aux variations des browniens $W^{(1)}$ et $W^{(3)}$. On développe aussi des simulations numériques afin de tester l'efficacité de ses stratégies de couverture.

On finit par conclure que le modèle CAPM peut nous renseigner sur divers relations entre la surface de volatilité implicite du stock et celle de l'indice. Ces relations pourraient être utilisées pour différentes applications allant de la couverture dynamique d'options à l'arbitrage.

0.4 Market making sur options

0.4.1 Stratégies optimale de market making d'une option (dimension 1)

Dans ce chapitre, on s'inspire du travail d'Avellaneda et Stoikov dans Avellaneda and Stoikov (2008), et on essaie d'établir un cadre théorique pour l'étude de stratégies de market-making sur options. On suppose ainsi que, le sous-jacent S a la dynamique suivante sous la probabilité $\mathcal{P}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma(y_t) dW_t^{(1)}, \\ dy_t &= a_R(y_t)dt + b_R(y_t)dW_t^{(2)}, \end{aligned}$$

où $W^{(1)}$ and $W^{(2)}$ sont deux mouvements Browniens sous $\mathcal{P}$ tel que $d\left\langle W^{(1)}, W^{(2)} \right\rangle_t = \rho_R dt$.
D'un autre côté, on suppose que la dynamique du sous-jacent S, sous la probabilité risque neutre $\mathcal{Q}$, peut être décrite comme suit:

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sigma(y_t) dW_t^{*,(1)}, \\ dy_t &= a_I(y_t)dt + b_I(y_t)dW_t^{*,(2)}, \end{aligned}$$

où $W^{*,(1)}$ and $W^{*,(2)}$ sont deux mouvements browniens sous $\mathcal{Q}$ tel que $d\left\langle W^{*,(1)}, W^{*,(2)} \right\rangle_t = \rho_I dt$.

De plus, on suppose que la mesure de probabilité $\mathcal{Q}$ n'est éventuellement pas absolument continue par rapport à $\mathcal{P}$. En effet, les paramètres régissant la dynamique de la volatilité instantanée sous la probabilité risque-neutre peuvent être misspécifiés, ce qui induit un écart entre la distribution implicite et la distribution historique du sous-jacent.

On suppose qu'un market maker d'options fournit de la liquidité sur une option de maturité T et de payoff $h(S_T)$. A chaque date $t \in [0, T]$, ce dernier envoie des ordres limites à l'achat et à la vente dans le carnet d'ordre. L'exécution de ces ordres limites dépend de la fréquence d'arrivée des ordres de marché. On suppose que, la probabilité λ^+ (respectivement λ^-) d'arrivée d'un ordre d'achat (respectivement ordre de vente) qui consomme un ordre limite de vente (respectivement d'achat) à une distance δ^+ (respectivement δ^-) du prix mid s'écrit comme suit:

$$\lambda^+(\delta^+) = \frac{A}{\left(B + (\delta^+)^{\frac{1}{\beta}}\right)^{\gamma}}, \qquad \lambda^-(\delta^-) = \frac{A}{\left(B + (\delta^-)^{\frac{1}{\beta}}\right)^{\gamma}},$$

où $A, B > 0$ et $\gamma > 1$.
L'arrivée d'ordres de marché est susceptible de changer l'inventaire et la richesse du market maker dans le cas où ses ordres limites se font exécuter. Ce dernier devrait donc bien choisir ses prix de quotation afin d'atteindre ses objectifs en terme de gain et de gestion de risque. On essaie ici de formuler le problème d'optimisation du market maker et de le résoudre en utilisant une approche de contrôle optimal stochastique.

Dans un premier temps, on considère le cas où le market maker est risque neutre. Ce dernier a une fonction d'utilité linéaire, et a donc pour objectif de maximiser l'espérance de sa richesse future. Sa fonction valeur s'écrit sous la forme:

$$u(t,s,y,q_1,x) = Sup_{\{(\delta_t^+,\delta_t^-)\in\mathcal{A}\}} E^{\mathcal{P}}(X_T + q_{1,T}h(S_T))|S_t = s, y_t = y, q_{1,t} = q_1, X_t = x),$$

où $\mathcal{A} = \mathbb{R}^+ \times \mathbb{R}^+$.

L'équation Hamilton-Jacobi-Bellmann associée à ce problème s'écrit:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u_0 + sup_{\{(\delta_t^+,\delta_t^-)\in\mathcal{A}\}} \left(J^+\left(\delta_t^+\right) + J^-\left(\delta_t^-\right)\right) = 0,$$

où les fonctions J^+ et J^- sont définies comme suit:

$$\begin{aligned} J^+(\delta_t^+) &= \lambda^+(\delta_t^+)\left(u_0(t,s,y,q_{1,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}+\delta_t^+)) - u_0(t,s,y,q_{1,t^-},x_{t^-})\right), \\ J^-(\delta_t^-) &= \lambda^-(\delta_t^-)\left(u_0(t,s,y,q_{1,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}-\delta_t^-)) - u_0(t,s,y,q_{1,t^-},x_{t^-})\right), \end{aligned}$$

et les opérateurs $\mathcal{L}_1$ et $\mathcal{L}_2$ sont donnés par:

$$\begin{aligned} \mathcal{L}_1 &= \mu S_t \frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(y_t)S_t^2\frac{\partial^2}{\partial S^2} + a_R(y_t)\frac{\partial}{\partial y} + \frac{1}{2}b_R^2(y_t)\frac{\partial^2}{\partial y^2} + \rho_R b_R(y_t)\sigma(y_t)S_t\frac{\partial^2}{\partial S\partial y}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial x}q_{2,t}\mu S_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}q_{2,t}^2\sigma(y_t)^2S_t^2 + \frac{\partial^2}{\partial x\partial S}q_{2,t}\sigma^2(y_t)S_t^2 + \frac{\partial^2}{\partial x\partial y}q_{2,t}\sigma(y_t)S_t b_R(y_t)\rho_R, \end{aligned}$$

De plus, u_0 vérifie la condition finale:

$$u_0(T,s,y,q_1,x) = x + q_1 h(s).$$

Sous des hypothèses de régularité et de croissance polynomiale, on montre que u_0, solution de l'équation HJB, coincide avec la fonction valeur u. On détermine aussi les contrôles optimaux $\delta_{L,*,t}^-$ et $\delta_{L,*,t}^+$ du market maker à la date t. En interprétant les formules analytiques de $\left(\delta_{L,*,t}^-,\delta_{L,*,t}^+\right)$, on voit bien que la stratégie optimale du market maker permet de profiter de l'inefficience de prix provenant du décalage entre la dynamique de S sous la probabilité $\mathcal{P}$ et sa dynamique sous la probabilité $\mathcal{Q}$.

Dans un deuxième temps, on étudie le cas où le market maker est averse au risque. Ce dernier vise à maximiser l'espérance de sa richesse future tout en minimisant sa variance. Sa fonction valeur s'écrit comme suit:

$$u^\epsilon(t,s,y,q_1,x) = Sup_{\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\}} E_{t,s,y,q_1,x}^{\mathcal{P}}\left(X_T + q_{1,T}h(S_T)\right) - \epsilon Var_{t,s,y,q_1,x}^{\mathcal{P}}\left(X_T + q_{1,T}h(S_T)\right),$$

L'équation HJB associée à ce problème peut être donnée ci-dessous:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\,u_0^\epsilon + \sup_{\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\}} \left(J^{-,\epsilon}(\delta_t^-,\delta_t^+) + J^{+,\epsilon}(\delta_t^-,\delta_t^+)\right) = \epsilon\left(q_{1,t}^2V_t + T_t\right),$$

où les quantités V_t et T_t sont données analytiquement, et les fonctions $J^{+,\epsilon}, J^{-,\epsilon}$ sont définies comme suit:

$$\begin{aligned} J^{+,\epsilon}(\delta^+) &= \lambda^+(\delta^+)\left(u_0^\epsilon(t,s,y,q_{1,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}+\delta^+)) - u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right), \\ J^{-,\epsilon}(\delta^-) &= \lambda^-(\delta^-)\left(u_0^\epsilon(t,s,y,q_{1,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}-\delta^-)) - u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right). \end{aligned}$$

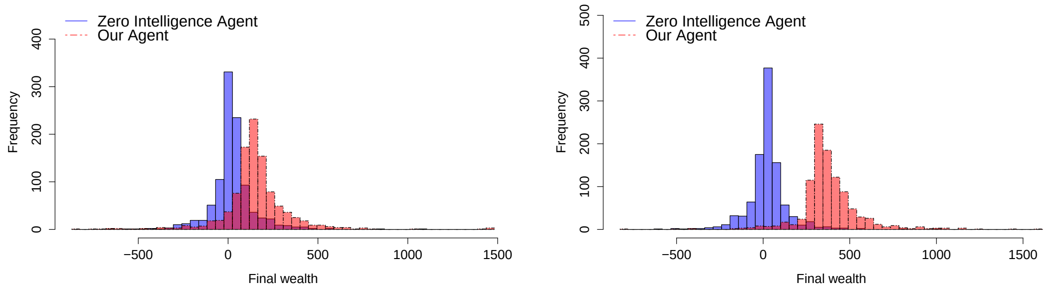
De plus, u_0^ϵ vérifie la condition finale $u_0^\epsilon(T,s,y,q_1,x) = x + q_1h(s)$. On montre alors que u_0^ϵ coincide avec u^ϵ si elle vérifie des conditions de régularité et de croissance polynomiale.

En faisant l'hypothèse que le paramètre ϵ est petit, on utilise une méthode de perturbation singulière pour résoudre l'équation HJB et on fournit des approximations analytiques pour les contrôles optimaux $\hat{\delta}_{*,t}^-$ et $\hat{\delta}_{*,t}^+$ à la date t. En interprétant les formules analytiques obtenues, on peut voir que le market maker adapte sa stratégie de quotation pour:

- profiter de l'inefficience de prix provenant du décalage entre la distribution implicite du sous-jacent et sa distribution historique

- alléger son inventaire d'options dans le but de diminuer l'écart type de sa richesse future.

Afin d'étayer notre étude théorique, on a comparé la performance de la stratégie optimale d'un market maker risque neutre à celle d'une stratégie classique utilisée couramment par des markets makers sur options. Cette stratégie, qu'on nomme "stratégie zéro-intelligence", consiste à placer des prix à l'achat et la vente symétriquement autour du prix mid de l'option dans le but de gagner $0.005 \times \vartheta_{BS}$ à chaque trade exécuté, où ϑ_{BS} représente le vega Black-Scholes de l'option considérée. En effectuant une étude comparative sur des données simulées, on trouve que la stratégie optimale donne de meilleurs résultats que la stratégie "zéro-intelligence", comme vient de l'illustrer les graphes suivants:



(b) Distribution de la richesse finale pour $\beta = 1$ (c) Distribution de la richesse finale pour $\beta = 0.5$

Figure 2: Comparaison entre la stratégie optimale et la stratégie "Zéro-intelligence"

On a aussi étudié l'effet le la misspécification des paramètres sur la stratégie optimale du market maker. En effet, on a vu à travers l'étude analytique réalisée, que dans le cas où la dynamique du sous-jacent sous la probabilité historique est différente de celle sous la probabilité risque-neutre, le market maker adapte sa stratégie de quotation afin de pouvoir bénéficier de l'inefficience du prix sur le marché. Afin d'appuyer les résultats de la partie analytique, on génère des simulations numériques dans lequels le sous-jacent suit le modèle de Heston. En effet, on suppose que S a la dynamique suivante sous la probabilité $\mathcal{P}$:

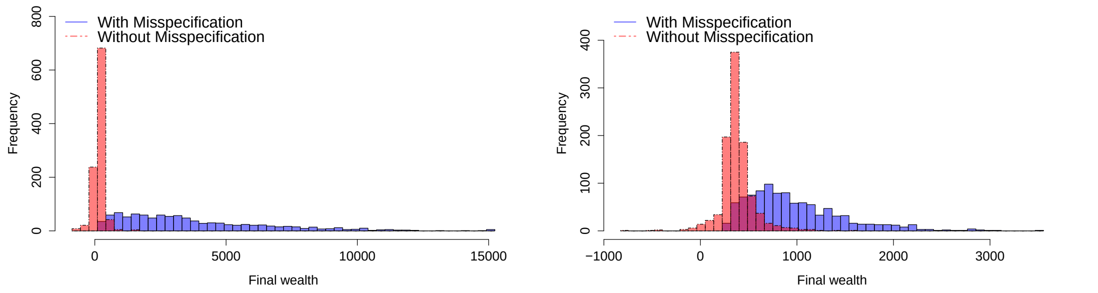
$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{y_t} dW_t^{(1)} \\ dy_t &= k_R(\theta_R - y_t)dt + \eta_R \sqrt{y_t} dW_t^{(2)} \end{aligned}$$

où $d\left\langle W^{(1)}, W^{(2)} \right\rangle_t = \rho_R dt$. De l'autre côté, le spot S a la dynamique suivante sous la probabilité risque-neutre $\mathcal{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{y_t} dW_t^{*,(1)} \\ dy_t &= k_I(\theta_I - y_t)dt + \eta_I \sqrt{y_t} dW_t^{*,(2)} \end{aligned}$$

où $d\left\langle W^{*,(1)}, W^{*,(2)} \right\rangle_t = \rho_I dt$.

Dans un premier exemple, on suppose que $\theta_R \neq \theta_I$, et que $k_R = k_I, \eta_R = \eta_I, \rho_R = \rho_I$, et on étudie l'effet de misspécification du paramètre θ sur la stratégie du market maker:



(a) Distribution de la richesse finale pour $\beta = 1$ (b) Distribution de la richesse finale pour $\beta = 0.5$

Figure 3: Comparaison des distributions de la richesse finale dans les cas où $\theta_R = \theta_I$ et $\theta_R \neq \theta_I$

On étudie par la suite d'autres exemples où $\rho_R \neq \rho_I$ ou aussi $\eta_R \neq \eta_I$.

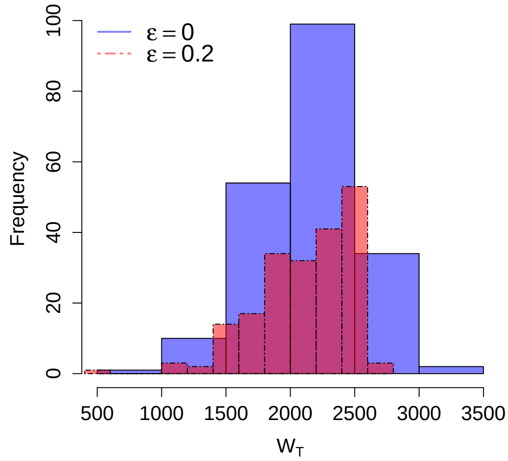
0.4.2 Stratégie optimale en dimension supérieure

Un market maker d'options a généralement un mandat pour fournir la liquidité sur un ensemble d'options ayant des sous-jacents différents. A titre d'exemple, ce dernier peut être censé émettre des quotations pour les options sur les actifs du CAC40, ou bien sur les titres appartenant au DAX,... Pour étudier ce problème, on a supposé que le market maker va classifier l'ensemble des options en paires, ensuite il va traiter chaque paire d'options séparément. Ainsi pour une paire donnée, le market maker va déterminer les prix à l'achat et à la vente qu'il va proposer pour chaque élément avec un objectif bien déterminé qui se traduit à travers sa fonction d'utilité.

Dans la première partie, on suppose que le market maker est risque neutre. Son objectif est de maximiser l'espérance de sa richesse future. On écrit ainsi l'équation HJB relative à ce problème d'optimisation, et on détermine après résolution les quantités $\left(C^{(i)}_{\mathcal{Q},t} - \delta^{-}_{i,L,t}\right)$ et $\left(C^{(i)}_{\mathcal{Q},\sqcup} + \delta^{+}_{i,L,t}\right)$ représentant respectivement les prix d'achat et de vente sur l'option $i \in \{1, 2\}$. On peut noter ici que les prix d'achat et de vente sur les deux options appartenant à la même paire sont indépendants, ce qui est normal car la fonction d'utilité ne contient aucun terme croisé.

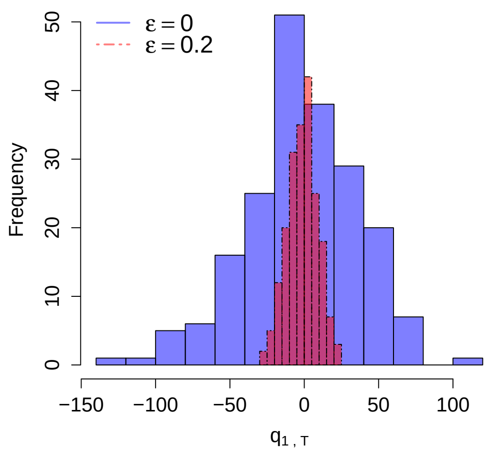
Dans la deuxième partie, on suppose que le market maker est averse au risque. Il a pour but de maximiser l'espérance de sa richesse future tout en minimisant sa variance provenant principalement du risque de volatilité de l'inventaire sur les deux options. On écrit l'équation HJB relative à ce nouveau problème d'optimisation, et on détermine par la suite les quantités $\left(C^{(i)}_{\mathcal{Q},\sqcup} - \delta^{-}_{i,*,t}\right)$ et $\left(C^{(i)}_{\mathcal{Q},\sqcup} + \delta^{+}_{i,*,t}\right)$ représentant respectivement les prix d'achat et de vente sur l'option $i \in \{1, 2\}$. Dans ce nouveau cas, les prix de quotation sur les deux options de la paire sont dépendants. En effet, en supposant que les sous jacents S_1 et S_2 ont des volatilité instantanées corrélées, la variance de la richesse finale contiendra un terme croisé dépendant de la covariance des risques de volatilité des deux options. Par conséquent, si le market maker accumule un inventaire sur l'une des options, il va ajuster ses quotations sur la deuxième option dans le but de réduire la variance de sa richesse.

Afin de supporter notre étude théorique, on génère des simulations Monte Carlo afin de tester les stratégies optimales relatives aux agents risque neutre et averses au risque respectivement. De plus, on montre expérimentalement que la stratégie optimale du market maker averse au risque, réduit notablement l'écart-type de la richesse finale comparé à celle du market maker risque neutre.

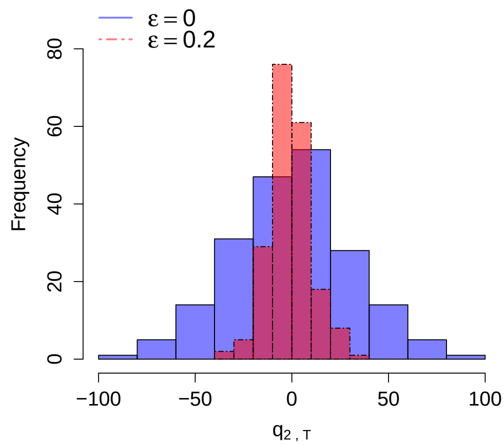


(a) Histograms of $\mathcal{W}_T$

Statistics on $\mathcal{W}_T$	Mean	StD
$\epsilon = 0$	2139.37	385.92
$\epsilon = 0.2$	2115.07	369.55
$\epsilon = 0.4$	2031.78	365.22



(b) Histograms of $q_{1,T}$



(c) Histograms of $q_{2,T}$

Statistics	Mean Of $q_{1,T}$	StD Of $q_{1,T}$	Mean Of $q_{2,T}$	StD Of $q_{2,T}$
$\epsilon = 0$	-0.81	37.65	-0.50	31.78
$\epsilon = 0.2$	-0.62	10.08	-0.24	10.90
$\epsilon = 0.4$	-0.64	9.40	0.04	9.47

0.5 Dynamique jointe de la surface de volatilité implicite et du spot

Dans cette partie, on s'intéresse à la dynamique jointe spot/volatilité implicite ainsi que de son utilité dans diverses applications comme la couverture dynamique d'options. En effet, si on pose un modèle de volatilité stochastique pour le sous-jacent S sous la probabilité $\mathcal{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_t dW_t^{(1)}, \\ d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)}, \end{aligned}$$

où les deux mouvements browniens $W^{(1)}$ et $W^{(2)}$ sont corrélés $\left(d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho dt\right)$, et les coefficients a et b ne dépendent pas du spot S mais uniquement du processus σ, alors la volatilité implicite $\Sigma(K, T-t)$ de l'option C de strike K et de maturité $T-t$ (vérifiant $P_{BS}\left(t, T, S_t, K, \Sigma(K, T-t)\right) = C(t, T, S_t, K, \sigma_t))$ est une fonction de la moneyness $\log\left(\frac{K}{S_t}\right)$. Dans ce cas, le delta Δ_t de l'option vérifie:

$$\Delta_t = \frac{\partial C}{\partial S} = \frac{\partial P_{BS}}{\partial S} + \frac{\partial P_{BS}}{\partial \Sigma}\frac{\partial \Sigma(K, T-t)}{\partial S}$$

Si on approxime la volatilité implicite $\Sigma(K, T-t)$ à l'ordre 2 en $\log\left(\frac{K}{S_t}\right)$ pour $|K - S_t| \ll S_t$ (au voisinage de la monnaie):

$$\Sigma(K, T-t) = \Sigma_{ATM}(T-t) + Skew_{T-t}\log\left(\frac{K}{S_t}\right) + Curv_{T-t}\log\left(\frac{K}{S_t}\right)^2 + o\left(\log\left(\frac{K}{S_t}\right)^2\right),$$

alors:

$$\frac{\partial \Sigma(K, T-t)}{\partial S} = \left(\frac{\partial \Sigma_{ATM}(T-t)}{\partial \log(S_t)} - Skew_{T-t}\right)\frac{1}{S_t} + O\left(\log(\frac{K}{S_t})\right).$$

On peut bien voir dans ce cas que la connaissance du terme $\frac{\partial \Sigma_{ATM}(T-t)}{\partial \log(S_t)}$ est cruciale pour le calcul de Δ_t afin d'effectuer une couverture dynamique efficiente des options.

Dans le but d'étudier la dynamique jointe spot/volatilité implicite, Bergomi a proposé dans Bergomi (2009), un modèle pour la dynamique du sous-jacent S ainsi que ses variances forwards instantanées:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{\xi_t^t}dW_t, \\ d\xi_t^T &= \xi_t^T w \sum_{i=1}^{n} \lambda_{i,t}^T dZ_t^{(i)}, \end{aligned}$$

où $d\left\langle Z^{(i)}, W\right\rangle_t = \rho_i dt$.

En définissant la quantité $\delta\xi_t = \xi_t^t - \xi_0^t = \xi_0^t w \int_0^t \sum_{i=1}^n \lambda_{i,s}^t dZ_s^{(i)} + O\left(w^2\right)$, l'auteur a introduit la function f:

$$f(\tau, t) = \frac{1}{d\tau}E(\frac{dS_\tau}{S_\tau}\delta\xi_t),$$

Il a ensuite introduit une quantité R_T, qu'il a appelée "Skew Stickiness Ratio", afin d'étudier la relation entre la dynamique jointe volatilité implicite à la monnaie/spot caractérisée à travers $d\left\langle \sigma_{ATM}(T), \log(S)\right\rangle_t$ et le skew implicite $\mathcal{S}_T = \frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}$:

$$R_T = \frac{d\left\langle \sigma_{ATM}(T), \log(S)\right\rangle_t}{d\left\langle \log(S)\right\rangle_t \mathcal{S}_T}$$

En utilisant des approximations à l'ordre 1 dans la vovol w, l'auteur montre que:

$$R_T = \frac{\int_0^T \xi_0^t dt}{\xi_0^0} \frac{\int_0^T f(0,u)du}{\int_0^T \int_0^t f(\tau,t)d\tau dt} + O\big(w^2\big)\,,$$

En se basant sur ces éléments, l'auteur a conclu que, dans le cadre de modèles de volatilité stochastique linéaire, l'indicateur R_T converge vers la valeur 2, sous la probabilité $\mathcal{Q}$, quand T tend vers 0.

0.5.1 Estimation du Skew Stickiness Ratio implicite $R_T^{\mathcal{Q}}$

Notre travail vise à étudier l'indicateur R_T et voir ce qu'il peut contenir comme informations concernant la dynamique jointe spot/volatilité implicite et sa relation avec le skew implicite. Au lieu de modéliser la volatilité instantanée du sous-jacent, on part d'un modèle dans lequel on spécifie:

- une paramétrisation du smile de volatilité implicite autour de la monnaie:

$$\forall K, |K-S_t| \ll S_t, \sigma_{BS,t}(K,T) = \sigma_{ATM,t}(T) + \mathcal{S}_T \log(\frac{K}{S_t}) + \mathcal{C}_T \log(\frac{K}{S_t})^2 + o\left(\log(\frac{K}{S_t})^2\right),$$

- la dynamique jointe spot/volatilité implicite à la monnaie:

$$d\sigma_{ATM,t}(T) = \alpha dt + R_T \mathcal{S}_T d\log(S_t) + \eta dZ_t,.$$

où $d\left\langle Z, \log(S)\right\rangle_t = 0$.

En utilisant cette modélisation, et en écrivant l'EDP de pricing vérifiée par le prix de l'option sous $\mathcal{Q}$, on propose une méthode qui permet d'estimer la quantité R_T sous la probabilité de pricing $\mathcal{Q}$. En effet, $R_T^{\mathcal{Q}}$ serait une solution de l'équation de second ordre suivante:

$$\frac{1}{2}\frac{\partial^2 C^L}{\partial \sigma_{ATM}^2}(R_T^{\mathcal{Q}})^2 \mathcal{S}_T^2 d\left\langle \log(S)\right\rangle_t + R_T^{\mathcal{Q}}\left(\frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} S_t \mathcal{S}_T d\left\langle \log(S)\right\rangle_t - \vartheta_t \mathcal{S}_T E^{\mathcal{Q}}(d\log(S_t))\right)$$
$$-\left(\Gamma_t - \frac{1}{2}\frac{\partial^2 C^L}{\partial \sigma_{ATM}^2}\eta^2\right)dt = 0.$$

où C^L est le prix d'une option de strike K_L et de maturité T, C^H est le prix d'une option de strike K_H et de maturité T, et Γ_t est connue explicitement. La quantité $R_T^{\mathcal{Q}}$ estimée est appelée "Skew stickiness ratio implicite".

Nous introduisons ensuite $R_T^{\mathcal{P}}$ comme étant la mesure de la quantité R_T sous la probabilité historique $\mathcal{P}$:

$$R_T^{\mathcal{P}} = \frac{d\left\langle \sigma_{ATM}(T), \log(S)\right\rangle_t}{\mathcal{S}_T d\left\langle \log(S)\right\rangle_t}.$$

En utilisant des données empiriques, on mène une étude comparative entre $R_T^{\mathcal{P}}$ et $R_T^{\mathcal{Q}}$. On observe, que pour les petites maturités, $R_T^{\mathcal{P}}$ a tendance à être généralement inférieure à $R_T^{\mathcal{Q}}$, ce qui peut pourrait être considéré comme un biais de marché.

A fin de pouvoir profiter de ce biais, on propose une stratégie de trading qui vise à monétiser l'écart $\left(R_T^{\mathcal{Q}} - R_T^{\mathcal{P}}\right)$ et on teste l'efficacité de cette stratégie sur des données réelles.

0.5.2 Limite du Skew Stickiness Ratio dans la limite des maturités courtes

En appliquant la méthode d'estimation proposée précédemment sur les données réelles d'options pour divers actifs, on remarque que la quantité $R_T^{\mathcal{Q}}$ peut dépasser la valeur 2 dans le cas limite où T tend vers 0. Cette observation, étant en discordance avec les résultats obtenus dans le cadre des modèles de volatilité stochastique linéaire, relève des questions sur le modèle adéquat permettant d'expliquer la dynamique du sous-jacent sous la probabilité risque-neutre $\mathcal{Q}$.

Pour essayer de répondre à cette question, les auteurs dans Vargas et al. (2013) ont proposé un modèle Garch asymmétrique afin de reproduire des valeurs de R_T supérieures à 2 quand $T \sim 0$. Ce modèle, à temps discret, modélise le rendement logarithmique du sous-jacent entre les dates t_i et t_{i+1} de la façon suivante:

$$\begin{aligned} r_i &= \log\left(\frac{S_{i+1}}{S_i}\right) = \sigma_i \epsilon_i, \\ \sigma_{i+1}^2 &= \bar{\sigma}^2 + a(\sigma_i^2 - \bar{\sigma}^2) + \nu\sigma_i^2 f(\epsilon_i). \end{aligned}$$

Relativement à ce modèle, la quantité R_T peut s'écrire sous la forme:

$$R_T = R_{T,Linear} \frac{\frac{Skewness_T}{6\sqrt{T}}}{Skew_T}$$

où la quantité $R_{T,Linear}$ coincide avec le Skew stickiness ratio dans les modèles de volatilité stochastique linéaire et $Skewness_T$ représente la skewness du rendement logarithmique $\log\left(\frac{S_T}{S_0}\right)$ sur une période de longueur T. Par conséquent, l'indicateur R_T peut atteindre une valeur supérieure à 2 pour une maturité $T \sim 0$ si $\frac{Skewness_T}{6\sqrt{T}}$ est supérieur à $Skew_T$ pour cette même maturité. Si f est une fonction non-linéaire, cet effet peut être reproduit pour une maturité de l'ordre de quelque jours. Cependant, on a remarqué que pour une maturité de l'ordre de 3 mois, le rapport $\frac{\frac{Skewness_T}{6\sqrt{T}}}{Skew_T}$ se stabilise à la valeur 1, pour différents choix de la fonction f, ce qui fait que R_T devient égal à $R_{T,Linear}$ et ne pourrait pas du coup dépasser la valeur 2. Par conséquent, le modèle Garch asymétrique ne fournit pas une justification théorique pour l'observation $R_T^{\mathcal{Q}} > 2$ quand $T = 3M$.

Afin de pouvoir justifier ce résultat, on propose un modèle à saut où la volatilité instantanée du sous-jacent est stochastique:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - \lambda k)dt + \sigma(y_t)\sqrt{1-\rho_t^2}dW_t^{(1)} + \sigma(y_t)\rho_t dW_t^{(2)} + (J_t - 1)dN_t, \\ dy_t &= b(y_t)dW_t^{(2)} \end{aligned}$$

où N_t est un processus de Poisson d'intensité λ, r est le taux sans-risque, J_t est une variable aléatoire positive et $k = E(J_t - 1)$. Dans ce cadre de modèle, on montre que le Skew stickiness ratio peut atteindre des valeurs supérieures à 2, du fait de la présence des sauts. En effet, on montre que sous l'hypothèse $E((J-1)^3) = E((J-1)^4)$:

$$\lim_{T\to 0} R_T = 2 + \frac{\lambda}{\sigma^2(y_t)}\left(\frac{E((J-1)^3)}{2} - E((J-1)^2)\right),$$

Introduction (In English)

The Black and Scholes model, presented in 1973, is one of the most important advances the financial industry has known. This model established the first theoretical foundations for the pricing and dynamic hedging of European options. It has been noticed since then that the hypothesis of constant volatility assumed in this model is too strong. Indeed, it was proven empirically that the instantaneous volatility of the underlying is time-varying and should then be considered as a stochastic process. In addition, the black and Scholes model is not able to reproduce all stylized facts observed in the option market.
We denote here $(C_{Market}(K_i, T_j))_{\{1\leq i\leq n, 1\leq j\leq p\}}$ the market prices of european calls with strikes $(K_i)_{1\leq i\leq n}$, and maturities $(T_j)_{1\leq j\leq p}$. Through the inversion of the Black-Scholes pricing formula, we obtain the implied volatilities $(\sigma(K_i, T_j))_{\{1\leq i\leq n, 1\leq j\leq p\}}$ verifying:

$$C_{Market}(K_i, T_j) \;=\; C_{Black-Scholes}\left(K_i, T_j, \sigma(K_i, T_j)\right).$$

We can then introduce the following notations:

- $(\sigma(K_i, T))_{\{1\leq i\leq n\}}$ represents the implied volatility smile of maturity T observed in the market.
- $(\sigma(K_i, T_j))_{\{1\leq i\leq n, 1\leq j\leq p\}}$ represents the implied volatility surface observed in the market.

The hypothesis of constant volatility in the Black-Scholes model leads to an implied volatility $\sigma(K, T)$ which is constant independently of the strike K and the maturity T. This result is not consistent with the observed market volatility surface. For this reason, it has been considered that the Black-Scholes model has some limits and is not sufficient to descrive the dynamics of the underlying under the risk-neutral probability measure $\mathcal{Q}$. Since then, many researchers studied this problem and suggested more sophisticated models for the dynamics of the underlying in order to justify the form of the implied volatility surface observed in the market. For example, Merton suggested a jump model in which the spot process S has the following dynamics:

$$\frac{dS_t}{S_t} \;=\; \mu dt + \sigma dW_t + (J-1)\, dN_t$$

where N is a Poisson process with intensity λ, and J is a positive random variable. Other researchers made the choice to concentrate on the dynamics of the instantaneous volatility process, and they proposed different models in the form:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma(y_t) dW_t^{(1)},\\ dy_t &= a(y_t)dt + b(y_t)dW_t^{(2)},\end{aligned}$$

where $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho dt$. The most known models among the family of stochastic volatility models are Heston (1993) and Hagan et al. (2002). The first objective of this class of models is to obtain an implied volatility surface which is similar to the one observed in the market. The second objective is to reproduce the joint dynamics of the spot and the implied volatility in order to provide successful hedging strategies.

More recently, other researchers adopt a new approach which consists in modeling directly the dynamics of the implied volatility surface. This may be advantageous as the implied volatility is an observable variable in contrast with the instantaneous volatility that needs to be estimated through filtering techniques or using high frequency data. The works of Schönbucher (1999) and Cont et al. (2002) represent good examples of this approach.

In this thesis, we study the underlying/option joint dynamics, and we try to understand the implication of these dynamics on diverse applications related to option trading. This thesis contains three independent chapters which can be read separately. Each chapter focuses on the effect of the spot/volatility joint dynamics on a concrete application.

In the first chapter of the thesis, we try to understand the implications of the joint dynamics between two assets on the relation between their respective implied volatility surfaces. We choose the Capital asset pricing model to describe the joint dynamics of a stock and its market index. Besides, we suppose that the instantaneous volatilities of the stock and the index are both stochastic in order to provide a realistic framework for option pricing. Then, we undertake a theoretical study in order to provide analytical approximations of the prices of European options on the stock and the index respectively. Next to that, we carry out an empirical study with the aim to test the validity of the theoretical results on real data, and we investigate the presence of a market inefficiency arising from the incoherence in the relation between the implied volatility surfaces of the stock and the index given the stock/index joint dynamics.

In the second chapter of the thesis, we focus on the subject of option market making. The difficulty of this task comes from the numerous risks supported by the market maker. These risks include the delta risk of the underlying, the volatility risk, the option liquidity risk, the inventory risk,... In order to study this problem, we build a theoretical framework which takes into account the spot/instantaneous volatility joint dynamics, the frequency of arrival of market orders into the order book,... By specifying the utility function of the market maker in terms of return maximization and variance reduction, we formulate her optimization problem and we use a stochastic control approach in order to determine her optimal quoting policy. The organization of this chapter can be given as follows. In the first part, we focus on the problem of making markets in one option only. First, we consider the case of a risk-neutral market maker and we determine analytic expressions of her optimal bid and ask quotes. Second, we turn towards the case of a risk averse market maker and we determine analytic approximations of her optimal quotes. In the second part of the chapter, we use a similar approach to study market making strategies on two options, and we point out the effect of the joint dynamics between the underlying assets on the optimal strategy.

In the third chapter of the thesis, we study the relation between the underlying return and the increment of the implied volatility. Let us introduce the function f as follows:

$$\begin{aligned} f : \mathbb{R}^+ \times \mathbb{R}^+ &\to \mathbb{R}^+, \\ (K,T) &\to \sigma(K,T). \end{aligned}$$

We get interested in the relation between the joint dynamics of the spot and the at-the-money implied volatility with maturity T on one hand, and the quantity $\frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}$ on the other. In order to explore this relation, we study the following quantity:

$$\mathcal{R}_T = \frac{d\left\langle \sigma(S,T), \log(S) \right\rangle_t}{d\left\langle \log(S) \right\rangle_t \frac{\partial \sigma(K,T)}{\partial \log(K)}_{|K=S_t}}$$

Using $\mathcal{R}_T$, we characterize the relation between the joint dynamics of the spot and the at-the-money implied volatility on one hand, and the implied "Skew" with maturity T defined

as $\frac{\partial\sigma(K,T)}{\partial\log(K)}_{|K=S_t}$. We also study the eventual presence of arbitrage opportunity which can arise if this relation doesn't hold.
For the rest of the report, we introduce the following notations:

- $(\Omega, \mathcal{F}, \mathbb{P})$ represents the probability space equipped with the filtration $\mathcal{F}$ satisfying the usual conditions, and the objective probability measure $\mathbb{P}$.
- $(\Omega, \mathcal{F}, \mathbb{Q})$ represents the probability space equipped with the filtration $\mathcal{F}$ satisfying the usual conditions, and the risk-neutral probability measure $\mathbb{Q}$ which is used for option pricing.

The results of the three chapters are summarized below.

0.6 Effect of the stock/index joint dynamics on the relation between their respective implied volatility surfaces

In this chapter, we focus on the effect of the joint dynamics between two underlying assets on the relation between their respective implied volatility surfaces. In order to deal with this problem, we use the Capital asset pricing model to describe the joint dynamics of a stock and its market index. Through this model, the return $R_S(t)$ of the stock S at date t can be written as a function of the return $R_I(t)$ of the index I at date t:

$$R_S(t) \quad = \quad \alpha + \beta R_I(t) + \epsilon(t),$$

where ϵ denotes the idiosyncratic part independent of the market return.

This model is frequently used in portfolio construction and risk management. In this work, we use the CAPM model in order to investigate the relations that may exist between the implied volatility surface of the stock and the one of the index.

0.6.1 Estimation of the parameter beta using option prices

The beta coefficient in the CAPM model characterizes the dependence between the asset return and the market index return. Thus, this parameter is crucial as it is a good indicator of the effect of the systematic risk on the asset return. The parameter beta can be estimated through the regression of the returns of the asset S on the returns of the index I:

$$\beta \quad = \quad \frac{cov(R_S, R_I)}{Var(R_I)}.$$

The estimation of the parameter β, using real financial data, on a sliding window of a given length shows that this parameter is time-varying. This finding is contradictory with the hypothesis of constant beta assumed in the model. For this reason, the estimation of the coefficient beta can be problematic. Indeed, the estimation of this parameter on a window of historical data gives a measure of the realization of this parameter on a past window. Whereas, the realized beta in the future can be significantly different from its past value.
Option prices usually traduce the expectation of market participants on the realization of several quantities in the future. For example, the implied variance $\sigma_I^2(K,T)$ of an European option with strike K and maturity T represents the market expectation of the future break-even variance $\sigma_{BEVL}^2(K,T)$ introduced by Dupire:

$$\sigma_{BEVL}^2(K,T) \quad = \quad \frac{E\left(\int_0^T \Gamma_{BS,t} S_t^2 \sigma_t^2 dt\right)}{E\left(\int_0^T \Gamma_{BS,t} S_t^2 dt\right)},$$

In our study, we follow an analogical reasoning in order to estimate the parameter beta under the risk-neutral measure through the use of option prices. In addition, we try to investigate whether this estimator is informative about the future realization of the parameter beta under the historical probability measure. The authors in Fouque and Kollman (2011) proposed a continuous-time capital asset pricing model in which the instantaneous volatility of the index I is stochastic:

$$\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + \sigma dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)},
\end{aligned}$$

where $W_t^{(3)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(4)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(4)} \end{pmatrix}$ is a $(\mathbb{P}, \mathcal{F})$ Brownian motion. Under the hypothesis $0 < \epsilon \ll 1$, a singular perturbation technique is performed with respect to the parameter ϵ in order to provide approximations for the prices of european options on the stock S and the index I. Afterward, the implied volatilities $\Sigma_I(K,T)$ and $\Sigma_S(K,T)$ of the index and the stock respectively are approximated as follows:

$$\begin{aligned}
\Sigma_I(K_I, T) &= b_I + a_I \frac{\log(\frac{F_I}{K_I})}{T}, \\
\Sigma_S(K_S, T) &= b_S + a_S \frac{\log(\frac{F_S}{K_S})}{T}.
\end{aligned}$$

The quantities F_I et F_S represent the forward prices for the maturity T for the index and the stock respectively. Using the analytic expressions of b_I, b_S, a_I, a_S, the parameter β can be approximated by $\hat{\beta}$:

$$\hat{\beta} = \left(\frac{a_S}{a_I}\right)^{\frac{1}{3}} \frac{b_S}{b_I}. \tag{1}$$

In order to improve the modeling of the joint dynamics of the stock and the index, we suppose that the idiosyncratic volatility of the stock is stochastic. The new model is then detailed as follows:

$$\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f_1(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + f_2(Z_t) dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)}, \\
dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt + \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{(4)},
\end{aligned}$$

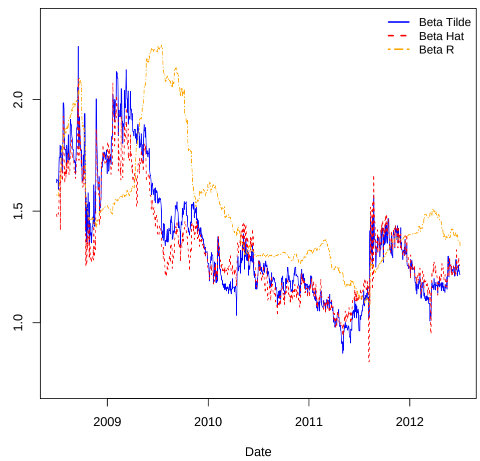
where $W_t^{(3)} = \rho_Y W_t^{(1)} + \sqrt{1-\rho_Y^2} W_t^{(5)}$, $W_t^{(4)} = \rho_Z W_t^{(2)} + \sqrt{1-\rho_Z^2} W_t^{(6)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(5)} \\ W^{(6)} \end{pmatrix}$ is a $(\mathbb{P}, \mathcal{F})$ Brownian motion.

Using a singular perturbation technique on the parameter ϵ, we give approximations for the prices of European options on the stock and the index. Afterward, we deduce analytic approximations of the implied volatilities and we obtain a new estimator of the parameter β:

$$\tilde{\beta} \quad = \quad \sqrt{\frac{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}}{(\bar{\sigma}_I^*)^2}} \left(1 + \frac{V_2^{S,\epsilon}}{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}} - \frac{V_2^{I,\epsilon}}{(\bar{\sigma}_I^*)^2} \right) + o(\epsilon)\,.$$

The estimator $\hat{\beta}$ becomes biased when the idiosyncratic volatility of the stock is stochastic. Thus, the advantage of the new estimator $\tilde{\beta}$ is to be a non-biased estimator of the parameter β.

Using the theoretical results provided so far, we conduct an empirical study in order to compare the implied beta $\tilde{\beta}$ estimated using option price, the historical beta β_H obtained using a linear regression on a window of historical data, and the forward beta β_F estimated on a forward window. The graph below gives the example for the Financial Select Sectoe ETF (XLF) versus the SPX index:



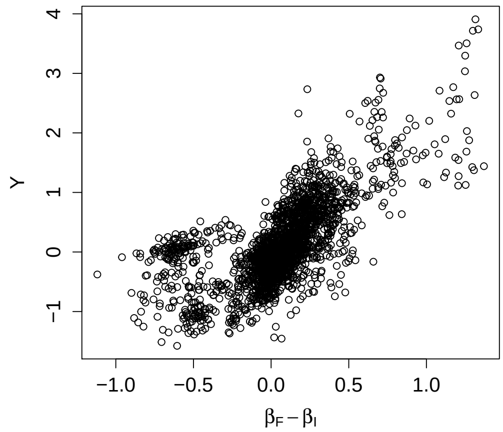
(d) XLF vs SPX

Figure 4: Comparison between $\tilde{\beta}$ (blue line) and $\hat{\beta}$ (red line)

The aim of this study is to investigate the predictive power of the different estimators on the value of the forward beta, thus we can deduce whether the implied beta $\tilde{\beta}$ and then option prices contain additional information concerning the forward beta. The results of the empirical study show that the historical beta gives generally the best prediction of the forward beta.

0.6.2 Application for arbitrage strategies

The results of the empirical study constitute a proof of the existence of inefficiencies in option prices. Indeed, the presence of differences between the implied beta and the forward beta, shows that there are temporary discrepancies between the risk-neutral pricing measure and the real-world probability measure. We propose an arbitrage strategy that aims to profit from misspecification of the parameter beta as the expected $P\&L$ of this strategy is proportional to the quantity $(\beta_{Realized} - \beta_{Implied})$. In the graph below, we plot the $P\&L$ of the strategy tested on $XLF - SPX$ as a function of $(\beta_{Realized} - \beta_{Implied})$:



(a) XLF vs SPX

The y-axis represents the daily $P\&L$ of the strategy while the x-axis represents the difference between the forward realized beta β_F and the implied beta β_I.

0.6.3 Application for option hedging

The Capital Asset Pricing Model offers a good setting to modelize the joint dynamics of the stock and the index. We tried to use this model framework in order to define hedging strategies for options on the stock S using instruments on the index I.

In the first part, we show that the implied beta enables to hedge partially the volatility risk of an option on the stock. Indeed, we can hedge an option on the stock with respect to variations of the process Y using options on the index.
Let $P^{S,\epsilon}$ be the price of an european option on the stock, $P^{I,\epsilon}$ the price of an european option on the index, Δ_S the delta of the stock option and Δ_I the delta of the index option. We define the portfolio X as follows:

$$X_t = P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} + \Delta_{I,t}\vartheta_t I_t - \Delta_{S,t} S_t,$$

with:

$$\vartheta_t = \beta^2 \frac{S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2 \frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}} + o(\epsilon)\,.$$

and we prove that this portfolio is insensible to variations of the process Y.

In the second part, we show that the parameter β can be used to hedge the delta of a stock option using index futures. Indeed, if we take a portfolio L as below:

$$L_t = P_t^{S,\epsilon} - \vartheta_t P^{I,\epsilon} - \varphi_{I,t} I_t,$$

where:

$$\varphi_{I,t} = \frac{\beta S_t}{I_t}\frac{\partial P^{S,\epsilon}}{\partial S} - \vartheta_t \frac{\partial P^{I,\epsilon}}{\partial I},$$

$$\vartheta_t = \beta^2 \frac{S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2 \frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}} + o(\epsilon)\,,$$

then L is made insensible to variations of $W^{(1)}$ et $W^{(3)}$.
We can conclude that the CAPM model can inform us about the relations between the implied volatility surfaces of the stock and the index respectively. These relations may be very useful in different applications ranging from dynamic hedging to arbitrage.

0.7 Market making on options

0.7.1 Optimal market making strategies in the one-dimensional case

In this chapter, we build on the work of Avellaneda and Stoikov in their seminal paper Avellaneda and Stoikov (2008). We attempt to establish a theoretical framework for market making strategies on options. In the beginning, we specify the dynamics of the asset under the objective measure $\mathcal{P}$ and the pricing measure $\mathcal{Q}$. We suppose that the underlying asset S follows a generic stochastic volatility model under $\mathcal{P}$. We assume also that the risk-neutral measure $\mathcal{Q}$, used by market practitioners to price options on this asset, is not necessarily absolutely continuous with respect to $\mathcal{P}$. This feature accounts for the eventual discrepancy between the implied distribution of the underlying asset and its historical distribution.

We suppose that there is an agent willing to provide liquidity on a traded option with maturity T and payoff $H(S_T)$. At each date $t \in [0, T]$, this market maker fixes the distances (δ_t^-, δ_t^+) and posts a bid quote $\mathcal{C}_t^b$ and an ask quote $\mathcal{C}_t^a$ around the market observable price $\mathcal{C}_{t,\mathcal{Q}}$:

$$\begin{aligned} \mathcal{C}_t^b &= \mathcal{C}_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-, \\ \mathcal{C}_t^a &= \mathcal{C}_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+ \end{aligned}$$

In order to model the probability of execution of these limit orders, we make an hypothesis on the analytic expressions of the probabilities of arrival of market orders. Indeed, the probability λ^+ (respectively λ^-) of arrival of a buy market order (respectively a sell market order) consuming an ask quote (respectively a bid quote) at a distance δ^+ (respectively δ^-) from the mid price, can be written as follows:

$$\lambda^+(\delta^+) = \frac{A}{\left(B + (\delta^+)^{\frac{1}{\beta}}\right)^{\gamma}}, \qquad \lambda^-(\delta^-) = \frac{A}{\left(B + (\delta^-)^{\frac{1}{\beta}}\right)^{\gamma}},$$

where $A, B > 0$ and $\gamma > 1$.

The arrival of market orders influences the inventory and the cash held by the Market maker in the case where her limit orders are executed. Meanwhile, the MM has the control of her bid and ask quotes, which in their turn affect the probabilities of arrival of market orders. It is then important for the Market maker to choose her control variables in the most optimal way in order to attain her objectives.

In the first part of the study, we consider the case of a risk-neutral market maker whose objective is to maximize the expectation of her terminal wealth at the maturity date of the option. Her value function writes:

$$u(t, s, y, q_1, x) = Sup_{\{(\delta_t^+, \delta_t^-) \in \mathcal{A}\}} E^{\mathcal{P}}(X_T + q_{1,T} h(S_T)) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x),$$

where $\mathcal{A}$ denotes the set of admissible values for the controls (δ_t^-, δ_t^+) and is equal to $\mathbb{R}^+ \times \mathbb{R}^+$.

This is a stochastic control problem whose associated Hamilton-Jacobi-Bellmann equation writes:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2) u_0 + sup_{\{(\delta_t^+, \delta_t^-) \in \mathcal{A}\}} \left(J^+\left(\delta_t^+\right) + J^-\left(\delta_t^-\right)\right) = 0,$$

where the functions J^+ and J^- are defined such that:

$$\begin{aligned} J^+(\delta_t^+) &= \lambda^+(\delta_t^+)\left(u_0(t,s,y,q_{1,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}+\delta_t^+))-u_0(t,s,y,q_{1,t^-},x_{t^-})\right), \\ J^-(\delta_t^-) &= \lambda^-(\delta_t^-)\left(u_0(t,s,y,q_{1,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}-\delta_t^-))-u_0(t,s,y,q_{1,t^-},x_{t^-})\right), \end{aligned}$$

and the differential operators $\mathcal{L}_1$ et $\mathcal{L}_2$ can be given as follows:

$$\begin{aligned} \mathcal{L}_1 &= \mu S_t\frac{\partial}{\partial S}+\frac{1}{2}\sigma^2(y_t)S_t^2\frac{\partial^2}{\partial S^2}+a_R(y_t)\frac{\partial}{\partial y}+\frac{1}{2}b_R^2(y_t)\frac{\partial^2}{\partial y^2}+\rho_R b_R(y_t)\sigma(y_t)S_t\frac{\partial^2}{\partial S\partial y}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial x}q_{2,t}\mu S_t+\frac{1}{2}\frac{\partial^2}{\partial x^2}q_{2,t}^2\sigma(y_t)^2S_t^2+\frac{\partial^2}{\partial x\partial S}q_{2,t}\sigma^2(y_t)S_t^2+\frac{\partial^2}{\partial x\partial y}q_{2,t}\sigma(y_t)S_t b_R(y_t)\rho_R, \end{aligned}$$

In addition, u_0 verifies the final condition:

$$u_0(T,s,y,q_1,x) = x+q_1h(s).$$

If the function u_0 is finite, smooth and has polynomial growth, we can prove that it coincides with the value function, and we deduce the optimal controls $\delta_{L,*,t}^-$ and $\delta_{L,*,t}^+$ of the Market Maker at date t.

In the second part of the study, we suppose that the market maker is risk-averse. She has the objective of maximizing the expectation of her final wealth while minimizing its variance. Her value function writes:

$$u^\epsilon(t,s,y,q_1,x) = Sup_{\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\}}E^{\mathcal{P}}_{t,s,y,q_1,x}\left(X_T+q_{1,T}h(S_T)\right)-\epsilon Var^{\mathcal{P}}_{t,s,y,q_1,x}\left(X_T+q_{1,T}h(S_T)\right),$$

This is a stochastic control problem whose Hamilton-Jacobi-Bellmann equation can be given in the following way:

$$\left(\partial_t+\mathcal{L}_1+\mathcal{L}_2\right)u_0^\epsilon+\sup_{\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\}}\left(J^{-,\epsilon}(\delta_t^-,\delta_t^+)+J^{+,\epsilon}(\delta_t^-,\delta_t^+)\right) = \epsilon\left(q_{1,t}^2V_t+T_t\right),$$

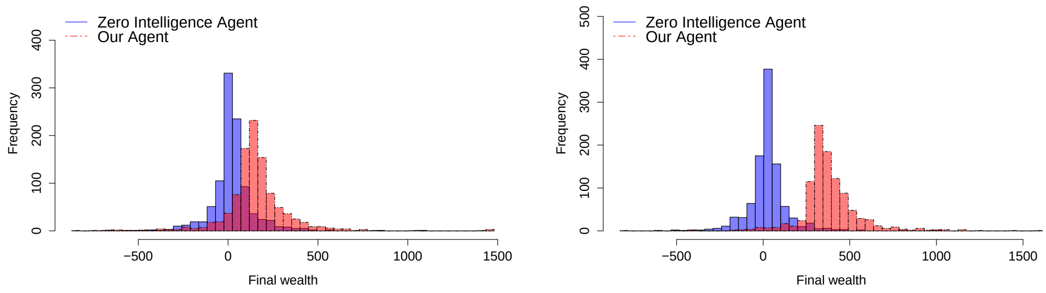
where the quantities V_t et T_t are determined analytically, and the functions $J^{+,\epsilon}$, $J^{-,\epsilon}$ are defined as follows:

$$\begin{aligned} J^{+,\epsilon}(\delta^+) &= \lambda^+(\delta^+)\left(u_0^\epsilon(t,s,y,q_{1,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}+\delta^+))-u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right), \\ J^{-,\epsilon}(\delta^-) &= \lambda^-(\delta^-)\left(u_0^\epsilon(t,s,y,q_{1,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}-\delta^-))-u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right), \end{aligned}$$

Besides, u_0^ϵ is subject to the final condition $u_0^\epsilon(T,s,y,q_1,x)=x+q_1h(s)$. Under smoothness and polynomial growth conditions on u_0^ϵ, it can be shown that it coincides with the value function u^ϵ.

Through the use of a singular perturbation technique on the parameter ϵ, we provide analytic approximations of the bid and ask distances $\left(\delta_{*,t}^-,\delta_{*,t}^+\right)$ at date t.

In order to support our theoretical study, we test whether the optimal strategy of a risk-neutral market maker gives better results than a classical strategy commonly used by practitioners. This strategy, which we denote "zero-intelligence strategy", consists in placing bid and ask quotes symmetrically around the mid price with the objective to pay in a margin equal to $0.005\times\vartheta_{BS}$ at each executed trade (ϑ_{BS} denotes the Black-Scholes vega of the considered option). We carry out a comparative study of these two strategies using Monte Carlo simulations and we conclude that the optimal strategy gives better results as it is shown in the graphs below:



(b) Distribution de la richesse finale pour $\beta = 1$ (c) Distribution de la richesse finale pour $\beta = 0.5$

Figure 5: Comparaison entre la stratégie optimale et la stratégie "Zéro-intelligence"

In addition, we study the effect of parameters misspecification on the optimal strategy of the market maker. Indeed, the analytical results of our theoretical study show that the market maker adapts her strategy in order to profit from price inefficiency which is due to the discrepancy between the implied and realized distributions of the underlying. In order to illustrate these results experimentally, we suppose that the underlying follows a Heston model under the objective probability measure $\mathcal{P}$:

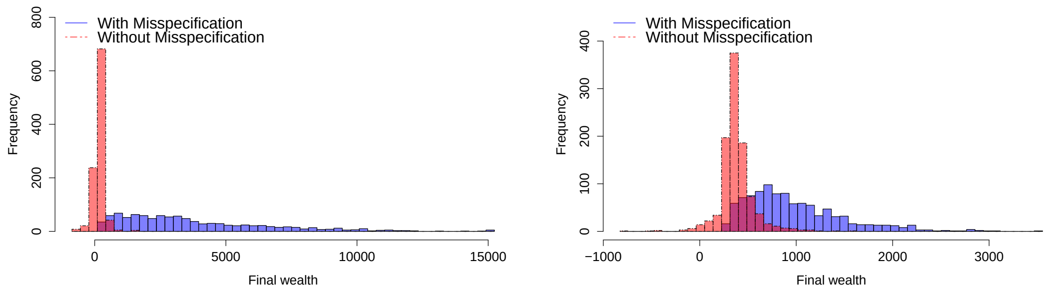
$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{y_t} dW_t^{(1)} \\ dy_t &= k_R(\theta_R - y_t)dt + \eta_R \sqrt{y_t} dW_t^{(2)} \end{aligned}$$

where $d\left\langle W^{(1)}, W^{(2)} \right\rangle_t = \rho_R dt$. Under the risk-neutral measure $\mathcal{Q}$, S has the following dynamics:

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{y_t} dW_t^{*,(1)} \\ dy_t &= k_I(\theta_I - y_t)dt + \eta_I \sqrt{y_t} dW_t^{*,(2)} \end{aligned}$$

where $d\left\langle W^{*,(1)}, W^{*,(2)} \right\rangle_t = \rho_I dt$.

We suppose here that $\theta_R \neq \theta_I$, while $(k_R, \eta_R, \rho_R) = (k_I, \eta_I, \rho_I)$. We can then compare the effect of the misspecification of the parameter θ on the optimal strategy of the market maker. By testing the optimal strategy using numerical simulations, we conclude that the market maker profits from price inefficiency in order to augment her gains:



(a) Distribution de la richesse finale pour $\beta = 1$ (b) Distribution de la richesse finale pour $\beta = 0.5$

Figure 6: Comparison of the distributions of the final wealth in the case $\theta_R = \theta_I$ and $\theta_R \neq \theta_I$

0.7.2 Optimal market making strategies in a higher dimension

A market maker has generally a mandate to provide liquidity on a set of options having different underlyings. For example, she may be supposed to set quotations for options on the $CAC40$ constituents, or on names of the DAX index,...
In order to consider this problem, we suppose that the market maker classifies his quoted options into pairs, and then treats each pair separately. Let us suppose that the two options, contained in a given pair, are indexed by $i \in \{1, 2\}$ and that their underlyings S_i have the following dynamics under $\mathcal{Q}$:

$$\begin{aligned} \frac{dS_{i,t}}{S_{i,t}} &= rdt + \sigma_i(y_{i,t})dW_t^{*,i}, \\ dy_{i,t} &= a_{i,I}(y_{i,t})dt + b_{i,I}(y_{i,t})dZ_t^{*,i}, \end{aligned}$$

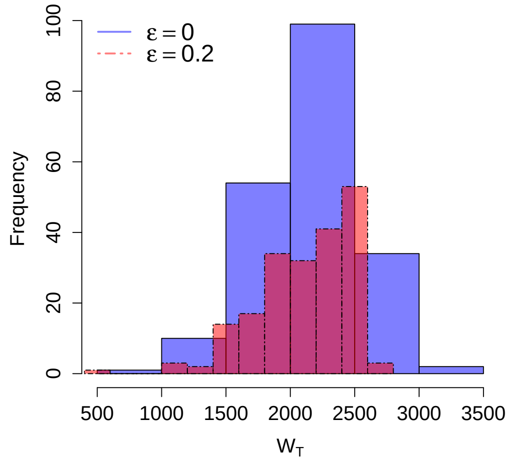
where $W^{*,i}$ and $Z^{*,i}$ are two $(\mathcal{Q}, \mathcal{F})$ Brownian motions such that $d\left\langle W^{*,i}, Z^{*,i}\right\rangle_t = \rho_{i,I}dt$. Besides, $d\left\langle W^{*,1}, W^{*,2}\right\rangle_t = \rho_{1,2,I}dt$ and $d\left\langle Z^{*,1}, Z^{*,2}\right\rangle_t = \tilde{\rho}_{1,2,I}dt$. The market maker will determine the bid and ask prices on each option of the pair depending on her utility function.

