Anomalous Diffusion & Fractional Random Walks

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Abstract

This work introduces some historical models of standard diffusion done mainly in the beginning of the 20^{th} century and other models of anomalous diffusion. First, we present two physical models of standard diffusion associated to Paul Langevin and Adolf Fick. Then, we show how these processes wan be interpreted as random walks of particles thanks to the work of Albert Einstein and the Fokker-Planck equation (also called Kolmogorov forward equation).
In the last part of the first section we show how some diffusion processes are not standard but anomalous. These anomalous processes are discussed in the second section were three different types of these processes are generated using different models. Firstly, fractional Brownian motion is generated by modifying the diffusion coefficient in the standard diffusion equation. After that, continuous time random walks are introduced and their master equation is computed. This master equation is used to generate subdiffuion using power law waiting time and superdiffusion using Lévy flights.

Key words: Random walk, Standard diffusion, Continuous time random walk, Anomalous diffusion.

Contents

1 Introduction

In the year 1827, the botanist Robert Brown observed under a microscope the behaviour of pollen grains suspended in water. He saw that the grains had a constant motion in water, and this behaviour was reproduced by all the small substances that can be suspended in water.
Robert Brown and his contemporaries failed to explain the reason of the movement they observed. But in 1905, Albert Einstein published a famous paper in "Annalen der Physik" where he came with a physical explanation of this phenomenon. Einstein came to these conclusions by using mathematical tools developed by both Thorvald N. Thiele in 1880 and Louis Bachelier in 1900[1].
The Brownian motion observed by Robert Brown and described later by Einstein is called standard diffusion. It is widely observed in nature and other scientific fields like economy, and it is characterized by a linear mean square displacement of the "particles". This Brownian motion isn't the only type of diffusion processes, there are other types of diffusion called anomalous diffusion where the mean square displacement is not linear. Such diffusion processes can be observed in fluids far from equilibrium as in the rotating annulus experiment where particles keep the same position for a random period of time, and their steps can be of the system scale[2].
This work is aims to summarize and present some research about standard and anomalous diffusion processes. It is largely theoretical and some python codes are used to produce simulations that help at the understanding of the theory. In order to achieve the expected goals we start by studying the historical models of standard. Then we tackle anomalous diffusion, and the different mathematical and physical concepts behind these models. The simulations use Monte-Carlo methods for generating random samples and the numerical results are validated using least squares regression and theoretical results.

2 Models for Standard Diffusion

2.1 Langevin's Equation

One way of treating Brownian Motion was imagined by Paul Langevin who described this motion with a stochastic differential equation.
Let's consider a particle of mass m performing a random walk inside a fluid. Langevin came with the following equation:

$$\mathrm{m}\ddot{x} = \text{-}a\dot{x} + \mathrm{F(t)}$$

the first term of the right side represents the friction force and F(t) is a random driving force.
We assume that: $<\mathrm{F(t)}> = 0$ and $<x\mathrm{F(t)}> = 0$.

By multiplying the previous equation with x we get:

$$\mathrm{m}x\ddot{x} = \text{-a}x\dot{x} + x\mathrm{F(t)}$$

$$\mathrm{m}\left(\frac{d(x\dot{x})}{dt} - \dot{x}^2\right) = \text{-a}x\dot{x} + x\mathrm{F(t)}$$

averaging this equation gives:

$$\mathrm{m}\frac{d}{dt}<x\dot{x}> = \mathrm{m}<\dot{x}^2> \text{ - a}<x\dot{x}>$$

using Boltzmann's Principle of Equipartition of Energy[4]: $\frac{1}{2}\mathrm{m}<\dot{x}^2> = \frac{1}{2}k_B\mathrm{T}$, we get the differential equation:

$$\frac{d}{dt}\left(\frac{1}{2}\frac{d}{dt}<x^2>\right) + \frac{a}{m}\left(\frac{1}{2}\frac{d}{dt}<x^2>\right) = \frac{k_BT}{m}$$

Since the mean square displacement is zero at t=0, the solution of the equation is:

$$\frac{1}{2}\frac{d}{dt}<x^2> = \frac{k_BT}{a}(1 \text{ - } \exp(\text{-}\frac{a}{m}t))$$

By integrating this equation we get:

$$<x^2> = \frac{2k_BT}{a}(t - \frac{1}{a}(1 - \exp(-\frac{a}{m}t))$$

For a time t $>>$ $(\frac{a}{m})^{-1}$:

$$<x^2> \sim \frac{2k_BT}{a}\ t$$

therefore:

$$<x^2> = 2\mathrm{D}t$$

with: D $= \frac{k_BT}{a}$.

