# VEGA $KT$ FOR LSV MODELS: AN AD APPROACH

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ABSTRACT. Following our papers [HL13, Nat22] on the computation of the Vega $KT$ under the local volatility model, we consider local stochastic volatility models in the present article. Our algorithm is based on the calculation of the local Vega through Monte-Carlo simulation and algorithmic differentiation. We illustrate the efficiency of our algorithm on various numerical experiments.

## 1. INTRODUCTION

In the Black-Scholes framework, the volatility $\sigma$ is a constant parameter, thus the computation of the Vega risk is easy and can be obtained using a "bump and recompute" method which consists in pricing the exotic option under consideration with volatilities $\sigma$ and $\sigma + \delta\sigma$. Once the Vega risk has been computed for our exotic option with a (path-dependent) payoff $F_{\bar{T}}$ and price $\mathcal{O}_0$ at $t = 0$, it can be cancelled using a single Vanilla payoff with price $C_0$ at $t = 0$ and weight $\omega$ such that

$$\frac{\partial \mathcal{O}_0}{\partial \sigma} + \omega \frac{\partial C_0}{\partial \sigma} = 0$$

In the local volatility model (in short LV) or local stochastic volatility models (in short LSV), the computation of this sensitivity is more challenging. By construction these models are perfectly calibrated to the full implied volatility surface $(\Sigma_{TK})_{T\in[0,\bar{T}],K\in\mathbb{R}_+}$ and the Vega hedge consists in building a portfolio of Vanilla calls $C(T,K)$ with weights $(\omega(T,K))_{T\in[0,\bar{T}],K\in\mathbb{R}_+}$ such that

$$\frac{\delta \mathcal{O}_0}{\delta \Sigma_{TK}} + \int_0^\infty dK' \int_0^{\bar{T}} dT' \omega(T',K') \frac{\delta C(T',K')}{\delta \Sigma_{TK}} = 0, \quad \forall (T,K) \in [0,\bar{T}] \times \mathbb{R}_+ \tag{1.1}$$

Here $\bar{T}$ denotes the maturity of our exotic option. In LV or LSV models, the dependence in the implied volatility surface is book-coded through the so-called local volatility function which is a deterministic function of the time and the spot: $\sigma_{\mathrm{loc}}(\cdot,\cdot) : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$. As the sensitivity in (1.1) is ill-defined (an infinitesimal bump of the implied volatility surface around a point $(T,K)$ leads to a static arbitrage), one should instead consider[1]:

$$\frac{\delta \mathcal{O}_0}{\delta \sigma^2_{\mathrm{loc}}(T,K)} + \int_0^\infty dK' \int_0^{\bar{T}} dT' \omega(T',K') \frac{\delta C(T',K')}{\delta \sigma^2_{\mathrm{loc}}(T,K)} = 0, \quad \forall (T,K) \in [0,\bar{T}] \times \mathbb{R}_+ \tag{1.3}$$

For the LV model, the computation of this sensitivity and the weight $\omega(\cdot,\cdot)$ have been obtained explicitly in [HL13, Nat22]. In the present paper, we consider the case of LSV models. In a LSV model, besides the dependence on the time and the spot, the instantaneous volatility depends also non-linearly on the joint probability density of the spot and the volatility. The corresponding diffusion is a non-linear SDE in the sense of McKean-Vlasov with a singular kernel. As a result, the computation of the sensitivities with respect to the local volatility function is much more difficult.

[1]For any $(u,y) \in [0,\bar{T}] \times \mathbb{R}_+$, $\frac{\delta \mathcal{T}}{\delta \sigma^2_{\mathrm{loc}}(u,y)}(\sigma^2_{\mathrm{loc}})$ refers to Gâteaux's derivative along the direction $(t,s) \to \delta(t-u)\delta(s-y)$, with $\delta$ the Dirac function, of the functional $\mathcal{T}$ at $\sigma^2_{\mathrm{loc}}$. It is defined as the unique element $\Delta(u,y)$ (if it exists) such that:

$$\lim_{\epsilon\to 0} \frac{\mathcal{T}(\sigma^2_{\mathrm{loc}} + \epsilon \delta\sigma^2_{\mathrm{loc}}) - \mathcal{T}(\sigma^2_{\mathrm{loc}})}{\epsilon} = \int_0^{\bar{T}} dt \int_0^\infty ds \Delta(t,s) \delta\sigma^2_{\mathrm{loc}}(t,s) \tag{1.2}$$

A more precise discussion on functional derivatives can be found in [CH53].

The paper is structured as follows: In Section 2, we recap quickly LSV models. Section 3 recalls the most important results about Vega hedging already derived in [HL13, Nat22]. Section 4 states our Theorem 4.2 concerning the *exact* and *explicit* computation of the sensitivity to the local volatility function. For efficiency, we derive some approximations, easy to compute. The technical proofs are reported in Appendix A. Finally we conclude with some numerical experiments highlighting the efficiency of our algorithm.

## 2. LSV model in a nutshell

A LSV model is described by the following diffusion:

$$dS_t = \sigma(t, S_t)\, a_t dW_t \tag{2.1}$$

$$da_t = b(a_t)\, dt + \Sigma(a_t)\left(\rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp}\right), \quad a_0 = 1, \quad d\left\langle W, W^{\perp}\right\rangle_t = 0 \tag{2.2}$$

where $S_t$ denotes the asset price at $t$ and $a_t$ is the instantaneous stochastic volatility. The leverage function $\sigma$ is

$$\sigma^2(t, K) = \frac{\sigma_{\mathrm{loc}}^2(t, K)}{\mathbb{E}\left[a_t^2 \mid S_t = K\right]}, \quad \sigma_{\mathrm{loc}}^2(t, K) := \frac{2\partial_t C(t, K)}{\partial_K^2 C(t, K)} \tag{2.3}$$

where $\sigma_{\mathrm{loc}}$ is the Dupire volatility that is implied from market prices of Vanilla options $(C(t, K))_{(t,K)\in\mathbb{R}_+^2}$ (see [Dupire94]). By construction, $\mathcal{L}\mathrm{aw}(S_t) = \mu(t, \cdot)$ where $\mu(t, K) := \partial_K^2 C(t, K)$. For the sake of simplicity, we have not included here deterministic rate, dividends and repo, although this can be easily incorporated.[2]. We also suppose that the process $(a_t)_t$ is only a 1-dimensional Itô's diffusion but our result can be easily extended to the multi-dimensional case. As the volatility coefficient depends on the conditional law $\mathcal{L}\mathrm{aw}(a_t|S_t)$ through the conditional expectation (2.3), this diffusion belongs to the class of non-linear McKean-Vlasov SDEs (see Chapter 10 in [GuyonHL]). The existence of these models was shown in a recent paper (see [Djete]). The non-linear Fokker-Plank PDE satisfied by the joint probability density $p(t, s, a)$ is

$$\begin{aligned}
\partial_t p(t, s, a) &= \frac{1}{2}\partial_s^2\left(\sigma_{\mathrm{loc}}^2(t, s)\frac{a^2\int_{\mathbb{R}} p(t, s, a')da'}{\int_{\mathbb{R}}(a')^2 p(t, s, a')da'}p(t, s, a)\right) \\
&\quad + \rho\partial_{sa}\left(\Sigma(a)\sigma_{\mathrm{loc}}(t, s)a\sqrt{\frac{\int_{\mathbb{R}} p(t, s, a')da'}{\int_{\mathbb{R}}(a')^2 p(t, s, a')da'}}p(t, s, a)\right) + (\mathcal{L}^{\perp})^* p(t, s, a)
\end{aligned}$$

with $(\mathcal{L}^{\perp})^*$ is the adjoint operator of $\mathcal{L}^{\perp}\cdot := b(a)\partial_a \cdot + \frac{1}{2}\Sigma(a)\partial_a^2\cdot$. The conditional expectation in equation (2.3) is computed using a 2D PDE implementation or the particle method for multi-factor LSV models (see Chapter 11 in [GuyonHL] for a review).

The model is discretized using an log-Euler scheme with diffusion dates $t_0 := 0 < \ldots < t_n := \bar{T}$ including payoff fixing dates:

$$\ln\frac{S_{t_{i+1}}}{S_{t_i}} = -\frac{1}{2}\sigma_S(t_i, S_{t_i})^2 a_{t_i}^2\delta t_i + \sigma_S(t_i, S_{t_i})a_{t_i}\sqrt{\delta t_i}G_i \tag{2.4}$$

where $(G_i)_{0\le i\le n-1}$ are independent normal random variables and $\delta t_i := t_{i+1} - t_i$. We have set $\sigma_S(t, s) := \sigma(t, s)/s$.

**Example 2.1** (Bergomi's model (see Chapter 7 in [Bergomi]))**.** *In our numerical experiments (see Section 5), we take:* $a_t = e^{\nu X_t}$ *where* $(X_t)_t$ *is an Ornstein-Uhlenbeck process:*

$$dX_t = -\lambda X_t dt + \left(\rho dW_t + \sqrt{1-\rho^2}dW_t^{\perp}\right), X_0 = 0$$

*This model depends on three parameters: a mean-reversion* $k \in \mathbb{R}_+$*, a spot/vol correlation* $\rho \in (-1, 1)$ *and a volatility-of-volatility* $\nu \in \mathbb{R}_+$*. In practice, they are calibrated to Break-even Vomma and Vanna* [HLLuc].

[2] As is well known (see for instance [HL2009]), cash/yield dividends and non-zero repo and interest rate can be included by introducing a continuous positive local martingale $M_t$ defined by $S_t = A(t)M_t + B(t)$ where the two functions $A(\cdot)$ and $B(\cdot)$ depend on the cash/yield discrete dividends, repo and interest rate. Market Vanilla options on $S_t$ with strike $K$ map into $A(t)$ Vanilla options on $M_t$ with strike $(K - B(t))/A(t)$, and the model is then built for the process $(M_t)_{0\le t\le\bar{T}}$.