In the first part, we suppose that the market maker is risk neutral. Her objective is to maximize the expectation of her terminal wealth. We write the Hamilton-Jacobi-Bellmann equation relative to this optimization problem and then we solve it in order to obtain $\delta^-_{i,L,t}, \delta^+_{i,L,t}$ which represent respectively the optimal distances of the bid and ask prices respectively to the mid price $C^{(i)}_{\mathcal{Q},t}$ of the option i. It can be noticed that the quotations of the two options belonging to the same pair are independent since the utility function of the market maker is linear and doesn't contain a crossed term.

In the second part, we suppose that the market maker is risk averse. She aims to maximize the expectation of her final wealth while reducing its variance due to the volatility risk of the option inventory. We determine the new HJB equation relative to the new optimization problem. By solving this equation, we provide the analytic expressions of $\delta^-_{i,*,t}, \delta^+_{i,*,t}$ denoting the distances of the bid and ask prices of the option i to its mid price $C^{(i)}_{\mathcal{Q},t}$. In this new case, the quotes on the two options depend on each other. Indeed, if the market maker accumulates an inventory in one of the two options, she will adjusts her quotes on the second option with the aim to reduce the variance of her final wealth.

In order to support our theoretical study, we generate Monte Carlo simulations in order to test the optimal strategies both in the cases of a risk-neutral agent and a risk averse agent. In addition, we show experimentally that the optimal strategy of the risk averse market maker reduces significantly the variance of the final wealth compared to the strategy of a risk neutral market maker.

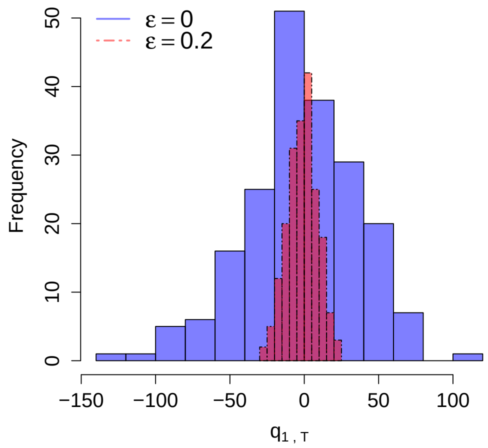
Let ϵ be the parameter controlling the degree of risk adversity of the market maker, $q_{1,T}$ be the final inventory of the option 1 at the maturity date T and $q_{2,T}$ be the final inventory of the option 2 at the maturity date T. The graph below represents the histogram of the final wealth $\mathcal{W}_T$ at the maturity date T:



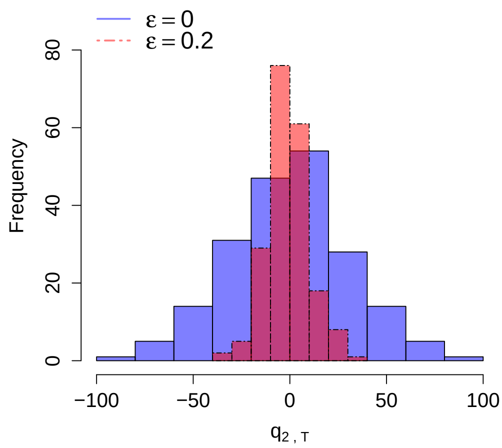
(a) Histograms of $\mathcal{W}_T$

The investigation of this graph shows experimentally how augmenting the parameter ϵ enables to reduce the variance of final wealth.

In addition, the two following histograms show the distribution of of $q_{1,T}$ and $q_{2,T}$ obtained through the numerical simulations:



(b) Histograms of $q_{1,T}$



(c) Histograms of $q_{2,T}$

0.8 Joint dynamics of the spot and the implied volatility surface

We study in this chapter the joint dynamics between the spot and the implied volatility, and we investigate its usefulness for different applications like option dynamic hedging. Indeed, if we suppose that the spot process S follows a stochastic volatility model:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_t dW_t^{(1)}, \\ d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)}, \end{aligned}$$

where $W^{(1)}$ et $W_t^{(2)}$ are two correlated Brownian motions $\left(d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho dt\right)$, and the coefficients a et b don't depend on the spot S but only on the process σ, then the implied volatility $\Sigma(K, T-t)$ of the option C with strike K and maturity $T-t$ (verifying $P_{BS}\left(t, T, S_t, K, \Sigma(K, T-t)\right) = C(t, T, S_t, K, \sigma_t))$ is a function of the moneyness $\log\left(\frac{K}{S_t}\right)$.
In this case, the delta Δ_t of the option writes:

$$\Delta_t = \frac{\partial C}{\partial S} = \frac{\partial P_{BS}}{\partial S} + \frac{\partial P_{BS}}{\partial \Sigma}\frac{\partial \Sigma(K, T-t)}{\partial S}$$

Let us suppose now that, for strikes K verifying the condition $|K - S_t| \ll S_t$ (near the money strikes), the implied volatility $\Sigma(K, T-t)$ can be approximated at order 2 in $\log\left(\frac{K}{S_t}\right)$:

$$\Sigma(K, T-t) = \Sigma_{ATM}(T-t) + Skew_{T-t}\log\left(\frac{K}{S_t}\right) + Curv_{T-t}\log\left(\frac{K}{S_t}\right)^2 + O\left(\log\left(\frac{K}{S_t}\right)^2\right),$$

then using this approximation, we obtain that:

$$\frac{\partial \Sigma(K, T-t)}{\partial S} = \left(\frac{\partial \Sigma_{ATM}(T-t)}{\partial \log(S_t)} - Skew_{T-t}\right)\frac{1}{S_t} + O\left(\log(\frac{K}{S_t})\right),$$

thus, the term $\frac{\partial \Sigma_{ATM}(T-t)}{\partial \log(S_t)}$ is crucial for the computation of Δ_t and for the realization of efficient dynamic hedging strategies.

The difficulty of the choice of the adequate model for the underlying's dynamics lies in the conciliation between the two objectives. The first objective is that the model, when calibrated to market data at a given date t, should enable to reproduce an implied volatility smile which is in concordance with the one observed in the market. This can be done by calibrating the model's parameters with the aim to minimize the difference between the theoretical volatility smile and the market smile. The second objective for the model is to reproduce the joint dynamics of the spot and the implied volatility. The conciliation between these two objectives is a difficult task.

In order to study this problem, Bergomi proposed in Bergomi (2009) a model for the dynamics of the underlying process S. In this model, Bergomi specified the dynamics of the instantaneous forward variances. The model can be described as follows:

$$\begin{aligned}
\frac{dS_t}{S_t} &= rdt + \sqrt{\xi_t^t}dW_t, \\
d\xi_t^T &= \xi_t^T w \sum_{i=1}^{n} \lambda_{i,t}^T dZ_t^{(i)},
\end{aligned}$$

where $d\left\langle Z^{(i)}, W\right\rangle_t = \rho_i dt$.
Let the quantity $\delta\xi_t = \xi_t^t - \xi_0^t = \xi_0^t w \int_0^t \sum_{i=1}^n \lambda_{i,s}^t dZ_s^{(i)} + O\left(w^2\right)$, the author introduced the function f:

$$f(\tau, t) = \frac{1}{d\tau}E(\frac{dS_\tau}{S_\tau}\delta\xi_t).$$

At order 1 in the vovol w, the quadratic covariation between the at-the-money implied volatility and the logarithm of the underlying writes:

$$d\left\langle \sigma_{ATM}(T), \log(S)\right\rangle_t = \frac{1}{2\sigma_{ATM,0}(T)T}\left(\int_0^T f(0,u)du\right)dt,$$

The author gave an approximation for the skew $\mathcal{S}_T$ at order 1 en w, using the skewness of the distribution of the returns $\log(\frac{S_T}{S_0})$:

$$\mathcal{S}_T = \frac{1}{2\sqrt{T}} \frac{\int_0^T \int_0^t f(\tau, t) d\tau dt}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}},$$

Using these informations, Bergomi introduced a new quantity R_T, which he called "Skew Stickiness Ratio", in order to study the relation between the joint dynamics of the spot/at-the-money volatility with maturity T and the implied Skew $\mathcal{S}_T$:

$$R_T = \frac{d \left\langle \sigma_{ATM}(T), \log(S) \right\rangle_t}{d \left\langle \log(S) \right\rangle_t \mathcal{S}_T}.$$

The Skew Stickiness Ratio R_T for the maturity T writes:

$$R_T = \frac{\int_0^T \xi_0^t dt}{\xi_0^0} \frac{\int_0^T f(0, u) du}{\int_0^T \int_0^t f(\tau, t) d\tau dt},$$

Based on these elements, the author concluded that in the setting of linear stochastic volatility models, the indicator R_T converges towards the value 2 when T tends to 0.

0.8.1 Estimation of the implied Skew stickiness ratio $\mathcal{R}_T^{\mathcal{Q}}$

Our work aims to study the indicator R_T and investigate the information it delivers on the spot/implied volatility joint dynamics and its relation with the implied skew. In our approach, we:

- specify a parametrization of the implied volatility smile for near the money strikes:

$$\forall K, |K - S_t| \ll S_t, \sigma_{BS,t}(K, T) = \sigma_{ATM,t}(T) + \mathcal{S}_T \log(\frac{K}{S_t}) + \mathcal{C}_T \log(\frac{K}{S_t})^2 + o\left(\log(\frac{K}{S_t})^2\right),$$

- the spot/at-the-money implied volatility joint dynamics:

$$d\sigma_{ATM,t}(T) = \alpha_t dt + R_T \mathcal{S}_T d \log(S_t) + \eta dZ_t, .$$

  where $d \left\langle Z, \log(S) \right\rangle_t = 0$.

Using the pricing PDE satisfied by the option under $\mathcal{Q}$ in our model, we propose a method which enables to estimate, at a given date t, the value of the quantity R_T under $\mathcal{Q}$. Indeed, we show that R_T is a solution of a second-order equation of the form:

$$\frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} (R_T^{\mathcal{Q}})^2 \mathcal{S}_T^2 d \left\langle \log(S) \right\rangle_t + R_T^{\mathcal{Q}} \left(\frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} S_t \mathcal{S}_T d \left\langle \log(S) \right\rangle_t - \vartheta_t \mathcal{S}_T E^{\mathcal{Q}}(d \log(S_t)) \right)$$
$$- \left(\Gamma_t - \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} \eta^2 \right) dt = 0.$$

where C^L is the price of an option with strike K_L and maturity T, C^H is the price of an option with strike K_H and maturity T, and Γ_t is known explicitly. The estimated value is denoted $R_T^{\mathcal{Q}}$ where $\mathcal{Q}$ denotes the risk-neutral probability measure used for option pricing.

In the second part, we introduce $R_T^{\mathcal{P}}$ as the measure of the quantity R_T under $\mathcal{P}$ and we effectuate an empirical study in order to compare the evolution of $R_T^{\mathcal{P}}$ and $R_T^{\mathcal{Q}}$ for several assets. The study shows that, in the limit of small maturities, the quantity $R_T^{\mathcal{P}}$ tends to be lower than the quantity $R_T^{\mathcal{Q}}$. In order to take profit from this market inefficiency, we propose a trading strategy that aims to monetize the spread $\left(R_T^{\mathcal{Q}} - R_T^{\mathcal{P}}\right)$ and we test the efficiency of this strategy on real data.

0.8.2 Limit of the Skew Stickiness Ratio for short maturities

We perform an empirical study, and we notice that the quantity R_T^Q can exceed the value 2 in the limit of short maturities. This empirical observation is in discordance with the properties of linear stochastic volatility models, and raises many questions concerning the theoretical explication of such result. It is then useful to find a model that proves that the value of R_T^Q can be higher than 2 when T tends to 0.

The authors in Vargas et al. (2013) suggested an asymmetric Garch model in order to reproduce values of R_T higher than 2 when $T \sim 0$. In this discrete-time model, the logarithmic return of the underlying between the dates t_i and t_{i+1} can be written as follows:

$$\begin{aligned} r_i &= \log\left(\frac{S_{i+1}}{S_i}\right) = \sigma_i \epsilon_i, \\ \sigma_{i+1}^2 &= \bar{\sigma}^2 + a(\sigma_i^2 - \bar{\sigma}^2) + \nu \sigma_i^2 f(\epsilon_i). \end{aligned}$$

Relatively to this model, the quantity R_T can be written in the following way:

$$R_T = R_{T,Linear} \frac{\frac{Skewness_T}{6\sqrt{T}}}{Skew_T}$$

where the quantity $R_{T,Linear}$ denotes the Skew stickiness ratio relative to linear stochastic volatility models and $Skewness_T$ represents the skewness of the logarithmic return $\log\left(\frac{S_T}{S_0}\right)$ on a period of length T. Consequently, the indicator R_T can have a value higher than 2 for a maturity $T \sim 0$ if $\frac{Skewness_T}{6\sqrt{T}}$ is superior to $Skew_T$.

If the function f is non-linear, the asymmetric Garch model can reproduce the desired effect for a maturity T of the order of several days. However, we noticed that for a maturity equal to 3 Months, and for different choices of the function f, the ratio $\frac{\frac{Skewness_T}{6\sqrt{T}}}{Skew_T}$ stabilizes at the value 1 which makes the quantity R_T equal to $R_{T,Linear}$. Therefore, the Garch model can not provide a theoretical justification of the observation $R_T^Q > 2$ when $T = 3M$.

In order to justify this result, we propose a jump model for the underlying dynamics, in which the instantaneous volatility is stochastic:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - \lambda k)dt + \sigma(y_t)\sqrt{1-\rho_t^2}dW_t^{(1)} + \sigma(y_t)\rho_t dW_t^{(2)} + (J_t - 1)dN_t, \\ dy_t &= b(y_t)dW_t^{(2)} \end{aligned}$$

where N_t is a Poisson process with intensity λ, r is the risk-free rate, J_t is a positive random variable and $k = E(J_t - 1)$. In this model, we prove that the Skew stickiness ratio may attain values higher than 2, due to the presence of jumps. Indeed, under the hypothesis $E((J-1)^3) = E((J-1)^4)$, we have:

$$\lim_{T\to 0} R_T = 2 + \frac{\lambda}{\sigma^2(y_t)}\left(\frac{E((J-1)^3)}{2} - E((J-1)^2)\right),$$

Chapter 1

Effect of the stock/index joint dynamics on the relation between their implied volatility surfaces

Note : A part of this chapter is accepted in the proceedings of the 8th Kolkata Econophysics conference.

1.1 Introduction

The capital asset pricing model is an econometric model that provides an estimation of an asset's return in function of the systematic risk that is expressed in the market risk. The latter model is an expansion of an earlier work of Markowitz on portfolio construction (see Markowitz (1952)). The CAPM model was considered to be an original and innovative model because it introduced the concept of systematic and specific risk. The parameter β, which is a key parameter in this model, enables to separate the stock risk into two parts: the first part represents the systematic risk contained in the market index, while the second part is the idiosyncratic risk that reflects the specific performance of the stock. The parameter beta in the capital asset pricing model is very useful for portfolio construction purposes (see Weston (1973), Blume and Friend (1973), Reilly and Brown (2011)), thus its estimation is a matter of interest and has been a subject of study for several authors in the last decades. This parameter was traditionally estimated using historical data of daily returns of the stock and the market index where it is obtained as the slope of the linear regression of stock returns on market index returns (see Galagedera (2007)). This approach is backward-looking as it estimates the realized value of the parameter in the past using historical data. The authors in Abdymomunov and Morley (2011) and Andersen et al. (2006) showed that the parameter β is not constant but rather time-varying. Thus, the value of the realized beta in the future can be remarkably different from its value in the past, and the backward-looking estimation may be inefficient.

In the recent literature, different authors have focused on the estimation of the beta coefficient using option data, which provides a different estimation method for this parameter. Indeed, whereas classical methods allow an historical estimation of this parameter, the option based estimation method enables to obtain a "forward looking" measure of this parameter. Thus, the obtained estimator represents the information contained in derivatives prices and then summarizes the expectation of market participants for the forward realization of this parameter.

In Christoffersen et al. (2008), Christoffersen, Jacobs and Vainberg provided an estimation of this parameter using the risk-neutral variance and skewness of the stock and the index. More recently, Fouque and Kollman proposed in Fouque and Kollman (2011) a continuous-time CAPM model in which the market index has a stochastic volatility driven by a fast mean-reverting process. Using a perturbation method, they managed to obtain an approximation of the beta parameter depending on the skews of implied volatilities of both the stock and the index. Fouque and Tashman introduced in Fouque and Tashman (2012) a "Stressed-Beta model" in which the parameter β can take two values depending on the market regime. Using this model, Fouque et *al* provided a method to price options on the index and the stock. This method enables also to estimate the parameter β based on options data. In Carr and Madan (2000), Carr and Madan used the CAPM model to price options on the stock when options on the index are liquid. Their approach didn't aim to estimate the parameter beta using option prices, but to price options on the stock given the parameter beta and options prices on the market index.

In this chapter, we deal with the estimation of the coefficient beta under the risk-neutral measure using options prices and we highlight the utility of the obtained estimator for diverse applications. This chapter is organized as follows. In the second section, we present the capital asset pricing model where the stock has a constant idiosyncratic volatility. We recall briefly the method presented in Fouque and Kollman (2011), which allows to estimate the parameter beta using implied volatility data. In the third section, we consider a new model in which the stock's idiosyncratic volatility is stochastic, this choice was motivated by several empirical studies which confirm the random character of the idiosyncratic volatility process. Consequently, the new model is more adapted to describe the joint dynamics of the stock and the market index. In the setting of the new model, we provide approximations for European option prices on both the stock and the index through the use of a perturbation technique. Afterward, we deduce an estimator of the parameter beta using option and underlying prices, and we call this estimator the implied beta. In the fourth section, we present some possible applications which emphasize the utility of the beta estimator. First, we investigate the capacity of the implied beta to predict the forward realized beta under the historic probability measure. We notice then that there are temporal discrepancies between the implied beta and the forward realized one. For this reason, we suggest an arbitrage strategy that aims to monetize the difference between the implied beta and the forward realized beta. Following that, we show in a second application that the parameter beta can be used for the purpose of hedging stock options using instruments on the index. We run Monte-Carlo simulations to illustrate the theoretical study and test numerically the hedging methods.

1.2 Model with constant idiosyncratic volatility

1.2.1 Presentation of the model

Consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ satisfies the usual conditions, and where $\mathbb{P}$ is the objective probability measure. The authors in Fouque and Kollman (2011) proposed a continuous time capital asset pricing model where the market index has a stochastic volatility. Under the probability measure P, the dynamics of the stock S and the index I are described as follows:

$$
\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + \sigma dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)},
\end{aligned}
$$

where $W_t^{(3)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(4)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(4)} \end{pmatrix}$ is a Wiener process under $\mathbb{P}$

The authors made the assumption that $0 < \epsilon \ll 1$ which implies that the process (Y), which drives the index volatility, is a fast mean-reverting Ornstein-Uhlenbeck process.

Let $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{\sigma} \\ \gamma(Y_t) \end{pmatrix}$ where the function γ denotes the volatility risk-premium. The probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, is defined in the following way:

$$\frac{dP^*}{dP}_{|\mathcal{F}_t} = exp\left(-\int_0^t \lambda'_u dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du\right),$$

Let $W^* = \begin{pmatrix} W^{*,(1)} \\ W^{*,(2)} \\ W^{*,(4)} \end{pmatrix}$ such that: $W_t^* = W_t + \int_0^t \lambda_u du$. Using Girsanov's theorem, it follows that W^* is a $(\mathbb{P}^*, \{\mathcal{F}_t\})$ Brownian motion.

Under continuity and boundedness conditions on the function γ, $\mathbb{P}^*$ is a risk-neutral probability measure under which the index and the stock have the following dynamics:

$$\begin{aligned} \frac{dI_t}{I_t} &= rdt + f(Y_t) dW_t^{*,(1)}, \\ \frac{dS_t}{S_t} &= rdt + \beta f(Y_t) dW_t^{*,(1)} + \sigma dW_t^{*,(2)}, \\ dY_t &= (\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\chi(Y_t))dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)}, \end{aligned}$$

where $\chi(Y_t) = \rho \frac{\mu_I - r}{f(Y_t)} + \sqrt{1-\rho^2}\gamma(Y_t)$ and $W_t^{*,(3)} = \rho W_t^{*,(1)} + \sqrt{1-\rho^2} W_t^{*,(4)}$.

1.2.2 Calibration of implied beta

Using a singular perturbation method with respect to the small parameter ϵ, the authors in Fouque and Kollman (2011) obtained an approximation $\tilde{P}^{I,\epsilon}(K_I, T)$ for the price of an European call on the index with strike K_I and maturity T, and an approximation $\tilde{P}^{S,\epsilon}(K_S, T)$ for the price of an European call on the stock with strike K_S and maturity T. Afterward, through the use of a Taylor expansion in $\sqrt{\epsilon}$ for the implied volatility of the stock and the index, they provided an approximation for the implied volatilities $\Sigma_I(K_I, T)$ and $\Sigma_S(K_S, T)$ of the index and the stock respectively:

$$\begin{aligned} \Sigma_I(K_I, T) &= b_I + a_I \frac{\log(\frac{F_I}{K_I})}{T}, \\ \Sigma_S(K_S, T) &= b_S + a_S \frac{\log(\frac{F_S}{K_S})}{T}. \end{aligned}$$

It should be precised that he quantities F_I and F_S denote the forward prices for maturity T of the index and the stock respectively, while the quantities b_I, a_I, b_S, a_S are functions of the model parameters. Thus, the parameter β can be approximated by $\hat{\beta}$ which is defined as :

$$\hat{\beta} = (\frac{a_S}{a_I})^{\frac{1}{3}} \frac{b_S}{b_I}. \tag{1.1}$$

1.2.3 Limits of the model

In the model described so far, the stock's idiosyncratic volatility is supposed to be constant and equal to σ. This hypothesis can be considered too strong, indeed the authors in Xu and Malkiel (2003), Campbell et al. (2001), Kearney and Potì (2008), Wei and Zhang (2006) conducted empirical studies on the idiosyncratic volatility and gave empirical evidence of the randomness of this process.

In order to have an idea about the magnitude of fluctuations of the idiosyncratic volatility, the graph of the parameter σ of the Financial Select Sector (named XLF) when projected on the SPX index is given below. The considered period ranges from 01/01/2008 to 31/12/2012. The parameter σ is obtained through the computation of the standard deviation of errors in the linear regression of the daily returns of the Financial Select Sector (XLF) on the daily returns of the SPX index using a sliding window of 1 month.

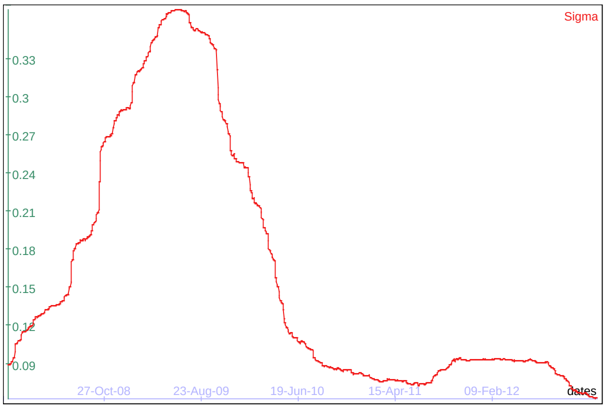


Figure 1.1: Evolution of the idiosyncratic volatility of XLF with respect to the SPX index

The inspection of the graph above shows that the assumption of constant idiosyncratic volatility can be strong, and thus can induce a misleading understanding of the joint dynamics of the stock and the index. Consequently, a new model will be introduced in the next section in order to account for this characteristic.

1.3 Model with stochastic idiosyncratic volatility

A new continuous-time capital asset pricing model is presented here. In this model setting, the stock's idiosyncratic volatility is driven by a fast mean-reverting Ornstein-Uhlenbeck process. The aim of this section is to derive approximations for European option prices on both the stock and the index, and to provide an estimator of the parameter β using option prices.

1.3.1 Presentation of the model

Under the historic probability measure $\mathbb{P}$, the stock and the index have the the following dynamics:

$$
\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f_1(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + f_2(Z_t) dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt + \frac{\nu_Y \sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)}, \\
dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt + \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{(4)},
\end{aligned}
$$

where $W_t^{(3)} = \rho_Y W_t^{(1)} + \sqrt{1-\rho_Y^2} W_t^{(5)}$, $W_t^{(4)} = \rho_Z W_t^{(2)} + \sqrt{1-\rho_Z^2} W_t^{(6)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(5)} \\ W^{(6)} \end{pmatrix}$ is a $(\mathbb{P}, \{\mathcal{F}_t\})$ Wiener process. It is supposed that $0 < \epsilon \ll 1$ and $0 < \frac{\epsilon}{\alpha} \ll 1$. This hypothesis implies that the processes (Y) and (Z) are fast mean-reverting Ornstein-Uhlenbeck processes.

Let the process λ be defined as $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f_1(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} \\ \gamma_1(Y_t) \\ \gamma_2(Z_t) \end{pmatrix}$ where the functions γ_1 and γ_2 denote the volatility risk premiums related to the processes Y and Z respectively. The probability measure P^*, equivalent to P, can then be defined as follows:

$$
\frac{dP^*}{dP} = exp\left(- \int_0^t \lambda_u' dW_u - \frac{1}{2} \int_0^t |\lambda_u|^2 du \right).
$$

Let $W_t^* = W_t + \int_0^t \lambda_u du$. It can be deduced, through the use of Girsanov's theorem, that W^* is a $(\mathbb{P}^*, \{\mathcal{F}_t\})$ Brownian motion. Thus, the dynamics of the stock and the index, and under $\mathbb{P}^*$, can be described in the following way:

$$
\begin{aligned}
\frac{dI_t}{I_t} &= r dt + f_1(Y_t) dW_t^{*,(1)}, \\
\frac{dS_t}{S_t} &= r dt + \beta f_1(Y_t) dW_t^{*,(1)} + f_2(Z_t) dW_t^{*,(2)}, \\
dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt - \frac{\nu_Y \sqrt{2}}{\sqrt{\epsilon}} \chi_1(Y_t) dt + \frac{\nu_Y \sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)}, \\
dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt - \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\epsilon}} \chi_2(Z_t) dt + \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{*,(4)},
\end{aligned}
$$

where: $\chi_1(Y_t) = \rho_Y \frac{\mu_I - r}{f_1(Y_t)} + \sqrt{1-\rho_Y^2} \gamma_1(Y_t)$ and $\chi_2(Z_t) = \rho_Z \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} + \sqrt{1-\rho_Z^2} \gamma_2(Z_t)$.

The processes $W^{*,(3)}$ and $W^{*,(4)}$ are Brownian motions under $\mathbb{P}^*$ such that:

$$
\begin{aligned}
W^{*,(3)} &= \rho_Y W^{*,(1)} + \sqrt{1-\rho_Y^2} W^{*,(5)}, \\
W^{*,(4)} &= \rho_Z W^{*,(2)} + \sqrt{1-\rho_Z^2} W^{*,(6)}.
\end{aligned}
$$

1.3.2 Pricing options on the index and the stock

Approximation formula for index option price

Let $P^{I,\epsilon}(K_I, T) = E^{P^*}(e^{-r(T-t)}(I_T - K_I)^+ | \mathcal{F}_t)$ be the price of an European call on the index with strike K_I and maturity T. The processes (I) and (Y) have the same dynamics as in the model with constant idiosyncratic volatility, which implies that the pricing of index options remains the same.

By means of a perturbation method with respect to the parameter ϵ, the authors obtained in Fouque and Kollman (2011) an approximation $\tilde{P}^{I,\epsilon}(K_I,T)$ for the price $P^{I,\epsilon}(K_I,T)$ at order 1 in ϵ. The proof of the approximation result is given in Fouque and Kollman (2011) and also in (1.6.1) for completeness. The results are recalled here:

$$\tilde{P}^{I,\epsilon} = \tilde{P}_0^{I,\epsilon} - (T-t)V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 \tilde{P}_0^{I,\epsilon}}{\partial I_t^2}), \tag{1.2}$$

where the quantities $\tilde{P}_0^{I,\epsilon}$ and $V_3^{I,\epsilon}$ are defined as:

$$\tilde{P}_0^{I,\epsilon} = P_{BS}^I(\bar{\sigma}_I^*), \tag{1.3}$$

$$V_2^{I,\epsilon} = -\frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_Y \left\langle \phi_I' \chi_1 \right\rangle_1, \tag{1.4}$$

$$V_3^{I,\epsilon} = \frac{\sqrt{\epsilon}}{\sqrt{2}}\rho_Y \nu_Y \left\langle \phi_I' f_1 \right\rangle_1, \tag{1.5}$$

$$(\bar{\sigma}_I^*)^2 = \left\langle f_1^2 \right\rangle_1 - 2V_2^{I,\epsilon}. \tag{1.6}$$

The operator $\langle . \rangle_1$ denotes the average with respect to the invariant distribution of the Ornstein-Uhlenbeck process (Y_1) whose dynamics are described as follows:

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t^{(3)}.$$

Besides, the function ϕ_I is defined as the solution of the following Poisson equation:

$$\mathcal{L}_0^I \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_1, \tag{1.7}$$

where $\mathcal{L}_0^I$ is the infinitesimal generator of the process (Y_1) and can be written:

$$\mathcal{L}_0^I = \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2}.$$

Approximation formula for stock option price

Let $P_t^{S,\epsilon}(K_S,T)$ be the price, at time t, of an European call on the stock with strike K_S and maturity T:

$$P_t^{S,\epsilon}(K_S,T) = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | \mathcal{F}_t).$$

The notation $P_t^{S,\epsilon}$ is used instead of $P_t^{S,\epsilon}(K_S,T)$ for simplification purposes.
Using a perturbation technique on the parameter ϵ, an approximation $\tilde{P}^{S,\epsilon}$ for the option's price $P_t^{S,\epsilon}$ is obtained. The approximation error is at order 1 in ϵ. The results can be detailed as follows:

Proposition 1.1.

$$\tilde{P}^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} - (T-t)V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\epsilon}}{\partial S_t^2}), \tag{1.8}$$

where the quantities $\tilde{P}_0^{S,\epsilon}$ *and* $V_3^{S,\epsilon}$ *are defined as:*

$$\tilde{P}_0^{S,\epsilon} = P_{BS}^S(t, S_t, \bar{\sigma}_S^*), \tag{1.9}$$

$$(\bar{\sigma}_S^*)^2 = \bar{\sigma}_S^2 - 2V_2^{S,\epsilon}, \tag{1.10}$$

$$V_2^{S,\epsilon} = -\frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^2 \nu_Y \left\langle \phi_I' \chi_1 \right\rangle_{1,2} + \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' \chi_2 \right\rangle_{1,2}), \tag{1.11}$$

$$V_3^{S,\epsilon} = \frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^3 \rho_Y \nu_Y \left\langle \phi_I' f_1 \right\rangle_{1,2} + \rho_Z \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' f_2 \right\rangle_{1,2}). \tag{1.12}$$

The operator $\langle . \rangle_{1,2}$ denotes the averaging with respect to the invariant distribution of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_t$ which has the following dynamics:

$$d\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \left(\begin{pmatrix} m_Y \\ m_Z \end{pmatrix} - \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} \right) dt + \sqrt{2} \begin{pmatrix} \nu_Y & 0 \\ 0 & \sqrt{\alpha}\nu_Z \end{pmatrix} d \begin{pmatrix} W_t^{(3)} \\ W_t^{(4)} \end{pmatrix}.$$

The function ϕ_{Idios} is the solution of the equation:

$$\mathcal{L}_0^S \phi_{Idios}(z) = f_2^2(z) - \langle f_2^2 \rangle_{1,2}.$$

where $\mathcal{L}_0^S$ is the infinitesimal generator of the two-dimensional Ornstein-Uhlenbeck process $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.

Proof. The price of an European call on the stock writes:

$$P_t^{S,\epsilon} = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | S_t = x, Y_t = y, Z_t = z).$$

Since the process (S, Y, Z) is markovian, the Feynman-Kac theorem (see Pham (2007)) can be used to prove that $\mathcal{L}^S P_t^{S,\epsilon} = 0$ where $\mathcal{L}^S$ is a differential operator expanded in powers of $\sqrt{\epsilon}$:

$$\mathcal{L}^S = \mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S,$$

and $\mathcal{L}_0^S, \mathcal{L}_1^S, \mathcal{L}_2^S$ are defined as follows:

$$\begin{aligned}
\mathcal{L}_0^S &= (m_Y - y)\frac{\partial}{\partial y} + \nu_Y^2 \frac{\partial^2}{\partial y^2} + \alpha(m_Z - z)\frac{\partial}{\partial z} + \alpha\nu_Z^2 \frac{\partial^2}{\partial z^2}, \\
\mathcal{L}_1^S &= -\nu_Y\sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \beta S_t f_1(y)\sqrt{2}\rho_Y\nu_Y \frac{\partial^2}{\partial S \partial y} - \nu_Z\sqrt{2\alpha}\chi_2(z)\frac{\partial}{\partial z} + S_t f_2(z)\sqrt{2\alpha}\rho_Z\nu_Z \frac{\partial^2}{\partial S \partial z}, \\
\mathcal{L}_2^S &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S}S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}S_t^2(\beta^2 f_1(y)^2 + f_2(z)^2).
\end{aligned}$$

The differential operator $\mathcal{L}_0^S$ is the infinitesimal generator of the two-dimensional Ornstein-Uhlenbeck process $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ which has the following dynamics:

$$d\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \left(\begin{pmatrix} m_Y \\ m_Z \end{pmatrix} - \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} \right) dt + \sqrt{2} \begin{pmatrix} \nu_Y & 0 \\ 0 & \sqrt{\alpha}\nu_Z \end{pmatrix} d \begin{pmatrix} W_t^{(3)} \\ W_t^{(4)} \end{pmatrix}.$$

The following notations are used:

- The operator $\langle . \rangle_1$ denotes the averaging with respect to the invariant distribution of the process $(Y_{1,t})_t$.
- The operator $\langle . \rangle_2$ denotes the averaging with respect to the invariant distribution of the process $(Y_{2,t})_t$.
- The operator $\langle . \rangle_{1,2}$ denotes the averaging with respect to the invariant distribution of $\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}_t$.

The option price $P^{S,\epsilon}$ is expanded in powers of $\sqrt{\epsilon}$:

$$P^{S,\epsilon} = \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i P_i^{S,\epsilon},$$

then, the expression $\mathcal{L}^S P_t^{S,\epsilon}$ can be written as follows:

$$(\mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S)(\sum_{i=0}^{\infty}(\sqrt{\epsilon})^i P_i^{S,\epsilon}) \quad = \quad 0.$$

By classifying the terms of the last equation by powers of $\sqrt{\epsilon}$, the terms of orders $(-2),(-1),0,1,2$ in $\sqrt{\epsilon}$ are obtained:

$$(-2) \quad : \quad \mathcal{L}_0^S P_0^{S,\epsilon} = 0, \tag{1.13}$$
$$(-1) \quad : \quad \mathcal{L}_1^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_1^{S,\epsilon} = 0, \tag{1.14}$$
$$(0) \quad : \quad \mathcal{L}_2^S P_0^{S,\epsilon} + \mathcal{L}_1^S P_1^{S,\epsilon} + \mathcal{L}_0^S P_2^{S,\epsilon} = 0, \tag{1.15}$$
$$(1) \quad : \quad \mathcal{L}_2^S P_1^{S,\epsilon} + \mathcal{L}_1^S P_2^{S,\epsilon} + \mathcal{L}_0^S P_3^{S,\epsilon} = 0, \tag{1.16}$$
$$(2) \quad : \quad \mathcal{L}_2^S P_2^{S,\epsilon} + \mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_0^S P_4^{S,\epsilon} = 0. \tag{1.17}$$

The term of order (-2) in $\sqrt{\epsilon}$ states that $\mathcal{L}_0^S P_0^{S,\epsilon} = 0$. Since the operator $\mathcal{L}_0^S$ contains only derivatives with respect to y and z, this equation can be solved by choosing $P_0^{S,\epsilon} = P_0^{S,\epsilon}(t,S_t)$ independent of Y_t and Z_t.

The term of order (-1) in $\sqrt{\epsilon}$ states that $\mathcal{L}_1^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_1^{S,\epsilon} = 0$. The differential operator $\mathcal{L}_1^S$ contains first and second order derivatives with respect to y and z, thus $\mathcal{L}_1^S P_0^{S,\epsilon} = 0$. The equation becomes then $\mathcal{L}_0^S P_1^{S,\epsilon} = 0$. The equation is satisfied for $P_1^{S,\epsilon} = P_1^{S,\epsilon}(t,S_t)$ independent of Y_t and Z_t.

The quantities $P_0^{S,\epsilon}$ and $P_1^{S,\epsilon}$ being independent of Y_t and Z_t, it can be stated that:

$$\mathcal{L}_0^S P_0^{S,\epsilon} = \mathcal{L}_1^S P_0^{S,\epsilon} = \mathcal{L}_0^S P_1^{S,\epsilon} = \mathcal{L}_1^S P_1^{S,\epsilon} = 0.$$

Since $\mathcal{L}_1^S P_1^{S,\epsilon} = 0$, the term of order 0 in $\sqrt{\epsilon}$ becomes:

$$\mathcal{L}_2^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_2^{S,\epsilon} = 0,$$

which is a Poisson equation for $P_2^{S,\epsilon}$ with respect to $\mathcal{L}_0^S$. The solvability condition for this equation is:

$$\left\langle \mathcal{L}_2^S P_0^{S,\epsilon} \right\rangle_{1,2} = \left\langle \mathcal{L}_2^S \right\rangle_{1,2} P_0^{S,\epsilon} = 0.$$

The average $\left\langle \mathcal{L}_2^S \right\rangle_{1,2}$ of the generator $\mathcal{L}_2^S$ verifies:

$$\left\langle \mathcal{L}_2^S \right\rangle_{1,2} \quad = \quad \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S_t}S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}S_t^2 \left\langle \beta^2 f_1^2(y) + f_2^2(z) \right\rangle_{1,2}.$$

It can be deduced that $\left\langle \mathcal{L}_2^S \right\rangle_{1,2} = \mathcal{L}_{BS}^S(\bar{\sigma}_S)$ where $\bar{\sigma}_S^2 = \beta^2 \left\langle f_1^2 \right\rangle_{1,2} + \left\langle f_2^2 \right\rangle_{1,2}$.

Consequently, $P_0^{S,\epsilon}$ is the solution of the following problem:

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma}_S)P_0^{S,\epsilon} &= 0, \\ P_0^{S,\epsilon}(T,S_T) &= h(S_T). \end{aligned}$$

It can be easily seen that the quantity $P_0^{S,\epsilon}$ represents the Black-Scholes price of the option with implied volatility equal to $\bar{\sigma}_S$:

$$P_0^{S,\epsilon} = P_{BS}^S(t,S_t,\bar{\sigma}_S).$$

The term of order 1 in $\sqrt{\epsilon}$ is a Poisson equation for $P_3^{S,\epsilon}$ with respect to $\mathcal{L}_0^S$, whose solvability condition is:

$$\left\langle \mathcal{L}_2^S \right\rangle_{1,2} P_1^{S,\epsilon} = -\left\langle \mathcal{L}_1^S P_2^{S,\epsilon} \right\rangle_{1,2} = \left\langle \mathcal{L}_1^S (\mathcal{L}_0^S)^{-1} (\mathcal{L}_2^S - \left\langle \mathcal{L}_2^S \right\rangle_{1,2}) \right\rangle_{1,2} P_0^{S,\epsilon}. \tag{1.18}$$

The quantity $P_1^{S,\epsilon}$ is the solution of the last equation with terminal condition $P_1^{S,\epsilon}(T, S_T) = 0$.

The function f_1 is independent of z and the function f_2 is independent of y, it follows that: $\left\langle f_1^2 \right\rangle_{1,2} = \left\langle f_1^2 \right\rangle_1$ and $\left\langle f_2^2 \right\rangle_{1,2} = \left\langle f_2^2 \right\rangle_2$. In addition, the function ϕ_I which is the solution of (1.7), doesn't depend on z. This implies:

$$\mathcal{L}_0^S \phi_I(y) = \mathcal{L}_0^I \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_{1,2}.$$

Let the function ϕ_{Idios} be the solution of the following equation:

$$\mathcal{L}_0^S \phi_{Idios}(z) = f_2^2(z) - \left\langle f_2^2 \right\rangle_{1,2}. \tag{1.19}$$

The function ϕ_{Idios} doesn't depend on y, thus it can be deduced:

$$\mathcal{L}_0^S(\beta^2 \phi_I(y) + \phi_{Idios}(z)) = \beta^2 (f_1^2(y) - \left\langle f_1^2 \right\rangle_{1,2}) + (f_2^2(z) - \left\langle f_2^2 \right\rangle_{1,2}).$$

Building on this, it can be obtained:

$$\mathcal{L}_1^S (\mathcal{L}_0^S)^{-1} (\mathcal{L}_2^S - \left\langle \mathcal{L}_2^S \right\rangle_{1,2}) = (\beta^2 \mathcal{L}_1^S \phi_I(y) + \mathcal{L}_1^S \phi_{Idios}(z)) \frac{1}{2} S_t^2 \frac{\partial^2}{\partial S_t^2}.$$

The development of the right-hand side of the previous equation yields:

$$\begin{aligned} \frac{1}{2}(\beta^2 \left\langle \mathcal{L}_1^S \phi_I(y) \right\rangle_{1,2} + \left\langle \mathcal{L}_1^S \phi_{Idios}(z) \right\rangle_{1,2}) = {} & (\frac{\beta^3 \nu_Y \rho_Y}{\sqrt{2}} \left\langle \phi_I' f_1 \right\rangle_{1,2} + \frac{\rho_Z \nu_Z \sqrt{\alpha}}{\sqrt{2}} \left\langle \phi_{Idios}' f_2 \right\rangle_{1,2}) S_t \frac{\partial}{\partial S_t} \\ & - (\frac{\beta^2 \nu_Y}{\sqrt{2}} \left\langle \phi_I' \chi_1 \right\rangle_{1,2} + \frac{\nu_Z \sqrt{\alpha}}{\sqrt{2}} \left\langle \phi_{Idios}' \chi_2 \right\rangle_{1,2}). \end{aligned}$$

Let the quantities $V_2^{S,\epsilon}$ and $V_3^{S,\epsilon}$ be defined as follows:

$$\begin{aligned} V_3^{S,\epsilon} &= \frac{\sqrt{\epsilon}}{\sqrt{2}} (\beta^3 \nu_Y \rho_Y \left\langle \phi_I' f_1 \right\rangle_{1,2} + \rho_Z \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' f_2 \right\rangle_{1,2}), \\ V_2^{S,\epsilon} &= -\frac{\sqrt{\epsilon}}{\sqrt{2}} (\beta^2 \nu_Y \left\langle \phi_I' \chi_1 \right\rangle_{1,2} + \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' \chi_2 \right\rangle_{1,2}). \end{aligned}$$

The equation (1.18) becomes:

$$\left\langle \mathcal{L}_2^S \right\rangle_{1,2} \sqrt{\epsilon} P_1^{S,\epsilon} = V_2^{S,\epsilon} S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2} + V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t} (S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}). \tag{1.20}$$

Consequently, $P_1^{S,\epsilon}$ is the solution of (1.20) with the final condition $P_1^{S,\epsilon}(T, S_T) = 0$. Since the differential operator $\left\langle \mathcal{L}_2^S \right\rangle_{1,2} = \mathcal{L}_{BS}^S(\bar{\sigma}_S)$ commits with the operators $\mathcal{D}_{1,S} = S_t \frac{\partial}{\partial S_t}$ and $\mathcal{D}_{2,S} = S_t^2 \frac{\partial^2}{\partial S_t^2}$, and that $\left\langle \mathcal{L}_2^S \right\rangle_{1,2} P_0^{S,\epsilon} = 0$, the solution to the last problem can be given explicitly as:

$$\sqrt{\epsilon} P_1^{S,\epsilon} = -(T-t)(V_2^{S,\epsilon} S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2} + V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t} (S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2})). \tag{1.21}$$

In order to check the validity of the solution, the following verification can be made:

$$\begin{aligned} \left\langle \mathcal{L}_2^S \right\rangle_{1,2} \sqrt{\epsilon} P_1^{S,\epsilon} &= (V_2^{S,\epsilon} \mathcal{D}_{2,S} P_0^{S,\epsilon} + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} P_0^{S,\epsilon}) \left\langle \mathcal{L}_2^S \right\rangle_{1,2} ((t-T)) \\ &\quad - (T-t)(V_2^{S,\epsilon} \mathcal{D}_{2,S} \left\langle \mathcal{L}_2^S \right\rangle_{1,2} (P_0^{S,\epsilon}) + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} \left\langle \mathcal{L}_2^S \right\rangle_{1,2} (P_0^{S,\epsilon})) \\ &= V_2^{S,\epsilon} \mathcal{D}_{2,S} P_0^{S,\epsilon} + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} P_0^{S,\epsilon}. \end{aligned}$$

By neglecting terms of order higher to 1 in $\sqrt{\epsilon}$, the stock option price can be approximated by $(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})$. In order to reduce the number of the parameters in the approximation, a second approximation is derived here. Let $\mathcal{L}_{BS}(\bar{\sigma}_S^*)$ be the Black-Scholes differential operator with volatility $\bar{\sigma}_S^*$:

$$(\bar{\sigma}_S^*)^2 = \bar{\sigma}_S^2 - 2V_2^{S,\epsilon}.$$

Let the quantity $\tilde{P}_0^{S,\epsilon}$ be introduced as the solution of the following problem:

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma}_S^*)\tilde{P}_0 &= 0, \\ \tilde{P}_0^{S,\epsilon}(T, S_T) &= (S_T - K_S)^+. \end{aligned}$$

It follows that $\tilde{P}_0^{S,\epsilon} = P_{BS}(t, S_t, \bar{\sigma}_S^*)$. Following that, the quantity $\tilde{P}_1^{S,\epsilon}$ is defined as the solution of the following problem:

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma}_S^*)\sqrt{\epsilon}\tilde{P}_1^{S,\epsilon} &= V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2}{\partial S_t^2}\tilde{P}_0^{S,\epsilon}), \\ \tilde{P}_1^{S,\epsilon}(T, S_T) &= 0. \end{aligned}$$

Using the same arguments as before, it can be deduced that $\tilde{P}_1^{S,\epsilon} = -(T - t)V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2}{\partial S_t^2}\tilde{P}_0^{S,\epsilon})$.

It can be proved that the option price $P^{S,\epsilon}$ can be approximated up to order 1 in $\sqrt{\epsilon}$ by $\tilde{P}^{S,\epsilon}$ which is defined as:

$$\tilde{P}^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}.$$

The proof of this result is detailed in (1.6.2).

■

1.3.3 Calibration of implied beta using options prices

In the last section, approximations for the prices of European options on the index and the stock are given respectively in (1.2) and (1.8). Using these results, an estimator of the parameter β is provided in this model setting.

Approximation formula of the implied volatility smile

For the purpose of the beta estimation, a Taylor expansion is carried out in (1.6.3) in order to provide approximations of the implied volatilities of the stock and the index. The following results are obtained:

Proposition 1.2. *The implied volatility of an European call on the index with strike K_I and maturity T can be approximated, at order 1 in $\sqrt{\epsilon}$, by $\Sigma_I(K_I, T)$:*

$$\Sigma_I(K_I, T) = b_I + a_I \frac{\log(\frac{F_I}{K_I})}{T}, \tag{1.22}$$

where $b_I = \bar{\sigma}_I^* - \frac{V_3^{I,\epsilon}}{2\bar{\sigma}_I^*}$, $a_I = \frac{V_3^{I,\epsilon}}{(\bar{\sigma}_I^*)^3}$ and $F_I = I_t e^{r(T-t)}$.
Likewise, the implied volatility of an European call on the stock with strike K_S and maturity T can be approximated, at order 1 in $\sqrt{\epsilon}$, by $\Sigma_S(K_S, T)$:

$$\Sigma_S(K, T) = b_S + a_S \frac{\log(\frac{F_S}{K_S})}{T}, \tag{1.23}$$

where $b_S = \bar{\sigma}_S^* - \frac{V_3^{S,\epsilon}}{2\bar{\sigma}_S^*}$, $a_S = \frac{V_3^{S,\epsilon}}{(\bar{\sigma}_S^*)^3}$ and $F_S = S_t e^{r(T-t)}$.

Comparison with the model with constant idiosyncratic volatility

The approximations of the smiles of the stock and the index, given in (1.22) and (1.23), are used here to estimate the parameter β. Indeed, based on the definitions of $V_3^{S,\epsilon}$ and $V_3^{I,\epsilon}$, it can be written that:

$$\begin{aligned}
V_3^{S,\epsilon} &= \beta^3 V_3^{I,\epsilon} + \frac{\sqrt{\epsilon}}{\sqrt{2}}\rho_Z \nu_Z \sqrt{\alpha}\left\langle \phi'_{Idios} f_2 \right\rangle_{1,2}, \\
\frac{V_3^{S,\epsilon}}{V_3^{I,\epsilon}} &= \beta^3 + \frac{\rho_Z \nu_Z \sqrt{\alpha}}{\rho_Y \nu_Y}\frac{\left\langle \phi'_{Idio} f_2 \right\rangle_{1,2}}{\left\langle \phi'_I f_1 \right\rangle_{1,2}}.
\end{aligned}$$

The estimator $\hat{\beta}$ proposed in Fouque and Kollman (2011) and introduced in (1.1) verifies:

$$\hat{\beta}^3 = \frac{V_3^{S,\epsilon}}{V_3^{I,\epsilon}} = \beta^3 + \frac{\rho_Z \nu_Z \sqrt{\alpha}}{\rho_Y \nu_Y}\frac{\left\langle \phi'_{Idio} f_2 \right\rangle_{1,2}}{\left\langle \phi'_I f_1 \right\rangle_{1,2}}.$$

it can be deduced that, in the case of stochastic idiosyncratic volatility, the quantity $\hat{\beta}$ is a biased estimator of the parameter β. Thus, it would be useful to provide an unbiased estimator of β in the new model setting.

Alternative method for the estimation

It can be recalled that:

$$\bar{\sigma}_S^2 = \beta^2 \bar{\sigma}_I^2 + \left\langle f_2^2 \right\rangle_{1,2}.$$

Using the relations between $\bar{\sigma}_I^2$ and $(\bar{\sigma}_I^*)^2$ as well as between $\bar{\sigma}_S^2$ and $(\bar{\sigma}_S^*)^2$, the following result is deduced:

$$\beta^2 = \frac{(\bar{\sigma}_S^*)^2 - \left\langle f_2^2 \right\rangle_{1,2}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\epsilon}} + \frac{2V_2^{S,\epsilon}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\epsilon}}.$$

Based on the smile approximation formula, given in (1.39), for an asset A denoting either the stock S or the index I, it can be stated:

$$b_A = \bar{\sigma}_A^* - \frac{1}{2}a_A(\bar{\sigma}_A^*)^2.$$

The latter second order equation in $\bar{\sigma}_A^*$ has two admissible solutions:

$$\begin{aligned}
x_1 &= \frac{1-\sqrt{1-2a_A b_A}}{a_A}, \\
x_2 &= \frac{1+\sqrt{1-2a_A b_A}}{a_A}.
\end{aligned}$$

Since $V_2^{A,\epsilon}$ and $V_3^{A,\epsilon}$ are of order 1 in $\sqrt{\epsilon}$, then a_A is of order 1 in $\sqrt{\epsilon}$ and $b_A = \bar{\sigma}_A^* + o(\sqrt{\epsilon})$. Thus, it can be deduced that the appropriate solution of the second-order equation is x_1, and that the quantities $\bar{\sigma}_S^*$ and $\bar{\sigma}_I^*$ can be written as below:

$$\bar{\sigma}_S^* = \frac{1-\sqrt{1-2a_S b_S}}{a_S} \quad , \quad \bar{\sigma}_I^* = \frac{1-\sqrt{1-2a_I b_I}}{a_I}.$$

Using previous results, the parameter β can be approximated using $\tilde{\beta}$:

$$\tilde{\beta} = \sqrt{\frac{(\frac{1-\sqrt{1-2a_S b_S}}{a_S})^2 - \left\langle f_2^2 \right\rangle_{1,2}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\epsilon}} + \frac{2V_2^{S,\epsilon}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\epsilon}}}. \tag{1.24}$$

Through the use of a Taylor expansion, it can be shown that $\tilde{\beta}$ writes:

$$\tilde{\beta} \quad = \quad \sqrt{\frac{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}}{(\bar{\sigma}_I^*)^2}} \left(1 + \frac{V_2^{S,\epsilon}}{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}} - \frac{V_2^{I,\epsilon}}{(\bar{\sigma}_I^*)^2} \right) + o(\epsilon)\,.$$

In order to compute the value of $\tilde{\beta}$, the quantities $\langle f_2^2 \rangle_{1,2}$ and $V_2^{S,\epsilon}$ can be estimated statistically using historical data from both option and underlying prices. The estimation procedure will be detailed in the next section.

Numerical simulations

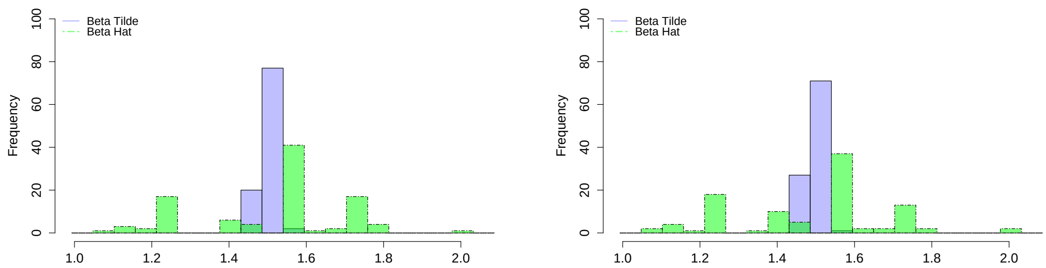
The accuracy of the estimators $\tilde{\beta}$ and $\hat{\beta}$ are tested in this subsection on simulated data. Indeed, Monte Carlo simulations of the CAPM model with stochastic idiosyncratic volatility are performed using the following parameters:

$$\begin{cases} \epsilon = 0.1, & \alpha = 1, \\ \rho_Y = -0.8, & \rho_Z = -0.5, \\ \nu_Y = 0.15, & \beta = 1.5, \\ r = 0, & \mu_S = \mu_I = 0, \\ \gamma_1 = 0, & \gamma_2 = 0, \end{cases}$$

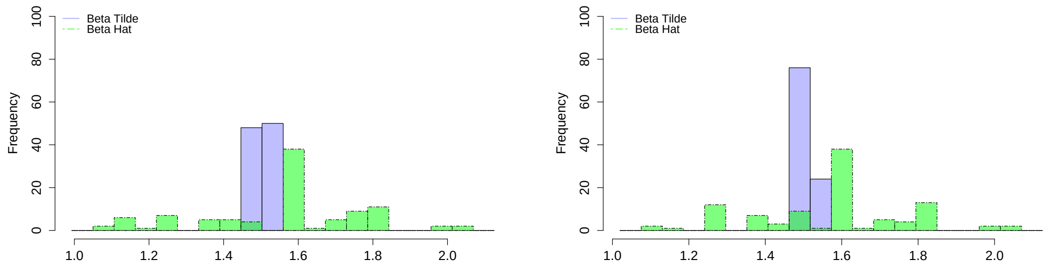
The choice of the parameters $r = \mu_S = \mu_I = 0$ and $\gamma_1 = \gamma_2 = 0$ enables to have $V_2^{S,\epsilon} = V_2^{I,\epsilon} = 0$. Consequently, the parameter $\tilde{\beta}$ in (1.24) becomes:

$$\tilde{\beta} \quad = \quad \sqrt{\frac{(\frac{1-\sqrt{1-2a_S b_S}}{a_S})^2 - \langle f_2^2 \rangle_{1,2}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2}}.$$

Several experiments are carried out here according to the value of the parameter ν_Z. For each value of $\nu_Z \in \{0.05, 0.15, 0.25, 0.4\}$, we launch 100 Monte Carlo pricing algorithms whose random number generators have different seeds. In each of the 100 pricing algorithms, we generate 20000 paths of the Brownian motions $\left(W_t^{*,(1)}, W_t^{*,(2)}, W_t^{*,(3)}, W_t^{*,(4)}\right)_{\{0 \le t \le T\}}$ as well as their antithetic. We deduce afterward the paths of the processes $(I_t, S_t, y_t, z_t)_{\{0 \le t \le T\}}$ and we price, using the Monte Carlo method, options on the stock and the index with different strikes and with maturity $T = 0.5$. Finally, we compute the estimators $\tilde{\beta}$ and $\hat{\beta}$ using (1.1) and (1.24) respectively, and we obtain the following histograms:



(a) $\nu_Z = 0.05$ (b) $\nu_Z = 0.15$



(c) $\nu_Z = 0.25$

(d) $\nu_Z = 0.4$

Figure 1.2: Histogram of the estimators $\tilde{\beta}$ and $\hat{\beta}$ for $\beta = 1.5$

The histograms show that the estimator $\tilde{\beta}$ insures a faster convergence towards the parameter β. Indeed, by using only 40000 antithetic paths in each Monte Carlo pricing algorithm, the bias of the estimator $\tilde{\beta}$ is lower than the one of $\hat{\beta}$.

