VOLATILITY MODELING HOMEWORK

MAARAF Oumayma †

Exercise 1: Dynamics of variance swap volatilities in the Heston model

1. Applying the Itô formula on $Y := \left(e^{kt}V_t\right)_{t\in[0,T]}$ yields :

$$\begin{aligned}\mathrm{d}Y_t &= kY_t\,\mathrm{d}t + e^{kt}\,\mathrm{d}V_t\\ &= ke^{kt}V^0\,\mathrm{d}t + \omega e^{kt}\sqrt{V_t}\,\mathrm{d}W_t^V\end{aligned}$$

So that :

$$\forall\ 0 \le t \le u \le T: \quad Y_u = Y_t + V^0\left(e^{ku} - e^{kt}\right) + \omega\int_t^u e^{ks}\sqrt{V_t}\,\mathrm{d}W_s^V$$

Thus :

$$\forall\ 0 \le t \le u \le T: \quad V_u = e^{-k(u-t)}V_t + V^0\left(1 - e^{-k(u-t)}\right) + \omega\int_t^u e^{-k(u-s)}\sqrt{V_t}\,\mathrm{d}W_s^V$$

By taking the conditional expectation on both sides and knowing that the local martingale part of V is in fact a true martingale, we get :

$$\forall\ 0 \le t \le u \le T: \quad \xi_t^u = e^{-k(u-t)}V_t + V^0\left(1 - e^{-k(u-t)}\right) = V^0 + e^{-k(u-t)}\left(V_t - V^0\right)$$

2. Let $t < T$, let's prove first that :

$$\mathbb{E}\left(\int_t^T V_u\,\mathrm{d}u \mid \mathcal{F}_t\right) = \int_t^T \mathbb{E}\left(V_u \mid \mathcal{F}_t\right)\mathrm{d}u \tag{1.1}$$

For this purpose, let X be a bounded $\mathcal{F}_t$-measurable r.v. We have :

$$\begin{aligned}\mathbb{E}\left(X\int_t^T V_u\,\mathrm{d}u\right) &= \mathbb{E}\left(\int_t^T XV_u\,\mathrm{d}u\right)\\ &\overset{\text{Fubini}}{=} \int_t^T \mathbb{E}\left(XV_u\right)\mathrm{d}u\\ &\overset{\text{Tower}}{=} \int_t^T \mathbb{E}\left(X\,\mathbb{E}\left(V_u \mid \mathcal{F}_t\right)\right)\mathrm{d}u\\ &\overset{\text{Fubini}}{=} \mathbb{E}\left(X\int_t^T \mathbb{E}\left(V_u \mid \mathcal{F}_t\right)\mathrm{d}u\right)\end{aligned}$$

Which concludes the proof of formula 1.1.

† o.maaraf@gmail.com

Now owing to formula 1.1 and qst 1, we have :

$$\begin{aligned}
\hat{\sigma}_T^2(t) &= \mathbb{E}\left(\frac{1}{T-t}\int_t^T V_u \,\mathrm{d}u \mid \mathcal{F}_t\right) \\
&= \frac{1}{T-t}\int_t^T \underbrace{\mathbb{E}\left(V_u \mid \mathcal{F}_t\right)}_{=\xi_t^u} \mathrm{d}u \\
&= \frac{1}{T-t}\left(V^0\left(T-t\right) + \left(V_t - V^0\right)\int_t^T e^{-k(u-t)}\,\mathrm{d}u\right) \\
&= V^0 + \frac{1-e^{-k(T-t)}}{k(T-t)}\left(V_t - V^0\right)
\end{aligned}$$

3. Denote for $t < T$, $\psi(t) := \frac{1-e^{-k(T-t)}}{k(T-t)}$. We have :

$$\mathrm{d}\hat{\sigma}_T^2(t) = \left(V_t - V^0\right)\mathrm{d}\psi(t) + k\psi(t)k(V^0 - V_t)\,\mathrm{d}t + \omega\psi(t)\sqrt{V_t}\,\mathrm{d}W_t^V$$

Now, applying the Itô formula on $x \mapsto \sqrt{x}$ yields :

$$\mathrm{d}\hat{\sigma}_T(t) = \mathrm{d}\sqrt{\hat{\sigma}_T^2(t)} = \frac{1}{2\hat{\sigma}_T(t)}\,\mathrm{d}\hat{\sigma}_T^2(t) + (\ldots)\,\underbrace{\mathrm{d}\langle\hat{\sigma}_T^2\rangle_t}_{\propto \mathrm{d}t}$$

Finally :

$$\mathrm{d}\hat{\sigma}_T(t) = (\ldots)\,\mathrm{d}t + \frac{\omega}{2}\frac{\psi(t)}{\hat{\sigma}_T(t)}\,\mathrm{d}W_t^V$$

Which is the desired dynamics.