*Note that this model is unique if we impose that the forward variance curve process $T \mapsto \zeta_t^T := \mathbb{E}_t[a_T^2]$ admits (1) a 1D Markov representation in terms of a process $X_t$ and (2) the dynamics of $\zeta_t^T$ is time-homogeneous. Indeed, we know from interest-rate modelling that the volatility $\sigma(t,T)$ of the forward variance curve should be a separable function in $t$ and $T$ and the homogeneity property implies that $\sigma(t,T)$ should be an exponential function. It turns out that $\zeta_t^T$ can be written as an explicit function*

$$\zeta_t^T = \zeta_0^T e^{2\nu e^{-k(T-t)} X_t - \frac{(2\nu)^2}{2}\mathbb{E}[e^{-2k(T-t)}X_t^2]} \tag{2.5}$$

*In particular, we have that the instantaneous volatility $a_t := \sqrt{\zeta_t^t}$ is given by*

$$a_t = \sqrt{\zeta_0^t} e^{\nu X_t - \nu^2 \mathbb{E}[X_t^2]} \tag{2.6}$$

*and the SDE (2.1) can be written equivalently as*

$$dS_t \quad = \quad \frac{\sigma_{\mathrm{loc}}(t,S_t)}{\sqrt{\mathbb{E}[e^{2\nu X_t}|S_t]}} e^{\nu X_t} dW_t \tag{2.7}$$

*Here the deterministic function $\zeta_0^t$ has been absorbed into the local volatility function $\sqrt{\zeta_0^t}\sigma(t,S_t) \to \sigma(t,S_t)$ without loss of generality.*

## 3. Vega hedging with Vanillas

3.1. **Vega hedging in continuous time.** By following the same reasoning employed in (Theorem 2.4, [HL13]), and taking into consideration that the Vega for Vanilla options is the same for both LV and LSV models once the local volatility is the same, one can show that the Vega projection on Vanilla calls $\omega$ is the result of a Fokker-Planck type operator applied (see 3.1) to the sensitivity w.r.t. the local volatility. Therefore, the computation of the Vega hedge boils down to the computation of the Vega sensitivity $\frac{\delta \mathcal{O}_0}{\delta \sigma_{\mathrm{loc}}^2(t,K)}$:

**Proposition 3.1** ([Dupire08, HL13]). *For all $(t,K) \in [0,\bar{T}] \times \mathbb{R}_+$, $\omega$ is given by*

$$\omega(t,K) = \left(\partial_t \cdot + \frac{1}{2}\partial_K^2 \sigma_{\mathrm{loc}}^2(t,K)\cdot\right)\left(\frac{2}{\mu(t,K)}\frac{\delta \mathcal{O}_0}{\delta \sigma_{\mathrm{loc}}^2(t,K)}\right)$$

3.2. **Vega hedging in discrete time.** The Vega hedge in a discrete-time setting (where the log-Euler discretization (2.4) is used) is obtained in [Nat22]. A reminder of the result is provided hereafter. We define $\tilde{\Gamma}_{\mathrm{loc}}$ as the solution of the following equation:

$$\frac{\delta \mathcal{O}_0}{\delta \sigma_{\mathrm{loc}}(t_i,S)} = \frac{1}{S}\mu(t_i,S)\int_0^{+\infty} dK\ \tilde{\Gamma}_{\mathrm{loc}}(t_{i+1},K)\ \mathcal{V}^{\mathrm{BS}}\left(S,K,\sigma_{\mathrm{loc,S}}^2(t_i,S)\delta t_k\right), \quad \forall k \in [\![0,n-1]\!] \tag{3.1}$$

with $\sigma_{\mathrm{loc,S}}(t,S) := \sigma_{\mathrm{loc}}(t,S)/S$, $\tilde{\Gamma}_{\mathrm{loc}}(t_{n+1},\cdot) = 0$, and $\mathcal{V}^{BS}(S_0,K,v)$ is the Black-Scholes Vega of a Vanilla call with initial spot $S_0$, strike $K$ and variance $v$.

**Proposition 3.2** ([Nat22]). *$\omega$ is given by*

$$\omega(t_i,K) = \mathcal{L}^d \tilde{\Gamma}_{\mathrm{loc}}(t_i,K) \tag{3.2}$$

*where $\mathcal{L}^d$ is defined by*

$$\mathcal{L}^d \cdot (t_k,K) = \int_0^{+\infty} dK' \cdot (t_{k+1},K')\partial_K^2 P_{\mathrm{BS}}(K,K',\sigma_{\mathrm{loc,S}}(t_k,S)^2\delta t_k) - \cdot(t_k,K)$$

*where $P_{\mathrm{BS}}(S_0,K,v)$ denotes the Black-Scholes price of a Vanilla call with initial spot $S_0$, strike $K$ and variance $v$.*

3.3. **Vega hedging with finite strikes.** In the market, we only have access to a limited set of strikes. Let us denote the implied volatility surface at our disposal $(\Sigma_{t_i,K_j})_{i\in[\![1,n]\!],j\in\mathcal{K}}$, and its interpolated version by $(\Sigma_{t_i,K})_{i,K\in\mathbb{R}_+}$. In this situation, we are looking for a strategy $(\omega_{i,j})_{i,j}$ such that:

$$\frac{\partial\mathcal{O}_0}{\partial\Sigma_{t_i,K_j}} + \sum_{k,m}\omega_{k,m}\frac{\partial C(t_k,K_m)}{\partial\Sigma_{t_i,K_j}} = 0, \quad (i,j)\in[\![1,n]\!]\times\mathcal{K} \tag{3.3}$$

The explicit formula of $(\omega_{i,j})_{i,j}$ is derived in [Guennoun], and we recall his result:

**Proposition 3.3** ([Guennoun])**.** *$(\omega_{i,j})_{i,j}$ verifying equation (3.3) is given by*

$$\omega_{i,j} \quad = \quad \int_0^{+\infty} dK\ \omega(t_i,K)\ \psi_{ij}(K), \quad (i,j)\in[\![1,n]\!]\times\mathcal{K} \tag{3.4}$$

*with for all $(i,j)\in[\![1,n]\!]\times\mathcal{K}$, $\psi_{i,j}$ is a kernel function defined by:*

$$\psi_{ij}(K) := \frac{\partial\Sigma_{t_i,K}}{\partial\Sigma_{t_i,K_j}}\frac{\mathcal{V}^{\mathrm{BS}}(S_0,K,\Sigma^2_{t_i,K}t_i)}{\mathcal{V}^{\mathrm{BS}}(S_0,K_j,\Sigma^2_{t_i,K_j}t_i)}, \quad \forall K\in\mathbb{R}_+$$

## 4. Explicit computation of the local Vega

**Definition 4.1.** *(Local Gamma) We define the local Gamma function $\Gamma_{\mathrm{loc}}$ for all $(t,K)\in\left[0,\bar{T}\right]\times\mathbb{R}_+$ by*

$$\Gamma_{\mathrm{loc}}(t,K) := \mathbb{E}\left[a_t^2\partial_{ss}\mathcal{O}_t + \frac{\rho a_t\Sigma(a_t)}{\sigma(t,S_t)}\partial_{sa}\mathcal{O}_t \mid S_t = K\right]$$

*with $\mathcal{O}_t := \mathbb{E}_t[F_{\bar{T}}]$.*

This local Gamma will be computed by AD (see Section 4.4).

4.1. **Operator $\mathcal{P}_t$.** Let $t\in[0,\bar{T}]$ and define the operator $\mathcal{P}_t : C^{(0,\infty,\infty)}\left(\left[0,\bar{T}\right]\times\mathbb{R}^2_+;\mathbb{R}\right)\to C^{(0,\infty)}\left(\left[0,\bar{T}\right]\times\mathbb{R}_+;\mathbb{R}\right)$ as follows:

$$\begin{aligned}\mathcal{P}_tF(u,y) := \frac{1_{t\geq u}}{\sigma(u,y)}\Bigg\{&\mathbb{E}\left[\frac{D_u\left[F(t,S_t,a_t)Y_t\right]}{Y_u}a_u \mid S_u = y\right]\\ &-\mathbb{E}\left[F(t,S_t,a_t)\frac{Y_t}{Y_u}a_u^2 \mid S_u = y\right]\partial_y\left(\sigma(u,y)\right)\Bigg\}\end{aligned} \tag{4.1}$$

where $D$ is the Malliavin derivative w.r.t. the Brownian $W$.. For completeness, we give the explicit expression of this Malliavin derivative[3]:

$$\begin{aligned}\mathbb{E}\left[\frac{D_u\left[F(t,S_t,a_t)Y_t\right]}{Y_u}a_u \mid S_u = y\right] &= \mathbb{E}\left[\frac{Y_t}{Y_u}a_u\left(\partial_SF(t,S_t,a_t)\left(A_uY_t+B_uX_t\right)+\partial_aF(t,S_t,a_t)B_uZ_t\right) \mid S_u = y\right]\\ &+ \mathbb{E}\left[\frac{F(t,S_t,a_t)\left(I_uC_t+A_uJ_t+B_uQ_t\right)}{Y_u}a_u \mid S_u = y\right]\end{aligned}$$

[3]A straightforward computation gives:

$$\begin{aligned}D_sS_t &= A_sY_t+B_sX_t, \quad B_s := \rho\frac{\Sigma(a_s)}{Z_s}, \quad A_s := \frac{1}{Y_s}\left(a_s\sigma(s,S_s)-\rho\frac{\Sigma(a_s)}{Z_s}X_s\right)\\ D_sa_t &= B_sZ_t\\ D_sY_t &= I_sC_t+A_sJ_t+B_sQ_t, \quad I_s := \frac{1}{C_s}\left(\partial_S\sigma(s,S_s)a_sY_s-A_sJ_s-B_sQ_s\right)\end{aligned}$$

where the processes $(Y_t, Z_t, X_t, C_t, J_t, Q_t)$ are:

$$\begin{aligned}
\frac{dY_t}{Y_t} &= a_t \partial_S \sigma(t, S_t) dW_t, \quad Y_0 := 1 \\
\frac{dZ_t}{Z_t} &= b'(a_t)dt + \Sigma'(a_t)\left(\rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp}\right), \quad Z_0 := 1 \\
dX_t &= \left(\sigma(t, S_t) Z_t + X_t a_t \partial_S \sigma(t, S_t)\right) dW_t, \quad X_0 := 0 \\
\frac{dC_t}{C_t} &= \partial_S^2 \sigma(t, S_t) Y_t a_t dW_t, \quad C_0 := 1 \\
dJ_t &= \left(\partial_S \sigma(t, S_t) J_t + \partial_S^2 \sigma(t, S_t) Y_t^2\right) a_t dW_t, \quad J_0 := 0 \\
dQ_t &= \left(\partial_S \sigma(t, S_t) Q_t a_t + \partial_S^2 \sigma(t, S_t) Y_t X_t a_t + \partial_S \sigma(t, S_t) Y_t Z_t\right) dW_t, \quad Q_0 := 0
\end{aligned}$$

4.2. **Operator $\mathcal{A}$.** Let us introduce the operator $\mathcal{A} : C^{(0,\infty)}\left([0,\bar{T}] \times \mathbb{R}_+; \mathbb{R}\right) \to C^{(0,\infty,\infty)}\left([0,\bar{T}] \times \mathbb{R}_+^2; \mathbb{R}\right)$ defined by:

$$\mathcal{A}\phi\,(t, s, a) := \frac{1}{2}\partial_s\left(\sigma^2(t,s)\,\phi(t,s)\left(1 - \frac{\sigma^2(t,s)}{\sigma_{\text{loc}}^2(t,s)}a^2\right)\right) \tag{4.2}$$

Note that $\mathcal{P}_t\mathcal{A}$ is then a linear operator on $C^{(0,\infty)}\left([0,\bar{T}] \times \mathbb{R}_+; \mathbb{R}\right)$. The operators $\mathcal{P}_t$ and $\mathcal{P}_t\mathcal{A}$ will be used below to establish a nice representation of the sensitivity $\frac{\delta\mathcal{O}_0}{\delta\sigma_{\text{loc}}^2(u,y)}$.