1.4 Applications for the estimation of the parameter β

In this section, we highlight the importance of the estimation of the parameter β under the risk-neutral measure $\mathbb{P}^*$. We present two main applications which emphasize the utility of this estimation for econometric studies or also for derivatives hedging. In the first subsection, we investigate whether the implied beta $\tilde{\beta}$ can predict the future value of the parameter β under the real-world probability measure $\mathbb{P}$. This question is legitimate to be posed since option-implied information under the risk-neutral pricing measure $\mathbb{P}^*$ may help to predict information under the physical measure $\mathbb{P}$. In the second subsection, we show that the estimator $\tilde{\beta}$ may be crucial for hedging purposes. Indeed, we point out that the quantity $\tilde{\beta}$ can be used in order to hedge the volatility or the delta risk of stock options using instruments on the index. Thus, we conclude that having a good estimation of the value of the parameter β under the risk-neutral measure $\mathbb{P}^*$ can be very useful from a risk-management perspective.

1.4.1 Prediction of forward beta

We use spot and option data of several US stocks and ETF between 01/01/2008 and 31/12/2012. The option prices have a maturity equal to $T = 0.5$ and a moneyness ranging between 80% to 120%. The data sample used in this empirical study includes the following instruments: Financial Select Sector (XLF), Energy Select Sector (XLE), Materials Select Sector (XLB), Technology Select Sector (XLK), Industrial Select Sector (XLI), Goldman Sachs (GS), Microsoft (MSFT), General Electric (GE), Google (GOOG), IBM (IBM), JP Morgan (JPM), Cisco Systems (CSCO), Bank Of America (BAC), Intel Corporation (INTC), Chevron Corporation (CVX), Caterpillar Inc(CAT), Apple Inc (AAPL), Alcoa Inc (AA).

For every date t of the sample, the estimators $\hat{\beta}(t)$ and $\tilde{\beta}(t)$ are computed according to (1.1) and (1.24) respectively. In order to compute $\tilde{\beta}$, the methodology given below is followed:

1. The quantities $\bar{\sigma}_I$ and $\bar{\sigma}_S$ are approximated by the historical volatilities of the market index and the stock respectively using underlying log-returns.

2. The implied volatilities $\Sigma_I(K_I, T)$ and $\Sigma_S(K_S, T)$ are regressed on the variables $\frac{\log(\frac{F_I(T)}{K_I})}{T-t}$ and $\frac{\log(\frac{F_S(T)}{K_S})}{T-t}$ respectively, thus the slopes a_S, a_I and the intercepts b_S, b_I are deduced.

3. From the estimated slope a_S, the intercept b_S and the effective volatility $\bar{\sigma}_S$, the following quantities can be calculated:

   - $\bar{\sigma}_S^* = \frac{1-\sqrt{1-2a_S b_S}}{a_S}$
   - $V_2^{S,\epsilon} = \frac{\bar{\sigma}_S^2-(\bar{\sigma}_S^*)^2}{2}$

4. Likewise, from the estimated slope a_I, the intercept b_I and the effective volatility $\bar{\sigma}_I$, the following quantities are computed :

   - $\bar{\sigma}_I^* = \frac{1-\sqrt{1-2a_I b_I}}{a_I}$
   - $V_2^{I,\epsilon} = \frac{\bar{\sigma}_I^2-(\bar{\sigma}_I^*)^2}{2}$

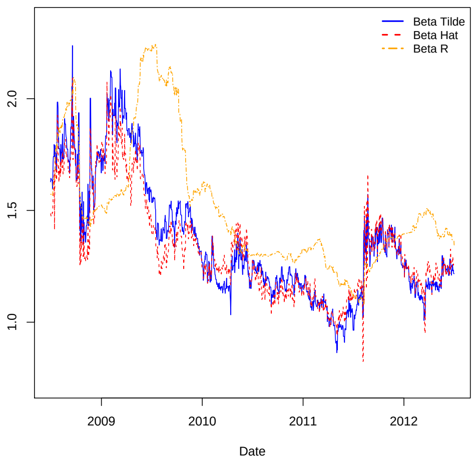
5. The time series of the idiosyncratic volatility $(f(Z_t))_t$ is obtained through the regression of the log-returns of the stock (S) on those of the index (I). Using these data, the parameters m_Z, ν_Z and ρ_Z are calibrated using the maximum likelihood method as suggested in Franco (2003). The quantity $\left\langle f_2^2 \right\rangle_{1,2}$ is then evaluated. For example, if f_2 denotes the exponential function, then:

$$\left\langle f_2^2 \right\rangle_{1,2} \quad = \quad e^{2m_Z+2\nu_Z^2}$$

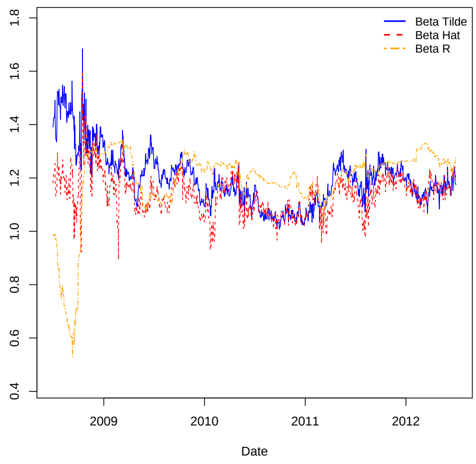
6. The estimator $\tilde{\beta}$ can finally be computed using (1.24).

Let the quantity β_H denote the historical measure of the parameter beta. Thus, at a given date t, $\beta_H(t)$ is defined as the slope of the linear regression of the stock log-returns on the index log-returns between $t-T$ and t. The estimator β_H is computed on a backward window of length T, and so can be compared to the estimators $\hat{\beta}(t)$ and $\tilde{\beta}(t)$ which are estimated using option prices with maturity T.

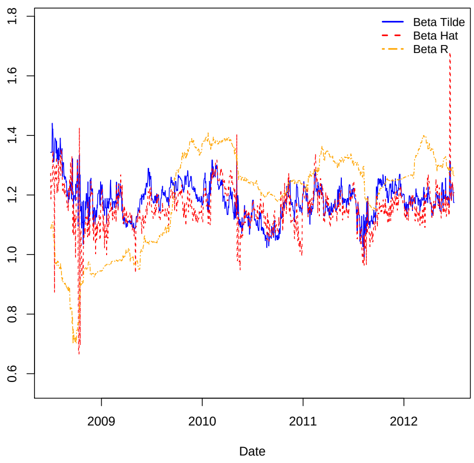
The graphs below represent the time series $(\hat{\beta}(t))_{t_0\leq t\leq t_N}$, $(\tilde{\beta}(t))_{t_0\leq t\leq t_N}$ and $(\beta_H(t))_{t_0\leq t\leq t_N}$ for $t_0 = 01/01/2008$ and $t_N = 31/12/2012$.



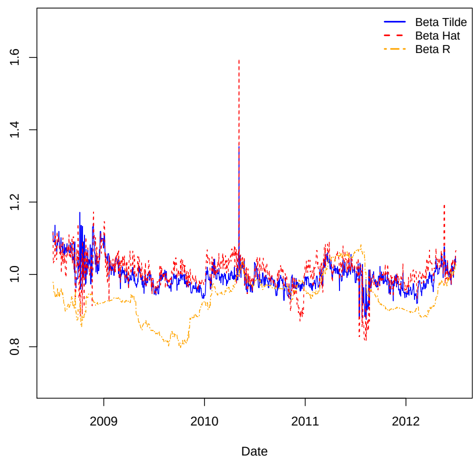
(a) XLF vs SPX



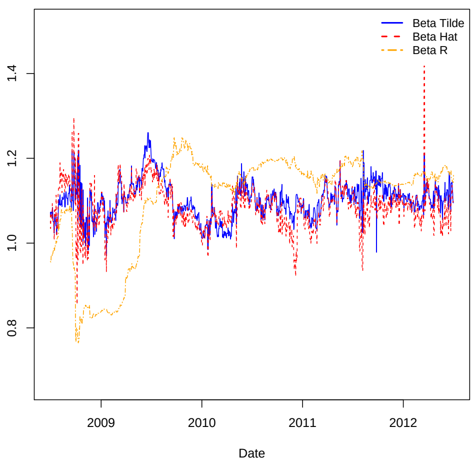
(b) XLE vs SPX



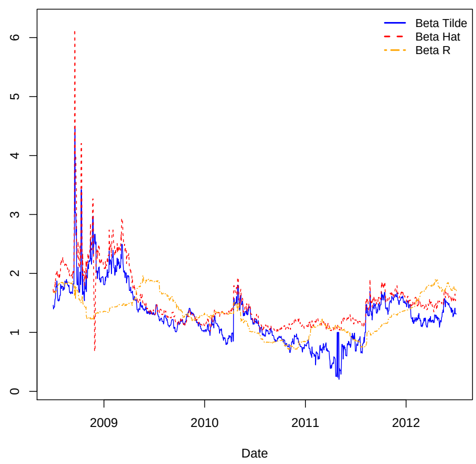
(c) XLB vs SPX



(d) XLK vs SPX

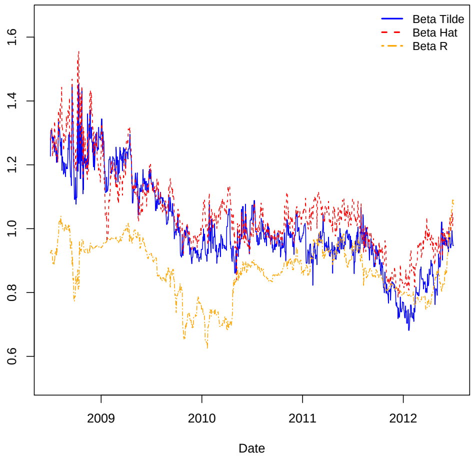


(e) XLI vs SPX

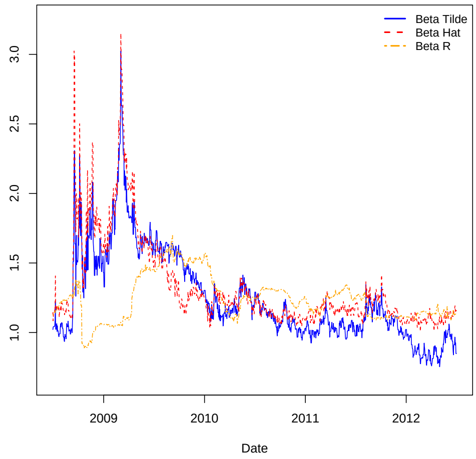


(f) GS vs SPX

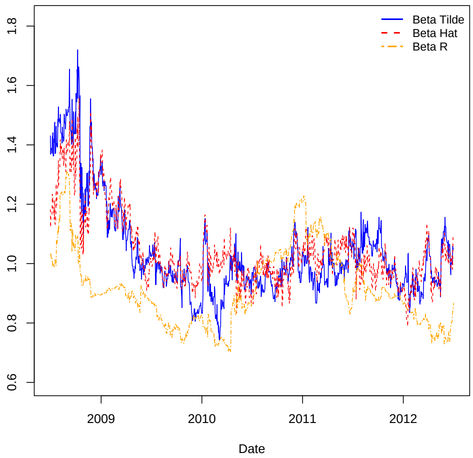
Figure 1.3: Comparison between $\tilde{\beta}$ (blue line) and $\hat{\beta}$ (red line)



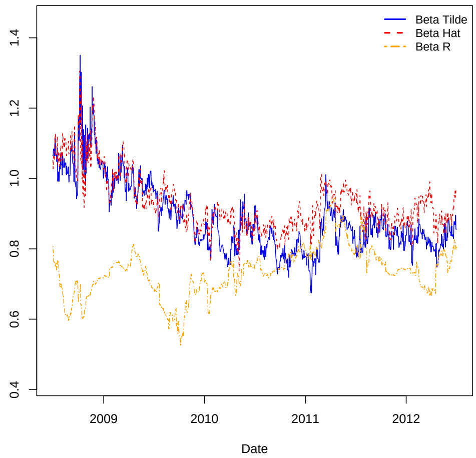
(a) MSFT vs SPX



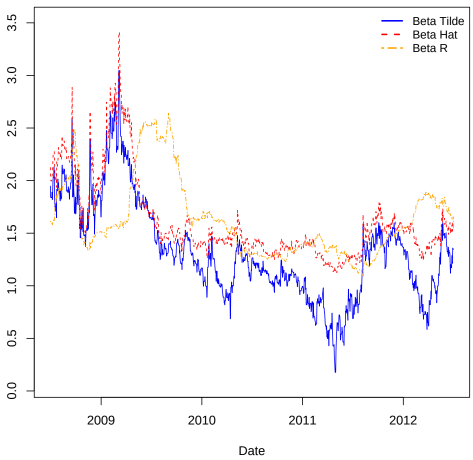
(b) GE vs SPX



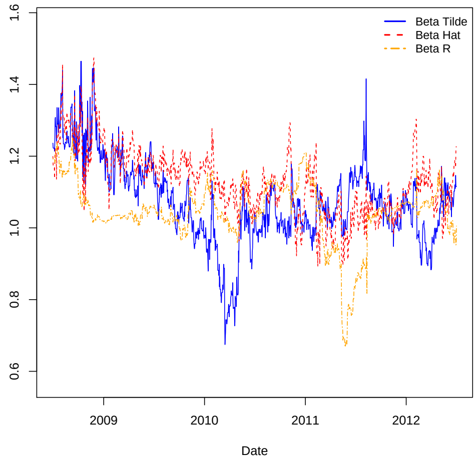
(c) GOOG vs SPX



(d) IBM vs SPX

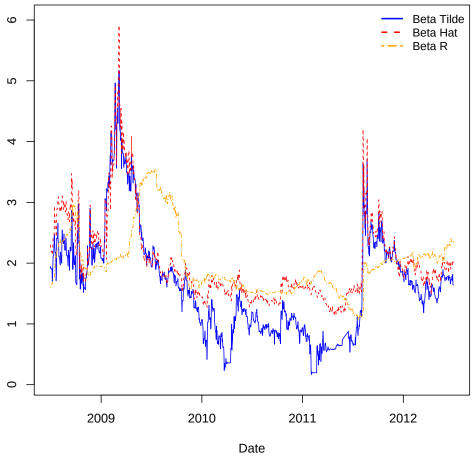


(e) JPM vs SPX

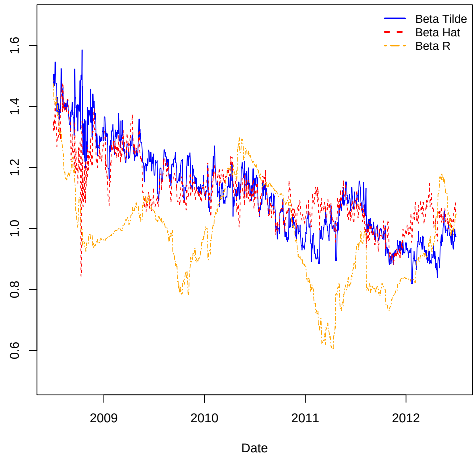


(f) CSCO vs SPX

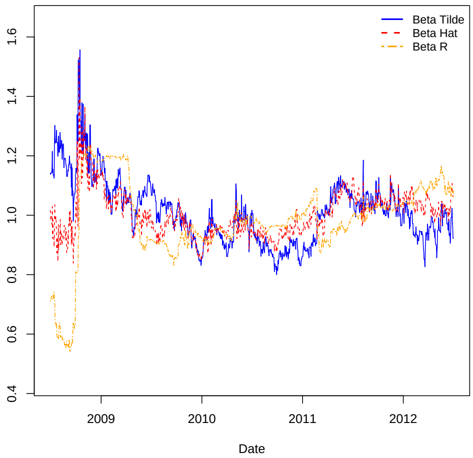
Figure 1.4: Comparison between $\tilde{\beta}$ (blue line) and $\hat{\beta}$ (red line)



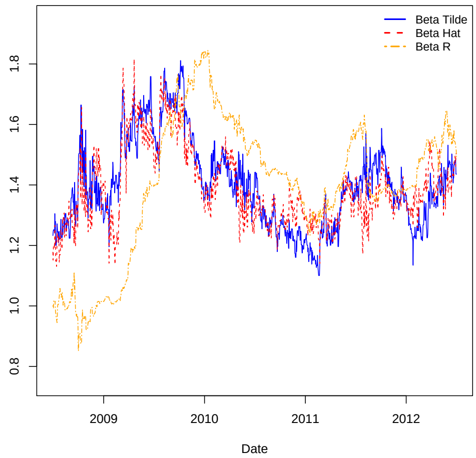
(a) BAC vs SPX



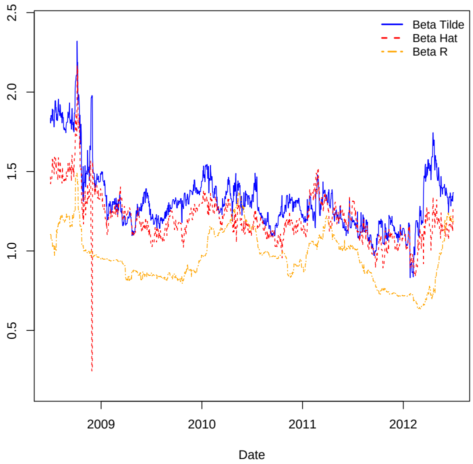
(b) INTC vs SPX



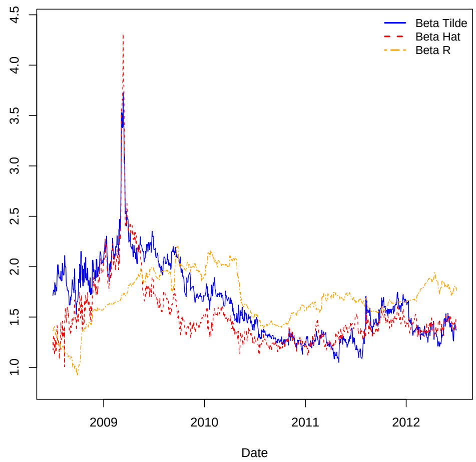
(c) CVX vs SPX



(d) CAT vs SPX



(e) AAPL vs SPX



(f) AA vs SPX

Figure 1.5: Comparison between $\tilde{\beta}$ (blue line) and $\hat{\beta}$ (red line)

Let the parameter β_F denote the forward realized beta, which is obtained as the slope of the linear regression of stock returns on index returns performed on a forward window with length T. In other words, $\beta_F(t) = \beta_H(t+T)$. It is then interesting to see how well the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H predict the quantity β_F. The predictive power of these estimators is tested using several statistical measures, the bias and the root-mean-square error are retained here:

Stock	$E(\hat{\beta} - \beta_F)$	$E(\tilde{\beta} - \beta_F)$	$E(\beta_H - \beta_F)$
XLF.US	-0.148	-0.121	0.054
XLE.US	-0.05	0.0094	-0.0313
XLB.US	-0.0231	0.0002	-0.0345
XLK.US	0.0679	0.0542	-0.0158
XLI.US	-0.0258	-0.0183	-0.0215
AAPL.US	0.2649	0.4073	0.0005
AA.US	-0.1814	-0.0738	-0.0333
BAC.US	-0.1414	-0.3557	0.0222
CAT.US	-0.0453	-0.0574	-0.0512
CSCO.US	0.0854	0.0216	-0.0073
CVX.US	0.0252	0.0366	-0.0187
GE.US	0.0375	-0.0534	0.0016
GOOG.US	0.1693	0.1742	0.0193
GS.US	0.1306	-0.0962	0.0113
IBM.US	0.2063	0.1575	-0.0233
INTC.US	0.1462	0.1451	-0.0069
JPM.US	-0.0415	-0.347	0.0383
MSFT.US	0.1717	0.1102	-0.0234

Table 1.1: Bias of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

Stock	$\sqrt{E((\frac{\hat{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\tilde{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\beta_H-\beta_F}{\beta_F})^2)}$
XLF.US	0.143	0.138	0.195
XLE.US	0.183	0.207	0.128
XLB.US	0.196	0.209	0.116
XLK.US	0.132	0.124	0.095
XLI.US	0.128	0.112	0.108
AAPL.US	0.41	0.547	0.252
AA.US	0.187	0.188	0.181
BAC.US	0.234	0.407	0.337
CAT.US	0.118	0.123	0.149
CSCO.US	0.159	0.162	0.147
CVX.US	0.198	0.238	0.165
GE.US	0.154	0.189	0.158
GOOG.US	0.277	0.31	0.165
GS.US	0.341	0.292	0.303
IBM.US	0.369	0.348	0.134
INTC.US	0.279	0.294	0.244
JPM.US	0.158	0.292	0.265
MSFT.US	0.297	0.27	0.163

Table 1.2: RMSE of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

It can be seen through the tables above that the historical beta estimator β_H is unbiased for all the considered stocks and has in average the lowest RMSE compared to the estimators $\hat{\beta}$ and $\tilde{\beta}$. It can also be noticed that the estimator $\tilde{\beta}$ has a significant positive bias for the stocks AAPL, GOOG, IBM, INTC, MSFT and a negative bias for XLF and BAC. Thus, the RMSE of $\tilde{\beta}$ for these stocks is higher than the one of β_H. It can be deduced that option market participants may have an expectation of the parameter β which is temporarily different from its value under the physical probability measure $\mathbb{P}$, this finding proves the existence of discrepancies between the risk-neutral measure and the physical measure.

1.4.2 Arbitrage strategy

Theoretical setting

Let us suppose in this section that all the parameters may have different values between the real-world probability measure and the risk-neutral probability measure. We want to construct a portfolio with the aim to take profit principally from the discrepancy between the implied beta and effective realized beta.

Let us denote $m_Y, \nu_Y, \rho_Y, m_Z, \nu_Z, \rho_Z, \beta$ the parameters of the model under the risk-neutral probability measure and $\bar{m}_Y, \bar{\nu}_Y, \bar{\rho}_Y, \bar{m}_Z, \bar{\nu}_Z, \bar{\rho}_Z, \bar{\beta}$ the parameters of the model under the real-world probability measure.

We introduce the new generators $\bar{\mathcal{L}}_I$ and $\bar{\mathcal{L}}_S$ as follows:

$$
\begin{aligned}
\bar{\mathcal{L}}^I &= \bar{\mathcal{L}}_2^I + \frac{1}{\sqrt{\epsilon}}\bar{\mathcal{L}}_1^I + \frac{1}{\epsilon}\bar{\mathcal{L}}_0^I,\\
\bar{\mathcal{L}}_0^I &= \frac{\partial}{\partial y}(\bar{m}_Y - y) + \bar{\nu}_Y^2\frac{\partial^2}{\partial y^2},\\
\bar{\mathcal{L}}_1^I &= \sqrt{2}\bar{\rho}_Y\bar{\nu}_Y I_t f_1(y)\frac{\partial^2}{\partial I \partial y},\\
\bar{\mathcal{L}}_2^I &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I_t}I_t - .) + \frac{1}{2}\frac{\partial^2}{\partial I_t^2}I_t^2 f_1(y)^2,
\end{aligned}
$$

and:

$$
\begin{aligned}
\bar{\mathcal{L}}^S &= \bar{\mathcal{L}}_2^S + \frac{1}{\sqrt{\epsilon}}\bar{\mathcal{L}}_1^S + \frac{1}{\epsilon}\bar{\mathcal{L}}_0^S,\\
\bar{\mathcal{L}}_0^S &= \frac{\partial}{\partial y}(\bar{m}_Y - y) + \bar{\nu}_Y^2\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\alpha(\bar{m}_Z - z) + \alpha\bar{\nu}_Z^2\frac{\partial^2}{\partial z^2},\\
\bar{\mathcal{L}}_1^S &= \frac{\partial^2}{\partial S \partial y}\bar{\beta}S_t f_1(y)\bar{\nu}_Y\sqrt{2}\bar{\rho}_Y + \frac{\partial^2}{\partial S \partial z}S_t f_2(z)\bar{\nu}_Z\sqrt{2\alpha}\bar{\rho}_Z,\\
\bar{\mathcal{L}}_2^S &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S}S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}S_t^2(\bar{\beta}^2 f_1(y)^2 + f_2(z)^2).
\end{aligned}
$$

Let us introduce a portfolio X containing at time t, an option on the stock with maturity date $\mathcal{D}$, a quantity $-n_{I,t}$ of options on the index with the same maturity date $\mathcal{D}$, a quantity $-\Delta_{t,S}$ of the stock, and a quantity $n_{I,t}\Delta_{I,t}$ of the index. The portfolio X evolves as follows:

$$
X_t = P(t,S_t,Y_t) - n_{I,t}P(t,I_t,Y_t) - \Delta_{t,S}S_t + n_{I,t}\Delta_{I,t}I_t,
$$

It can be recalled here that $\mathcal{L}^I P(t,I_t,Y_t) = 0$ and $\mathcal{L}^S P(t,S_t,Y_t) = 0$, thus the portfolio (X) has the following dynamics under the real-world probability measure P:

$$
\begin{aligned}
dX_t = {}& ((\bar{\mathcal{L}}^S - rS_t\frac{\partial}{\partial S_t}) - (\mathcal{L}^S - rS_t\frac{\partial}{\partial S_t}))P(t,S_t,Y_t) - n_I((\bar{\mathcal{L}}^I - rI_t\frac{\partial}{\partial I_t}) - (\mathcal{L}^I - rI_t\frac{\partial}{\partial I_t}))P(t,I_t,Y_t)\\
&+ \frac{\partial P(t,S_t,Y_t)}{\partial S}dS_t - n_{I,t}\frac{\partial P(t,I_t,Y_t)}{\partial I_t}dI_t\\
&+ \frac{\partial P(t,S_t,Y_t)}{\partial y}\frac{\bar{\nu}_Y\sqrt{2}}{\sqrt{\epsilon}}dW_t^{(3)} + \frac{\partial P(t,S_t,Y_t)}{\partial z}\frac{\bar{\nu}_Z\sqrt{2\alpha}}{\sqrt{\epsilon}}dW_t^{(4)} - n_I\frac{\partial P(t,I_t,Y_t)}{\partial y}\frac{\bar{\nu}_Y\sqrt{2}}{\sqrt{\epsilon}}dW_t^{(3)}\\
&- \Delta_{t,S}dS_t + n_{I,t}\Delta_{t,I}dI_t.
\end{aligned}
$$

The quantities n_I, Δ_I, Δ_S are fixed as follows:

$$
n_I = \frac{\frac{\partial P(t,S_t,Y_t)}{\partial y}}{\frac{\partial P(t,I_t,Y_t)}{\partial y}}, \quad \Delta_{t,S} = \frac{\partial P_S}{\partial S_t}, \quad \Delta_{t,I} = \frac{\partial P_I}{\partial I_t}. \tag{1.25}
$$

It was proved previously that:

$$
\begin{aligned}
P(t,I_t,Y_t) &= P_0^{I,\epsilon}(t,I_t) + \sqrt{\epsilon}P_1^{I,\epsilon}(t,I_t) + \epsilon P_2^{I,\epsilon}(t,I_t,Y_t) + o(\epsilon)\\
P(t,S_t,Y_t) &= P_0^{S,\epsilon}(t,S_t) + \sqrt{\epsilon}P_1^{S,\epsilon}(t,S_t) + \epsilon P_2^{S,\epsilon}(t,S_t,Y_t) + o(\epsilon)
\end{aligned}
$$

then $n_I = \dfrac{\frac{\partial P_2^{S,\epsilon}(t,S_t,Y_t)}{\partial y}}{\frac{\partial P_2^{I,\epsilon}(t,I_t,Y_t)}{\partial y}} + o(\epsilon)$.

Using the analytic expressions of $P_2^{S,\epsilon}$ and $P_2^{I,\epsilon}$, it can be deduced:

$$
n_{I,t} = \beta^2\frac{S_t^2\frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2\frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}}.
$$

The dynamics of X can be written, using (1.25), in the following way:

$$
\begin{aligned}
dX_t &= \frac{1}{2}S_t^2 f_1^2(y_t)(\bar\beta^2-\beta^2)\frac{\partial^2 P^{S,\epsilon}}{\partial S_t^2}dt + \left(\alpha(\bar m_Z - m_Z)\frac{\partial P_2^{S,\epsilon}}{\partial z} + \alpha(\bar\nu_Z^2-\nu_Z^2)\frac{\partial^2 P_2^{S,\epsilon}}{\partial z^2}\right)dt\\
&+\left((\bar m_Y - m_Y)\frac{\partial P_2^{S,\epsilon}}{\partial y} + (\bar\nu_Y^2-\nu_Y^2)\frac{\partial^2 P_2^{S,\epsilon}}{\partial y^2}\right)dt\\
&+\sqrt{\epsilon}\left(\sqrt{2}\nu_Y\chi_1(y_t)\frac{\partial P_2^{S,\epsilon}}{\partial y} + \sqrt{2}S_t f_1(y_t)(\bar\beta\bar\nu_Y\bar\rho_Y - \beta\nu_Y\rho_Y)\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial y}\right)dt\\
&+\sqrt{\epsilon}\left(\sqrt{2\alpha}\nu_Z\chi_2(z_t)\frac{\partial P_2^{S,\epsilon}}{\partial z} + \sqrt{2\alpha}S_t f_2(z_t)(\bar\rho_Z\bar\nu_Z - \rho_Z\nu_Z)\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial z}\right)dt\\
&-n_I\sqrt{\epsilon}\left(I_t f_1(y_t)\sqrt{2}\left(\bar\rho_Y\bar\nu_Y - \rho_Y\nu_Y\right)\frac{\partial^2 P_2^{I,\epsilon}}{\partial I\partial y} + \nu_Y\sqrt{2}\chi_1(y_t)\frac{\partial P_2^{I,\epsilon}}{\partial y}\right)dt\\
&-n_{I,t}\left((\bar m_Y - m_Y)\frac{\partial P_2^{I,\epsilon}}{\partial y} + (\bar\nu_Y^2-\nu_Y^2)\frac{\partial^2 P_2^{I,\epsilon}}{\partial y^2}\right)dt\\
&+\nu_Z\sqrt{2\alpha}\sqrt{\epsilon}\frac{\partial P_2^{S,\epsilon}}{\partial z}dW_t^{(4)} + o(\epsilon)\,.
\end{aligned}
$$

Notice here that:

$$
\frac{\frac{\partial^2 P_2^{S,\epsilon}}{\partial y^2}}{\frac{\partial^2 P_2^{I,\epsilon}}{\partial y^2}} = \beta^2\frac{S_t^2\frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2\frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}} = n_{I,t} + o(\epsilon)\,,
$$

hence, it follows:

$$
\begin{aligned}
dX_t &= \frac{1}{2}S_t^2 f_1^2(y_t)(\bar\beta^2-\beta^2)\frac{\partial^2 P^{S,\epsilon}}{\partial S_t^2}dt + \left(\alpha(\bar m_Z - m_Z)\frac{\partial P_2^{S,\epsilon}}{\partial z} + \alpha(\bar\nu_Z^2-\nu_Z^2)\frac{\partial^2 P_2^{S,\epsilon}}{\partial z^2}\right)dt\\
&+\sqrt{\epsilon}\left(\sqrt{2}S_t f_1(y_t)(\bar\beta\bar\nu_Y\bar\rho_Y - \beta\nu_Y\rho_Y)\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial y}\right)dt\\
&+\sqrt{\epsilon}\left(\sqrt{2\alpha}\nu_Z\chi_2(z_t)\frac{\partial P_2^{S,\epsilon}}{\partial z} + \sqrt{2\alpha}S_t f_2(z_t)(\bar\rho_Z\bar\nu_Z - \rho_Z\nu_Z)\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial z}\right)dt\\
&-n_I\sqrt{\epsilon}\left(I_t f_1(y_t)\sqrt{2}\left(\bar\rho_Y\bar\nu_Y - \rho_Y\nu_Y\right)\frac{\partial^2 P_2^{I,\epsilon}}{\partial I\partial y}\right)dt\\
&+\nu_Z\sqrt{2\alpha}\sqrt{\epsilon}\frac{\partial P_2^{S,\epsilon}}{\partial z}dW_t^{(4)} + o(\epsilon)\,.
\end{aligned}
$$

Under the assumption:

$$
\mathcal{H}: m_Y = \bar m_Y, \quad \nu_Y = \bar\nu_Y, \quad \rho_Y = \bar\rho_Y, \quad \nu_Z = \bar\nu_Z, \quad \rho_Z = \bar\rho_Z,
$$

the previous relation becomes:

$$
\begin{aligned}
dX_t &= \frac{1}{2}S_t^2 f_1^2(y_t)(\bar\beta^2-\beta^2)\frac{\partial^2 P^{S,\epsilon}}{\partial S_t^2}dt + \sqrt{\epsilon}\left(\sqrt{2}S_t f_1(y_t)\nu_Y\rho_Y(\bar\beta-\beta)\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial y}\right)dt\\
&+\sqrt{\epsilon}\left(\sqrt{2\alpha}\nu_Z\chi_2(z_t)\frac{\partial P_2^{S,\epsilon}}{\partial z}\right)dt + \nu_Z\sqrt{2\alpha}\sqrt{\epsilon}\frac{\partial P_2^{S,\epsilon}}{\partial z}dW_t^{(4)} + o(\epsilon)\,.
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
E(dX_t|\mathcal{F}_t) &= (\bar{\beta}-\beta)\left(\frac{1}{2}S_t^2 f_1^2(y_t)(\bar{\beta}+\beta)\frac{\partial^2 P^{S,\epsilon}}{\partial S_t^2}+\sqrt{\epsilon}\sqrt{2}S_t f_1(y_t)\nu_Y\rho_Y\frac{\partial^2 P_2^{S,\epsilon}}{\partial S\partial y}\right)dt+O\big(\sqrt{\epsilon}\big)\,, \\
Var(dX_t|\mathcal{F}_t) &= 2\epsilon\alpha\nu_Z^2\left(\frac{\partial P_2^{S,\epsilon}}{\partial z}\right)^2 dt.
\end{aligned}
$$

Empirical study

In this subsection, we carry out an empirical study in order to test the suggested strategy on real data. It is important to mention here that some inevitable approximations are used when testing the strategy. Indeed, in the strategy described so far, we hold 3-Months options from the inception date to the maturity date, and we delta-hedge them in a daily basis. Let $\mathcal{I}_S(K_S,T)$ and $\mathcal{I}_I(K_I,T)$ be the implied volatilities of the stock and index options respectively:

$$
P^{S,\epsilon}(t,K_S,T)=P_{BS}^S(t,S_t,\mathcal{I}_S(K_S,T),T),\quad P^{I,\epsilon}(t,K_I,T)=P_{BS}^I(t,I_t,\mathcal{I}_I(K_I,T),T).
$$

The implied volatilities are obtained using interpolation methods since we don't dispose of market implied volatilities for all strikes and maturities. The procedure can be explained in the following way. We hold a $\mathcal{N}$-Months option from the inception date t to the maturity date $t+\mathcal{N}\ Months$. We suppose that the $\mathcal{N}$ following months contain $\sum_{i=1}^{\mathcal{N}}\mathcal{D}_i$ days, which means that each month $i\in\{1,..,N\}$ has $\mathcal{D}_i$ days. Let $0\le L\le\sum_{i=1}^{\mathcal{N}}\mathcal{D}_i$, then at the date $t+L\ Days$, we deal with an option whose time to maturity is equal to $T=(\sum_{i=1}^{\mathcal{N}}\mathcal{D}_i-L)$ Days. At this exact date ($t+L\ Days$), we don't dispose of data corresponding to the volatility smile with maturity $T=(\sum_{i=1}^{\mathcal{N}}\mathcal{D}_i-L)$ Days, but we observe two implied volatility smiles with maturities T_{Min} and T_{Max} such that $T_{Min}<T<T_{Max}$. Thus, we can deduce an approximation of the implied volatility smile for the maturity T through an interpolation method.
Once the implied volatilities $\mathcal{I}_S(K_S,T)$ and $\mathcal{I}_I(K_I,T)$ are approximated, we compute the option prices $P^{S,\epsilon}(t,K_S,T)$ and $P^{I,\epsilon}(t,K_I,T)$, in addition to the option deltas $\Delta_{S,t}$ and $\Delta_{I,t}$:

$$
\begin{aligned}
\Delta_{S,t} &= \frac{\partial P_{BS}^S}{\partial S_t}+\frac{\partial P_{BS}^S}{\partial\mathcal{I}_S}\frac{\partial\mathcal{I}_S}{\partial S_t}, \\
\Delta_{I,t} &= \frac{\partial P_{BS}^I}{\partial I_t}+\frac{\partial P_{BS}^I}{\partial\mathcal{I}_I}\frac{\partial\mathcal{I}_I}{\partial I_t}.
\end{aligned}
$$

We can recall here that $\mathcal{I}_S(K_S,T)$ and $\mathcal{I}_I(K_I,T)$ can be approximated at order 1 in $\sqrt{\epsilon}$ by $\Sigma_S(K_S,T)$ and $\Sigma_I(K_I,T)$ which are introduced in (1.23) and (1.22) respectively. Thus, $\frac{\partial\mathcal{I}_S}{\partial S_t}=O(\epsilon)$ and $\frac{\partial\mathcal{I}_I}{\partial I_t}=O(\epsilon)$ and then the quantities $\Delta_{S,t}$ and $\Delta_{I,t}$ can be approximated at order 1 in $\sqrt{\epsilon}$ by $\frac{\partial P_{BS}^S}{\partial S_t}$ and $\frac{\partial P_{BS}^I}{\partial I_t}$ respectively.

We introduce the following notations:

- $Y_t=X_{t+\mathcal{N}_{Months}}-X_t$ denotes the $P\&L$ of the portfolio X started at the inception date t and closed at the maturity date $t+\mathcal{N}_{Months}$ (it is needless to mention that Y_t is not $\mathcal{F}_t$-measurable but is rather $\mathcal{F}_{t+\mathcal{N}_{Months}}$-measurable).

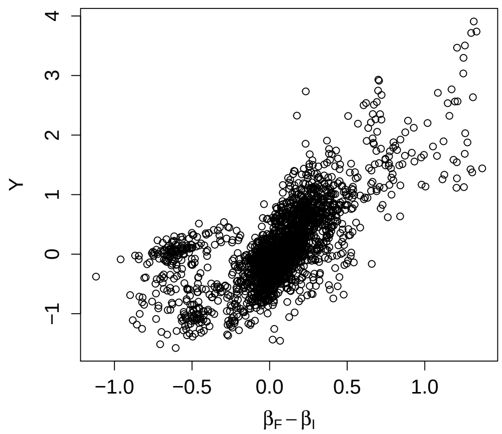
- $\beta_{F,t}(\mathcal{N}_{Months})$ denotes the realized beta between t and $t+\mathcal{N}_{Months}$, it is obtained through the linear regression of the stock log-returns on the index log-returns between t and $t+\mathcal{N}_{Months}$.

- $\beta_{I,t}(\mathcal{N}_{Months})$ is the implied beta estimated at time t from $\mathcal{N}$-Months maturity options.

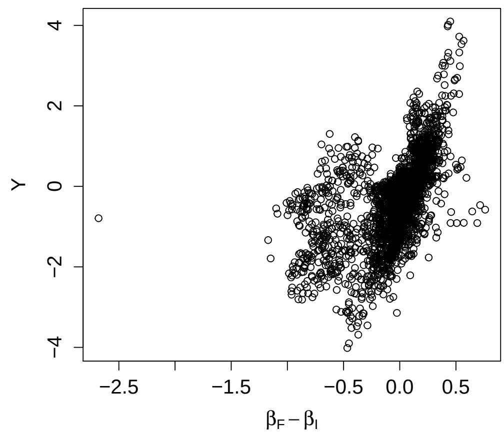
The main purpose of the back-test is to show at which extent the $P\&L$ Y is explained by $\beta_F(\mathcal{N}_{Months}) - \beta_I(\mathcal{N}_{Months})$. This means that we want to see the part of the desired $P\&L$ $E(dX_t|\mathcal{F}_t)$ in the total $P\&L$ dX_t. The residual part $dX_t - E(dX_t|\mathcal{F}_t)$ is explained at first place by the random part of dX_t, and in the second place by the additional terms in case where the hypothesis $\mathcal{H}$ doesn't hold.

In the practical implementation of the back-test, the portfolio X is launched every day but is not necessarily conserved until the maturity date of the options (in $\mathcal{N}$ months). Indeed, we unwind the portfolio before the maturity of the options if the spot processes (S or I) have moved for more than 10% since the setting of the trade, because in this case the quantity n_I of index options is not correct anymore and has to be changed.

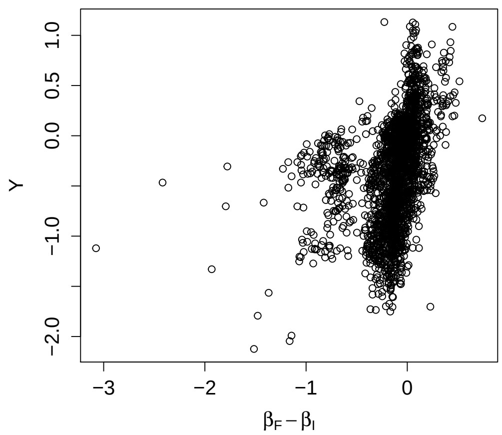
The strategy is tested on 6-Months options ($\mathcal{N} = 6$) between 01/01/2005 and 01/08/2014. We give here the graphs of Y_t as a function of $(\beta_{F,t}(\mathcal{N}_{Months}) - \beta_{I,t}(\mathcal{N}_{Months}))$ for some examples:



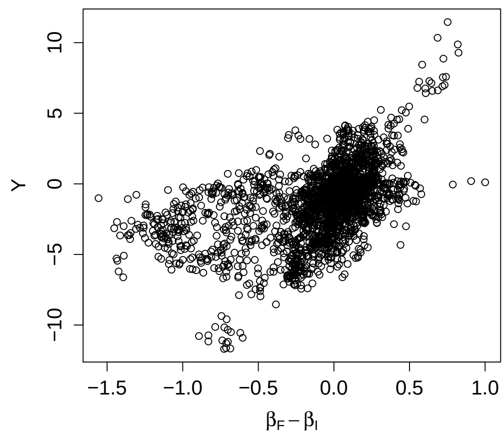
(a) XLF vs SPX



(b) XLB vs SPX



(c) XLK vs SPX



(d) XLE vs SPX

Figure 1.6: The portfolio $P\&L$ as a function of the discrepancy term $(\beta_F - \beta_I)$.

Statistics On PnL	Regression slope	R-squared
XLB-SPX	1.9701656	0.3172391
XLF-SPX	1.4595975	0.4812934
XLE-SPX	3.5216376	0.2596000
XLK-SPX	0.7545623	0.1543600

1.4.3 Hedging of options on the stock by instruments on the index

The capital asset pricing model offers a practical framework where the stock dynamics are linked to those of the market index. This section aims to show that the parameter beta may be very useful to hedge the delta risk or the volatility risk of a stock option using instruments on the index.

Hedging volatility risk

The natural way to hedge the volatility risk of a stock option is to use other stock options with a longer maturity. Nevertheless, if options on this stock are not sufficiently liquid, it won't be possible to carry out this hedging strategy. In this case, it may be judicious to use options on the index in order to perform the hedging.

Let X be the value of the portfolio containing an option on the stock, a quantity $-\vartheta$ of an option on the index, a quantity $-\Delta_S$ of the stock and a quantity Δ_I of the index. The value of this portfolio at time t is $X_t = P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} + \Delta_{I,t} I_t - \Delta_{S,t} S_t$ and X_t satisfies the following SDE:

$$\begin{aligned}
dX_t = &\left(\mathcal{L}^S P_t^{S,\epsilon} + r P_t^{S,\epsilon}\right) dt + \frac{\partial P_t^{S,\epsilon}}{\partial S}\left(dS_t - rS_t dt\right) + \frac{\partial P_t^{S,\epsilon}}{\partial y}\frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)} + \frac{\partial P_t^{S,\epsilon}}{\partial z}\frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{*,(4)} \\
&- \vartheta_t\left(\mathcal{L}^I P_t^{I,\epsilon} + r P_t^{I,\epsilon}\right) dt - \vartheta_t \frac{\partial P_t^{I,\epsilon}}{\partial I}\left(dI_t - rI_t dt\right) - \vartheta_t \frac{\partial P_t^{I,\epsilon}}{\partial y}\frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)} \\
&- \Delta_{S,t} dS_t + \Delta_{I,t} dI_t.
\end{aligned}$$

By rearranging the terms, the last equation becomes:

$$\begin{aligned}
dX_t = &\, r\left(P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} + \vartheta_t \frac{\partial P_t^{I,\epsilon}}{\partial I} I_t - \frac{\partial P_t^{S,\epsilon}}{\partial S} S_t\right) dt + \left(\frac{\partial P_t^{S,\epsilon}}{\partial S} - \Delta_{t,S}\right) dS_t + \left(\Delta_{I,t} - \vartheta_t \frac{\partial P_t^{I,\epsilon}}{\partial I}\right) dI_t \\
&+ \left(\frac{\partial P_t^{S,\epsilon}}{\partial y} - \vartheta_t \frac{\partial P_t^{I,\epsilon}}{\partial y}\right)\frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)} + \frac{\partial P_t^{S,\epsilon}}{\partial z}\frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{*,(4)},
\end{aligned}$$

If the parameters $(\vartheta_t, \Delta_{S,t}, \Delta_{I,t})$ are chosen in the following way:

$$\vartheta_t = \frac{\frac{\partial P^{S,\epsilon}}{\partial y}}{\frac{\partial P^{I,\epsilon}}{\partial y}} \quad , \quad \Delta_{S,t} = \frac{\partial P^{S,\epsilon}}{\partial S_t} \quad , \quad \Delta_{I,t} = \vartheta_t \frac{\partial P^{I,\epsilon}}{\partial I_t},$$

then the portfolio X is delta-hedged continuously in the stock and the index, and is made insensitive to the variations of the process y. It follows that:

$$\begin{aligned}
dX_t &= r\left(P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} + \Delta_{I,t} I_t - \Delta_{S,t} S_t\right) dt + \frac{\partial P_t^{S,\epsilon}}{\partial z}\frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{*,(4)}, \\
&= rX_t dt + \frac{\partial P_t^{S,\epsilon}}{\partial z}\frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{*,(4)},
\end{aligned}$$

which means that the risk of the volatility of the index is canceled, and the only remaining risk comes from the idiosyncratic volatility.

The hedging parameter ϑ_t can be obtained through the computation of the terms $\frac{\partial P^{S,\epsilon}}{\partial y}$ and $\frac{\partial P^{I,\epsilon}}{\partial y}$ using Monte-Carlo simulations. This computation method is time-consuming, so it may be interesting to have a closed-form approximation for this hedging ratio. For this purpose, the quantities $\frac{\partial P^{S,\epsilon}}{\partial y}$ and $\frac{\partial P^{I,\epsilon}}{\partial y}$ should be approximated at order 1 in ϵ as explained below:

$$\begin{aligned} P(t,I_t,Y_t) &= P_0^{I,\epsilon}(t,I_t) + \sqrt{\epsilon}P_1^{I,\epsilon}(t,I_t) + \epsilon P_2^{I,\epsilon}(t,I_t,Y_t) + o(\epsilon) \\ P(t,S_t,Y_t,Z_t) &= P_0^{S,\epsilon}(t,S_t) + \sqrt{\epsilon}P_1^{S,\epsilon}(t,S_t) + \epsilon P_2^{S,\epsilon}(t,S_t,Y_t,Z_t) + o(\epsilon) \end{aligned}$$

where:

$$\begin{aligned} P_2^{I,\epsilon}(t,I_t,Y_t) &= -\frac{1}{2}I_t^2\frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}\phi_I(Y_t), \\ P_2^{S,\epsilon}(t,S_t,Y_t,Z_t) &= -\frac{1}{2}S_t^2\frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}\left(\beta^2\phi_I(Y_t) + \phi_{Idios}(Z_t)\right). \end{aligned}$$

Consequently, the quantity ϑ_t writes:

$$\vartheta_t = \beta^2\frac{S_t^2\frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2\frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}} + o(\epsilon).$$

Since $P_0^{S,\epsilon}$ and $P_0^{I,\epsilon}$ have known closed-form expressions, the quantity ϑ_t can be approximated analytically by $\hat{\vartheta}_t = \beta^2\frac{S_t^2\frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}}{I_t^2\frac{\partial^2 P_0^{I,\epsilon}}{\partial I_t^2}}$, the approximation error is at order 1 in ϵ.

Delta and Vega Hedging

The lack of liquidity or the presence of transaction costs on the stock can make the delta-hedging of the stock option costly. Therefore, it may be useful to have an alternative hedging strategy in circumstances where there are trading constraints on the stock. The setting of the CAPM model enables the decomposition of the stock risk into two parts: the index risk and the idiosyncratic risk. It is then reasonable in this framework to hedge the delta of a stock option using the index.

Let L be the portfolio containing an option on the stock, a quantity $-\vartheta$ of an option on the index and a quantity $-\varphi_I$ of the index. We have then $L_t = P_t^{S,\epsilon} - \vartheta_t P^{I,\epsilon} - \varphi_{I,t}I_t$. It follows that:

$$\begin{aligned} dL_t = r\left(P_t^{S,\epsilon} - \vartheta_t P_t^{I,\epsilon} - \varphi_{I,t}I_t\right)dt + \left(\frac{\partial P_t^{S,\epsilon}}{\partial S}\beta S_t f_1(y_t) - \vartheta_t\frac{\partial P_t^{I,\epsilon}}{\partial I}I_t f_1(y_t) - \varphi_{I,t}I_t f_1(y_t)\right)dW_t^{*,(1)} \\ + \left(\frac{\partial P_t^{S,\epsilon}}{\partial y} - \vartheta_I\frac{\partial P_t^{I,\epsilon}}{\partial y}\right)\frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}dW_t^{*,(3)} + \frac{\partial P_t^{S,\epsilon}}{\partial S}S_t f_2(z_t)dW_t^{*,(2)} + \frac{\partial P_t^{S,\epsilon}}{\partial z}\frac{\nu_Z\sqrt{2}}{\sqrt{\epsilon}}dW_t^{*,(4)}. \end{aligned}$$

By taking $(\vartheta_t, \varphi_{I,t})$ such that:

$$\vartheta_t = \frac{\frac{\partial P^{S,\epsilon}}{\partial y}}{\frac{\partial P^{I,\epsilon}}{\partial y}} \quad , \quad \varphi_{I,t} = \frac{\beta S_t}{I_t}\frac{\partial P^{S,\epsilon}}{\partial S} - \vartheta_t\frac{\partial P^{I,\epsilon}}{\partial I},$$

the risks related to the index and its volatility are hedged. It follows that:

$$dL_t \quad = \quad rL_t dt + \frac{\partial P^{S,\epsilon}}{\partial S} S_t f_2(z_t) dW_t^{*,(2)} + \frac{\partial P^{S,\epsilon}}{\partial z} \frac{\nu_Z \sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(4)}.$$

Here again, the computation of the quantities ϑ_t and $\varphi_{I,t}$ can be made easier through the estimation of the parameter β. Indeed, $\varphi_{I,t}$ and $\varphi_{I,t}$ can be approximated respectively by $\hat{\vartheta}_t$ and $\hat{\varphi}_{I,t} = \frac{\beta S_t}{I_t} \frac{\partial P^{S,\epsilon}}{\partial S} - \hat{\vartheta}_t \frac{\partial P^{I,\epsilon}}{\partial I}$.

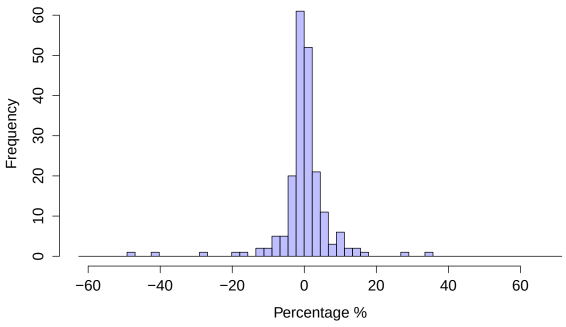
Numerical simulations

We perform in this subsection numerical simulations in order to test the hedging strategies proposed so far. First, we generate 200 paths of the processes $(I_t), (S_t), (y_t), (z_t)$ for $t \in [0, T]$ with a time step equal to $\delta = \frac{1}{256}$. On each of these simulated paths indexed by $i \in \{1, .., 200\}$, and for $k \in [0, \frac{T}{\delta}]$, we generate 50000 paths of $(I_t^{(i)}, S_t^{(i)}, y_t^{(i)}, z_t^{(i)})_{\{t \in [k\delta, T]\}}$ with a time step equal to δ and we compute the quantities $\Delta_{S,k\delta}^{(i)}, \Delta_{I,k\delta}^{(i)}, \left(P_{k\delta}^{S,\epsilon}\right)^{(i)}, \left(P_{k\delta}^{I,\epsilon}\right)^{(i)}$ using a Monte Carlo method. The quantities $\vartheta_{k\delta}$ and $\varphi_{I,k\delta}$ are approximated using $\hat{\vartheta}_{k\delta}$ and $\hat{\varphi}_{I,k\delta}$ respectively. We can then obtain $X_{k\delta}$ denoting the portfolio value at time $k\delta$, and we deduce the final value X_T at the maturity date T. It should be precised here that $\tilde{\beta}$ is used instead of β for the computation of $\hat{\vartheta}$ and $\hat{\varphi}_I$, so that the error of estimation is accounted in the hedging error.

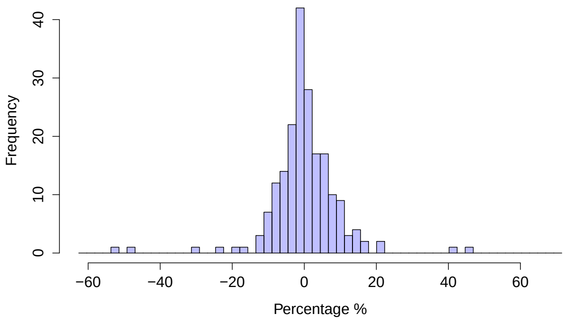
For these numerical simulations, the following parameters are used:

$$\begin{cases} \epsilon = 0.1, & \alpha = 1, \\ \rho_Y = -0.8, & \rho_Z = -0.5, \\ \nu_Y = 0.15, & \nu_Z = 0.05, \\ r = 0, & \mu_I = 0, \\ \beta = 1.5, & \mu_S = 0, \\ \gamma_1 = 0, & \gamma_2 = 0, \\ y_0 = \log(0.4), & z_0 = \log(0.2), \\ I_0 = 1000, & S_0 = 50, \end{cases}$$

It can be precised here, that since the hedging strategies are tested on 200 independent paths, the simulations are done in parallel using the computing cluster of Ecole Centrale Paris. The histograms of the hedging errors $\left(\frac{X_T - X_0}{X_0} \times 100\right)$ in the case of volatility risk hedging and $\left(\frac{L_T - L_0}{L_0} \times 100\right)$ in the case of Delta and volatility risk hedging, are given below:



(a) Histogram of $\frac{X_T - X_0}{X_0} \times 100$



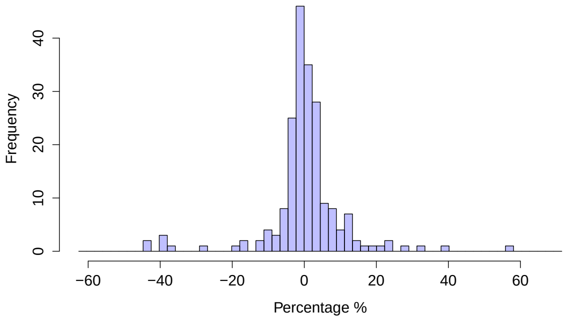
(b) Histogram of $\frac{L_T - L_0}{L_0} \times 100$

Figure 1.7: Histograms of hedging errors for $T = 1\ Month$

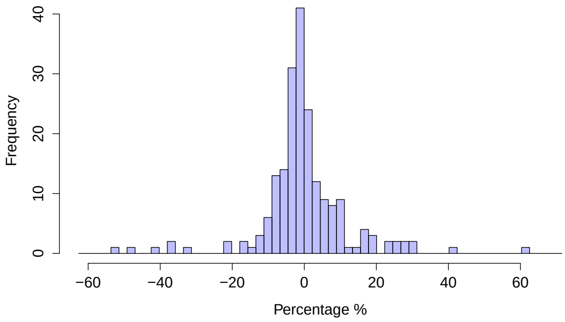
Statistics of Hedging 1M stock options	Median	Mean	Std	Skewness	Kurtosis
$100 \times \frac{X_T - X_0}{X_0}$	-0.04	0.00	7.51	-1.61	15.87
$100 \times \frac{L_T - L_0}{L_0}$	-0.43	0.07	9.64	-0.63	10.26

Table 1.3: Statistics of hedging errors for $T = 1\ Month$

The same procedure is repeated here in order to perform the numerical simulations for $T = 3$ $Months$.



(a) Histogram of $\frac{X_T - X_0}{X_0} \times 100$



(b) Histogram of $\frac{L_T - L_0}{L_0} \times 100$

Figure 1.8: Histograms of hedging errors for $T = 3\ Months$

Statistics of Hedging 3M stock options	Median	Mean	Std	Skewness	Kurtosis
$100 \times \frac{X_T - X_0}{X_0}$	0.07	0.29	10.11	-0.30	8.08
$100 \times \frac{L_T - L_0}{L_0}$	-1.34	-0.15	12.36	0.07	6.33

Table 1.4: Statistics of hedging errors for $T = 3\ Months$

1.5 Conclusion

We presented in this chapter a continuous time capital asset pricing model where the index and the stock have both stochastic volatilities. Through the use of a perturbation technique, we provided approximations of the prices of European options on the index and the stock and we proposed an unbiased estimator $\tilde{\beta}$ for the parameter beta under the risk-neutral pricing measure $\mathbb{P}^*$. We conducted an empirical study and showed that the estimator $\tilde{\beta}$, when compared to the classical estimator β_H, doesn't insure a better prediction of the future realized β under the physical measure $\mathbb{P}$. This proves again that there are some discrepancies between the pricing measure $\mathbb{P}^*$ and the real-world measure $\mathbb{P}$. Besides, we showed that the estimator $\tilde{\beta}$ can be very useful when it comes to hedging stock options using instruments on the index. This approach, considered here as a relative-hedging method, can be very useful when the instruments needed for perfect replication are not liquid.

It would also be quite interesting to extend our study to the use of the implied beta for applications on portfolio construction on option and underlying assets. We keep this subject for a future work.

