Corridor Variance Swap Spread

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Corridor Variance Swap Spread

Two assets:

S_1: typically SPX
S_2: Asian or European index, NKY, HSI, SX5E

Payoff at maturity T:

$$\frac{1}{T}\int_0^T \left[\sigma_{2,t}^2 - \bar{\sigma}_2^2\right] 1_{L \le S_{2,t} \le U}\, dt - \frac{1}{T}\int_0^T \left[\sigma_{1,t}^2 - \bar{\sigma}_1^2\right] 1_{L \le S_{2,t} \le U}\, dt$$

for $0 \le L < U \le \infty$. The fixed volatilities $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are typically chosen to make each leg of the swap spread have zero value at its inception.

The realized variance is accrued only when the level of the corridor index is within a predefined corridor. Vega exposures in specific range of market levels.

Sell Side: Natural hedge for the vega exposure of bank's structured product business, especially products with barriers, such as autocallable notes

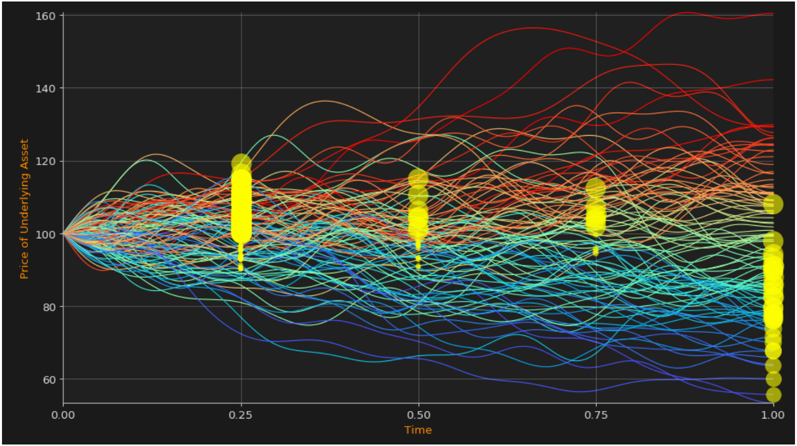
Buy Side: Construct relative value volatility trade in specific ranges of market levels

Autocallable Notes

- Popular structured product in Asian markets
- The investor receives a higher coupon
- Autocalled (the product terminates early) if the underlying hits a set target (typically 100% - 110% of the spot)
- Down-and-in ATM put option with barrier set at 60% - 70% of the spot
- The investor is effectively selling the embedded option to the issuer. The option premium provides the additional yield.
- The issuer has to short other options to hedge the long vega exposure, which puts downward pressure on implied volatilities in the listed options market.
- Digital risks of the barriers.

Autocallable Notes

Autocallable Note Payoff Structure



- Autocall level: 100, observed quarterly
- Coupon: 5% Quarterly, Coupon Barrier: 90
- Conversion Barrier: 75, observed daily

Variance Swap

Payoff at maturity T: $\frac{1}{T}\int_0^T \sigma_t^2\,dt - \bar{\sigma}^2$

The fixed volatility $\bar{\sigma}$ is chosen to make the swap have zero value at its inception.

In the absence of jumps, the payoff of a variance swap can be perfectly replicated by a static portfolio of vanilla options and a dynamic position in the underlying:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad \text{(diffusion process, no jumps)}$$

$$d\log S_t = \frac{1}{S_t}dS_t - \frac{1}{2}\sigma_t^2 dt$$

$$\int_0^T \sigma_t^2\,dt = -2\log\frac{S_T}{S_0} + \int_0^T \frac{2}{S_t}dS_t \qquad \text{(replication formula)}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma_t^2\,dt\right] = \mathbb{E}^{\mathbb{Q}}\left[-2\log\frac{S_T}{S_0}\right] + 2\int_0^T (r_t - q_t)\,dt = \mathbb{E}^{\mathbb{Q}}\left[-2\log\frac{S_T}{F_{0,T}}\right]$$

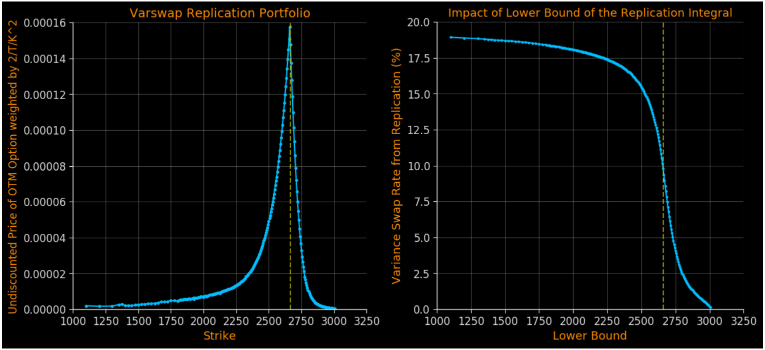
$$= \int_0^{F_{0,T}} \frac{2}{K^2}\mathrm{Put}(K,T)dK + \int_{F_{0,T}}^{\infty} \frac{2}{K^2}\mathrm{Call}(K,T)dK = \int_0^{\infty}\mathrm{OTMF}(K,T)dK$$

where $\mathrm{Call}(K,T) = \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right]$, $\mathrm{Put}(K,T) = \mathbb{E}^{\mathbb{Q}}\left[(K - S_T)^+\right]$ and $\mathrm{OTMF}\,(K,T) = \mathrm{Call}(K,T)$ if $K \geq F_{0,T}$ else $\mathrm{Put}(K,T)$.

Variance Swap Replication

$$\bar{\sigma}^2 = \frac{1}{T}\left[\int_0^{F_{0,T}} \frac{2}{K^2}\mathrm{Put}(K,T)dK + \int_{F_{0,T}}^{\infty} \frac{2}{K^2}\mathrm{Call}(K,T)dK\right]$$

SPX Index, as of 03-20-2018, expiry 04-20-2018, $\bar{\sigma} = 18.9\%$



Variance Swap and Smile Extrapolation

- Options are only traded in a finite range of strikes on the market
- Variance Swap replication is very sensitive to the prices/vols of deep out-of-the-money options, especially puts.
- Market quotes of variance swap rates can be used to select the extrapolation of implied volatilities outside of the range of traded strikes.