4.3. **Main result.** For use below, we define the function $F$ for all $(t, s, a) \in [0, \bar{T}] \times \mathbb{R}_+^2$ by:

$$F\,(t, s, a) := \partial_s\left\{\sigma^2(t,s)\Gamma_{\text{loc}}(t,s)\left(\frac{\sigma^2(t,s)}{\sigma_{\text{loc}}^2(t,s)}a^2 - 1\right)\right\}$$

**Theorem 4.2.**

**(1)** *Let $(u, y) \in [0, \bar{T}] \times \mathbb{R}_+$, we have*

$$\begin{aligned}
\frac{2\frac{\delta\mathcal{O}_0}{\delta\sigma_{\text{loc}}^2(u,y)}}{\mu(u,y)} = \left(\frac{\sigma(u,y)}{\sigma_{\text{loc}}(u,y)}\right)^2 &\Bigg\{\Gamma_{\text{loc}}(u,y) \\
&- \frac{1}{2}\sum_{i=0}^{\infty}\int_0^{\bar{T}} dt_1 \cdots \int_0^{t_i} dt_{i+1}\mathcal{P}_{t_{i+1}}\mathcal{A}\mathcal{P}_{t_i}\cdots\mathcal{A}\mathcal{P}_{t_1}F(u,y)\Bigg\}
\end{aligned} \tag{4.3}$$

*The sum of iterated time-ordered integrals in equation (4.3) converges absolutely by the bound*

$$\left|\int_0^{\bar{T}} dt_1 \cdots \int_0^{t_i} dt_{i+1}\mathcal{P}_{t_{i+1}}\mathcal{A}\mathcal{P}_{t_i}\cdots\mathcal{A}\mathcal{P}_{t_1}F(u,y)\right| \le \frac{(\bar{T}-u)^n}{n!}C^n$$

*for a finite constant $C$.*

**(2)** *The sum in equation (4.3) can be written as the solution of a linear flow:*

$$\frac{2\frac{\delta\mathcal{O}_0}{\delta\sigma_{\text{loc}}^2(u,y)}}{\mu(u,y)} = \left(\frac{\sigma(u,y)}{\sigma_{\text{loc}}(u,y)}\right)^2\left(\Gamma_{\text{loc}}(u,y) - \frac{1}{2}\int_0^{\bar{T}} dt\, X_t^t(u,y)\right)$$

*where $X_\cdot^t$ is the unique solution of the linear flow on $C^{(0,\infty)}([0,\bar{T}] \times \mathbb{R}_+; \mathbb{R})$:*

$$\frac{dX_s^t}{ds} = (\mathcal{P}_s\mathcal{A})\,(X_s^t), \quad X_0^t := \mathcal{P}_t F \tag{4.4}$$

*Proof.* From Proposition A.1 detailed in the appendix,

$$2\frac{\delta\mathcal{O}_0}{\delta\sigma_{\text{loc}}^2(u,y)} = \left(\frac{\sigma(u,y)}{\sigma_{\text{loc}}(u,y)}\right)^2\Gamma_{\text{loc}}(u,y)\,\mu(u,y) - \int_0^{\bar{T}} dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma_{\text{loc}}^2(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t\right]}{\delta\sigma_{\text{loc}}^2(u,y)}\right]$$

Using Lemma A.3, the second term in the equation above can be rewritten as

$$2\frac{\delta \mathcal{O}_0}{\delta \sigma^2_{\mathrm{loc}}(u,y)} = \left(\tfrac{\sigma(u,y)}{\sigma_{\mathrm{loc}}(u,y)}\right)^2 \Gamma_{\mathrm{loc}}(u,y)\,\mu(u,y) - \textstyle\int_0^{\bar{T}} dt \mathbb{E}\left[F(t,S_t,a_t)\, S^{u,y}_t\right]$$

where $S^{u,y}_t := \frac{\delta S_t}{\delta \sigma^2_{\mathrm{loc}}(u,y)}$. The dynamics of $S^{u,y}_t$ is provided in Lemma A.4. Finally by iterating the equation in Lemma A.5, we get our final result **(1)** and **(2)** (see Lemma A.6). □

**Remark 4.3** (Expansion in the volatility-of-volatility)**.** *The sum in equation (4.3) can be seen as an expansion in volatility-of-volatility. If we take for example* $\Sigma(a) = \nu a$*, the* $n$*-iterated integral is of order* $(\nu^2(\bar{T}-u))^n$*.*

4.4. **Computing $\Gamma_{\mathrm{loc}}$ by AD.** From Proposition (A.2), we have that

$$\Gamma_{\mathrm{loc}}(u,y) = \frac{1}{\mu(u,y)\sigma(u,y)}\frac{\delta \mathcal{O}_0}{\delta\sigma(u,y)}$$

The derivative $\frac{\delta \mathcal{O}_0}{\delta\sigma(u,y)}$ can then be computed by algorithmic differentiation as in [Nat22]:

**Proposition 4.4.** *The local Gamma at* $t_i$ *can be written as:*

(4.5)
$$\mu(t_i,S)\sigma(t_i,S)\Gamma_{\mathrm{loc}}(t_i,S) = \mathbb{E}_0\left[S_{t_{i+1}}\left(-a^2_{t_i}\sigma_S(t_i,S_{t_i})\delta t_i + \sqrt{\delta t_i}a_{t_i}G_i\right)\overline{S_{t_{i+1}}}\delta(S_{t_i}-S)\right], \quad i \in [[0,n-1]]$$

*where* $\bar{S}_.$ *are given by the following recurrence formula:*

$$\begin{cases} \overline{S_{t_n}} = \partial_{S_{t_n}} F_{\bar{T}} \\ \overline{S_{t_i}} = \partial_{S_{t_i}} F_{\bar{T}} + \partial_{S_{t_i}} S_{t_{i+1}}\overline{S_{t_{i+1}}}, \quad \forall i \in [[0,n-1]] \end{cases}$$

*with*

$$\partial_{S_{t_i}} S_{t_{i+1}} = \frac{S_{t_{i+1}}}{S_{t_i}}\left[1 + \partial_{S_{t_i}}\sigma_S(t_i,S_{t_i})S_{t_i}\left(-a^2_{t_i}\sigma_S(t_i,S_{t_i})\delta t_i + \sqrt{\delta t_i}a_{t_i}G_i\right)\right]$$

$\partial_{S_{t_i}} F_{\bar{T}}$ *denotes the payoff derivative with respect to the spot at observation date* $t_i$*. If* $t_i$ *is not a payoff fixing date then* $\partial_{S_{t_i}} F_{\bar{T}} := 0$*.*

The proof is straightforward and left to the reader.

**Remark 4.5.** *In practice* $\partial_{S_{t_i}} F_{\bar{T}}$ *and* $\partial_{S_{t_i}}\sigma_S(t_i,S_{t_i})$ *are computed by finite differences. In the case where the payoff function is not differentiable, this provides a natural regularization.*

4.5. **Approximation.**

**Corollary 4.6** (See Appendix A for the proof.)**.** *Let* $(u,y) \in [0,\bar{T}] \times \mathbb{R}_+$*, we have*

$$\frac{2\frac{\delta\mathcal{O}_0}{\delta\sigma^2_{\mathrm{loc}}(u,y)}}{\mu(u,y)} = \left(\frac{\sigma(u,y)}{\sigma_{\mathrm{loc}}(u,y)}\right)^2\left[\Gamma_{\mathrm{loc}}(u,y) - \frac{1}{2}\left(\bar{T}-u\right)\Upsilon(u,y) + O((\bar{T}-u)^2)\right] \tag{4.6}$$

*where*

$$\Upsilon(u,y) := -\partial^2_y\left(\sigma^2(u,y)\Gamma_{\mathrm{loc}}(u,y)\right)\left(\frac{\sigma_{\mathrm{loc}}(u,y)}{\sigma(u,y)}\right)^2 + \partial^2_y\left(\frac{\sigma^4(u,y)}{\sigma^2_{\mathrm{loc}}(u,y)}\Gamma_{\mathrm{loc}}(u,y)\right)\mathbb{E}[a^4_u|S_u=y]$$
$$+ \frac{2\rho}{\sigma(u,y)}\partial_y\left(\frac{\sigma^4(u,y)}{\sigma^2_{\mathrm{loc}}(u,y)}\Gamma_{\mathrm{loc}}(u,y)\right)\mathbb{E}\left[a^2_u\Sigma(a_u) \mid S_u = y\right]$$

4.6. **Recipe.** Our current implementation of the calculation of the Vega hedging weights from the AD local Vega is as follows:

- We compute $\Gamma_{\mathrm{loc}}$ by Monte-Carlo using Proposition 4.4. In practise the Dirac $\delta(S_{t_i}-S)$ is regularized using a kernel.
- Then we use our approximation (4.6) to get $\frac{\delta \mathcal{O}_0}{\delta \sigma^2_{\mathrm{loc}}(u,y)}$. $\mathbb{E}[a_u^4|S_u=y]$ and $\mathbb{E}[a_u^2\Sigma(a_u)|S_u=y]$ are computed by PDE or particle method during the calibration phase. As an approximation for the Bergomi model for which $\Sigma(a)=\nu a$ valid at order one in the volatility-of-volatility $\nu$, we use

$$\begin{aligned}\mathbb{E}[a_u^4|S_u=y] &\approx \left(\frac{\sigma_{\mathrm{loc}}(u,y)}{\sigma(u,y)}\right)^4\\ \mathbb{E}[a_u^2\Sigma(a_u)|S_u=y] &\approx \nu\left(\frac{\sigma_{\mathrm{loc}}(u,y)}{\sigma(u,y)}\right)^3\end{aligned}$$

- We compute the local Gamma $\bar{\Gamma}_{\mathrm{loc}}$ at each $t_{k+1}$ using Equation (3.1) on each point of the local volatility grid. In practice, we use the following approximation, which is simpler to compute:

$$\bar{\Gamma}(t_{k+1},S) = \frac{\frac{\delta \mathcal{O}_0}{\delta \sigma^2_{\mathrm{loc}}(t_k,S)}}{2\mu(t_k,S)\delta t_k} + O(\delta t_k) \tag{4.7}$$

- We apply then Equation (3.2) to compute the density $\omega(t_k,K)$ on each point of the local volatility grid. The integrals are computed as discrete sums over the local volatility grid.
- Finally we use Equation (3.4) to compute the discrete weights $\omega_{ij}$. Again the integrals are computed as discrete sums over the local volatility grid.

