COMPUTATION OF BREAK-EVEN FOR LV AND LSV MODELS

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Abstract. We obtain some new (Malliavin) probabilistic representations for the so-called break-even in Local Volatility (LV) and Local Stochastic Volatility (LSV) models for which we get some approximations extending the formulas derived in [1] in the ATM case. These break-even correspond to the instantaneous volatility of the implied volatility processes and their correlations with the spot. The accuracy of theses approximations are checked against a MC pricer. The ATM BE involve the at-the-money skew for the associated naked SVM in the case of a LSV model. This function is learnt using a low-complexity neural network. As explained in [1], these formulas allow to calibrate LSV models on historical Vanna and Vomma break-even.

1. Break-even: definitions

Let us consider the price $C_t(T,K) := \mathbb{E}_t^{\mathbb{P}}[\frac{S_T}{S_t} - K)^+]$ at t of a call option with strike K and maturity T produced by an arbitrage-free model – say LV or LSV model.

Definition 1.1 (Implied volatility)**.** *We define the implied volatility σ_t^{TK} of strike K and maturity T as the constant volatility $\sigma = \sigma_t^{TK}$ such that*

$$C_t(T,K) = \mathrm{BS}(S_t := 1, K, T-t, \sigma)$$

where BS *denotes the Black-Scholes formula with spot S_t, strike K, maturity $T-t$ and volatility σ.*

Given a diffusive model, we are interested in computing the dynamics of the implied volatility, more precisely the instantaneous spot/volatility and volatility/volatility correlations:

Definition 1.2 (Vomma/Vanna BE)**.** *The BE spot-volatility and the BE vol-vol are defined by*

$$\begin{aligned} \mathrm{B}_t^S(T,K) &:= d\langle \ln \sigma_\cdot^{TK}, \ln S\rangle_t \\ \mathrm{B}_t(T,K,T',K') &:= d\langle \ln \sigma_\cdot^{TK}, \ln \sigma_\cdot^{T'K'}\rangle_t \end{aligned}$$

2. Implied volatility dynamics for LV and LSV

2.1. **LV.** We consider a LV model for which

$$dS_t = \sigma(t, S_t)dW_t$$

For use below, we set $\sigma_t := \sigma(t,S_t)$ and $\sigma_t' = \partial_S \sigma(t,S_t)$. For the sake of simplicity, we will assume no dividends, zero repo and interest rate so that the spot price S.

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is a martingale.[1] The function σ is calibrated to market prices of Vanillas using Dupire's formula [2] and is assumed to be smooth.

2.1.1. *Probabilistic representation.*

Theorem 2.1. *Set*

$$\Sigma_t^{TK} := \frac{1}{K^2(T-t)(\sigma_t^{T,K})^2} \frac{1}{S_t} \int_t^T ds\, \mathbb{E}_t \left[\sigma_s^2 \frac{Y_T}{Y_s} \left(\frac{\sigma_s'}{\sigma_s} Y_s - \frac{1}{S_t} \right) | \frac{S_T}{S_t} = K \right]$$

with the tangent processes:

$$dY_s = \sigma_s' Y_s dW_t, \quad Y_t := 1$$

Then,

$$\begin{aligned} B_t^S(T,K) &= \Sigma_t^{TK} \sigma_t^2 \\ \mathrm{B}_t(T,K,T',K') &= \Sigma_t^{TK} \Sigma_t^{T'K'} \sigma_t^2 \end{aligned}$$

All the proofs are reported in Appendix A.

2.2. **LSV.** We consider a LSV1F model:

$$\begin{aligned} dS_t &= \sigma_t a_t dW_t, \quad a_t := e^{\nu X_t} \\ dX_t &= -kX_t dt + \left(\rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp} \right) \end{aligned}$$

where $\sigma_t := \sigma(t, S_t)$. This model depends on three parameters: a mean-reversion k, a spot/vol correlation ρ and a volatility-of-volatility ν. The function $\sigma(t,S)$ is calibrated on market prices of Vanillas using a 2D PDE implementation or the particle method (see [3] for a review).