1.6 Appendices

1.6.1 Appendix 1: Pricing options on the index

Let $P_t^{I,\epsilon} = E^P(h(I_T)|I_t = x, Y_t = y)$ be the price of a european option on the index with payoff $h(I_T)$ and maturity T.

$$P_t^{I,\epsilon} = P^{I,\epsilon}(t, I_t, Y_t).$$

Since the process (I, Y) is markovian, applying the Feynman-Kac theorem yields:

$$\mathcal{L}^I P_t^{I,\epsilon} = 0,$$

where $\mathcal{L}^I$ is a differential operator whose elements can be classified by powers of $\sqrt{\epsilon}$:

$$\begin{aligned}
\mathcal{L}^I &= \mathcal{L}_2^I + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^I + \frac{1}{\epsilon}\mathcal{L}_0^I, \\
\mathcal{L}_0^I &= \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1^I &= -\nu_Y \sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \sqrt{2}\rho_Y \nu_Y I_t f_1(y)\frac{\partial^2}{\partial I \partial y}, \\
\mathcal{L}_2^I &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I_t} I_t - .) + \frac{1}{2}\frac{\partial^2}{\partial I_t^2} I_t^2 f_1(y)^2.
\end{aligned}$$

The differential operator $\mathcal{L}_0^I$ represents the infinitesimal generator of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$ which has the following dynamics:

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t.$$

The price $P^{I,\epsilon}$ can be expanded in powers of $\sqrt{\epsilon}$:

$$P^{I,\epsilon} = \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i P_i^{I,\epsilon}.$$

Next to that, the term $\mathcal{L}^I P_t^{I,\epsilon} = 0$ can be expanded and its elements can be classified by powers of $\sqrt{\epsilon}$. The terms of orders $-2, -1, 0$ and 1 in $\sqrt{\epsilon}$ are written below:

$$\begin{aligned}
(-2) &: \mathcal{L}_0^I P_0^{I,\epsilon} = 0, \\
(-1) &: \mathcal{L}_1^I P_0^{I,\epsilon} + \mathcal{L}_0^I P_1^{I,\epsilon} = 0, \\
(0) &: \mathcal{L}_2^I P_0^{I,\epsilon} + \mathcal{L}_1^I P_1^{I,\epsilon} + \mathcal{L}_0^I P_2^{I,\epsilon} = 0, \\
(1) &: \mathcal{L}_2^I P_1^{I,\epsilon} + \mathcal{L}_1^I P_2^{I,\epsilon} + \mathcal{L}_0^I P_3^{I,\epsilon} = 0,
\end{aligned}$$

The term $P_0^{I,\epsilon}$ is a solution of $\mathcal{L}_0^I P_0^{I,\epsilon} = 0$ with final condition $P_0^{I,\epsilon}(T, I_T, y_T) = h(I_T)$ (h is independent of y_T). By solving this equation, it can be found that:

$$P_0(t, I_t, y_t) = C_1(t, I_t)\int_0^y e^{\frac{u^2}{2\mu_Y^2} - \frac{u m_Y}{\mu^2}} du + C_2(t, I_t).$$

If C_1 is not the null function then the solution diverges when $y \to +\infty$. Nevertheless, in the case of a call option, the option price is bounded ($0 \leq P(t, I_t) \leq I_t$). Then the quantity C_1 has to be null, and then P_0 is independent of y.
The term of order (-1) in $\sqrt{\epsilon}$ yields $\mathcal{L}_1^I P_0^{I,\epsilon} + \mathcal{L}_0^I P_1^{I,\epsilon} = 0$ which reduces to $\mathcal{L}_0^I P_1^{I,\epsilon} = 0$ (since P_0

doesn't depend on y). Using the same reasoning as before, it can proved that $P_1^{I,\epsilon} = P_1^{I,\epsilon}(t, I_t)$ which is independent of y. Consequently:

$$\begin{aligned}\mathcal{L}_0^I P_0^{I,\epsilon} &= \mathcal{L}_1^I P_0^{I,\epsilon} = 0,\\ \mathcal{L}_0^I P_1^{I,\epsilon} &= \mathcal{L}_1^I P_1^{I,\epsilon} = 0.\end{aligned}$$

Using $\mathcal{L}_1^I P_1^{I,\epsilon} = 0$, the term of order 0 in $\sqrt{\epsilon}$ becomes:

$$\mathcal{L}_0^I P_2^{I,\epsilon} + \mathcal{L}_2^I P_0^{I,\epsilon} = 0.$$

This is a Poisson equation for $P_2^{I,\epsilon}$. Its solvability condition is :

$$\left\langle \mathcal{L}_0^I P_2^{I,\epsilon} \right\rangle_1 + \left\langle \mathcal{L}_2^I P_0^{I,\epsilon} \right\rangle_1 = 0,$$

where the operator $\langle . \rangle_1$ is the average with respect to the invariant distribution $N(m_Y, \nu_Y^2)$ of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$.
Since $\mathcal{L}_0^I$ is the infinitesimal generator of the process (Y_1), $\left\langle \mathcal{L}_0^I P_2^{I,\epsilon} \right\rangle_1 = 0$. Then, the solvability condition reduces to:

$$\left\langle \mathcal{L}_2^I \right\rangle_1 P_0^{I,\epsilon} = 0. \tag{1.26}$$

The operator $\left\langle \mathcal{L}_2^I \right\rangle_1$ has the following form:

$$\left\langle \mathcal{L}_2^I \right\rangle_1 = \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I} I - .) + \frac{1}{2}\frac{\partial^2}{\partial I^2} I^2 \left\langle f_1^2 \right\rangle_1 .$$

Consequently $\left\langle \mathcal{L}_2^I \right\rangle_1 = \mathcal{L}_{BS}(\bar{\sigma}_I)$ where $\bar{\sigma}_I^2 = \left\langle f_1^2 \right\rangle_1$. The term $P_0^{I,\epsilon}$ is the solution of the following problem :

$$\begin{aligned}\mathcal{L}_{BS}(\bar{\sigma}_I) P_0^{I,\epsilon} &= 0,\\ P_0^{\epsilon}(T, I_T) &= h(I_T).\end{aligned}$$

Therefore, $P_0^{I,\epsilon} = P_{BS}(t, I_t, \bar{\sigma}_I)$ meaning that $P_0^{I,\epsilon}$ is the Black-Scholes price of the index option with implied volatility equal to $\bar{\sigma}_I$.
As a result, the term $P_2^{I,\epsilon}$ can be written as $P_2^{I,\epsilon} = -(\mathcal{L}_0^I)^{-1}(\mathcal{L}_2^I - \left\langle \mathcal{L}_2^I \right\rangle_1) P_0^{I,\epsilon}$. The term of order 1 in $\sqrt{\epsilon}$ is a poisson equation for $P_3^{I,\epsilon}$. Its solvability condition is :

$$\begin{aligned}\left\langle \mathcal{L}_2^I P_1^{I,\epsilon} \right\rangle_1 &= -\left\langle \mathcal{L}_1^I P_2^{I,\epsilon} \right\rangle_1, & (1.27)\\ \left\langle \mathcal{L}_2^I \right\rangle_1 P_1^{I,\epsilon} &= \left\langle \mathcal{L}_1^I (\mathcal{L}_0^I)^{-1} (\mathcal{L}_2^I - \left\langle \mathcal{L}_2^I \right\rangle_1) \right\rangle_1 P_0^{I,\epsilon}. & (1.28)\end{aligned}$$

Let ϕ_I the solution of the following Poisson equation:

$$\mathcal{L}_0 \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_1 . \tag{1.29}$$

Since the difference term between the differential operator $\mathcal{L}_2^I$ and its average is:

$$\mathcal{L}_2^I - \left\langle \mathcal{L}_2^I \right\rangle_1 = \frac{1}{2}(f_1^2(y) - \left\langle f_1^2 \right\rangle_1) I_t^2 \frac{\partial^2}{\partial I_t^2},$$

it can be written that:

$$(\mathcal{L}_0^I)^{-1}(\mathcal{L}_2 - \left\langle \mathcal{L}_2 \right\rangle_1) = \frac{1}{2}\phi_I(y) I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

By applying the operator $\mathcal{L}_1$ to the last equation, it follows that:

$$\mathcal{L}_1^I(\mathcal{L}_0^I)^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle_1) = (-\nu_Y \sqrt{2} \left\langle \chi_1 \phi_I' \right\rangle_1 + \sqrt{2} \rho_Y \nu_Y \left\langle f_1 \phi_I' \right\rangle_1 I_t \frac{\partial}{\partial I_t}) \frac{1}{2} I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

Let the quantities $V_2^{I,\epsilon}$ and $V_3^{I,\epsilon}$ be defined as below:

$$\begin{aligned} V_2^{I,\epsilon} &= -\frac{\sqrt{\epsilon}}{\sqrt{2}} \nu_Y \left\langle \phi_I' \chi_1 \right\rangle_1, \\ V_3^{I,\epsilon} &= \frac{\sqrt{\epsilon}}{\sqrt{2}} \rho_Y \nu_Y \left\langle \phi_I' f \right\rangle_1. \end{aligned}$$

Using $V_2^{I,\epsilon}$ and $V_3^{I,\epsilon}$, the equation (1.28) becomes:

$$\left\langle \mathcal{L}_2^I \right\rangle_1 \sqrt{\epsilon} P_1^{I,\epsilon} = V_2^{I,\epsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t} (I_t^2 \frac{\partial^2 P_0}{\partial I_t^2}).$$

Therefore, $P_1^{I,\epsilon}$ is the solution of the following problem :

$$\begin{aligned} \left\langle \mathcal{L}_2^I \right\rangle_1 \sqrt{\epsilon} P_1^{I,\epsilon} &= V_2^{I,\epsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t} (I_t^2 \frac{\partial^2 P_0}{\partial I_t^2}), & (1.30) \\ P_1^{I,\epsilon}(T, I_T) &= 0. & (1.31) \end{aligned}$$

In order to simplify the notations, the following differential operators are defined:

$$\begin{aligned} \mathcal{D}_{1,I} &= I_t \frac{\partial}{\partial I_t}, \\ \mathcal{D}_{2,I} &= I_t^2 \frac{\partial^2}{\partial I_t^2}. \end{aligned}$$

Using the fact that $\left\langle \mathcal{L}_2^I \right\rangle_1 = \mathcal{L}_{BS}(\bar{\sigma}_I)$ commits with $\mathcal{D}_{1,I}$ and $\mathcal{D}_{2,I}$, and that $\left\langle \mathcal{L}_2^I \right\rangle_1 P_0 = 0$, the solution to the last problem can be given explicitly by:

$$\sqrt{\epsilon} P_1^{I,\epsilon} = -(T-t)(V_2^{I,\epsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t} (I_t^2 \frac{\partial^2 P_0}{\partial I_t^2})).$$

By neglecting terms of order higher or equal to 2 in $\sqrt{\epsilon}$, the option's price can be approximated by $(P_0^{I,\epsilon} + \sqrt{\epsilon} P_1^{I,\epsilon})$. As it was proven by the authors in Fouque and Kollman (2011), it is possible to carry out a parameter reduction method and approximate $P^{I,\epsilon}$ by the following formula:

$$P^{I,\epsilon} \sim \tilde{P_0}^{I,\epsilon} + \sqrt{\epsilon} \tilde{P_1}^{I,\epsilon},$$

such as:

$$\begin{aligned} \tilde{P_0}^{I,\epsilon} &= P_{BS}(\bar{\sigma}_I^*), & (1.32) \\ (\bar{\sigma}_I^*)^2 &= \bar{\sigma}_I^2 - 2 V_2^{I,\epsilon}, & (1.33) \\ \sqrt{\epsilon} \tilde{P_1}^{I,\epsilon} &= -(T-t) V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t} (I_t^2 \frac{\partial^2 \tilde{P}_0}{\partial I_t^2}). & (1.34) \end{aligned}$$

1.6.2 Appendix 2: Accuracy of the approximation

It can be seen here that:

$$\mathcal{L}_{BS}(\bar{\sigma}_S^*) = \mathcal{L}_{BS}(\bar{\sigma}_S) - V_2^{S,\epsilon} S_t^2 \frac{\partial^2}{\partial S_t^2}. \tag{1.35}$$

Using (1.35), it can be proved that:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S)(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon}) &= -V_2^{S,\epsilon} S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\epsilon}}{\partial S_t^2}, \\
(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})(T, S_T) &= 0.
\end{aligned}$$

The source term is $O(\sqrt{\epsilon})$ because of $V_2^{S,\epsilon}$, then the difference term $(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})$ is also $O(\sqrt{\epsilon})$. Consequently, it follows that:

$$|P^{S,\epsilon} - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})| \le |P^{S,\epsilon} - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})| + |(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})|,$$

The first term $|P^{S,\epsilon,\delta} - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})|$ is already $o(\epsilon)$. Therefore, the second term should be studied. To simplify the notations, the error term $\mathcal{R}$ is introduced as following:

$$\mathcal{R} = (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}).$$

Besides the differential operators $\mathcal{H}_\epsilon$ and $\mathcal{H}_\epsilon^*$ are defined:

$$\begin{aligned}
\mathcal{H}_\epsilon &= V_2^{S,\epsilon}\mathcal{D}_{2,S} + V_3^{S,\epsilon}\mathcal{D}_{1,S}\mathcal{D}_{2,S}, \\
\mathcal{H}_\epsilon^* &= V_3^{S,\epsilon}\mathcal{D}_{1,S}\mathcal{D}_{2,S}.
\end{aligned}$$

Using the previous notations, the quantity $\mathcal{L}_{BS}(\bar{\sigma}_S)\mathcal{R}$ can be computed:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S)\mathcal{R} &= \mathcal{L}_{BS}(\bar{\sigma}_S)((P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})), \\
&= \mathcal{H}_\epsilon P_0^{S,\epsilon} - (\mathcal{L}_{BS}(\bar{\sigma}_S^*) + V_2^{S,\epsilon}\mathcal{D}_{2,S})(\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}), \\
&= \mathcal{H}_\epsilon P_0^{S,\epsilon} - \mathcal{H}_\epsilon^* \tilde{P}_0^{S,\epsilon} - V_2^{S,\epsilon}\mathcal{D}_{2,S}(\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}), \\
&= \mathcal{H}_\epsilon^*(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon}) - V_2^{S,\epsilon}\mathcal{D}_{2,S}(\tilde{P}_0^{S,\epsilon} - P_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}).
\end{aligned}$$

Knowing that:

- $(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})$ is $O(\sqrt{\epsilon})$.
- $\mathcal{H}_\epsilon^*$ is $O(\sqrt{\epsilon})$.
- $V_2^{S,\epsilon}\mathcal{D}_{2,S}$ is $O(\sqrt{\epsilon})$.
- $\sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}$ is $O(\sqrt{\epsilon})$.

and additionally $\mathcal{R}(T) = 0$, then it follows that $\mathcal{R} = O(\epsilon)$. This concludes the derivation of the following result:

$$P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon} + O(\epsilon).$$

So up to order 1 in $\sqrt{\epsilon}$, the option price $P^{S,\epsilon}$ can be approximated by $\tilde{P}^{S,\epsilon}$ which is defined as:

$$\tilde{P}^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}.$$

The estimation error obtained, when approximating $P^{S,\epsilon}$ by $\tilde{P}^{S,\epsilon}$, is at order 1 in ϵ. Indeed, by neglecting terms of order higher to 1 in $\sqrt{\epsilon}$, the term $(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})$ is obtained as an approximation of the price $P(t, S_t, Y_t)$. It is then important to show that this approximation is of order 1 in ϵ meaning that:

$$|P(t, S_t, Y_t) - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})| \leq C\epsilon.$$

The proof of this property is given in Jean-Pierre et al. (2000) in the case where the payoff h is smooth. A summary of this proof is given here in order to make this chapter self contained.

Let us introduce the quantity $Z^{S,\epsilon}$ which verifies that:

$$P(t, S_t, Y_t) \quad = \quad P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} + \epsilon P_2^{S,\epsilon} + \epsilon^{\frac{3}{2}} P_3^{S,\epsilon} - Z^{S,\epsilon}.$$

Since $\mathcal{L}^S P(t, S_t, Y_t) = 0$, it follows that:

$$\mathcal{L}^S Z^{S,\epsilon} \quad = \quad \mathcal{L}_S(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} + \epsilon P_2^{S,\epsilon} + \epsilon^{\frac{3}{2}} P_3^{S,\epsilon}).$$

The differential operator $\mathcal{L}^S$ can be written as $\mathcal{L}^S = \mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S$. By developing $\mathcal{L}^S Z^{S,\epsilon}$ and regrouping the terms by orders of $\sqrt{\epsilon}$, it follows that:

$$\begin{aligned}
\mathcal{L}^S Z^{S,\epsilon} = \frac{1}{\epsilon}\mathcal{L}_0^S P_0^{S,\epsilon} + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0^S P_1^{S,\epsilon} + \mathcal{L}_1^S P_0^{S,\epsilon}) + (\mathcal{L}_0^S P_2^{S,\epsilon} + \mathcal{L}_1^S P_1^{S,\epsilon} + \mathcal{L}_2^S P_0^{S,\epsilon}) \\
+ \sqrt{\epsilon}(\mathcal{L}_0^S P_3^{S,\epsilon} + \mathcal{L}_1^S P_2^{S,\epsilon} + \mathcal{L}_2^S P_1^{S,\epsilon}) + \epsilon(\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{3}{2}}\mathcal{L}_2^S P_3^{S,\epsilon}.
\end{aligned}$$

The terms $P_0^{S,\epsilon}$, $P_1^{S,\epsilon}$ and $P_2^{S,\epsilon}$ are chosen to nullify the first four terms in the previous equation, therefore:

$$\mathcal{L}^S Z^{S,\epsilon} = \epsilon(\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{3}{2}}\mathcal{L}_2^S P_3^{S,\epsilon},$$

and $Z^{S,\epsilon}$ satisfies the final condition:

$$Z^{S,\epsilon}(T, S_T, Y_T, Z_T) \quad = \quad \epsilon P_2(T, S_T, Y_T, Z_T) + \epsilon^{\frac{3}{2}} P_3(T, S_T, Y_T, Z_T).$$

Using the Feynman-Kac theorem, it follows that:

$$\begin{aligned}
Z^{S,\epsilon}(t, x, y, z) = \epsilon E(e^{-r(T-t)}(P_2(T, S_T, Y_T, Z_T) + \epsilon^{\frac{1}{2}} P_3(T, S_T, Y_T, Z_T)) \\
- \int_t^T e^{-r(u-t)}((\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{1}{2}}\mathcal{L}_2^S P_3^{S,\epsilon})(u, S_u, Y_u, Z_u)du | S_t = x, Y_t = y, Z_t = z).
\end{aligned}$$

Under assumptions on the smoothness of the payoff function h and boundedness of the functions χ_1 and χ_2, the term $Z^{S,\epsilon}$ is at most linearly growing in $|y|$ and $|z|$ and then $Z^{S,\epsilon}(t, x, y, z) = O(\epsilon)$ The demonstration of the accuracy of the approximation for a non smooth payoff h (as in the case of a call option) is derived in Papanicolaou et al. (2003).

1.6.3 Appendix 3: Approximation of the implied volatility

the method developed here was suggested in Fouque and Kollman (2011) for the model with constant idiosyncratic volatility.

The symbol A is used to denote either the stock S or the index I. The price of an European option $P^{A,\epsilon}$ on the asset A can be written as:

$$P^{A,\epsilon} \quad = \quad \tilde{P}_0^{A,\epsilon} - (T-t)V_3^{A,\epsilon} A_t \frac{\partial}{\partial A_t}(A_t^2 \frac{\partial^2 \tilde{P}_0^{A,\epsilon}}{\partial A_t^2}) + o(\epsilon)\,, \tag{1.36}$$

where $\tilde{P}_0^{A,\epsilon}$ is defined as:

$$\tilde{P}_0^{A,\epsilon} \quad = \quad P_{BS}(t, A_t, \bar{\sigma}_A^*).$$

The term $P^{A,\epsilon}$ could represent the price of the option on the index $P^{I,\epsilon}$ if $A = I$ or the price of the option on the stock $P^{S,\epsilon}$ in the case where $A = S$.

Let I_A be the implied volatility associated to the asset's option price $P^{A,\epsilon}$ meaning that $P^{A,\epsilon} = P_{BS}(t, A_t, I_A(K_A, T))$. An expansion of $I_A(K, T)$ could be made around $\bar{\sigma}_A^*$ in powers of $\sqrt{\epsilon}$:

$$I_A(K_A, T) = \bar{\sigma}_A^* + \sqrt{\epsilon} I_1(K_A, T) + O(\epsilon)\,.$$

Using Taylor's formula, it follows that:

$$P^{A,\epsilon} = P_{BS}(t, A_t, \bar{\sigma}_A^*) + \frac{\partial P_{BS}}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\epsilon} I_1 + o(\epsilon)\,. \tag{1.37}$$

Combining the equations (1.36) and (1.37) gives:

$$\frac{\partial P_{BS}(t, A_t, \bar{\sigma}_A^*)}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\epsilon} I_1(K, T) = -(T-t) V_3^{A,\epsilon} A_t \frac{\partial}{\partial A_t}(A_t^2 \frac{\partial^2 \tilde{P}_0}{\partial A_t^2}). \tag{1.38}$$

Performing simple computations on the derivatives of the Black-Scholes price yields that:

$$\mathcal{D}_{2,A} \tilde{P}_0 \quad = \quad \frac{1}{\bar{\sigma}_A^*(T-t)} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

By applying then the operator $\mathcal{D}_{1,A}$ to the last equation, it can be obtained that:

$$\mathcal{D}_{1,A} \mathcal{D}_{2,A} \tilde{P}_0 \quad = \quad \frac{A_t}{\bar{\sigma}_A^*(T-t)} \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

Using closed-form formulas of Black-Scholes greeks, it can be written that:

$$A_t \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*) = -\frac{d_2}{\bar{\sigma}_A^* \sqrt{T-t}} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

The equation (1.38) can be written as:

$$\sqrt{\epsilon} I_1(K_A, T) = -\frac{V_3^{A,\epsilon}}{\bar{\sigma}_A^*} \frac{A_t \frac{\partial^2 \tilde{P}_0}{\partial A_t \partial \bar{\sigma}_A^*}}{\frac{\partial \tilde{P}_0}{\partial \bar{\sigma}_A^*}}.$$

Then, it is straightforward that:

$$\sqrt{\epsilon} I_1(K_A, T) = \frac{V_3^{A,\epsilon} d_2(K_A, T)}{(\bar{\sigma}_A^*)^2 \sqrt{T-t}},$$

where:

$$d_2(K_A, T) \quad = \quad \frac{\log(\frac{A_t e^{r(T-t)}}{K_A}) - \frac{(\bar{\sigma}_A^*)^2}{2}(T-t)}{\bar{\sigma}_A^* \sqrt{T-t}}.$$

The implied volatility can then be approximated using the following formula:

$$I_A(K_A, T) \quad = \quad \bar{\sigma}_A^* - \frac{V_3^{A,\epsilon}}{2\bar{\sigma}_A^*} + \frac{V_3^{A,\epsilon}}{(\bar{\sigma}_A^*)^3} \frac{\log(\frac{F_A(T)}{K_A})}{T-t}. \tag{1.39}$$

Then, the following smile approximation can be obtained:

$$I_A(K_A,T) \quad = \quad b_A + a_A \frac{\log(\frac{F_A(T)}{K_A})}{T-t},$$

with:

$$\begin{aligned} b_A &= \bar{\sigma}_A^* - \frac{V_3^{A,\epsilon}}{2\bar{\sigma}_A^*}, \\ a_A &= \frac{V_3^{A,\epsilon}}{(\bar{\sigma}_A^*)^3}. \end{aligned}$$

Chapter 2

Optimal market making of options in the one-dimensional case

Note : A part of this chapter is submitted to Market Microstructure and Liquidity.

2.1 Introduction

Market makers have a mandate to provide liquidity on some securities that they choose in advance. Their role consists in continuously setting bid and ask quotes on the instruments in which they make markets and are obligated to buy and sell at their displayed bid and offer prices. Market makers have a fundamental and important role in financial markets since they obviously increase the liquidity of their quoted instruments. Indeed, by setting their quotes in the order book, they increase its depth and they provide liquidity for impatient agents who are willing to cross the bid-ask spread to get their orders executed immediately. In periods of imbalance between buy and sell orders, market makers could hold a significant directional position for a while and accumulate a non negligible inventory. Thus, they may be required to bear a market risk which could cause significant losses. To be paid for such risks, market makers expect a positive profit since they set bid and ask quotes around the mid price, and then they offer to buy (respectively sell) at a bid price (respectively ask price) which is lower (respectively higher) than the mid price.

This chapter deals with the problem of options market making. There have not been many studies devoted to market making in options despite the importance of this subject. However, we can find, in the recent literature, several works focusing on the problem of market making and high frequency trading of stocks through the use of a stochastic optimization method. Avellaneda and Stoikov get inspired from the work of Ho and Stoll in Ho and Stoll (1981), and addressed in Avellaneda and Stoikov (2008) the problem of high frequency market making of a stock. In their model framework, the stock price process diffuses as an arithmetic Brownian motion in the intraday time-scale, and the market maker aims to determine the optimal quotes in order to maximize her expected exponential utility from terminal wealth. The authors solved the optimization problem using a stochastic control approach and provided approximations of the optimal quotes near the terminal time T using an expansion in the inventory parameter. Following that, Guéant, Lehalle and Fernandez-Tapia dealt in Guéant et al. (2012) with the problem of optimally setting limit orders for the purpose of portfolio liquidation. The authors provided analytic expressions for optimal bid quotes (respectively ask quotes) in order to liquidate a short portfolio position (respectively a long portfolio position). In Guéant et al. (2013), they used a similar technique to deal with the problem of market making with inventory risk. They provide expressions for the optimal bid and ask quotes in the case where the market maker stops setting ask quotes (respectively bid quotes) if her inventory exceeds some negative level $-\mathcal{R}$ (respectively some positive level $\mathcal{R}$). This risk constraint allows the authors to provide closed-form formula in order to solve the problem analytically. Next to that, Fodra and Labadie extended in Fodra and Labadie (2012) the work of Avellaneda and Stoikov to the case where the stock price process has a drift term and a stochastic volatility. They provided the optimal bid and ask quotes in the case where the utility function is linear, but the solution is not valid under all stock models. Indeed, the value function is finite when the stock is an Ornstein-Uhlenbeck process for example but is not defined when the stock is a geometric Brownian motion. In Fodra and Labadie (2013), they extended their previous work by perturbing the utility function with a constraint on the accumulated inventory and they provided an approximation of the solution of the problem.

The authors in Cartea et al. (2013) addressed the problem of model ambiguity and its effect on the optimal market making strategy. In their case of study, the ambiguity includes the choice of the model for the stock dynamics or also for the arrival rate of market orders. Using an entropic penalization, the authors solved the optimization problem and showed how the market maker adjusts her quotes to reduce adverse selection costs. Following that, the authors in Nyström et al. (2014) dealt with the market making problem in the case of ambiguity concerning the correct probability distribution of the asset. Since the asset price process follows a Bachelier model, the uncertainty concerns the drift and the volatility. The authors formulated a "worst-case" stochastic optimal control problem where the risk-averse market maker aims to maximize her exponential utility function from terminal wealth in the worst possible scenario. After obtaining analytic expressions for the optimal bid and ask quotes, the authors interpreted the effect of the uncertainty on the optimal strategy. In particular, they pointed out that as the uncertainty increases, the market maker has to adapt her quotes in order to reduce her inventory.

In Stoikov and Sağlam (2009), Stoikov and Sağlam proposed a mean variance framework to study the problem of optimal market making in options. At the beginning, the authors delivered the optimal bid and ask quotes on the option in the case of a complete market where the volatility of the stock is constant and the option can be perfectly hedged by trading the stock. Afterward, they removed the hypothesis of constant volatility and they supposed that the stock and its instantaneous volatility remain constant during the trading session but may have large overnight moves. Using a dynamic programming method, they provide analytic expressions for the optimal quotes using a discrete time grid.

In this chapter, the problem of optimal market making of options in a continuous time setting is addressed. The dynamics of the instantaneous volatility of the stock are specified in order to deliver a realistic framework for option pricing. Thus, the dimension of the market making problem increases as the instantaneous volatility represents an additional state variable. In addition, it is supposed here that the market maker hedges continuously her option inventory by trading the liquid stock which makes her hold an inventory in the stock. We aim, through this study, to determine the optimal market making strategy on options in the setting of a generic stochastic volatility model. In the second section, we define the joint dynamics of the spot and its instantaneous volatility under both the real-world probability measure and the pricing measure. In the third section, we present several forms of the market impact function in the option market. Conditional to this function, the intensity of arrival of market orders is determined analytically for the purpose of determination of the optimal strategy. In the fourth section, we consider that the market maker is risk neutral and aims to maximize the expectation of her wealth at the maturity date of the option. Thus, we determine analytic expressions for the optimal quotes through the use of a stochastic control approach. In the fifth section, the market maker is supposed to be risk averse, she aims to maximize the expectation of her final wealth while reducing its variance. We propose, in this case, a mean-variance framework to determine the optimal quoting policy. Indeed, the utility function of the market maker is penalized by an additional term which depends on the variance of final wealth. By means of an asymptotic expansion in the penalty parameter, we provide approximations of the optimal quotes and show how they are affected by the different variables. In the final section, we generate Monte Carlo simulations in order to test numerically the performance of the strategy. Through these simulations, we compare the performances of the optimal market making strategy and a zero-intelligence one. Next to that, we show the effects of the misspecification of the parameters on both the performance of the strategy and the accumulated inventory.

2.2 Model setup

Consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ satisfies the usual conditions, and where $\mathcal{P}$ denotes the real-world probability measure. Under $\mathcal{P}$, the spot process S has the following dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(y_t) dW_t^{(1)}, \tag{2.1}$$

$$dy_t = a_R(y_t)dt + b_R(y_t)dW_t^{(2)}, \tag{2.2}$$

where $W^{(1)}$ and $W^{(2)}$ are two $(\mathcal{P}, \mathcal{F})$ Brownian motions such that $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho_R dt$. The functions a_R, b_R satisfy sufficient conditions which ensure the existence of a strong solution to (2.2), besides $\forall T > 0$:

$$E^{\mathcal{P}}\left(\int_0^T \sigma(y_t)^2 dt\right) < +\infty,$$

$$E^{\mathcal{P}}\left(\int_0^T \left(a_R^2(y_t) + b_R^2(y_t)\right) dt\right) < +\infty,$$

Suppose that there is an European option with maturity T and payoff function $h(S_T)$ which is traded in the option market. Let the quantity $C_{\mathcal{P}}$ be defined as:

$$C_{\mathcal{P}}(t, S_t, y_t) = E^{\mathcal{P}}\left(h(S_T)|\mathcal{F}_t\right). \tag{2.3}$$

The quantity $C_{\mathcal{P}}$ doesn't represent the option price under $\mathcal{P}$ since $\mathcal{P}$ is not a risk-neutral probability measure. Nevertheless, this quantity will be useful in the sequel.

Market participants price options under a risk-neutral probability measure $\mathcal{Q}$ different from $\mathcal{P}$. To make the model framework as general as possible, it is supposed here that market agents could have a misleading view of the dynamics of the process (y), and then they may misspecify the parameters characterizing its distribution. This feature, which was discussed and investigated in Jena and Tankov (2011), accounts for the discrepancy between the physical measure and the pricing measure.

Let the interest rate be equal to r. It is supposed that, under the pricing measure $\mathcal{Q}$, the process S evolves as follows:

$$\frac{dS_t}{S_t} = r dt + \sigma(y_t) dW_t^{*,(1)}, \tag{2.4}$$

$$dy_t = a_I(y_t)dt + b_I(y_t)dW_t^{*,(2)}, \tag{2.5}$$

where $W^{*,(1)}$ and $W^{*,(2)}$ are two $(\mathcal{Q}, \mathcal{F})$ Brownian motions such that $d\left\langle W^{*,(1)}, W^{*,(2)}\right\rangle_t = \rho_I dt$.

The equation (2.5) has a strong solution under sufficient conditions on a_I and b_I, and $\forall T > 0$:

$$E^{\mathcal{Q}}\left(\int_0^T \sigma(y_t)^2 dt\right) < +\infty,$$
$$E^{\mathcal{Q}}\left(\int_0^T \left(a_R^2(y_t) + b_R^2(y_t)\right) dt\right) < +\infty.$$

The option price $C_{\mathcal{Q}}$ under the risk-neutral probability measure $\mathcal{Q}$ writes:

$$C_{\mathcal{Q}}(t, S_t, y_t) = e^{-r(T-t)} E^{\mathcal{Q}}(h(S_T)|\mathcal{F}_t). \tag{2.6}$$

Suppose that there is an agent who makes markets in this option and who trades with limit orders in the form of bid and ask quotes around the option's mid price $C_{\mathcal{Q}}$. In addition, the agent of interest trades continuously in the liquid stock in order to keep the option inventory delta-hedged. At a given time t, this market maker sets an ask price C_t^a and a bid price C_t^b such that:

$$\begin{aligned} C_t^a &= C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+, \\ C_t^b &= C_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-, \end{aligned}$$

where δ_t^- and δ_t^+ are positive quantities representing respectively the bid distance and the ask distance of the market maker. Those distances influence the option inventory held by the market maker since they affect the arrival rate of market buy and sell orders.

The notations defined here are going to be used in the rest of the study:

- $q_{1,t}$ denotes the option inventory held by the market maker at time t.
- $q_{2,t}$ is the stock inventory held by the market maker at time t.
- X_t is the cash held by the market maker at time t.
- $C_{\mathcal{Q}}(t, S_t, y_t)$ is the mid price of the option at time t.
- $\Delta_t = \Delta(t, S_t, y_t)$ is the option's delta at time t.
- $\mathcal{W}_t$ denotes the wealth of the market maker at time t.

The arrival of liquidity-consuming orders to the order book are modeled by two independent Poisson processes: N^+ for buy orders consuming the ask quotes, and N^- for sell orders consuming the bid quotes. Therefore, the dynamics of the process q_1 can be described as following:

$$dq_{1,t} = dN_t^- - dN_t^+. \tag{2.7}$$

The market maker delta-hedges continuously her option position. Thus, her inventory in stock $q_{2,t}$ at time t verifies the relation $q_{2,t} = -q_{1,t}\Delta_t$, and then:

$$dq_{2,t} = -\Delta_t dq_{1,t} - q_{1,t} d\Delta_t - d\left\langle q_1, \Delta\right\rangle_t = -\Delta_t dN_t^- + \Delta_t dN_t^+ - q_{1,t} d\Delta_t. \tag{2.8}$$

The cash process (X) of the market maker is affected by three factors. Indeed, if her ask quote gets executed, she receives the ask price $(C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+)$. On the other hand, if her bid quote gets executed, she pays the bid price $(C_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-)$. In addition to that, the continuous trading in the stock, for the purpose of delta-hedging, induces a continuous variation in the cash process. Thus, the process (X) has the following dynamics:

$$dX_t = (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+) dN_t^+ - (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-) dN_t^- + q_{2,t} dS_t. \tag{2.9}$$

The wealth of the market maker at time t can be computed as follows:

$$\mathcal{W}_t = X_t + q_{1,t} C_{\mathcal{Q}}(t, S_t, y_t). \tag{2.10}$$

The market maker aims to provide liquidity in the traded option while staying delta-hedged. She wants to choose optimally her strategy in order to maximize her expected utility from terminal wealth. In order to formulate the optimization problem, a crucial step is to choose a model for the intensities of the processes N^+ and N^-.

2.3 Intensity of arrivals of market orders

Several works have been recently carried out to study the intensity of arrival of market-orders in the order book of a stock and then deduce the probability of execution of limit orders. Indeed, in Omura et al. (2000), the authors pointed out that the execution probability of limit orders is affected by the liquidity on the opposite side of the order book and also the bid-ask spread. In addition, in Lo et al. (2002), the authors used survival analysis in order to estimate the conditional distribution of limit-order execution times as a function of different variables such as the limit price, the order size, and current market conditions (volatility, bid-ask spread). In particular, the authors showed empirically that the larger is the distance between the mid-quote price and the limit price, the longer is the expected time-to-execution.

In our study, we will neglect the effects of the volatility and the bid-ask spread on the probability of execution of limit orders. Indeed, we use similar assumptions as in Avellaneda and Stoikov (2008) and we suppose that:

- the distribution density $f_{\mathcal{V}}$ of market orders size is a power-law density:

$$\forall x > 0, f_{\mathcal{V}}(x) = \frac{\gamma L^\gamma}{(L+x)^{\gamma+1}},$$

  where $\gamma > 1$.

- the price impact ΔP following a market order of size $\mathcal{V}$ writes:

$$\Delta P = K \mathcal{V}^\beta.$$

- the intensity of arrival of market orders to the order book is constant

Under theses hypothesis, we prove in (2.8.1), that the intensity of arrival of a market order consuming a limit order situated at a distance δ from the mid-price has the form:

$$\lambda(\delta) = \frac{A}{\left(B + \delta^{\frac{1}{\beta}}\right)^\gamma},$$

where $A, B > 0$.

2.4 Optimization problem for a risk-neutral market maker

The market maker aims to maximize her expected utility from terminal wealth at the option's maturity date T. This terminal wealth consists of the sum of the cash amount and the market value of the option inventory. The objective of the market maker is to determine, at a given time t, the optimal distances $(\delta^+_{L,*,t}, \delta^-_{L,*,t})$ solution of the following problem:

$$(\delta^+_{L,*,t}, \delta^-_{L,*,t}) = ArgSup_{\{\delta^+_t, \delta^-_t\}} E^{\mathcal{P}}(U(X_T + q_{1,T} h(S_T)) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x).$$

where U is the utility function.

In this section, the market maker is considered to be risk-neutral, thus the utility function U is linear:

$$U(T, S_T, y_T, q_{1,T}, X_T) = X_T + q_{1,T} h(S_T), \tag{2.11}$$

where:

$$X_T = \int_0^T (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+) dN_t^+ - \int_0^T (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-) dN_t^- + \int_0^T q_{2,t} dS_t.$$

The problem associated with the linear utility function is considered here. The cases corresponding to $\beta = \frac{1}{2}$ (square root market impact) and $\beta = 1$ (linear market impact) are studied separately and the optimal bid and ask quotes are provided analytically in each case.

In order to solve the optimization problem, a stochastic control approach is used. The value function of the market maker can be defined in the following way:

$$u(t, s, y, q_1, x) = Sup_{\{(\delta_t^+, \delta_t^-) \in \mathcal{A}\}} E^{\mathcal{P}}(X_T + q_{1,T} h(S_T)) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x),$$

where $\mathcal{A}$ denotes the set of admissible values for the controls (δ_t^-, δ_t^+) and is equal to $\mathbb{R}^+ \times \mathbb{R}^+$.

Let the differential operators $\mathcal{L}_1, \mathcal{L}_2$ be introduced below:

$$\begin{aligned}
\mathcal{L}_1 &= \mu S_t \frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(y_t) S_t^2 \frac{\partial^2}{\partial S^2} + a_R(y_t) \frac{\partial}{\partial y} + \frac{1}{2} b_R^2(y_t) \frac{\partial^2}{\partial y^2} + \rho_R b_R(y_t) \sigma(y_t) S_t \frac{\partial^2}{\partial S \partial y}, \\
\mathcal{L}_2 &= \frac{\partial}{\partial x} q_{2,t} \mu S_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_{2,t}^2 \sigma(y_t)^2 S_t^2 + \frac{\partial^2}{\partial x \partial S} q_{2,t} \sigma^2(y_t) S_t^2 + \frac{\partial^2}{\partial x \partial y} q_{2,t} \sigma(y_t) S_t b_R(y_t) \rho_R,
\end{aligned}$$

Let u_0 be the solution of the following Hamilton-Jacobi-Bellman equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2) u_0 + sup_{\{(\delta_t^+, \delta_t^-) \in \mathcal{A}\}} \left(J^+ \left(\delta_t^+\right) + J^- \left(\delta_t^-\right) \right) = 0, \tag{2.12}$$

where the functions J^+ and J^- are defined as follows:

$$\begin{aligned}
J^+(\delta_t^+) &= \lambda^+(\delta_t^+) \left(u_0(t, s, y, q_{1,t^-} - 1, x_{t^-} + (C_{\mathcal{Q}} + \delta_t^+)) - u_0(t, s, y, q_{1,t^-}, x_{t^-}) \right), \\
J^-(\delta_t^-) &= \lambda^-(\delta_t^-) \left(u_0(t, s, y, q_{1,t^-} + 1, x_{t^-} - (C_{\mathcal{Q}} - \delta_t^-)) - u_0(t, s, y, q_{1,t^-}, x_{t^-}) \right),
\end{aligned}$$

and u_0 is subject to the final condition:

$$u_0(T, s, y, q_1, x) = x + q_1 h(s).$$

If u_0 is smooth, finite and has polynomial growth, we show similarly as it was done in Pham (2007), that it coincides with the value function u, that is $u_0(t, s, y, q_1, x) = u(t, s, y, q_1, x)$.

2.4.1 The case where $\beta = \frac{1}{2}$ (square root market impact)

The intensities of arrivals of market orders can be written as follows:

$\forall \delta^+, \delta^- \geq 0$, $\lambda^+(\delta^+) = \frac{A}{\left(B + (\delta^+)^2\right)^\gamma}$ and $\lambda^-(\delta^-) = \frac{A}{\left(B + (\delta^-)^2\right)^\gamma}$,

where $A, B \geq 0$ and $\gamma > 1$.

Analytic solution

Proposition 2.1. *Let* $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)$.

The optimal controls $(\delta^+_{L,*,t}, \delta^-_{L,*,t})$ *of the market maker at time* t *are:*

$$\delta^+_{L,*,t} = \frac{\sqrt{\gamma^2 M_0^2(t,s,y) + B(2\gamma-1)} - \gamma M_0(t,s,y)}{2\gamma-1}, \tag{2.13}$$

$$\delta^-_{L,*,t} = \frac{\gamma M_0(t,s,y) + \sqrt{\gamma^2 M_0^2(t,s,y) + B(2\gamma-1)}}{2\gamma-1}, \tag{2.14}$$

and the value function is:

$$u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right), \tag{2.15}$$

where:

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right).$$

and:

$$\begin{aligned} J_0(t,s,y) = \lambda^+(\delta^+_{L,*,t})&\left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1}\right) \\ + \lambda^-(\delta^-_{L,*,t})&\left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1}\right). \end{aligned}$$

Proof. Since the utility function is linear as given in (2.11), we will look for a solution of the form:

$$u(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\theta_1(t,s,y), \tag{2.16}$$

Let $f_0^+ = J^+$. Using (2.16), the function f_0^+ can be written explicitly as:

$$f_0^+(\delta^+) = \lambda^+(\delta^+)(\delta^+ + M_0(t,s,y)),$$

where $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \theta_1(t,s,y)$.
In order to determine $\delta^+_{L,*,t} = ArgMax_{\{x>0\}} f_0^+(x)$, the derivative of the function f_0^+ is computed as follows:

$$(f_0^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B + (\delta^+)^2}\left(\left(\delta^+\right)^2(1-2\gamma) - 2\gamma M_0(t,s,y)\,\delta^+ + B\right).$$

The function $(f_0^+)'$ vanishes at the points x_1^+ and x_2^+:

$$x_1^+ = \frac{-\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}, \qquad x_2^+ = \frac{-\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1},$$

Since $\gamma \geq 1$ and $B > 0$:

$$\gamma|M_0| \leq \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)},$$

which implies that $x_1^+ < 0$ and $x_2^+ > 0$.

It can be recalled here that $1-2\gamma < 0$. Therefore, it can be deduced that the function f_0^+ reaches its maximum on R^+ at x_2^+. It follows that:

$$\delta^+_{L,*,t} = \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} - \gamma M_0}{2\gamma - 1}$$

and:

$$f_0^+(\delta^+_{L,*,t}) = \lambda^+(\delta^+_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1} \right). \tag{2.17}$$

Let $f_0^- = J^-$. Using the form of the value function suggested in (2.16), the function f_0^- writes:

$$f_0^-(\delta^-) = \lambda^-(\delta^-)(\delta^- - M_0(t,s,y)).$$

In order to determine $\delta^-_{L,*,t} = ArgMax_{\{x>0\}} f_0^-(x)$, the derivative of the function f_0^- is computed:

$$(f_0^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + (\delta^-)^2} \left((1-2\gamma)(\delta^-)^2 + 2\gamma M_0 \delta^- + B \right).$$

The function $(f_0^-)'$ changes sign and becomes null at the two following points:

$$x_1^- = \frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}, \qquad x_2^- = \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1},$$

Using the same reasoning as before, it can be proved that $x_1^- < 0$ and $x_2^- > 0$ and that:

$$\delta^-_{L,*,t} = \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}.$$

The function f_0^- evaluated at its maximum gives:

$$f_0^-(\delta^-_{L,*,t}) = \lambda^-(\delta^-_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1} \right). \tag{2.18}$$

Finally, using (2.17) and (2.18), $J_0(t,s,y) = f_0^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t})$ can be written as follows:

$$\begin{aligned} J_0(t,s,y) = {} & \lambda^+(\delta^+_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1} \right) \\ & + \lambda^-(\delta^-_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1} \right). \end{aligned}$$

and it is straightforward to see that J_0 is independent of q_1.

The equation (2.12) becomes:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u + J_0(t,s,y) = 0.$$

For this equation to be solved, the terms are sorted by powers of q_1. The functions θ_0, θ_1 and J_0 are independent of q_1 (they depend only on t, s, y). The terms obtained after the classification are either of order 0 or 1 in q_1. By nullifying these terms, we obtain the following equations:

$$\begin{aligned} (0) &: (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0, \\ (1) &: (\partial_t + \mathcal{L}_1)\theta_1 - \Delta_t \mu S_t = 0, \end{aligned}$$

The function θ_1 is the solution of the (1)-term (at order 1 in q_1):

$$(\partial_t + \mathcal{L}_1)\theta_1 - \Delta_t \mu S_t \quad = \quad 0,$$

and is subject to the final condition $\theta_1(T,s,y) = h(s)$. Using the Feynman-Kac formula, the following is obtained:

$$\theta_1(t,s,y) = E^{\mathcal{P}}_{t,s,y}(h(S_T)) - \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du),$$

The quantity $M_0(t,s,y)$ can now be computed explicitly:

$$M_0(t,s,y) \quad = \quad C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du).$$

Finally, the function θ_0 is the solution of the following equation:

$$(\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) \quad = \quad 0,$$

and satisfies also the final condition $\theta_0(T,s,y) = 0$. By the use of the Feynman-Kac formula:

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right),$$

As a conclusion, $u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du)\right)$ is the solution of the HJB equation (2.12).

Using a similar approach as in Theorem 3.5.2 in Pham (2007), we give in (2.8.5) the details of the verification theorem in our case of study. In this verification theorem, we show that it is sufficient to prove that u_0 is finite, smooth and has a polynomial growth.

By using the concavity of the square root function, we obtain:

$$|J_0(t,s,y)| \leq \frac{2A}{B^\gamma}\frac{(2\gamma-1)|M_0| + \sqrt{B(2\gamma-1)}}{2\gamma-1},$$

We assume here that the traded option is a call or a put, then $\exists\, C_1 > 0$ such that:

$$max\left(C_{\mathcal{P}}(t,s,y), C_{\mathcal{Q}}(t,s,y)\right) \leq C_1(1+S_t)$$

and we also have $|\Delta(t,s,y)| \leq 1$. It follows that $\exists M_1 > 0$, $\forall u < T$, $|M_0(u,S_u,y_u)| \leq M_1(1+S_u)$. Using this result, we can state that:

$$\begin{aligned}
|E_{t,s,y}\left(J_0(u,S_u,y_u)\right)| &\leq E_{t,s,y}\left|J_0(u,S_u,y_u)\right|, \\
&\leq \frac{2A}{B^\gamma(2\gamma-1)}\left((2\gamma-1)\left(M_1 + M_1 S_t e^{\mu(u-t)}\right) + \sqrt{B(2\gamma-1)}\right),
\end{aligned}$$

and then:

$$|\theta_0(t,s,y)| \leq \frac{2A}{B^\gamma}\left(\left(M_1 + \sqrt{\frac{B}{2\gamma-1}}\right)(T-t) + M_1 S_t \frac{e^{\mu(T-t)}-1}{\mu}\right).$$

We can now write the following statement:

$$|u_0(t,s,y,q_1,x)| \leq |x| + |\theta_0(t,s,y)| + |q_1|\left(C_{\mathcal{P}}(t,s,y) + S_t e^{\mu(T-t)}\right),$$

It can be seen that $\exists C_2 > 0$ such that:

$$|u_0(t,s,y,q_1,x)| \leq C_2\left(1 + x^2 + s^2 + y^2 + q_1^2\right),$$

and then u_0 is finite and has a quadratic growth.

Besides, u_0 is smooth since it is a combination of smooth functions. Consequently, the function u_0 coincides with the value function. ■

Interpretation of the strategy

The quantity $C_{\mathcal{P}}(t,s,y)$ represents the expected payoff of the option under the physical measure $\mathcal{P}$. In addition, the quantity $\mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right) = \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)dS_u\right)$ represents the cost of delta-hedging of the option due to the presence of a trend in the dynamics of the underlying. Thus, the indifference price for the option under the probability measure $\mathcal{P}$ is $C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)$. Thus, it can be deduced that $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right)$ represents the difference between the price of the option under the risk-neutral probability $\mathcal{Q}$ and its indifference price under the historic probability $\mathcal{P}$.

Let f, g be the functions defined as:

$$f(x) = \frac{\gamma x + \sqrt{\gamma^2 x^2 + B(2\gamma-1)}}{2\gamma-1}, \quad g(x) = \frac{\sqrt{\gamma^2 x^2 + B(2\gamma-1)} - \gamma x}{2\gamma-1}.$$

By simple derivation, it can be shown:

$$\begin{aligned}
\frac{\partial \delta^-_{L,*,t}}{\partial M_0} &= f'(M_0) = \frac{\gamma}{2\gamma-1}\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + \gamma M_0}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}} \geq 0,\\
\frac{\partial \delta^+_{L,*,t}}{\partial M_0} &= g'(M_0) = \frac{\gamma}{2\gamma-1}\frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}} \leq 0,
\end{aligned}$$

It follows that the bid distance $\delta^-_{L,*,t} = f(M_0)$ is an increasing function of M_0 while the ask distance $\delta^+_{L,*,t} = g(M_0)$ is a decreasing function of M_0. Indeed, if M_0 increases, it becomes more rational for the market maker to sell the option. Thus, she lowers her bid quote ($\delta^-_{L,*,t}$ increases) and also her ask quote ($\delta^+_{L,*,t}$ decreases). In this way, she is more likely to get her ask quote executed and she lowers at the same time the probability of execution of her bid quote.

Following the same reasoning, if M_0 decreases, it gets more profitable to buy the option. Thus, the market maker highers both her bid quote ($\delta^-_{L,*,t}$ decreases) and her ask quote ($\delta^+_{L,*,t}$ increases).

In the particular case where $M_0 = 0$, the market price of the option is equal to its indifference price under $\mathcal{P}$. In this case $\delta^-_{L,*,t} = \delta^+_{L,*,t} = \frac{B}{\sqrt{2\gamma-1}}$. This means that the bid quote C^b_t and the ask quote C^a_t are symmetric around the mid price $C_{\mathcal{Q}}$, and the market-maker has no directional bets.

It can be deduced through the expressions of the optimal quotes $\delta^-_{L,*,t}$ and $\delta^+_{L,*,t}$ that, in the case of option mispricing, the market maker adjusts her bid and ask quotes in an optimal way. Indeed, she becomes more aggressive in the side that makes her profit from the mispricing term M_0 and more conservative in the opposite side. As a result, she insures her role of a liquidity provider while trying to profit from the mispricing induced by the misspecification of the parameters.

It is now interesting to study the impact of the parameters on the spread between the optimal bid and ask quotes. For this purpose, the bid-ask spread $\mathcal{S}_{L,*,t}$ of the market maker is defined in the following way: $\mathcal{S}_{L,*,t} = \delta^-_{L,*,t} + \delta^+_{L,*,t}$. Using analytic expressions of $\delta^-_{L,*,t}$ and $\delta^+_{L,*,t}$, it can be deduced that $\mathcal{S}_{L,*,t} = 2\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}$. By simple derivation with respect to the variable γ, it can be obtained:

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = -\frac{2\left(B\left(2\gamma-1\right) + \gamma M_0^2\right)}{\left(2\gamma-1\right)^2\sqrt{\gamma^2 M_0^2 + B\left(2\gamma-1\right)}} < 0.$$

This means that the bid-ask spread $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ. Indeed when γ increases, the intensity of arrivals of market orders decreases and the probability of execution of a quote at a distance δ from the mid price decreases. Consequently, the market maker contracts her bid-ask spread and places her quotes closer to the mid price. In addition, the spread $\mathcal{S}_{L,*,t}$ increases with the parameter B and doesn't depend on the parameter A. It can also be noticed that $\mathcal{S}_{L,*,t}$ is an increasing function of $|M_0|$, which means that the market maker widens her spread when the gap between the indifference price and the market price becomes significant.

2.4.2 The case where $\beta = 1$ (linear market impact)

The intensities of arrivals of market orders can be written as follows:

$\forall \delta^+, \delta^- \geq 0,\ \lambda^+(\delta^+) = \frac{A}{(B+\delta^+)^\gamma}$ and $\lambda^-(\delta^-) = \frac{A}{(B+\delta^-)^\gamma}$. The optimization problem can be solved similarly.

Analytic solution

Proposition 2.2. *Let* $\mathcal{S} = \frac{B}{\gamma}$, *and* $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta_u S_u du\right)$.

The optimal controls $(\delta^+_{L,*}, \delta^-_{L,*})$ *of the market maker are:*

$$\delta^+_{L,*,t} = \left(\frac{B - \gamma M_0(t,s,y)}{\gamma - 1}\right)^+, \tag{2.19}$$

$$\delta^-_{L,*,t} = \left(\frac{B + \gamma M_0(t,s,y)}{\gamma - 1}\right)^+, \tag{2.20}$$

and the value function is:

$$u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta_u S_u du\right)\right), \tag{2.21}$$

where:

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right),$$

and:

$$J_0(t,s,y) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-M_0(t,s,y))^{\gamma-1}} - \frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma (B+M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S},\mathcal{S}] \\ \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-M_0(t,s,y))^{\gamma-1}} + \frac{A}{(B-C)^\gamma}(-M_0(t,s,y)) & \text{if} \quad M_0(t,s,y) \in]-\infty,-\mathcal{S}] \\ -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma (B+M_0(t,s,y))^{\gamma-1}} + \frac{A}{B^\gamma}(M_0(t,s,y)) & \text{if} \quad M_0(t,s,y) \in [\mathcal{S},+\infty[\end{cases}$$

The proof is given in Appendix (2.8.2).

Interpretation of the strategy

Based on the results stated in (2.2), the following derivatives can be computed:

$$\frac{\partial \delta^+_{L,*,t}}{\partial M_0} = -\frac{\gamma}{\gamma-1} 1_{\{M_0 \leq \mathcal{S}\}} \leq 0,$$
$$\frac{\partial \delta^-_{L,*,t}}{\partial M_0} = \frac{\gamma}{\gamma-1} 1_{\{M_0 \geq -\mathcal{S}\}} \geq 0,$$

It can then be noticed that:

1. The distance $\delta^+_{L,*,t}$ of the ask-quote to the mid price is a decreasing function of the mispricing term M_0 (as it is the case for $\beta = \frac{1}{2}$).

2. The distance $\delta^-_{L,*,t}$ of the bid quote to the mid price is an increasing function of the mispricing term M_0 (same finding as for $\beta = \frac{1}{2}$).

The same conclusions drawn in the case where $\beta = \frac{1}{2}$ concerning the effect of the mispricing term M_0 on the optimal bid and ask quotes are conserved when $\beta = 1$.
The spread $\mathcal{S}_{L,*,t} = \delta^+_{L,*,t} + \delta^-_{L,*,t}$ writes:

$$\mathcal{S}_{L,*,t} = \begin{cases} \frac{2B}{\gamma-1} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S}, \mathcal{S}], \\ \frac{B-\gamma M_0(t,s,y)}{\gamma-1} & \text{if} \quad M_0(t,s,y) \leq -\mathcal{S}, \\ \frac{B+\gamma M_0(t,s,y)}{\gamma-1} & \text{if} \quad M_0(t,s,y) \geq \mathcal{S}, \end{cases}$$

By differentiating the last expression with respect to the variable γ, it follows that:

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = \begin{cases} -\frac{2B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S}, \mathcal{S}], \\ \frac{M_0(t,s,y)-B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \leq -\mathcal{S}, \\ -\frac{M_0(t,s,y)+B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \geq \mathcal{S}, \end{cases}$$

It can be seen that $\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} \leq 0$ and then $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ (same finding for $\beta = \frac{1}{2}$).

2.5 Optimization problem for a risk-averse market maker

The profit of the market maker comes from the bid-ask spread received when market orders hits her quotes and trades occur. Since the times of arrivals of market orders are uncertain, there is a risk on the value of future wealth and also on future inventory. In order to risk-manage her strategy, the market maker may solve at each time t an optimization problem in order to determine the optimal distances $\left(\delta^-_{t,*}, \delta^+_{t,*}\right)$ which allow her to maximize the expectation of her future wealth at the maturity date T while minimizing its variance.
In the following, to make the HJB equation possible to solve, the variance of final wealth is going to be approximated. Indeed, under the assumption that $\delta^+, \delta^- \ll C_{\mathcal{Q}}$, we can write:

$$X_T \sim \int_t^T C_{\mathcal{Q}}(u, S_u, y_u) dN^+_u - \int_t^T C_{\mathcal{Q}}(u, S_u, Y_u) dN^-_u + \int_0^T q_{2,u} dS_u.$$

Supposing that N^+ and N^- are independent Poisson processes, it can be deduced that the conditional variance at time t of X_T is:

$$V(X_T|\mathcal{F}_t) \sim E^{\mathcal{P}}\left(\int_t^T C^2_{\mathcal{Q}}(u, S_u, y_u)\left(\lambda^+_u + \lambda^-_u\right) du + \int_t^T q^2_{1,u} \Delta^2_u \sigma^2(y_u) S^2_u du\right),$$

Besides:

$$V(q_{1,T} h(S_T)|\mathcal{F}_t) \sim E^{\mathcal{P}}\left(\int_t^T E^{\mathcal{P}}\left(h^2(S_T)|\mathcal{F}_u\right)\left(\lambda^+_u + \lambda^-_u\right) du\right),$$

and:

$$Cov(X_T, q_{1,T} h(S_T)) \sim -E^{\mathcal{P}}\left(\int_t^T C_{\mathcal{Q}}(u, S_u, y_u) C_{\mathcal{P}}(u, S_u, y_u)\left(\lambda^+_u + \lambda^-_u\right) du\right)$$

Then $Var\left(X_T + q_{1,T}h(S_T)\right)$ can be approximated by $E^{\mathcal{P}}\left(\int_t^T \left(q_{1,u}^2 V_u + T_u\right) du\right)$ where:

$$
\begin{aligned}
T_t &= \left(C_{\mathcal{Q}}^2 + E^{\mathcal{P}}\left(h^2(S_T)|\mathcal{F}_t\right) - 2C_{\mathcal{Q}}C_{\mathcal{P}}\right)\left(\lambda_{L,t}^+ + \lambda_{L,t}^-\right),\\
V_t &= \Delta_t^2 \sigma^2(y_t) S_t^2.
\end{aligned}
$$

The quantities λ_u^+ and λ_u^-, which depend on δ_u^+ and δ_u^- respectively, are replaced by $\lambda_{L,u}^+ = \lambda^+(\delta_{L,*,u}^+)$ and $\lambda_{L,u}^- = \lambda^-(\delta_{L,*,u}^-)$ in order to simplify the resolution of the problem.