Example I - Mixed Lognormals

For a given maturity T and $\mathbb{E}\left[X_T\right] = F$, the risk-neutral density function ϕ of X_T is given by

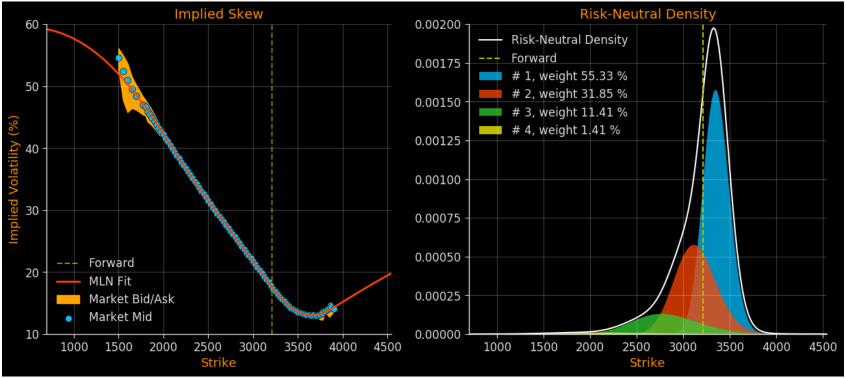
$$\phi(x) = \sum_{i=1}^{n} w_i \phi_i(x), \quad \text{with } \phi_i = \text{pdf of Lognormal} \left(\log(\mu_i F) - \frac{1}{2}\sigma_i^2 T, \sigma_i^2 T\right)$$

$$\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} w_i \mu_i = 1$$

$$\text{Call}(K) = \mathbb{E}\left[(X_T - K)^+\right] = \sum_{i=1}^{n} w_i \text{BlackCall}(K, T, \mu_i F, \sigma_i)$$

Variance Swap and Smile Extrapolation

Calibrate $(w_i, \mu_i, \sigma_i)_{1 \le i \le n}$ to vanilla option smiles.



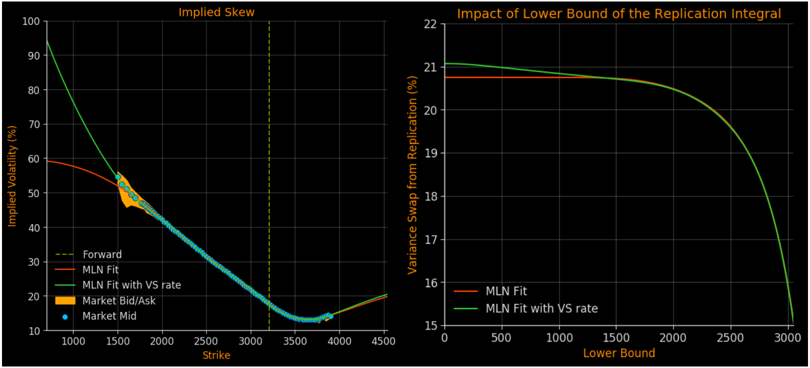
SX5E Index, as of 03-23-2018, expiry 06-15-2018

Var swap rate from mixed lognormal fit	20.75 %
Var swap rate mid market quote	21.07 %

Variance Swap and Smile Extrapolation

Calibrate $(w_i, \mu_i, \sigma_i)_{1 \le i \le n}$ to both vanilla option smiles and var swap rate.

$$\mathrm{VS}^2 = \sum_{i=1}^{n} w_i \left(\sigma_i^2 - \frac{2}{T} \log \mu_i \right)$$

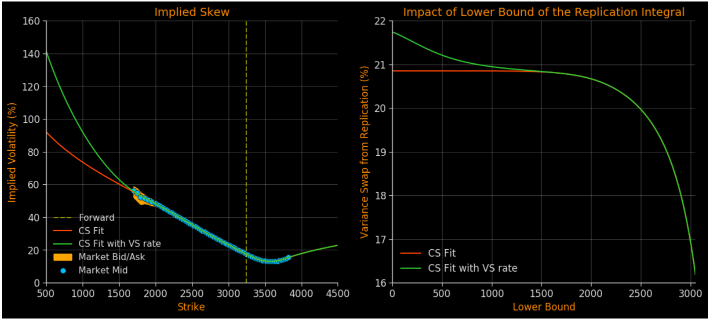


SX5E Index, as of 03-23-2018, expiry 06-15-2018

Variance Swap and Smile Extrapolation

Example II - Cubic Spline Regression

- The total implied variance function $x \mapsto \hat{\sigma}^2(x)T$ where $x = \log(K/F)$ is represented by a cubic spline, best fit to the vanilla smile, and linearly extrapolated for low and high strikes.
- Use var swap rate to find slopes for linear extrapolation at low and high strikes.



SX5E Index, as of 03-23-2018, expiry 06-15-2018

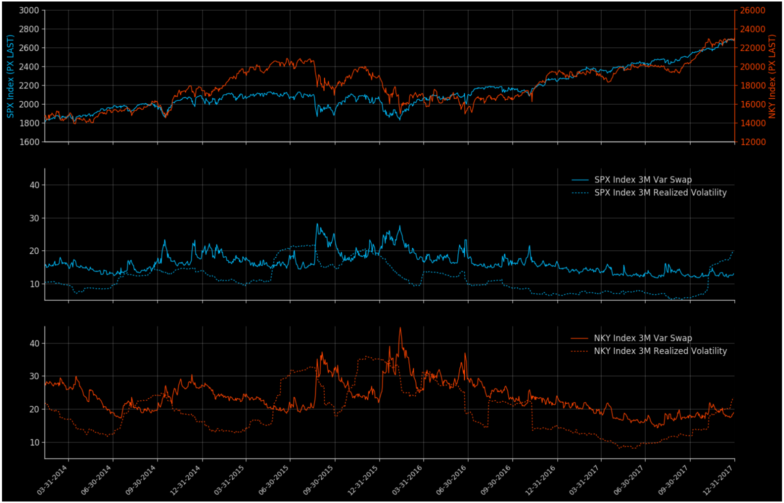
Var swap rate from cubic spline regression fit	20.85 %
Var swap rate mid market quote	21.73 %

Variance Spreads

Relative value volatility trades

- There exists a premium of implied volatility to realized volatility
- Buy variance on index with low premium (such as Asian or European indices) and sell variance on index with high premium (typically SPX)
- More stable P&L

Variance Spreads



Single-Asset Corridor Variance Swap

Payoff at maturity T:

$$\frac{1}{T}\int_0^T \left[\sigma_t^2 - \bar{\sigma}^2\right] 1_{L \le S_t \le U}\, dt, \quad 0 \le L < U \le \infty$$

The fixed volatility $\bar{\sigma}$ is chosen to make the swap have zero value at its inception.