2.2 Fick's Law and the Diffusion Equation

Let's consider particle diffusion along the x-direction in a 3-dimensional space. Particle conservation implies that the time variation of the density $c(x,t)$ inside an elementary parallelepipedal volume is equal to the inflow minus the outflow of particles.

$$c(x,t+dt)dxdydz = c(x,t)dxdydz + q(x,t)dydzdt - q(x+dx,t)dydzdt$$

$$\frac{c(x,t+dt)-c(x,t)}{dt} = -\frac{q(x+dx,t)-q(x,t)}{dx}$$

in the limit $dt \to 0$ and $dx \to 0$:

$$\frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x}$$

Fick's Law[5] stats that $q(x,t) = -D(x)\frac{\partial c}{\partial x}$ where D is the diffusion coefficient. Thus, the diffusion equation takes the form:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}(\mathrm{D}(x)\frac{\partial c}{\partial x})$$

for a constant diffusion coefficient $D(x) = D$:

$$\frac{\partial c}{\partial t} = \text{-D}\frac{\partial^2 c}{\partial x^2}$$

Green's solution $G(x,t)$ of this diffusion equation with the initial condition $G(x,t=0) = \delta(x)$ can be obtained using spatial Fourier transform:

$$\frac{\partial G}{\partial t} = -D\frac{\partial^2 G}{\partial x^2}$$

$$\frac{\partial \hat{G}(q,t)}{\partial t} = -Dq^2\hat{G}(q,t)$$

since $\hat{\delta}(q) = 1$:

$$\hat{G}(q,t) = e^{-Dq^2t}$$

The inverse Fourier transform leads to Green's solution of the diffusion equation:

$$G(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

The associated mean square displacement is:

$$< x^2 >= \int_{-\infty}^{+\infty} x^2 G(x,t)dx.$$

$$<x^2> = 2\mathrm{D}t$$

2.3 Einstein's Approach

In 1905, Einstein introduced a new method of getting the diffusion equation[6]. Let's consider a 1-dimensional random walk where the probability of being at position x at time t is $P(x,t)$, and where the random walker takes steps at discrete time Δt and the jump length is described by the probability density function $\lambda_{\Delta x}$. The master equation that describes the evolution of the probability density function P is:

$$P(x,t) = \int_{-\infty}^{+\infty} \lambda_{\Delta x}(\Delta x) P(x - \Delta x, t - \Delta t) d\Delta x.$$

We consider the function P as analytic, therefore:

$$\forall \Delta x, \Delta t : P(x - \Delta x, t - \Delta t) = P(x,t) - \Delta t \partial_t P(x,t) - \Delta x \partial_x P(x,t) + \frac{\Delta x^2}{2} \partial_x^2 P(x,t) + ...$$

Also $\forall i \geq 3$: $<\Delta x^i> = 0$,
The master equation becomes:

$$\frac{\partial P}{\partial t} = \frac{<\Delta x^2>}{2\Delta t} \frac{\partial^2 P}{\partial x^2} - \frac{<\Delta x>}{\Delta t} \frac{\partial P}{\partial x}$$

We assume that $\lambda_{\Delta x}$ is symmetric then $<\Delta x> = 0$, consequently:

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

where $D = \frac{<\Delta x^2>}{2\Delta t}$.

As shown previously this diffusion equation has a Gaussian solution, and the mean square displacement is proportional to time.

2.4 Fokker-Planck Equation

The Fokker-Planck equation (also called Kolmogorov forward equation) is a more general diffusion equation than those introduced previously.
Similarly to Einstein's approach we consider a 1-dimensional random walk, but in this case the probability density function of step length $\lambda_{\Delta x}$ depends on both Δx and x. Consequently we use the notation $\lambda_{\Delta x,x}$. The master equation describing the evolution of P is:

$$P(x,t) = \int_{-\infty}^{+\infty} \lambda_{\Delta x,x}(\Delta x, x - \Delta x) P(x - \Delta x, t - \Delta t) d\Delta x.$$

By assuming that both functions $\lambda_{\Delta x,x}$ and $P(x - \Delta x, t - \Delta t)$ are analytic after evaluating the integrals we find the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2}(D(x)P(x,t)) - \frac{\partial}{\partial x}(v(x)P(x,t))$$

where $v(x) = \frac{\mu_{\Delta x}(x)}{\Delta t}$ and $D(x) = \frac{<\Delta x^2>(x)}{2\Delta t}$.