2.2.1. *Calibration of the LSV on BE.* Following [1], the LSV parameters are calibrated on BE. The procedure involves an optimization. At each step of the gradient descent where the LSV parameters are changed, one needs to calibrate $\sigma(t,S)$ (using a 2D PDE or particle method) and then compute the BE using a Monte-Carlo simulation. The resulting optimization is very quite-consuming. In order to circumvent this drawback, approximations for the ATM-BE have been derived in [1]. In this section, we give some probabilistic representations for these BE. Then, we derive some approximations, applicable for any strike.

2.2.2. *Probabilistic representation.*

[1] As is well-known (see for instance [5]), cash/yield dividends and non-zero repo and interest rate can be included by introducing a continuous positive local martingale X_t defined by $S_t = A(t)X_t + B(t)$ where the two functions $A(\cdot)$ and $B(\cdot)$ depend on the cash/yield discrete dividends, repo and interest rate.

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Theorem 2.2. *We set*

$$\begin{aligned}
\Sigma_t^{TK,(0)} &:= \frac{1}{K^2(T-t)(\sigma_t^{T,K})^2}\frac{1}{S_t}\int_t^T ds \\
&\mathbb{E}_t\left[\left(\frac{\sigma'_s}{\sigma_s}Y_s - \frac{1}{S_t}\right)(\sigma_s a_s)\left(\frac{Y_T}{Y_s}\left(\sigma_s a_s - e^{k(s-t)}\rho Z_s\right) + e^{k(s-t)}\rho Z_T\right)|\frac{S_T}{S_t} = K\right] \\
\Sigma_t^{TK,(1)} &:= \frac{1}{K^2(T-t)(\sigma_t^{T,K})^2}\frac{1}{S_t^2}\int_t^T ds \\
&\mathbb{E}_t\left[\left(\frac{\sigma'_s}{\sigma_s}Z_s + \nu e^{-k(s-t)}\right)(\sigma_s a_s)\left(\frac{Y_T}{Y_s}\left(\sigma_s a_s - e^{k(s-t)}\rho Z_s\right) + e^{k(s-t)}\rho Z_T\right)|\frac{S_T}{S_t} = K\right]
\end{aligned}$$

with the tangent processes:

$$\begin{aligned}
dY_s &= \sigma'_s Y_s a_s dW_s, \quad Y_t = 1 \\
dZ_s &= (\sigma'_s Z_s + \sigma_s \nu e^{-k(s-t)})a_s dW_s, \quad Z_t = 0
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{B}_t(T,K,T',K') &= \sigma_t^2 a_t^2 \Sigma_t^{TK,(0)}\Sigma_t^{T'K',(0)} + \Sigma_t^{TK,(1)}\Sigma_t^{T'K',(1)} \\
&+ \rho\sigma_t a_t\left(\Sigma_t^{TK,(0)}\Sigma_t^{T'K',(1)} + \Sigma_t^{TK,(1)}\Sigma_t^{T'K',(0)}\right) \\
B_t^S(T,K) &= \Sigma_t^{TK,(0)}\sigma_t^2 a_t^2 + \rho\Sigma_t^{TK,(1)}\sigma_t a_t
\end{aligned}$$

The extension of this result to multi-factor LSV models is straightforward and therefore not considered.

Remark 2.3. *For $\nu = 0$, we reproduce our previous result (2.1) as $Z_s = 0$ in this case.*

3. Brownian bridge approximation for LV and LSV Break-even

The probabilistic representations of the BE involve a conditioning with respect to S_T. It is therefore natural to use a Brownian bridge approximation for evaluating these conditional expectations.

3.1. Break-even approximation for LV.

Using Theorem 2.1, we derive an approximation for the dynamics of σ_t^{TK}:

Corollary 3.1.

$$\Sigma_t^{TK} \approx \frac{1}{K^2 S_t}\frac{\int_t^T ds\,\left(\partial_S \ln\sigma_{\mathrm{Dup}}(s,S_s^*) + \frac{1}{S_s^*} - \frac{1}{S_t}\right)(S_s^*)^2\sigma_{\mathrm{Dup}}^2(s,S_s^*)}{(T-t)(\sigma_t^{T,K})^2} \tag{3.1}$$

where $\sigma_{\mathrm{Dup}}(t,S) := \sigma(t,S)/S$ and

$$\ln\frac{S_s^*}{S_t} := \frac{s-t}{T-t}\ln\frac{K}{S_t}$$

Remark 3.2 (ATM case)**.** *The above formula can be further simplified for $K = 1$. We use an approximation for σ_t^{TK} – see for example [4]:*