- Range-limited vega exposure
- Excellent alternatives for investors who wish to long volatility and have a range-bound view on the index.

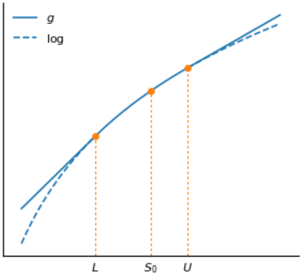
To replicate corridor variance swap, we define

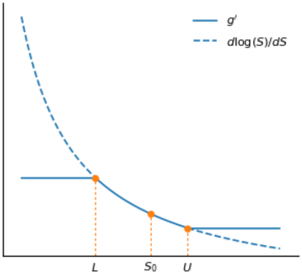
$$g(S) = \begin{cases} \log(L) + \frac{1}{L}(S-L) & \text{if } S < L \\ \log(S) & \text{if } L \le S \le U \\ \log(U) + \frac{1}{U}(S-U) & \text{if } S > U \end{cases}$$

$$g'(S) = \begin{cases} \frac{1}{L} & \text{if } S < L \\ \frac{1}{S} & \text{if } L \le S \le U \\ \frac{1}{U} & \text{if } S > U \end{cases} \quad \text{and} \quad g''(S) = \begin{cases} 0 & \text{if } S < L \\ -\frac{1}{S^2} & \text{if } L \le S \le U \\ 0 & \text{if } S > U \end{cases}$$

Single-Asset Corridor Variance Swap

The function g is equal to log function inside the corridor $[L, U]$ and linearly extrapolated beyond the corridor.





$$dg(S_t) = g'(S_t)dS_t + \frac{1}{2}g''(S_t)S_t^2\sigma_t^2 dt = g'(S_t)dS_t - \frac{1}{2}\sigma_t^2 1_{L \le S_t \le U} dt$$

Single-Asset Corridor Variance Swap

$$\int_0^T \sigma_t^2 1_{L\le S_t\le U}\,dt = \int_0^T 2g'(S_t)dS_t - 2\left[g(S_T) - g(S_0)\right]$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T 2g'(S_t)dS_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T 2\left(\frac{S_t}{L}1_{S_t<L} + 1_{L\le S_t\le U} + \frac{S_t}{U}1_{S_t>U}\right)(r_t - q_t)\,dt\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T 2\left(-\frac{(L-S_t)^+}{L} + 1 + \frac{(S_t-U)^+}{U}\right)(r_t - q_t)\,dt\right] \\
&= \int_0^T \left[-\frac{2}{L}\mathrm{Put}(L,t) + \frac{2}{U}\mathrm{Call}(U,t)\right](r_t - q_t)\,dt + 2\int_0^T (r_t - q_t)\,dt
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma_t^2 1_{L\le S_t\le U}\,dt\right] &= \int_0^T \left[-\frac{2}{L}\mathrm{Put}(L,t) + \frac{2}{U}\mathrm{Call}(U,t)\right](r_t - q_t)\,dt \\
&+ \int_L^U \frac{2}{K^2}\mathrm{OTMF}\left(K,T\right)dK - 2\left(g\left(F_{0,T}\right) - g\left(S_0\right) - \int_0^T (r_t - q_t)\,dt\right)
\end{aligned}$$

Note that the last term in the right-hand side of the equation vanishes if $L \le S_0 \le U$.

The payoff of a corridor variance swap can be replicated by vanilla options with strikes within the corridor at the final maturity, a continuous stream of vanilla options of intermediate maturities, and a dynamic portfolio in the underlying S.

Two-Asset Corridor Variance Swap

Two legs of a corridor variance swap spread:

Leg 1:	$\frac{1}{T}\int_0^T \left[\sigma_{2,t}^2 - \bar{\sigma}_2^2\right] 1_{L \le S_{2,t} \le U}\, dt$	Single-Asset Corridor Var Swap
Leg 2:	$\frac{1}{T}\int_0^T \left[\sigma_{1,t}^2 - \bar{\sigma}_1^2\right] 1_{L \le S_{2,t} \le U}\, dt$	Two-Asset Corridor Var Swap

The challenge of valuation/risk management mainly comes from the two-asset leg.

Notations

$$\mathrm{VS}_i^2(T) = \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma_{i,t}^2\, dt\right] \qquad \text{Vanilla var swap on } i\text{-th asset}$$

$$\mathrm{VS}_{i,j}^2(L,U,T) = \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma_{i,t}^2 1_{L \le S_{j,t} \le U}\, dt\right] \qquad \text{Two-asset corridor var swap}$$

$$\mathrm{VS}_i^2\,(L,U,T) = \mathrm{VS}_{i,i}^2\,(L,U,T) \qquad \text{Single-asset corridor var swap}$$

Consistency Conditions

1. As $S_2 \to S_1$, $\mathrm{VS}_{1,2}(L,U,T) \to \mathrm{VS}_1(L,U,T)$
2. As $L \downarrow 0$ and $U \uparrow \infty$, $\mathrm{VS}_{1,2}(L,U,T) \uparrow \mathrm{VS}_1(T)$
3. As $U \downarrow L$, $\mathrm{VS}_{1,2}(L,U,T) \downarrow 0$

Two-Asset Corridor Variance Swap - Cross Local Variance

Question: What is the risk exposure of a two-asset corridor var swap?

$$\mathbb{E}\left[\int_0^T \sigma_{1,t}^2 1_{L\leq S_{2,t}\leq U} dt\right] = \int_0^T \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 | S_{2,t}\right] 1_{L\leq S_{2,t}\leq U}\right] dt$$

The key is to evaluate the *cross local variance*:

$$\boxed{\mathbb{E}\left[\left.\sigma_{1,t}^2\right| S_{2,t}\right]}$$

Two simple cases:

1. S_1 and S_2 are identical:

$$\mathbb{E}\left[\left.\sigma_{1,t}^2\right| S_{2,t}\right] = \sigma_{1,\mathrm{loc}}^2\left(t, S_{1,t}\right)$$

