Glossary

French	English
sommet	vertex
arête	edge
arc	arc
boucle	self-loop
graphe orienté	directed graph
degré sortant	outgoing degree
degré entrant	incoming degree
chaîne	chain
chemin	path

Review: incidence matrix vertices-arcs (directed case)

Recall that we assume that the graph is simple (no self-loops, no multiple edges).

$n = \#$ of vertices $\qquad m = \#$ of edges

Enumerate the vertices and the arcs:

$v_1, v_2, \dots, v_n \qquad e_1, e_2, \dots, e_m$

$$A = \underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{m \text{ columns}} \Big\} n \text{ rows}$$

v_i is represented by the i-th row, $i = 1, \dots, n$.

e_k is represented by the k-th column, $k = 1, \dots, m$.

If $e_k = (v_i, v_j)$, we define

$$a_{ik} = 1 \qquad a_{jk} = -1.$$

(We put a 1 in the origin of the arc, and a -1 in the destination)

Every column has exactly a 1 and a -1. The rest of the entries are 0.

Obs. $\delta_+(v_i) = \#$ of 1's in the i-th row.

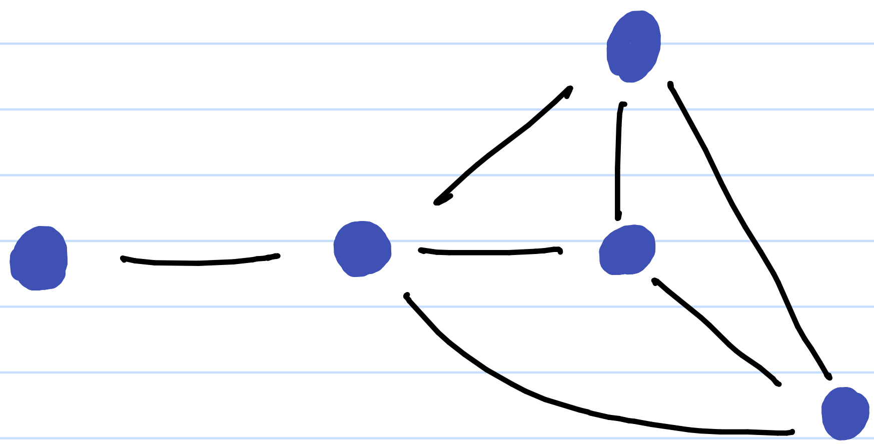
$\delta_-(v_i) = \#$ of -1's in the i-th row.

$\delta(v_i) = \#$ of non-zero elements in the i-th row.

Incidence matrix for the non-directed case

If $e_k = \{v_i, v_j\}$, we set $a_{ik} = 1$ and $a_{jk} = 1$.

Exe. Find the incidence matrix for the following graph:



How does the matrix change if you switch the enumeration of two vertices? And of two edges?

Other representation: adjacency matrix

$$A = \begin{array}{c} \overbrace{\qquad\qquad\qquad}^{n \text{ columns}} \\ \left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) \end{array} \Bigg\} n \text{ rows}$$

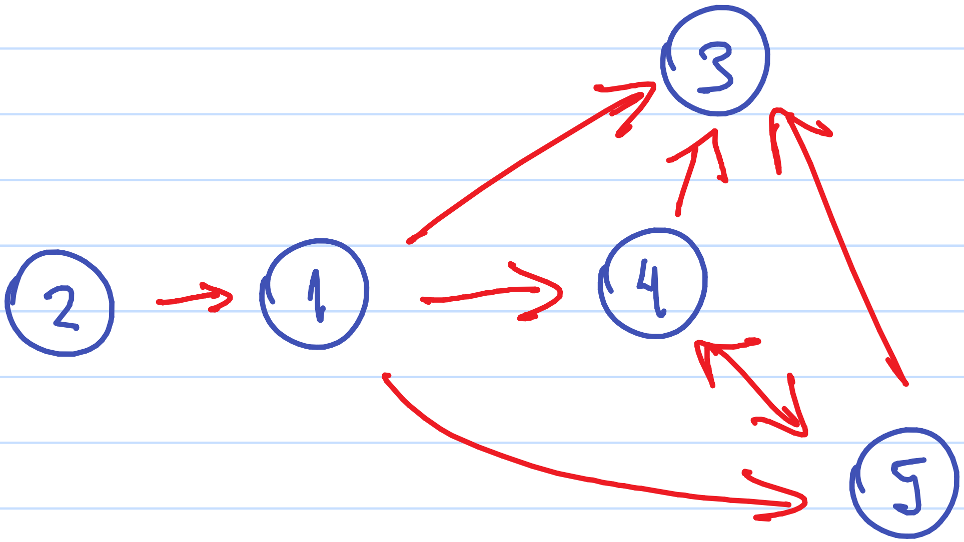
Directed case: if $e_k = (v_i, v_j)$, set $a_{ij} = 1$ (and set the rest of the entries as 0).