The value function u^ϵ of the market maker is defined as follows:

$$
u^\epsilon(t,s,y,q_1,x) = Sup_{\left\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\right\}} E_{t,s,y,q_1,x}^{\mathcal{P}}\left(H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T)\right),
$$

where the quantity H^ϵ writes:

$$
H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T) = X_T + q_{1,T}h(S_T) - \epsilon\int_t^T \left(q_{1,u}^2 V_u + T_u\right) du,
$$

Let u_0^ϵ be the solution of the following HJB equation:

$$
\left(\partial_t + \mathcal{L}\right) u_0^\epsilon + sup_{\left\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\right\}} J^\epsilon(\delta_t^-,\delta_t^+) = \epsilon\left(q_{1,t}^2 V_t + T_t\right), \tag{2.22}
$$

The operator $\mathcal{L}$ is defined such that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, and the term J^ϵ satisfies $J^\epsilon = J^{-,\epsilon} + J^{+,\epsilon}$ where the functions $J^{+,\epsilon}, J^{-,\epsilon}$ are defined in the following way:

$$
\begin{aligned}
J^{+,\epsilon}(\delta^+) &= \lambda^+(\delta^+)\left(u_0^\epsilon(t,s,y,q_{1,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}+\delta^+)) - u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right),\\
J^{-,\epsilon}(\delta^-) &= \lambda^-(\delta^-)\left(u_0^\epsilon(t,s,y,q_{1,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}-\delta^-)) - u_0^\epsilon(t,s,y,q_{1,t^-},x_{t^-})\right),
\end{aligned}
$$

If u_0^ϵ is smooth, finite and has polynomial growth, then it coincides with the value function u^ϵ (see (2.8.6)).
The objective of the market maker, at a given date t, is to determine $(\delta_{*,t}^+, \delta_{*,t}^-)$ solution of the following problem:

$$
(\delta_{*,t}^+, \delta_{*,t}^-) = Argsup_{\left\{\delta_t^+,\delta_t^-\right\}} E^{\mathcal{P}}\left(H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T)|S_t = s, y_t = y, q_{1,t} = q_1, X_t = x\right).
$$

in order to set optimal bid and ask quotes in the option order book.

The problem is going to be solved in the cases where $\beta = \frac{1}{2}$ and $\beta = 1$.

2.5.1 The case where $\beta = \frac{1}{2}$ (square-root market impact)

Analytic approximation

Proposition 2.3. *The optimal controls $(\delta_{*,t}^+, \delta_{*,t}^-)$ of the market maker can be approximated at order 1 in ϵ by $(\hat{\delta}_{*,t}^-, \hat{\delta}_{*,t}^+)$ which are defined as:*

$$
\hat{\delta}_{*,t}^+ = \frac{-\gamma M^+ + \sqrt{\gamma^2 (M^+)^2 + B(2\gamma-1)}}{2\gamma-1}, \tag{2.23}
$$

$$
\hat{\delta}_{*,t}^- = \frac{\gamma M^- + \sqrt{\gamma^2 (M^-)^2 + B(2\gamma-1)}}{2\gamma-1}, \tag{2.24}
$$

where the quantities $M^+(t,s,y,q_1)$ and $M^-(t,s,y,q_1)$ are defined as:

$$
\begin{aligned}
M^+(t,s,y,q_1) &= M_0(t,s,y) + \epsilon M_1(t,s,y,q_1),\\
M^-(t,s,y,q_1) &= M_0(t,s,y) + \epsilon M_2(t,s,y,q_1),
\end{aligned}
$$

and $M_1(t,s,y,q_1)$ and $M_2(t,s,y,q_1)$ are given explicitly:

$$\begin{aligned} M_1(t,s,y,q_1) &= -\theta_3(t,s,y) + (1-2q_1)\theta_4(t,s,y), \\ M_2(t,s,y,q_1) &= -\theta_3(t,s,y) - (1+2q_1)\theta_4(t,s,y), \end{aligned}$$

with:

$$\begin{aligned} \theta_4(t,s,y) &= -E^{\mathcal{P}}_{t,s,y}\left(\int_t^T V_u du\right), \\ \theta_3(t,s,y) &= -2E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \theta_4(u,s_u,y_u)\left(\lambda^+(\delta^+_{L,*,u}) - \lambda^-(\delta^-_{L,*,u})\right) du\right), \end{aligned}$$

The approximation error is at order 2 in ϵ:

$$\begin{aligned} |\delta^+_{*,t} - \hat{\delta}^+_{*,t}| = O\left(\epsilon^2\right), \\ |\delta^-_{*,t} - \hat{\delta}^-_{*,t}| = O\left(\epsilon^2\right). \end{aligned}$$

Proof. Let u_0^ϵ be the solution of the HJB equation (2.22). Under the assumption that $\epsilon \sim 0$, an asymptotic expansion technique can be performed with respect to the parameter ϵ:

$$u_0^\epsilon(t,s,y,q_1,x) = x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t,s,y,q_1). \tag{2.25}$$

If $\epsilon = 0$, then the case of a linear utility function is recovered. This implies that $u_0^0(t,s,y,q_1,x) = x + v_0(t,s,y,q_1) = u_0(t,s,y,q_1,x)$ where u_0 is the function defined in (2.15). Therefore, it is assumed that v_0 has the following form:

$$v_0(t,s,y,q_1) = \theta_0(t,s,y) + q_1\theta_1(t,s,y).$$

Since the utility function contains a constraint on the square of the inventory q_1 in the traded option, it is assumed that v_1 has the following form:

$$v_1(t,s,y,q_1) = \theta_2(t,s,y) + q_1\theta_3(t,s,y) + q_1^2\theta_4(t,s,y),$$

In order to solve the HJB equation, the jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated.

Let $f^+ = J^{+,\epsilon}$, the function f^+ writes:

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+)(u(t,s,y,q_1-1,x+(c+\delta^+)) - u(t,s,y,q_1,x)), \\ &= \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y) + \epsilon M_1(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right), \end{aligned}$$

where:

$$\begin{aligned} M_1(t,s,y,q_1) &= v_1(t,s,y,q_1-1) - v_1(t,s,y,q_1), \\ &= -\theta_3(t,s,y) + (1-2q_1)\theta_4(t,s,y), \end{aligned}$$

and:

$$R^+(t,s,y,q_1) = \sum_{k=2}^{+\infty} \epsilon^{k-2}\left(v_k\left(t,s,y,q_1-1\right) - v_k\left(t,s,y,q_1\right)\right)$$

Let $M^+(t,s,y,q_1) = M_0(t,s,y) + \epsilon M_1(t,s,y,q_1)$. In order to determine $\delta^+_{*,t} = ArgSup_{\{x\geq 0\}} f^+(x)$, the derivative $(f^+)'$ is computed:

$$(f^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B+(\delta^+)^2}\left(\left(\delta^+\right)^2(1-2\gamma) - 2\gamma\left(M^+\left(t,s,y\right) + \epsilon^2 R^+(t,s,y,q_1)\right)\delta^+ + B\right).$$

The derivative function $(f^+)'$ has the two following zeros x_1^+ and x_2^+:

$$\begin{aligned} x_1^+ &= \frac{-\gamma\left(M^+ + \epsilon^2 R^+\right) - \sqrt{\gamma^2\left(M^+ + \epsilon^2 R^+\right)^2 + B(2\gamma-1)}}{2\gamma-1} < 0, \\ x_2^+ &= \frac{-\gamma\left(M^+ + \epsilon^2 R^+\right) + \sqrt{\gamma^2\left(M^+ + \epsilon^2 R^+\right)^2 + B(2\gamma-1)}}{2\gamma-1} > 0. \end{aligned}$$

It can be deduced that $\delta^+_{*,t} = x_2^+$, and then using Taylor's expansion, we deduce:

$$\delta^+_{*,t} = \frac{-\gamma M^+ + \sqrt{\gamma^2 (M^+)^2 + B(2\gamma-1)}}{2\gamma-1} + O\left(\epsilon^2\right).$$

Performing Taylor expansion with respect to the parameter ϵ yields:

$$\delta^+_{*,t} = \delta^+_{L,*,t} + \epsilon\left(-\frac{\gamma}{2\gamma-1}M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right) + O\left(\epsilon^2\right).$$

It is then useful to write $f^+(\delta^+_{*,t})$ as the sum of $f_0^+(\delta^+_{L,*,t})$ plus a perturbation term due to the parameter ϵ. This will be of great use in the resolution of the HJB equation.
Noticing that $f^+(x) = f_0^+(x) + \epsilon\lambda^+(x)M_1(t,s,y,q_1) + O\left(\epsilon^2\right)$ and using Taylor's expansion, enables to obtain the following result:

$$\begin{aligned} f^+(\delta^+_{*,t}) &= f^+(\delta^+_{L,*,t}) + (f^+)'(\delta^+_{L,*,t})(\delta^+_{*,t} - \delta^+_{L,*,t}) + O\left((\delta^+_{*,t} - \delta^+_{L,*,t})^2\right), \\ &= f_0^+(\delta^+_{L,*,t}) + \epsilon\lambda^+(\delta^+_{L,*,t})M_1 + \epsilon\left(-\frac{\gamma}{2\gamma-1}M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right)(f^+)'(\delta^+_{L,*,t}) + O\left(\epsilon^2\right). \end{aligned}$$

Thus, since $(f^+)'(x) = (f_0^+)'(x) + O(\epsilon)$ and $(f_0^+)'(\delta^+_{L,*,t}) = 0$, it can be deduced:

$$f^+(\delta^+_{*,t}) = f_0^+(\delta^+_{L,*,t}) + \epsilon\lambda^+(\delta^+_{L,*,t})M_1 + O\left(\epsilon^2\right),$$

Let $f^- = J^{-,\epsilon}$. Using the form of the value function suggested in (2.25), the function f^- writes:

$$\begin{aligned} f^-(\delta^-) &= \lambda^-(\delta^-)(u(t,s,y,q_1+1,x-(c-\delta^-)) - u(t,s,y,q_1,x)), \\ &= \lambda^-(\delta^-)\left(\delta^- - \left(M_0(t,s,y) + \epsilon M_2(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)\right) \end{aligned}$$

where:

$$\begin{aligned} M_2(t,s,y,q_1) &= -\left(v_1(t,s,y,q_1+1) - v_1(t,s,y,q_1)\right), \\ &= -\theta_3(t,s,y) - (1+2q_1)\theta_4(t,s,y), \end{aligned}$$

and:

$$R^-(t,s,y,q_1) = -\sum_{k=2}^{+\infty} \epsilon^{k-2}\left(v_k\left(t,s,y,q_1+1\right) - v_k\left(t,s,y,q_1\right)\right).$$

The quantity $M^-(t,s,y,q_1) = M_0(t,s,y) + \epsilon M_2(t,s,y,q_1)$ is introduced to simplify the notations.
By differentiating f^-, it can be shown that:

$$(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + (\delta^-)^2}\left((1-2\gamma)(\delta^-)^2 + 2\gamma\left(M^- + \epsilon^2 R^-\right)\delta^- + B\right).$$

Thus, $(f^-)'$ gets null in the two following points:

$$\begin{aligned}
x_1^- &= \frac{\gamma\left(M^- + \epsilon^2 R^-\right) - \sqrt{\gamma^2\left(M^- + \epsilon^2 R^-\right)^2 + B(2\gamma-1)}}{2\gamma-1}, \\
x_2^- &= \frac{\gamma\left(M^- + \epsilon^2 R^-\right) + \sqrt{\gamma^2\left(M^- + \epsilon^2 R^-\right)^2 + B(2\gamma-1)}}{2\gamma-1}.
\end{aligned}$$

Since $x_1^- < 0$, $x_2^- > 0$ and $\gamma > 1$, then $\delta_{*,t}^- = ArgSup_{\{x\geq 0\}} f^-(x) = x_2^-$. It follows that:

$$\delta_{*,t}^- = \frac{\gamma M^- + \sqrt{\gamma^2 (M^-)^2 + B(2\gamma-1)}}{2\gamma-1} + O\big(\epsilon^2\big).$$

Through the use of Taylor expansion with respect to the parameter ϵ, the last relation yields:

$$\delta_{*,t}^- = \delta_{L,*,t}^- + \epsilon\left(\frac{\gamma}{2\gamma-1} M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right) + O\big(\epsilon^2\big).$$

The quantity $f^-(\delta_{*,t}^-)$ can be written as the sum of $f_0^-(\delta_{L,*,t}^-)$ plus a perturbation term due to the parameter ϵ. Indeed, since:

$$f^-(x) = f_0^-(x) - \epsilon\lambda^-(x) M_2(t,s,y,q_1) + O\big(\epsilon^2\big),$$

then, by using Taylor's expansion, it follows:

$$\begin{aligned}
f^-(\delta_{*,t}^-) &= f^-(\delta_{L,*,t}^-) + (f^-)'(\delta_{L,*,t}^-)(\delta_{*,t}^- - \delta_{L,*,t}^-) + O\Big((\delta_{*,t}^- - \delta_{L,*,t}^-)^2\Big), \\
&= f_0^-(\delta_{L,*,t}^-) - \epsilon\lambda^-(\delta_{L,*,t}^-) M_2 + \epsilon\left(\frac{\gamma}{2\gamma-1} M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right)(f^-)'(\delta_{L,*,t}^-) + O\big(\epsilon^2\big).
\end{aligned}$$

It can be recalled at this stage that $(f^-)'(x) = (f_0^-)'(x) + O(\epsilon)$, and $(f_0^-)'(\delta_{L,*,t}^-) = 0$. Thus, the last equation becomes:

$$f^-(\delta_{*,t}^-) = f_0^-(\delta_{L,*,t}^-) - \epsilon\lambda^-(\delta_{L,*,t}^-) M_2 + O\big(\epsilon^2\big).$$

Now that the terms $f^+(\delta_{*,t}^+)$ and $f^-(\delta_{*,t}^-)$ are approximated separately, the quantity $J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = f^-(\delta_{*,t}^-) + f^+(\delta_{*,t}^+)$ can be written as below:

$$\begin{aligned}
J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) &= f_0^+(\delta_{L,*,t}^+) + \epsilon M_1 \lambda^+(\delta_{L,*,t}^+) + f_0^-(\delta_{L,*,t}^-) - \epsilon M_2 \lambda^-(\delta_{L,*,t}^-) + O\big(\epsilon^2\big), \\
&= J_0(\delta_{L,*,t}^-, \delta_{L,*,t}^+) + \epsilon\Big(M_1\lambda^+(\delta_{L,*,t}^+) - M_2\lambda^-(\delta_{L,*,t}^-)\Big) + O\big(\epsilon^2\big).
\end{aligned}$$

By rearranging the terms of $J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+)$ by powers of ϵ, it can be obtained:

$$J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = J_0(t,s,y) + \epsilon J_1(t,s,y,q_1) + O\big(\epsilon^2\big),$$

where $J_1(t,s,y,q_1) = J_{1,0}(t,s,y) + q_1 J_{1,1}(t,s,y)$ and:

$$\begin{aligned}
J_{1,0}(t,s,y) &= \lambda^+(\delta_{L,*,t}^+)(-\theta_3 + \theta_4) - \lambda^-(\delta_{L,*,t}^-)(-\theta_3 - \theta_4), \\
J_{1,1}(t,s,y) &= -2\theta_4\Big(\lambda^+(\delta_{L,*,t}^+) - \lambda^-(\delta_{L,*,t}^-)\Big).
\end{aligned}$$

The HJB equation (2.22) can be regrouped by terms according to their order in ϵ. The term at order 0 in ϵ leads to the equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1\theta_1) + J_0(t,s,y) = 0,$$

with the final conditions:

$$\theta_0(T,s,y)=0, \quad \theta_1(T,s,y)=h(s).$$

The functions θ_0, θ_1 are then deduced:

$$\begin{aligned}
\theta_1(t,s,y) &= C_{\mathcal{P}}(t,s,y)-\mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right), \\
\theta_0(t,s,y) &= E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right).
\end{aligned}$$

The term at order 1 in ϵ, gives the following equation:

$$(\partial_t+\mathcal{L})(\theta_2+q_1\theta_3+q_1^2\theta_4)+J_1(t,s,y) \quad = \quad q_{1,t}^2 V_t+T_t,$$

The terms of the last equation can be sorted by their orders in q_1. The term of order 2 in q_1 gives the following equation for θ_4:

$$(\partial_t+\mathcal{L}_1)\theta_4(t,s,y) \quad = \quad V_t.$$

Using the final condition $\theta_4(T,s,y)=0$, it can be deduced that:

$$\theta_4(t,s,y) \quad = \quad -E^{\mathcal{P}}_{t,s,y}\left(\int_t^T V_u du\right).$$

The function θ_3 is the solution of the equation resulting from the term of order 1 in q_1:

$$(\partial_t+\mathcal{L}_1)\theta_3(t,s,y)+J_{1,1}(t,s,y) \quad = \quad 0,$$

and it also satisfies the final condition $\theta_3(T,s,y)=0$, it follows that:

$$\theta_3(t,s,y) \quad = \quad E^{\mathcal{P}}_{t,s,y}\left(\int_t^T J_{1,1}(u,s_u,y_u)du\right).$$

Finally, the function θ_2 is the solution of the equation related to the term at order 0 in q_1:

$$(\partial_t+\mathcal{L}_1)\theta_2(t,s,y)+J_{1,0}(t,s,y)=T_t,$$

and is subject to the final condition $\theta_2(T,s,y)=0$. Using again the Feynman-Kac formula, the following is obtained:

$$\theta_2(t,s,y) \quad = \quad E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \left(J_{1,0}-T\right)(u,s_u,y_u)du\right).$$

Let $\mathcal{R}^\epsilon(t,s,y,q_1)=\sum_{k=2}^{+\infty}\epsilon^k v_k(t,s,y,q_1)$ and $\tilde{u}^\epsilon(t,s,y,q_1,x)=x+v_0(t,s,y)+\epsilon v_1(t,s,y,q_1)$, then the function u^ϵ, writes:

$$u^\epsilon(t,s,y,q_1,x) \quad = \quad \tilde{u}^\epsilon(t,s,y,q_1)+\mathcal{R}^\epsilon(t,s,y,q_1),$$

It has been previously shown, through the use of Taylor expansion of the terms $f^+(\delta^+_{*,t})$ and $f^-(\delta^-_{*,t})$, that:

$$J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) \quad = \quad J_0(t,s,y)+\epsilon J_1(t,s,y,q_1)+O\big(\epsilon^2\big),$$

Let $g_\epsilon(t,s,y,q_1)=J^\epsilon(\delta^-_{*,t},\delta^+_{*,t})-(J_0(t,s,y)+\epsilon J_1(t,s,y,q_1))$, then (2.22) writes:

$$(\partial_t+\mathcal{L}_1+\mathcal{L}_2)\left(\tilde{u}^\epsilon(t,s,y,q_1,x)+\mathcal{R}^\epsilon(t,s,y,q_1)\right)+\left(J_0(t,s,y)+\epsilon J_1(t,s,y)+g_\epsilon(t,s,y,q_1)\right)=\epsilon\left(q_{1,t}^2V_t+T_t\right).$$

It follows that:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\mathcal{R}^\epsilon(t,s,y,q_1) + g_\epsilon(t,s,y,q_1) \quad = \quad 0.$$

Using the final condition $\mathcal{R}^\epsilon(T, S_T, y_T, q_{1,T}) = 0$, it can be obtained through the use of the Feynman-Kac formula:

$$\mathcal{R}^\epsilon(t,s,y,q_1) \quad = \quad E^{\mathcal{P}}_{t,s,y,q_1}\left(\int_t^T g_\epsilon(u, s_u, y_u, q_{1,u})du\right),$$

which implies that $\mathcal{R}^\epsilon(t,s,y,q_1) = O(\epsilon^2)$ and $|u^\epsilon(t,s,y,q_1,x) - \tilde{u}^\epsilon(t,s,y,q_1,x)| = O(\epsilon^2)$.
The verification theorem can be done as detailed in (2.8.6). Indeed, for u_0^ϵ to coincide with the value function u^ϵ, it is sufficient to prove that it's smooth, finite and satisfies a polynomial growth condition.
Under the assumption that the function σ^2 has quadratic growth, we can prove that $\exists C_4 > 0$ such that $|\theta_4(t,s,y)| \leq C_4\left(1 + s^4 + y^4\right)$.
On the other hand, using the analytic expression of θ_3, it can be proved that $\exists C_3 > 0$, $|\theta_3(t,s,y)| \leq C_3\left(1 + s^4 + y^4\right)$. In addition, since the functions λ^+ and λ^- are bounded on $\mathbb{R}^+$, then $\exists C_T > 0$, $|T_t| \leq C_T\left(1 + s^4\right)$, which implies that $\exists C_2 > 0$, $|\theta_2(t,s,y)| \leq C_2\left(1 + s^4 + y^4\right)$.
Based on these results, it follows that $\exists \tilde{C} > 0$, $\tilde{u}^\epsilon(t,s,y,q_1,x) \leq \tilde{C}\left(1 + s^8 + y^8 + q_1^4 + x^2\right)$, and then u^ϵ is finite and satisfies a polynomial growth condition.

In addition, the functions $\mathcal{R}^\epsilon$ and $\tilde{u}^\epsilon$ are smooth. Consequently, u^ϵ is smooth and coincides with the value function.

■

Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), indeed:

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial M_0} &= \frac{\partial \hat{\delta}^+_{*,t}}{\partial M^+}\frac{\partial M^+}{\partial M_0} = g'(M^+) < 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial M_0} &= \frac{\partial \hat{\delta}^-_{*,t}}{\partial M^-}\frac{\partial M^-}{\partial M_0} = f'(M^-) > 0,
\end{aligned}$$

It can then be deduced that the market maker still adjusts her quotes depending on the term M_0.

The new feature here is the dependence of the distances $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$ on the inventory q_1. In order to study this characteristic, the partial derivatives of $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$ with respect to the variable q_1 are computed:

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t,s,y)\frac{\gamma}{2\gamma - 1}g'(M^+) < 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t,s,y)\frac{\gamma}{2\gamma - 1}f'(M^-) > 0,
\end{aligned}$$

It can be deduced from the expressions above that, when the option inventory q_1 increases, the market maker posts more aggressive ask quotes and more conservative bid quotes in order to reduce her inventory. Thus, the market maker, being risk-averse, adjusts her quoting policy in order not to accumulate a large inventory to avoid being too exposed to market moves.

The bid-ask spread of the market maker writes $\mathcal{S}_{*,t} = \hat{\delta}^-_{*,t} + \hat{\delta}^+_{*,t}$. Using Taylor expansion, $\mathcal{S}_{*,t}$ can be approximated as following:

$$\begin{aligned}
\mathcal{S}_{*,t} &= 2\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1} + \epsilon\frac{\gamma}{2\gamma-1}\left[(M_2 - M_1) + \frac{M_0\gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}(M_1+M_2)\right], \\
&= \mathcal{S}_{L,*,t} + \epsilon\frac{\gamma}{2\gamma-1}\left[-2\theta_4 + 2\frac{M_0\gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}(-\theta_3 - 2q_1\theta_4)\right],
\end{aligned}$$

The term $(M_2 - M_1)$ is a positive quantity that increases the bid-ask spread. Indeed, havin a constraint on the variance of her final wealth, the risk-averse market maker wants to have a bigger margin in order to be paid for the risk she bears. Therefore, the bid-ask spread is widened. The term after depends on q_1 and M_0, its effect on $\mathcal{S}_{*,t}$ is not straightforward. It is easier to see the effects of q_1 directly on $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$.

2.5.2 The case where $\beta = 1$ (linear market impact)

Analytic approximation

Proposition 2.4. *The optimal controls $(\delta^+_{*,t}, \delta^-_{*,t})$ of the market maker can be approximated at order 1 in ϵ by $(\hat{\delta}^-_{*,t}, \hat{\delta}^+_{*,t})$ which are defined as:*

$$\hat{\delta}^+_{*,t} = \begin{cases} \frac{B - \gamma M^+(t,s,y,q_1)}{\gamma - 1} & \text{if} \quad M^+(t,s,y) \leq \mathcal{S}, \\ 0 & \text{if} \quad M^+(t,s,y) \geq \mathcal{S}, \end{cases} \tag{2.26}$$

$$\hat{\delta}^-_{*,t} = \begin{cases} \frac{B + \gamma M^-(t,s,y,q_1)}{\gamma - 1} & \text{if} \quad M^-(t,s,y) \geq -\mathcal{S}, \\ 0 & \text{if} \quad M^-(t,s,y) \leq -\mathcal{S}, \end{cases} \tag{2.27}$$

and the error of approximation is at order 2 in ϵ:

$$\begin{aligned}
|\hat{\delta}^+_{*,t} - \delta^+_{*,t}| &= O(\epsilon^2), \\
|\hat{\delta}^-_{*,t} - \delta^-_{*,t}| &= O(\epsilon^2).
\end{aligned}$$

The proof of this Proposition and the value function are given in Appendix (2.8.3).

Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), indeed:

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial M_0} &= -\frac{\gamma}{\gamma-1}1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial M_0} &= \frac{\gamma}{\gamma-1}1_{\{M^- \geq -\mathcal{S}\}} \geq 0,
\end{aligned}$$

The proxies $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$ of the optimal distances depend on the inventory q_1. In order to understand the effect of q_1 on $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$, the first derivatives with respect to the variable q_1 can be computed explicitly:

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial q_1} &= 2\epsilon\theta_4(t,s,y)\frac{\gamma}{\gamma-1}1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t,s,y)\frac{\gamma}{\gamma-1}1_{\{M^- \geq -\mathcal{S}\}} \geq 0,
\end{aligned}$$

The first derivatives highlights the effect of the option inventory q_1 on the distances $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$. Indeed, if q_1 increases, the market maker lowers her bid and ask quotes with the aim to cut down her option inventory.

2.6 Numerical Simulations (The case of a risk neutral market maker)

In this section, Monte Carlo simulations are performed in order to test the performance of the market making strategies stated previously. It is supposed in this section that the spot process follows a Heston model which means that under the real-world probability measure $\mathcal{P}$, the spot process S has the following dynamics:

$$\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{y_t} dW_t^{(1)} \\
dy_t &= k_R(\theta_R - y_t)dt + \eta_R \sqrt{y_t} dW_t^{(2)}
\end{aligned}$$

where $d\left\langle W^{(1)}, W^{(2)} \right\rangle_t = \rho_R dt$.

Market participant price options on the stock under a risk-neutral pricing measure $\mathcal{Q}$, under which the spot process S has the following dynamics:

$$\begin{aligned}
\frac{dS_t}{S_t} &= r dt + \sqrt{y_t} dW_t^{*,(1)} \\
dy_t &= k_I(\theta_I - y_t)dt + \eta_I \sqrt{y_t} dW_t^{*,(2)}
\end{aligned}$$

where $d\left\langle W^{*,(1)}, W^{*,(2)} \right\rangle_t = \rho_I dt$.
The functions a_R, b_R and σ are:

$$\begin{aligned}
a_R(y_t) &= k_R\left(\theta_R - y_t\right), \\
b_R(y_t) &= \eta_R \sqrt{y_t}, \\
\sigma(y_t) &= \sqrt{y_t},
\end{aligned}$$

The term $C_{\mathcal{Q}}(t, s, y)$ is the option price in the Heston model (see Heston (1993)). The term $C_{\mathcal{P}} = E^{\mathcal{P}}((S_T - K)^+ | \mathcal{F}_t)$ can be computed explicitly as explained in Appendix (2.8.4).

The first part of the numerical study is devoted to the comparison between the optimal strategy in the case of a linear utility function, and a zero-intelligence strategy. In the second part, the effect of the misspecification of the parameters is studied. It can be seen how that could affect the performance of the strategy as well as the inventory of the market maker.

The numerical simulations are performed as following. We consider here that the traded option has a maturity equal to 3 Months ($T = 0.25$) and a strike equal to 100 ($K = 100$). For each of the following cases, we fix the set of parameters $(r, k_I, \theta_I, \eta_I, \rho_I)$ which characterizes the stock dynamics under the risk-neutral probability $\mathcal{Q}$ as well as the set of the parameters $(\mu, k_R, \theta_R, \eta_R, \rho_R)$ characterizing the dynamics under the real-world probability measure $\mathcal{P}$. We simulate 1000 paths of the spot and instantaneous variance processes $(S_t, v_t)_{\{0 \le t \le T\}}$ starting from $S_0 = 100$ and $v_0 = 0.04$. It is considered here that there is 6 trading hours per day and that the market maker refreshes the quotes every 5 minutes. This means that there is $12 \times 6 = 72$ points per day. Since there is approximately 64 business days in a 3 months period, this amounts to $64 \times 72 = 4608$ points per simulated path. At each point, the quantities $C_{\mathcal{Q}}(t, s, y)$ and $C_{\mathcal{P}}(t, s, y)$ are computed using a Fast Fourier Transform method. This simulation task is numerically consuming and was performed using the computing cluster at Ecole Centrale Paris.

For each simulated path, the optimal market making strategy is used from the inception date ($t = 0$) until the maturity date of the option ($t = T$). We then performed a statistical study on the results obtained upon the test of the optimal strategy on 1000 independent paths.

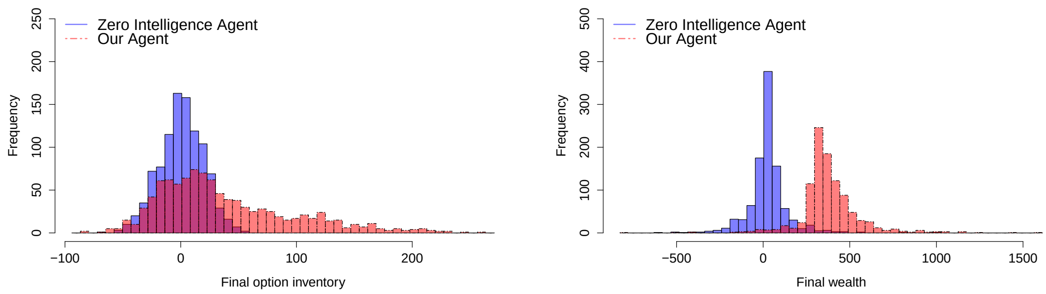
2.6.1 Comparison with a classical market making strategy

Let the implied volatility $\Sigma_t(K,T)$ of the call option be defined such that $C_{\mathbb{Q}}(t,S_t,y_t) = P_{BS}\left(t,S_t,\Sigma_t(K,T)\right)$, where P_{BS} denotes the Black-Scholes option price formula. In addition, let $\vartheta_{BS} = \frac{\partial P_{BS}(t,S_t,\Sigma_t(K,T))}{\partial \Sigma_t}$ be the Black-Scholes vega of the option. In this subsection, we suppose that there is an agent who places equidistant bid and ask quotes denoted $C^b_{ZI,t}$ and $C^a_{ZI,t}$ respectively:

$$
\begin{aligned}
C^a_{ZI,t} &= C_{\mathbb{Q}}(t,S_t,y_t) + \delta_{ZI,t},\\
C^b_{ZI,t} &= C_{\mathbb{Q}}(t,S_t,y_t) - \delta_{ZI,t},
\end{aligned}
$$

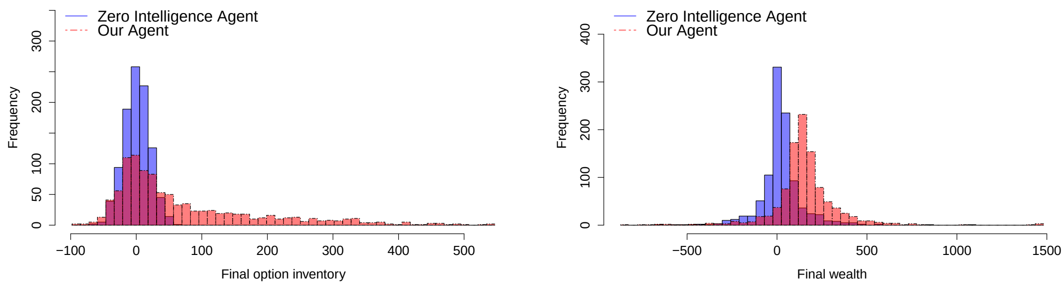
where $\delta_{ZI,t} = 0.005 \times \vartheta_{BS}$. This means that the agent attempts to earn $0.005\vartheta_{BS}$ for each trade. We denote this classical strategy the "zero-intelligence" strategy.

It is supposed here that there is no misspecification of the model parameters: $(k_R,\theta_R,\eta_R,\rho_R) = (k_I,\theta_I,\eta_I,\rho_I) = (4, 0.04, 0.5, -0.4)$. Besides, the stock doesn't have a drift under the real world-probability measure ($\mu = 0$) and the risk-free rate is null ($r = 0$). The results of the numerical simulations can be summarized as follows:



(a) Final value of q_1 (b) Final value of Wealth

Figure 2.1: Statistics for $\beta = 0.5$



(a) Final value of q_1 (b) Final value of Wealth

Figure 2.2: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and with zero-intelligence	1.00	0.80	19.05	-0.14	-0.02
$\beta = 1$ and with optimal quotes	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and with zero-intelligence	1.00	1.13	19.90	-0.07	0.00

Table 2.1: Statistics on final option inventory $q_{1,T}$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and with zero-intelligence	21.34	25.01	116.44	0.82	11.05
$\beta = 1$ and with optimal quotes	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and with zero-intelligence	17.60	23.32	121.75	0.86	11.51

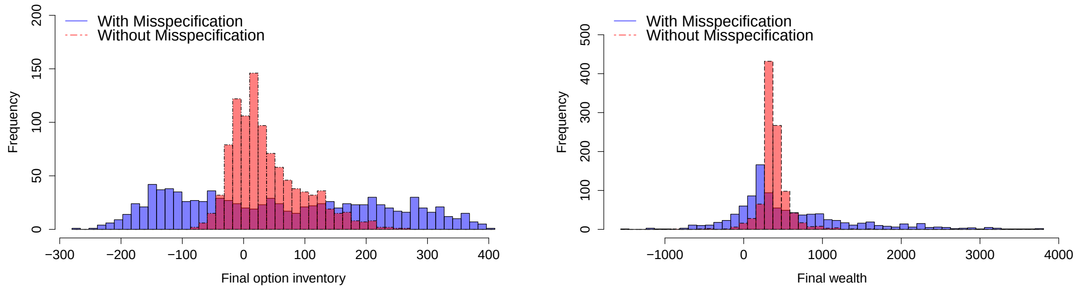
Table 2.2: Statistics on final wealth $\mathcal{W}_T$

The numerical simulations show that the strategy using the optimal quotes performs better than the zero-intelligence strategy. The explication is straightforward: the optimal strategy has the advantage to use the information on the liquidity of the option and the arrival rate of market orders. Thus, the quotes are placed in an optimal way inside the order book.

2.6.2 Effect of the misspecification of parameters

Misspecification of the parameter ρ

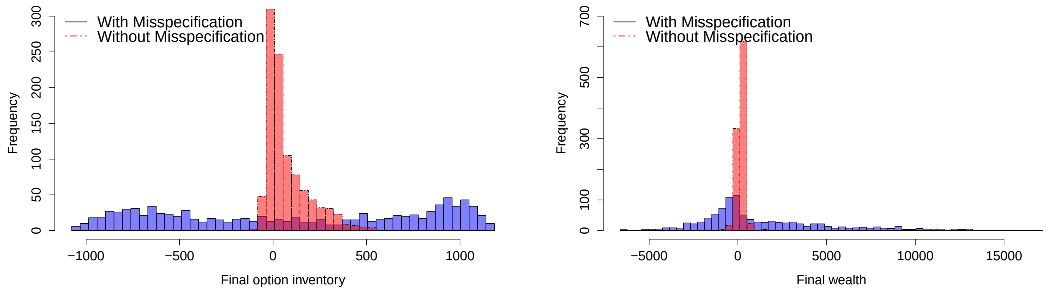
Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.5, -0.9)$. The statistics of 1000 simulations are given below:



(a) Final value of q_1

(b) Final value of Wealth

Figure 2.3: Statistics for $\beta = 0.5$



(a) Final value of q_1 (b) Final value of Wealth

Figure 2.4: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	23.00	38.41	58.66	0.97	0.62
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	42.50	57.30	165.14	0.19	-1.16
$\beta = 1$ and $\rho_R = \rho_I$	29.00	72.942	114.13	1.51	1.93
$\beta = 1$ and $\rho_R \neq \rho_I$	105.50	112.38	699.43	-0.049	-1.48

Table 2.3: Statistics on final option inventory $q_{1,T}$

The authors in Forde and Jacquier (2009) gave an approximation of the implied volatility in Heston model for near the money strikes and small maturities (which corresponds to our case of study $T = 0.25$):

$$\Sigma(K,T) = \sqrt{y_0}\left(1 + \frac{1}{4}\frac{\rho\eta}{y_0}\log(\frac{K}{S_0}) + \left(\frac{1}{24} - \frac{5}{48}\rho^2\right)\frac{\eta^2}{y_0^2}\log(\frac{K}{S_0})^2 + O\left(\log(\frac{K}{S_0})^3\right)\right) \quad (2.28)$$

The analytic approximation (2.28) points out the effect of the parameter ρ on the option price when $T \sim 0$. Indeed, at first order in $\log(\frac{K}{S_t})$, the implied volatility $\Sigma_t(K,T)$ is an increasing function of ρ if $K > S_t$ and a decreasing function if $K < S_t$. Consequently, in each simulation, the evolution of the moneyness $\left(\log(\frac{K}{S_t})\right)$ of the option determines the quantity $C_{\mathcal{P}}(t,s,y) - C_{\mathcal{Q}}(t,s,y)$ and influences the aggressiveness of the quotes on either the bid or the ask side. It can be noticed that the distribution of the option inventory $q_1(T)$ at the maturity date T is more dispersed in the case where $\rho_R \neq \rho_I$.

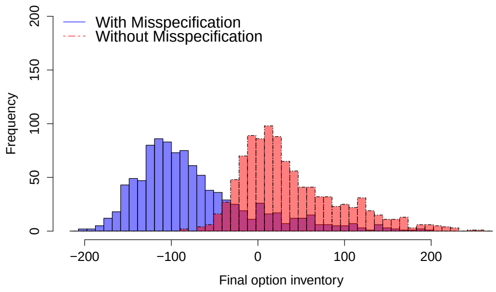
Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	356.89	379.63	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	331.58	584.06	772.01	1.29	2.01
$\beta = 1$ and $\rho_R = \rho_I$	144.54	154.90	187.97	0.74	12.24
$\beta = 1$ and $\rho_R \neq \rho_I$	102.22	1485.31	3780.57	1.20	1.26

Table 2.4: Statistics on final wealth $\mathcal{W}_T$.

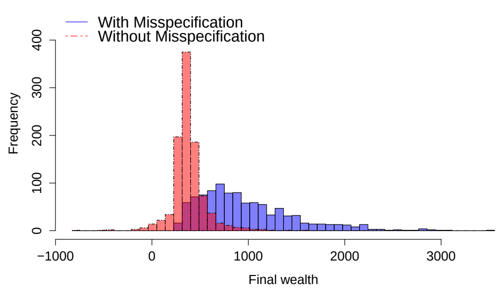
The statistics show that the final wealth in the case where $\rho_R = -0.4$ and $\rho_I = -0.9$ is on average higher than in the case where $\rho_R = \rho_I = -0.4$. Indeed, the quoting policy of the market maker is adapted in order to profit from the misspecification of the parameter ρ.

Misspecification of the parameter θ

Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.0625, 0.5, -0.4)$.

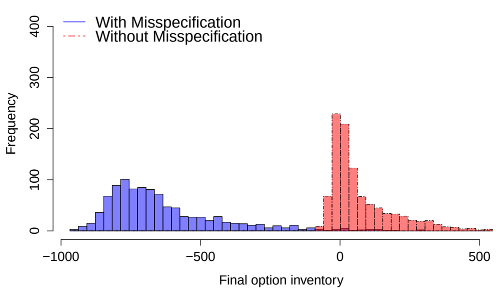


(a) Final value of q_1

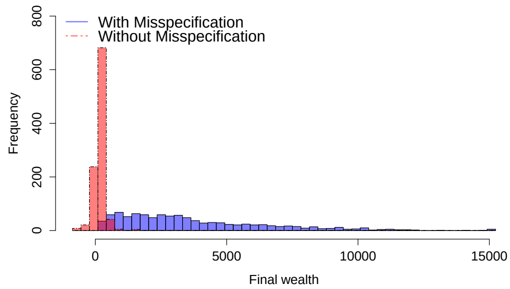


(b) Final value of Wealth

Figure 2.5: Statistics for $\beta = 0.5$



(a) Final value of q_1



(b) Final value of Wealth

Figure 2.6: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	-90.50	-73.81	68.76	1.22	1.45
$\beta = 1$ and $\theta_R = \theta_I$	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and $\theta_R \neq \theta_I$	-693.00	-636.17	204.57	1.51	2.52

Table 2.5: Statistics on final option inventory $q_{1,T}$.

Since $\theta_R < \theta_I$, we have $C_{\mathcal{P}}(t, S_t, y_t) < C_{\mathcal{Q}}(t, S_t, y_t)$. The market maker posts aggressive ask quotes and conservative bid quotes. Therefore, she is more likely to finish with a short option position, which is confirmed by the simulations.

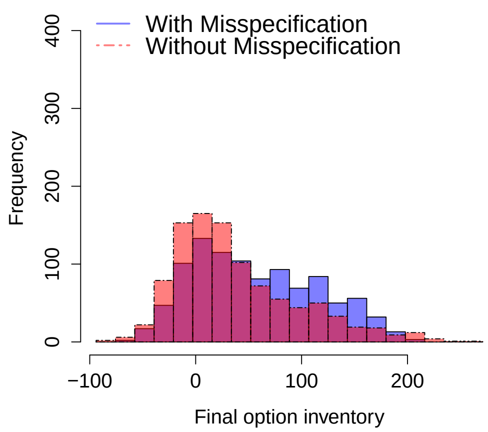
Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	856.03	972.96	504.59	1.26	1.91
$\beta = 1$ and $\theta_R = \theta_I$	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and $\theta_R \neq \theta_I$	3029.61	3737.93	2834.59	1.18	1.26

Table 2.6: Statistics on final wealth $\mathcal{W}_T$

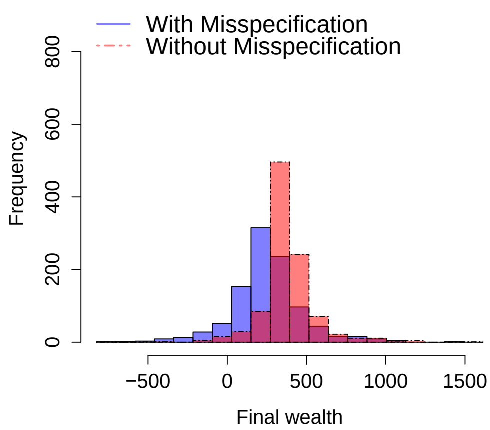
The final wealth in the case where the parameter θ doesn't have the same values under the historical measure $\mathcal{P}$ and the pricing measure $\mathcal{Q}$ ($\theta_R = 0.04, \theta_I = 0.0625$) is in average higher than in the default case ($\theta_R = \theta_I = 0.04$). The numerical simulations support the results of the theoretical study and show experimentally how the optimal strategy enables the market maker to take advantage from the parameter misspecification in order to augment her profit.

Misspecification of the parameter η

In this subsection, Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.7, -0.4)$. The statistics of 1000 simulations are given below:

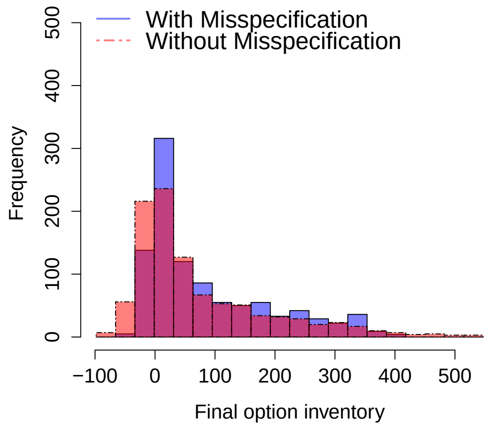


(a) Final value of q_1

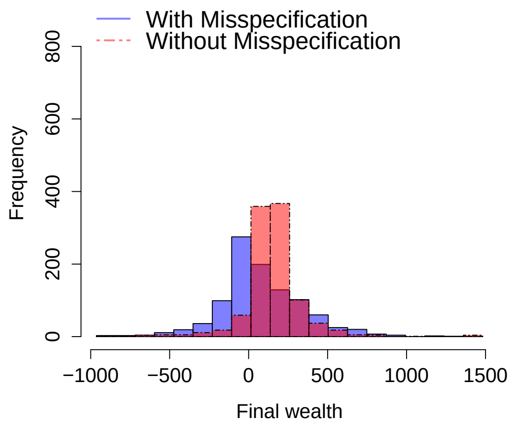


(b) Final value of Wealth

Figure 2.7: Statistics for $\beta = 0.5$



(a) Final value of q_1



(b) Final value of Wealth

Figure 2.8: Statistics for $\beta = 1$

It can be seen through the approximation (2.28) that the effect of the parameter η on the implied volatility $\Sigma(K, T)$ of the option and then its price depends on its moneyness. Thus, at first order in $\log(\frac{K}{S_t})$, if $\rho < 0$, an increase of the parameter η would increase the option price if $K < S_t$, and decrease it if $K > S_t$. Therefore, the aggressiveness of the quotes of the market maker on either the bid or the ask side of the order book will depend on the path of the process S in each simulation.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\eta_R = \eta_I$	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and $\eta_R \neq \eta_I$	48.00	57.00	58.72	0.34	-0.83
$\beta = 1$ and $\eta_R = \eta_I$	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and $\eta_R \neq \eta_I$	38.50	86.20	105.06	1.13	0.19

Table 2.7: Statistics on final option inventory $q_{1,T}$.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\eta_R = \eta_I$	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\eta_R \neq \eta_I$	251.15	250.49	227.15	0.22	2.98
$\beta = 1$ and $\eta_R = \eta_I$	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and $\eta_R \neq \eta_I$	34.57	81.90	269.94	0.44	2.27

Table 2.8: Statistics on final wealth $\mathcal{W}_T$.

The general remark here is that the final wealth $\mathcal{W}_T$ in the case $\eta_R \neq \eta_I$ has a lower average and a higher standard deviation than in the case $\eta_R = \eta_I$ (for both $\beta = \frac{1}{2}$ and $\beta = 1$). This observation can, at first glance, be contrary to intuition, since the misspecification of a parameter gives the opportunity to the market maker to profit from price inefficiency. Although, there is a simple explication for this result. The increase of the parameter η under the pricing measure raised the volatility risk of the option which is engendered by the stochastic process y. Since this risk wasn't hedged, the variance of the final wealth $\mathcal{W}_T$ increased and the numerical simulations showed that the rise of the volatility risk prevented the market maker from augmenting her gain.

2.7 Conclusion

In this paper, we established a framework for option market making. Using a stochastic control approach, we obtained analytic expressions for optimal bid and ask quotes in the case of a risk-neutral market maker. Next to that, we considered the case of a risk-averse market maker who attempts to reduce the uncertainty of her final wealth. We posed the optimization problem in a mean-variance framework and used a singular perturbation technique in order to provide approximations for the optimal quotes. Through the use of Monte Carlo simulations, we supported the findings of the theoretical study and we showed the impact of parameters misspecification on the final wealth and inventory of the market maker. We noticed also that, keeping the volatility risk non-hedged can be risky for the market maker. Thus, we will show in a future work that the market maker can provide liquidity on options on several assets while trying to cancel the aggregated volatility risk.

2.8 Appendices

2.8.1 Appendix 1: Intensity of arrivals of market orders

In order to determine the form of the functions λ^+ and λ^-, we need to specify the distribution function of the size of market orders and also the market impact following the execution of a market order.

Let $f_{\mathcal{V}}$ denote the density distribution of the size of market orders in absolute value of their cash amount. Several studies proved that this density decays as a power law (see Avellaneda and Stoikov (2008)). We will suppose here that $f_{\mathcal{V}}$ is a power-law density:

$$\forall x > 0, f_{\mathcal{V}}(x) = \frac{\gamma L^{\gamma}}{(L+x)^{\gamma+1}}. \tag{2.29}$$

From a practical point of view, there is a strictly positive lower bound x_{Min} for x which corresponds to the option price. Nevertheless, it is supposed in the theoretical case that the density $f_{\mathcal{V}}$ is positive for $0 \leq x \leq x_{Min}$.

In the other hand, the market impact has been studied by different authors in the econophysics literature and it is widely accepted that the change in price ΔP following a market order of size $\mathcal{V}$ has the following form:

$$\Delta P = K\mathcal{V}^{\beta}. \tag{2.30}$$

There are two values of β which are supported by different researchers: $\beta = 1$ which corresponds to a linear market impact and $\beta = \frac{1}{2}$ which corresponds to a square root market impact.

The probability that a bid quote (respectively ask quote) placed at a distance δ^- (respectively δ^+) from the mid price gets executed is equal to the probability that a sell market order (respectively buy market order) with volume $\mathcal{V}$ triggers a market impact which is higher or equal to δ^- (respectively δ^+). We suppose, as in Avellaneda and Stoikov (2008), that the intensity of arrival of market orders is constant and equal to F. Then, for $\delta \geq 0$:

$$\begin{aligned}
\lambda(\delta) &= F \times P(\Delta P \geq \delta), \\
&= F \times P(\mathcal{V}^{\beta} \geq \frac{\delta}{K}), \\
&= F \times P\left(\mathcal{V} \geq \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}\right), \\
&= F \int_{\left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}}^{+\infty} f_{\mathcal{V}}(x)dx, \\
&= F \frac{L^{\gamma}}{(L + \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}})^{\gamma}}.
\end{aligned}$$

Therefore, the intensity of occurrence of the event that a market order consumes a limit order at a distance δ from the mid price can be written as follows:

$$\lambda(\delta) = \frac{A}{(B + \delta^{C})^{\gamma}},$$

where: $A = FK^{\frac{\gamma}{\beta}}L^{\gamma}$, $B = LK^{\frac{1}{\beta}}$ and $C = \frac{1}{\beta}$.

For $A = 1\ Tick^2 s^{-1}$, $B = 5\ Tick$, $C = 1$, $\gamma = 2$ and $Tick = 0.01$, the following intensities λ^- and λ^+ are obtained:

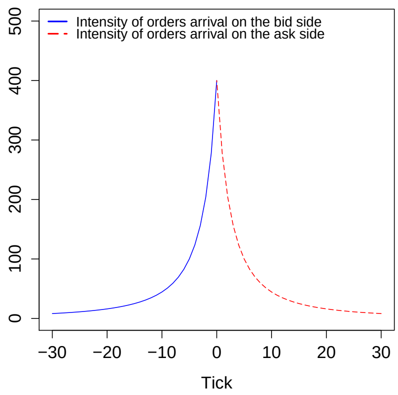


Figure 2.9: Shape of the functions λ^+ (red line) and λ^- (blue line)

Remark: Several empirical studies proved that $\gamma > 1$. Indeed, for the authors in Gopikrishnan et al. (2000), $\gamma = 1.53$. In Maslov and Mills (2001), Maslow and Mills propose the value $\gamma = 1.4$. On the other hand, Gabaix et *al* in Gabaix et al. (2005) suggested $\gamma = 1.5$ in the French market.

2.8.2 Appendix 2: Solution in the risk-neutral framework ($\beta = 1$)

The utility function U given in (2.11) is linear. Therefore, we make the following Ansatz for u:

$$u(t,s,y,q_1,x) \quad = \quad x + \theta_0(t,s,y) + q_1\theta_1(t,s,y), \tag{2.31}$$

Let $f_0^+ = J^+$, this function represents the jump part due to the execution of the ask quote. Using (2.31), the function f_0^+ can be written explicitly as:

$$f_0^+(\delta^+) \quad = \quad \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y)\right),$$

where $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \theta_1(t,s,y)$.

In order to determine $\delta^+_{L,*,t} = ArgMax_{\{x\geq 0\}} f_0^+(x)$, the derivative of the function f_0^+ is computed:

$$(f_0^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B+\delta^+}\left(\delta^+\left(1-\gamma\right) + B - \gamma M_0\left(t,s,y\right)\right),$$

If $M_0(t,s,y) \geq \frac{B}{\gamma}$, then $\forall\ \delta^+ \geq 0$, $(f_0^+)'(\delta^+) \leq -\frac{(\gamma-1)\delta^+}{B+\delta^+}\lambda^+(\delta^+) < 0$. It follows that f_0^+ is decreasing on $[0,+\infty[$ and then $\delta^+_{L,*,t} = 0$.
If $M_0(t,s,y) \leq \frac{B}{\gamma}$, the function $(f_0^+)'$ changes sign on the interval $[0,+\infty[$ and becomes null at $x^+ = \frac{B-\gamma M_0(t,s,y)}{\gamma-1}$. The table of variations yields that $\delta^+_{L,*,t} = x^+$.
In conclusion, $\delta^+_{L,*,t}$ can be determined as follows:

$$\delta^+_{L,*,t} \quad = \quad \left(\frac{B-\gamma M_0(t,s,y)}{\gamma-1}\right)^+,$$

and:

$$f_0^+(\delta^+_{L,*,t}) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma(B-M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \leq \frac{B}{\gamma} \\ \frac{A}{B^\gamma}M_0(t,s,y) & \text{if} \quad M_0(t,s,y) \geq \frac{B}{\gamma} \end{cases} \tag{2.32}$$

The same approach can be applied to the function $f_0^- = J^-$. Indeed, using the form of the value function suggested in (2.31), the function f_0^- can be written as follows:

$$f_0^-(\delta^-) \quad = \quad \lambda^-(\delta^-)(\delta^- - M_0(t,s,y)).$$

In order to determine $\delta^-_{L,*,t} = ArgMax_x f_0^-(x)$, the derivative of the function f_0^- is computed:

$$(f_0^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B+\delta^-}((1-\gamma)\delta^- + (B+\gamma M_0(t,s,y))).$$

If $M_0(t,s,y) \leq -\frac{B}{\gamma}$, then $\forall\, \delta^- \geq 0$, $(f_0^-)'(\delta^-) < 0$. Consequently, the function (f_0^-) is decreasing on $[0,+\infty[$ and $\delta^-_{L,*,t} = 0$. On the other hand, if $M_0(t,s,y) \geq -\frac{B}{\gamma}$, the function $(f_0^-)'$ changes sign in the interval $[0,+\infty[$ and becomes null at $x^- = \frac{B+\gamma M_0(t,s,y)}{\gamma-1}$. Since the function f_0^- is increasing on $[0,x^-]$ and decreasing on $[x^-,+\infty[$, then $\delta^-_{L,*,t} = x^-$.
Consequently, $\delta^-_{L,*,t}$ writes:

$$\delta^-_{L,*,t} \quad = \quad \left(\frac{B+\gamma M_0(t,s,y)}{\gamma-1}\right)^+,$$

and:

$$f_0^-(\delta^-_{L,*,t}) = \begin{cases} -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma(B+M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \geq -\frac{B}{\gamma} \\ -\frac{A}{B^\gamma}M_0(t,s,y) & \text{if} \quad M_0(t,s,y) \leq -\frac{B}{\gamma} \end{cases} \tag{2.33}$$

In order to simplify the notations, the following quantities are introduced: $\mathcal{S} = \frac{B}{\gamma}$ and $J_0(t,s,y) = f_0^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t})$.
The equation (2.12) becomes:

$$\mathcal{H} : (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\,(x + \theta_0(t,s,y) + q_1\theta_1(t,s,y)) + J_0(t,s,y) = 0$$

The terms of the HJB equation are sorted by powers of q_1:

$$\begin{aligned} (0) \quad &: \quad (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0, \\ (1) \quad &: \quad (\partial_t + \mathcal{L}_1)\theta_1 - \mu\Delta_t S_t = 0, \end{aligned}$$

Using the final conditions and applying the Feynman-Kac formula yields:

$$\begin{aligned} \theta_1(t,s,y) \quad &= \quad C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right), \\ \theta_0(t,s,y) \quad &= \quad E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right). \end{aligned}$$

Based on these results, the quantity $M_0(t,s,y)$ can be deduced:

$$M_0(t,s,y) \quad = \quad C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right),$$

and $u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right)$ is the solution of the HJB equation (2.12).

The function u_0 coincides with the value function if it is smooth, finite and has a quadratic growth.
In order to prove the quadratic growth of u_0, we start by studying the function θ_0:

- If $M_0(t,s,y) \in [-\mathcal{S},\mathcal{S}]$, then:

$$(\frac{\gamma+1}{\gamma}B)^{-(\gamma-1)} \leq (B - M_0(t,s,y))^{-(\gamma-1)} \leq (\frac{\gamma-1}{\gamma}B)^{-(\gamma-1)},$$
$$(\frac{\gamma+1}{\gamma}B)^{-(\gamma-1)} \leq (B + M_0(t,s,y))^{-(\gamma-1)} \leq (\frac{\gamma-1}{\gamma}B)^{-(\gamma-1)},$$

Then $\exists M > 0$ such that $|J_0(t,s,y)| \leq M$.

- If $M_0(t,s,y) \geq \mathcal{S}$, then:

$$E^{\mathcal{P}}_{t,s,y}\left(|J_0(u,S_u,y_u)|\right) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B+\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E^{\mathcal{P}}_{t,s,y}\left(|M_0(u,S_u,y_u)|\right),$$

- If $M_0(t,s,y) \leq -\mathcal{S}$ then:

$$E^{\mathcal{P}}_{t,s,y}\left(|J_0(u,S_u,y_u)|\right) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E\left(|M_0(u,S_u,y_u)|\right),$$

We assume here that the traded option is a call or a put, then $\exists C_2' > 0$, $E\left(|M_0(t,s,y)|\right) \leq C_2'(1+s)$. It follows that $\exists C_2'' > 0$, such that $|\theta_0(t,s,y)| \leq C_2''(1+s)$. Using the following relation $\forall a,b \in \mathbb{R}$, $ab \leq \frac{a^2+b^2}{2}$, we can prove that $\exists C_2''' > 0$ such that:

$$|u_0(t,s,y,q_1,x)| \leq C_2'''\left(1 + x^2 + s^2 + y^2 + q_1^2\right)$$

which implies that u_0 is finite and has a quadratic growth.
Recall here that θ_0 is the solution of the equation $(\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0$ with the final condition $\theta_0(T,s,y) = 0$. Since the function J_0 is continuous (J_0 is at least $C^{0,0,0}$), then θ_0 is smooth. Consequently, the function u_0 is also smooth and it coincides with the value function.