2. S_1 and S_2 are independent:

$$\mathbb{E}\left[\left.\sigma_{1,t}^2\right| S_{2,t}\right] = \mathbb{E}\left[\sigma_{1,t}^2\right] = \mathbb{E}\left[\sigma_{1,\mathrm{loc}}^2\left(t, S_{1,t}\right)\right] \quad \text{(const)}$$

Two-Asset Corridor Variance Swap - Cross Local Variance

As difference of two down-variance payoffs (lower bound=0):

$$\mathbb{E}\left[\int_0^T \sigma_{1,t}^2 1_{L\le S_{2,t}\le U}\,dt\right] = \mathbb{E}\left[\int_0^T \sigma_{1,t}^2 1_{S_{2,t}\le U}\,dt\right] - \mathbb{E}\left[\int_0^T \sigma_{1,t}^2 1_{S_{2,t}\le L}\,dt\right]$$
$$=: V_{1,2}(U,T) - V_{1,2}(L,T)$$

where $V_{1,2}(K,T)$ is the undiscounted value of the down-variance payoff:

$$V_{1,2}(K,T) = \int_0^T \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 | S_{2,t}\right] 1_{S_{2,t}\le K}\right] dt$$

$$\frac{\partial V_{1,2}}{\partial K} = \int_0^T \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 | S_{2,t}\right]\delta\left(S_{2,t}-K\right)\right] dt = \int_0^T \mathbb{E}\left[\sigma_{1,t}^2\, | S_{2,t} = K\,\right] \frac{\partial^2 C_2}{\partial K^2}(K,t)dt$$

$$\mathbb{E}\left[\sigma_{1,T}^2 \middle| S_{2,T} = K\right] = \frac{\frac{\partial^2 V_{1,2}}{\partial T \partial K}}{\frac{\partial^2 C_2}{\partial K^2}} \quad \text{where } C_2(K,T) = \mathbb{E}\left[\left(S_{2,T}-K\right)^+\right].$$

The cross local variance $\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t} = K\right]$ can be locked in by trading two-asset corridor variance swaps.

The similar analysis can be applied to single-asset corridor variance swaps where we can lock in the local variance $\sigma_{\mathrm{loc}}^2(t,S_t) = \mathbb{E}\left[\sigma_t^2 | S_t\right]$.

Two-Asset Corridor Variance Swap LV + LV

Basket Local Volatility Model (LV+LV)

$$\frac{dS_{1,t}}{S_{1,t}} = (r_{1,t} - q_{1,t})\,dt + \sigma_{1,\text{loc}}(t, S_{1,t})\,dW^S_{1,t}$$

$$\frac{dS_{2,t}}{S_{2,t}} = (r_{2,t} - q_{2,t})\,dt + \sigma_{2,\text{loc}}(t, S_{2,t})\,dW^S_{2,t}$$

$$dW^S_{1,t}dW^S_{2,t} = \rho^{S_1 S_2}_t\,dt$$

$$\text{VS}^2_{1,2}(L, U; \text{LV+LV}) = \frac{1}{T}\int_0^T \mathbb{E}\left[\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_{1,t}) \middle| S_{2,t}\right] 1_{L \le S_{2,t} \le U}\right] dt$$

Pros

- Simple multi-asset model, consistent with vanilla smile of each asset
- The correlation ρ_t can be calibrated to the market prices of basket options or other products with exposure to S_1 v.s. S_2 correlation.

Cons: What about exotic risks such as volatility-of-volatility risk, (cross) spot/volatility correlation risk?

Two-Asset Corridor Variance Swap SLV + LV

One-Factor SLV Model + Local Vol Model (SLV+LV)

$$\begin{aligned}
\frac{dS_{1,t}}{S_{1,t}} &= (r_{1,t} - q_{1,t})\, dt + \sigma_{1,\text{loc}}\,(t, S_{1,t})\, \frac{\sqrt{V_{1,t}}}{\sqrt{\mathbb{E}\left[\, V_{1,t} \middle|\, S_{1,t}\right]}} dW_{1,t}^{S} \\
dV_{1,t} &= a_1(t, V_{1,t})dt + b_1(t, V_t)dW_{1,t}^{V} \\
\frac{dS_{2,t}}{S_{2,t}} &= (r_{2,t} - q_{2,t})\, dt + \sigma_{2,\text{loc}}\,(t, S_{2,t})\, dW_{2,t}^{S}
\end{aligned}$$

where

$$dW_{1,t}^{S} dW_{2,t}^{S} = \rho_t^{S_1 S_2}\, dt, \quad dW_{1,t}^{S} dW_{1,t}^{V} = \rho_t^{S_1 V_1}\, dt, \quad dW_{2,t}^{S} dW_{1,t}^{V} = \rho_t^{S_2 V_1}\, dt.$$

Three correlations need to be specified and they must satisfy the positive-definiteness constraint:

$$1 - \left(\rho_t^{S_1 S_2}\right)^2 - \left(\rho_t^{S_1 V_1}\right)^2 - \left(\rho_t^{S_2 V_1}\right)^2 + 2\rho_t^{S_1 V_1} \rho_t^{S_1 S_2} \rho_t^{S_2 V_1} > 0 \qquad \text{(D)}$$

$$\begin{tikzcd}
S_1 \arrow[r, dash] \arrow[d, dash] & S_2 \arrow[dl, dash] \\
V_1 &
\end{tikzcd}$$

The SDE for $S_{1,t}$ is a McKean SDE and can be efficiently simulated by particle method [GL].

We assume that the two models (SLV+LV and LV+LV) have the same local vol functions $\sigma_{1,\text{loc}}$ and $\sigma_{2,\text{loc}}$, and the same spot/spot correlation $\rho_t^{S_1 S_2}$.

We shall compare the following three functions:

1. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_{1,t})\,\frac{V_{1,t}}{\mathbb{E}[V_{1,t}|S_{1,t}]}\,\middle|\, S_{2,t}\right]$ in SLV + LV
2. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_{1,t})\,\middle|\, S_{2,t}\right]$ in SLV + LV
3. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_{1,t})\,\middle|\, S_{2,t}\right]$ in LV + LV

Note that (2) and (3) are not the same in general even if $S_{1,t}$ and $S_{2,t}$ have the same marginals and the same correlation $\rho_t^{S_1 S_2}$ drives $W^S_{1,t}$ and $W^S_{2,t}$, the joint distribution of $S_{1,t}$ and $S_{2,t}$ are not the same in SLV+LV and LV+LV due to stochastic volatility $V_{1,t}$ and cross spot-vol correlation $\rho^{S_2 V_1}$ in SLV+LV.