$$(T-t)(\sigma_t^{TK})^2 \approx \int_t^T ds\sigma^{\mathrm{Dup}}(s,S_s^*)^2$$

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By differentiating by K at $K=1$, we obtain:

$$(T-t)\partial_K \sigma_t^{T,1}\sigma_t^{T,1} \approx S_t \int_t^T ds\, \sigma_{\rm Dup}(s,S_t)\partial_S \sigma_{\rm Dup}(s,S_t)\frac{s-t}{(T-t)}$$

and therefore,

$$\partial_s((s-t)^2\partial_K \sigma_t^{s,1}\sigma_t^{s,1}) = (s-t)\sigma_{\rm Dup}(s,S_t)S_t\partial_S\sigma_{\rm Dup}(s,S_t)$$

Plugging this expression in (3.1) implies that

$$\Sigma_t^{T,1} \approx \frac{\partial_K \sigma_t^{T,1}}{\sigma_t^{T,1}} + \frac{1}{T-t}\int_t^T ds \left(\partial_K \sigma_t^{s,1}\right)\frac{\sigma_t^{s,1}}{(\sigma_t^{T,1})^2} \tag{3.2}$$

Note that the formula, derived in ([1], Equation (2.84)), is slightly different:

$$\Sigma_t^{T,1} \approx \frac{\partial_K \sigma_t^{T,1}}{\sigma_t^{T,1}} + \frac{1}{T-t}\int_t^T ds \left(\partial_K \sigma_t^{s,1}\right)\frac{\sigma_t^{s,1}}{(\sigma_t^{T,1})^2}\frac{\sigma^{\rm Dup}(s,S_0)^2}{(\sigma_t^{s,1})^2}$$

3.2. **Break-even approximation for LSV.** Using Theorem 2.2, we derive an approximation for the dynamics of σ_t^{TK}.

Corollary 3.3.

$$\Sigma_t^{TK,(0)} \approx \frac{1}{K^2}\frac{1}{S_t}\frac{\int_t^T ds\left(\partial_S \ln\sigma_{\rm Dup}(s,S_s^*) - \partial_S\ln\sigma_{\rm SVM}(s,S_s^*) + \frac{1}{S_s^*} - \frac{1}{S_t}\right)(S_s^*)^2\sigma_{\rm Dup}(s,S_s^*)^2}{(T-t)(\sigma_t^{T,K})^2}$$

$$\Sigma_t^{TK,(1)} \approx \frac{\nu}{K^2S_t^2}\frac{\int_t^T ds\, e^{-k(s-t)}(S_s^*)^2\sigma_{\rm Dup}(s,S_s^*)^2}{(T-t)(\sigma_t^{T,K})^2}$$

where $\sigma_{\rm SVM}(s,S) := \sqrt{\mathbb{E}[a_s^2|S_s=S]}$ and $\sigma_{\rm Dup}(s,S) := \sigma(s,S)\sigma_{\rm SVM}(s,S)$.

Remark 3.4. *For $K=1$, proceeding as in Remark 3.2, this can be simplified into:*

$$\Sigma_t^{T1,(0)} \approx \frac{(\sigma_{LV}^T)'}{\sigma_t^{T,1}} + \frac{1}{T-t}\int_t^T ds(\sigma_{\rm LV}^s)'\frac{\sigma_t^{s,1}}{(\sigma_t^{T,1})^2} \tag{3.3}$$

$$\Sigma_t^{T1,(1)} \approx \nu\int_t^T e^{-k(s-t)}ds\frac{\partial_s((\sigma_t^{s,1})^2(s-t))}{(T-t)(\sigma_t^{T,1})^2} \tag{3.4}$$

where $(\sigma_{\rm LV}^s)' := \partial_K\sigma_t^{s,1} - \mathcal{S}_s^{\rm SV}$. $\mathcal{S}_s^{\rm SV}$ is the at-the-money skew generated by the naked SV model. Note that the formula (3.3), derived in ([1], Equation (12.48)), is slightly different:

$$\Sigma_t^{T1,(0)} \approx \frac{(\sigma_{LV}^T)'}{(\sigma_t^{T,1})} + \frac{1}{T-t}\int_t^T ds(\sigma_{\rm LV}^s)'\frac{\sigma_t^{s,1}}{(\sigma_t^{T,1})^2}\frac{\sigma^{\rm Dup}(s,S_0)^2}{(\sigma_t^{s,1})^2} \tag{3.5}$$

We will show in our numerical experiments that our formula (3.2) is more accurate.