2.8.3 Appendix 3: Solution in the mean-variance framework ($\beta = 1$)

Let u^ϵ be the solution of the HJB equation (2.22). Under the assumption that $\epsilon \sim 0$, an asymptotic expansion technique is performed with respect to the parameter ϵ:

$$u^\epsilon(t,s,y,q_1,x) \quad = \quad x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t,s,y,q_1),$$

Given the form of the utility function, the following Ansatz on v_0 and v_1 is made :

$$\begin{aligned} v_0(t,s,y,q_1) &= \theta_0(t,s,y) + q_1\theta_1(t,s,y), \\ v_1(t,s,y,q_1) &= \theta_2(t,s,y) + q_1\theta_3(t,s,y) + q_1^2\theta_4(t,s,y), \end{aligned}$$

In order to solve the HJB equation, the Jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated.
Let $f^+ = J^{+,\epsilon}$, the function f^+ writes:

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+)(u(t,s,y,q_1-1,x+(c+\delta^+)) - u(t,s,y,q_1,x)), \\ &= \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y) + \epsilon M_1(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right), \end{aligned}$$

Let $M^+(t,s,y,q_1) = M_0(t,s,y,q_1) + \epsilon M_1(t,s,y,q_1)$. By differentiating f^+, it can be shown that:

$$(f^+)'(\delta^+) \quad = \quad \frac{\lambda^+(\delta^+)}{B+\delta^+}\left(\delta^+(1-\gamma) + B - \gamma M^+(t,s,y,q_1) - \gamma\epsilon^2 R^+(t,s,y,q_1)\right),$$

In order to determine $\delta^+_{*,t} = ArgMax_{\{x \geq 0\}} f^+(x)$, two cases should be distinguished:

- $M^+(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1) \geq \mathcal{S}$: In this case, $\forall\ \delta^+ \geq 0$, $(f^+)'(\delta^+) \leq 0$, then the function f^+ is decreasing on the interval $[0,+\infty[$ and $\delta^+_{*,t} = 0$.
- $M^+(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1) \leq \mathcal{S}$: The function $(f^+)'$ changes its sign on $[0,+\infty[$ and gets null in x^+:

$$x^+ \quad = \quad \frac{B - \gamma\left(M^+(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right)}{\gamma - 1}.$$

  Since $\gamma > 1$ then $\delta^+_{*,t} = x^+$.

In conclusion, $\delta^+_{*,t}$ writes:

$$\delta^+_{*,t} \quad = \quad \left(\frac{B - \gamma\left(M^+(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right)}{\gamma - 1}\right)^+,$$

and, using Taylor expansion, it follows:

$$\delta^+_{*,t} = \delta^+_{L,*,t} - \epsilon\frac{\gamma}{\gamma-1}M_1(t,s,y,q_1)1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}} + O\big(\epsilon^2\big)\,.$$

In order to solve the HJB equation, it is useful to write $f^+(\delta^+_{*,t})$ as the sum of $f^+_0(\delta^+_{L,*,t})$ plus a correction that depends on the parameter ϵ.
We have:

$$f^+(x) \quad = \quad f^+_0(x) + \epsilon\lambda^+(x)M_1(t,s,y,q_1) + O\big(\epsilon^2\big)\,,$$

then, through the use of Taylor expansion, one can write:

$$\begin{aligned} f^+(\delta^+_{*,t}) &= f^+(\delta^+_{L,*,t}) + (f^+)'(\delta^+_{L,*,t})(\delta^+_{*,t} - \delta^+_{L,*,t}) + O\Big((\delta^+_{*,t} - \delta^+_{L,*,t})^2\Big)\,, \\ &= f^+_0(\delta^+_{L,*,t}) + \epsilon\lambda^+(\delta^+_{L,*,t})M_1 - \epsilon(f^+)'(\delta^+_{L,*,t})\frac{\gamma}{\gamma-1}M_1 1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}} + O\big(\epsilon^2\big)\,. \end{aligned}$$

Since $(f^+)'(x) = (f^+_0)'(x) + O(\epsilon)$, the last equation becomes:

$$f^+(\delta^+_{*,t}) \quad = \quad f^+_0(\delta^+_{L,*,t}) + \epsilon M_1\left(\lambda^+(\delta^+_{L,*,t}) - \frac{\gamma}{\gamma-1}(f^+_0)'(\delta^+_{L,*,t})1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}}\right) + O\big(\epsilon^2\big)\,,$$

It can be recalled at this stage that $(f^+_0)'(\delta^+_{L,*,t}) = \frac{B-\gamma M_0}{B}\lambda^+(0)1_{\{M_0 > \mathcal{S}\}}$.
Notice that if $M_0 \in \left[min\left(\mathcal{S}, \mathcal{S} - \epsilon M_1 - \epsilon^2\mathcal{R}^+\right), max\left(\mathcal{S}, \mathcal{S} - \epsilon M_1 - \epsilon^2\mathcal{R}^+\right)\right]$, then $|\frac{B-\gamma M_0}{B}\lambda^+(0)| = O(\epsilon)$. This means:

$$(f^+_0)'(\delta^+_{L,*,t}) \quad = \quad \frac{B - \gamma M_0}{B}\lambda^+(0)1_{\{M^+ + \epsilon^2 R^+ > \mathcal{S}\}} + O(\epsilon)\,,$$

and then:

$$f^+(\delta^+_{*,t}) \quad = \quad f^+_0(\delta^+_{L,*,t}) + \epsilon M_1(t,s,y,q_1)\lambda^+(\delta^+_{L,*,t}) + O\big(\epsilon^2\big)\,.$$

The optimal bid distance $\delta^-_{*,t}$ can be determined using the same method. Indeed, let $f^- = J^{-,\epsilon}$, the function f^- writes:

$$\begin{aligned} f^-(\delta^-) &= \lambda^-(\delta^-)(u(t,s,y,q_1+1,x-(c-\delta^-)) - u(t,s,y,q_1,x)), \\ &= \lambda^-(\delta^-)\left(\delta^- - \left(M_0(t,s,y) + \epsilon M_2(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)\right). \end{aligned}$$

Let $M^-(t,s,y,q_1) = M_0(t,s,y) + \epsilon M_2(t,s,y,q_1)$. Differentiating f^- yields:

$$(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B+\delta^-}\left(\delta^-(1-\gamma) + B + \gamma M^-(t,s,y,q_1) + \gamma\epsilon^2 R^-(t,s,y,q_1)\right).$$

Afterward, it can be seen that if $M^- + \epsilon^2 R^- < -\mathcal{S}$, then $\delta^-_{*,t} = 0$, whereas if $M^- + \epsilon^2 R^- \geq -\mathcal{S}$, then $(f^-)'$ changes its sign on $[0,+\infty[$ and gets null in $x^- = \frac{B+\gamma M^-(t,s,y,q_1)+\gamma\epsilon^2 R^-(t,s,y,q_1)}{\gamma-1} = \delta^-_{*,t}$. So, the quantity $\delta^-_{*,t}$ writes:

$$\delta^-_{*,t} = \left(\frac{B+\gamma\left(M^-(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)}{\gamma-1}\right)^+$$

which implies that:

$$\delta^-_{*,t} = \delta^-_{L,*,t} + \epsilon\frac{\gamma}{\gamma-1}M_2(t,s,y,q_1)1_{\{M^-+\epsilon^2R^-\geq-\mathcal{S}\}} + O(\epsilon^2).$$

The quantity $f^-(\delta^-_{*,t})$ can be written as the sum of $f^-_0(\delta^-_{L,*,t})$ plus a correction term due to the parameter ϵ. Indeed:

$$f^-(x) = f^-_0(x) - \epsilon\lambda^-(x)M_2(t,s,y,q_1) + O(\epsilon^2),$$

Then, using Taylor's expansion:

$$\begin{aligned} f^-(\delta^-_{*,t}) &= f^-(\delta^-_{L,*,t}) + (f^-)'(\delta^-_{L,*,t})(\delta^-_{*,t} - \delta^-_{L,*,t}) + O\Big((\delta^-_{*,t} - \delta^-_{L,*,t})^2\Big), \\ &= f^-_0(\delta^-_{L,*,t}) - \epsilon\lambda^-(\delta^-_{L,*,t})M_2 + \epsilon(f^-)'(\delta^-_{L,*,t})\frac{\gamma}{\gamma-1}M_2 1_{\{M^-+\epsilon^2R^-\geq-\mathcal{S}\}} + O(\epsilon^2). \end{aligned}$$

Using the relation $(f^-)'(x) = (f^-_0)'(x) + O(\epsilon)$ implies:

$$f^-(\delta^-_{*,t}) = f^-_0(\delta^-_{L,*,t}) + \epsilon M_2(t,s,y,q_1)\left(-\lambda^-(\delta^-_{L,*,t}) + \frac{\gamma}{\gamma-1}(f^-_0)'(\delta^-_{L,*,t})1_{\{M^-+\epsilon^2R^-\geq-\mathcal{S}\}}\right) + O(\epsilon^2).$$

We have also $(f^-_0)'(\delta^-_{L,*,t}) = \left(\frac{B+\gamma M_0}{B}\lambda^-(0)\right)1_{\{M_0<-\mathcal{S}\}}$. Following the same method, it can be shown that if $M_0 \in \left[min(-\mathcal{S}, -\mathcal{S} - \epsilon M_2 - \epsilon^2\mathcal{R}^-), max(-\mathcal{S}, -\mathcal{S} - \epsilon M_2 - \epsilon^2\mathcal{R}^-)\right]$, then $|\frac{B+\gamma M_0}{B}\lambda^-(0)| = O(\epsilon)$. Therefore, it can be deduced that:

$$(f^-_0)'(\delta^-_{L,*,t}) = \frac{B+\gamma M_0}{B}\lambda^-(0)1_{\{M^-+\epsilon^2R^-<-\mathcal{S}\}} + O(\epsilon).$$

and:

$$f^-(\delta^-_{*,t}) = f^-_0(\delta^-_{L,*,t}) - \epsilon M_2(t,s,y,q_1)\lambda^-(\delta^-_{L,*,t}) + O(\epsilon^2),$$

Now that the terms $f^+(\delta^+_{*,t})$ and $f^-(\delta^-_{*,t})$ are computed separately, the term $J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) = f^+(\delta^+_{*,t}) + f^-(\delta^-_{*,t})$ is deduced:

$$\begin{aligned} J(\delta^-_{*,t},\delta^+_{*,t}) &= f^+_0(\delta^+_{L,*,t}) + \epsilon M_1\lambda^+(\delta^+_{L,*,t}) + f^-_0(\delta^-_{L,*,t}) - \epsilon M_2\lambda^-(\delta^-_{L,*,t}) + O(\epsilon^2), \\ &= J_0(\delta^-_{L,*,t},\delta^+_{L,*,t}) + \epsilon M_1(t,s,y,q_1)\lambda^+(\delta^+_{L,*,t}) - \epsilon M_2(t,s,y,q_1)\lambda^-(\delta^-_{L,*,t}) + O(\epsilon^2), \end{aligned}$$

The terms of $J(\delta^-_{*,t},\delta^+_{*,t})$ are classified according to their power in ϵ:

$$J(\delta^-_{*,t},\delta^+_{*,t}) = J_0(t,s,y) + \epsilon J_1(t,s,y,q_1) + O(\epsilon^2),$$

where $J_1(t,s,y,q_1) = J_{1,0}(t,s,y) + q_1 J_{1,1}(t,s,y)$ and:

$$\begin{aligned} J_{1,0}(t,s,y) &= \lambda^+(\delta^+_{L,*,t})(-\theta_3+\theta_4) - \lambda^-(\delta^-_{L,*,t})(-\theta_3-\theta_4), \\ J_{1,1}(t,s,y) &= -2\theta_4\left(\lambda^+(\delta^+_{L,*,t}) - \lambda^-(\delta^-_{L,*,t})\right), \end{aligned}$$

The HJB equation can be separated into several terms according to the order of the parameter ϵ. By nullifying the term of order 0 in ϵ, it can be obtained that:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1\theta_1) + J_0 = 0,$$

with the final conditions:

$$\theta_0(T,s,y) = 0, \quad \theta_1(T,s,y) = h(s).$$

The functions θ_0 and θ_1 are equivalent to those found in the case of a linear utility function without inventory constraints, thus:

$$\begin{aligned} \theta_1(t,s,y) &= C_{\mathcal{P}}(t,s,y) - \mu E_{t,s,y}\left(\int_t^T \Delta_u S_u du\right) \\ \theta_0(t,s,y) &= E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right), \end{aligned}$$

The term of order 1 in ϵ leads to the following equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(\theta_2 + q_1\theta_3 + q_1^2\theta_4) + J_1(t,s,y) = q_1^2 V + T,$$

with the final conditions:

$$\theta_2(T,s,y) = 0, \quad \theta_3(T,s,y) = 0, \quad \theta_4(T,s,y) = 0.$$

The functions θ_2, θ_3 and θ_4 are:

$$\begin{aligned} \theta_2(t,s,y) &= E^{\mathcal{P}}_{t,s,y}\left(\int_t^T (J_{1,0} - T)(u,s_u,y_u)du\right), \\ \theta_3(t,s,y) &= E^{\mathcal{P}}_{t,s,y}(\int_t^T J_{1,1}(u,s_u,y_u)du), \\ \theta_4(t,s,y) &= -E^{\mathcal{P}}_{t,s,y}(\int_t^T V_u du). \end{aligned}$$

It can be demonstrated, as it was done in the case $\beta = \frac{1}{2}$, that the function u^ϵ is smooth, finite and has polynomial growth. Besides, u^ϵ can be approximated at order 1 in ϵ by $\tilde{u}^\epsilon(t,s,y,q_1,x) = x + v_0(t,s,y) + \epsilon v_1(t,s,y,q_1)$.

Remark:

The optimal ask quote $\delta^+_{*,t}$ can be approximated at order 1 in ϵ by $\hat{\delta}^+_{*,t}$:

$$\hat{\delta}^+_{*,t} = \left(\frac{B - \gamma M^+(t,s,y,q_1)}{\gamma - 1}\right)^+.$$

Indeed, if $M^+(t,s,y) \in \left[Min\left(\mathcal{S} - \epsilon^2 R^+, \mathcal{S}\right), Max\left(\mathcal{S} - \epsilon^2 R^+, \mathcal{S}\right)\right]$, then $|\frac{B-\gamma M^+(t,s,y,q_1)}{\gamma-1}| = O\big(\epsilon^2\big)$ and consequently:

$$|\hat{\delta}^+_{*,t} - \delta^+_{*,t}| = O\big(\epsilon^2\big),$$

Similarly, it can be seen that if $M^- \in \left[Min\left(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-\right), Max\left(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-\right)\right]$, then $|\frac{B+\gamma M^-}{\gamma-1}| = O\big(\epsilon^2\big)$. Thus, the optimal bid quote $\delta^-_{*,t}$ can be approximated at order 1 in ϵ by $\hat{\delta}^-_{*,t}$:

$$\hat{\delta}^-_{*,t} = \left(\frac{B + \gamma M^-}{\gamma - 1}\right)^+,$$

and the approximation error is at order 2 in ϵ:

$$|\hat{\delta}^-_{*,t} - \delta^-_{*,t}| = O\big(\epsilon^2\big)$$

2.8.4 Appendix 4: Solving the Heston PDE with a trend on the underlying

We aim to determine here an analytic expression for the quantity $C(t,S_t,y_t) = E^{\mathcal{P}}\left((S_T-K)^+\right)$ in the Hestom model. In this model, $C(t,S_t,y_t)$ is the solution of the following equation:

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial S}\mu S_t+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}y_tS_t^2+\frac{\partial C}{\partial y}k(\theta-y_t)+\frac{1}{2}\frac{\partial^2 C}{\partial y^2}\eta^2y_t+\frac{\partial^2 C}{\partial S\partial y}\rho S_ty_t\eta=0,$$

with the final condition: $C(T,S_T,y_T)=(S_T-K)^+$.
We are going to use for this purpose the same approach as in Heston (1993) and Gatheral (2006).

Let $x_t=\log(\frac{S_t}{K})$ and $P(t,x_t,y_t)=C(t,S_t,y_t)$, then P is the solution of the following equation:

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial x}(\mu-\frac{y_t}{2})+\frac{1}{2}\frac{\partial^2 P}{\partial x^2}y_t+\frac{\partial P}{\partial y}k(\theta-y_t)+\frac{1}{2}\frac{\partial^2 P}{\partial y^2}\eta^2y_t+\frac{\partial^2 P}{\partial x\partial y}\rho y_t\eta=0, \tag{2.34}$$

By analogy with the Black-Scholes formula, the guessed solution of the previous equation has the form:

$$P(t,x_t,y_t) \;=\; K\left(e^{x_t}P_1-e^{-\mu(T-t)}P_2\right), \tag{2.35}$$

By inserting (2.35) in (2.34), it can be obtained that:

$$Ke^xT_1(t,x,y)-Ke^{-\mu(T-t)}T_2(t,x,y) \;=\; 0,$$

where:

$$\begin{aligned}T_1(t,x,y)=\;&\frac{\partial P_1}{\partial t}+(\mu-\frac{y_t}{2})(P_1+\frac{\partial P_1}{\partial x})+\frac{y_t}{2}(2\frac{\partial P_1}{\partial x}+P_1+\frac{\partial^2 P_1}{\partial x^2})+k(\theta-y_t)\frac{\partial P_1}{\partial y}+\frac{\eta^2y_t}{2}\frac{\partial^2 P_1}{\partial y_t^2}\\&+\rho\eta y_t(\frac{\partial^2 P_1}{\partial x\partial y}+\frac{\partial P_1}{\partial y})),\end{aligned}$$

and:

$$T_2(t,x,y) \;=\; \mu P_2+\frac{\partial P_2}{\partial t}+(\mu-\frac{y_t}{2})\frac{\partial P_2}{\partial x}+\frac{y_t}{2}\frac{\partial^2 P_2}{\partial x^2}+k(\theta-y_t)\frac{\partial P_2}{\partial y}+\frac{\eta^2y_t}{2}\frac{\partial^2 P_2}{\partial y_t^2}+\rho\eta y_t\frac{\partial^2 P_2}{\partial x\partial y}$$

In order to solve the equation (2.34), it is possible to choose P_1 and P_2 in (2.35) such that $T_1(t,x,y)=T_2(t,x,y)=0$. Let $\tau=T-t$, it follows that for $j\in\{1,2\}$, P_j satisfies the following equation:

$$-\frac{\partial P_j}{\partial \tau}+\mu P_j+(\mu+u_jy_t)\frac{\partial P_j}{\partial x}+\frac{y_t}{2}\frac{\partial^2 P_j}{\partial x^2}+(a-b_jy_t)\frac{\partial P_j}{\partial y}+\frac{\eta^2y_t}{2}\frac{\partial^2 P_j}{\partial y_t^2}+\rho\eta y_t\frac{\partial^2 P_j}{\partial x\partial y}=0$$

where $u_1=\frac{1}{2}$, $u_2=-\frac{1}{2}$, $a=k\theta$, $b_1=k-\rho\eta$, $b_2=k$.
Using a Fourier transform method, let $\tilde{P}_j(\tau,u,y)$ be defined as following:

$$\tilde{P}_j(\tau,u,y)=\int_{-\infty}^{+\infty}e^{-iux}P_j(\tau,x,y)dx$$

Evaluating $\tilde{P}_j(\tau,u,y)$ at $\tau=0$ gives:

$$\begin{aligned}\tilde{P}_j(0,u,y) &= \int_{-\infty}^{+\infty}e^{-iux}P_j(0,x,y)dx,\\&= \int_{-\infty}^{+\infty}e^{-iux}1_{\{x>0\}}dx,\\&= \frac{1}{iu}\end{aligned}$$

By doing necessary calculations, it can be found that:

$$-\frac{\partial \tilde{P}_j}{\partial \tau} + \mu \tilde{P}_j + (\mu + u_j y_t) iu \tilde{P}_j - u^2 \frac{y_t}{2} \tilde{P}_j + (a - b_j y_t) \frac{\partial \tilde{P}_j}{\partial y} + \frac{\eta^2 y_t}{2} \frac{\partial^2 \tilde{P}_j}{\partial y_t^2} + \rho \eta y_t iu \frac{\partial \tilde{P}_j}{\partial y} = 0$$

Let:

$$\beta' = b_j - \rho \eta iu, \qquad \gamma' = \frac{\eta^2}{2}, \qquad \alpha' = -\frac{u^2}{2} + iuu_j,$$

Using the new notations, $\tilde{P}_j$ satisfies the following equation:

$$-\frac{\partial \tilde{P}_j}{\partial \tau} + a \frac{\partial \tilde{P}_j}{\partial y} + y_t (\alpha' \tilde{P}_j - \beta' \frac{\partial \tilde{P}_j}{\partial y} + \gamma' \frac{\partial^2 \tilde{P}_j}{\partial y_t^2}) + (1 + iu) \mu \tilde{P}_j = 0, \tag{2.36}$$

Now by substituting $\tilde{P}_j(\tau, u, y)$ by:

$$\begin{aligned} \tilde{P}_j(\tau, u, y) &= \exp(C(u,\tau)\theta + D(u,\tau) y_t) \tilde{P}_j(0, u, y), \\ &= \frac{1}{iu} \exp(C(u,\tau)\theta + D(u,\tau) y_t) \end{aligned}$$

and by using the relations:

$$\frac{\partial \tilde{P}_j}{\partial \tau} = (\theta \frac{\partial C}{\partial \tau} + y_t \frac{\partial D}{\partial \tau}) \tilde{P}_j, \quad \frac{\partial \tilde{P}_j}{\partial y} = D(u,\tau) \tilde{P}_j, \quad \frac{\partial^2 \tilde{P}_j}{\partial y^2} = D(u,\tau)^2 \tilde{P}_j,$$

the equation (2.36) becomes:

$$\left[-\theta \frac{\partial C}{\partial \tau} + aD + (1 + iu)\mu \right] + y_t \left[\alpha' - \beta' D + \gamma' D^2 - \frac{\partial D}{\partial \tau} \right] = 0$$

The two terms at order 0 and 1 in y_t should then be null, which gives the two following equations:

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= \frac{a}{\theta} D + \frac{1 + iu}{\theta} \mu, \\ \frac{\partial D}{\partial \tau} &= \alpha' - \beta' D + \gamma' D^2, \end{aligned}$$

Let:

$$r_+ = \frac{\beta' + \sqrt{\beta'^2 - 4\alpha'\gamma'}}{2\gamma'}, \qquad r_- = \frac{\beta' - \sqrt{\beta'^2 - 4\alpha'\gamma'}}{2\gamma'}, \qquad g = \frac{r_-}{r_+}.$$

Integrating with the terminal conditions $C(u, 0) = 0$ and $D(u, 0) = 0$ gives:

$$\begin{aligned} D(u,\tau) &= r_- \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}, \\ C(u,\tau) &= \frac{1 + iu}{\theta} \mu\tau + k \left[r_- \tau - \frac{2}{\eta^2} \ln(\frac{1 - ge^{-d\tau}}{1 - g}) \right] \end{aligned}$$

and finally:

$$P_j(\tau, x, y_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} Re(\tilde{P}_j(\tau, u, y_t) e^{iux}) du,$$

2.8.5 Appendix 5: Verification theorem

Let $\delta = (\delta^-, \delta^+)$ be an admissible control process, and consider the following processes:

$$\begin{aligned}
dq_{1,t} &= dN_t^- - dN_t^+, \\
dX_t &= (C_\mathcal{Q}(t, S_t, y_t) + \delta_t^+)dN_t^+ - (C_\mathcal{Q}(t, S_t, Y_t) - \delta_t^-)dN_t^- + q_{2,t}dS_t, \\
\frac{dS_t}{S_t} &= \mu dt + \sigma(y_t)dW_t^{(1)}, \\
dy_t &= a_R(y_t)dt + b_R(y_t)dW_t^{(2)},
\end{aligned}$$

where N^- and N^+ are Poisson processes with intensities λ^- and λ^+ respectively, where $\forall \tau \geq 0$:

$$\lambda_\tau^+ = \lambda^+(\delta_\tau^+), \quad \lambda_\tau^- = \lambda^-(\delta_\tau^-).$$

Let $t_N = T \wedge \{\tau > t, |S_\tau - s| \geq n\} \wedge \{|y_\tau - y| \geq n\} \wedge \{\left|N_\tau^+ - N_t^+\right| \geq n\} \wedge \{\left|N_\tau^- - N_t^-\right| \geq n\}$, and u_0 be the solution of the HJB equation (2.12). Since u_0 is smooth, we have:

$$\begin{aligned}
&u_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n}) = u_0(t, s, y, q_1, x) + \int_{\tau=t}^{t_n} (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau)d\tau \\
&+ \int_{\tau=t}^{t_n} \frac{\partial u}{\partial s} S_\tau \sigma(y_\tau)dW_\tau^{(1)} + \frac{\partial u_0}{\partial y} b_R(y_\tau)dW_\tau^{(2)} - \frac{\partial u_0}{\partial x} q_{1,\tau}\Delta_\tau S_\tau \sigma(y_\tau)dW_\tau^{(1)} \\
&+ \int_{\tau=t}^{t_n} \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_\mathcal{Q} + \delta^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^-(\delta^-)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_\mathcal{Q} - \delta^-)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_\mathcal{Q} + \delta^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) dM_\tau^+ \\
&+ \int_{\tau=t}^{t_n} \lambda^-(\delta^-)\left(u(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_\mathcal{Q} - \delta^-)) - u(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) dM_\tau^-
\end{aligned}$$

where M^+ and M^- are the compensated processes associated to N^+ and N^- respectively. Using the polynomial growth of u, as well as the the fact that the functions λ^+ and λ^- are bounded, the local martingales in the previous equation are martingales. Thus, by taking the expectation on both sides of the last equation, we obtain:

$$\begin{aligned}
&E\left(u_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = u_0(t, s, y, q_1, x) + E\left(\int_{\tau=t}^{t_n} (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau)d\tau\right) \\
&+ E\left(\int_{\tau=t}^{t_n} \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_\mathcal{Q} + \delta^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right) \\
&+ E\left(\int_{\tau=t}^{t_n} \lambda^-(\delta^-)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_\mathcal{Q} - \delta^-)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right)
\end{aligned}$$

Again using the polynomial growth of u_0, we can deduce that u_0 is integrable on $[0, t_n]$ and by applying the dominated convergence theorem, it can be obtained that:

$$lim_{n\to+\infty} E\left(u_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = E\left(u_0(T, S_T, y_T, q_{1,T}, X_T)\right),$$

and the equation becomes:

$$
\begin{aligned}
&E\left(u_0(T,S_T,y_T,q_{1,T},X_T)\right)=u_0(t,s,y,q_1,x)+E\left(\int_{\tau=t}^{T}\left(\partial_\tau+\mathcal{L}_1+\mathcal{L}_2\right)u_0(\tau,s_\tau,y_\tau,q_{1,\tau},x_\tau)d\tau\right)\\
&+E\left(\int_{\tau=t}^{T}\lambda^+(\delta^+)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}-1,x_{\tau^-}+C_Q+\delta^+)-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)d\tau\right)\\
&+E\left(\int_{\tau=t}^{T}\lambda^-(\delta^-)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}+1,x_{\tau^-}-(C_Q-\delta^-))-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)d\tau\right)
\end{aligned}
$$

Recalling that u_0 is the solution of the HJB equation (2.12), we have for $(\delta_t^-,\delta_t^+)\in\mathcal{A}$:

$$
\begin{aligned}
&\left(\partial_t+\mathcal{L}_1+\mathcal{L}_2\right)u_0+\lambda^+(\delta^+)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}-1,x_{\tau^-}+C_Q+\delta_t^+)-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)\\
&+\lambda^-(\delta^-)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}+1,x_{\tau^-}-(C_Q-\delta_t^-))-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)\leq 0,
\end{aligned}
$$

which means that $E\left(U(T,S_T,y_T,q_{1,T},X_T)\right)\leq u_0(t,s,y,q_1,x)$ and then $u(t,s,y,q_1,x)=Sup_{(\delta^-,\delta^+)\in\mathcal{A}}E\left(U(T,S_T,y_T,q_{1,T},X_T)\right)\leq u_0(t,s,y,q_1,x)$.

In addition, since u_0 is solution of (2.12), then for $(\delta_t^-,\delta_t^+)=(\delta_{t,*}^-,\delta_{t,*}^+)$ we have:

$$
E\left(U(T,S_T^{(\delta_{t,*}^-,\delta_{t,*}^+)},y_T^{(\delta_{t,*}^-,\delta_{t,*}^+)},q_{1,T}^{(\delta_{t,*}^-,\delta_{t,*}^+)},X_T^{(\delta_{t,*}^-,\delta_{t,*}^+)})\right)=u_0(t,s,y,q_1,x),
$$

and then $u_0(t,s,y,q_1,x)\leq Sup_{(\delta_t^-,\delta_t^+)\in\mathcal{A}}E\left(U(T,S_T^{(\delta_t^-,\delta_t^+)},y_T^{(\delta_t^-,\delta_t^+)},q_{1,T}^{(\delta_t^-,\delta_t^+)},X_T^{(\delta_t^-,\delta_t^+)})\right)=u(t,s,y,q_1,x)$.

Finally, we conclude that $u_0(t,s,y,q_1,x)=u(t,s,y,q_1,x)$.

2.8.6 Appendix 6: Verification theorem

Let $\delta=(\delta^-,\delta^+)$ be an admissible control process, and consider the following processes:

$$
\begin{aligned}
dq_{1,t} &= dN_t^- - dN_t^+,\\
dX_t &= (C_Q(t,S_t,y_t)+\delta_t^+)dN_t^+ - (C_Q(t,S_t,Y_t)-\delta_t^-)dN_t^- + q_{2,t}dS_t,\\
\frac{dS_t}{S_t} &= \mu dt+\sigma(y_t)dW_t^{(1)},\\
dy_t &= a_R(y_t)dt+b_R(y_t)dW_t^{(2)},
\end{aligned}
$$

where N^- and N^+ are Poisson processes with intensities λ^- and λ^+ respectively, where $\forall\tau\geq 0$:

$$
\lambda_\tau^+=\lambda^+(\delta_\tau^+),\quad \lambda_\tau^-=\lambda^-(\delta_\tau^-).
$$

Let $t_N = T \wedge \{\tau > t, |S_\tau - s| \geq n\} \wedge \{|y_\tau - y| \geq n\} \wedge \{|N^+_\tau - N^+_t| \geq n\} \wedge \{|N^-_\tau - N^-_t| \geq n\}$, and u^ϵ_0 be the solution of the HJB equation (2.22). Since u^ϵ_0 is smooth, we have:

$$
\begin{aligned}
&u^\epsilon_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n}) = u^\epsilon_0(t, s, y, q_1, x) + \int_{\tau=t}^{t_n} (\partial_\tau + \mathcal{L}_1 + \mathcal{L}_2)\, u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau) d\tau \\
&+ \int_{\tau=t}^{t_n} \frac{\partial u^\epsilon_0}{\partial s} S_\tau \sigma(y_\tau) dW^{(1)}_\tau + \frac{\partial u^\epsilon_0}{\partial y} b_R(y_\tau) dW^{(2)}_\tau - \frac{\partial u^\epsilon_0}{\partial x} q_{1,\tau} \Delta_\tau S_\tau \sigma(y_\tau) dW^{(1)}_\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^+(\delta^+) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_Q + \delta^+) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^-(\delta^-) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_Q - \delta^-)) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^+(\delta^+) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_Q + \delta^+) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) dM^+_\tau \\
&+ \int_{\tau=t}^{t_n} \lambda^-(\delta^-) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_Q - \delta^-)) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) dM^-_\tau
\end{aligned}
$$

where M^+ and M^- are the compensated processes associated to N^+ and N^- respectively. Using the polynomial growth of u^ϵ_0, as well as the the fact that the functions λ^+ and λ^- are bounded, the local martingales in the previous equation are martingales. Thus, by taking the expectation on both sides of the last equation, we obtain:

$$
\begin{aligned}
&E\left(u^\epsilon_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = u^\epsilon_0(t, s, y, q_1, x) + E\left(\int_{\tau=t}^{t_n} (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau) d\tau \right) \\
&+ E\left(\int_{\tau=t}^{t_n} \lambda^+(\delta^+) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_Q + \delta^+) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \right) \\
&+ E\left(\int_{\tau=t}^{t_n} \lambda^-(\delta^-) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_Q - \delta^-)) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \right)
\end{aligned}
$$

Again using the polynomial growth of u^ϵ_0, we can deduce that u^ϵ_0 is integrable on $[0, t_n]$ and by applying the dominated convergence theorem, it can be obtained that:

$$
lim_{n \to +\infty} E\left(u^\epsilon_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = E\left(u^\epsilon_0(T, S_T, y_T, q_{1,T}, X_T)\right),
$$

and the equation becomes:

$$
\begin{aligned}
&E\left(u^\epsilon_0(T, S_T, y_T, q_{1,T}, X_T)\right) = u^\epsilon_0(t, s, y, q_1, x) + E\left(\int_{\tau=t}^{T} (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau) d\tau \right) \\
&+ E\left(\int_{\tau=t}^{T} \lambda^+(\delta^+) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_Q + \delta^+) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \right) \\
&+ E\left(\int_{\tau=t}^{T} \lambda^-(\delta^-) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_Q - \delta^-)) - u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) d\tau \right)
\end{aligned}
$$

Recalling that u^ϵ_0 is the solution of the HJB equation (2.22), we have for $(\delta^-_t, \delta^+_t) \in \mathcal{A}$:

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u^\epsilon_0 + \lambda^+(\delta^+) \left(u^\epsilon_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_Q + \delta^+_t) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) \\
&+ \lambda^-(\delta^-) \left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_Q - \delta^-_t)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-}) \right) \leq \epsilon (q_1^2 V_t + T_t),
\end{aligned}
$$

which means that $E\left(U(T, S_T, y_T, q_{1,T}, X_T) - \epsilon \int_{\tau=t}^{T} \left(q^2_{1,\tau} V_\tau + T_\tau \right) d\tau \right) \leq u^\epsilon_0(t, s, y, q_1, x)$ and by taking the supremum with respect to (δ^-_t, δ^+_t), we obtain $u^\epsilon(t, s, y, q_1, x) \leq u^\epsilon_0(t, s, y, q_1, x)$.

In addition, since u_0^ϵ is solution of (2.22), then for $(\delta_t^-, \delta_t^+) = (\delta_{t,*}^-, \delta_{t,*}^+)$ we have:

$$E\left(U(T, S_T^{(\delta_{t,*}^-,\delta_{t,*}^+)}, y_T^{(\delta_{t,*}^-,\delta_{t,*}^+)}, q_{1,T}^{(\delta_{t,*}^-,\delta_{t,*}^+)}, X_T^{(\delta_{t,*}^-,\delta_{t,*}^+)}) - \epsilon \int_{\tau=t}^{T} \left(q_{1,\tau}^2 V_\tau^{(\delta_{t,*}^-,\delta_{t,*}^+)} + T_\tau^{(\delta_{t,*}^-,\delta_{t,*}^+)}\right) d\tau\right) = u_0^\epsilon(t,s,y,q_1,x),$$

and then $u_0^\epsilon(t,s,y,q_1,x) \le u^\epsilon(t,s,y,q_1,x)$.

Finally, we conclude that $u_0^\epsilon(t,s,y,q_1,x) = u^\epsilon(t,s,y,q_1,x)$.

Chapter 3

Optimal market making of options in the two-dimensional case

3.1 Introduction

In the previous chapter, we addressed the problem of market making of an option on a given underlying asset. We considered that the market maker provides liquidity in this option while continuously delta-hedging her inventory. Two scenarios were considered in this study depending on the behavior of the market maker. In the first one, the market maker is risk neutral and wants to maximize the expectation of her final wealth. Through the use of optimal stochastic control, we provided analytic expressions of the optimal quotes and we determined the optimal strategy. In the second scenario, the market maker is risk averse, thus she also aims to reduce the variance of her final wealth. In this case, we provided approximations of the optimal quotes as a function of the penalty parameter that characterize the risk aversion of the market maker. It is important to recall that the instantaneous volatility of the underlying is stochastic. Thus, it is not possible to hedge all the risk of the option by trading the underlying. Indeed, the option inventory has a residual volatility risk which may cause significant variations of the wealth process. This risk source should then be taken into account.

From a practical point of view, an option market-maker has a mandate to provide liquidity on several options which have different underlying assets. Thus, the problem of market making can be studied in the multidimensional case where the agent sets bid and ask quotes on different options. In this chapter, we will present a new method for risk-management of the inventory in the case where the market maker provides liquidity in different options. The structure of this chapter is as follows. In the second section, we introduce the model framework, in particular the dynamics of the underlying assets and the probability of execution of limit orders in the option's order book. Afterward, we study two approaches a market maker can adopt as a liquidity provider. Indeed, in the third section, we consider that the market maker is risk neutral and wants to determine the optimal quotes on the options with the aim of maximizing the expectation of her final wealth. Using optimal stochastic control, we determine analytic expressions of the optimal quotes and we interpret the optimal strategy. In the fourth section, we suppose that the market maker is risk averse. She aims to set the optimal quotes in a mean-variance utility framework. We provide approximations of the optimal quotes and the value function of the market maker and we point out the effect of the option inventory on the optimal strategy of the market maker.

3.2 Model setup

We consider here an option market maker operating on two options C_1 and C_2 having respectively the underlying assets S_1 and S_2. Under the real-world probability measure $\mathcal{P}$, the spot process $(S_i)_{i\in\{1,2\}}$ has the following dynamics:

$$\frac{dS_{i,t}}{S_{i,t}} = \mu_i dt + \sigma_i(y_{i,t})dW_t^i, \tag{3.1}$$

$$dy_{i,t} = a_{i,R}(y_{i,t})dt + b_{i,R}(y_{i,t})dZ_t^i, \tag{3.2}$$

where W^i and Z^i are $(\mathcal{P},\mathcal{F})$-Brownian motions such that $d\left\langle W^i, Z^i\right\rangle_t = \rho_{i,R}dt$. In addition, the two stocks S_1 and S_2 as well as the volatility processes y_1 and y_2 are correlated, indeed $d\left\langle W^1, W^2\right\rangle_t = \rho_{1,2,R}dt$ and $d\left\langle Z^1, Z^2\right\rangle_t = \tilde{\rho}_{1,2,R}dt$.

Market participants price options under a risk-neutral probability measure $\mathcal{Q}$. Under this measure, S_1 and S_2 have the following dynamics:

$$\frac{dS_{i,t}}{S_{i,t}} = rdt + \sigma_i(y_{i,t})dW_t^{*,i}, \tag{3.3}$$

$$dy_{i,t} = a_{i,I}(y_{i,t})dt + b_{i,I}(y_{i,t})dZ_t^{*,i}, \tag{3.4}$$

where $W^{*,i}$ and $Z^{*,i}$ are two $(\mathcal{Q},\mathcal{F})$-Brownian motions such that $d\left\langle W^{*,i}, Z^{*,i}\right\rangle_t = \rho_{i,I}dt$. Besides, $d\left\langle W^{*,1}, W^{*,2}\right\rangle_t = \rho_{1,2,I}dt$ and $d\left\langle Z^{*,1}, Z^{*,2}\right\rangle_t = \tilde{\rho}_{1,2,I}dt$
It is clear that the probability measure $\mathcal{Q}$ may not be absolutely continuous with respect to $\mathcal{P}$.

An European option on the asset S_i with payoff $h_i(S_{i,T})$ at the maturity T has a price $C_{\mathcal{Q}}^{(i)}$ under the risk-neutral probability measure $\mathcal{Q}$ which is given as follows:

$$C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) = e^{-r(T-t)}E^{\mathcal{Q}}\left(h_i(S_{i,T})|\mathcal{F}_t\right). \tag{3.5}$$

We suppose that there are European options on the assets S_1 and S_2 which have a maturity equal to T and payoff functions equal to $h_1(S_{1,T})$ and $h_2(S_{2,T})$ respectively. The option market maker under consideration provides liquidity on these two options and proposes continuously bid and ask quotes for the option $C^{(i)}$ denoted respectively C_i^b and C_i^a. Indeed, at a given time t, the Market Maker chooses her optimal bid and ask distances $\delta_{i,t}^-$ and $\delta_{i,t}^+$ from the mid-price $C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t})$ and sets her bid and ask quotes as follows:

$$C_{i,t}^a = C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) + \delta_{i,t}^+, \tag{3.6}$$

$$C_{i,t}^b = C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) - \delta_{i,t}^-, \tag{3.7}$$

In addition, the Market Maker trades continuously in the liquid stocks in order to delta-hedge her option inventory.
The notations defined here are going to be used in the rest of the study:

- $q_{i,t}$ is the inventory at time t in the option $C^{(i)}$.
- $\ell_{i,t}$ is the inventory at time t in the stock S_i.
- X_t is the cash held by the market maker at time t.
- $C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t})$ is the mid price of the option $C^{(i)}$ at time t.
- $\Delta_{i,t} = \Delta_i(t, S_{i,t}, y_{i,t})$ is the delta of the option $C^{(i)}$ at time t.

The trading activity of the market maker depends on the rate of arrival of liquidity-consuming market orders at her quoted prices. These market orders on the option $C^{(i)}$ are modeled by two independent Poisson processes: N_i^+ for buy orders consuming the ask quotes, and N_i^- for sell orders consuming the bid quotes. Therefore, the inventory q_i on the option $C^{(i)}$ has the following dynamics:

$$dq_{i,t} \quad = \quad dN_{i,t}^- - dN_{i,t}^+. \tag{3.8}$$

The market maker delta-hedges continuously her option inventory q_i, and holds at each date t, a quantity $\ell_{i,t} = -q_{i,t}\Delta_{i,t}$ of the stock S_i. The process ℓ_i evolves as follows:

$$\begin{aligned} d\ell_{i,t} &= -\Delta_{i,t}dq_{i,t} - q_{i,t}d\Delta_{i,t} - d\langle q_i, \Delta_i\rangle_t\,, & (3.9)\\ &= -\Delta_{i,t}dN_{i,t}^- + \Delta_{i,t}dN_{i,t}^+ - q_{i,t}d\Delta_{i,t}. & (3.10)\end{aligned}$$

When the ask quote of the market maker on the option $C^{(i)}$ gets executed, the cash process X augments by $(C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) + \delta_{i,t}^+)$. Likewise, when her bid quote is executed, the cash process diminishes by $(C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) - \delta_{i,t}^-)$. In addition, the continuous trading in the stocks S_1 and S_2 for the purpose of delta-hedging induces a continuous variation in the cash process. This, the cash process X evolves as follows:

$$dX_t \quad = \quad \sum_{i=1}^{2}\Big((C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) + \delta_{i,t}^+)dN_{i,t}^+ - (C_{\mathcal{Q}}^{(i)}(t, S_{i,t}, y_{i,t}) - \delta_{i,t}^-)dN_{i,t}^- + \ell_{i,t}dS_{i,t}\Big). \tag{3.11}$$

The counting process N_i^+ of market buy orders and the counting process N_i^- of market sell orders are Poisson processes which have the intensities λ_i^+ and λ_i^- respectively. We assume here that λ_i^+ and λ_i^- have the following forms:

$$\lambda_i^+(\delta_i^+) \quad = \quad \frac{A_i}{\Big(B_i + \big(\delta_i^+\big)^{C_i}\Big)^{\gamma_i}}, \qquad \lambda_i^-(\delta_i^-) = \frac{A_i}{\Big(B_i + \big(\delta_i^-\big)^{C_i}\Big)^{\gamma_i}},$$

where $A_1, A_2, B_1, B_2 \geq 0$ and $\gamma_1, \gamma_2 > 1$.

The Market Maker controls only the bid and ask distances $\delta_{i,t}^-$ and $\delta_{i,t}^+$ on which depends the arrival rate of market orders. She wants to optimize her choice of the control variables in order to maximize her expected utility from terminal wealth at the options maturity date T. The wealth $\mathcal{W}_t$ at time t is the sum of the cash amount and the market value of the options inventories, it can be written as $\mathcal{W}_t = X_t + q_{1,t}C_{\mathcal{Q}}^{(1)}(t, S_{1,t}, y_{1,t}) + q_{2,t}C_{\mathcal{Q}}^{(2)}(t, S_{2,t}, y_{2,t})$. So at time t, the market maker aims to determine $\Big(\delta_{1,*,t}^+, \delta_{1,*,t}^-, \delta_{2,*,t}^+, \delta_{2,*,t}^-\Big)$ which solve the following problem:

$$\begin{aligned}\Big(\delta_{1,*,t}^+, \delta_{1,*,t}^-, \delta_{2,*,t}^+, \delta_{2,*,t}^-\Big) = &\underset{\{(\delta_1^+,\delta_1^-,\delta_2^+,\delta_2^-)\in\mathcal{A}\}}{ArgSup} \quad E^{\mathcal{P}}(U(\mathcal{W}_T)|\forall i \in \{1,2\}\, S_{i,t} = s_i, y_{i,t} = y_i, q_{i,t} = q_i,\\ &X_t = x),\end{aligned}$$

where U is the utility function.

3.3 Linear utility function

We suppose here that the utility function U of the market maker is the identity function:

$$U(\mathcal{W}_T) \quad = \quad \mathcal{W}_T, \tag{3.12}$$

In order to solve the optimization problem, a stochastic control approach is used. The value function of the market maker is:

$$u(t, s_1, y_1, s_2, y_2, q_1, q_2, x) = \underset{\{(\delta^+_{1,t},\delta^-_{1,t},\delta^+_{2,t},\delta^-_{2,t})\in\mathcal{A}\}}{Sup} E^{\mathcal{P}}(\mathcal{W}_T|S_{1,t} = s_1, y_{1,t} = y_1, S_{2,t} = s_2, y_{2,t} = y_2,$$
$$q_{1,t} = q_1, q_{2,t} = q_2, X_t = x),$$

where $\mathcal{A}$ denotes the set of admissible values for the controls $(\delta^+_1, \delta^-_1, \delta^+_2, \delta^-_2)$ and is equal to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.
We define the differential operators $\mathcal{L}_1, \mathcal{L}_2$ as follows:

$$\mathcal{L}_1 = \left(\sum_{i=1}^{2} \mu_i S_{i,t}\frac{\partial}{\partial S_i}\right) + \frac{1}{2}\left(\sigma_1^2(y_{1,t})S_{1,t}^2\frac{\partial^2}{\partial S_1^2} + \sigma_2^2(y_{2,t})S_{2,t}^2\frac{\partial^2}{\partial S_2^2} + 2\sigma_1(y_{1,t})\sigma_2(y_{2,t})S_{1,t}S_{2,t}\rho_{1,2,R}\frac{\partial^2}{\partial S_1\partial S_2}\right)$$
$$+ \left(\sum_{i=1}^{2} a_{i,R}(y_{i,t})\frac{\partial}{\partial y_i}\right) + \frac{1}{2}\left(b_{1,R}^2(y_{1,t})\frac{\partial^2}{\partial y_1^2} + b_{2,R}^2(y_{2,t})\frac{\partial^2}{\partial y_2^2} + 2b_{1,R}(y_{1,t})b_{2,R}(y_{2,t})\tilde{\rho}_{1,2,R}\frac{\partial^2}{\partial y_1\partial y_2}\right),$$

and:

$$\mathcal{L}_2 = \left(\sum_{i=1}^{2} \frac{\partial}{\partial x}\ell_{i,t}\mu_i S_{i,t}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\ell_{1,t}^2\sigma_1^2(y_{1,t})S_{1,t}^2 + \ell_{2,t}^2\sigma_2^2(y_{2,t})S_{2,t}^2 + 2\ell_{1,t}\ell_{2,t}S_{1,t}S_{2,t}\sigma_1(y_{1,t})\sigma_2(y_{2,t})\rho_{1,2,R}\right)$$
$$+ \sum_{i=1}^{2}\frac{\partial^2}{\partial x\partial S_i}\left(\ell_{i,t}\sigma_i^2(y_{i,t})S_{i,t}^2 + (\ell_{1,t} + \ell_{2,t} - \ell_{i,t})\sigma_1(y_{1,t})\sigma_2(y_{2,t})S_{1,t}S_{2,t}\rho_{1,2,R}\right)$$
$$+ \frac{\partial^2}{\partial x\partial y_1}\left(\ell_{1,t}\sigma_1(y_{1,t})S_{1,t}b_{1,R}(y_{1,t})\rho_{1,R} + \ell_{2,t}\sigma_2(y_{2,t})S_{2,t}b_{1,R}(y_{1,t})\chi_{1,R}\right)$$
$$+ \frac{\partial^2}{\partial x\partial y_2}\left(\ell_{2,t}\sigma_2(y_{2,t})S_{2,t}b_{2,R}(y_{2,t})\rho_{2,R} + \ell_{1,t}\sigma_1(y_{1,t})S_{1,t}b_{2,R}(y_{2,t})\chi_{2,R}\right),$$

where $\left\langle Z^{(1)}, W^{(2)}\right\rangle_t = \chi_{1,R}dt$ and $\left\langle Z^{(2)}, W^{(1)}\right\rangle_t = \chi_{2,R}dt$.

Let u_0 be the solution of the following Hamilton-Jacobi-Bellman equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\,u_0 + \underset{\{(\delta^+_{1,t},\delta^-_{1,t},\delta^+_{2,t},\delta^-_{2,t})\in\mathcal{A}\}}{Sup}\left(J_1^+(\delta_1^+) + J_1^-(\delta_1^-) + J_2^+(\delta_2^+) + J_2^-(\delta_2^-)\right) = 0 \quad (3.13)$$

where the functions J_1^+, J_2^+, J_1^- and J_2^- are defined by:

$$J_1^+(\delta_{1,t}^+) = \lambda_1^+(\delta_{1,t}^+)\Big(u_0(t,s,y,q_{1,t^-}-1,q_{2,t^-},x_{t^-}+(C^{(1)}_{\mathcal{Q},t}+\delta^+_{1,t})) - u_0(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\Big),$$
$$J_1^-(\delta_{1,t}^-) = \lambda_1^-(\delta_{1,t}^-)\Big(u_0(t,s,y,q_{1,t^-}+1,q_{2,t^-},x_{t^-}-(C^{(1)}_{\mathcal{Q},t}-\delta^-_{1,t})) - u_0(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\Big),$$
$$J_2^+(\delta_{2,t}^+) = \lambda_2^+(\delta_{2,t}^+)\Big(u_0(t,s,y,q_{1,t^-},q_{2,t^-}-1,x_{t^-}+(C^{(2)}_{\mathcal{Q},t}+\delta^+_{2,t})) - u_0(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\Big),$$
$$J_2^-(\delta_{2,t}^-) = \lambda_2^-(\delta_{2,t}^-)\Big(u_0(t,s,y,q_{1,t^-},q_{2,t^-}+1,x_{t^-}-(C^{(2)}_{\mathcal{Q},t}-\delta^-_{2,t})) - u_0(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\Big),$$

and u_0 satisfies the final condition:

$$u_0(T,s,y,q_1,q_2,x) = x + q_1h_1(s_1) + q_2h_2(s_2).$$

It can be proved that, if u_0 is smooth, finite and has polynomial growth then it coincides with the value function u.
We consider here the case of a square-root market impact ($\beta = \frac{1}{2}$). Under this hypothesis, the intensities of arrivals of market orders have the following forms:

In the case where ($\beta = \frac{1}{2}$), we obtain the following results:

Proposition 3.1. *For $i \in \{1,2\}$, let:*

$$M_{i,0}(t,s_i,y_i) = C_{\mathcal{Q}}^{(i)}(t,s_i,y_i) - \left(C_{\mathcal{P}}^{(i)}(t,s_i,y_i) - \mu_i E^{\mathcal{P}}\left(\int_t^T \Delta_{i,u} S_{i,u} du\right)\right),$$

where $C_{\mathcal{P}}^{(i)}(t,s_i,y_i) = E_{t,s_i,y_i}^{\mathcal{P}}\left(h_i\left(S_{i,T}\right)|\mathcal{F}_t\right)$.

The optimal bid and ask distances $(\delta_{i,L,t}^{-}, \delta_{i,L,t}^{+})$ on the option $C^{(i)}$ at time t are:

$$\delta_{i,L,t}^{+} = \frac{\sqrt{\gamma_i^2 M_{i,0}^2(t,s_i,y_i) + B_i(2\gamma_i - 1)} - \gamma_i M_{i,0}(t,s_i,y_i)}{2\gamma_i - 1}, \tag{3.14}$$

$$\delta_{i,L,t}^{-} = \frac{\gamma_i M_{i,0}(t,s_i,y_i) + \sqrt{\gamma_i^2 M_{i,0}^2(t,s_i,y_i) + B_i(2\gamma_i - 1)}}{2\gamma_i - 1}, \tag{3.15}$$

and the value function is:

$$\begin{aligned} u(t,s_1,y_1,s_2,y_2,q_1,q_2,x) = x + \theta_0(t,s_1,y_1,s_2,y_2) + q_1\left(C_{\mathcal{P}}^{(1)}(t,s_1,y_1) - E^{\mathcal{P}}\left(\int_t^T \mu_1 \Delta_{1,u} S_{1,u} du\right)\right) \\ + q_2\left(C_{\mathcal{P}}^{(2)}(t,s_2,y_2) - E^{\mathcal{P}}\left(\int_t^T \mu_2 \Delta_{2,u} S_{2,u} du\right)\right), \end{aligned}$$

where:

$$\theta_0(t,s_1,y_1,s_2,y_2) = E_{t,s_1,y_1,s_2,y_2}\left(\int_t^T J_0(u,S_{1,u},y_{1,u},S_{2,u},y_{2,u}) du\right).$$

and:

$$\begin{aligned} J_0(t,s_1,y_1,s_2,y_2) = \sum_{i=1}^{2} \lambda_i^{+}(\delta_{i,L,t}^{+}) \left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)} + M_{i,0}(\gamma_i - 1)}{2\gamma_i - 1}\right) \\ + \lambda_i^{-}(\delta_{i,L,t}^{-}) \left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)} + M_{i,0}(1 - \gamma_i)}{2\gamma_i - 1}\right). \end{aligned}$$

The proof is detailed in Appendix (3.7.1)

3.4 Linear utility function with a penalty on the variance

In this section, we consider that the market maker is risk averse. She wants to reduce the variance of her final wealth caused by the volatility risk. This variance term is equal to $E^{\mathcal{P}}(\mathcal{V}_t(T))$ where:

$$\begin{aligned} \mathcal{V}_t(T) = \int_t^T q_{1,u}^2 \left(\frac{\partial C_1^{\mathcal{P}}}{\partial y_1}\right)^2 b_{1,R}^2(y_{1,u}) du + \int_t^T q_{2,u}^2 \left(\frac{\partial C_2^{\mathcal{P}}}{\partial y_2}\right)^2 b_{2,R}^2(y_{2,u}) du \\ + 2\int_t^T q_{1,u} q_{2,u} \frac{\partial C_1^{\mathcal{P}}}{\partial y_1} \frac{\partial C_2^{\mathcal{P}}}{\partial y_2} b_{1,R}(y_{1,u}) b_{2,R}(y_{2,u}) \tilde{\rho}_{1,2,R} du. \end{aligned}$$

Let $\mathcal{H}^{\epsilon}$ be the function defined as follows:

$$\mathcal{H}_t^{\epsilon}(T) = X_T + q_{1,T} h_1(S_{1,T}) + q_{2,T} h_2(S_{2,T}) - \epsilon \mathcal{V}_t(T).$$

We introduce the value function u^ϵ of the Market Maker:

$$u^\epsilon(t,s,y,q_1,q_2,x) = \underset{\{(\delta^+_{1,t},\delta^-_{1,t},\delta^+_{2,t},\delta^-_{2,t})\in\mathcal{A}\}}{Sup} E^{\mathcal{P}}\left(\mathcal{H}_t(T)\right).$$

We define u_0^ϵ as the solution of the following HJB equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\, u^\epsilon + \underset{\{(\delta^+_{1,t},\delta^-_{1,t},\delta^+_{2,t},\delta^-_{2,t})\in\mathcal{A}\}}{Sup} J^\epsilon(\delta^+_{1,t},\delta^-_{1,t},\delta^+_{2,t},\delta^-_{2,t}) = \epsilon\mathcal{C}(t,s_1,y_1,s_2,y_2,q_1,q_2) \tag{3.16}$$

where:

$$\begin{aligned}\mathcal{C}(t,s_1,y_1,s_2,y_2,q_1,q_2) &= q_1^2\left(\frac{\partial C_1^{\mathcal{P}}}{\partial y_1}\right)^2 b^2_{1,R}(y_{1,t}) + q_2^2\left(\frac{\partial C_2^{\mathcal{P}}}{\partial y_2}\right)^2 b^2_{2,R}(y_{2,t})\\ &\quad + 2q_1q_2\frac{\partial C_1^{\mathcal{P}}}{\partial y_1}\frac{\partial C_2^{\mathcal{P}}}{\partial y_2} b_{1,R}(y_{1,t})b_{2,R}(y_{2,t})\tilde{\rho}_{1,2,R},\end{aligned}$$

and $J^\epsilon = \sum_{i=1}^2 J_i^{-,\epsilon} + J_i^{+,\epsilon}$, where the functions $J_i^{+,\epsilon}, J_i^{-,\epsilon}$ are given as follows:

$$\begin{aligned}
J_1^{+,\epsilon}(\delta^+_{1,t}) &= \lambda_1^+(\delta_1^+)\left(u_0^\epsilon(t,s,y,q_{1,t^-}-1,q_{2,t^-},x_{t^-}+(C_{\mathcal{Q}}^{(1)}+\delta^+_{1,t})) - u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\right),\\
J_1^{-,\epsilon}(\delta^-_{1,t}) &= \lambda_1^-(\delta_1^-)\left(u_0^\epsilon(t,s,y,q_{1,t^-}+1,q_{2,t^-},x_{t^-}-(C_{\mathcal{Q}}^{(1)}-\delta^-_{1,t})) - u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\right),\\
J_2^{+,\epsilon}(\delta^+_{2,t}) &= \lambda_2^+(\delta_2^+)\left(u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-}-1,x_{t^-}+(C_{\mathcal{Q}}^{(2)}+\delta^+_{2,t})) - u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\right),\\
J_2^{-,\epsilon}(\delta^-_{2,t}) &= \lambda_2^-(\delta_2^-)\left(u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-}+1,x_{t^-}-(C_{\mathcal{Q}}^{(2)}-\delta^-_{2,t})) - u_0^\epsilon(t,s,y,q_{1,t^-},q_{2,t^-},x_{t^-})\right).
\end{aligned}$$

In addition, u_0^ϵ satisfies the final condition:

$$u_0^\epsilon(T,s_1,y_1,s_2,y_2,q_1,q_2,x) = x + q_1h_1(s_1) + q_2h_2(s_2).$$

It can be shown here that if u_0^ϵ is smooth, finite and has polynomial growth, then it coincides with the value function u^ϵ.
We obtain the following results in the case $\beta = \frac{1}{2}$.