Example 1.

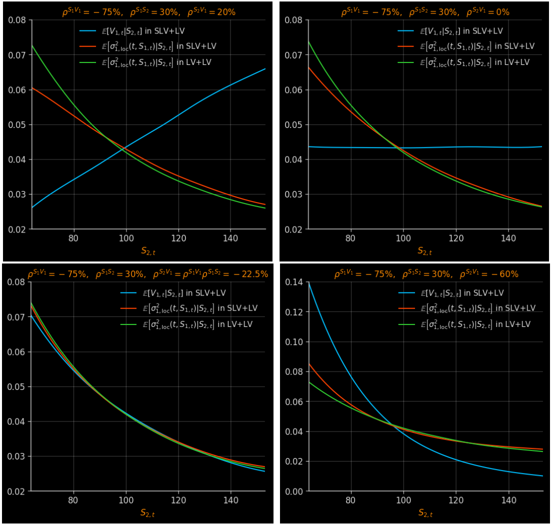
$$(\text{SLV+LV}) \begin{cases} \dfrac{dS_{1,t}}{S_{1,t}} = \sqrt{V_{1,t}} dW^S_{1,t} \\ dV_{1,t} = \kappa_1 \left(\theta_1 - V_{1,t}\right) dt + \omega_1 \sqrt{V_{1,t}} dW^V_{1,t} \\ \dfrac{dS_{2,t}}{S_{2,t}} = \sigma_{2,\text{loc}}\left(t, S_{2,t}\right) dW^S_{2,t} \end{cases} \quad \text{with} \quad \begin{aligned} dW^S_{1,t} dW^V_{1,t} &= \rho^{S_1 V_1} dt \\ dW^S_{1,t} dW^S_{2,t} &= \rho^{S_1 S_2} dt \\ dW^S_{2,t} dW^V_{1,t} &= \rho^{S_2 V_1} dt \end{aligned}$$

$$\text{LV+LV} \begin{cases} \dfrac{dS_{1,t}}{S_{1,t}} = \sigma_{1,\text{loc}}\left(t, S_{1,t}\right) dW^S_{1,t} \\ \dfrac{dS_{2,t}}{S_{2,t}} = \sigma_{2,\text{loc}}\left(t, S_{2,t}\right) dW^S_{2,t} \end{cases} \quad \text{with} \quad dW^S_{1,t} dW^S_{2,t} = \rho^{S_1 S_2} dt$$

Numerical values of model parameters:

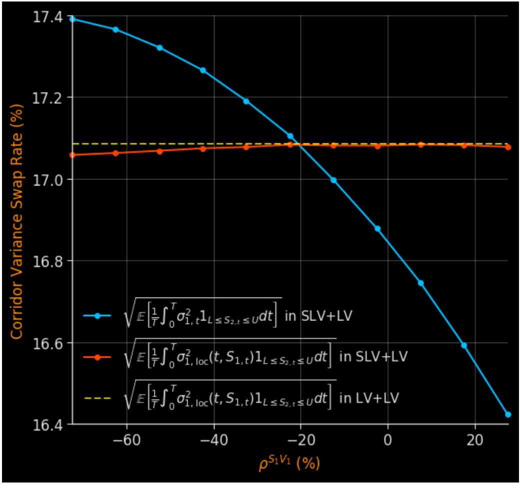
$S_{1,0} = 100,\ S_{2,0} = 100,\ \rho^{S_1 S_2} = 0.3$
$V_{1,0} = 0.02,\ \kappa_1 = 3,\ \theta_1 = 0.05,\ \omega_1 = 0.5,\ \rho^{S_1 V_1} = -0.75$
$\sigma_{1,\text{loc}}$ is the local vol surface in Heston model $\left(V_0 = V_{1,0}, \kappa = \kappa_1, \theta = \theta_1, \omega = \omega_1, \rho = \rho^{S_1 V_1}\right)$
$\sigma_{2,\text{loc}} \equiv 0.2$
The positive definiteness constraints (D) gives $-0.856 \le \rho^{S_2 V_1} \le 0.406$
$\rho^{S_2 V_1}_* = \rho^{S_1 S_2} \rho^{S_1 V_1} = -0.225.$

Two-Asset Corridor Variance Swap SLV+LV vs LV+LV



$t = 0.5$

Two-Asset Corridor Variance Swap SLV+LV vs LV+LV



Two-Asset Corridor Variance Swap SLV+LV vs LV+LV

Question: In SLV + LV, when do we have

$$\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right] = \mathbb{E}\left[\sigma_{1,\text{loc}}^2\left(t, S_{1,t}\right) \middle| S_{2,t}\right]?$$

One sufficient condition that makes the above equality hold is

$$\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{1,t}, S_{2,t}\right] = \mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{1,t}\right] \tag{A}$$

Assume (A) holds. Then

$$\begin{aligned}\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right] &= \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{1,t}, S_{2,t}\right] \middle| S_{2,t}\right] = \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{1,t}\right] \middle| S_{2,t}\right] \\ &= \mathbb{E}\left[\sigma_{1,\text{loc}}^2\left(t, S_{1,t}\right) \middle| S_{2,t}\right]\end{aligned}$$

The assumption (A) says that given $S_{1,t}$, the other variable $S_{2,t}$ provides no additional information in estimating $\sigma_{1,t}^2$.

In other words, the dependence of $\sigma_{1,t}^2$ on $S_{2,t}$ is only through the dependence of $\sigma_{1,t}^2$ on $S_{1,t}$ and then the dependence of $S_{1,t}$ on $S_{2,t}$.

