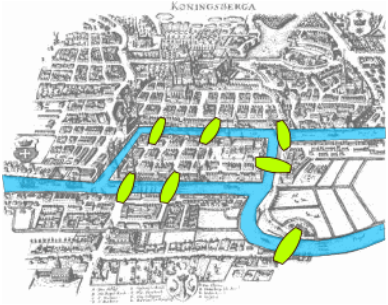
En 1766, Euler résout le problème dit des 7 ponts de la ville de Königsberg : à savoir "est-il possible de suivre un chemin qui emprunte chaque pont une fois et une seule et revienne au point de départ ?"



Plan de la ville de Königsberg à l'époque d'Euler, et ses 7 ponts au dessus de la rivière Pregolia (source : *Wikipedia*).

Exercice 3.1.1. Modéliser le problème ci dessus sous forme de graphe.

Def. Let $G=(V,E)$ be a non-directed graph.

1. An **Eulerian chain** is a chain that is simple and passes through all the edges.
2. An **Eulerian cycle** is a cycle that passes through all the edges.
3. G is called **Eulerian** (resp. **semi-Eulerian**) if it admits an Eulerian cycle (resp. Eulerian chain).

Intuition. A semi-Eulerian graph can be drawn without taking the pencil off the paper. In the Eulerian case, we finish the drawing where we started.

Thm. Let $G=(V,E)$ be a non-directed, connected graph without self-loops.

1. G is Eulerian $\iff$ $\delta(v)$ is even $\forall v \in V$.
2. G is semi-Eulerian but not Eulerian $\iff$ there are exactly two vertices with odd degree.

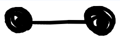
Proof. 1. $\Rightarrow$) Suppose $\exists$ an Eulerian cycle $(v_0, v_1, \dots, v_q = v_0)$.
If $v \neq v_0$, $\delta(v)$ is even because we count twice every time the cycle visits it (one edge on arrival and one on departure).
For a similar reason, $\delta(v_0)$ is even.

2. $\Rightarrow$) Suppose $\exists$ an Eulerian-chain that is not a cycle (it's not closed). Call $d \in V$ the departure vertex of this Eulerian-chain, and $a \in V$, the arriving one.
If $v \in V \setminus \{d, a\}$, then $\delta(v)$ is even for the same reason than before. For a similar reason, $\delta(d)$ and $\delta(a)$ are odd.

1. G is Eulerian $\iff$ $\delta(v)$ is even $\forall v \in V$.

2. G is semi-Eulerian but not Eulerian $\iff$ there are exactly two vertices with odd degree.

$\Leftarrow$) As usual, call $n = \# V$ and $m = \# E$. We prove 1. $\Leftarrow$) and 2. $\Leftarrow$) both at once by induction on m.

If $m=1$, then $n=2$, and in this case it's obviously true since the graph is simply .

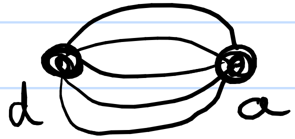
If $m=2$, we have two posibilities:

 $\rightsquigarrow$ Eulerian

 $\rightsquigarrow$ semi-Eulerian

Assume 1. $\Leftarrow$) and 2. $\Leftarrow$) are true for $1, 2, \dots, m$, and let $G = (V, E)$ with $\# E = m+1$.

Case 1. Suppose G has exactly two vertices with odd degree, say d and a.

If $V = \{d, a\}$, it's trivial (the graph in this case is ), so we can assume that $V \neq \{d, a\}$. Hence, without loss of generality, $\exists \{d, w\} \in E$ with $w \neq a$ (if not, switch the roles of d and a). Consider the partial graph (V, E') with $E' = E \setminus \{d, w\}$. In this partial graph, $\delta'(d)$ is even and $\delta'(w)$ is odd.