4. MC computation of BE

The computation of BE for our LSV model uses the above approximations (see Equations (3.3), (3.4)). This formula depends on the at-the-money skew $\mathcal{S}_\cdot^{\rm SV}$ generated by the naked SV model. These approximations are checked against a Monte-Carlo simulation. In [1], the naked SV ATMF skew is estimated using an asymptotic formula. As shown in our numerical experiments, the resulting approximation is not accurate enough and fails when the volatility of volatility raises. It

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is therefore better to use a MC engine in order to compute the naked SV ATMF skew[2].

4.1. **Turbo-charging the naked Stochastic Vol skew computation with neural network.** The use of a MC engine is a very time consuming tool when integrating it in the optimization problem involved for the calibration of the LSV parameters. To get rid of this time-consuming problem while keeping the quality of the MC engine estimation of the naked SV ATMF skew, we trained a Neural Network (with one layer) to *learn* the naked SV ATMF skew formula.
The *randomly generated* inputs are

- a volatility of volatility $\nu \in [0.8; 4.5]$
- a mean reversion speed $\kappa \in [0.5; 8]$
- a correlation spot/vol $\rho \in (-1; 1)$
- and a term-structure of ATMF implied volatilities through a set of seven discrete ATMF implied volatilities at the expiries $\{0.25, 0.5, 1, 2, 5, 7, 10\}$ defined thanks to a parametric arbitrage-free formula

$$T \times \hat{\sigma}_T^2 = \int_0^T (\sigma_\infty + (\sigma_0 - \sigma_\infty)e^{-t/\tau})^2 dt$$

  where the three parameters σ_0, σ_∞ and τ are randomly generated in $[0.1, 0.6]$, $[0.1, 0.6]$ and $[0.25, 5]$.

The three LSV parameters plus the seven discrete ATMF implied volatilities make a input set of dimension 10.

The outputs, computed with a MC engine, are 15 ATMF skews at the expiries $\{1/52, 1/12, 0.25, 0.75, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ under the naked SV 1F model. These skews are computed using a Monte-Carlo algorithm enhanced with the mixing solution where the computation of a vanilla option requires only the simulation of the volatility process (OU). Equipped with these data, we trained a neural network with one single hidden layer made of 20 nodes. The results are very accurate (see Figure 1).

4.2. **Computation of BE.** In our numerical examples, we calibrate $\xi_0^{t_i}$ on the following ATMF implied volatility term-structure depicted in Figure 2. Thanks to a linear interpolation in variance of the ATMFVols (and a flat extrapolation on the left side), we are able to derive a continuous function $\hat{\sigma}_T^2 T$. From this, we easily derive a piecewise constant function $\overline{\sigma}(t)$, such as $\hat{\sigma}_T^2 T = \int_0^T \overline{\sigma}^2(t)dt$. Finally, all the integrands (see Equations 3.3, 3.4) are computed with a 20th-order Gauss-Legendre quadrature.
Then, we will compute the BE volatility and correlation for the SX5E as of the 18/05/2020. We compare three methods:

(1) *MC*: we use a brute force MC.
(2) *SV Skew Approx – LB*: based on the asymptotics formulas (see Equations 3.5, 3.4) and an asymptotic formula for the skew of the naked SV.
(3) *SV Skew NN*: we use formula (3.3,3.4) and the skew $\mathcal{S}_t^{SV}$ is computed using our NN approximation, presented above.

[2]As first noticed by Lorenzo Bergomi (private communication).