Non-directed case: if $e_k = \{v_i, v_j\}$, set $a_{ij} = 1$. In this case we have a symmetric matrix: $a_{ij} = a_{ji}$ $\forall i,j$.

Exe. Compute the adjacency matrix associated to the graph of the previous slide.

Exe. How can the total # of arcs (directed case) and the total # of edges (non-directed case) be computed in terms of the adjacency matrix? And $\delta_+(v)$, $\delta_-(v)$ and $\delta(v)$ for a given $v \in V$?

Vector coding from the adjacency matrix: the directed case



Let's codify this graph in two vectors:

T_v of dimension $n+1$

T_E of dimension m.

T_v is defined as

$T_v(1)=1$ by convention

$T_v(i)=T_v(i-1)+\delta_+(i-1)$, $i=2,\dots,n+1$

In this case:

$T_v=(1,4,5,5,7,9)$.

For $v\in V$, let $N_+(v)=\{v' : (v,v')\in E\}$ the outgoing neighbourhood of v.

T_E is defined as

$$T_E=(\underbrace{\quad\quad}_{N_+(1)}\ \underbrace{\quad\quad}_{N_+(2)}\cdots\underbrace{\quad\quad}_{N_+(n)}).$$

In this case, $N_+(3)=\phi$

$$T_E=(\underbrace{3,4,5}_{N_+(1)},\underbrace{1}_{N_+(2)},\underbrace{3,5}_{N_+(4)},\underbrace{3,4}_{N_+(5)}).$$

Note. In the subject notes, the numbering of the indices of the vectors starts at zero. This is a minor difference that does not affect the way these vectors are defined.

Exe. Let $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

be the adjacency matrix of a directed graph. Obtain T_V and T_E from A (without passing through the graph).

The non-directed case

$V=\{1,\dots,n\}$.

T_V (of dimension $n+1$) is:

$T_V(1)=1$ by convention

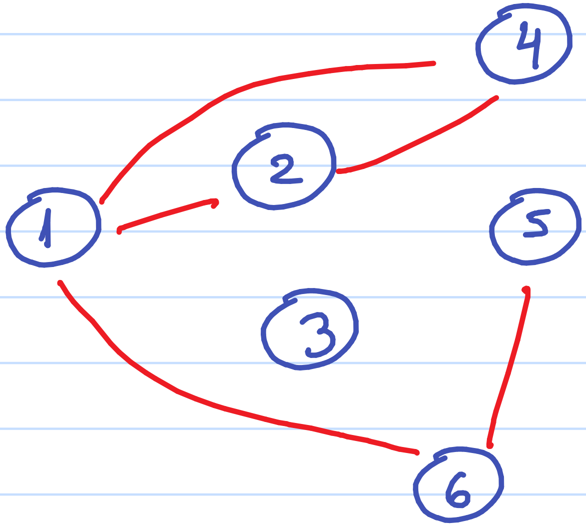
$T_V(i)=T_V(i-1)+\delta(i-1),\ i=2,\dots,n+1.$

you count the degrees instead of the outgoing degrees

T_E (of dimension $2m$) is:

$$T_E=(\underbrace{\qquad\qquad}_{\text{neighbours of }1}\ \underbrace{\qquad\qquad}_{\text{neighbours of }2}\ \dots\ \underbrace{\qquad\qquad}_{\text{neighbours of }n}).$$

Exe. Obtain T_V and T_E for the graph



Solution:

$T_V=(1,4,6,6,8,9,11)$

$T_E=(2,4,6,1,4,1,2,6,1,5).$

Exe. In the directed case, for the three representations (incidence matrix, adjacency matrix and vectorial) calculate:

i. the storage cost

ii. the necessary time to know if certain arc (i,j) is in the graph.