1. G is Eulerian $\iff$ $\delta(v)$ is even $\forall v \in V$.

2. G is semi-Eulerian but not Eulerian $\iff$ there are exactly two vertices with odd degree.

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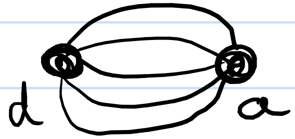
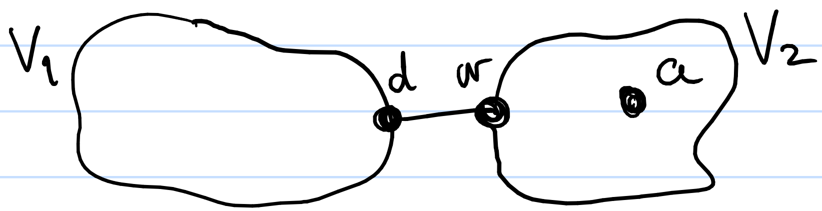


), so we can assume that $V \neq \{d, a\}$. Hence, without loss of generality, $\exists \{d, w\} \in E$ with $w \neq a$ (if not, switch the roles of d and a). Consider the partial graph (V, E') with $E' = E \setminus \{d, w\}$. In this partial graph, $\delta'(d)$ is even and $\delta'(w)$ is odd.

Subcase 1.a. If (V, E') is connected, by induct. hypothesis $\exists$ an Eulerian-chain, say $(w = v_0, v_1, \dots, v_q = a)$. Then the chain $(d, w = v_0, v_1, \dots, v_q = a)$ is an Eulerian-chain in G.

Subcase 1.b. Suppose (V, E') is disconnected, and let V_1 and V_2 be its connected components.



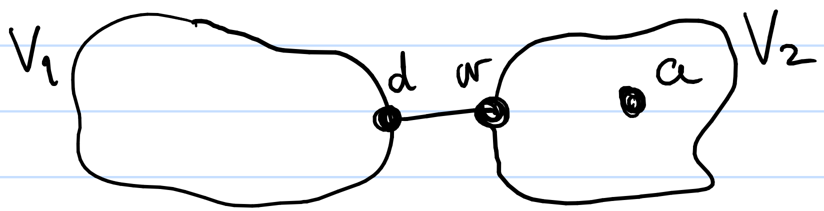
1. G is Eulerian $\iff$ $\delta(v)$ is even $\forall v \in V$.

2. G is semi-Eulerian but not Eulerian $\iff$ there are exactly two vertices with odd degree.

a needs to be in V_2 because $\sum_{v \in V_2} \delta'(v)$ is even.

Subcase 1.a. If (V, E') is connected, by induct. hypothesis $\exists$ an Eulerian-chain, say $(w = v_0, v_1, \dots, v_q = a)$. Then the chain $(d, w = v_0, v_1, \dots, v_q = a)$ is an Eulerian-chain in G.

Subcase 1.b. Suppose (V, E') is disconnected, and let V_1 and V_2 be its connected components.



Let (V_1, E_1) (resp. (V_2, E_2)) be the subgraph induced by V_1 (resp. V_2) in (V, E'). By induct. hypothesis, $\exists$ an Eulerian-cycle in (V_1, E_1), say $(d = v_0, v_1, \dots, v_q = v_0 = d)$. Also by induct. hypothesis, $\exists$ an Eulerian chain in (V_2, E_2), say $(w = w_0, w_1, \dots, w_r = a)$.

Hence

$(d = v_0, v_1, \dots, v_q = v_0 = d, w = w_0, w_1, \dots, w_r = a)$

is an Eulerian-chain in (V, E).

1. G is Eulerian $\iff$ $\delta(v)$ is even $\forall v \in V$.

2. G is semi-Eulerian but not Eulerian $\iff$ there are exactly two vertices with odd degree.

Case 2 (read as an exercise)

Suppose $\delta(v)$ is even $\forall v \in V$.
Let $\{d,a\} \in E$, and let $E' = E \setminus \{\{d,a\}\}$. The subgraph (V,E') is connected; otherwise we contradict that the sum of the degrees is even.
We can then apply the induct. hypothesis: $\exists$ an Eulerian chain, say $(a=v_0, v_1, \dots, v_q = d)$.
Then, $(a=v_0, v_1, \dots, v_q = d, a)$ is an Eulerian-cycle. $\blacksquare$

a needs to be in V_2 because $\sum_{v \in V_2} \delta'(v)$ is even.