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Model1F - Parameters

nu	k1	rhoS1	Stationary Vol
243%	2.87	-94%	101.2%

ATMF Volatility

T	0.25	0.50	1.00	2.00	5.00	7.00	10.00
CAC	24%	23%	22%	20%	19%	20%	20%

T	Bergomi	MC	NN
0.02	-112%	-114%	-113%
0.08	-105%	-96%	-98%
0.25	-91%	-74%	-74%
0.50	-73%	-59%	-60%
0.75	-61%	-50%	-51%
1.00	-52%	-43%	-44%
2.00	-32%	-29%	-29%
3.00	-23%	-21%	-22%
4.00	-18%	-17%	-17%
5.00	-15%	-14%	-14%
6.00	-12%	-12%	-12%
7.00	-11%	-10%	-10%
8.00	-9%	-9%	-9%
9.00	-8%	-9%	-8%
10.00	-8%	-8%	-8%

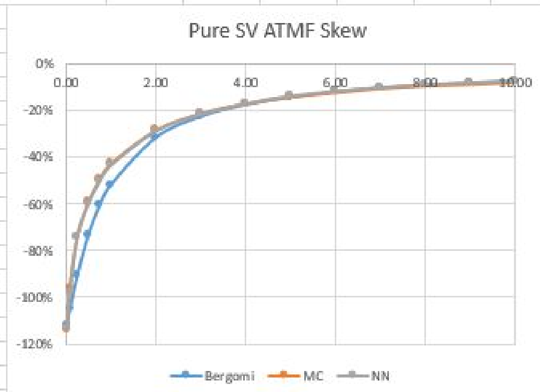

FIGURE 1. Approximation of the ATMF Skews of the naked SV 1F model with a neural network.

T	ATMF Vol
0.25	40.00%
0.50	33.00%
1.00	26.00%
2.00	22.00%
5.00	21.00%
7.00	21.00%
10.00	21.00%

FIGURE 2. ATMF implied volatility term-structure used in our numerical experiments.

4.3. **Numerical examples.** The results are represented in Figures 3. Our numerical examples show that our new approximation is close to the BE generated by MC.

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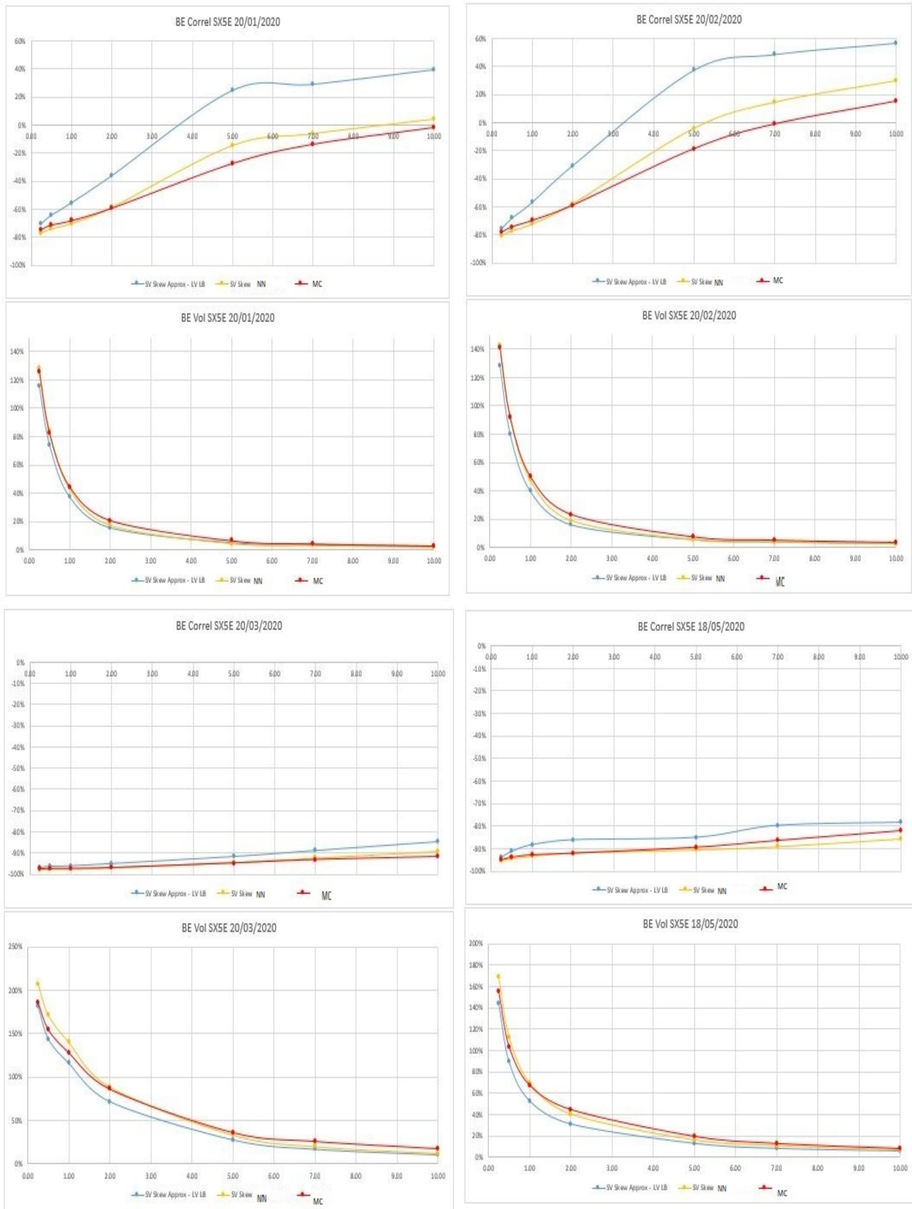