2.5 Simulation of Standard Random Walk

In this paragraph we simulate both 1-dimensional and 2-dimensional random walks. The previous paragraphs show that these two random walks are standard diffusion processes, and consequently we expect the mean square displacement to be a linear function of time.
The linear shape of the MSD is not enough to conclude on the validity of the simulations, therfore we use a least squares regression to calculate α and show that $\alpha = 1$ with an error of 10^{-3}.

2.5.1 1-Dimensional Random Walk

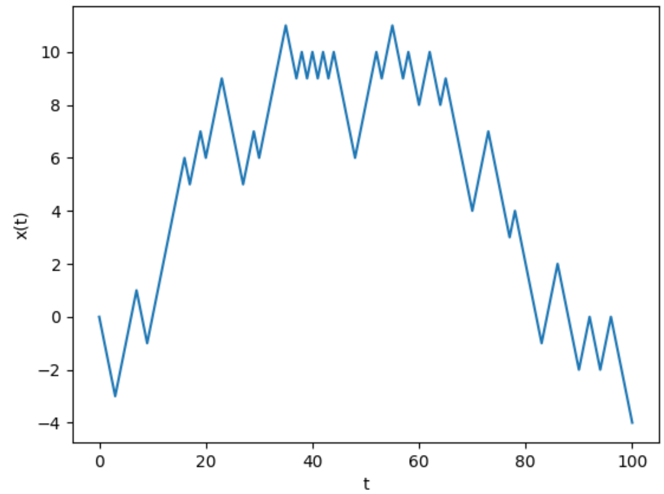

Figure 1: Position as a function of time

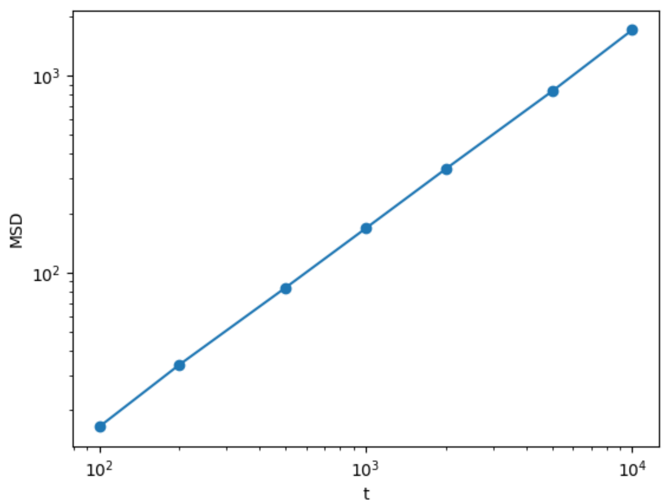

Figure 2: MSD as a function of time in the log scale

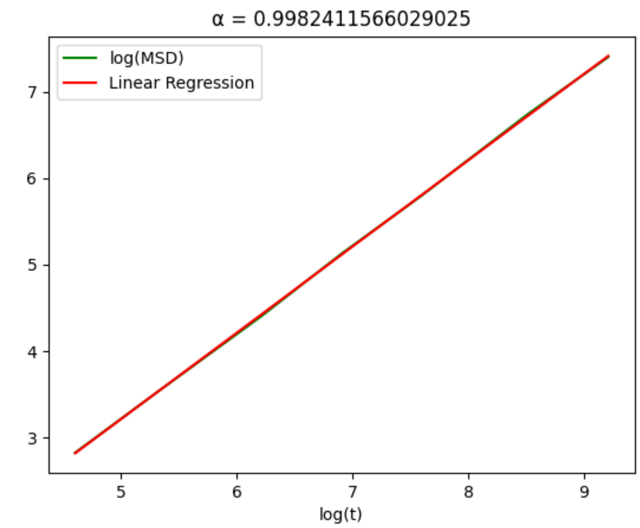

Figure 3: MSD and the corresponding regression as functions of time in the log scale