Let $\left(W_t^1, W_t^2, W_t^3\right)$ be the 3-dimensional uncorrelated standard Brownian motion. In the SLV+LV model, if we let

$$\left.\begin{aligned} dW_{1,t}^S &= dW_t^1 \\ dW_{2,t}^S &= \rho_t^{S_1 S_2} dW_t^1 + \sqrt{1-\left(\rho_t^{S_1 S_2}\right)^2}\, dW_t^2 \\ dW_{1,t}^V &= \rho_t^{S_1 V_1} dW_t^1 + \sqrt{1-\left(\rho_t^{S_1 V_1}\right)^2}\, dW_t^3 \end{aligned}\right\} \implies \quad dW_{2,t}^S dW_{1,t}^V = \rho_t^{S_1 S_2} \rho_t^{S_1 V_1} dt.$$

If we choose

$$\rho_t^{S_2 V_1} = \rho_t^{S_1 S_2} \rho_t^{S_1 V_1} \tag{C}$$

Numerical results show that

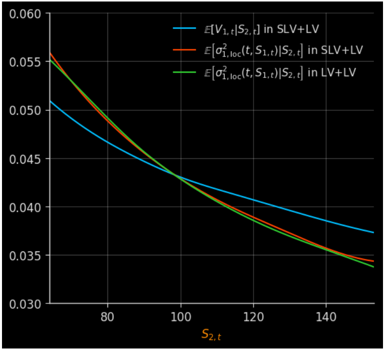
$$\mathbb{E}\left[\sigma_{1,\mathrm{loc}}^2\left(t, S_{1,t}\right) \middle| S_{2,t}\right] \text{ in SLV + LV } \approx \mathbb{E}\left[\sigma_{1,\mathrm{loc}}^2\left(t, S_{1,t}\right) \middle| S_{2,t}\right] \text{ in LV + LV}$$

However, $\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right]$ in SLV+LV may differ even if (C) holds.

Example 1a.

Same model parameters as in Example 1, except that the correlations now are time-varying:

$$\rho_t^{S_1 V_1} = -0.75, \quad \rho_t^{S_1 S_2} = 0.3 \text{ if t} \leq 0.25 \text{ else } 0, \quad \rho_t^{S_2 V_1} = \rho_t^{S_1 S_2} \rho_t^{S_1 V_1}$$



$\mathbb{E}\left[V_{1,t}|S_{2,t}\right]$ in SLV+LV differs markedly from $\mathbb{E}\left[\sigma^2_{1,\mathrm{loc}}\left(t, S_{1,t}\right)\middle| S_{2,t}\right]$ in SLV+LV and LV+LV.

Two-Asset Corridor Variance Swap SLV+LV vs LV+LV

Example 2.

$$\text{SLV} + \text{LV} \quad \begin{cases} \dfrac{dS_{1,t}}{S_{1,t}} = \bar{\sigma}_1 \dfrac{\sqrt{V_{1,t}}}{\sqrt{\mathbb{E}\left[V_{1,t}|S_{1,t}\right]}} \, dW_{1,t}^S \\ dV_{1,t} = \kappa\left(\theta - V_{1,t}\right) dt + \omega \sqrt{V_{1,t}} \, dW_{1,t}^V \\ \dfrac{dS_{2,t}}{S_{2,t}} = \bar{\sigma}_2 \, dW_{2,t}^S \end{cases} \quad \text{with} \quad \begin{array}{l} dW_{1,t}^S dW_{2,t}^S = \rho^{S_1 S_2} \, dt \\ dW_{1,t}^S dW_{1,t}^V = \rho^{S_1 V_1} \, dt \\ dW_{2,t}^S dW_{1,t}^V = \rho^{S_2 V_1} \, dt \end{array}$$

Here $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are constants. In particular, the corresponding local vol functions are constants: $\sigma_{1,\text{loc}}\left(t, S_{1,t}\right) = \bar{\sigma}_1$ and $\sigma_{2,\text{loc}}\left(t, S_{2,t}\right) = \bar{\sigma}_2$.

$$\text{LV} + \text{LV} \quad \begin{cases} \dfrac{dS_{1,t}}{S_{1,t}} = \bar{\sigma}_1 \, dW_{1,t}^S \\ \dfrac{dS_{2,t}}{S_{2,t}} = \bar{\sigma}_2 \, dW_{2,t}^S \end{cases} \quad \text{with} \quad dW_{1,t}^S dW_{2,t}^S = \tilde{\rho}_t^{S_1 S_2} \, dt.$$

Since $\bar{\sigma}_1$ is constant, the correlation $\tilde{\rho}_t^{S_1 S_2}$ has no impact on $\text{VS}_{1,2}(L, U, T)$ in LV+LV.

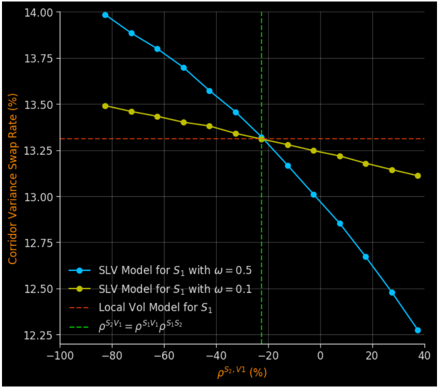
Two-Asset Corridor Variance Swap SLV+LV vs LV+LV

Numerical Values of model parameters:

$$\bar{\sigma}_1 = 0.15,\ \bar{\sigma}_2 = 0.2,\ S_{1,0} = 100,\ S_{2,0} = 100, L = 70,\ U = 110,\ T = 1$$

$$V_0 = 0.0165,\ \kappa = 3,\ \theta = 0.05,\ \omega = 0.5 \text{ and } 0.1, \rho^{S_1V_1} = -0.75,\ \rho^{S_1S_2} = 0.3$$

LV+LV is unable to capture the vol-of-vol risk, cross spot/vol correlation risk.



Two-Asset Corridor Variance Swap - Approximation I

Postulate:

$$\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right] \approx \alpha_t + \beta_t \sigma_{2,\mathrm{loc}}^2\left(t, S_{2,t}\right) \tag{PI}$$

Motivation: if $\sigma_{1,t}^2 = \alpha_t + \beta_t \sigma_{2,t}^2 + \epsilon_t$ and $\mathbb{E}\left[\epsilon_t | S_{2,t}\right] = 0$, then the postulate holds.

$$\begin{aligned}\mathrm{VS}_{1,2}^2(L,U,T) &= \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma_{1,t}^2 1_{L \le S_{2,t} \le U}\, dt\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right] 1_{L \le S_{2,t} \le U}\right] dt \\ &\approx \frac{1}{T}\int_0^T \alpha_t \mathbb{E}\left[1_{L \le S_{2,t} \le U}\right] dt + \frac{1}{T}\int_0^T \beta_t \mathbb{E}\left[\sigma_{2,\mathrm{loc}}^2(t, S_{2,t}) 1_{L \le S_{2,t} \le U}\right] dt\end{aligned}$$

If $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$ are both constants, then the formula reduces to

$$\mathrm{VS}_{1,2}^2(L,U,T) \approx \alpha\, \mathrm{RA}_2(L,U,T) + \beta\, \mathrm{VS}_2^2\left(L,U,T\right)$$

The parameters α and β can be estimated from historical data (regressing $\sigma_{1,t}^2$ on $\sigma_{2,t}^2$) or calibrated to the current market prices of relevant instruments.