FIGURE 3. Break-even ATM Vanna and Vomma as a function of the maturity: MC versus approximations.

APPENDIX A. PROOFS

Proof of theorem 2.1. Using Itô's lemma on $C_t(T,K) := \mathbb{E}_t\left[\left(\frac{S_T}{S_t} - K\right)^+\right]$:

$$dC_t(T,K) \quad = \quad \mathbb{E}_t\left[1\left(\frac{S_T}{S_t} > K\right)\left(Y_T - \frac{S_T}{S_t}\right)\right]\frac{dS_t}{S_t}$$

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where $Y_T := \frac{\partial S_T}{\partial S_t}$ given by

$$dY_u = \sigma'_u Y_u dW_u, \quad Y_t := 1$$

Then using the Clark-Ocone formula:

$$dC_t(T,K) = \int_t^T ds \mathbb{E}_t \left[\mathbb{E}_s \left[\frac{D_s S_T}{S_t} \delta\left(\frac{S_T}{S_t} - K \right) \right] \mathbb{E}_s \left[(D_s Y_T - \frac{D_s S_T}{S_t}) \right] \right] \frac{dS_t}{S_t}$$

where D_s denotes the Malliavin derivative. Note that $D_s S_.$, $D_s Y_.$ are martingales and

$$D_s S_T = \sigma_s \frac{Y_T}{Y_s}, \quad D_s S_s = \sigma_s, \quad D_s Y_s = \sigma'_s Y_s \tag{A.1}$$

Therefore,

$$dC_t(T,K) = \int_t^T ds \mathbb{E}_t \left[\delta\left(\frac{S_T}{S_t} - K \right) \sigma_s \frac{Y_T}{Y_s} \left(D_s Y_s - \frac{D_s S_s}{S_t} \right) \right] \frac{dS_t}{S_t} \tag{A.2}$$

By plugging the formulas (A.1) in (A.2), we obtain the dynamics of call prices $C_t(T,K)$ as

$$\frac{dC_t(T,K)}{\partial_K^2 C_t(T,K)} = \int_t^T ds \mathbb{E}_t \left[\sigma_s^2 \left(\frac{\sigma'_s}{\sigma_s} Y_s - \frac{1}{S_t} \right) \frac{Y_T}{Y_s} | \frac{S_T}{S_t} = K \right] \frac{dS_t}{S_t^2}$$

We conclude as

$$\frac{\partial_\sigma \mathrm{BS}(1,K,T-t,\sigma)}{\partial_K^2 \mathrm{BS}(1,K,T-t,\sigma)} = K^2 \sigma (T-t)$$

and

$$\frac{dC_t(T,K)}{\partial_K^2 C_t(T,K)} = K^2 \sigma_t^{TK} (T-t) d\sigma_t^{TK}$$

□

Proof of Corollary 3.1.
(1) If $T-t$ is small, we take

$$Y_s \approx 1$$

Therefore,

$$\mathbb{E}_t \left[\sigma_s^2 \frac{Y_T}{Y_s} \left(\frac{\sigma'_s}{\sigma_s} Y_s - \frac{1}{S_t} \right) | \frac{S_T}{S_t} = K \right] \approx \mathbb{E}_t \left[\sigma_s^2 \left(\frac{\sigma'_s}{\sigma_s} - \frac{1}{S_t} \right) | \frac{S_T}{S_t} = K \right]$$