2.5.2 2-Dimensional Random Walk

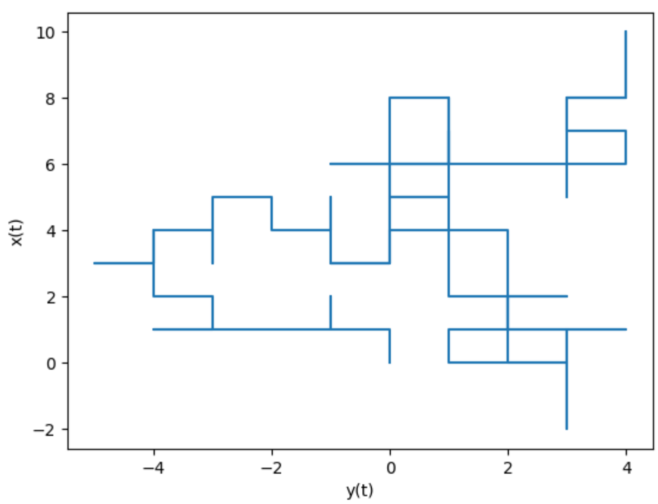

Figure 4: Position as a function of time

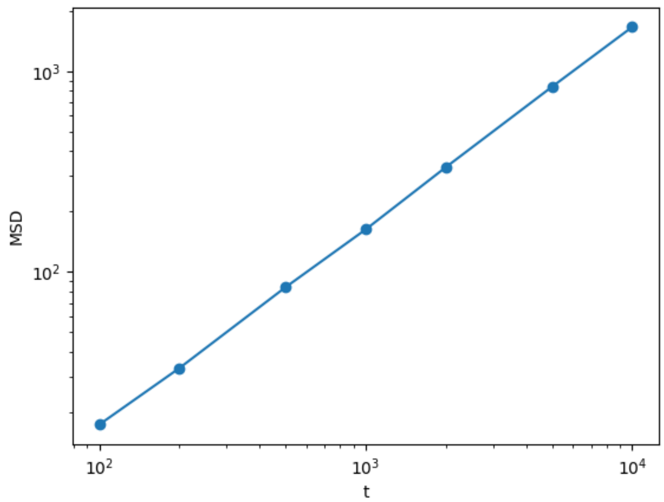

Figure 5: MSD as a function of time in the log scale

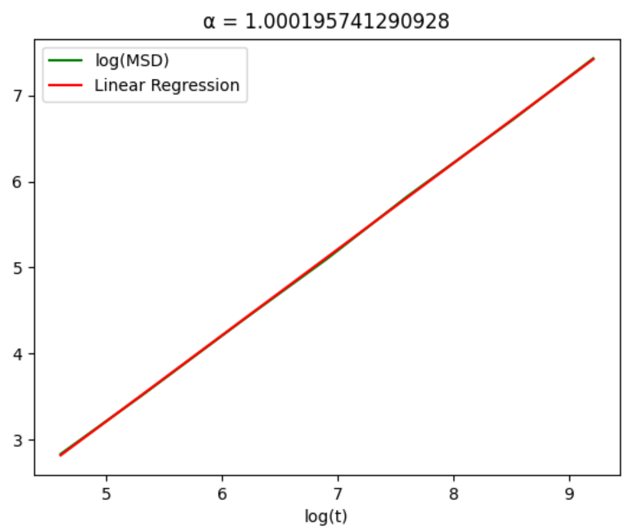

Figure 6: MSD and the corresponding regression as functions of time in the log scale

2.5.3 Conclusion

Both figures 3 and 6 show the mean square displacement of a random walker as a function of time in the log-log scale in the 1-dimensional and 2-dimensional cases respectively. This function is linear in both cases, and the regression coefficient is equal to one as predicted theoretically. Consequently, the simulations are valide.

2.6 Random walks and Percolation

One of the natural processes where random walks and diffusion appear is percolation. In this section we show a simple example of water percolation and a second where a random walk is realized in a square lattice.
The environment that water passes through (soil for instance) can be represented by a graph. Every graph is characterized by vertices and lines between these vertices. In this section, we construct a graph where the

nature of an edge between two vertices, whether it is open (can be used to go from one vertex to another) or closed, depends on the result of a Bernoulli law sampling with parameter p. We say that a link is open with a probability p and closed with a probability 1-p.
In this first example we simulate the percolation of water in a 2-dimensional heterogeneous environment that will be represented by a 50x50 graph where the white grids are the open edges and black grids are the closed.