Two-Asset Corridor Variance Swap - Approximation I

For a given set of maturities $T_1 < \cdots < T_n$, we assume α_t and β_t take constant values between each T_i and T_{i+1}.

Then we can calibrate α_t and β_t, if for each T_i, there exists a set of lower and upper bounds (L_i, U_i) such that the prices of the following instruments are available:

1. Vanilla var swaps, $\mathrm{VS}_1(T_i)$ and $\mathrm{VS}_2(T_i)$
2. Range accural and single-asset corridor var swap on S_2, $\mathrm{RA}_2\,(L_i, U_i, T_i)$ and $\mathrm{VS}_2\,(L_i, U_i, T)$
3. Two-asset corridor var swap $\mathrm{VS}_{1,2}(L_i, U_i, T)$

Note that $\mathrm{VS}_1(T_i)$, $\mathrm{VS}_2(T_i)$, $\mathrm{RA}_2\,(L_i, U_i, T_i)$ and $\mathrm{VS}_2\,(L_i, U_i, T)$ can be easily computed (say, by replication) from an implied vol surface or local volatility model calibrated to the market prices of vanilla call and put options.

Now let $T = T_1$ and $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$ for $0 \le t \le T$, then we have

$$\mathrm{VS}^2_{1,2}\,(L, U, T) = \alpha \mathrm{RA}_2(L, U, T) + \beta \mathrm{VS}^2_2(L, U, T) \tag{A1}$$

Letting $L = 0$ and $U = \infty$ gives another constraint

$$\mathrm{VS}^2_1(T) = \alpha + \beta \mathrm{VS}^2_2(T) \tag{A2}$$

We can solve equations (A1) and (A2) to obtain α and β.
The values of α and β at further maturities can be obtained by bootstrap.

Two-Asset Corridor Variance Swap - Approximation I

Example 3. How good is the approximation?

$$
\begin{aligned}
\frac{dS_{1,t}}{S_{1,t}} &= \sqrt{V_{1,t}}\,dW^S_{1,t} \\
dV_{1,t} &= \kappa_1\left(\theta_1 - V_{1,t}\right)dt + \omega_1\sqrt{V_{1,t}}\,dW^V_{1,t} \\
\frac{dS_{2,t}}{S_{2,t}} &= \sigma_{2,\mathrm{loc}}\left(t, S_{2,t}\right)dW^S_{2,t}
\end{aligned}
\quad \text{with} \quad
\begin{aligned}
dW^S_{1,t}dW^V_{1,t} &= \rho^{S_1V_1}dt \\
dW^S_{1,t}dW^S_{2,t} &= \rho^{S_1S_2}dt \\
dW^S_{2,t}dW^V_{1,t} &= \rho^{S_2V_1}dt
\end{aligned}
$$

Numerical values of model parameters:

$S_{1,0} = 100,\ S_{2,0} = 100,\ \rho^{S_1S_2} = 0.3,\ \rho^{S_2V_1} = -0.5$
$V_{1,0} = 0.02,\ \kappa_1 = 3,\ \theta_1 = 0.05,\ \omega_1 = 0.5,\ \rho^{S_1V_1} = -0.75$
$\sigma_{2,\mathrm{loc}}$ is the local vol surface in another Heston model with parameters $V_0 = 0.04$, $\kappa = 1$, $\theta = 0.06$, $\omega = 0.25$ and $\rho = -0.85$

For a list of maturities $T_i = \frac{i}{12}$, $i = 1, \cdots, 12$, and corridor $L = 70$ and $U = 110$, the above model is used to compute the values of $\mathrm{VS}_1(T_i)$, $\mathrm{VS}_2(T_i)$, $\mathrm{RA}_2(L, U, T_i)$, $\mathrm{VS}_2(L, U, T_i)$ and $\mathrm{VS}_{1,2}(L, U, T_i)$. Then we calibrate piece-wise constants α_t and β_t to these values by bootstrapping.

Two-Asset Corridor Variance Swap - Approximation I

We apply the approximation formula with calibrated α_t and β_t to other corridors.

$\mathrm{VS}_{1,2}\left(L, U, T\right)$ from simulation

L/U	56/88	63/99	70/110	77/121	84/132
$T = 0.25$	5.67	12.63	16.67	16.88	16.53
$T = 0.50$	8.41	14.00	17.56	17.98	17.29
$T = 0.75$	9.81	14.62	17.82	18.36	17.61
$T = 1.00$	10.61	14.88	17.81	18.43	17.72

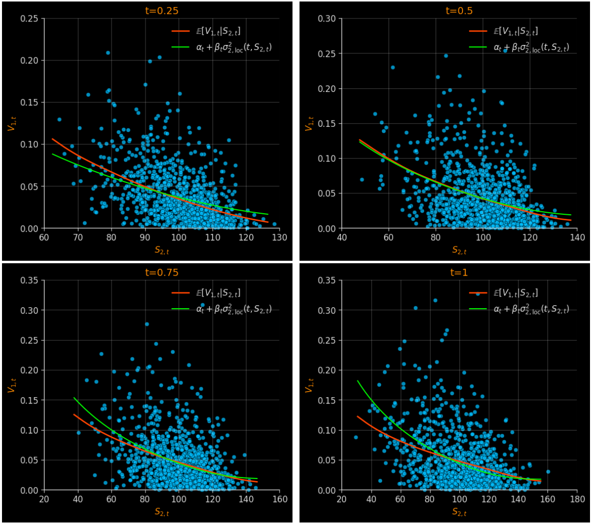
$\mathrm{VS}_{1,2}\left(L, U, T\right)$ from approximation