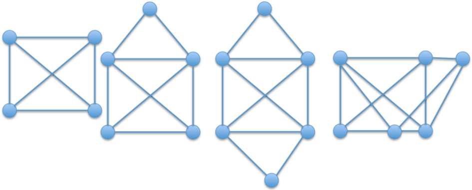
Let (V_1, E_1) (resp. (V_2, E_2)) be the subgraph induced by V_1 (resp. V_2) in (V,E'). By induct. hypothesis, $\exists$ an Eulerian-cycle in (V_1,E_1), say $(d=v_0, v_1, \dots, v_q = v_0 = d)$. Also by induct. hypothesis, $\exists$ an Eulerian chain in (V_2,E_2), say $(w=w_0, w_1, \dots, w_r = a)$.

Hence

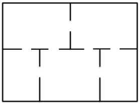
$(d=v_0, v_1, \dots, v_q = v_0 = d, w = w_0, w_1, \dots, w_r = a)$

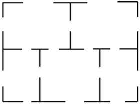
is an Eulerian-chain in (V,E).

Exercice 3.1.2. Est-il possible de tracer les figures suivantes sans lever la crayon et sans passer deux fois sur le même trait.

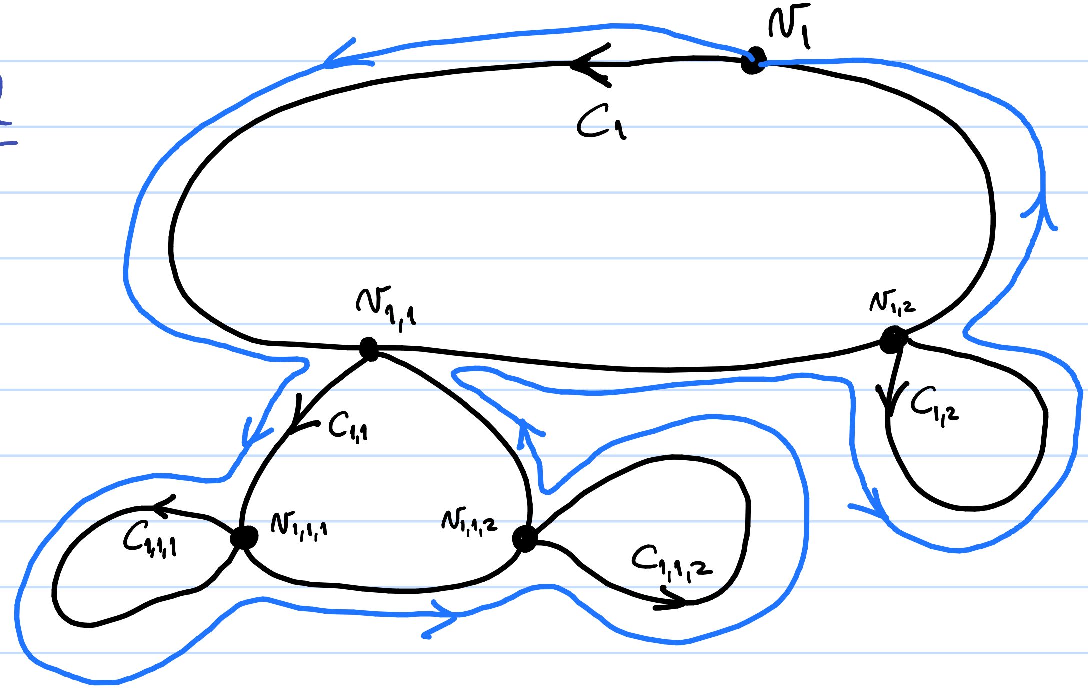


Exercice 3.1.3. Est-il possible de se promener dans ces maisons en passant une et une seule fois par chacune des ouvertures ?





Recipe to find an Eulerian-cycle

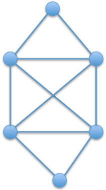


- Start a simple chain in an arbitrary vertex v_1, until not being possible to move anymore. The result is a cycle C_1.
- Remove the edges of C_1 from the graph.
- If the vertices from C_1 has no remaining edges, C_1 is an Eulerian cycle.
- If not, let $v_{1,1}$ be the 1st vertex in C_1 with remaining edges. Start a simple chain in $v_{1,1}$ until not being able to move anymore. Call $C_{1,1}$ the cycle that results.
- Remove the edges of $C_{1,1}$ from the graph.
- Continue until having no vertices with remaining edges...