## 5. Numerical results

All numerical results are expressed in units of Vega. In order to avoid numerical convergence effects, our AD Vega $KT$ results are computed with a huge number $5\times 10^6$ Monte-Carlo trials, well-above the proper number needed for achieving convergence. For comparison, we give also the AD Vega $KT$ for the LV model, calibrated to the same implied volatility (SX5E as of 11/10/2022, with no rate/repo and dividends) as our LSV model. We have used the one-factor Bergomi model, detailed in Example 2.1, where $\nu=1$, $k=3$ and $\rho=-0.86$ that are typical parameters calibrated to break-even [HLLuc].

5.1. **Vanilla options.** We first consider Vanilla options, seen as a strip of Call options. For such options, the Vega KT should be localized on the Call options. In particular, the Vega KT for the LV and LSV models should coincide.

5.1.1. *Call option.* We present the result for an 1Y ATM call, see Figure 1. We obtain that the Vega risk is localized on the strike 100% at 1Y maturity as expected.

5.1.2. *Call spread option.* We then consider an 1Y Call Spread option $95\%-105\%$, see Figure 2. We obtain that the Vega risk is localized on the two strikes $95\%-105\%$ and 1Y maturity as expected.

5.2. **Path-dependent options.** We present the results for path-dependent options. For such options, the Vega KT for the LV and LSV models should not coincide but we expect that the results should be closed. Indeed when $\nu$ is small, the LV and LSV dynamics are closed, resulting in a similar Vega risk. Note that we have chosen $\nu=1.0$ which is not a small volatility-of-volatility.

- 1Y ATM Put option with 6 months Up And Out barrier at level 80%, $\left(1-\frac{S_{t_i}}{S_0}\right)^+\prod_{t_i\in(6M,1Y)}1_{\frac{S_{t_i}}{S_0}<0.8}$ see Figure 3
- 5Y ATM Asian Put option with monthly Asian $(1-\frac{1}{N}\sum_{i=1}^N\frac{S_{t_i}}{S_0})^+$, see Figure 4.
- 5Y Call Lookback Max 130% - monthly Lookback dates $(\max_{0\le t_i\le T}\frac{S_{t_i}}{S_0}-1.3)^+$, see Figure 5.

- Autocallable, see Figure 6.

## 6. Conclusion

In this paper, we have presented a method for the construction of the hedging portfolio for an exotic trade, in a LSV model using Monte Carlo method and AD. The precise relation between local volatility sensitivities and vanilla hedges is precisely stated in Theorem 4.2. Our method is applicable for multi-asset option and therefore is highly relevant for practical purposes as LSV models are commonly used among practitioners.

## Appendix A. Proofs

**Proposition A.1.** *The sensitivity w.r.t.* $\sigma_{\text{loc}}$ *at* $(u,y)\in[0,\bar{T}]\times\mathbb{R}_+$ *can be expressed as follows:*

$$\begin{aligned}2\frac{\delta\mathcal{O}_0}{\delta\sigma^2_{\text{loc}}(u,y)} &= \left(\frac{\sigma(u,y)}{\sigma_{\text{loc}}(u,y)}\right)^2\Gamma_{\text{loc}}(u,y)\,\mu(u,y)\\ &\quad-\int_0^{\bar{T}}dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\,\frac{\delta\mathbb{E}\left[a_t^2\mid S_t\right]}{\delta\sigma^2_{\text{loc}}(u,y)}\right]\end{aligned}\tag{A.1}$$

*Proof.* Let $F_{\bar{T}}$ be the payoff associated to $\mathcal{O}_0$. We consider then $\mathcal{O}(t,s,a)=\mathbb{E}\left[F_{\bar{T}}\mid S_t=s,a_t=a\right]$ the price of the exotic option at $t\in[0,\bar{T}]$ conditioned to $(S_t=s,a_t=a)$. $\mathcal{O}$ can also potentially depend on some additional path-dependent variables that we do not display. It holds that

$$\partial_t\mathcal{O}(t,s,a)+\frac{1}{2}\frac{\sigma^2_{\text{loc}}(t,s)}{\mathbb{E}\left[a_t^2\mid S_t=s\right]}a^2\partial_{ss}\mathcal{O}(t,s,a)+\mathcal{L}^{\perp}\mathcal{O}(t,s,a)+\frac{\rho\sigma_{\text{loc}}(t,s)\,a\Sigma(a)}{\left(\mathbb{E}\left[a_t^2\mid S_t=s\right]\right)^{\frac{1}{2}}}\partial_{sa}\mathcal{O}(t,s,a)=0\tag{A.2}$$

By differentiating equation A.2 with respect to $\sigma^2_{\text{loc}}$ at $(u,y)$, we get

$$\begin{aligned}&(\partial_t+\mathcal{L})\frac{\delta\mathcal{O}}{\delta\sigma^2_{\text{loc}}(u,y)}(t,s,a)+\frac{1}{2}\frac{\sigma^2(t,s)}{\sigma^2_{\text{loc}}(t,s)}\delta(t-u)\,\delta(s-y)\,a^2\partial_{ss}\mathcal{O}(t,s,a)\\ -\frac{1}{2}\frac{\sigma^4(t,s)}{\sigma^2_{\text{loc}}(t,s)}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=s\right]}{\delta\sigma^2_{\text{loc}}(u,y)}a^2\partial_{ss}\mathcal{O}(t,s,a)&+\frac{\rho\sigma(t,s)}{2\sigma^2_{\text{loc}}(t,s)}\delta(t-u)\,\delta(s-y)\,a\Sigma(a)\partial_{sa}\mathcal{O}(t,s,a)\\ &-\frac{1}{2}\frac{\rho\sigma^3(t,s)}{\sigma^2_{\text{loc}}(t,s)}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=s\right]}{\delta\sigma^2_{\text{loc}}(u,y)}a\Sigma(a)\partial_{sa}\mathcal{O}(t,s,a)\end{aligned}$$

where $\mathcal{L}$ is the infinitesimal operator associated to the Itô diffusion $(S_t,a_t)_{t\in[0,\bar{T}]}$. Taking into account that $\frac{\delta\mathcal{O}}{\delta\sigma^2_{\text{loc}}(u,y)}\left(\bar{T},s,a\right)=0$ at maturity, we can write, using Feynman-Kac theorem:

$$\begin{aligned}2\frac{\delta\mathcal{O}_0}{\delta\sigma^2_{\text{loc}}(u,y)}&=\mathbb{E}\left[\int_0^{\bar{T}}\frac{\sigma^2(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\left\{a_t^2\partial_{ss}\mathcal{O}_t+\frac{\rho}{2\sigma(t,S_t)}a_t\Sigma(a_t)\partial_{sa}\mathcal{O}_t\right\}\delta(t-u)\,\delta(s-y)\,dt\right]\\ &\quad-\mathbb{E}\left[\int_0^{\bar{T}}\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\left\{a_t^2\partial_{ss}\mathcal{O}_t+\frac{\rho}{2\sigma(t,S_t)}a_t\Sigma(a_t)\partial_{sa}\mathcal{O}_t\right\}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=\cdot\right]}{\delta\sigma^2_{\text{loc}}(u,y)}(S_t)\,dt\right]\end{aligned}$$

Here $\mathcal{O}_t:=\mathcal{O}(t,S_t,a_t)$. Thanks to Bayes formula, we deduce that

$$\begin{aligned}2\frac{\delta\mathcal{O}_0}{\delta\sigma^2_{\text{loc}}(u,y)}&=\mathbb{E}\left[\frac{\sigma^2(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\left\{a_t^2\partial_{ss}\mathcal{O}_t+\frac{\rho}{2\sigma(t,S_t)}a_t\Sigma(a_t)\partial_{sa}\mathcal{O}_t\right\}dt\mid S_t=y\right]\mu(u,y)\,1_{\bar{T}\geq u}\\ &\quad-\int_0^{\bar{T}}dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\left\{a_t^2\partial_{ss}\mathcal{O}_t+\frac{\rho}{2\sigma(t,S_t)}a_t\Sigma(a_t)\partial_{sa}\mathcal{O}_t\right\}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=\cdot\right]}{\delta\sigma^2_{\text{loc}}(u,y)}(S_t)\right]\end{aligned}$$

Finally, it suffices to condition on $S_t$ in order to get our formula for the sensitivity. ☐

**Proposition A.2.** *The sensitivity w.r.t.* $\sigma$ *at* $(u,y)\in[0,\bar{T}]\times\mathbb{R}_+$ *can be expressed as follows:*

$$\frac{\delta\mathcal{O}_0}{\delta\sigma(u,y)}=\Gamma_{\text{loc}}(u,y)\mu(u,y)\sigma(u,y)$$

*Proof.* The PDE satisfied by $\mathcal{O}$ can be written in terms of $\sigma(\cdot,\cdot)$ as:

$$\partial_t \mathcal{O}(t,s,a) + \frac{1}{2}\sigma^2(t,s)a^2\partial_{ss}\mathcal{O}(t,s,a) + \mathcal{L}^{\perp}\mathcal{O}(t,s,a) + \rho\sigma(t,s)\,a\Sigma(a)\partial_{sa}\mathcal{O}(t,s,a) = 0 \tag{A.3}$$

By following closely the derivation in Proposition A.1, we get

$$2\frac{\delta\mathcal{O}_0}{\delta\sigma^2(u,y)} = \Gamma_{\text{loc}}(u,y)\,\mu(u,y)\,1_{\bar{T}\geq u}$$

from which we derive our result. □

**Lemma A.3.** *The second term in equation A.1 can be rewritten as*

$$\int_0^{\bar{T}} dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t\right]}{\delta\sigma^2_{\text{loc}}(u,y)}\right] = \int_0^{\bar{T}} dt\mathbb{E}\left[F(t,S_t,a_t)\,S_t^{u,y}\right]$$

*where* $S_t^{u,y} := \frac{\delta S_t}{\delta\sigma^2_{\text{loc}}(u,y)}$ *and $F$ was defined in Theorem 4.2.*

