Path-Dependent Volatility

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1. the 2-factor Markovian PDV model

For $t \in [0, T]$, the 2-f PDV reads :

$$\begin{aligned}
\mathrm{d}S_t &= S_t \sigma\left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{1,t} &= -\lambda_1 R_{1,t}\,\mathrm{d}t + \lambda_1 \sigma\left(R_{1,t}, R_{2,t}\right) \mathrm{d}W_t \\
\mathrm{d}R_{2,t} &= \lambda_2 \left(\sigma\left(R_{1,t}, R_{2,t}\right)^2 - R_{2,t}\right) \mathrm{d}t
\end{aligned}$$

s.t $\sigma : (R_1, R_2) \mapsto \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2}$.

Let $X := (S, R_1, R_2)^\top$ be our state process. Then, X is a markovian-diffusion with dynamics :

$$\mathrm{d}X_t = \mu\left(t, X_t\right) \mathrm{d}t + \delta\left(t, X_t\right) \mathrm{d}W_t$$

with, for $(t, x) = (t, (S, R_1, R_2)) \in [0, T] \times \mathbb{R}^3$:

$$\mu(t,x) := \begin{pmatrix} 0 \\ -\lambda_1 R_1 \\ \lambda_2\left(\sigma\left(R_1, R_2\right)^2 - R_2\right) \end{pmatrix}, \quad \delta(t,x) = \begin{pmatrix} S\sigma\left(R_1, R_2\right) & 0 & 0 \\ 0 & \lambda_1 \sigma\left(R_1, R_2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The infinitesimal generator of X, denoted by $\mathcal{L}^X$, is defined for $u \in \mathcal{C}_c^{1,2}\left((0,T) \times \mathbb{R}^3\right)$ by

$$\mathcal{L}^X u := -\left(\lambda_1 R_1 \partial_{R_1} u + \lambda_2 R_2 \partial_{R_2} u\right) + \lambda_2 \sigma^2 \partial_{R_2} u + \frac{\sigma^2}{2}\left(S^2 \partial^2_{S,S} u + \lambda_1^2 \partial^2_{R_1,R_1} u\right)$$

The dynamics of the vol reads :

$$\mathrm{d}\sigma_t = \overbrace{\Big(\underbrace{-\beta_1 \lambda_1 R_{1,t}}_{\text{leverage effect}} + \underbrace{\frac{\beta_2 \lambda_2}{2} \frac{\sigma_t^2 - R_{2,t}}{\sqrt{R_{2,t}}}}_{\text{vol clustering}}\Big)}^{=\kappa\left(t,(S_u)_{u\le t},(\sigma_u)_{u\le t}\right) \text{ jointly path-dependent drift} \Rightarrow \text{strong Zumbach effect}} \mathrm{d}t + \underbrace{\beta_1 \lambda_1}_{\text{vol of vol constant}} \sigma_t \,\mathrm{d}W_t$$

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