Exercise 2: One-factor lognormal forward instantaneous variance models

2.1. One-factor Bergomi model

1. Let $0 \le t \le u$. Noting that X is a weiner process, we have :

$$\mathbb{E}\left(X_t\right) = 0 \quad ; \quad \mathrm{Var}\left(X_t\right) = \mathbb{E}\left(X_t^2\right) = \int_0^t e^{-2k(t-s)}\,\mathrm{d}s = \frac{1-e^{-2kt}}{2k}$$

Thus :

$$\begin{aligned}
\xi_t^u &= \xi_0^u \exp\left(\omega\int_0^t e^{-k(u-s)}\,\mathrm{d}Z_s - \frac{1}{2}\int_0^t e^{-2k(u-s)}\,\mathrm{d}s\right) \\
&= \xi_0^u \exp\left(\omega e^{-k(u-t)}\int_0^t e^{-k(t-s)}\,\mathrm{d}Z_s - \frac{\omega^2}{2}e^{-2k(u-t)}\int_0^t e^{-2k(t-s)}\,\mathrm{d}s\right) \\
&= \xi_0^u \exp\left(\omega e^{-k(u-t)}X_t - \frac{\omega^2}{2}e^{-2k(u-t)}\mathrm{Var}\left(X_t\right)\right) \\
&= \xi_0^u f^u\left(t, X_t\right)
\end{aligned}$$

2. Knowing that, $f^t \in \mathcal{C}^{1,2}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$, we have by Itô's formula :

$$\begin{aligned}
\mathrm{d}f^t(t,X_t) &= \left(\partial_t f^t(t,X_t) - kX_t\partial_x f^t(t,X_t) + \frac{1}{2}\partial_{x,x}^2 f^t(t,X_t)\right)\mathrm{d}t + \partial_x f^t(t,X_t)\,\mathrm{d}Z_t \\
&= f^t(t,X_t)\left(\left(-\frac{\omega^2}{2}e^{-2kt} - k\omega X_t + \frac{\omega^2}{2}\right)\mathrm{d}t + \omega\,\mathrm{d}Z_t\right)
\end{aligned}$$

Now by the representation $V_t = \xi_0^t f^t(t, X_t)$ along with the smoothness of $t \mapsto \xi_0^t$ we get, using Itô's formula :

$$\begin{aligned}
\mathrm{d}V_t &= \partial_t \xi_0^t f^t(t, X_t)\,\mathrm{d}t + \xi_0^t\,\mathrm{d}f^t(t, X_t) \\
&= \frac{\partial_t \xi_0^t}{\xi_0^t} V_t\,\mathrm{d}t + V_t \left(\left(\frac{\omega^2}{2}(1 - e^{-2kt}) - k\omega X_t \right) \mathrm{d}t + \omega\,\mathrm{d}Z_t \right) \\
&= \underbrace{V_t \left(\frac{\partial_t \xi_0^t}{\xi_0^t} + \frac{\omega^2}{2}(1 - e^{-2kt}) - k\omega X_t \right)}_{=\beta_t} \mathrm{d}t + \underbrace{\omega V_t}_{=\nu_t} \mathrm{d}Z_s
\end{aligned}$$

We first observe that the unlike the Heston model, where the diffusion of V is proportional to the instantaneous volatility, in the 1F Bergomi V is proportional to the instantaneous variance (i.e its self). One can also remark that the ξ_t^u does not admit a markov representation unless we impose the structural restriction :

$$\partial_u \xi_0^u = k \left(V^0 - \xi_0^u \right)$$

So that the Heston model is not able to generate general term structures of VS volatilities. However, on can prove (using simple calculations) that for a flat initial term-structure of VS variances, for long maturities, the instantaneous volatility of $\hat{\sigma}_T(t)$ decays like $\frac{1}{T-t}$ similarly to the Heston model.