*Proof.* First observe that $\mathbb{E}\left[a_t^2 \mid S_t = s\right]$ can be written as:

$$\mathbb{E}\left[a_t^2 \mid S_t = s\right] = \frac{\mathbb{E}\left[a_t^2\delta(S_t - s)\right]}{\mathbb{E}\left[\delta(S_t - s)\right]} \tag{A.4}$$

thus, for all $t \geq u$:

$$\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t = s\right]}{\delta\sigma^2_{\text{loc}}(u,y)} = \frac{1}{\mu(t,s)}\left\{\mathbb{E}\left[a_t^2\delta'(S_t - s)\,S_t^{u,y}\right] - \frac{\sigma^2_{\text{loc}}(t,s)}{\sigma^2(t,s)}\mathbb{E}\left[\delta'(S_t - s)\,S_t^{u,y}\right]\right\} \tag{A.5}$$

where $\delta'$ should be understood as the derivative of the Dirac function in the distribution sense. We obtain then:

$$\begin{aligned}
&\int_0^{\bar{T}} dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t\right]}{\delta\sigma^2_{\text{loc}}(u,y)}\right] \\
&= \int_0^{\bar{T}} dt\int_0^{\infty} dK\frac{\sigma^4(t,K)}{\sigma^2_{\text{loc}}(t,K)}\Gamma_{\text{loc}}(t,K)\left\{-\mathbb{E}\left[a_t^2\partial_K\delta(S_t - K)\,S_t^{u,y}\right] + \frac{\sigma^2_{\text{loc}}(t,K)}{\sigma^2(t,K)}\mathbb{E}\left[\partial_K\delta(S_t - K)\,S_t^{u,y}\right]\right\}
\end{aligned}$$

and, using an integration by parts in the distribution sense yields

$$\begin{aligned}
\int_0^{\bar{T}} dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t\right]}{\delta\sigma^2_{\text{loc}}(u,y)}\right] &= \int_0^{\bar{T}} dt\int_0^{\infty} dK\partial_K\left(\frac{\sigma^4(t,K)}{\sigma^2_{\text{loc}}(t,K)}\Gamma_{\text{loc}}(t,K)\right)\mathbb{E}\left[a_t^2\delta(S_t - K)\,S_t^{u,y}\right] \\
&\quad - \int_0^{\bar{T}} dt\int_0^{\infty} dK\partial_K\left(\sigma^2(t,K)\,\Gamma_{\text{loc}}(t,K)\right)\mathbb{E}\left[\delta(S_t - K)\,S_t^{u,y}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{\bar{T}} dt\mathbb{E}\left[\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\frac{\delta\mathbb{E}\left[a_t^2 \mid S_t\right]}{\delta\sigma^2_{\text{loc}}(u,y)}\right] &= \int_0^{\bar{T}} dt\mathbb{E}\left[\partial_S\left(\frac{\sigma^4(t,S_t)}{\sigma^2_{\text{loc}}(t,S_t)}\Gamma_{\text{loc}}(t,S_t)\right)\mathbb{E}\left[a_t^2 S_t^{u,y} \mid S_t\right]\right] \\
&\quad - \int_0^{\bar{T}} dt\mathbb{E}\left[\partial_S\left(\sigma^2(t,S_t)\,\Gamma_{\text{loc}}(t,S_t)\right)\mathbb{E}\left[S_t^{u,y} \mid S_t\right]\right]
\end{aligned}$$

This implies our result once we use our definition of the function $F$ stated in Theorem 4.2. □

**Lemma A.4.** *The process* $\left(S_t^{u,y}\right)_t$ *can be expressed as*

$$\begin{aligned}
S_t^{u,y} = Y_t\int_0^t \frac{a_s}{2Y_s}&\left(\frac{\sigma(u,y)}{\sigma^2_{\text{loc}}(u,y)}\delta(S_s - y)\,\delta(s - u) - \frac{\sigma^3(s,S_s)}{\sigma^2_{\text{loc}}(s,S_s)}\frac{\delta\mathbb{E}\left[a_s^2 \mid S_s = .\right]}{\delta\sigma^2_{\text{loc}}(u,y)}(S_s)\right) \\
&\left(dW_s - a_s\partial_S\left(\sigma(t,S_s)\right)ds\right)
\end{aligned}$$

*where* $\frac{\delta\mathbb{E}\left[a_s^2|S_s=.\right]}{\delta\sigma_{\text{loc}}^2(u,y)}$ *is given by Equation (A.5). The tangent process* $(Y_t)_t$ *is defined by*

$$dY_t = \partial_S\left(\sigma\left(t,S_t\right)\right)a_tY_tdW_t,\ Y_0 := 1$$

*Proof.* Let us find a process $(\Sigma_t)_t$ of the form $d\Sigma_t = \mu\left(t,\Sigma_t\right)dt + \beta\left(t,\Sigma_t\right)dW_t$, such that $S_t^{u,y} = Y_t\Sigma_t$. On one hand, we have by applying Itô's formula:

$$dS_t^{u,y} = \left(Y_t\mu\left(t,\Sigma_t\right) + \partial_S\left(\sigma\left(t,S_t\right)\right)a_tY_t\beta\left(t,\Sigma_t\right)\right)dt + \left(\beta\left(t,\Sigma_t\right)Y_t + S_t^{u,y}\partial_S\left(\sigma\left(t,S_t\right)\right)a_t\right)dW_t$$

On the other hand

$$dS_t^{u,y} = \left(\frac{1}{2}\frac{\sigma\left(t,S_t\right)}{\sigma_{\text{loc}}^2\left(t,S_t\right)}\delta\left(S_t-y\right)\delta\left(t-u\right) + \partial_S\left(\sigma\left(t,S_t\right)\right)S_t^{u,y} - \frac{1}{2}\frac{\sigma^3\left(t,S_t\right)}{\sigma_{\text{loc}}^2\left(t,S_t\right)}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=.\right]}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}\left(S_t\right)\right)a_tdW_t$$

hence by identification:

$$\beta\left(t,\Sigma_t\right) = \frac{1}{2Y_t}\left(\frac{\sigma\left(t,S_t\right)}{\sigma_{\text{loc}}^2\left(t,S_t\right)}\delta\left(S_t-y\right)\delta\left(t-u\right) - \frac{\sigma^3\left(t,S_t\right)}{\sigma_{\text{loc}}^2\left(t,S_t\right)}\frac{\delta\mathbb{E}\left[a_t^2\mid S_t=.\right]}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}\left(S_t\right)\right)a_t \tag{A.6}$$

and

$$\mu\left(t,\Sigma_t\right) = -\partial_S\left(\sigma\left(t,S_t\right)\right)a_t\beta\left(t,\Sigma_t\right)$$

which concludes our proof. □

**Lemma A.5.** *Let* $\tilde{F}$ *be in* $C^{(0,1,1)}([0,\bar{T}]\times\mathbb{R}_+^2;\mathbb{R})$. *It holds that*

$$\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)S_t^{u,y}\right] = \frac{1}{2}\mu(u,y)\left(\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}\left(u,y\right)}\right)^2\mathcal{P}_t\tilde{F}(u,y) + \int_0^t ds\mathbb{E}\left[\mathcal{A}\mathcal{P}_t\tilde{F}(s,S_s,a_s)S_s^{u,y}\right] \tag{A.7}$$

*where the operators* $\mathcal{P}_t$ *and* $\mathcal{A}$ *are defined in Section 4.*

*Proof.* By plugging the representation of $S_t^{u,y}$ established in Lemma A.4, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)S_t^{u,y}\right] &= \mathbb{E}\Bigg[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s}{2Y_s}\Bigg(\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}^2\left(u,y\right)}\delta\left(S_s-y\right)\delta\left(s-u\right) \\
&\quad -\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\text{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}\left(S_s\right)\Bigg)\left(dW_s - a_s\partial_S\left(\sigma\left(t,S_s\right)\right)ds\right)\Bigg] \\
&= \mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s}{2Y_s}\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}^2\left(u,y\right)}\delta\left(S_s-y\right)\delta\left(s-u\right)dW_s\right] \\
&\quad -\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s}{2Y_s}\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\text{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}\left(S_s\right)dW_s\right] \\
&\quad -\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s^2}{2Y_s}\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}^2\left(u,y\right)}\delta\left(S_s-y\right)\delta\left(s-u\right)\partial_S\left(\sigma\left(t,S_s\right)\right)ds\right] \\
&\quad +\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s^2}{2Y_s}\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\text{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}\left(S_s\right)\partial_S\left(\sigma\left(t,S_s\right)\right)ds\right]
\end{aligned} \tag{A.8}$$

We simplify each term of the expression above independently. We set

$$\chi_t^{(0)}\left(u,y\right) := \mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)\frac{Y_t}{Y_u}a_u^2\mid S_u=y\right]1_{t\geq u} \tag{A.9}$$

$$\chi_t^{(1)}\left(u,y\right) := \mathbb{E}\left[\frac{D_u\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\right]}{Y_u}a_u\mid S_u=y\right]1_{t\geq u} \tag{A.10}$$

*First term:* By applying Clark-Ocone's formula (see [Nualart95]), we get

$$\begin{aligned}
&\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t \frac{a_s}{2Y_s}\frac{\sigma\left(u,y\right)}{\sigma_{\mathrm{loc}}^2\left(u,y\right)}\delta\left(S_s-y\right)\delta\left(s-u\right)dW_s\right]\\
&=\mathbb{E}\left[\int_0^t \frac{D_s\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\right]}{2Y_s}\frac{\sigma\left(u,y\right)}{\sigma_{\mathrm{loc}}^2\left(u,y\right)}a_s\delta\left(S_s-y\right)\delta\left(s-u\right)ds\right]\\
&=\frac{1}{2}\frac{\sigma\left(u,y\right)}{\sigma_{\mathrm{loc}}^2\left(u,y\right)}\chi_t^{(1)}(u,y)\mu(u,y)
\end{aligned}\tag{A.11}$$

*Second term:* Using the same idea as the one used to simplify the first term, we have

$$\begin{aligned}
&\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t \frac{a_s}{2Y_s}\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2\left(u,y\right)}\left(S_s\right)dW_s\right]\\
&=\mathbb{E}\left[\int_0^t \frac{D_s\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\right]}{2Y_s}\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}a_s\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2\left(u,y\right)}\left(S_s\right)ds\right]\\
&=\frac{1}{2}\int_0^t\mathbb{E}\left[\chi_t^{(1)}\left(s,S_s\right)\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2\left(u,y\right)}\left(S_s\right)\right]ds
\end{aligned}$$

Next, by using the same arguments as in the proof of Lemma A.3, we can write

$$\begin{aligned}
\int_0^t\mathbb{E}\left[\chi_t^{(1)}\left(s,S_s\right)\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}\frac{\delta\mathbb{E}\left[a_s^2\mid S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2\left(u,y\right)}\left(S_s\right)\right]ds &= \frac{1}{2}\int_0^t\mathbb{E}\left[\partial_S\left(\chi_t^{(1)}\left(s,S_s\right)\frac{\sigma^3\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}\right)a_s^2S_s^{u,y}\right]ds\\
&-\frac{1}{2}\int_0^t\mathbb{E}\left[\partial_S\left(\chi_t^{(1)}\left(s,S_s\right)\sigma\left(s,S_s\right)\right)S_s^{u,y}\right]ds
\end{aligned}\tag{A.12}$$