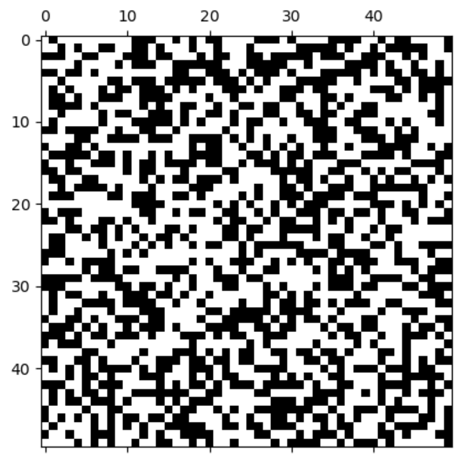

Figure 7: The grid of a random graph for p = 0.55

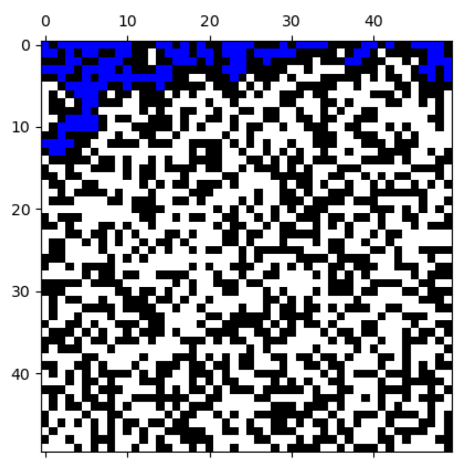

Figure 8: The grid of the same graph after water percolation from the top

The last figure shows water diffuses in the grid without reaching the bottom. We say that the water doesn't percolate.
Let's change the probability p and see how it impacts the results.

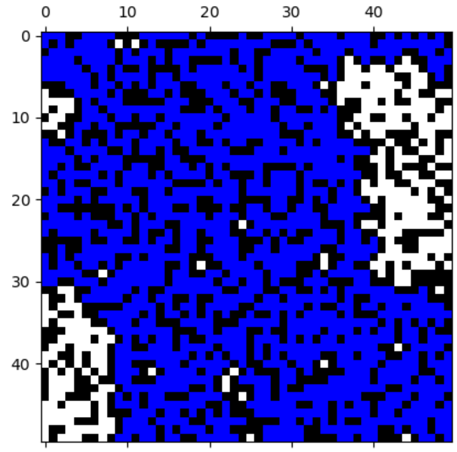

Figure 9: The grid of a random graph for p = 0.65 after water percolation from the top

Unlike the previous case, the water here reaches the bottom. It percolates!
As it can be naturally predicted, the percolation of water depends on the probability p. It can be shown that if p is greater than a critic probability p_c than there will be a percolation in the grid, and the value of p_c depends only on the dimension d of space that contains the graph not the size of this graph (in the case we have: d=2 and p_c=0.592746)[7].

Let's keep the same type of lattices. This time we take a random walk from the center of the lattice and we see whether there is a percolation or not. This random walker is unbiased he has the same probability to go in every direction, and he takes a step of length 1 in each second (dt=1) to the right, the left, up or down. An additional parameter this time that we didn't need in the first example is the duration of the random walk that we denote by t.

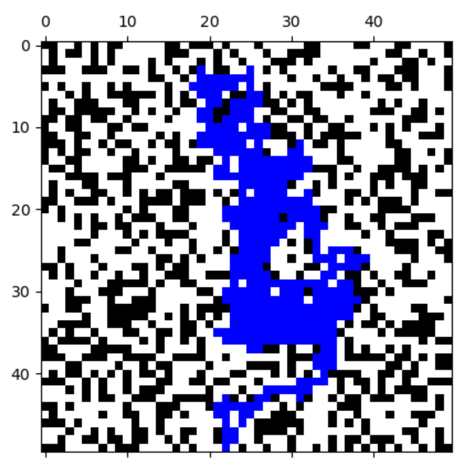

Figure 10: 50x50 lattice at the end of a random walk of a duration of 16,000 seconds

This random walker diffuses and it is obvious that diffusion depends not only on p_c but also on t. It is clear that for small values of t there won't be any percolation, but the results from the previous paragraphs make us think about a relation between the mean square displacement of the random walker and time. The mean

square displacement gives us an idea about the distance the random walker crosses after time t and implicitly this distance helps to determine whether there is percolation or not.
In the case of such random walk on a lattice, the mean square displacement is:

$$<x^2> \sim t^{2/d_w}$$

d_w is the fractal dimension of the random walker, it is approximately equal to $3d_f/2$ where d_f is the fractal dimension of the lattice[8].