For instance, the Eulerian-cycle obtained in the drawing is

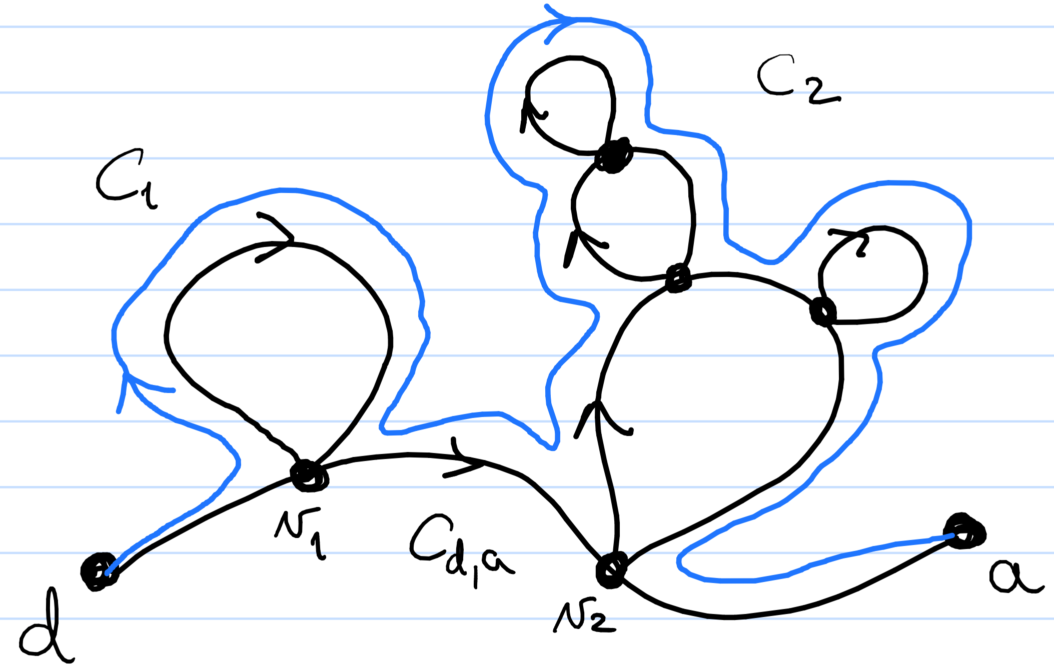
$$v_1 \longrightarrow v_{1,1} \longrightarrow v_{1,1,1} \xrightarrow{C_{1,1,1}} v_{1,1,1} \longrightarrow v_{1,1,2}$$
$$\xrightarrow{C_{1,1,2}} v_{1,1,2} \longrightarrow v_{1,1} \longrightarrow v_{1,2} \xrightarrow{C_{1,2}} v_{1,2} \longrightarrow v_1 .$$

Exe. Appy this algorithm to the following graph:



Recipe to find an Eulerian chain

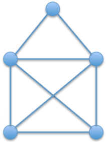
- Start a simple chain in d until not being able ~~to~~ move anymore. This generates a chain $C_{d,a}$ that finishes at a.
- Remove the edges of this chain from the graph.
- Use the previous recipe in the remaining graph with roots in the vertices of $C_{d,a}$ that still have edges.



In this example, the obtained Eulerian-chain is

$$d \to N_1 \xrightarrow{C_1} N_1 \longrightarrow N_2 \xrightarrow{C_2} N_2 \to a.$$

Exe. Appy this algorithm to the following graph:



Graphes eulériens : cas orientés

On a la une condition similaire pour les circuits et chemins eulériens :

Théorème 3.1.3

Un graphe orienté connexe admet

- un circuit eulérienne si et seulement si $\forall v \in V, \delta^+(v) = \delta^-(v)$
- un chemin eulérien si et seulement si $\forall v \in V, \delta^+(v) = \delta^-(v)$, sauf pour 2 sommets, un de ces sommets de degré impair a un degré sortant de plus que de degré entrant et l'autre sommet de degré impair a un degré sortant de moint que de degré entrant.