*Third term:*

$$\begin{aligned}
&\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s^2}{2Y_s}\frac{\sigma(u,y)}{\sigma_{\mathrm{loc}}^2(u,y)}\delta\left(S_s-y\right)\delta\left(s-u\right)\partial_S\left(\sigma\left(t,S_s\right)\right)ds\right]\\
&=\frac{1}{2}\frac{\sigma(u,y)}{\sigma_{\mathrm{loc}}^2(u,y)}\partial_y\left(\sigma\left(u,y\right)\right)\chi_t^{(0)}(u,y)\mu\left(u,y\right)
\end{aligned}\tag{A.13}$$

*Last term:* Finally, we follow the same lines as that for Lemma A.3:

$$\begin{aligned}
&\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)Y_t\int_0^t\frac{a_s^2}{2Y_s}\frac{\sigma^3(s,S_s)}{\sigma_{\mathrm{loc}}^2(s,S_s)}\frac{\delta\mathbb{E}\left[a_s^2|S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2(u,y)}\left(S_s\right)\partial_S\left(\sigma\left(s,S_s\right)\right)ds\right]\\
&=\frac{1}{2}\int_0^t\mathbb{E}\left[\chi_t^{(0)}\left(s,S_s\right)\frac{\sigma^3(s,S_s)}{\sigma_{\mathrm{loc}}^2(s,S_s)}\partial_S\left(\sigma\left(t,S_s\right)\right)\frac{\delta\mathbb{E}\left[a_s^2|S_s=.\right]}{\delta\sigma_{\mathrm{loc}}^2(u,y)}\left(S_s\right)\right]ds\\
&=\frac{1}{2}\int_0^t\mathbb{E}\left[\partial_S\left(\chi_t^{(0)}\left(s,S_s\right)\frac{\sigma^3(s,S_s)}{\sigma_{\mathrm{loc}}^2(s,S_s)}\partial_S\left(\sigma\left(s,S_s\right)\right)\right)a_s^2S_s^{u,y}\right]ds\\
&-\frac{1}{2}\int_0^t\mathbb{E}\left[\partial_S\left(\chi_t^{(0)}\left(s,S_s\right)\sigma\left(s,S_s\right)\partial_S\left(\sigma\left(s,S_s\right)\right)\right)S_s^{u,y}\right]ds
\end{aligned}\tag{A.14}$$

Plugging A.11, A.12, A.13, A.14 into A.8

$$\begin{aligned}
\mathbb{E}\left[\tilde{F}\left(t,S_t,a_t\right)S_t^{u,y}\right] &= \frac{1}{2}\left(\frac{\sigma\left(u,y\right)}{\sigma_{\mathrm{loc}}\left(u,y\right)}\right)^2\mathcal{P}_t\tilde{F}(u,y)\mu\left(u,y\right)\\
&+\frac{1}{2}\int_0^t ds\mathbb{E}\left[\tilde{F}^{(1)}\left(s,S_s,a_s\right)S_s^{u,y}\right]
\end{aligned}$$

where

$$\tilde{F}^{(1)}\left(s,S_s,a_s\right)=\partial_S\left(\chi_t^{(1)}\left(s,S_s\right)-\chi_t^{(0)}\left(s,S_s\right)\partial_S\left(\sigma\left(t,S_s\right)\right)\right)\sigma\left(s,S_s\right)\left\{1-\frac{\sigma^2\left(s,S_s\right)}{\sigma_{\mathrm{loc}}^2\left(s,S_s\right)}a_s^2\right\}$$

Finally, from our definition of the operators $\mathcal{A}$ and $\mathcal{P}_t$, we get our result. □

**Lemma A.6.** *Let $\tilde{F}$ be in $C^{(0,\infty,\infty)}([0,\bar{T}]\times\mathbb{R}_+^2;\mathbb{R})$.*
**(1)**

$$\begin{aligned}\mathbb{E}[\tilde{F}(T,S_T,a_T)S_T^{u,y}] &= \frac{1}{2}\left(\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}\left(u,y\right)}\right)^2\mu\left(u,y\right)\Bigg( \\ &\mathcal{P}_T\tilde{F}(u,y)+\sum_{i=0}^{\infty}\int_0^T dt_1\cdots\int_0^{t_i}dt_{i+1}P_{t_{i+1}}\mathcal{A}\cdots\mathcal{P}_{t_1}\mathcal{A}P_T\tilde{F}(u,y)\Bigg)\end{aligned}$$

**(2)**

$$\mathbb{E}[\tilde{F}(T,S_T,a_T)S_T^{u,y}] = \frac{1}{2}\left(\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}\left(u,y\right)}\right)^2\mu\left(u,y\right)X_T(u,y)$$

*where $X_t$ is the solution of the linear flow on $C^{(0,\infty)}([0,\bar{T}]\times\mathbb{R}_+;\mathbb{R})$:*

$$\frac{dX_t}{dt}=\left(\mathcal{P}_t\mathcal{A}\right)\left(X_t\right),\quad X_0=\mathcal{P}_T\tilde{F} \tag{A.15}$$

*Proof.* **(1)** By iterating the equation in Lemma A.5, we get our result.
**(2)** The sum of iterated time-ordered integrals can be seen as the solution of the flow A.15. □

*Proof of Corollary 4.6.* $F$ is of the form

$$F\left(t,s,a\right)=A\left(t,s\right)+B\left(t,s\right)a^2$$

where $A$ and $B$ are defined by follows:

$$B\left(t,s\right)=\partial_S\left(\frac{\sigma^4\left(t,s\right)}{\sigma_{\text{loc}}^2\left(t,s\right)}\Gamma_{\text{loc}}\left(t,s\right)\right),\ A\left(t,s\right)=-\partial_S\left(\sigma^2\left(t,s\right)\Gamma_{\text{loc}}\left(t,s\right)\right)$$

By plugging this representation in $\chi_t^{(0)}$ and $\chi_t^{(0)}$ (see Equations (A.9), (A.10) ) and assuming that $t$ is close to $u$:

$$\begin{aligned}\chi_t^{(0)}\left(u,y\right) &= \mathbb{E}\left[F\left(t,S_t,a_t\right)\frac{Y_t}{Y_u}a_u^2\mid S_u=y\right] \\ &= \mathbb{E}\left[\left\{A\left(t,S_t\right)+B\left(t,S_t\right)a_t^2\right\}\frac{Y_t}{Y_u}a_u^2\mid S_u=y\right] \\ &= \mathbb{E}\left[A\left(t,S_t\right)\frac{Y_t}{Y_u}a_u^2\mid S_u=y\right]+\mathbb{E}\left[B\left(t,S_t\right)\frac{Y_t}{Y_u}a_u^2a_t^2\mid S_u=y\right] \\ &= A\left(u,y\right)M_2\left(u,y\right)+B\left(u,y\right)M_4\left(u,y\right)+O(t-u)\end{aligned}$$

where $M_n\left(u,y\right):=\mathbb{E}[a_u^n|S_u=y]$ and

$$\begin{aligned}\chi_t^{(1)}\left(u,y\right) &= \mathbb{E}\left[\frac{D_u\left[F\left(t,S_t,a_t\right)Y_t\right]}{Y_u}a_u\mid S_u=y\right] \\ &= \left(\partial_yA\left(u,y\right)M_2\left(u,y\right)+\partial_yB\left(u,y\right)M_4\left(u,y\right)\right)\sigma\left(u,y\right) \\ &+ 2B\left(u,y\right)\mathbb{E}\left[a_u^2\Sigma\left(a_u\right)\rho\mid S_u=y\right]+A\left(u,y\right)\partial_y\left(\sigma\left(u,y\right)\right)M_2\left(u,y\right) \\ &+ B\left(u,y\right)\partial_y\left(\sigma\left(u,y\right)\right)M_4\left(u,y\right)+O(t-u)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{P}_tF\left(u,y\right) &= \partial_yA\left(u,y\right)M_2\left(u,y\right)+\partial_yB\left(u,y\right)M_4\left(u,y\right) \\ &+ \frac{2B\left(u,y\right)\rho}{\sigma\left(u,y\right)}\mathbb{E}\left[a_u^2\Sigma\left(a_u\right)\mid S_u=y\right]+O(t-u)\end{aligned}$$

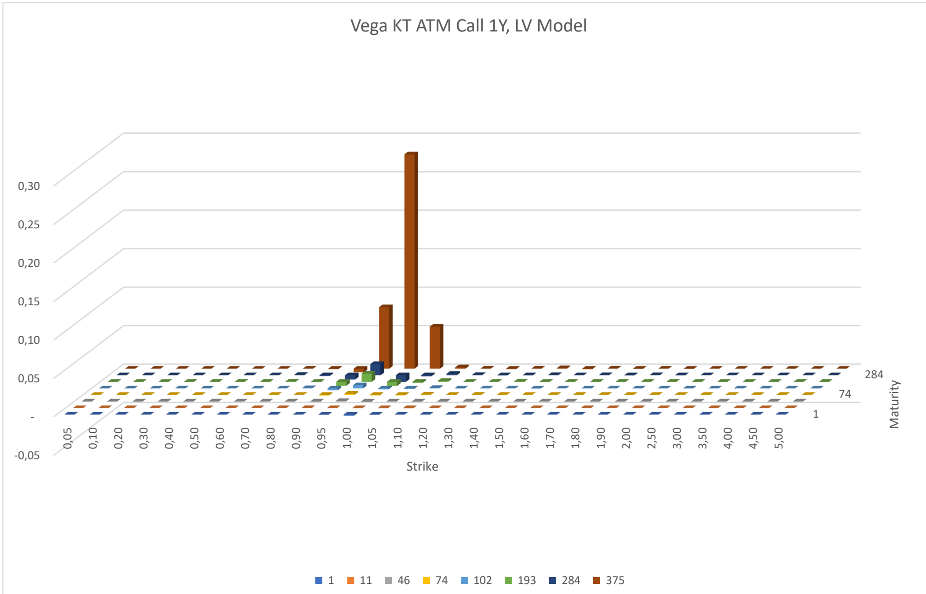
From Theorem 4.2,

$$2\frac{\delta\mathcal{O}_0}{\delta\sigma_{\text{loc}}^2\left(u,y\right)}=\frac{\mu\left(u,y\right)}{2}\left(\frac{\sigma\left(u,y\right)}{\sigma_{\text{loc}}\left(u,y\right)}\right)^2\left\{2\Gamma_{\text{loc}}\left(u,y\right)1_{\bar{T}\geq u}-\int_u^{\bar{T}}dt\mathcal{P}_tF\left(u,y\right)+O((\bar{T}-u)^2)\right\}$$

Finally, $\int_u^{\bar{T}} dt \mathcal{P}_t F(u, y) = \mathcal{P}_u F(u, y)(\bar{T} - u) + O((\bar{T} - u)^2)$.