This mean square displacement is not a linear function of time. This diffusion process is not standard, we call it anomalous diffusion.

3 Anomalous Diffusion

3.1 Fractional Brownian Motion

In standard diffusion the mean square displacement is proportional to time $< x^2 >\sim t^{\alpha}$ where $\alpha = 1$. Many experiments of diffusion show that the mean square displacement can be proportional to a fractional power law of time, in this case $\alpha \neq 1$.
One way of modeling fractional diffusion is by replacing the diffusion coefficient D in the diffusion equation by a time dependent coefficient $D(t) = \alpha t^{\alpha} D$. In this case the diffusion equation is:

$$\frac{\partial c}{\partial t} = \alpha t^{\alpha} D \frac{\partial^2 c}{\partial x^2}$$

This equation is associated to the probability density function of a fractional Brownian motion[9]. The solution of this equation is the Gaussian distribution:

$$\mathrm{G}(x,\mathrm{t}) = \frac{1}{\sqrt{4\pi D t^{\alpha}}}\, e^{-\frac{x^2}{4Dt^{\alpha}}}$$

and the mean square displacement associated is:

$$< x^2 > = 2\mathrm{D}t^{\alpha}$$

In this paragraph we simulate a fractional random walk where $\alpha = 1/2$. In order to generate random numbers depending on non-uniform probability distributions the Inverse Transform Sampling is used. This method will be used also in the following paragraphs.