Définition 3.2.1

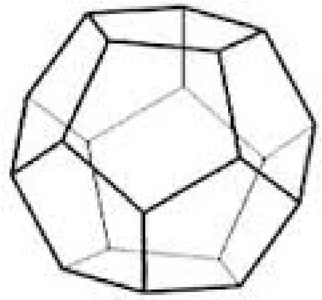
- Une chaîne est **hamiltonienne** si elle passe par tous les sommets une fois et une seule.
- Un cycle est **hamiltonien** si c'est un cycle élémentaire comptant autant d'arêtes que de sommets dans G.
- Un graphe est **hamiltonien** (resp. **semi-hamiltonien**) s'il est possible de trouver un cycle hamiltonien (resp. une chaîne hamiltonienne).

Contrairement aux graphes eulériens, il n'y a pas de caractérisation simple des graphes hamiltoniens.

Knight's tour problem

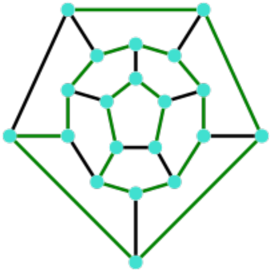
Can a knight cover the hole chessboard visiting each site exactly once?

Exercice 3.2.2. Jeu de Hamilton (1859) : trouver une chaîne hamiltonienne dans un dodécaèdre.



▷ **Solution**

On peut représenter ce graphe dans le plan et parcourir les nœuds de l'extérieur vers l'intérieur



Quelques critères

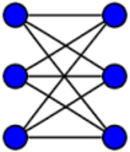
Proposition 3.2.2

- Si $\exists v \in V$ tel que $\delta(v) = 1$ et $n > 1$ alors le graphe n'est pas hamiltonien.
- Si $\exists v \in V$ tel que $\delta(v) = 2$ alors les deux arêtes incidentes à v appartiennent à tout cycle hamiltonien.
- K_n est hamiltonien.

Définition 3.3.1 – Graphe biparti

Un graphe est biparti si il existe une partition $\{V_1, V_2\}$ de V telle que, pout toute arête $e = \{v_1, v_2\}$, $\{v_1, v_2\} \cap V_1$ et $\{v_1, v_2\} \cap V_2$ sont des singletons.

Exemple 3.3.1. $K_{3,3}$: on note $K_{i,j}$ un graphe biparti complet, c'est à dire, tel que $\#V_1 = i$, $\#V_2 = j$ et tout sommet de V_1 est relié à tout sommet de V_2.



Proposition 3.4.1

Si $G = (V, E)$ est biparti et si $|\#V_1 - \#V_2| > 1$ alors G n'est ni hamiltonien, ni semi-hamiltonien.

Review: partial order

Def. A binary relation R defined in a set A is said to be a **partial order** if:

1. xRx $\forall x \in A$ (reflexibity);
2. xRy and $yRx \Rightarrow x=y$ (antisymmetry);
3. xRy and $yRz \Rightarrow xRz$ (transitivity).

A set with a partial order is called a **partially ordered set.**

Obs. Since a partial order is not symmetric, xRy is not the same as yRx.

Notation. For a partial order R, we will write $\leq$ instead of R.

Examples.

1. The relation $\leq$ in $\mathbb{R}$ is a partial order.
2. The inclusion $\subseteq$ between subsets is a partial order.

Def. Let $(A, \leq)$ be a partially ordered set. An element $m \in A$ is said to be **maximal** if there are no larger elements but itself. In other words, if $x \geq m$ then $x = m$.

Example. $S = \{1,2,3\}$,

$A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$, with the partial order $\subseteq$.

$$\begin{tikzcd}[column sep=small]
\{1,2\} & & & \{2,3\} & \\
\{1\} \arrow[u, dash] & & \{2\} \arrow[ull, dash] \arrow[ur, dash] & & \{3\} \arrow[ul, dash] \\
& & \phi \arrow[ull, dash] \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

$\{1,2\}$ and $\{2,3\}$ are maximal elements.

This example shows that there could be more than one maximal element. But, there is always a maximal element?

Thm. If A is a finite non-empty partially ordered set, then there is a maximal element.