2.2. Rough Bergomi model

1. Let $0 \leq t \leq u$, owing to Itô isometry we have :

$$\mathbb{E}\left(|X_t^u|^2 \right) = \int_0^t (u - s)^{2H-1}\,\mathrm{d}s = \frac{u^{2H} - (u-t)^{2H}}{2H}$$

By noting that $\mathbb{E}\left(X_t^u\right) = 0$, we have $\mathrm{Var}\left(X_t^u\right) = \mathbb{E}\left(|X_t^u|^2\right)$. Now, plugging the formula of K into the expression of ξ_t^u yields the desired representation.

2. For $t \geq 0$, we have (assuming that $t \mapsto \xi_0^t$ is differentiable and positive) :

$$\mathrm{d}V_t = \frac{\partial_t \xi_0^t}{\xi_0^t} V_t\,\mathrm{d}t + \nu V_t\,\mathrm{d}X_t^t$$

Now finding the dynamics of X_t^t is a bit tricky. To illustrate this, let $g : \mathbb{R}_+^2 \to \mathbb{R}$ be s.t for all $t \geq 0$, $g(t, .) \in L^2(\mathbb{R}_+)$, so that the process :

$$\forall\, t \in \mathbb{R}_+ : \quad Y_t := \int_0^t g(t, s)\,\mathrm{d}Z_s$$

is well defined. Now suppose that g satisfies some regularity assumptions so that the stochastic Leibniz formula is applicable and we get :

$$\mathrm{d}Y_t = \left(\int_0^t \partial_t g(t, s)\,\mathrm{d}Z_s \right) \mathrm{d}t + f(t, t)\,\mathrm{d}Z_t$$

In our context, the function g is defined by $g(t, s) = (t - s)^{H - \frac{1}{2}}$ which explodes to $+\infty$ when s approaches to t. To overcome this problem, we propose to modulate the power-law kernel by a log term to control its singularity. We introduce the following family of parametrized kernels, defined, for every $\lambda > 0$, by :

$$\forall \theta \geq 0 : \quad K_\lambda(\theta) = \omega \theta^{H - \frac{1}{2}} \left(\lambda \log(\frac{1}{\theta}) \vee 1 \right)^{-2} = \begin{cases} \omega(-\lambda)^{-2} \theta^{H - \frac{1}{2}} \log(\theta)^{-2}; & \theta \in [0, e^{-\frac{1}{\lambda}}] \\ K(\theta) & \theta > e^{-\frac{1}{\lambda}} \end{cases}$$

Now, noting that $K_\lambda \underset{\lambda\to+\infty}{\to} K$, that $K'_\lambda \underset{\lambda\to+\infty}{\to} K'$ and owing to the fact that $\lim_{\theta\to 0^+} K_\lambda(\theta) = 0$, we get by using the above Libneiz formula that :

$$\mathrm{d}X_t^t = \left(H - \frac{1}{2}\right)\left(\int_0^t (t-s)^{H-\frac{3}{2}}\,\mathrm{d}Z_s\right)\mathrm{d}t$$

Finally the dynamics of V reads :

$$\mathrm{d}V_t = V_t\left(\frac{\partial_t \xi_0^t}{\xi_0^t} + \left(H-\frac{1}{2}\right)\int_0^t (t-s)^{H-\frac{3}{2}}\,\mathrm{d}Z_s\right)\mathrm{d}t$$

We observe that the process V is positive and of finite-variation!

2.3. Heston model as a variance curve model

Using the formula of ξ_t^u in Exercice 1-qst1 and the fact that $V_t = \xi_t^t$, Itô's formula yields :

$$\mathrm{d}\xi_t^u = ke^{-k(u-t)}\left(V_t - V^0\right)\mathrm{d}t - ke^{-k(u-t)}(V_t - V^0)\,\mathrm{d}t + \omega e^{-k(u-t)}\sqrt{V_t}\,\mathrm{d}W_t^V = \omega e^{-k(u-t)}\sqrt{\xi_t^t}\,\mathrm{d}W_t^V$$

Exercise 3: Two-factor Bergomi model

1. Let $0 \le t \le u$. We introduce the following function :

$$\nu_\xi(t,u) = \omega\alpha_\theta\left((1-\theta)\,e^{-k_1(u-t)} + \rho_{1,2}\theta e^{-k_2(u-t)},\ \sqrt{1-\rho_{1,2}^2}\theta e^{-k_2(u-t)}\right)^\top$$