(2) The conditional expectation $\mathbb{E}[\cdot | S_T = K]$ can be approximated by a Brownian bridge approximation where the expectation is evaluated only through the deterministic path $s \mapsto S_s^*$ with

$$\ln \frac{S_s^*}{S_t} = \frac{s-t}{T-t} \left(\ln \frac{K}{S_t} \right) \tag{A.3}$$

In particular for $K = S_t$, $S_s^* = S_t$.
(3) We obtain

$$\mathbb{E}_t [\sigma_s^2 \left(\frac{\sigma'_s}{\sigma_s} - \frac{1}{S_t} \right) | \frac{S_T}{S_t} = K] \approx \sigma(s, S_s^*)^2 \left(\partial_S \ln \sigma(s, S_s^*) - \frac{1}{S_t} \right)$$

This implies our result.

□

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Proof of Theorem 2.2. Using Itô's lemma on $C_t(T,K) := \mathbb{E}_t\left[\left(\frac{S_T}{S_t}-K\right)^+\right]$:

$$\begin{aligned}
dC_t(T,K) &= \mathbb{E}_t\left[1\left(\frac{S_T}{S_t}>K\right)\left(Y_T-\frac{S_T}{S_t}\right)\right]\frac{dS_t}{S_t}\\
&+ \mathbb{E}_t\left[1\left(\frac{S_T}{S_t}>K\right)\frac{Z_T}{S_t}\right](\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp})
\end{aligned}$$

where $Y_T := \frac{\partial S_T}{\partial S_t}$ and $Z_T := \frac{\partial S_T}{\partial X_t}$ with

$$\begin{aligned}
dY_u &= \sigma_u' Y_u a_u dW_u, \quad Y_t = 1\\
dZ_u &= \left(\sigma_u' Z_u + \sigma_u \nu e^{-k(u-t)}\right) a_u dW_u, \quad Z_t = 0
\end{aligned}$$

From the Clark-Ocone formula, we have

$$\begin{aligned}
dC_t(T,K) &= \int_t^T ds\mathbb{E}_t\left[\mathbb{E}_s[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_sS_T}{S_t}\right]\mathbb{E}_s\left[(D_sY_T-\frac{D_sS_T}{S_t})\right]\frac{dS_t}{S_t}\\
&+ \int_t^T ds\mathbb{E}_t\left[\mathbb{E}_s\left[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_s^{\perp}S_T}{S_t}\right]\mathbb{E}_s\left[(D_s^{\perp}Y_T-\frac{D_s^{\perp}S_T}{S_t})\right]\right]\frac{dS_t}{S_t}\\
&+ \int_t^T ds\mathbb{E}_t\left[\mathbb{E}_s\left[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_sS_T}{S_t}\right]\mathbb{E}_s\left[\frac{D_sZ_T}{S_t}\right]\right](\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp})\\
&+ \int_t^T ds\mathbb{E}_t\left[\mathbb{E}_s\left[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_s^{\perp}S_T}{S_t}\right]\mathbb{E}_s\left[\frac{D_s^{\perp}Z_T}{S_t}\right]\right](\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp})
\end{aligned}$$

with the Malliavin derivatives:

$$\begin{aligned}
D_sS_u &= \frac{Y_u}{Y_s}\left(\sigma_s a_s - e^{k(s-t)}\rho Z_s\right) + e^{k(s-t)}\rho Z_u, \quad D_sS_s = \sigma_s a_s\\
D_s^{\perp}S_u &= \sqrt{1-\rho^2}e^{k(s-t)}\left(Z_u - \frac{Z_s}{Y_s}Y_u\right), \quad D_s^{\perp}S_s = 0
\end{aligned}$$

