On some theoretical proprieties of the two factor PDV model

Recall the dynamics of the two factor PDV model, for all $t \in [0, T]$:

$$
\begin{aligned}
\mathrm{d}S_t &= S_t \sigma \left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{1,t} &= -\lambda_1 R_{1,t} \,\mathrm{d}t + \lambda_1 \sigma \left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{2,t} &= \lambda_2 \left(\sigma \left(R_{1,t}, R_{2,t}\right)^2 - R_{2,t}\right) \mathrm{d}t
\end{aligned}
$$

where $\sigma : (R_1, R_2) \mapsto \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2}$.

R_1 and R_2 can be defined equivalently (denote $\sigma_{.} := \sigma(R_{1,.}, R_{2,.})$) by :

$$
\begin{aligned}
R_{1,t} &= R_{1,0} + \int_0^t \lambda_1 e^{-\lambda_1 (t-u)} \sigma_u \,\mathrm{d}W_u \\
R_{2,t} &= R_{2,0} + \int_0^t \lambda_2 e^{-\lambda_2 (t-u)} \sigma_u^2 \,\mathrm{d}u
\end{aligned}
$$

We know that $R := (R_1, R_2)$ is a Markov diffusion with Feller-Dynkin operator given by :

$$
L_R := -\lambda_1 R_1 \partial_{R_1} + \lambda_2 \left(\sigma^2 - R_2\right) \partial_{R_2} + \frac{1}{2} \lambda_1^2 \sigma^2 \partial^2_{R_1, R_2}
$$

We donte by $L_R^\dagger$ its adjoint operator defined by :

$$
L_R^\dagger := \lambda_1 R_1 \partial_{R_1} - \lambda_2 \left(\sigma^2 - R_2\right) \partial_{R_2} + \frac{1}{2} \lambda_1^2 \sigma^2 \partial^2_{R_1, R_2}
$$

For $0 \leq s \leq t \leq T$, where $T > 0$, we know that, if it exists, the transition probability density function of $(R_u^{s,r})_{s \leq u \leq r}$ denoted by $p_{s,t}(r, .)$ satisfies the Fokker-Plank equation :

$$
\partial_t p_{s,t}(r, .) = L_R^\dagger p_{s,t}(r, .), \quad p_{s,s}(r, .) = \delta(x - y)
$$

Let $(\kappa_n, \Psi_n)_{n \in \mathbb{N}}$ be the sequence of eigenvalues and eigenvectors of $L_R^\dagger$ and suppose for simplicity that $s = 0$. Then, $R \mapsto p_t(r, R)$ can be written as :

$$
\forall\ R \in \mathbb{R}^2 : \quad p_t(r, R) = \sum_{n=0}^{+\infty} \alpha_n e^{\kappa_n t} \Psi_n(R)
$$

Where (α_n) is a real-valued coordinates sequence and it is taken so that

$$
\int_{\mathbb{R}^2} p_t(r, R) \,\mathrm{d}R = 1
$$

Questions

1. How can we determine (κ_n, Ψ_n) ?

2. Any ideas on how can we determine the conditional distribution $g(R_t) \mid S_t = S$ for $t \in (0,T)$, $g : \mathbb{R}^2 \overset{\text{Borel}}{\longrightarrow} \mathbb{R}$ and $S \in \mathbb{R}_+$?

We focus on the process R because if one denotes $X := \log(S)$, Then for $t \in [0,T]$, $X_t = F\left((R_u)_{u \le t}\right)$, so that the prices of vanilla/exotic derivatives on e^X are fully determined by the laws of the process R.

Our main goal is to determine some expansions of VIX futures/vanilla options on VIX prices in this model.