L/U	56/88	63/99	70/110	77/121	84/132
$T = 0.25$	5.66	12.50	16.67	16.89	16.53
$T = 0.50$	8.55	14.03	17.56	17.93	17.22
$T = 0.75$	10.10	14.75	17.82	18.24	17.43
1.00	11.03	15.10	17.81	18.23	17.42

approximation $-$ simulation

L/U	56/88	63/99	70/110	77/121	84/132
$T = 0.25$	-0.01	-0.13	0.0	0.00	0.00
$T = 0.50$	0.14	0.03	0.0	-0.05	-0.07
$T = 0.75$	0.29	0.13	0.0	-0.12	-0.17
$T = 1.00$	0.43	0.21	0.0	-0.19	-0.30

Two-Asset Corridor Variance Swap - Approximation I



Two-Asset Corridor Variance Swap - Approximation I

The accuracy can be improved with more market quotes. For example, if there are market quotes for two sets of corridors (L, U) and $(\tilde{L}, \tilde{U})$ (typically (70%, 110%) and (70%,130%) of the spot), we can modify the postulate (PI) as follows:

$$\mathbb{E}\left[\sigma_{1,t}^2 \middle| S_{2,t}\right] \approx \alpha_t + \left(\beta_t + \gamma_t \frac{S_{2,t}}{S_{2,0}}\right) \sigma_{2,\mathrm{loc}}^2\left(t, S_{2,t}\right) \tag{PIa}$$

If $\alpha_t \equiv \alpha$, $\beta_t \equiv \beta$ and $\gamma_t \equiv \gamma$, for $0 \leq t \leq T$, then

$$\begin{aligned}
\mathrm{VS}_{1,2}^2(L,U,T) &= \alpha \mathrm{RA}_2(L,U,T) + \beta \mathrm{VS}_2^2\left(L,U,T\right) + \gamma \mathrm{GS}_2^2\left(L,U,T\right) \\
\mathrm{VS}_{1,2}^2(\tilde{L},\tilde{U},T) &= \alpha \mathrm{RA}_2(\tilde{L},\tilde{U},T) + \beta \mathrm{VS}_2^2\left(\tilde{L},\tilde{U},T\right) + \gamma \mathrm{GS}_2^2\left(L,U,T\right) \\
\mathrm{VS}_1^2(T) &= \alpha + \beta \mathrm{VS}_2^2(T) + \gamma \mathrm{GS}_2^2(T)
\end{aligned}$$

where

$$\mathrm{GS}_i^2(L,U,T) = \mathbb{E}\left[\frac{1}{T}\int_0^T \frac{S_{i,t}}{S_{i,0}} \sigma_{i,t}^2 1_{L \leq S_{i,t} \leq U}\, dt\right]$$

is the corridor Gamma swap on S_i with corridor (L, U) and $\mathrm{GS}_i(T) = \mathrm{GS}_i(0, \infty, T)$ is the vanilla Gamma swap.

The single-asset corridor Gamma swap can be perfectly replicated by static positions of vanilla options (replicating truncated and linearly extrapolated payoff $S_T \log S_T$) and dynamically rebalanced delta-hedging portfolio. In particular, if $r_t \equiv q_t$, then

$$\mathrm{GS}_i^2(L,U,T) = \frac{1}{T}\int_L^U \frac{2}{K} \mathrm{OTM}(K,T)dK.$$

For a list of maturities $\{T_i\}$, we shall assume α_t, β_t and γ_t take on constant values between T_i and T_{i+1} so that their values can be calibrated by bootstrapping.

Two-Asset Corridor Variance Swap - Approximation I

- Simple and fast calibration and pricing, no Monte Carlo simulation needed
- Establishes explicit dependence on volatilities and correlations, well-suited for risk management
- Once the values of α_t, β_t (and γ_t) are calibrated, the approximation formula (PI) (or (PIa)) for the cross local variance can be used to price two-asset corridor var swaps with other corridors (for deals entered earlier when the corridors were set relative to the spot at the time of the trade), or even other general cross weighted variance payoffs, i.e.,

$$\mathbb{E}\left[\frac{1}{T}\int_0^T \sigma_{1,t}^2 w\left(S_{2,t}\right)dt\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[\mathbb{E}\left[\left.\sigma_{1,t}^2\right| S_{2,t}\right] w\left(S_{2,t}\right)\right]dt$$

$$= \mathbb{E}\left[\frac{1}{T}\int_0^T \left(\alpha_t + \left(\beta_t + \gamma_t \frac{S_{2,t}}{S_{2,0}}\right)\sigma_{2,\mathrm{loc}}^2\left(t, S_{2,t}\right)\right) w\left(S_{2,t}\right)dt\right]$$

The last expectation can be easily evaluated by replication, PDE or Monte Carlo simulation in the local vol model.

Two-Asset Corridor Variance Swap - Approximation II

Postulate: The variance index $S_{1,t}$ follows a local volatility dynamics.

$$\frac{dS_{1,t}}{S_{1,t}} = \sigma_{1,\text{loc}}\left(t, S_{1,t}\right) dW_{1,t}$$

Then

$$\begin{aligned}
\text{VS}^2_{1,2}(L,U) &= \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma^2_{1,\text{loc}}\left(t, S_{1,t}\right) 1_{L \le S_{2,t} \le U}\, dt\right] \\
&= \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma^2_{1,\text{loc}}\left(t, S_{1,t}\right) \mathbb{E}\left[1_{L \le S_{2,t} \le U} \,\middle|\, S_{1,t}\right]\right] \\
&= \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma^2_{1,\text{loc}}(t, S_{1,t}) \left(\Phi_t\left(U|S_{1,t}\right) - \Phi_t\left(L|S_{1,t}\right)\right) dt\right]
\end{aligned}$$

where

$$\Phi_t(K|S_{1,t}) = \mathbb{E}\left[1_{S_{2,t} \le K} \,\middle|\, S_{1,t}\right]$$

is the conditional CDF of $S_{2,t}$ given $S_{1,t}$.

In this case we price the two-asset corridor variance swap as a single-asset weighted variance swap on S_1 with weight $\Phi_t\left(U|S_{1,t}\right) - \Phi_t\left(L|S_{1,t}\right)$.

The conditional CDF Φ_t may be derived from the copula of $S_{1,t}$ and $S_{2,t}$, which in turn can be calibrated to the market prices of basket payoffs $S_{1,t}$ and $S_{2,t}$ if available.

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