$\square$

## Appendix B. Figures



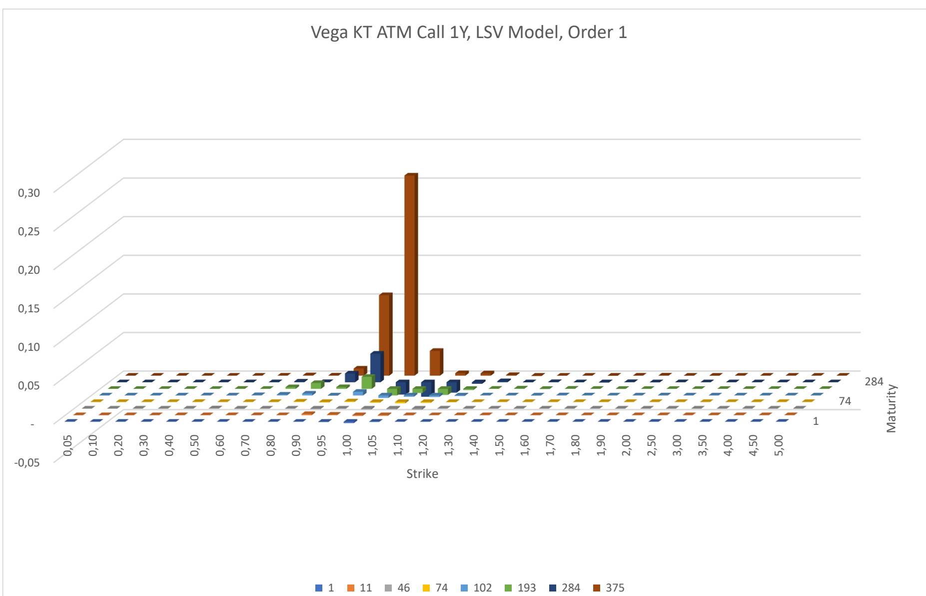
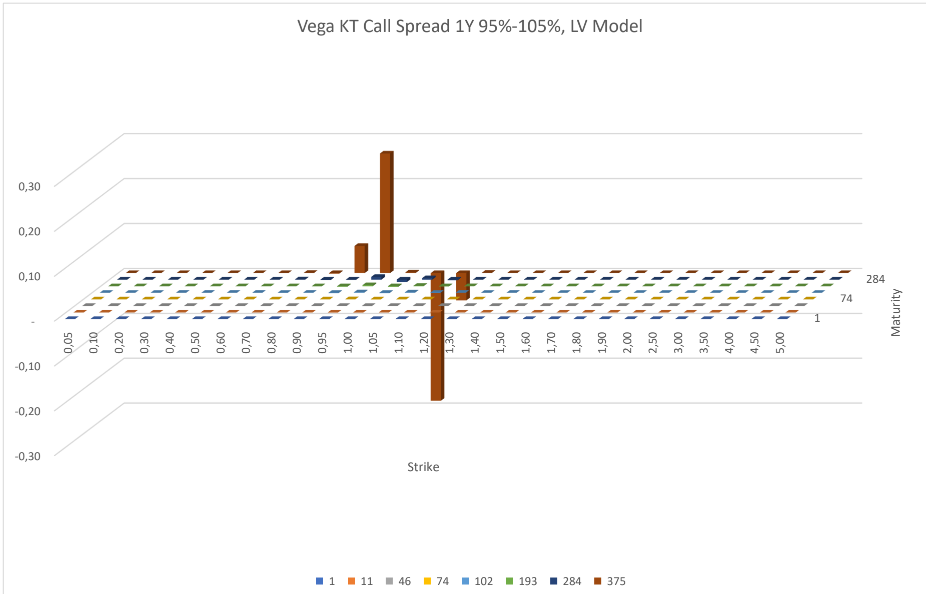




Figure 1. Vega *KT* AD Call ATM 1Y. Top: LV. Bottom: LSV.



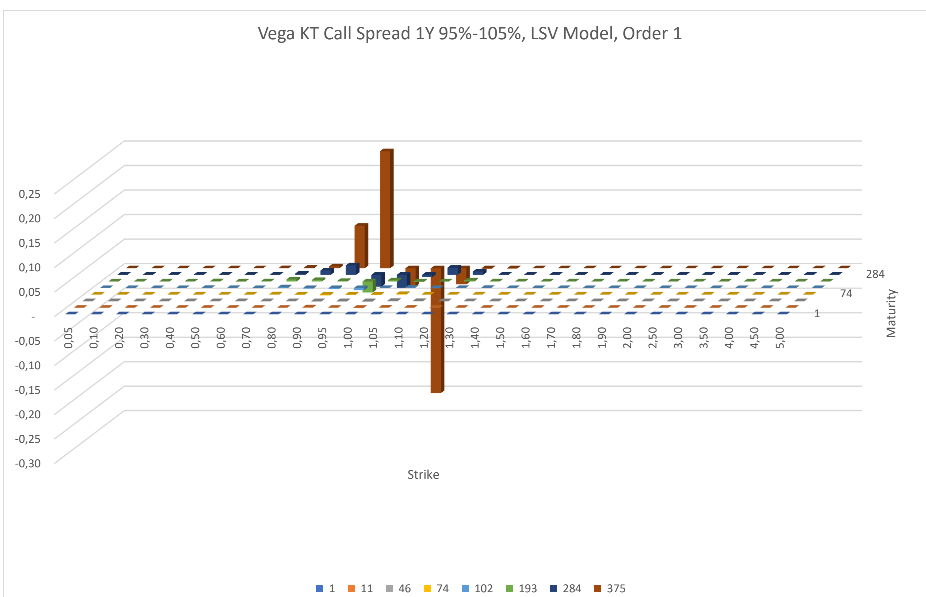
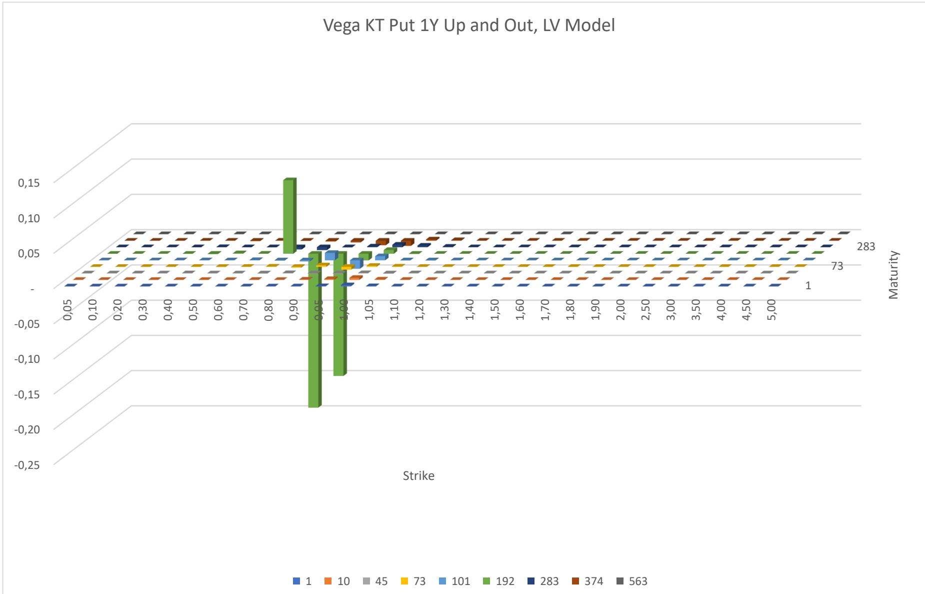




Figure 2. Vega $KT$ AD CallSpread. Top: LV. Bottom: LSV.



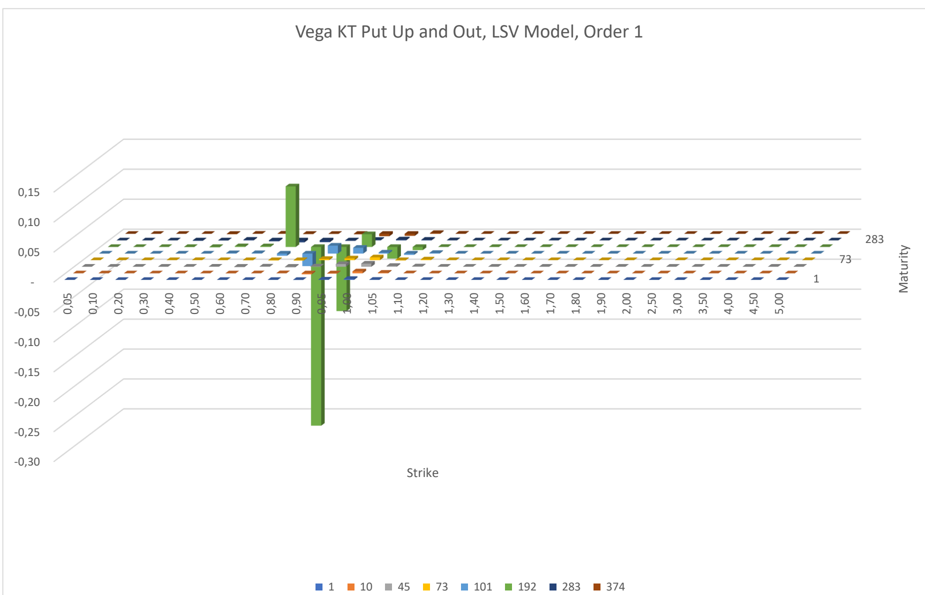
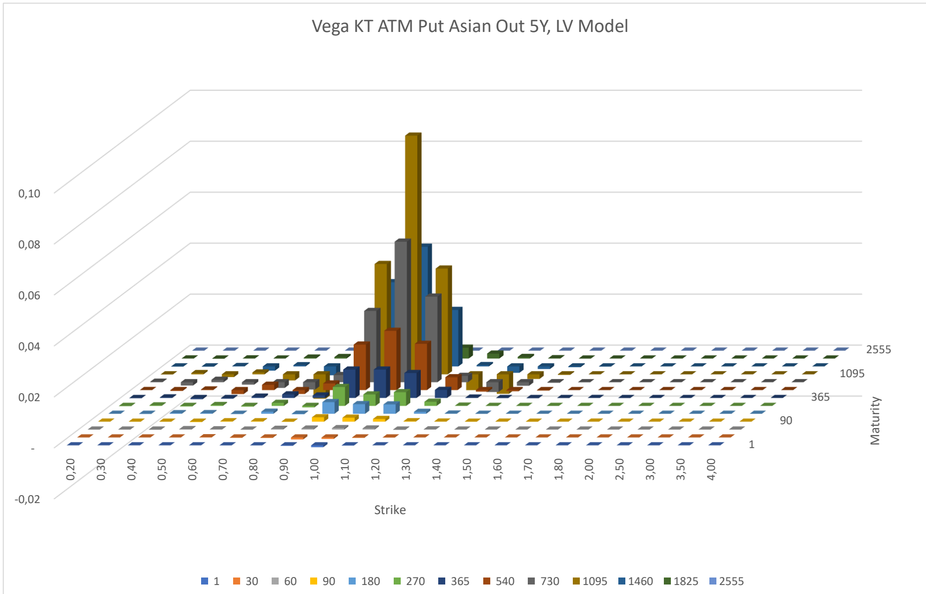




Figure 3. Vega $KT$ AD 1Y ATM Short Put 6 months Up and Out at level 80%. Top: LV, Bottom: LSV.



