**MASTERS IN MATHEMATICS FOR FINANCE AND DATA**
**ECOLE DES PONTS PARISTECH – UNIVERSITÉ GUSTAVE EIFFEL**
**VOLATILITY MODELING**
**HOMEWORK**
**DUE DATE: FEBRUARY 13, 2023 AT BEGINNING OF CLASS**

## Exercise 1: Dynamics of variance swap volatilities in the Heston model

We consider the Heston model

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - q_t)\,dt + \sqrt{V_t}\,dW_t^S \\ dV_t &= k\left(V^0 - V_t\right)dt + \omega\sqrt{V_t}\,dW_t^V \\ d\langle W^S, W^V\rangle_t &= \rho\,dt\end{aligned}$$

where

- $W^S$, $W^V$ are two correlated standard Brownian motions,
- $k > 0$ is the mean reversion rate,
- $V_0 > 0$ is the initial variance,
- $V^0 > 0$ is the long-term variance,
- $\rho \in [-1, 1]$ is the spot-vol correlation,
- $\omega > 0$ is the volatility of variance.

We denote by $\mathcal{F}_t = \sigma(W_s^S, W_s^V, s \le t)$ the filtration generated by $(W^S, W^V)$. For $t \le u$, we denote by $\xi_t^u = \mathbb{E}[V_u|\mathcal{F}_t]$ the instantaneous forward variances.

(1) Prove that for $t \le u$, $\xi_t^u = V^0 + e^{-k(u-t)}(V_t - V^0)$. (Admit that the process $(\int_0^t e^{ks}\sqrt{V_s}dW_s^V, t \ge 0)$ is a true martingale.)

(2) Deduce that for $t < T$ variance swap variances $\hat{\sigma}_T^2(t) := \mathbb{E}\left[\frac{1}{T-t}\int_t^T V_s\,ds\middle|\mathcal{F}_t\right]$ are given by

$$\hat{\sigma}_T^2(t) = V^0 + \frac{1 - e^{-k(T-t)}}{k(T-t)}(V_t - V^0).$$

(3) Deduce that the dynamics of variance swap volatilities reads as:

$$d\hat{\sigma}_T(t) = \cdots dt + \frac{\omega}{2}\frac{1 - e^{-k(T-t)}}{k(T-t)}\frac{\sqrt{V_t}}{\hat{\sigma}_T(t)}\,dW_t^V$$

where $\cdots dt$ denotes some drift term that we do not ask you to compute.
This proves that, for a flat initial term-structure of variance swap variances ($V_t = V^0$), for long maturities $T - t \gg \frac{1}{k}$, the instantaneous volatility of $\hat{\sigma}_T(t)$ decays like $\frac{1}{T-t}$. This is not in line with the power-law decay $\frac{1}{(T-t)^\alpha}$ with $\alpha \approx 0.35$ that we observe in equity markets.

## Exercise 2: One-factor lognormal forward instantaneous variance models

We consider one-factor lognormal forward instantaneous variance models of the form

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - q_t)\,dt + \sqrt{\xi_t^t}\,dW_t \\ \xi_t^u &= \xi_0^u \exp\left(\int_0^t K(u-s)\,dZ_s - \frac{1}{2}\int_0^t K(u-s)^2\,ds\right), \quad u \geq t \\ d\langle W, Z\rangle_t &= \rho\,dt\end{aligned}$$

where $W$, $Z$ are two correlated standard Brownian motions and $r_t$, $q_t$ and the kernel $K : \mathbb{R}_+ \to \mathbb{R}_+$ are deterministic. We denote by $V_t := \xi_t^t$ the instantaneous (lognormal) variance of $S_t$.

(1) **One-factor Bergomi model.** Assume that $K(\theta) = \omega \exp(-k\theta)$ with $\omega \geq 0$, $k > 0$.