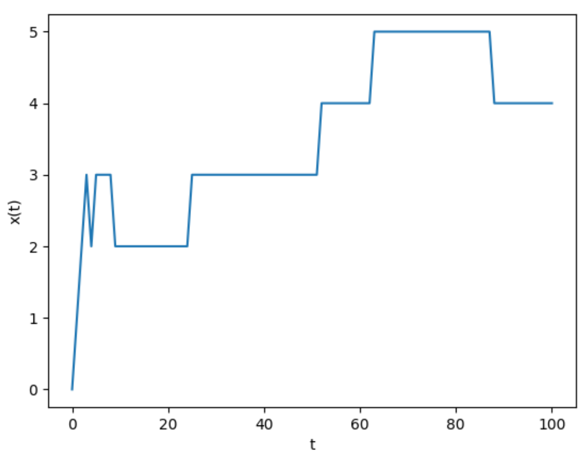

Figure 11: Position as a function of time

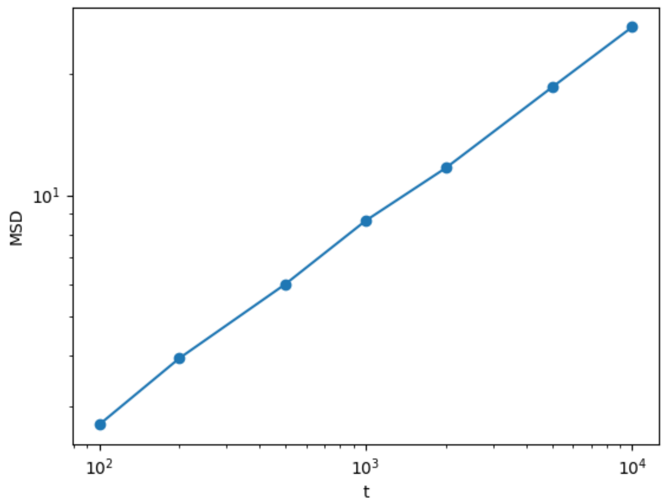

Figure 12: MSD as a function of time in the logarithmic scale

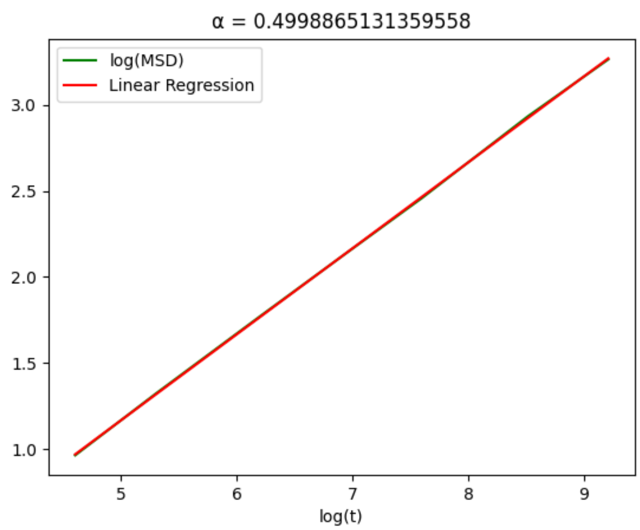

Figure 13: MSD and the corresponding regression as functions of time in the log scale

The result obtained from the linear regression shows that as expected $< x^2 >\sim t^{1/2}$. Thus the simulations are correct.

Remark[10]: Another way of representing the diffusion coefficient is by using fractional derivatives:

$$\mathrm{D(t)} = {}_0D_t^{1-\alpha}(\Gamma(\alpha)D)$$

where ${}_0D_t^{1-\alpha}$ is Riemann-Liouville left-sided integral of order $1-\alpha$.
Hence, the previous diffusion equation can be expressed as follows:

$$\frac{\partial c}{\partial t} = {}_0D_t^{1-\alpha}(\Gamma(\alpha)D)\frac{\partial^2 c}{\partial x^2}$$

3.2 Continuous Time Random Walk

In the first section have been demonstrated that a random walk characterized by a small step and a small waiting time leads to a standard diffusion. In order to model anomalous diffusion we should allow the random

walker to take large steps (steps up to the system size in a finite system and infinite steps in an unbounded system).
In this section we are going to introduce a more general random walk where both step length and waiting time are picked from two different probability densities. These walks are called Continuous Time Random Walks (CTRWs). Before defining the probability density of a CTRW let's introduce the concept of turning-points in a random walk.
A point is called turning-point if the random walker begins a new step after reaching it. The distribution of the turning-points $Q(x,t)$ is given by the equation:

$$Q(x,t) = \int_{-\infty}^{+\infty}\int_0^t \lambda_{\Delta x,\Delta t}(\Delta x,\Delta t)Q(x-\Delta x,t-\Delta t)d\Delta t d\Delta x + \delta(t)P(x,t=0)$$

where the first term on the right side describes a completed random walk and the second one is the initial condition [11].
In the waiting model of a CTRW, the probability for a random walker to be at position x at time t is:

$$P(x,t) = \int_0^t \Phi(\Delta t)Q(x,t-\Delta t)d\Delta t$$

where:

$$\Phi(t) = 1 - \int_0^t\int_{-\infty}^{+\infty} \lambda_{\Delta x,\Delta t}(\Delta x,\Delta t)d\Delta x d\Delta t$$

$\Phi(t)$ is the probability of not making a step during a period of time equal to t, and it is called the survival probability.
By combining the previous equations we get the CTRW master equation:

$$P(x,t) = \int_0^t\int_{-\infty}^{+\infty} \lambda_{\Delta x,\Delta t}(x-\Delta x,t-\Delta t)P(\Delta x,\Delta t)d\Delta x d\Delta t + \Phi(\Delta t)P(x,t=0)$$

In the waiting model for CTRW space and time are independent[12], thus we can consider:
$\lambda_{\Delta x,\Delta t}(x,t) = \lambda_{\Delta x}(x)\lambda_{\Delta t}(t)$, and consequently the master equation becomes:

$$P(x,t) = \int_0^t\int_{-\infty}^{+\infty} \lambda_{\Delta x}(x-\Delta x)\lambda_{\Delta t}(t-\Delta t)P(\Delta x,\Delta t)d\Delta x d\Delta t + \Phi(\Delta t)P(x,t=0)$$