Solution.

	Storage cost	Time to know if $(i,j) \in E$
Incidence	$n \cdot m$	$O(m)$
Adjacency	$n \cdot n$	$O(1)$
Vectorial	$n+1+m$	$O(1+\max_{i=1,\dots,n} \delta(i))$

Weighted graphs

A graph is called weighted if each arc or edge is associated with a positive number (or cost).

Exe. Adapt the three representations to take into acount weighted graphs.

Solution. Incidence matrix: multiply each column by the weight of the arc/ edge it represents.

Adjacency matrix: multiply each entry by the associated weight.

Vectorial representation: you need a weight vector of size m (directed case) or $2m$ (non-directed) associated to T_E.

Non-directed graphs: chains

Def. A chain in a non-directed graph (V, E) is a sequence of vertices $v_0, v_1, \dots, v_q$ such that $\{v_i, v_{i+1}\} \in E \quad \forall i = 0, 1, \dots, q-1$.

In other words, successive vertices in the chain are adjacent.

We say that q is the lenght of the chain, and we say that the chain connects v_0 and v_q.

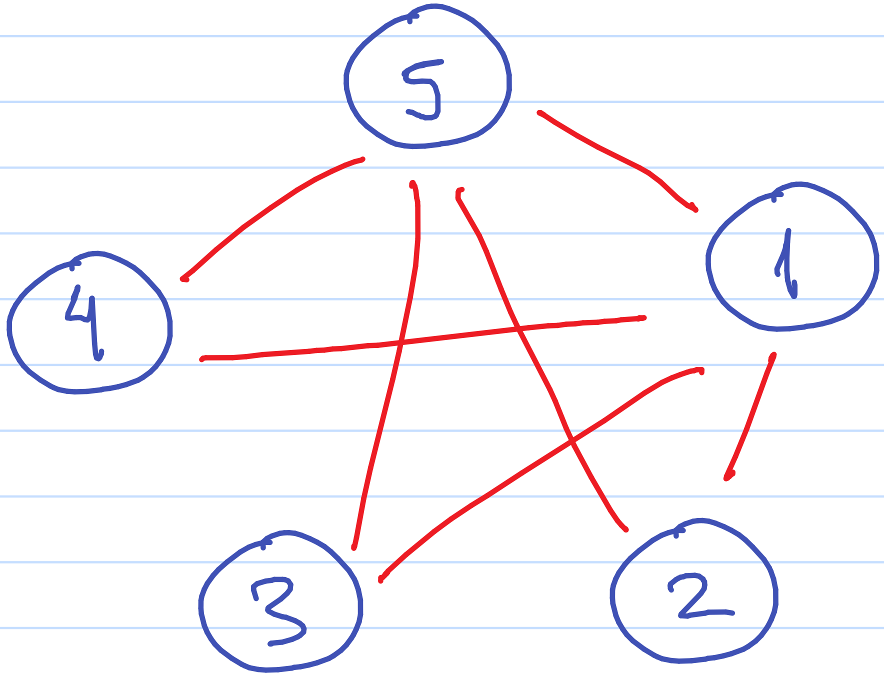
Def. A chain is elementary if all its vertices are different, simple if all its edges are different, and closed if $v_0 = v_q$.

A chain that is closed and simple is called a cycle.

A cycle is elementary if the vertices are diffent with the exception of the first and the last ones.

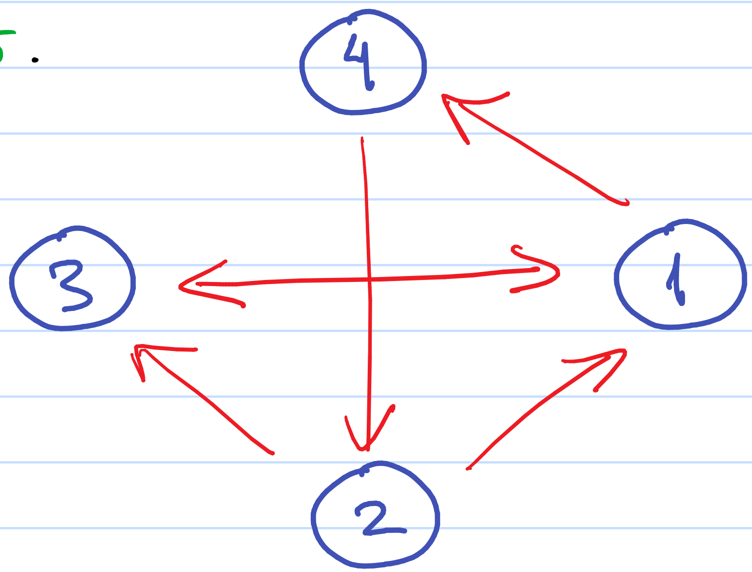
Examples.