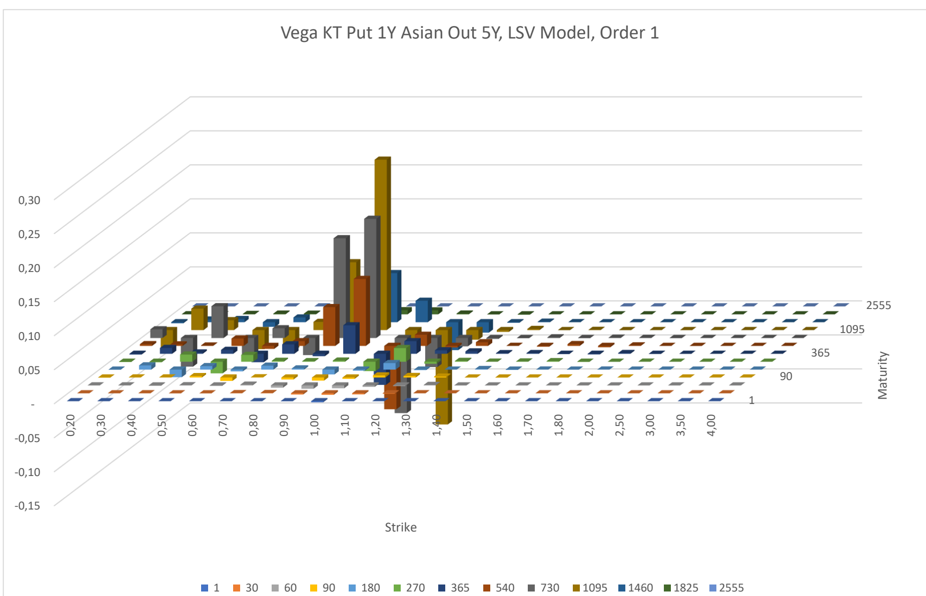
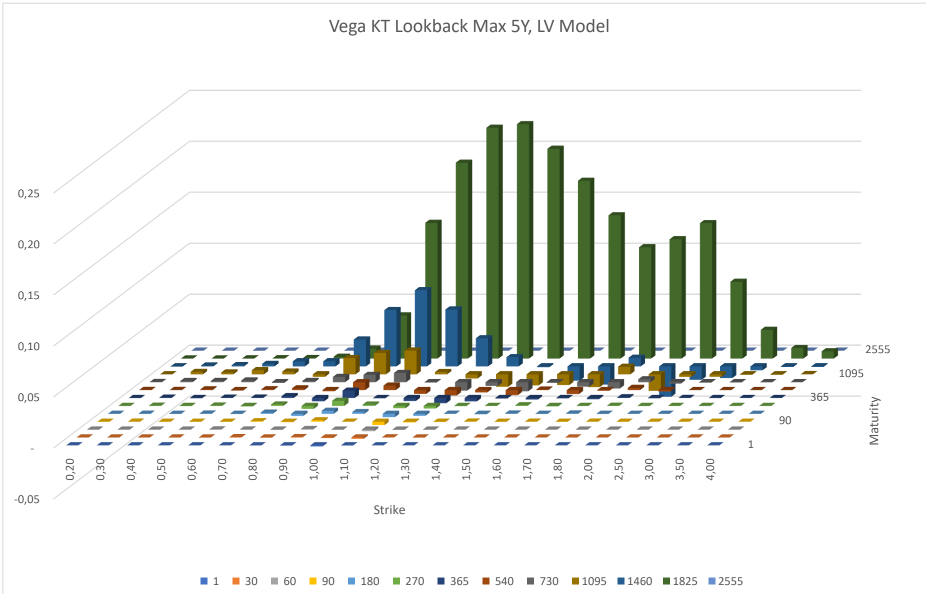




FIGURE 4. Vega $KT$ AD 5Y ATM Put monthly Asian. Top: LV, Bottom: LSV.



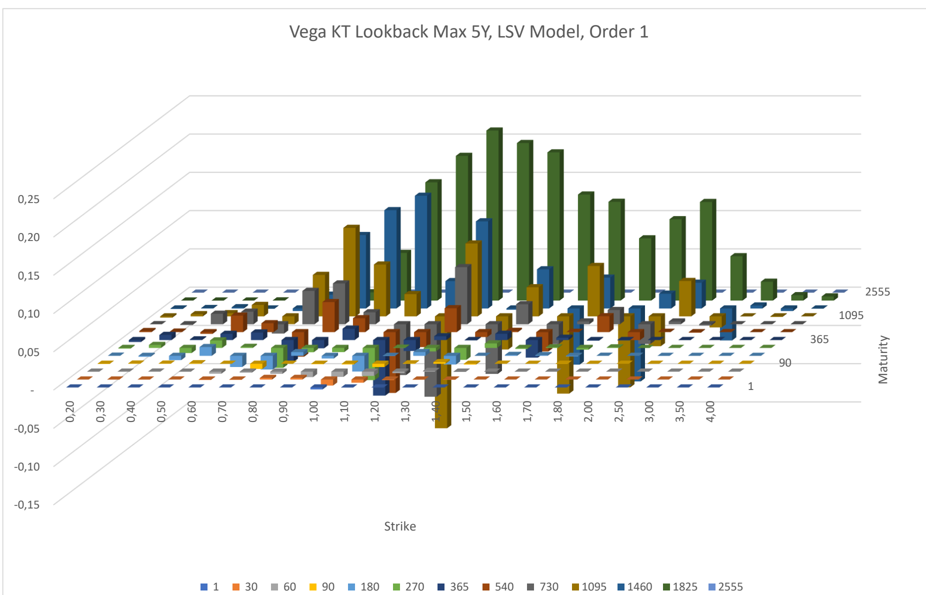
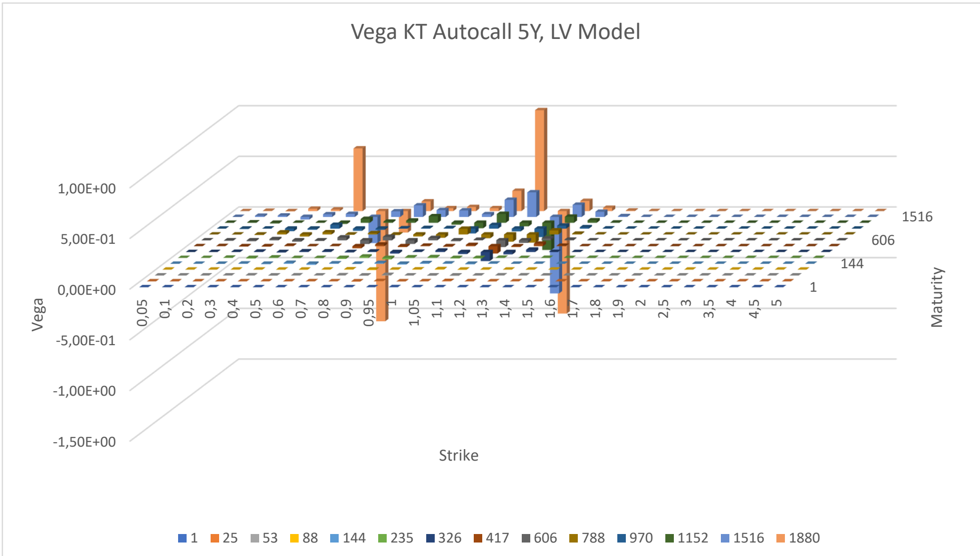




FIGURE 5. Vega $KT$ AD 5Y Call Lookback Max 130% - monthly Lookback dates. Top: LV, Bottom: LSV.



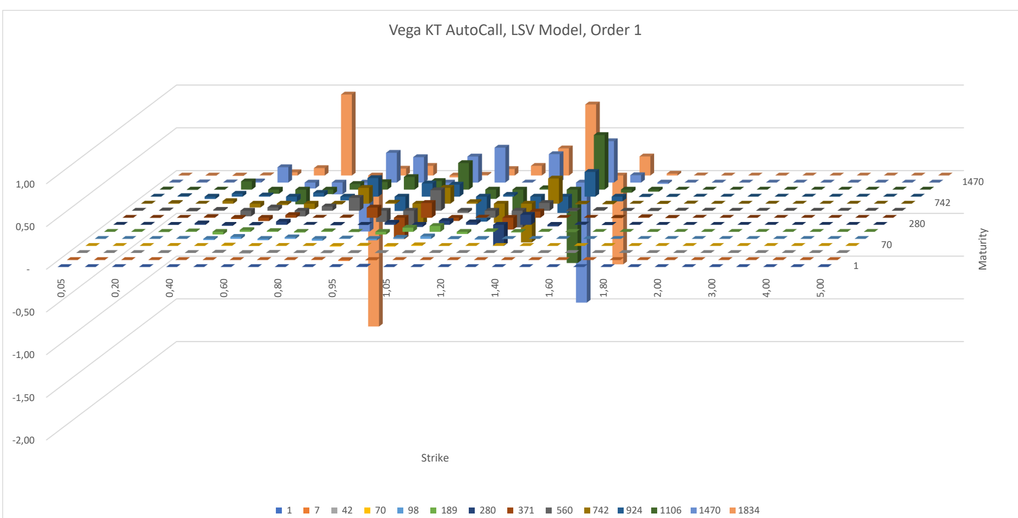




FIGURE 6. Vega $KT$ AD Autocallable. Top: LV, Bottom: LSV.

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