Let $W^{1,\perp}$ be a standard BM independant of W^1 s.t :

$$W^2 = \rho_{1,2}W^1 + \sqrt{1-\rho_{1,2}^2}W^{1,\perp}$$

So that the process defined by $\hat{W} := \left(W^1, W^{1,\perp}\right)^T$ is a two dimensional standard BM and the dynamics of ξ^u reads :

$$\mathrm{d}\xi_t^u = \xi_t^u \nu_\xi(t,u)^\top\,\mathrm{d}\hat{W}_t$$

Thus, ξ^u is the exponential martingale of the process $\int_0^\cdot \nu_\xi(s,u)^\top\,\mathrm{d}\hat{W}_s$:

$$\xi_t^u = \xi_0^u \exp\left(\int_0^t \nu_\xi(s,u)^\top\,\mathrm{d}\hat{W}_s - \frac{1}{2}\int_0^t \|\nu_\xi(s,u)\|_2^2\,\mathrm{d}s\right) \tag{3.1}$$

Now, noting that :

$$\begin{aligned}\|\nu_\xi(s,u)\|_2^2 &= \omega^2\alpha_\theta^2\left(\left((1-\theta)\,e^{-k_1(u-t)} + \rho_{1,2}\theta e^{-k_2(u-t)}\right)^2 + (1-\rho_{1,2}^2)\theta^2 e^{-2k_2(u-t)}\right)\\ &= \omega^2\alpha_\theta^2\left((1-\theta)^2\,e^{-2k_1(u-t)} + \rho_{1,2}^2\theta e^{-2k_2(u-t)} + 2\theta(1-\theta)e^{-(k_1+k_2)(u-t)} + (1-\rho_{1,2}^2)\theta^2 e^{-2k_2(u-t)}\right)\end{aligned}$$

Now, it suffices to rearrange terms in formula 3.1 to get the desired markovian representation of ξ^u.

2. Noting that $f^t \in \mathcal{C}^{1,2,2}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$, we have by Itô's formula (for the sake of notation, we omit writing the variables of the function):

$$\begin{aligned}\mathrm{d}f^t(t,X_t^1,X_t^2) &= \left(\partial_t - k_1X_t^1\partial_{x_1} - k_2X_t^2\partial_{x_2} + \frac{1}{2}\Delta + \rho_{1,2}\partial^2_{x_1,x_2}\right)f^t(.)\,\mathrm{d}t + \sum_{i=1}^2 \partial_{x_i}f^t(.)\,\mathrm{d}W_t^i\\ &= f^t(t,X_t^1,X_t^2)\left(\mu(t,X_t^1,X_t^2)\,\mathrm{d}t + \omega\alpha_\theta\left((1-\theta)\,\mathrm{d}W_t^1 + \theta\,\mathrm{d}W_t^2\right)\right)\end{aligned}$$

Where :

$$\mu(t, X_t^1, X_t^2) := -\frac{\omega^2 \alpha_\theta^2}{2} \left(\sum_{i=1}^{2} \theta_i e^{-k_i t} \right)^2 - \omega \alpha_\theta \sum_{i=1}^{2} \theta_i k_i X_t^i + \frac{\omega^2 \alpha_\theta^2}{2} (\theta_1 + \theta_2)^2$$

s.t $\theta_1 := 1 - \theta$ and $\theta_2 := \theta$.

Thus :

$$\begin{aligned} \mathrm{d}V_t &= \frac{\partial_t \xi_0^t}{\xi_0^t} V_t \,\mathrm{d}t + \xi_0^t \,\mathrm{d}f^t(t, X_t^1, X_t^2) \\ &= V_t \left(\frac{\partial_t \xi_0^t}{\xi_0^t} + \mu(t, X_t^1, X_t^2) \right) \mathrm{d}t + \omega \alpha_\theta V_t \sum_{i=1}^{2} \theta_i \,\mathrm{d}W_t^i \end{aligned}$$

We notice that dynamics of V can be written in terms of the BM $\hat{W}$ introduced in qst 1 in same fashion as ξ^u, so that V is a brownian-driven diffusion. Thus, V is a Markov process.

□