Proposition 3.2. *Let the functions $\vartheta_{1,1}$, $\vartheta_{2,2}$ and $\vartheta_{1,2}$ be defined as follows:*

$$\begin{aligned}
\vartheta_{1,1}(t,s_1,y_1) &= -E^{\mathcal{P}}_{t,s_1,y_1}\left(\int_t^T b^2_{1,R}(y_{1,u})\left(\frac{\partial C^{(1)}}{\partial y_1}\right)^2(u,s_{1,u},y_{1,u})du\right),\\
\vartheta_{2,2}(t,s_2,y_2) &= -E^{\mathcal{P}}_{t,s_2,y_2}\left(\int_t^T b^2_{2,R}(y_{2,u})\left(\frac{\partial C^{(2)}}{\partial y_2}\right)^2(u,s_{2,u},y_{2,u})du\right),\\
\vartheta_{1,2}(t,s_1,y_1,s_2,y_2) &= -E^{\mathcal{P}}_{t,s_1,y_1,s_2,y_2}\left(\int_t^T \tilde{\rho}_{1,2,R}b_{1,R}(y_{1,u})b_{2,R}(y_{2,u})\frac{\partial C^{(1)}}{\partial y_1}\frac{\partial C^{(2)}}{\partial y_2}du\right),
\end{aligned}$$

For $i\in\{1,2\}$, let $\mathcal{D}_{i,L}\lambda_i(u,S_{i,u},y_{i,u}) = \lambda_i^+(\delta^+_{i,L,u}) - \lambda_i^-(\delta^-_{i,L,u})$. The functions ϑ_1, ϑ_2 are then defined as follows:

$$\begin{aligned}
\vartheta_1(t,s_1,y_1,s_2,y_2) &= E^{\mathcal{P}}_{t,s,y}\left(\int_t^T\left(-2\vartheta_{1,1}\mathcal{D}_{1,L}\lambda_1 - 2\vartheta_{1,2}\mathcal{D}_{2,L}\lambda_2\right)(u,S_{1,u},y_{1,u},S_{2,u},y_{2,u})du\right),\\
\vartheta_2(t,s_1,y_1,s_2,y_2) &= E^{\mathcal{P}}_{t,s,y}\left(\int_t^T\left(-2\vartheta_{2,2}\mathcal{D}_{2,L}\lambda_2 - 2\vartheta_{1,2}\mathcal{D}_{1,L}\lambda_1\right)(u,S_{1,u},y_{1,u},S_{2,u},y_{2,u})du\right).
\end{aligned}$$

For $i \in \{1,2\}$, the optimal controls $(\delta^+_{i,*,t}, \delta^-_{i,*,t})$ on the option $C^{(i)}$ can be approximated at order 1 in ϵ by $(\hat{\delta}^-_{i,*,t}, \hat{\delta}^+_{i,*,t})$:

$$\hat{\delta}^+_{i,*,t} = \frac{-\gamma M_i^+ + \sqrt{\gamma_i^2 (M_i^+)^2 + B_i(2\gamma_i - 1)}}{2\gamma_i - 1}, \tag{3.17}$$

$$\hat{\delta}^-_{i,*,t} = \frac{\gamma M_i^- + \sqrt{\gamma_i^2 (M_i^-)^2 + B_i(2\gamma_i - 1)}}{2\gamma_i - 1}, \tag{3.18}$$

where the quantities $M_i^+(t, s_1, y_1, s_2, y_2, q_1, q_2)$ and $M_i^-(t, s_1, y_1, s_2, y_2, q_1, q_2)$ are defined by:

$$\begin{aligned} M_i^+(t, s_1, y_1, s_2, y_2, q_1, q_2) &= M_{i,0}(t, s_i, y_i) + \epsilon M_{i,1}(t, s_1, y_1, s_2, y_2, q_1, q_2), \\ M_i^-(t, s_1, y_1, s_2, y_2, q_1, q_2) &= M_{i,0}(t, s_i, y_i) + \epsilon M_{i,2}(t, s_1, y_1, s_2, y_2, q_1, q_2), \end{aligned}$$

and $M_{i,1}(t, s_1, y_1, s_2, y_2, q_1, q_2)$ and $M_{i,2}(t, s_1, y_1, s_2, y_2, q_1, q_2)$ are given explicitly as follows:

$$\begin{aligned} M_{i,1}(t, s_1, y_1, s_2, y_2, q_1, q_2) &= -\vartheta_i + (1 - 2q_i)\vartheta_{i,i} - 2q_{\bar{i}}\vartheta_{1,2}, \\ M_{i,2}(t, s_1, y_1, s_2, y_2, q_1, q_2) &= -\vartheta_i - (1 + 2q_i)\vartheta_{i,i} - 2q_{\bar{i}}\vartheta_{1,2}, \end{aligned}$$

and $\bar{i}$ denotes the index different from i (if $i = 1$ then $\bar{i} = 2$ and if $i = 2$ then $\bar{i} = 1$).
The approximation error is at order 2 in ϵ:

$$\begin{aligned} \left| \delta^+_{*,t} - \hat{\delta}^+_{*,t} \right| &= O\left(\epsilon^2\right), \\ \left| \delta^-_{*,t} - \hat{\delta}^-_{*,t} \right| &= O\left(\epsilon^2\right). \end{aligned}$$

The proof is detailed in (3.7.2).

3.5 Numerical simulations

We perform Monte Carlo simulations in order to support the results of the theoretical study. We suppose in this section that the processes S_1 and S_2 follow a Heston model. Indeed, under $\mathcal{P}$, $\forall i \in \{1,2\}$ the spot process S_i has the following dynamics:

$$\begin{aligned} \frac{dS_{i,t}}{S_{i,t}} &= \mu_i dt + \sqrt{y_{i,t}} dW^{(1)}_{i,t} \\ dy_{i,t} &= k_{i,R}(\theta_{i,R} - y_{i,t})dt + \eta_{i,R}\sqrt{y_{i,t}} dW^{(2)}_{i,t} \end{aligned}$$

where $d\left\langle W^{(1)}_i, W^{(2)}_i \right\rangle_t = \rho_{i,R} dt$. In addition, $d\left\langle W^{(1)}_1, W^{(1)}_2 \right\rangle_t = \rho_{1,2,R} dt$ and $d\left\langle W^{(2)}_1, W^{(2)}_2 \right\rangle_t = \tilde{\rho}_{1,2,R} dt$.

Market participant price options on the stock under a risk-neutral pricing measure $\mathcal{Q}$, under which the spot process S has the following dynamics:

$$\begin{aligned} \frac{dS_{i,t}}{S_{i,t}} &= r dt + \sqrt{y_{i,t}} dW^{*,(1)}_{i,t} \\ dy_{i,t} &= k_{i,I}(\theta_{i,I} - y_{i,t})dt + \eta_{i,I}\sqrt{y_{i,t}} dW^{*,(2)}_{i,t} \end{aligned}$$

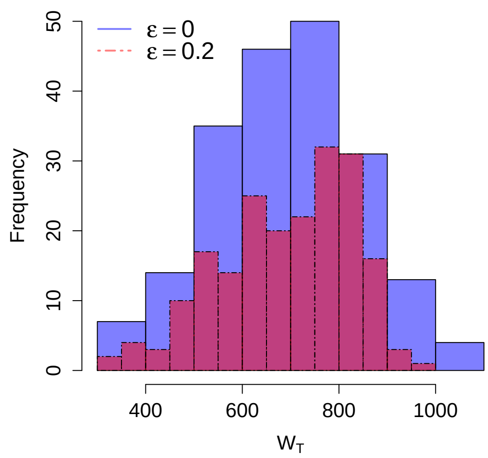
where $d\left\langle W^{*,(1)}, W^{*,(2)} \right\rangle_t = \rho_I dt$, $d\left\langle W^{*,(1)}_1, W^{*,(1)}_2 \right\rangle_t = \rho_{1,2,I} dt$ and $d\left\langle W^{*,(2)}_1, W^{*,(2)}_2 \right\rangle_t = \tilde{\rho}_{1,2,I} dt$.
The term $C_{\mathcal{Q}}(t, s_i, y_i)$ is the option price in the Heston model (see Heston (1993)). The term

$C_{\mathcal{P}} = E^{\mathcal{P}}((S_T - K)^+|\mathcal{F}_t)$ can be computed explicitly as explained in our previous work.

The numerical simulations are performed as following. In the first part, we perform the numerical simulations for a maturity $T = 1M$ and then for $T = 3M$. In both cases, we suppose that the parameters under the risk-neutral measure $\mathcal{Q}$ and the physical measure $\mathcal{P}$ are the same, thus $\forall i \in \{1, 2\}$, $(r, k_{i,I}, \theta_{i,I}, \eta_{i,I}, \rho_{i,I}) = (\mu_i, k_{i,R}, \theta_{i,R}, \eta_{i,R}, \rho_{i,R})$ and $(\rho_{1,2,R}, \tilde{\rho}_{1,2,R}) = (\rho_{1,2,I}, \tilde{\rho}_{1,2,I})$ We simulate 200 paths of the spot and instantaneous variance processes $(S_{i,t}, y_{i,t})_{\{0 \le t \le T\}}$ starting from $S_{1,0} = 100$, $y_{1,0} = 0.09$, $S_{2,0} = 50$ and $y_{2,0} = 0.04$. It is considered here that there are 6 trading hours per day and that the market maker refreshes the quotes every 5 minutes. This means that there is $12 \times 6 = 72$ points per day. Since there is approximately 64 business days in a 3 months period, this amounts to $64 \times 72 = 4608$ points per simulated path. At each point, the quantities $C^{(i)}_{\mathcal{Q}}(t, s_i, y_i)$ and $C^{(i)}_{\mathcal{P}}(t, s_i, y_i)$ are computed using a Fast Fourier Transform method. This simulation task is numerically consuming and was performed using the computing cluster at Ecole Centrale Paris.

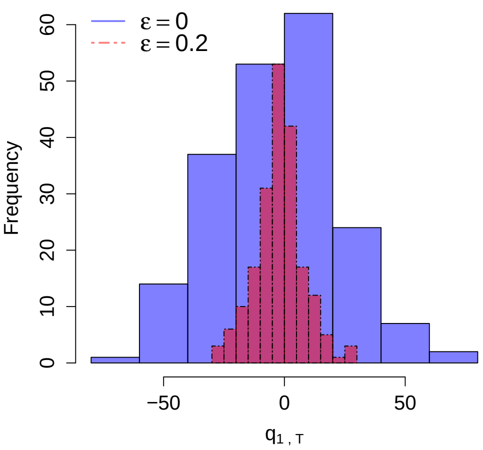
For each simulated path, the optimal market making strategy is used from the inception date $(t = 0)$ until the maturity date of the option $(t = T)$. We then performed a statistical study on the results obtained upon the test of the optimal strategy on 200 independent paths.

3.5.1 One-month maturity options

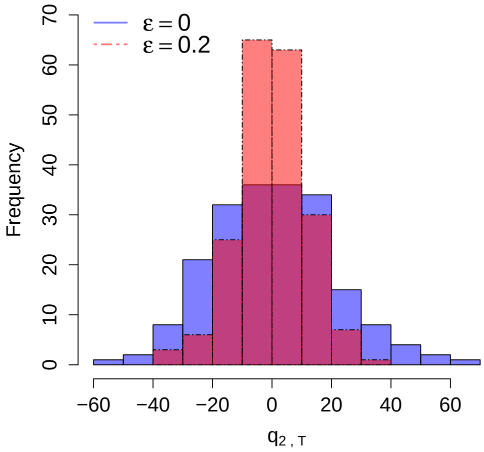


(a) Histograms of $\mathcal{W}_T$

Statistics on $\mathcal{W}_T$	Mean of $\mathcal{W}_T$	StD of $\mathcal{W}_T$
epsilon $= 0$	694.81	151.40
epsilon $= 0.1$	685.40	137.65
epsilon $= 0.2$	691.93	133.90
epsilon $= 0.4$	689.35	132.99

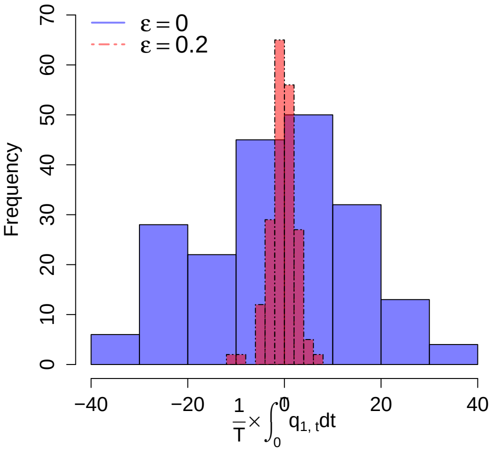


(b) Histograms of $q_{1,T}$

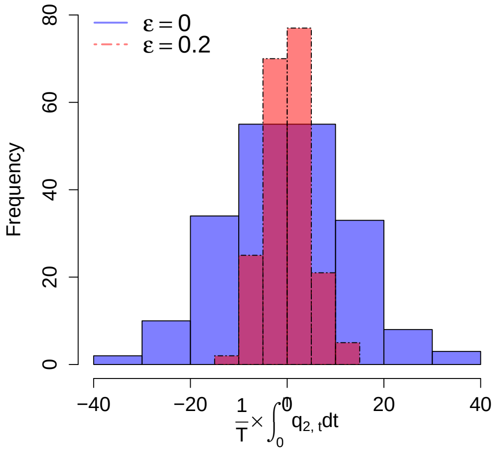


(c) Histograms of $q_{2,T}$

Statistics	Mean Of $q_{1,T}$	StD Of $q_{1,T}$	Mean Of $q_{2,T}$	StD Of $q_{2,T}$
epsilon = 0	-2.54	25.16	0.68	20.22
epsilon = 0.2	-1.52	9.94	0.36	11.62
epsilon = 0.4	-1.50	8.97	0.26	10.08



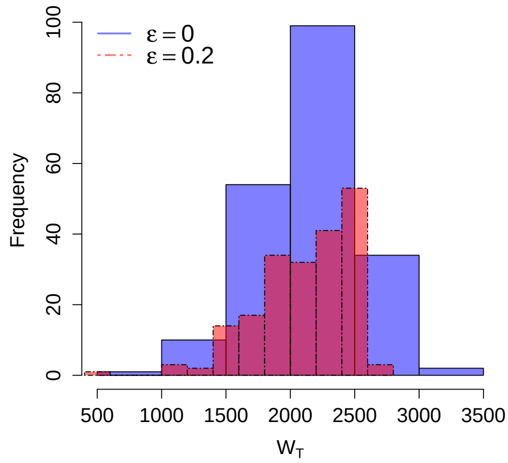
(d) Histograms of $\frac{1}{T}\int_0^T q_{1,t}dt$



(e) Histograms of $\frac{1}{T}\int_0^T q_{2,t}dt$

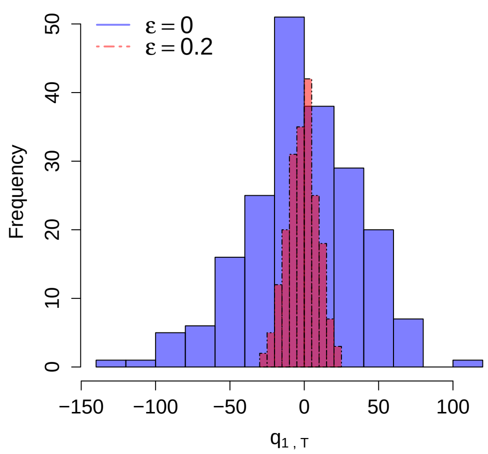
Statistics	Mean of $\frac{1}{T}\int_0^T q_{1,t}dt$	StD of $\frac{1}{T}\int_0^T q_{1,t}dt$	Mean of $\frac{1}{T}\int_0^T q_{2,t}dt$	StD of $\frac{1}{T}\int_0^T q_{2,t}dt$
epsilon = 0	-0.84	15.75	0.12	13.23
epsilon = 0.2	-0.36	2.67	0.02	4.66
epsilon = 0.4	-0.35	2.19	-0.06	3.50

3.5.2 Three-month maturity options

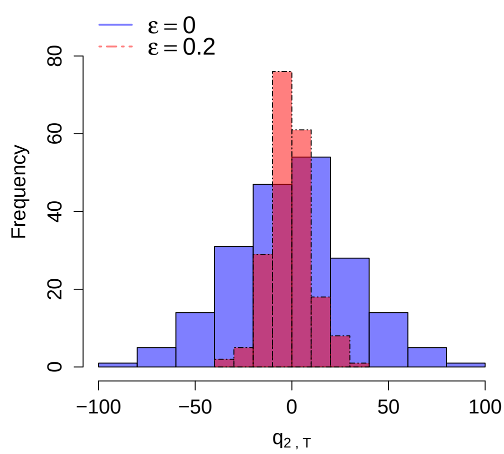


(f) Histograms of $\mathcal{W}_T$

Statistics on $\mathcal{W}_T$	Mean	StD
epsilon $= 0$	2139.37	385.92
epsilon $= 0.2$	2115.07	369.55
epsilon $= 0.4$	2031.78	365.22

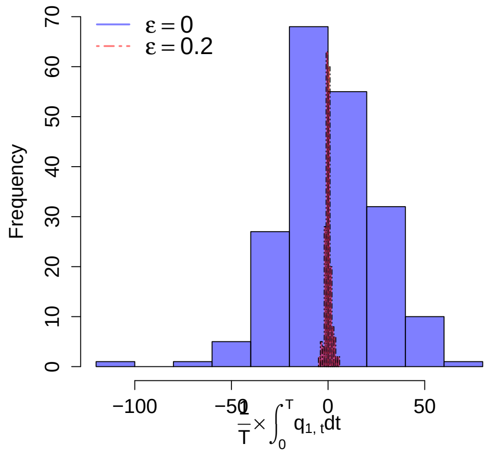


(g) Histograms of $q_{1,T}$

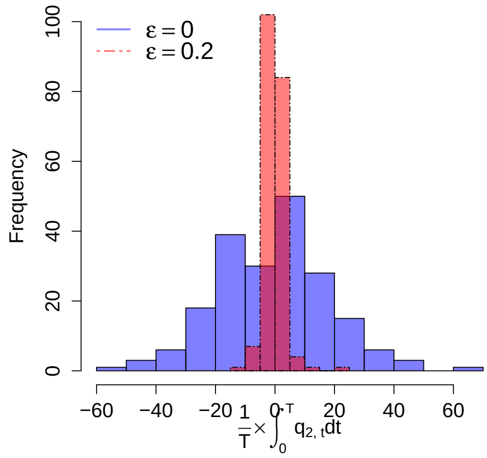


(h) Histograms of $q_{2,T}$

Statistics	Mean Of $q_{1,T}$	StD Of $q_{1,T}$	Mean Of $q_{2,T}$	StD Of $q_{2,T}$
epsilon = 0	-0.81	37.65	-0.50	31.78
epsilon = 0.2	-0.62	10.08	-0.24	10.90
epsilon = 0.4	-0.64	9.40	0.04	9.47



(i) Histograms of $\frac{1}{T}\int_0^T q_{1,t}dt$



(j) Histograms of $\frac{1}{T}\int_0^T q_{2,t}dt$

Statistics	Mean of $\frac{1}{T}\int_0^T q_{1,t}dt$	StD of $\frac{1}{T}\int_0^T q_{1,t}dt$	Mean of $\frac{1}{T}\int_0^T q_{2,t}dt$	StD of $\frac{1}{T}\int_0^T q_{2,t}dt$
epsilon = 0	0.62	24.12	-0.52	18.59
epsilon = 0.2	0.04	1.57	-0.10	3.07
epsilon = 0.4	0.03	1.41	-0.03	2.36

3.6 Conclusion

In this chapter, we established a model for market-making on two options on different assets. This framework is useful as it enables to hedge the volatility risk of the options in the book of the market-maker. The numerical simulations highlight the effect of the penalty parameter on the reduction of the standard deviation of the final wealth.

3.7 Appendix

3.7.1 Appendix 1: Proof in the case of a linear utility function

Let u_0 be the solution of (3.13). We suppose that u_0 has the following form:

$$u(t,s_1,y_1,s_2,y_2,q_1,q_2,x) = x+\theta_0(t,s_1,y_1,s_2,y_2)+q_1\theta_1(t,s_1,y_1,s_2,y_2)+q_2\theta_2(t,s_1,y_1,s_2,y_2),$$

The function J_i^+ can be written explicitly as:

$$J_i^+(x) = \lambda_i^+(x)\left(x+M_{i,0}(t,s_1,y_1,s_2,y_2)\right), \tag{3.19}$$

where $M_{i,0}(t,s_1,y_1,s_2,y_2)=C_Q^{(i)}(t,s_i,y_i)-\theta_i(t,s_1,y_1,s_2,y_2)$. The derivative of the function J_i^+ writes:

$$(J_i^+)'(x) = \frac{\lambda_i^+(x)}{B_i+x^2}\left(x^2\left(1-2\gamma_i\right)-2\gamma_i M_{i,0}x+B_i\right).$$

The function $(J_i^+)'$ vanishes in the points x_1^+ and x_2^+:

$$x_1^+ = \frac{-\gamma M_{i,0}-\sqrt{\gamma^2 M_{i,0}^2+B_i(2\gamma_i-1)}}{2\gamma_i-1}, \qquad x_2^+ = \frac{-\gamma_i M_{i,0}+\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}}{2\gamma_i-1},$$

Since $\gamma_i \geq 1$ and $B_i>0$, then $x_1^+<0$ and $x_2^+>0$. It can be deduced that this function reaches its maximum in x_2^+ and then:

$$\delta_{i,L,t}^+ = \frac{\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}-\gamma_i M_{i,0}}{2\gamma_i-1}.$$

It follows that:

$$J_i^+(\delta_{i,L,t}^+) = \lambda_i^+(\delta_{i,L,t}^+)\left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}+M_{i,0}(\gamma_i-1)}{2\gamma_i-1}\right), \tag{3.20}$$

Using the same approach as before, the function J_i^- can be written as follows:

$$J_i^-(x) = \lambda_i^-(x)(x-M_{i,0}(t,s_1,y_1,s_2,y_2)). \tag{3.21}$$

In order to determine $\delta_{i,L,t}^- = \underset{\{x>0\}}{ArgSup} J_i^-(x)$, the derivative of the function J_i^- is computed:

$$(J_i^-)'(x) = \frac{\lambda_i^-(x)}{B_i+x^2}\left((1-2\gamma_i)x^2+2\gamma_i M_{i,0}x+B_i\right).$$

The function $(J_i^-)'$ changes its sign and gets null in the two following points:

$$x_1^- = \frac{\gamma_i M_{i,0}-\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}}{2\gamma_i-1}, \qquad x_2^- = \frac{\gamma_i M_{i,0}+\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}}{2\gamma_i-1},$$

Using the same reasoning as before, it can be proved that $x_1^-<0$, $x_2^->0$ and:

$$\delta_{i,L,t}^- = \frac{\gamma_i M_{i,0}+\sqrt{\gamma_i^2 M_{i,0}^2+B_i(2\gamma_i-1)}}{2\gamma_i-1}.$$

The function J_i^- evaluated at its maximum writes:

$$J_i^-(\delta_{i,L,t}^-) = \lambda_i^-(\delta_{i,L,t}^-)\left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)} + M_{i,0}(1-\gamma_i)}{2\gamma_i - 1}\right). \tag{3.22}$$

Finally, using (3.20) and (3.22), $f_L(t, s_1, s_2, y_1, y_2) = \sum_{i=1}^2 J_i^-(\delta_{i,L,t}^-) + J_i^+(\delta_{i,L,t}^+)$ can be written as follows:

$$\begin{aligned} f_L(t, s_1, s_2, y_1, y_2) = \sum_{i=1}^{2} \lambda_i^-(\delta_{i,L,t}^-)&\left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)} + M_{i,0}(1-\gamma_i)}{2\gamma_i - 1}\right) \\ + \lambda_i^+(\delta_{i,L,t}^+)&\left(\frac{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)} + M_{i,0}(\gamma_i - 1)}{2\gamma_i - 1}\right). \end{aligned}$$

It can be noticed that the function $f_L(t, s_1, s_2, y_1, y_2)$ is independent of the parameters q_1 and q_2.

The equation (3.13) becomes:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u + f_L(t, s, y) = 0.$$

For this equation to be solved, the terms are classified by powers of q_1 and q_2. It can be recalled here that the functions $\theta_0, \theta_1, \theta_2$ and f_L are independent of q_1 and q_2. The terms obtained after the classification are either of order $(0,0), (1,0)$ or $(0,1)$ in (q_1, q_2). By nullifying these terms, it can be obtained that:

$$\begin{aligned} (0,0) &: (\partial_t + \mathcal{L}_1)\theta_0 + f_L(t, s, y) = 0, \\ (1,0) &: (\partial_t + \mathcal{L}_1)\theta_1 - \Delta_{1,t}\mu_1 S_{1,t} = 0, \\ (0,1) &: (\partial_t + \mathcal{L}_1)\theta_2 - \Delta_{2,t}\mu_2 S_{2,t} = 0. \end{aligned}$$

The term θ_1 is the solution of the $(1,0)$-term (at order 1 in q_1 and order 0 in q_2):

$$\begin{aligned} (\partial_t + \mathcal{L}_1)\theta_1 - \Delta_{1,t}\mu_1 S_{1,t} &= 0, \\ \theta_1(T, s_1, y_1, s_2, y_2) &= h_1(s_1). \end{aligned}$$

Using the Feynman-Kac formula, it can be obtained:

$$\theta_1(t, s_1, y_1) = E_{t,s_1,y_1}^{\mathcal{P}}(h_1(S_{1,T})) - E^{\mathcal{P}}\left(\int_t^T \mu_1 \Delta_{1,u} S_{1,u} du\right).$$

Solving now the equation corresponding to the $(0,1)$-order term (at order 0 in q_1 and order 1 in q_2) yields:

$$\begin{aligned} (\partial_t + \mathcal{L}_1)\theta_2 - \Delta_{2,t}\mu_2 S_{2,t} &= 0, \\ \theta_2(T, s_1, y_1, s_2, y_2) &= h_2(s_2). \end{aligned}$$

Using the Feynman-Kac theorem yields:

$$\theta_2(t, s_2, y_2) = E_{t,s_2,y_2}^{\mathcal{P}}(h_2(S_{2,T})) - E^{\mathcal{P}}\left(\int_t^T \mu_2 \Delta_{2,u} S_{2,u} du\right).$$

The quantities $M_{1,0}$ and $M_{2,0}$ can then be computed explicitly:

$$\begin{aligned} M_{1,0}(t, s_1, y_1) &= C_{\mathcal{Q}}(t, s_1, y_1) - \left(C_{\mathcal{P}}(t, s_1, y_1) - E^{\mathcal{P}}\left(\int_t^T \mu_1 \Delta_{1,u} S_{1,u} du\right)\right), \\ M_{2,0}(t, s_2, y_2) &= C_{\mathcal{Q}}(t, s_2, y_2) - \left(C_{\mathcal{P}}(t, s_2, y_2) - E^{\mathcal{P}}\left(\int_t^T \mu_2 \Delta_{2,u} S_{2,u} du\right)\right), \end{aligned}$$

The term θ_0 is the solution of the following equation:

$$\begin{aligned}(\partial_t + \mathcal{L}_1)\theta_0 + f_L(t,s_1,s_2,y_1,y_2) &= 0,\\ \theta_0(T,s,y) &= 0.\end{aligned}$$

By applying the Feynman-Kac formula:

$$\theta_0(t,s_1,y_1,s_2,y_2) = E_{t,s_1,y_1,s_2,y_2}\left(\int_t^T f_L(u,S_{1,u},S_{2,u},y_{1,u},y_{2,u})du\right).$$

Consequently, the function u_0, solution of the HJB equation (3.13), has the following form:

$$u_0(t,s_1,y_1,s_2,y_2,q_1,q_2,x) = x + \theta_0(t,s_1,s_2,y_1,y_2) + q_1\theta_1(t,s_1,y_1) + q_2\theta_2(t,s_2,y_2). \tag{3.23}$$

It can be checked that u_0 is smooth, finite and has polynomial growth. Consequently, it coincides with the value function u.

3.7.2 Appendix 2: Proof in the case of a mean-variance framework

Let u_0^ϵ be the solution of the HJB equation (3.16). Under the assumption that $\epsilon \sim 0$, a singular perturbation technique can be performed with respect to the parameter ϵ:

$$u_0^\epsilon(t,s_1,y_1,s_2,y_2,q_1,q_2,x) = x + \sum_{k=0}^{+\infty}\epsilon^k v_k(t,s_1,y_1,s_2,y_2,q_1,q_2), \tag{3.24}$$

If $\epsilon = 0$ then the case of a linear utility function without inventory constraints is recovered. This implies that $u_0^0(t,s_1,y_1,s_2,y_2,q_1,q_2,x) = x + v_0(t,s_1,y_1,s_2,y_2,q_1,q_2) = u_0(t,s_1,y_1,s_2,y_2,q_1,q_2,x)$ where u_0 is the value function defined in (3.23). Therefore, it is assumed that v_0 has the following form:

$$v_0(t,s_1,y_1,s_2,y_2,q_1,q_2) = \theta_0(t,s_1,y_1,s_2,y_2) + q_1\theta_1(t,s_1,y_1,s_2,y_2) + q_2\theta_2(t,s_1,y_1,s_2,y_2).$$

Furthermore, due to the form of the function $\mathcal{H}_t^\epsilon(T)$, it is assumed that v_1 has the following form:

$$\begin{aligned}v_1(t,s_1,y_1,s_2,y_2,q_1,q_2) = \vartheta_0(t,s_1,y_1,s_2,y_2) + \sum_{i=1}^{2} q_i\vartheta_i(t,s_1,y_1,s_2,y_2)\\ + q_1^2\vartheta_{1,1}(t,s_1,y_1,s_2,y_2) + q_2^2\vartheta_{2,2}(t,s_1,y_1,s_2,y_2) + 2q_1q_2\vartheta_{1,2}(t,s_1,y_1,s_2,y_2),\end{aligned}$$

The function $J_i^{+,\epsilon}$ writes:

$$J_i^{+,\epsilon}(x) = \lambda^+(x)\left(x + M_{i,0}(t,s_i,y_i) + \epsilon M_{i,1}(t,s_1,y_1,s_2,y_2,q_1,q_2) + \epsilon^2 R_i^+(t,s_1,y_1,s_2,y_2,q_1,q_2)\right),$$

where:

$$M_{i,1}(t,s_1,y_1,s_2,y_2,q_1,q_2) = -\vartheta_i + (1-2q_i)\vartheta_{i,i} - 2q_{\bar{i}}\vartheta_{1,2},$$

and:

$$R_i^+(t,s_1,y_1,s_2,y_2,q_1,q_2) = \sum_{k=2}^{+\infty}\epsilon^{k-2}\left(v_k\left(t,s_1,y_1,s_2,y_2,q_i-1,q_{\bar{i}}\right) - v_k\left(t,s_1,y_1,s_2,y_2,q_i,q_{\bar{i}}\right)\right)$$

Let $M_i^+(t,s_1,y_1,s_2,y_2,q_1,q_2) = M_0(t,s,y) + \epsilon M_{i,1}(t,s_1,y_1,s_2,y_2,q_1,q_2)$. It can be shown that the function $(J_i^{+,\epsilon})$ attains its maximum in :

$$\delta_{i,*,t}^+ = \frac{-\gamma_i\left(M_i^+ + \epsilon^2 R_i^+\right) + \sqrt{\gamma_i^2\left(M_i^+ + \epsilon^2 R_i^+\right)^2 + B_i(2\gamma_i - 1)}}{2\gamma_i - 1}.$$

Using Taylor's expansion with respect to the parameter ϵ:

$$\delta^+_{i,*,t} = \delta^+_{i,L,t} + \epsilon \left(-\frac{\gamma_i}{2\gamma_i - 1} M_{i,1} + \frac{\gamma_i^2 M_{i,0} M_{i,1}}{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)}} \right) + O(\epsilon^2) . \tag{3.25}$$

It is then useful to write $J_i^{+,\epsilon}(\delta^+_{i,*,t})$ as the sum of $J_i^+(\delta^+_{i,L,t})$ plus a perturbation term due to the parameter ϵ. Using Taylor's expansion, it can be obtained that:

$$J_i^{+,\epsilon}(\delta^+_{i,*,t}) = J_i^{+,\epsilon}(\delta^+_{i,L,t}) + (J_i^{+,\epsilon})'(\delta^+_{i,L,t})(\delta^+_{i,*,t} - \delta^+_{i,L,t}) + o\Big((\delta^+_{i,*,t} - \delta^+_{i,L,t})^2\Big) ,$$

Using the relation (3.25), it follows that:

$$J_i^{+,\epsilon}(\delta^+_{i,*,t}) = J_i^+(\delta^+_{i,L,t}) + \epsilon \left(-\frac{\gamma_i}{2\gamma_i - 1} M_{i,1} + \frac{\gamma_i^2 M_{i,0} M_{i,1}}{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)}} \right) (J_i^{+,\epsilon})'(\delta^+_{i,L,t})$$
$$+ \epsilon \lambda_i^+(\delta^+_{i,L,t}) M_{i,1} + O(\epsilon^2) .$$

Since $(J_i^{+,\epsilon})'(x) = (J_i^+)'(x) + O(\epsilon)$, and $(J_i^+)'(\delta^+_{i,L,t}) = 0$, it can be deduced that:

$$J_i^{+,\epsilon}(\delta^+_{i,*,t}) = J_i^+(\delta^+_{i,L,t}) + \epsilon \lambda_i^+(\delta^+_{i,L,t}) M_{i,1} + O(\epsilon^2) ,$$

Using the same reasoning, the function $J_i^{-,\epsilon}$ is given explicitly as follows:

$$J_i^{-,\epsilon}(\delta^-) = \lambda_i^-(\delta^-) \left(\delta^- - \left(M_{i,0}(t,s,y) + \epsilon M_{i,2}(t,s,y,q_1,q_2) + \epsilon^2 R_i^-(t,s,y,q_1,q_2)\right)\right)$$

where:

$$M_{i,2}(t,s_1,y_1,s_2,y_2,q_1,q_2) = -\vartheta_i - (1+2q_i)\vartheta_{i,i} - 2q_{\bar{i}}\vartheta_{1,2},$$

and:

$$R_i^-(t,s_1,y_1,s_2,y_2,q_1,q_2) = -\left(\sum_{k=2}^{+\infty} \epsilon^{k-2} \left(v_k(t,s_1,y_1,s_2,y_2,q_i+1,q_{\bar{i}}) - v_k(t,s,y,q_i,q_{\bar{i}})\right) \right)$$

Let $M_i^-(t,s_1,y_1,s_2,y_2,q_1,q_2) = M_{i,0}(t,s_1,y_1,s_2,y_2) + \epsilon M_{i,2}(t,s_1,y_1,s_2,y_2,q_1,q_2)$. The function $J_i^{-,\epsilon}$ attains its maximum in:

$$\delta^-_{i,*,t} = \frac{\gamma_i \left(M_i^- + \epsilon^2 R_i^-\right) + \sqrt{\gamma_i^2 (M_i^- + \epsilon^2 R_i^-)^2 + B_i(2\gamma_i - 1)}}{2\gamma_i - 1}.$$

Using Taylor's expansion with respect to the parameter ϵ, it can be shown that:

$$\delta^-_{i,*,t} = \frac{\gamma_i M_i^- + \sqrt{\gamma_i^2 (M_i^-)^2 + B_i(2\gamma_i - 1)}}{2\gamma_i - 1} + O(\epsilon^2) .$$

and:

$$\delta^-_{i,*,t} = \delta^-_{i,L,t} + \epsilon \left(\frac{\gamma_i}{2\gamma_i - 1} M_{i,2} + \frac{\gamma_i^2 M_{i,0} M_{i,2}}{\sqrt{\gamma_i^2 M_{i,0}^2 + B_i(2\gamma_i - 1)}} \right) + O(\epsilon^2) \tag{3.26}$$

The quantity $J_i^{-,\epsilon}(\delta^-_{i,*,t})$ can be written as the sum of $J_i^-(\delta^-_{i,L,t})$ plus a perturbation term due to the parameter ϵ. Indeed, following the same reasoning as before, we prove that:

$$J_i^{-,\epsilon}(\delta^-_{i,*,t}) = J_i^-(\delta^-_{i,L,t}) - \epsilon \lambda_i^-(\delta^-_{i,L,t}) M_{i,2} + O(\epsilon^2) ,$$

Now that the terms $J_i^{-,\epsilon}(\delta_{i,*,t}^{-})$ and $J_i^{+,\epsilon}(\delta_{i,*,t}^{+})$ are approximated separately, the term:

$$f^{\epsilon}(t,s_1,y_1,s_2,y_2,q_1,q_2)=\sum_{i=1}^{2}J_i^{-,\epsilon}(\delta_{i,*,t}^{-})+J_i^{+,\epsilon}(\delta_{i,*,t}^{+}),$$

can be written as below:

$$f^{\epsilon}(t,s_1,y_1,s_2,y_2,q_1,q_2) \;=\; f_L(t,s_1,y_1,s_2,y_2)+\epsilon\sum_{i=1}^{2}\Big(M_{i,1}\lambda_i^{+}(\delta_{i,L,t}^{+})-M_{i,2}\lambda_i^{-}(\delta_{i,L,t}^{-})\Big)+O\big(\epsilon^2\big)\,,$$

By rearranging the terms of f^{ϵ} by powers of ϵ, q_1, q_2, it can be obtained:

$$f^{\epsilon}(t,s_1,y_1,s_2,y_2,q_1,q_2) \;=\; f_L(t,s_1,y_1,s_2,y_2)+\epsilon g(t,s_1,y_1,s_2,y_2,q_1,q_2)+O\big(\epsilon^2\big)\,,$$

where $g(t,s_1,y_1,s_2,y_2,q_1,q_2) = g_0(t,s_1,y_1,s_2,y_2) + q_1g_1(t,s_1,y_1,s_2,y_2) + q_2g_2(t,s_1,y_1,s_2,y_2)$ and:

$$\begin{aligned}
g_1(t,s_1,y_1,s_2,y_2) &= -2\vartheta_{1,1}\left(\lambda_1^{+}(\delta_{1,L,t}^{+})-\lambda_1^{-}(\delta_{1,L,t}^{-})\right)-2\vartheta_{1,2}\left(\lambda_2^{+}(\delta_{2,L,t}^{+})-\lambda_2^{-}(\delta_{2,L,t}^{-})\right),\\
g_2(t,s_1,y_1,s_2,y_2) &= -2\vartheta_{2,2}\left(\lambda_2^{+}(\delta_{2,L,t}^{+})-\lambda_2^{-}(\delta_{2,L,t}^{-})\right)-2\vartheta_{1,2}\left(\lambda_1^{+}(\delta_{1,L,t}^{+})-\lambda_1^{-}(\delta_{1,L,t}^{-})\right),\\
g_0(t,s_1,y_1,s_2,y_2) &= \sum_{i=1}^{2}\left(-\vartheta_i\left(\lambda_i^{+}(\delta_{i,L,t}^{+})-\lambda_i^{-}(\delta_{i,L,t}^{-})\right)+\vartheta_{i,i}\left(\lambda_i^{+}(\delta_{i,L,t}^{+})+\lambda_i^{-}(\delta_{i,L,t}^{-})\right)\right).
\end{aligned}$$

The HJB equation (3.16) can be regrouped by terms according to their order in ϵ. The term of order 0 in ϵ leads to the equation:

$$(\partial_t+\mathcal{L}_1+\mathcal{L}_2)(x+\theta_0+q_1\theta_1+q_2\theta_2)+f_L(t,s_1,y_1,s_2,y_2) \;=\; 0,$$

with the final conditions:

$$\theta_0(T,s_1,y_1,s_2,y_2)=0,\quad \theta_1(T,s_1,y_1)=h_1(s_1),\quad \theta_2(T,s_2,y_2)=h_2(s_2).$$

The functions θ_0, θ_1 and θ_2 are then deduced:

$$\begin{aligned}
\theta_0(t,s_1,y_1,s_2,y_2) &= E_{t,s,y}\left(\int_t^T f_L(u,s_{1,u},y_{1,u},s_{2,u},y_{2,u})du\right),\\
\theta_1(t,s_1,y_1) &= C_{\mathcal{P}}^{(1)}(t,s_1,y_1)-E^{\mathcal{P}}\left(\int_t^T \mu_1\Delta_{1,u}S_{1,u}du\right),\\
\theta_2(t,s_2,y_2) &= C_{\mathcal{P}}^{(2)}(t,s_2,y_2)-E^{\mathcal{P}}\left(\int_t^T \mu_2\Delta_{2,u}S_{2,u}du\right),
\end{aligned}$$

The term of order 1 in ϵ, gives the following equation:

$$(\partial_t+\mathcal{L}_1+\mathcal{L}_2)v_1+g(t,s_1,y_1,s_2,y_2,q_1,q_2) \;=\; \mathcal{C}(t,s_1,y_1,s_2,y_2,q_1,q_2).$$

The function $\vartheta_{1,1}$ is solution of the following problem:

$$\begin{aligned}
(\partial_t+\mathcal{L}_1)\vartheta_{1,1} &= b_{1,R}^2(y_1)\left(\frac{\partial C^{(1)}}{\partial y_1}\right)^2,\\
\vartheta_{1,1}(T,s_1,y_1) &= 0.
\end{aligned}$$

It can be deduced that:

$$\vartheta_{1,1}(t,s_1,y_1) \;=\; -E_{t,s_1,y_1}^{\mathcal{P}}\left(\int_t^T b_{1,R}^2(y_{1,u})\left(\frac{\partial C^{(1)}}{\partial y_1}\right)^2(u,s_{1,u},y_{1,u})du\right),$$

The function $\vartheta_{2,2}$ is solution of the following problem:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_1)\vartheta_{2,2} &= b_{2,R}^2(y_2)\left(\frac{\partial C^{(2)}}{\partial y_2}\right)^2, \\
\vartheta_{2,2}(T, s_2, y_2) &= 0,
\end{aligned}
$$

this implies:

$$
\vartheta_{2,2}(t, s_2, y_2) = -E^{\mathcal{P}}_{t,s_2,y_2}\left(\int_t^T b_{2,R}^2(y_{2,u})\left(\frac{\partial C^{(2)}}{\partial y_2}\right)^2 (u, s_{2,u}, y_{2,u})du\right),
$$

The function $\vartheta_{1,2}$ is solution of the following problem:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_1)\vartheta_{1,2} &= \tilde{\rho}_{1,2,R} b_{1,R}(y_1) b_{2,R}(y_2)\frac{\partial C^{(1)}}{\partial y_1}\frac{\partial C^{(2)}}{\partial y_2}, \\
\vartheta_{1,2}(T, s_1, y_1, s_2, y_2) &= 0,
\end{aligned}
$$

It follows that:

$$
\vartheta_{1,2}(t, s_1, y_1, s_2, y_2) = -E^{\mathcal{P}}_{t,s_1,y_1,s_2,y_2}\left(\int_t^T \tilde{\rho}_{1,2,R} b_{1,R}(y_{1,u}) b_{2,R}(y_{2,u})\frac{\partial C^{(1)}}{\partial y_1}\frac{\partial C^{(2)}}{\partial y_2}du\right),
$$

The function ϑ_1 is solution of the following problem:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_1)\vartheta_1 + g_1(t, s_1, y_1, s_2, y_2) &= 0, \\
\vartheta_1(T, s_1, y_1, s_2, y_2) &= 0,
\end{aligned}
$$

which implies:

$$
\vartheta_1(t, s_1, y_1, s_2, y_2) = E^{\mathcal{P}}_{t,s,y}\left(\int_t^T g_1(u, s_{1,u}, y_{1,u}, s_{2,u}, y_{2,u})du\right),
$$

The function ϑ_2 is solution of the following problem:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_1)\vartheta_2 + g_2(t, s_1, y_1, s_2, y_2) &= 0, \\
\vartheta_2(T, s_1, y_1, s_2, y_2) &= 0,
\end{aligned}
$$

and then using again the Feynman-Kac formula, we obtain:

$$
\vartheta_2(t, s_1, y_1, s_2, y_2) = E^{\mathcal{P}}_{t,s,y}\left(\int_t^T g_2(u, s_{1,u}, y_{1,u}, s_{2,u}, y_{2,u})du\right),
$$

Using the same approach, we deduce that:

$$
\vartheta_0(t, s_1, y_1, s_2, y_2) = E^{\mathcal{P}}_{t,s,y}\left(\int_t^T g_0(u, s_{1,u}, y_{1,u}, s_{2,u}, y_{2,u})du\right),
$$

As we did in the last chapter, we can demonstrate that u_0^ϵ, solution of the HJB equation, coincides with the value function. In addition, u_0^ϵ can be approximated at order 1 in ϵ by $\tilde{u}_0^\epsilon = x + v_0(t, s_1, y_1, s_2, y_2, q_1, q_2) + \epsilon v_1(t, s_1, y_1, s_2, y_2, q_1, q_2)$.

Chapter 4

Joint dynamics of the spot and the implied volatility surface

Note: A part of this chapter will be submitted to the special issue of The Journal of Banking and Finance on the theme "Recent developments in financial econometrics and applications".

4.1 Introduction

The leverage effect is a well known feature in the equity markets, it has been deeply studied and documented by several authors (see Bekaert and Wu (2000), Bouchaud et al. (2001), Bollerslev et al. (2006), Ciliberti et al. (2009)). This feature consists in the increase of the volatility following negative returns of the underlying, which explains the negative skewness of the underlying returns. The leverage effect influences considerably the derivatives prices. Indeed, the volatility smile in the equity option market has the particularity to be skewed, which means that the implied volatility is a decreasing function of the strike. Furthermore, the increment of the at-the-money volatility is negatively correlated with the log-return of the underlying, and this property is called "the implied leverage effect". The understanding of this property is a subject of interest since it is crucial to perform an efficient option delta-hedging strategy.

In Bergomi (2009), the author showed that the rate of decay of the at-the-money forward skew is linked to the covariance between the at-the-money volatility increment and the underlying log-return. He introduced a quantity called the Skew Stickiness Ratio (SSR) in order to quantify the relation between these two features. In this chapter, we build on the work done in Bergomi (2009) and we conduct a study in order to understand the information contained in the Skew Stickiness Ratio. The structure of this chapter is as follows. In the second section, we propose a model-free approach in order to estimate the SSR implied by option prices under the risk-neutral probability measure $\mathcal{Q}$. In the third section, we carry out an empirical study which aims to compare the realized SSR under the objective measure $\mathcal{P}$ with the implied SSR under $\mathcal{Q}$. Following that, we define an arbitrage strategy which enables to monetize the discrepancy between these two quantities. In the fourth section, we recall that, in the framework of linear stochastic volatility models, the implied SSR has the limit 2 in the case of short maturities. Since this property is contradictory with empirical findings, we introduce a stochastic volatility model with jumps in order to justify that the implied SSR can exceed the value 2 in the case of short maturities.

4.2 A model-free approach for implied Skew Stickiness Ratio estimation

Let $\sigma_{BS,t}(K,T)$ denote the implied volatility at time t for strike K and residual maturity T. We focus in this study on short maturity options ($0 < T \ll 1$), and we assume that the risk-free interest rate is null ($r = 0$). We make the two following hypotheses:

- The implied volatility, can be approximated for near the money strikes, by a quadratic function of the log-moneyness $\log(\frac{K}{S_t})$. Indeed, $\forall K, |K - S_t| \ll S_t$,

$$\sigma_{BS,t}(K,T) = \sigma_{ATM,t}(T) + \mathcal{S}_T \log(\frac{K}{S_t}) + \mathcal{C}_T \log(\frac{K}{S_t})^2 + o\Big(\log(\frac{K}{S_t})^2\Big). \quad (4.1)$$

- The joint dynamics between the underlying S and the at-the-money implied volatility $\sigma_{ATM}(T)$ can be described as follows:

$$d\sigma_{ATM,t}(T) = \alpha_t dt + R_T \mathcal{S}_T d\log(S_t) + \eta dZ_t, \quad (4.2)$$

  where Z is a Brownian motion such that $d\langle Z, \log(S)\rangle_t = 0$.

The quantity R_T, used in (4.2), was introduced by Bergomi in Bergomi (2009). It is called the "Skew stickiness ratio" and is a function of the residual maturity T. The quantity R_T can be obtained using (4.2) in the following way:

$$R_T = \frac{d\langle \sigma_{ATM}(T), \log(S)\rangle_t}{\mathcal{S}_T d\langle \log(S)\rangle_t}. \quad (4.3)$$

Using the implied volatility smile parametrization (4.1), and the spot/at-the money implied volatility joint dynamics given in (4.2), we can obtain the following relation:

$$d\sigma_{BS,t}(K,T) = \mathcal{C}_T d\langle \log(S)\rangle_t + \Big((R_T - 1)\,\mathcal{S}_T + 2\mathcal{C}_T \log(\frac{S_t}{K})\Big) d\log(S_t) + \eta dZ_t + O\Big(\log(\frac{K}{S_t})^2\Big),$$

which simplifies into:

$$d\sigma_{BS,t}(K,T) = \mathcal{C}_T d\langle \log(S)\rangle_t + (R_T - 1)\mathcal{S}_T d\log(S_t) + \eta dZ_t + O\Big(\log(\frac{K}{S_t})\Big).$$

If $R_T = 1$, then $E(d\sigma_{BS,t}(K,T)|d\log(S_t)) = O(\mathcal{C}_T dt)$. This means that for $\mathcal{C}_T \sim 0$, the implied volatility for a given strike K remains in average constant even after a move in the underlying S. This property is called by practitioners the sticky strike rule.
On the other hand, if $R_T = 0$ then $E(d\sigma_{BS,t}(K,T)|d\log(S_t)) = -\mathcal{S}_T d\log(S_t) + O\Big(\log(\frac{K}{S_t})\Big) + O(\mathcal{C}_T dt)$. In this particular case, the implied volatility is in average a constant function of the moneyness $\frac{K}{S_t}$. This attribute is known among practitioners by the sticky-delta rule.

We aim here to estimate the value of R_T, under the risk-neutral measure $\mathcal{Q}$, using option prices with maturity T.

Let $C(t, S_t, K, T, \sigma_{ATM,t}(T-t), \mathcal{S}_{T-t}, \mathcal{C}_{T-t}) = E^{\mathcal{Q}}(e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t)$ be the price at date t of an European call option with residual maturity $(T-t)$ and strike K.
We suppose that $0 \leq T - t \ll 1$ and we introduce the following simplifying assumption $\mathcal{H}^*$ defined as follows:

- $\mathcal{H}^*$: For $T-t \sim 0$, the quantities $\sigma_{ATM}, \mathcal{S}, \mathcal{C}$ don't depend on $(T-t)$, and then $\sigma_{BS}(K, T-t)$ has no time dependence.

Under $\mathcal{H}^*$, we have $d\mathcal{S}_{T-t} = d\mathcal{C}_{T-t} = 0$, and the application of Itô's Lemma yields:

$$dC_t = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}d\left\langle S\right\rangle_t + \frac{\partial C}{\partial \sigma_{ATM}}d\sigma_{ATM,t} + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma_{ATM}^2}d\left\langle \sigma_{ATM}\right\rangle_t + \frac{\partial^2 C}{\partial S\partial \sigma_{ATM}}d\left\langle S,\sigma_{ATM}\right\rangle_t.$$

The implied volatility corresponding to this option is equal to $\sigma_{BS}(K, T-t)$, which gives:

$$C(t, S_t, K, T-t, \sigma_{ATM,T-t}, \mathcal{S}_{T-t}, \mathcal{C}_{T-t}) = P_{BS}(t, S_t, K, T, \sigma_{BS}(K, T-t)).$$

Recall that $r = 0$, then it follows:

$$\frac{\partial P_{BS}}{\partial t} + \frac{1}{2}\sigma_{BS,t}^2(K, T-t)S_t^2\frac{\partial^2 P_{BS}}{\partial S^2} \quad = \quad 0.$$

Under the hypothesis $\mathcal{H}^*$, we have $\frac{\partial P_{BS}}{\partial t} = \frac{\partial C}{\partial t}$. This relation implies that:

$$dC_t = -\frac{1}{2}\sigma_{BS,t}^2(K, T-t)S_t^2\frac{\partial^2 P_{BS}}{\partial S^2}dt + \frac{\partial C}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}d\left\langle S\right\rangle_t + \frac{\partial C}{\partial \sigma_{BS}}d\sigma_{ATM,t} + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma_{ATM}^2}d\left\langle \sigma_{ATM}\right\rangle_t + \frac{\partial^2 C}{\partial S\partial \sigma_{ATM}}d\left\langle S,\sigma_{ATM}\right\rangle_t.$$

In order to simplify the expressions, the following quantities x, τ and d_1 are introduced:

$$x = \log(\frac{K}{S_t}) \quad , \quad \tau = T-t \quad , \quad d_{1,t} = \frac{-x + \frac{1}{2}\sigma_{BS,t}^2(K,\tau)\tau}{\sigma_{BS,t}(K,\tau)\sqrt{\tau}}.$$

As explained in appendix (4.6.1), for near the money options ($x \sim 0$), high order derivatives of the option price C can be approximated using Black-Scholes greeks. Indeed, the second order partial derivative of C with respect to the spot price S writes:

$$\frac{\partial^2 C}{\partial S^2} \quad = \quad \frac{n(d_1)}{\sigma_{ATM}S_t\sqrt{\tau}}\left(1 - 3x\alpha + x^2(5\alpha^2 - \frac{5}{2}\beta)\right) + O\big(x^3\big) + O\big(\sqrt{\tau}\big). \tag{4.4}$$

where α and β are functions of σ_{ATM}:

$$\alpha(\sigma_{ATM}) = \frac{\mathcal{S}_{T-t}}{\sigma_{ATM}} \quad , \quad \beta(\sigma_{ATM}) = \frac{2\mathcal{C}_{T-t}}{\sigma_{ATM}}.$$

In addition, the second order partial derivative of C with respect to σ_{ATM} can be approximated using $d_{1,t}$, σ_{ATM}, x and τ:

$$\frac{\partial^2 C}{\partial \sigma_{ATM}^2} \quad = \quad \frac{S_t n(d_{1,t})}{\sigma_{ATM,t}^3\sqrt{\tau}}x^2 + O\big(x^3\big) + O\big(\sqrt{\tau}\big), \tag{4.5}$$

and finally the approximation of $\frac{\partial^2 C}{\partial S\partial \sigma_{ATM}}$ can be given as follows:

$$\frac{\partial^2 C}{\partial S\partial \sigma_{ATM}} \quad = \quad \frac{n(d_{1,t})}{\sigma_{ATM,t}^2\sqrt{\tau}}\left(x - x^2(2\alpha - \sigma_{ATM,t}\alpha')\right) + O\big(\sqrt{\tau}\big) + O\big(x^3\big). \tag{4.6}$$

Let K_L be a strike inferior to the spot value S_t, and K_H be the ATM strike ($K_H = S_t$). We define a portfolio X that contains, at time t, a quantity equal to (-1) unity of the option C^L with strike K_L and maturity T and a quantity equal to $(+n_{t,H})$ of the option C^H with strike K_H and maturity T. The portfolio is delta-hedged continuously, and evolves as follows:

$$dX_t \quad = \quad n_{t,H}\left(dC_t^H - \Delta_{t,H}dS_t\right) - \left(dC_t^L - \Delta_{t,L}dS_t\right)$$

Using (4.5) and (4.6), it can be deduced that the terms $\frac{\partial^2 C^H}{\partial \sigma_{ATM}^2}$ and $\frac{\partial^2 C^H}{\partial S \partial \sigma_{ATM}}$ are at order 1 in $\sqrt{\tau}$, and then can be neglected for $\tau \sim 0$.
In order to cancel the spot gamma sensitivity of the portfolio, the quantity $n_{t,H}$ is chosen as follows:

$$n_{t,H} = \frac{\frac{\partial^2 C^L}{\partial S^2}}{\frac{\partial^2 C^H}{\partial S^2}}.$$

Thus, the portfolio X has the following dynamics:

$$\begin{aligned} dX_t = {} & \frac{1}{2} S_t^2 \left(\sigma_{BS}^2(K_L, T) \frac{\partial^2 P_{BS}^L}{\partial S^2} - n_{t,H} \sigma_{BS}^2(K_H, T) \frac{\partial^2 P_{BS}^H}{\partial S^2} \right) dt + \left(n_{t,H} \frac{\partial C^H}{\partial \sigma_{ATM}} - \frac{\partial C^L}{\partial \sigma_{ATM}} \right) d\sigma_{ATM,t} \\ & - \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} d \langle \sigma_{ATM} \rangle_t - \frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} R_T \mathcal{S}_t S_t d \langle \log(S) \rangle_t . \end{aligned}$$

In order to simplify the notations, we introduce the following quantities:

$$\begin{aligned} \Gamma_t &= \frac{1}{2} S_t^2 \left(\sigma_{BS}^2(K_L, T) \frac{\partial^2 P_{BS}^L}{\partial S^2} - n_{t,H} \sigma_{BS}^2(K_H, T) \frac{\partial^2 P_{BS}^H}{\partial S^2} \right), \\ \vartheta_t &= n_{t,H} \frac{\partial C^H}{\partial \sigma_{ATM}} - \frac{\partial C^L}{\partial \sigma_{ATM}}. \end{aligned}$$

Under the risk-neutral probability measure $\mathcal{Q}$, we have $E(dX_t) = rX_t dt = 0$, then:

$$\frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} R_T^{\mathcal{Q}} S_t \mathcal{S}_T d \langle \log(S) \rangle_t = \Gamma_t dt + \vartheta_t E^{\mathcal{Q}}(d\sigma_{ATM,t}) - \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} d \langle \sigma_{ATM} \rangle_t . \quad (4.7)$$

Based on (4.2), it can be deduced that $d \langle \sigma_{ATM} \rangle_t = R_T^2 \mathcal{S}_T^2 d \langle \log(S) \rangle_t + \eta^2 d \langle Z \rangle_t$. Thus, we obtain the following equation:

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} (R_T^{\mathcal{Q}})^2 \mathcal{S}_T^2 d \langle \log(S) \rangle_t + R_T^{\mathcal{Q}} \left(\frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} S_t \mathcal{S}_T d \langle \log(S) \rangle_t - \vartheta_t \mathcal{S}_T E^{\mathcal{Q}}(d \log(S_t)) \right) \\ - \left(\Gamma_t - \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} \eta^2 \right) dt = 0. \end{aligned}$$

Under the simplifying assumptions that $\eta \sim 0$ and $E^{\mathcal{Q}}(d \log(S_t)) = 0$, the last equation becomes:

$$\frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} (R_T^{\mathcal{Q}})^2 \mathcal{S}_T^2 d \langle \log(S) \rangle_t + \frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} S_t \mathcal{S}_T R_T^{\mathcal{Q}} d \langle \log(S) \rangle_t - \Gamma_t dt = 0. \quad (4.8)$$

The resolution of the equation (4.8) enables to obtain the value of $R_T^{\mathcal{Q}}$:

$$R_T^{\mathcal{Q}} = \frac{-\frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} S_t \mathcal{S}_T d \langle \log(S) \rangle_t + \sqrt{D_t}}{\frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} (\mathcal{S}_T)^2 d \langle \log(S) \rangle_t}, \quad (4.9)$$

where D_t represents the discriminant of the equation (4.8):

$$D_t = \left(\frac{n(d_1^L) x_L}{\sigma_{ATM}^2 \sqrt{\tau}} \right)^2 S_t^2 \mathcal{S}_T^2 d \langle \log(S) \rangle_t^2 + 2 \Gamma_t dt \left(\frac{S_t n(d_1^L)}{\sigma_{ATM}^3 \sqrt{\tau}} x_L^2 \right) \mathcal{S}_T^2 d \langle \log(S) \rangle_t ,$$

4.3 On the question of arbitraging the Skew Stickiness Ratio

Let $R_T^{\mathcal{P}}$ be the value of the quantity R_T under the historic probability measure $\mathcal{P}$, $R_T^{\mathcal{P}}$ can be determined using the linear regression of the daily increments of the ATM volatility with maturity T on the daily log-returns of the spot process S:

$$d\sigma_{ATM,t}(T) \quad = \quad R_T \mathcal{S}_T d\log(S_t) + \eta dZ_t,$$

then $R_T^{\mathcal{P}}$ can be estimated as follows:

$$R_{t,T}^{\mathcal{P}} \quad = \quad \frac{\sum_{i=t-L}^{t} \left(\sigma_{ATM,i}(T) - \sigma_{ATM,i-1}(T)\right)\left(\log(S_i) - \log(S_{i-1})\right)}{\mathcal{S}_{t,T}\left(\sum_{i=t-L}^{t} \left(\log(S_i) - \log(S_{i-1})\right)^2\right)}.$$

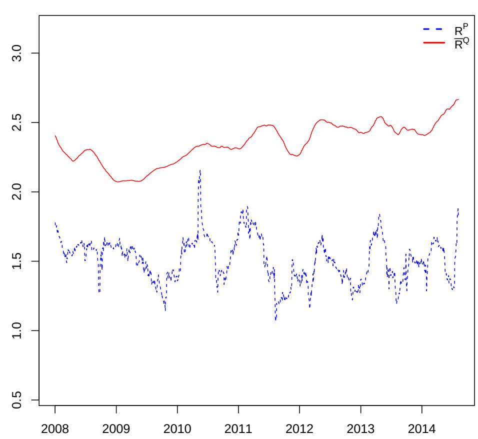
In order to to make this measure less noisy, $R_{t,T}^{\mathcal{P}}$ is computed in the the empirical study as follows:

$$R_{t,T}^{\mathcal{P}} \quad = \quad \frac{\sum_{i=t-L}^{t} \left(\sigma_{ATM,i}(T) - \sigma_{ATM,i-1}(T)\right)\left(\log(S_i) - \log(S_{i-1})\right)}{\left(\frac{1}{L+1}\sum_{i=t-L}^{t} \mathcal{S}_{i,T}\right)\left(\sum_{i=t-L}^{t} \left(\log(S_i) - \log(S_{i-1})\right)^2\right)}, \tag{4.10}$$

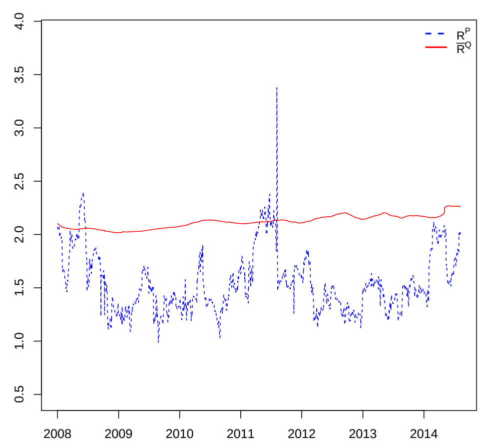
We study the historic evolution of $R_T^{\mathcal{P}}$ and $R_T^{\mathcal{Q}}$ and investigate the presence of discrepancies between these two quantities.