Note that $D_sY., D_sS., D_s^{\perp}Y., D_s^{\perp}S., D_sZ.$ and $D_s^{\perp}Z.$ are martingales and therefore

$$\begin{aligned}
dC_t(T,K) = \int_t^T ds\mathbb{E}_t[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_sS_T}{S_t}(D_sY_s-\frac{D_sS_s}{S_t})]\frac{dS_t}{S_t}\\
+ \int_t^T ds\mathbb{E}_t[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_s^{\perp}S_T}{S_t}(D_s^{\perp}Y_s-\frac{D_s^{\perp}S_s}{S_t})]\frac{dS_t}{S_t}\\
+ \int_t^T ds\mathbb{E}_t[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_sS_T}{S_t}\frac{D_sZ_s}{S_t}](\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp})\\
+ \int_t^T ds\mathbb{E}_t[\delta\left(\frac{S_T}{S_t}-K\right)\frac{D_s^{\perp}S_T}{S_t}\frac{D_s^{\perp}Z_s}{S_t}](\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp})
\end{aligned} \tag{A.4}$$

We have

$$\begin{aligned}
D_sY_s &= \sigma_s' Y_s a_s, \quad D_s^{\perp}Y_s = 0\\
D_sZ_s &= \sigma_s' Z_s a_s + \sigma_s a_s \nu e^{-k(s-t)}
\end{aligned}$$

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By plugging these formulas in (A.4), we obtain:

$$
\begin{aligned}
&\frac{dC_t(T,K)}{\partial_K^2 C_t(T,K)} = \\
&\int_t^T ds\mathbb{E}_t\left[\left(\frac{Y_T}{Y_s}\left(\sigma_s a_s - e^{k(s-t)}\rho Z_s\right) + e^{k(s-t)}\rho Z_T\right)\sigma_s a_s\left(\frac{\sigma_s'}{\sigma_s}Y_s - \frac{1}{S_t}\right)|\frac{S_T}{S_t} = K\right]\frac{dS_t}{S_t^2} \\
+ \quad &\frac{1}{S_t^2}\int_t^T ds\mathbb{E}_t\left[\left(\frac{Y_T}{Y_s}\left(\sigma_s a_s - e^{k(s-t)}\rho Z_s\right) + e^{k(s-t)}\rho Z_T\right)\sigma_s a_s\left(\frac{\sigma_s'}{\sigma_s}Z_s + \nu e^{-k(s-t)}\right)|\frac{S_T}{S_t} = K\right] \\
&(\rho dW_t + \sqrt{1-\rho^2}dW_t^{\perp})
\end{aligned}
$$

□

Proof of Corollary 3.3. **(1)** By assuming that $T-t$ is small, we have for the tangent process:

$$Y_s \approx 1, \quad Z_s \approx 0$$

This implies that:

$$
\begin{aligned}
\frac{dC_t(T,K)}{\partial_K^2 C_t(T,K)} &\approx \int_t^T ds\mathbb{E}_t[\left(\frac{\sigma_s'}{\sigma_s} - \frac{1}{S_t}\right)(\sigma_s a_s)^2|\frac{S_T}{S_t} = K]\frac{dS_t}{S_t^2} \\
&+ \frac{\nu}{S_t^2}\int_t^T e^{-k(s-t)}ds\mathbb{E}_t[(\sigma_s a_s)^2|\frac{S_T}{S_t} = K]\left(\rho dW_t + \sqrt{1-\rho^2}dW_t^{\perp}\right)
\end{aligned}
$$

(2) The conditional expectation can be replaced by a most-likely path approximation (S_s^*) (see (A.3)). We have

$$
\begin{aligned}
I_0 &:= \mathbb{E}_t[\left(\frac{\sigma_s'}{\sigma_s} - \frac{1}{S_t}\right)(\sigma_s a_s)^2|\frac{S_T}{S_t} = K] \approx \left(\frac{\sigma_s'}{\sigma_s}(s, S_s^*) - \frac{1}{S_t}\right)\mathbb{E}_t[(\sigma_s a_s)^2|S_s^*] \\
I_1 &:= \mathbb{E}_t[(\sigma_s a_s)^2|\frac{S_T}{S_t} = K] \approx \mathbb{E}_t[(\sigma_s a_s)^2|S_s^*]
\end{aligned}
$$

(3) Finally, we use that by definition $\mathbb{E}_t[(\sigma_s a_s)^2|S_s^*] := (S_s^*)^2\sigma_{\mathrm{Dup}}(s, S_s^*)^2$.

□

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