- A closed non-elementary chain: $(1, 5, 2, 1, 3, 5, 1)$
- A simple, not closed, not elementary chain: $(2, 1, 3, 5, 4, 1)$.
- An elementary cycle: $(1, 2, 5, 1)$.
- A non-elementary cycle: $(1, 2, 5, 1, 3, 5, 4, 1)$.



Directed graph

The analogous of the chains are called paths. In this case, the arc $(v_i, v_{i+1}) \in E$ $\forall i = 0, 1, \dots, q-1$, and we say the path goes from v_0 to v_q.
The rest of the definitions (lenght, elementary and closed) are analogous. A closed simple path is called a circuit.



Examples.

- A non-closed, elementary path:
  $(1, 4, 2, 3)$.
- A non-closed, non-elementary path:
  $(1, 3, 1, 4)$.
- An elementary circuit:
  $(1, 4, 2, 3, 1)$.
- A non-elementary circuit:
  $(1, 3, 1, 4, 2, 1)$.

Connection and strongly connection

Def. A non-directed graph is connected if, for every pair of distinct vertices, there's a chain that connects them.

Def. A directed graph is strongly connected if, $\forall$ pair of distinct vertices $v, v' \in V$, there exists a path from v to v'.

Def. Let $G=(V,E)$ be a directed graph. Its induced graph is the non-directed graph obtained by forgetting the directions in the arcs. In other words, every arc $(v,v') \in E$ is replaced by the edge $\{v,v'\}$.

Def. A directed graph is said to be connected if its induced graph is.

Obs. For a directed graph,

$$\text{Strongly connected} \Rightarrow \text{Connected.}$$

Exe. Exhibit a directed graph that is connected but not strongly connected.

Reminder: equivalence relation

Def. A binary relation in a set A is a subset $\Gamma \subset A \times A$. If $(x,y) \in \Gamma$, we say that x is related to y, and write xRy.

Def. The binary relation R is an equivalence relation if:

1. xRx $\forall x \in A$ (reflexibity);
2. $xRy \Rightarrow yRx$ (symmetry);
3. xRy and $yRz \Rightarrow xRz$ (transitivity).

Def. Let A be a set with an equivalence relation R. For $x \in A$, its equivalence class is defined as

$$[x] = \{y \in A : yRx\}.$$

Obs. By refexibity, $x \in [x]$ $\forall x \in A$.

Prop. Let A be a set with an equivalence relation R. Then two equivalence classes are either disjoint or equal.

Proof. Suppose $[x]$ and $[y]$ are such that $[x]\cap[y]=\phi$, and hence $\exists\, z \in [x]\cap[y]$. We'll show that $[x]\subset[y]$. Let $w\in[x]$. We have

$$\underset{\substack{\uparrow \\ w\in[x]}}{wRx}\,,\quad \underset{\substack{\uparrow \\ z\in[x]}}{xRz}\,,\quad \underset{\substack{\uparrow \\ z\in[y]}}{zRy} \underset{\substack{\nwarrow \\ \text{transitivity}}}{\Longrightarrow} wRy \Longrightarrow w\in[y].$$

Analogously one can show that $[y]\subset[x]$, and hence $[x]=[y]$ as desired. $\blacksquare$

Corollary. The family of equivalence classes constitutes a partition of A.

Equivalence relations in graphs

Let $G=(V,E)$. We define the following binary relations in V:

- Non-directed case: for every $v \in V$, set vRv, and, for every pair distinct vertices $v, w \in V$, set $vRw \iff$ they are connected by a chain.

- Directed case: for every $v \in V$, set vRv, and, for every pair distinct vertices $v, w \in V$, set $vRw \iff \exists$ a path from v to w and a path from w to v.

Exe. Prove that these binary relations are in fact equivalence relations.