4.3.1 Empirical study

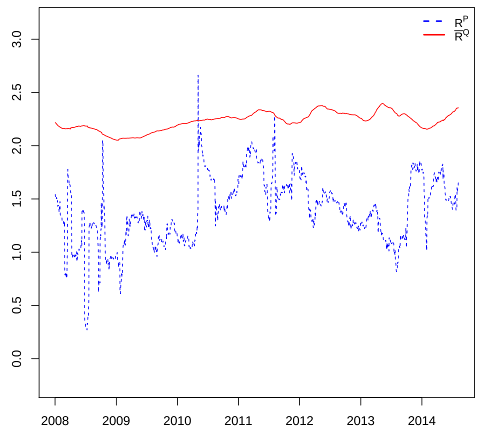
We carry out an empirical study on the skew stickiness ratio using the results provided in (4.9) and (4.10). We use historical data from option markets and from underlying price series, and we conduct the study on several assets including the SPX index, SX5E index, Financial Select Sector (XLF), Technology Select Sector (XLK). We focus here on $3M$ options ($T = \frac{3}{12}$) which is consistent with the hypothesis of short maturity ($T \sim 0$). The hypothesis $\mathcal{H}^*$ is supposed to be satisfied, and the quantity $R_T^{\mathcal{Q}}$ is obtained through the resolution of the equation (4.8). We dispose of data ranging from 01/01/2007 to 07/08/2014. For every day i, we estimate $R_{i,T}^{\mathcal{Q}}$ and $R_{i,T}^{\mathcal{P}}$ solutions of (4.9) and (4.10) respectively. In order to have two quantities calculated on the same window of data and then easily comparable, we introduce the quantity $\bar{R}_{i,T}^{\mathcal{Q}} = \sum_{j=i-L}^{i} R_{j,T}^{\mathcal{Q}}$. The following graphs give the evolution of $\bar{R}_{i,T}^{\mathcal{Q}}$ and $R_{i,T}^{\mathcal{P}}$ using the window parameter $L = 50$ Days:



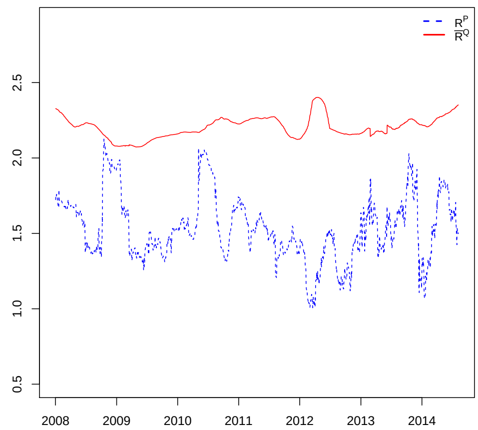
(k) SPX



(l) XLF



(m) XLK



(n) SX5E

The inspection of the graphs above show that the quantity $\bar{R}^{\mathcal{Q}}_T$ can be significantly larger than the value of 2. This finding is contradictory with the characteristics of linear stochastic volatility models. Indeed, Bergomi demonstrated in Bergomi (2009), that in the case of linear stochastic volatility models, the quantity $\bar{R}^{\mathcal{Q}}_T$ converges to the value 2 in the limit of small maturities. In addition, he showed that, in the case of time-homogeneous models and a flat term-structure of variance, $\bar{R}^{\mathcal{Q}}_T$ is restricted to the interval [1, 2]. This empirical discrepancy was pointed out by the authors in Vargas et al. (2013) who justified it by the existence of a non-linear leverage effect in equity markets which can not be captured using linear stochastic volatility models. Thus, the empirical study confirms the necessity of use of non-linear models on some assets like the SPX index for which the average value of $\bar{R}^{\mathcal{Q}}_T$ is the highest among the other examples.

In addition to that, it can be noticed that $R^{\mathcal{P}}_T$ is more oscillatory and unstable compared to $\bar{R}^{\mathcal{Q}}_T$, and that there is generally a discrepancy between the values of these two quantities. Bergomi pointed out in Bergomi (2009) that $R^{\mathcal{P}}_T$ is inferior to 2 which is, in the case of linear stochastic volatility models, the limit of $R^{\mathcal{Q}}_T$ when T tends to 0. This is true in average (but not always) and raises the question of arbitraging the spread $(R^{\mathcal{Q}}_T - R^{\mathcal{P}}_T)$. In Bergomi (2009), the author tried to establish a delta-hedged option portfolio whose return is proportional to the spread $(2 - R^{\mathcal{P}}_T)$. In the next paragraph, we build on this work and we define a slightly different strategy which aims to take profit from the discrepancy between $R^{\mathcal{Q}}_T$ and $R^{\mathcal{P}}_T$.

4.3.2 Taking advantage of the Skew Stickiness Ratio discrepancy

Under the real-world probability measure $\mathcal{P}$, the portfolio X, defined previously, has the following dynamics:

$$
\begin{aligned}
dX_t = \Gamma_t dt &+ \left(n_{t,H} \frac{\partial C^H}{\partial \sigma_{ATM}} - \frac{\partial C^L}{\partial \sigma_{ATM}} \right) d\sigma_{ATM,t} \\
&- \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma^2_{ATM}} \left((R^{\mathcal{P}}_T)^2 \mathcal{S}^2_T d \left\langle \log(S) \right\rangle_t + \eta^2 dt \right) - \frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} R^{\mathcal{P}}_T \mathcal{S}_T S_t d \left\langle \log(S) \right\rangle_t .
\end{aligned}
$$

Using the definition of $R_T^{\mathcal{Q}}$ given in (4.7), it can be deduced that:

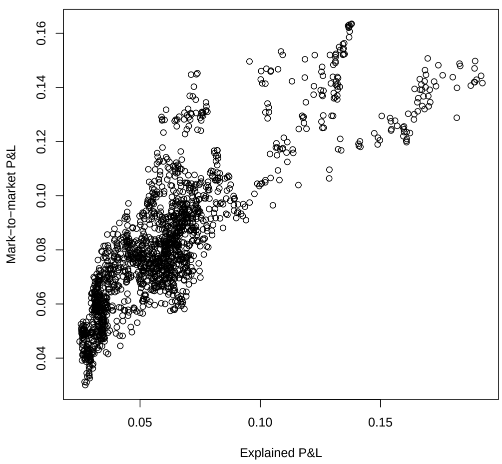
$$dX_t = \left(n_{t,H} \frac{\partial C^H}{\partial \sigma_{ATM}} - \frac{\partial C^L}{\partial \sigma_{ATM}} \right) \left(d\sigma_{ATM,t} - E^{\mathcal{Q}} \left(d\sigma_{ATM,t} \right) \right)$$
$$+ \frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} \left((R_T^{\mathcal{Q}})^2 - (R_T^{\mathcal{P}})^2 \right) \mathcal{S}_T^2 d \left\langle \log(S) \right\rangle_t + \frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} \left(R_T^{\mathcal{Q}} - R_T^{\mathcal{P}} \right) \mathcal{S}_T S_t d \left\langle \log(S) \right\rangle_t .$$

Then, we deduce that:

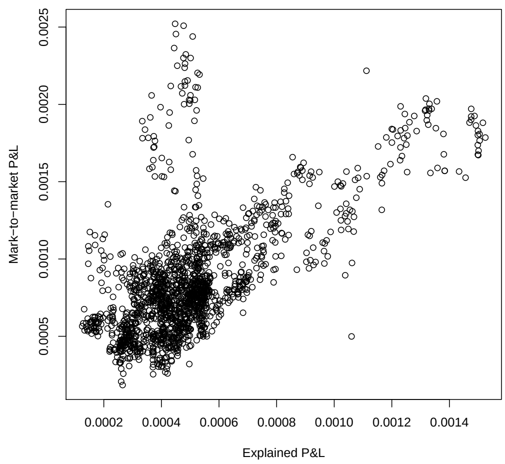
$$E(dX_t) \quad = \quad \left(R_T^{\mathcal{Q}} - R_T^{\mathcal{P}} \right) \left(\frac{1}{2} \frac{\partial^2 C^L}{\partial \sigma_{ATM}^2} \left(R_T^{\mathcal{Q}} + R_T^{\mathcal{P}} \right) \mathcal{S}_T^2 + \frac{\partial^2 C^L}{\partial S \partial \sigma_{ATM}} \mathcal{S}_T S_t \right) d \left\langle \log(S) \right\rangle_t$$

which is a function of the spread $\left(R_T^{\mathcal{Q}} - R_T^{\mathcal{P}} \right)$.

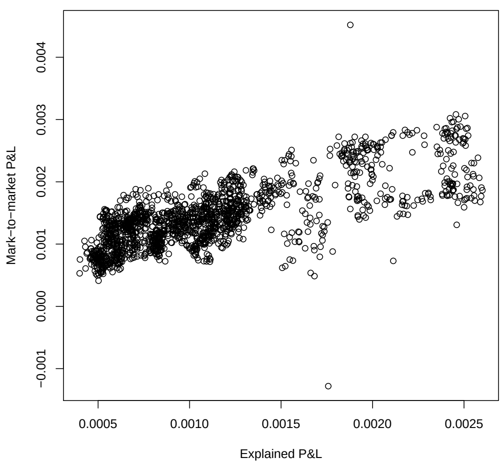
In the following, we run a backtest of a strategy which replicates the portfolio X. The strategy consists in selling every day a 3 months option with moneyness 95%, buying the quantity n_H of options with moneyness 100%, and doing the necessary delta-hedging. The portfolio is unwound the next day and started again. The scatter plots shows the daily market return of the portfolio (without transaction costs) as a function of the theoretical return given by the analytic expression of dX_t.



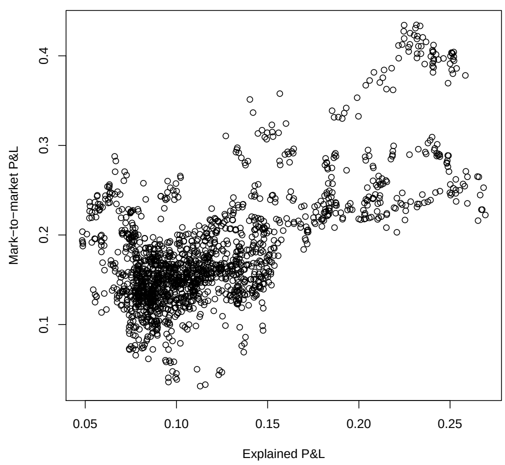
(o) SPX



(p) XLF



(q) XLK



(r) SX5E

Statistics on PnL explanation	Regression slope	R-squared
SPX	0.704	0.699
SX5E	0.932	0.462
XLF	1.02	0.495
XLK	0.694	0.546

The scatter plots above and the R^2 of the regressions show how well themarket return matches the theoretical return. It should be pointed out here that the difference between these two quantities can be explained by several factors:

- the quantities $\frac{\partial^2 C}{\partial S \partial \sigma_{ATM}}$, $\frac{\partial^2 C}{\partial S^2}$ and $\frac{\partial^2 C}{\partial \sigma_{ATM}^2}$ are not exact but approximated at order 2 in x and 0 in τ
- the hypothesis $\mathcal{H}^*$ was admitted in order to perform the computations, which is not guaranteed to be true in the realistic case.

4.4 Limit of the Skew Stickiness Ratio for short maturities

It was clear through the empirical study that R_T^Q can exceed the value of 2 when $T \sim 0$. This empirical finding is in discordance with the theoretical results established by Bergomi in Bergomi (2009) for the class of linear stochastic volatility models. In order to explain this phenomenon, the authors in Vargas et al. (2013) proposed an asymmetric Garch model which accounts for the non-linear leverage effect in the equity market. This model can produce values of R_T^Q superior to 2 for a maturity T of the order of several days. Meanwhile, for a 3 Months maturity ($T = 0.25$), which is the maturity considered in the empirical study, the quantity R_T^Q produced by the asymmetric Garch model can not be superior to 2. This is due to the fact that the implied at-the-money skew nearly coincides with the quantity $\frac{Skewness_T}{6\sqrt{T}}$ when T is of the order of several months (it can be precised here that $Skewness_T$ represents the skewness of $\log(\frac{S_T}{S_0})$). Consequently, the asymmetric Garch model doesn't justify theoretically the empirical observation $R_T^Q > 2$ for $T = 0.25$.

In order to support theoretically this empirical observation, we propose here a model which enables to obtain a value of R_T^Q superior to 2 when $T \sim 0$.

We suppose that, under the risk-neutral probability measure Q, the spot process S has the following dynamics:

$$\frac{dS_t}{S_t} = (r - \lambda k)dt + \sigma_t \sqrt{1-\rho_t^2} dW_t^{(1)} + \sigma_t \rho_t dW_t^{(2)} + (J_t - 1)dN_t$$

where N_t is a Poisson process with intensity λ, r is the risk-free rate, J_t is a random positive variable and $k = E(J_t - 1)$. The instantaneous volatility $\sigma_t = \sigma(Y_t)$ is a deterministic function of the stochastic process Y which evolves as follows:

$$dY_t = b_t dW_t^{(2)} \tag{4.11}$$

where $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = 0$.
It can be mentioned here that b_t and ρ_t can also depend on Y_t, that is:

$$b_t = b(Y_t) \quad , \quad \rho_t = \rho(Y_t)$$

and:

$$\int_0^t b_s^2 ds < \infty$$

Let $P_t = e^{-r(T-t)}E^{\mathcal{Q}}((S_T - K)^+)$ be the price of an European call option on S with strike K and maturity T. Under the assumption that b and λ are small, the following expansion for the option price P can be made:

$$P = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b^i \lambda^j P_{i,j}$$

Through the use of a singular perturbation technique in b and λ, the option price P can be approximated at order 1 in b and λ by $\hat{P}$ as proved in (4.6.2):

$$\hat{P}(t, S_t, y_t) = P_{0,0} + bP_{1,0} + \lambda P_{0,1}, \tag{4.12}$$

where:

$$P_{0,0}(t, S_t, y_t) = P_{BS}(t, S_t, \sigma(y_t), K, T), \tag{4.13}$$

$$P_{1,0}(t, S_t, y_t) = \frac{T-t}{2} S_t \sigma(y_t)\sigma'(y_t)\rho_t \frac{\partial^2 P_{0,0}}{\partial S \partial \sigma}, \tag{4.14}$$

$$P_{0,1}(t, S_t, y_t) = (T-t)(\phi(P_{0,0}) - S_t k \frac{\partial P_{0,0}}{\partial S}), \tag{4.15}$$

where the operator ϕ is defined as:

$$\begin{aligned} \phi(P) &= E(P(t, S.J, y)) - P(t, S, y) \\ &= \int_0^{\infty} P(t, S \times j, y) f_J(j) dj - P(t, S, y), \end{aligned}$$

and f_J is the probability density function of the random variable J.

Let $I_t(K,T)$ be the implied volatility of P, that is $I_t(K,T)$ is defined such that $P_t^{K,T} = P_{BS}\left(t, S_t, K, T, I_t(K,T)\right)$. In order to approximate $I_t(K,T)$ at first order in b and λ, a Taylor expansion of the implied volatility $I_t(K,T)$ is carried out in (4.6.3) and we prove that:

$$I_t(K,T) = \hat{I}_t(K,T) + O\big(b^2 + \lambda^2 + b\lambda\big),$$

where $\hat{I}_t(K,T)$ is defined as follows:

$$\hat{I}_t(K,T) = d(\rho, b, \lambda, T-t) + a(\rho, b, \lambda, T-t)\log(\frac{K}{F_{t,T}}) + c(\rho, b, \lambda, T-t)\log(\frac{K}{F_{t,T}})^2,$$

the quantities a, c and d are defined as follows:

$$\begin{aligned} a(\rho, b, \lambda, T-t) &= \frac{b_t\rho_t}{2}\frac{\sigma'(y)}{\sigma(y)} + \lambda_t \frac{E((J-1)^3)}{6\sigma^3(y)(T-t)} - \frac{\lambda_t E((J-1)^4)}{6\sigma^3(y)(T-t)}, \\ c(\rho, b, \lambda, T-t) &= \frac{\lambda_t E((J-1)^4)}{24\sigma^5(y)(T-t)^2}, \end{aligned}$$

and:

$$\begin{aligned} d(\rho, b, \lambda, T-t) = \sigma(y) + b_t\rho_t\sigma'(y)\frac{\sigma(y)}{4}(T-t) + \lambda_t \frac{E((J-1)^2)}{2\sigma(y)} - \lambda_t \frac{E((J-1)^3)}{4\sigma(y)} \\ + \frac{\lambda_t}{24} E((J-1)^4)(\frac{15}{4\sigma(y)} - \frac{1}{\sigma^3(y)(T-t)}) \end{aligned}$$

In the absence of jumps ($\lambda = 0$), the quantity $R_T^{\mathcal{Q}}$ writes:

$$R_T^{\mathcal{Q}} = \frac{\sigma'(y_t)\sigma(y_t)b_t\rho_t + \frac{b_t\rho_t}{4}(T-t)\left(\sigma'(y_t)^2 + \sigma(y_t)\sigma''(y_t)\right)\sigma(y_t)b_t\rho_t}{\frac{b_t\rho_t}{2}\sigma'(y_t)\sigma(y_t)},$$

then: $\underset{T\to 0}{lim} R_T = 2$.

Suppose now that the intensity of jumps is strictly positive ($\lambda > 0$). For the ATM skew $a(\rho, b, \lambda, T-t)$ not to be infinite when $T = t$, it can be supposed that $E((J-1)^3) = E((J-1)^4)$. Thus, $R_T^{\mathcal{Q}}$ has the following limit when $T \to 0$:

$$\underset{T\to 0}{lim} R_T = 2 + \frac{\lambda}{\sigma^2(y_t)}\left(\frac{E((J-1)^3)}{2} - E((J-1)^2)\right),$$

Consequently, according to this model, R_T can take a value superior to 2 when $T \to 0$.

4.5 Conclusion

In this chapter, we conducted a study on the Skew-Stickiness Ratio. We provided a model-free approach which allows to measure the SSR under the risk-neutral pricing measure $\mathcal{Q}$ for a given maturity T. We also pointed out that the historical value of the SSR under $\mathcal{P}$ can be different from its value under $\mathcal{Q}$ for short maturities. Thus, we suggested a trading strategy which aims to monetize the difference between $R_T^{\mathcal{Q}}$ and $R_T^{\mathcal{P}}$. We tested the suggested strategy on real data in order to show that the assumptions made in the theoretical study are not too strong and that the market return of the trading strategy is well explained by the model. We focused then on the empirical observation that $R_T^{\mathcal{Q}}$ may exceed the value 2 in the limit of short maturities. Since the asymmetric Garch model fails to reproduce this result for $T = 0.25$, we proposed a stochastic volatility model with Jumps that can reproduce this result.

4.6 Appendix

4.6.1 Appendix 1

We have:

$$\sigma_{BS}(K,T) = \sigma_{ATM}\left(1+\alpha(\sigma_{ATM})\log(\frac{K}{S_t})+\frac{1}{2}\beta(\sigma_{ATM})\left(\log(\frac{K}{S_t})\right)^2\right)+o\left(\log(\frac{K}{S_t})^2\right).$$

where $\alpha\left(\sigma_{ATM}(T)\right) = \frac{\mathcal{S}_T}{\sigma_{ATM}(T)}$ and $\beta\left(\sigma_{ATM}(T)\right) = \frac{2\mathcal{C}_T}{\sigma_{ATM}(T)}$. Then:

$$\frac{\partial^2 Q}{\partial S^2} = \frac{\partial^2 P_{BS}}{\partial S^2}+2\frac{\partial^2 P_{BS}}{\partial S\partial\sigma_{BS}}\frac{\partial\sigma_{BS}}{\partial S}+\frac{\partial^2 P_{BS}}{\partial\sigma_{BS}^2}(\frac{\partial\sigma_{BS}}{\partial S})^2+\frac{\partial P_{BS}}{\partial\sigma_{BS}}\frac{\partial^2\sigma_{BS}}{\partial S^2},$$

Then:

$$S_t^2\frac{\partial^2 Q}{\partial S^2} = \frac{S_t n(d_1)}{\sigma_{BS}\sqrt{\tau}}\left(1+2\sqrt{\tau}d_2\sigma_{ATM}(\alpha+\beta x)+d_1 d_2\tau\sigma_{ATM}^2(\alpha+\beta x)^2+\sigma_{BS}\tau\sigma_{ATM}(\alpha+x\beta+\beta)\right),$$

Recall that $d_1 = \frac{-x+\frac{1}{2}\sigma_{BS}^2\tau}{\sigma_{BS}\sqrt{\tau}}$ and $d_2 = d_1-\sigma_{BS}\sqrt{\tau}$, then the expression can be simplifies as follows:

$$\begin{aligned}
S_t^2\frac{\partial^2 Q}{\partial S^2} &= \frac{S_t n(d_1)}{\sigma_{BS}\sqrt{\tau}}\left(1-2x(\alpha+\beta x)(1-\alpha x-\frac{1}{2}\beta x^2)+x^2\alpha^2+o\big(x^3\big)\right)+O\big(\sqrt{\tau}\big),\\
&= \frac{S_t n(d_1)}{\sigma_{BS}\sqrt{\tau}}\left(1-2x\alpha(\sigma_{ATM})+x^2(3\alpha^2(\sigma_{ATM})-2\beta(\sigma_{ATM}))+o\big(x^3\big)\right)+O\big(\sqrt{\tau}\big),\\
&= \frac{S_t n(d_1)}{\sigma_{ATM}\sqrt{\tau}}\left(1-2x\alpha+x^2(3\alpha^2-2\beta)\right)\left(1-x\alpha-\frac{1}{2}\beta x^2\right)+o\big(x^3\big)+O\big(\sqrt{\tau}\big),\\
&= \frac{S_t n(d_1)}{\sigma_{ATM}\sqrt{\tau}}\left(1-3x\alpha(\sigma_{ATM})+x^2(5\alpha^2(\sigma_{ATM})-\frac{5}{2}\beta(\sigma_{ATM}))\right)+o\big(x^3\big)+O\big(\sqrt{\tau}\big),
\end{aligned}$$

The quantity $\frac{\partial^2 Q}{\partial\sigma_{ATM}^2}$ can also be written using partial derivatives of P_{BS}:

$$\frac{\partial^2 Q}{\partial\sigma_{ATM}^2} = \frac{\partial^2 P_{BS}}{\partial\sigma_{BS}^2}(\frac{\partial\sigma_{BS}}{\partial\sigma_{ATM}})^2+\frac{\partial P_{BS}}{\partial\sigma_{BS}}\frac{\partial^2\sigma_{BS}}{\partial\sigma_{ATM}^2}$$

Using the expressions of d_1 and d_2, we deduce that:

$$\begin{aligned}
\frac{1}{2}\sigma_{ATM}^2\frac{\partial^2 Q}{\partial\sigma_{ATM}^2} &= \frac{S_t n(d_1)d_1 d_2\sqrt{\tau}}{\sigma_{BS}}(\frac{\partial\sigma_{BS}}{\partial\sigma_{ATM}})^2+O\big(\sqrt{\tau}\big),\\
&= \frac{S_t n(d_1)\sigma_{ATM}^2}{2\sigma_{BS}^3\sqrt{\tau}}x^2+O\big(x^3\big)+O\big(\sqrt{\tau}\big),\\
&= \frac{S_t n(d_1)}{2\sigma_{ATM}\sqrt{\tau}}x^2+O\big(x^3\big)+O\big(\sqrt{\tau}\big),
\end{aligned}$$

Using the same method, we provide analytic approximation for $\frac{\partial^2 Q}{\partial S\partial\sigma_{ATM}}$ up to order 2 in x and 0 in τ. Indeed:

$$\frac{\partial^2 Q}{\partial S\partial\sigma_{ATM}} = \frac{\partial^2 P_{BS}}{\partial S\partial\sigma_{BS}}\frac{\partial\sigma_{BS}}{\partial\sigma_{ATM}}+\frac{\partial^2 P_{BS}}{\partial\sigma_{BS}^2}\frac{\partial\sigma_{BS}}{\partial\sigma_{ATM}}\frac{\partial\sigma_{BS}}{\partial S}+\frac{\partial P_{BS}}{\partial\sigma_{BS}}\frac{\partial^2\sigma_{BS}}{\partial S\partial\sigma_{ATM}}.$$

Then:

$$
\begin{aligned}
S\sigma_{ATM}\frac{\partial^2 Q}{\partial S \partial \sigma_{ATM}} &= \frac{S_t n(d_1)}{\sigma_{ATM}\sqrt{\tau}}\left(\frac{\sigma_{ATM}^2}{\sigma_{BS}^2}x - \frac{\sigma_{ATM}^3}{\sigma_{BS}^3}x^2(\alpha(\sigma_{ATM}) + x\beta(\sigma_{ATM}))\right)\frac{\partial \sigma_{BS}}{\partial \sigma_{ATM}} + O(\sqrt{\tau}),\\
&= \frac{S_t n(d_1)}{\sigma_{ATM}\sqrt{\tau}}\left((1-2\alpha(\sigma_{ATM})x)x - (1-3\alpha x)x^2(\alpha + x\beta)\right)\frac{\partial \sigma_{BS}}{\partial \sigma_{ATM}} + O(\sqrt{\tau}),\\
&= \frac{S_t n(d_1)}{\sigma_{ATM}\sqrt{\tau}}\left(x - x^2(2\alpha(\sigma_{ATM}) - \sigma_{ATM}\alpha'(\sigma_{ATM}))\right) + O(\sqrt{\tau}) + O(x^3).
\end{aligned}
$$

4.6.2 Appendix 2

Let P be the price of an European option on the stock S with maturity T and payoff $H(S_T)$:

$$P(t, S_t, Y_t, K, T) = e^{-r(T-t)}E^{\mathcal{Q}}(H(S_T)|F_t).$$

Using Itô's lemma:

$$\mathcal{L}P = 0,$$

where the operator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}P = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial S}S_t(r-\lambda k) + \frac{1}{2}S_t^2\sigma^2(y)\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}b^2 + \frac{\partial^2 P}{\partial S \partial y}S_t\sigma(y)b_t\rho_t + \lambda\phi(P) - rP.$$

where the operator ϕ is defined as:

$$
\begin{aligned}
\phi(P) &= E(P(t, S.J, y)) - P(t, S, y)\\
&= \int_0^\infty P(t, Sj, y)f_J(j)dj - P(t, S, y),
\end{aligned}
$$

and f_J is the probability density function of the random variable J. In order to separate the terms in $\mathcal{L}$ by their orders in b and λ, the following differential operators are introduced:

$$
\begin{aligned}
\mathcal{L}_{0,0} &= \frac{\partial}{\partial t} + r(S_t\frac{\partial}{\partial S_t} - .) + \frac{1}{2}\sigma^2(y)S_t^2\frac{\partial^2}{\partial S_t^2},\\
\mathcal{L}_{1,0} &= S_t\sigma(y)\rho_t\frac{\partial^2}{\partial S_t \partial y},\\
\mathcal{L}_{2,0} &= \frac{1}{2}\frac{\partial^2}{\partial y^2},\\
\mathcal{L}_{0,1} &= \phi(.) - kS_t\frac{\partial}{\partial S_t},
\end{aligned}
$$

which implies:

$$\mathcal{L}P = \left(\mathcal{L}_{0,0} + b\mathcal{L}_{1,0} + b^2\mathcal{L}_{2,0} + \lambda\mathcal{L}_{0,1}\right)P = 0 \tag{4.16}$$

Under the assumption that b and λ are small, the option price P can be expanded in powers of b and λ:

$$P = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b^i\lambda^j P_{i,j} \tag{4.17}$$

By insertion of (4.17) in (4.16) and regroupment of terms by their powers in b and λ, the following equations are obtained:

$$
\begin{aligned}
(0,0) &: \mathcal{L}_{0,0}P_{0,0} = 0,\\
(1,0) &: \mathcal{L}_{0,0}P_{1,0} + \mathcal{L}_{1,0}P_{0,0} = 0,\\
(0,1) &: \mathcal{L}_{0,0}P_{0,1} + \mathcal{L}_{0,1}P_{0,0} = 0.
\end{aligned}
$$

Using the zero-order term $(0,0)$, it can be seen that $P_{0,0}$ is the solution of the equation:

$$\mathcal{L}_{BS}(\sigma(y))P_{0,0} \quad = \quad 0,$$

with the final condition:

$$P_{0,0}(T, S_T, Y_T) \quad = \quad H(S_T).$$

Then, $P_{0,0}$ is the Black-Scholes price of the option with implied volatility equal to $\sigma(y)$:

$$P_{0,0}(t, S_t, Y_t) = P_{BS}(t, S_t, \sigma(y_t), K, T).$$

The equation of the $(1,0)$-term (at order 1 in b and 0 in λ) shows that $P_{1,0}$ verifies:

$$\mathcal{L}_{BS}\left(\sigma(y)\right)P_{1,0} \quad = \quad -S_t\sigma(y)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial y},$$

with the final condition:

$$P_{1,0}(T, S_T, Y_T) \quad = \quad 0.$$

In order to determine $P_{1,0}$, we start by computing the term $S_t\sigma(y)\frac{\partial^2}{\partial S\partial y}\mathcal{L}_{BS}(\sigma(y))$:

$$S_t\sigma(y)\frac{\partial^2}{\partial S\partial y}\mathcal{L}_{BS}(\sigma(y)) \quad = \quad \mathcal{L}_{BS}(\sigma(y))S_t\sigma(y)\frac{\partial^2}{\partial S\partial y} + \sigma^2(y)\sigma'(y)S_t\frac{\partial}{\partial S_t}(S_t^2\frac{\partial^2}{\partial S_t^2})$$

Let $R(t, S_t, y_t)$ be defined as follows:

$$R(t, S_t, y_t) \quad = \quad (T-t)S_t\sigma(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial y} - \frac{(T-t)^2}{2}\sigma^2(y_t)\sigma'(y_t)\rho_t\mathcal{D}_{1,S}\mathcal{D}_{2,S}P_{0,0}$$

where $\mathcal{D}_{1,S}$ and $\mathcal{D}_{2,S}$ denote the two following differential operators:

$$\mathcal{D}_{1,S} = S_t\frac{\partial}{\partial S_t} \quad , \quad \mathcal{D}_{2,S} = S_t^2\frac{\partial^2}{\partial S_t^2}$$

Then:

$$\begin{aligned}\mathcal{L}_{BS}(\sigma(y))R(t, S_t, Y_t) = &-S_t\sigma(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial y}\\ &+(T-t)\left(S_t\sigma(y)\frac{\partial^2}{\partial S\partial y}\mathcal{L}_{BS}\left(\sigma(y)\right)P_{0,0} - \rho_t\sigma^2(y)\sigma'(y)\mathcal{D}_{1,S}\mathcal{D}_{2,S}P_{0,0}\right)\\ &+(T-t)\sigma^2(y_t)\sigma'(y_t)\rho_t\mathcal{D}_{1,S}\mathcal{D}_{2,S}P_{0,0}\\ &-\frac{(T-t)^2}{2}\sigma^2(y_t)\sigma'(y_t)\rho_t\mathcal{D}_{1,S}\mathcal{D}_{2,S}\mathcal{L}_{BS}\left(\sigma(y)\right)P_{0,0}.\end{aligned}$$

Since $\mathcal{L}_{BS}\left(\sigma(y)\right)P_{0,0} = 0$, it can be deduced that:

$$\mathcal{L}_{BS}(\sigma(y))R(t, S_t, y_t) \quad = \quad -S_t\sigma(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial y}$$

and $R(T, S_T, y_T) = 0$. Then, it follows that:

$$P_{1,0}(t, S_t, y_t) \quad = \quad R(t, S_t, y_t)$$

$$P_{1,0}(t, S_t, Y_t) \quad = \quad (T-t)S_t\sigma(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial y} - \frac{(T-t)^2}{2}\sigma^2(y_t)\sigma'(y_t)\rho_t S_t\frac{\partial}{\partial S_t}(S_t^2\frac{\partial^2 P_{0,0}}{\partial S_t^2}).$$

Using the relation between the Gamma and the Vega in the Black-Scholes model, it follows that:

$$\begin{aligned}P_{1,0}(t,S_t,Y_t) &= (T-t)S_t\sigma(y_t)\sigma'(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial\sigma(y)} - \frac{(T-t)^2}{2}\sigma^2(y_t)\sigma'(y_t)\rho_t\frac{S_t}{\sigma(y)(T-t)}\frac{\partial^2 P_{0,0}}{\partial S\partial\sigma(y)}\\ &= \frac{T-t}{2}S_t\sigma(y_t)\sigma'(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial\sigma(y)}\end{aligned}$$

Finally, $P_{0,1}$ satisfies the equation in the $(0,1)$-term (of order 1 in λ and 0 in b):

$$\mathcal{L}_{BS}\left(\sigma(y)\right)P_{0,1} = S_t k\frac{\partial P_{0,0}}{\partial S} - \phi(P_{0,0}),$$

with the final condition:

$$P_{0,1}(T,S_T,Y_T) = 0.$$

Then, since the operator $\mathcal{L}_{BS}\left(\sigma(y)\right)$ commits with the operators ϕ and $S\frac{\partial}{\partial S}$, it follows:

$$P_{0,1}(t,S_t,Y_t) = (T-t)(\phi(P_{0,0}) - \frac{\partial P_{0,0}}{\partial S}S_t k).$$

As a conclusion, the option price P can be approximated at order 1 in b and λ by $\hat{P}$:

$$\hat{P}(t,S_t,y_t) = P_{0,0} + bP_{1,0} + \lambda P_{0,1},$$

where:

$$\begin{aligned}P_{0,0}(t,S_t,y_t) &= P_{BS}(t,S_t,\sigma(y_t),K,T),\\ P_{1,0}(t,S_t,y_t) &= \frac{T-t}{2}S_t\sigma(y_t)\sigma'(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial\sigma},\\ P_{0,1}(t,S_t,y_t) &= (T-t)(\phi(P_{0,0}) - S_t k\frac{\partial P_{0,0}}{\partial S}).\end{aligned}$$

4.6.3 Appendix 3

The option price $P^{K,T}$ can be approximated at order 1 in b and λ as:

$$P^{K,T} = P_{0,0} + bP_{1,0} + \lambda P_{0,1} + O\big(b^2+\lambda^2+b\lambda\big).$$

Let $I_t(K,T)$ be the implied volatility of $P^{K,T}$, which means that $P_t^{K,T} = P_{BS}(t,S_t,I_t(K,T),K,T)$. Recall that if $b=\lambda=0$, then the constant volatility model is recovered and $I_t(K,T)=\sigma(y_t)$, therefore $I_t(K,T)$ can be written at first order in b and λ as:

$$I_t(K,T) = \sigma(y) + bI_{1,t}(K,T) + \lambda I_{2,t}(K,T) + O\big(b^2+\lambda^2+b\lambda\big)$$

Thus, we can perform the following Taylor development:

$$P_t^{K,T} = P_{BS}(t,S_t,\sigma(y_t)) + \frac{\partial P_{BS}}{\partial\sigma}_{|\sigma(y)}(bI_{1,t}(K,T) + \lambda I_{2,t}(K,T)) + O\big(b^2+\lambda^2+b\lambda\big)$$

Since $P_{0,0} = P_{BS}(t,S_t,\sigma(y),K,T)$, then by equalizing terms which have the same order in b and λ, it can be deduced that:

$$\begin{aligned}\frac{\partial P_{BS}}{\partial\sigma}_{|\sigma(y_t)}I_{1,t}(K,T) &= P_{1,0} = \frac{T-t}{2}S_t\sigma(y_t)\sigma'(y_t)\rho_t\frac{\partial^2 P_{0,0}}{\partial S\partial\sigma}\\ \frac{\partial P_{BS}}{\partial\sigma}_{|\sigma(y_t)}I_{2,t}(K,T) &= P_{0,1} = (T-t)(\phi(P_{0,0}) - kS_t\frac{\partial P_{0,0}}{\partial S})\end{aligned}$$

The term $P_{0,0}$ is a Black-Scholes price, then $\frac{\partial^2 P_{0,0}}{\partial S \partial \sigma}$ and $\frac{\partial P_{BS}}{\partial \sigma}$ have the following analytic expressions:

$$\begin{aligned}
\frac{\partial^2 P_{0,0}}{\partial S \partial \sigma(y)} &= -\frac{n(d_1) d_2}{\sigma(y)} \\
\frac{\partial P_{0,0}}{\partial \sigma(y)} &= S_t n(d_1) \sqrt{T-t}
\end{aligned}$$

where:

$$\begin{aligned}
d_1 &= \frac{\log(\frac{F_{t,T}}{K}) + \frac{\sigma(y)^2}{2}(T-t)}{\sigma(y)\sqrt{T-t}} \\
d_2 &= \frac{\log(\frac{F_{t,T}}{K}) - \frac{\sigma(y)^2}{2}(T-t)}{\sigma(y)\sqrt{T-t}} \\
F_{t,T} &= S_t e^{r(T-t)}
\end{aligned}$$

By doing necessary calculations, it can be deduced that:

$$I_{1,t}(K,T) = \frac{\rho_t}{2} \frac{\sigma'(y)}{\sigma(y)} \log(\frac{K}{F_{t,T}}) + \rho_t \sigma'(y) \frac{\sigma(y)}{4}(T-t)$$

We aim now to give the expression of $I_2(K,T)$. Recall that:

$$P_{0,1}(t, S_t, y_t) = (T-t)\left(\phi(P_{0,0}) - k S_t \frac{\partial P_{0,0}}{\partial S}\right)$$

Conditional on the variable J, and under the hypothesis that $(J-1) \sim 0$, the quantity $P_{0,0}(S_t J)$ can be written as:

$$\begin{aligned}
P_{0,0}(S_t J) = P_{0,0}(S_t) &+ S_t(J-1)\frac{\partial P_{0,0}}{\partial S}(S_t) + \frac{1}{2} S_t^2 (J-1)^2 \frac{\partial^2 P_{0,0}}{\partial S^2}(S_t) + \frac{1}{6} S_t^3 (J-1)^3 \frac{\partial^3 P_{0,0}}{\partial S^3}(S_t) \\
&+ \frac{1}{24} S_t^4 (J-1)^4 \frac{\partial^4 P_{0,0}}{\partial S^4}(S_t) + O\big((J-1)^5\big)
\end{aligned}$$

Then, we can take the expectation of the previous equation with respect to the distribution of J, and we deduce that:

$$\begin{aligned}
E(P_{0,0}(S_t J)) = P_{0,0}(S_t) &+ S_t k \frac{\partial P_{0,0}}{\partial S}(S_t) + \frac{1}{2} S_t^2 E((J-1)^2) \frac{\partial^2 P_{0,0}}{\partial S^2}(S_t) \\
&+ \frac{1}{6} S_t^3 E((J-1)^3) \frac{\partial^3 P_{0,0}}{\partial S^3}(S_t) + \frac{1}{24} S_t^4 E((J-1)^4) \frac{\partial^4 P_{0,0}}{\partial S^4}(S_t) + O\big(E((J-1)^5)\big)
\end{aligned}$$

We can then write that:

$$\begin{aligned}
I_{2,t}(K,T) = (T-t) &\frac{\frac{1}{2}\frac{\partial^2 P_{0,0}}{\partial S^2}(S_t) S_t^2 E((J-1)^2) + \frac{1}{6}\frac{\partial^3 P_{0,0}}{\partial S^3}(S_t) S_t^3 E((J-1)^3) + \frac{1}{24}\frac{\partial^4 P_{0,0}}{\partial S^4}(S_t) S_t^4 E((J-1)^4)}{\frac{\partial P_{BS}}{\partial \sigma(y)}} \\
&+ O\big(E((J-1)^5)\big)
\end{aligned}$$

Here again, since $P_{0,0}$ is a Black-Scholes option price, the quantities $\frac{\partial^3 P_{0,0}}{\partial S^3}$ and $\frac{\partial^4 P_{0,0}}{\partial S^4}$ have analytic expressions:

$$\frac{\partial^3 P_{0,0}}{\partial S^3}(S_t) = -\frac{n(d_1)}{\sigma(y)\sqrt{T-t} S_t^2}(1 + \frac{d_1}{\sigma(y)\sqrt{T-t}})$$

and:

$$\frac{\partial^4 P_{0,0}}{\partial S^4}(S_t) = \frac{1}{\sigma(y)\sqrt{T-t}}(\frac{d_1 n(d_1)}{S_t^2}\frac{1}{S_t\sigma(y)\sqrt{T-t}} + \frac{2n(d_1)}{S_t^3})$$
$$- \frac{1}{\sigma^2(y)(T-t)}(\frac{(1-d_1^2)n(d_1)}{S_t^2}\frac{1}{S_t\sigma(y)\sqrt{T-t}} - \frac{2d_1 n(d_1)}{S_t^3})$$

Then using the definition of d_1, it can be deduced that:

$$I_{2,t}(K,T) = \frac{E((J-1)^2)}{2\sigma(y)} - \frac{E((J-1)^3)}{4\sigma(y)} + \frac{E((J-1)^3)}{6\sigma^3(y)(T-t)}\log(\frac{K}{F_{t,T}})$$
$$+ \frac{E((J-1)^4)}{24}(-\frac{4\log(\frac{K}{F_t})}{\sigma^3(y)(T-t)} + \frac{\log(\frac{K}{F_t})^2}{\sigma^5(y)(T-t)^2} + \frac{15}{4\sigma(y)} - \frac{1}{\sigma^3(y)(T-t)}) + O\big(E((J-1)^5)\big)$$

In conclusion, the implied volatility $I_t(K,T)$ writes $I_t(K,T) = \hat{I}_t(K,T) + O\big(b^2+\lambda^2+b\lambda\big)$ where:

$$\hat{I}_t(K,T) \quad = \quad d(\rho,b,\lambda,T-t) + a(\rho,b,\lambda,T-t)\log(\frac{K}{F_{t,T}}) + c(\rho,b,\lambda,T-t)\log(\frac{K}{F_{t,T}})^2$$

and the functions a, c and d are summarized as:

$$\begin{aligned} a(\rho,b,\lambda,T-t) &= \frac{b_t\rho_t}{2}\frac{\sigma'(y)}{\sigma(y)} + \lambda_t\frac{E((J-1)^3)}{6\sigma^3(y)(T-t)} - \frac{4\lambda_t E((J-1)^4)}{24\sigma^3(y)(T-t)}, \\ c(\rho,b,\lambda,T-t) &= \frac{\lambda_t E((J-1)^4)}{24\sigma^5(y)(T-t)^2}. \end{aligned}$$

$$d(\rho,b,\lambda,T-t) = \sigma(y) + b_t\rho_t\sigma'(y)\frac{\sigma(y)}{4}(T-t) + \lambda_t\frac{E((J-1)^2)}{2\sigma(y)} - \lambda_t\frac{E((J-1)^3)}{4\sigma(y)}$$
$$+ \frac{\lambda_t}{24}E((J-1)^4)(\frac{15}{4\sigma(y)} - \frac{1}{\sigma^3(y)(T-t)})$$

4.6.4 Appendix 4

At order 0 in x_L, the hedging ratio n_H writes:

$$n_H \quad = \quad \frac{n(d_{1,L})}{n(d_{1,H})}$$

then:

$$\begin{aligned} \Gamma_t &= \frac{1}{2}S_t^2\sigma_{BS}^2(K_L,T)\left(\frac{\partial^2 P_{BS}^L}{\partial S^2} - \frac{\sigma_{BS}^2(K_H,T)}{\sigma_{BS}^2(K_L,T)}\frac{n(d_{1,L})}{n(d_{1,H})}\frac{\partial^2 P_{BS}^H}{\partial S^2}\right), \\ &= \frac{1}{2}S_t^2\sigma_{BS}^2(K_L,T)\left(\frac{\partial^2 P_{BS}^L}{\partial S^2} - \frac{\sigma_{BS}(K_H,T)}{\sigma_{BS}(K_L,T)}\frac{\frac{\partial^2 P_{BS}^L}{\partial S^2}}{\frac{\partial^2 P_{BS}^H}{\partial S^2}}\frac{\partial^2 P_{BS}^H}{\partial S^2}\right), \\ &= \frac{1}{2}S_t^2\sigma_{BS}^2(K_L,T)\left(\frac{\partial^2 P_{BS}^L}{\partial S^2} - \frac{\sigma_{BS}(K_H,T)}{\sigma_{BS}(K_L,T)}\frac{\partial^2 P_{BS}^L}{\partial S^2}\right), \\ &= \frac{1}{2}S_t^2\sigma_{BS}(K_L,T)\frac{\partial^2 P_{BS}^L}{\partial S^2}\left(\sigma_{BS}(K_L,T) - \sigma_{BS}(K_H,T)\right), \end{aligned}$$

Thus, the discriminant D_t of the quadratic equation can be approximated at order 2 in x_L as following:

$$\begin{aligned} D_t &= \left(\frac{n(d_1^L)x_L}{\sigma_{ATM}^2\sqrt{\tau}}\right)^2 S_t\mathcal{S}_T^2 d\left\langle \log(S)\right\rangle_t^2 + 2\Gamma_t dt\left(\frac{S_t n(d_1^L)}{\sigma_{ATM}^3\sqrt{\tau}}x_L^2\right)\mathcal{S}_T^2 d\left\langle \log(S)\right\rangle_t, \\ &= \frac{n(d_1^L)^2 x_L^2 S_t^2 \mathcal{S}_T^2}{\tau\sigma_{ATM}^3} d\left\langle \log(S)\right\rangle_t dt\left(\sigma_{BS}(K_L,T) - \sigma_{BS}(K_H,T) + \frac{d\left\langle \log(S)\right\rangle_t}{\sigma_{ATM}(T)dt}\right). \end{aligned}$$

For $T \sim 0$, we can make the approximation $d\left\langle \log(S)\right\rangle_t = \sigma_{ATM}^2(T)dt$ and then $D_t \geq 0$ if $\sigma_{BS}(K_L,T) - \sigma_{BS}(K_H,T) + \sigma_{ATM}(T) \geq 0$.

General Conclusion and Outlooks

In this thesis, we studied several problems related to option trading. In the first part of this work, we showed how the joint dynamics between two assets may influence the relations between their options. Under the risk-neutral probability measure, and in the framework of a continuous-time CAPM model with stochastic volatilities, we studied the relations between two options on the stock and the index respectively. Based on this study, we concluded that the information contained in option prices can be very useful for many purposes including arbitrage strategies or also risk management. In the second part of our work, we focused on the problem of market making in options. We formulated an optimization problem and presented a mathematical approach which enables to provide liquidity in the option market in an optimal way. Using optimal stochastic control, we determined the optimal strategy depending on the utility function of the market maker. In particular, we pointed out that good assessment of the dynamics of the underlying under the historical measure and also under the risk-neutral measure, enables the market maker to optimize the profit generated by the optimal strategy. Next to that, we focused on market making strategies on two options on two correlated assets. After solving the optimization problem by using stochastic control, we showed that the joint dynamics of the two assets affects the bid and ask quotes on the two traded options. In the third part of the study, we focused on the joint dynamics between the spot price and the implied volatility surface. Through the use of a quantity called "Skew stickiness ratio", we tried to understand the relation between the implied skew and the spot/at-the-money implied volatility joint dynamics. We pointed out that the understanding of this relation is crucial in many applications like in option hedging or also in arbitrage strategies.

A lot remains to be done and we propose here several possible extensions of our work. One first direction of research is to extend the work we did on option market making by directly modeling the joint dynamics of the spot and the implied volatility surface. In fact, the instantaneous volatility of the spot has to be filtered or estimated using High Frequency data, and this can imply an estimation error that affects the optimal strategy. On the other hand, the implied volatility is an observable variable on the option market, and then such a modeling approach may be more advantageous from a practical perspective. The difficulty here is to find a good model for the dynamics of the implied volatility surface which is consistent with the absence of arbitrage opportunities and which reproduces the known empirical facts observed on the market. We can refer to the work done in Schönbucher (1999) and Cont et al. (2002) in order to extend our work. We suggest also, as a second research direction, to extend the study we did on the skew stickiness ratio to the curvature of the implied volatility smile. Indeed, we can follow the same approach in order to establish a relation between the curvature of the implied volatility smile and the second moment of the increment of the at-the-money implied volatility. Following that, we can test this relation on data in order to investigate whether there is a discrepancy between the risk-neutral measure and the real-world measure.

Bibliography

Azamat Abdymomunov and James Morley. Time variation of capm betas across market volatility regimes. *Applied Financial Economics*, 21(19):1463–1478, 2011.

Torben G Andersen, Tim Bollerslev, Francis X Diebold, and Ginger Wu. Realized beta: Persistence and predictability. *Advances in econometrics*, 20:1–39, 2006.

Marco Avellaneda and Sasha Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.

Geert Bekaert and Guojun Wu. Asymmetric volatility and risk in equity markets. *Review of Financial Studies*, 13(1):1–42, 2000.

Lorenzo Bergomi. Smile dynamics iv. *Risk*, 12:94–100, 2009.

Marshall E Blume and Irwin Friend. A new look at the capital asset pricing model. *The journal of finance*, 28(1):19–34, 1973.

Tim Bollerslev, Julia Litvinova, and George Tauchen. Leverage and volatility feedback effects in high-frequency data. *Journal of Financial Econometrics*, 4(3):353–384, 2006.

Jean-Philippe Bouchaud, Andrew Matacz, and Marc Potters. Leverage effect in financial markets: The retarded volatility model. *Physical Review Letters*, 87(22):228701, 2001.

John Y Campbell, Martin Lettau, Burton G Malkiel, and Yexiao Xu. Have individual stocks become more volatile? an empirical exploration of idiosyncratic risk. *The Journal of Finance*, 56(1):1–43, 2001.

Peter Carr and Dilip Madan. Factor models for option pricing. *forthcoming, Asia Pacific Financial Markets*, 2000.

Álvaro Cartea, Ryan Donnelly, and Sebastian Jaimungal. Robust market making. *Available at SSRN 2310645*, 2013.

Peter Christoffersen, Kris Jacobs, and Gregory Vainberg. Forward-looking betas. *manuscript, McGill University*, 2008.

Stefano Ciliberti, Jean-Philippe Bouchaud, and Marc Potters. Smile dynamics: A theory of the implied leverage effect. *Wilmott Journal*, 1(2):87–94, 2009.

Rama Cont, José Da Fonseca, et al. Dynamics of implied volatility surfaces. *Quantitative finance*, 2(1):45–60, 2002.

Pietro Fodra and Mauricio Labadie. High-frequency market-making with inventory constraints and directional bets. *arXiv preprint arXiv:1206.4810*, 2012.

Pietro Fodra and Mauricio Labadie. High-frequency market-making for multi-dimensional markov processes. *arXiv preprint arXiv:1303.7177*, 2013.

Martin Forde and Antoine Jacquier. Small-time asymptotics for implied volatility under the heston model. *International Journal of Theoretical and Applied Finance*, 12(06):861–876, 2009.

Jean-Pierre Fouque and Eli Kollman. Calibration of stock betas from skews of implied volatilities. *Applied Mathematical Finance*, 18(2):119–137, 2011.

Jean-Pierre Fouque and Adam P Tashman. Option pricing under a stressed-beta model. *Annals of Finance*, 8(2-3):183–203, 2012.

José Carlos Garcıa Franco. Maximum likelihood estimation of mean reverting processes. *Real Options Practice*, 2003.

Xavier Gabaix, Parameswaran Gopikrishnan, Vasiliki Plerou, and H Eugene Stanley. Institutional investors and stock market volatility. 2005.

Don UA Galagedera. An alternative perspective on the relationship between downside beta and capm beta. *Emerging Markets Review*, 8(1):4–19, 2007.

Jim Gatheral. *The volatility surface: a practitioner's guide*, volume 357. John Wiley & Sons, 2006.

Parameswaran Gopikrishnan, Vasiliki Plerou, Xavier Gabaix, and H Eugene Stanley. Statistical properties of share volume traded in financial markets. *Physical Review E*, 62(4):R4493, 2000.

Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Optimal portfolio liquidation with limit orders. *SIAM Journal on Financial Mathematics*, 3(1):740–764, 2012.

Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Dealing with the inventory risk: a solution to the market making problem. *Mathematics and financial economics*, 7(4):477–507, 2013.

Patrick S Hagan, Deep Kumar, Andrew S Lesniewski, and Diana E Woodward. Managing smile risk. *Wilmott Magazine*, page 249, 2002.

Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.

Thomas Ho and Hans R Stoll. Optimal dealer pricing under transactions and return uncertainty. *Journal of Financial economics*, 9(1):47–73, 1981.

Fouque Jean-Pierre, George Papanicolaou, and K Ronnie Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, 2000.

Rudra P Jena and Peter Tankov. Arbitrage opportunities in misspecified stochastic volatility models. *SIAM Journal on Financial Mathematics*, 2(1):317–341, 2011.

Colm Kearney and Valerio Potì. Have european stocks become more volatile? an empirical investigation of idiosyncratic and market risk in the euro area. *European Financial Management*, 14(3):419–444, 2008.

Andrew W Lo, A Craig MacKinlay, and June Zhang. Econometric models of limit-order executions. *Journal of Financial Economics*, 65(1):31–71, 2002.

Harry Markowitz. Portfolio selection*. *The journal of finance*, 7(1):77–91, 1952.

Sergei Maslov and Mark Mills. Price fluctuations from the order book perspective - empirical facts and a simple model. *Physica A: Statistical Mechanics and its Applications*, 299(1):234–246, 2001.

Kaj Nyström, Ould Aly, Sidi Mohamed, and Changyong Zhang. Market making and portfolio liquidation under uncertainty. *International Journal Of Theoretical and Applied Finance*, 17 (5), 2014.

Keiichi Omura, Yasuhiko Tanigawa, and Jun Uno. Execution probability of limit orders on the tokyo stock exchange. *Available at SSRN 252588*, 2000.

George Papanicolaou, Jean-Pierre Fouque, Knut Solna, and Ronnie Sircar. Singular perturbations in option pricing. *SIAM Journal on Applied Mathematics*, 63(5):1648–1665, 2003.

Huyên Pham. *Optimisation et contrôle stochastique appliqués à la finance*, volume 61. Springer, 2007.

Frank K Reilly and Keith Cates Brown. *Investment analysis and portfolio management.* Cengage Learning, 2011.

Philipp J Schönbucher. A market model for stochastic implied volatility. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 357(1758):2071–2092, 1999.

Sasha Stoikov and Mehmet Sağlam. Option market making under inventory risk. *Review of Derivatives Research*, 12(1):55–79, 2009.

Vincent Vargas, Tung-Lam Dao, and Jean-Philippe Bouchaud. Skew and implied leverage effect: smile dynamics revisited. *arXiv preprint arXiv:1311.4078*, 2013.

Steven X Wei and Chu Zhang. Why did individual stocks become more volatile?*. *The Journal of Business*, 79(1):259–292, 2006.

J Fred Weston. Investment decisions using the capital asset pricing model. *Financial Management*, pages 25–33, 1973.

Yexiao Xu and Burton G Malkiel. Investigating the behavior of idiosyncratic volatility*. *The Journal of Business*, 76(4):613–645, 2003.