(a) Prove that $\xi_t^u$ admits the one-dimensional Markov representation: $\xi_t^u = \xi_0^u f^u(t, X_t)$ with

$$f^u(t,x) = \exp\left(\omega e^{-k(u-t)}x - \frac{\omega^2}{2}e^{-2k(u-t)}\mathrm{Var}(X_t)\right), \quad \mathrm{Var}(X_t) = \frac{1-e^{-2kt}}{2k}$$

where the Ornstein-Uhlenbeck process $X_t := \int_0^t e^{-k(t-s)}dZ_s$ satisfies the Markov dynamics

$$dX_t = -kX_t\,dt + dZ_t, \quad X_0 = 0.$$

(b) (One-factor Bergomi model written as a traditional one-factor stochastic volatility model.) We assume that $u \mapsto \xi_0^u$ is differentiable. Write the dynamics of $V_t$, i.e., find $\beta_t$ and $\nu_t$ such that $dV_t = \beta_t dt + \nu_t dZ_t$. How does the dynamics of $V_t$ compare with the dynamics of the instantaneous variance in the Heston model?

(2) **Rough Bergomi model.** Assume that $K(\theta) = \nu\theta^{H-\frac{1}{2}}$ with $\nu > 0$, $H \in \left(0, \frac{1}{2}\right)$.

(a) Prove that

$$\xi_t^u = \xi_0^u \exp\left(\nu X_t^u - \frac{\nu^2}{2}\mathrm{Var}(X_t^u)\right)$$

where

$$X_t^u = \int_0^t (u-s)^{H-\frac{1}{2}}\,dZ_s, \qquad \mathrm{Var}(X_t^u) = \frac{u^{2H} - (u-t)^{2H}}{2H}.$$

Note that this is not a Markov representation as to simulate all the $\xi_t^u$ there are as many processes $X_t^u$ to simulate as there are forward maturities $u$.

(b) Try to write down the dynamics of $V_t$ in the form of an Itô process $dV_t = \beta_t dt + \nu_t dZ_t$. What do you observe?

(3) **Heston model as a variance curve model.** Using Exercise 1, prove that in the Heston model the dynamics of forward variances reads

$$d\xi_t^u = \omega e^{-k(u-t)}\sqrt{\xi_t^t}\,dW_t^V.$$

Note however that the Heston model cannot accomodate any arbitrary initial term-structure $(\xi_0^u, u \geq 0)$.

## Exercise 3: Two-factor Bergomi model

We consider the two-factor Bergomi model

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - q_t)\,dt + \sqrt{\xi_t^t}\,dW_t \\ \frac{d\xi_t^u}{\xi_t^u} &= \omega\alpha_\theta\left((1-\theta)e^{-k_1(u-t)}dW_t^1 + \theta e^{-k_2(u-t)}dW_t^2\right), \quad u \geq t \\ \alpha_\theta &= \left((1-\theta)^2 + \theta^2 + 2\rho_{12}\theta(1-\theta)\right)^{-1/2}\end{aligned}$$

where $r_t$ and $q_t$ are deterministic, $W, W^1, W^2$ are three correlated standard Brownian motions, $d\langle W^1, W^2\rangle_t = \rho_{12}\,dt$, $\omega \geq 0$, $k_1, k_2 > 0$, and $\theta \in [0,1]$.

(1) Prove that $\xi_t^u$ admits the two-dimensional Markov representation:

$$
\begin{aligned}
\xi_t^u &= \xi_0^u f^u(t, X_t^1, X_t^2) \\
f^u(t, x_1, x_2) &= \exp\left(\omega\alpha_\theta\left((1-\theta)e^{-k_1(u-t)}x_1 + \theta e^{-k_2(u-t)}x_2\right) - \frac{\omega^2}{2}\chi(t,u)\right) \\
\chi(t,u) &= \alpha_\theta^2\left((1-\theta)^2 e^{-2k_1(u-t)}\mathrm{Var}(X_t^1) + \theta^2 e^{-2k_2(u-t)}\mathrm{Var}(X_t^2)\right. \\
&\quad \left. + 2\theta(1-\theta)e^{-(k_1+k_2)(u-t)}\mathrm{Cov}(X_t^1, X_t^2)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
dX_t^i &= -k_i X_t^i\,dt + dW_t^i, \quad X_0^i = 0, \quad i \in \{1,2\} \\
\mathrm{Var}(X_t^i) &= \frac{1-e^{-2k_i t}}{2k_i} \\
\mathrm{Cov}(X_t^1, X_t^2) &= \rho_{12}\frac{1-e^{-(k_1+k_2)t}}{k_1+k_2}.
\end{aligned}
$$

(2) We assume that $u \mapsto \xi_0^u$ is differentiable. Write down the dynamics of the instantaneous variance $V_t = \xi_t^t$. Is $(V_t)$ a Markov process?