Using Fourier-Laplace transform of the CTRW master equation (Fourier transform in space and Laplace transform in time) gives:

$$\hat{P}(q,u) = \hat{\lambda}_{\Delta x}(q)\hat{\lambda}_{\Delta t}(u)\hat{\hat{P}}(q,u) + \hat{\Phi}(u)\hat{P}(q,0)$$

where q is the Fourier variable and u is the Laplace variable.
The Laplace transform of the survival probability can be easily simplified to:

$$\hat{\Phi}(u) = \frac{1}{u} - \frac{\hat{\lambda}_{\Delta t}(u)}{u}$$

Finally, the CTRW master equation is:

$$u\hat{\hat{P}}(q,u) = u\hat{\lambda}_{\Delta x}(q)\hat{\lambda}_{\Delta t}(u)\hat{\hat{P}}(q,u) + (1-\hat{\lambda}_{\Delta t}(u))\hat{P}(q,0)$$

3.3 Generating Fractional Subdiffusion using Power Law Waiting Time

One way of generating fractional subdiffusion is using a Pareto waiting time density:

$$\forall t \in [\tau,\infty],\ \lambda_{\Delta t}(t) = \frac{\alpha\tau^{\alpha}}{t^{1+\alpha}}$$

where $0 < \alpha < 1$.
The asymptotic Laplace transform for this density can be determined using a Tauberian theorem:

$$\hat{\lambda}_{\Delta t}(u) \sim 1 - \Gamma(1-\alpha)\tau^{\alpha}u^{\alpha}$$

Also, we consider that the step length density has the following asymptotic expansion:

$$\hat{\lambda}_{\Delta x}(q) \sim 1 - \frac{q^2\sigma^2}{2} + O(q^4)$$

where: $\sigma^2 = \int r^2\lambda(r)dr$.

After replacing these expressions in the CTRW master equation we get:

$$u\hat{\hat{P}}(q,u) - \hat{P}(q,0) = -\frac{q^2\sigma^2}{2\tau^\alpha\Gamma(1-\alpha)}u^{1-\alpha}\hat{\hat{P}}(q,u)$$

and by using the inverse Fourier-Laplace transform:

$$\frac{\partial P(x,t)}{\partial t} = D\mathcal{L}^{-1}(u^{1-\alpha}\frac{\partial^2 \hat{P}(x,u)}{\partial x^2})$$

where $D = \frac{\sigma^2}{2\tau^\alpha\Gamma(1-\alpha)}$.

A simple representation of this equation can be obtained by using fractional calculus:

$$u^{1-\alpha}\frac{\partial^2 \hat{P}(x,u)}{\partial x^2} = \mathcal{L}({}_0D_t^{1-\alpha}\frac{\partial^2 P(x,t)}{\partial x^2}) + ({}_0D_t^{-\alpha}\frac{\partial^2 \hat{P}(x,t)}{\partial x^2}\mid_{t=0})$$

where ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative of order α and ${}_0D_t^{-\alpha}$ is a fractional integral of order α. The integral is equal to zero [13], therefore we get the fractional subdiffusion equation:

$$\frac{\partial P(x,t)}{\partial t} = \mathrm{D}({}_0D_t^{-\alpha}\frac{\partial^2 P(x,t)}{\partial x^2})$$

Green's solution for this fractional subdiffusion equation with the initial condition G(x,t=0)=$\delta(x)$ can be written using Fox H functions [14], and the mean square displacement resulting in this case is:

$$< x^2 >= \frac{2D}{\Gamma(1+\alpha)}t^\alpha$$

Now we are simulate a fractional subdiffusive random walk where $\alpha = 1/2$.

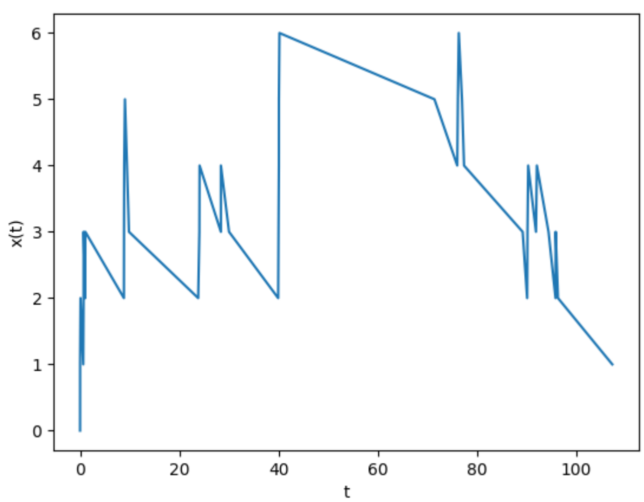

Figure 14: Position as a function of time

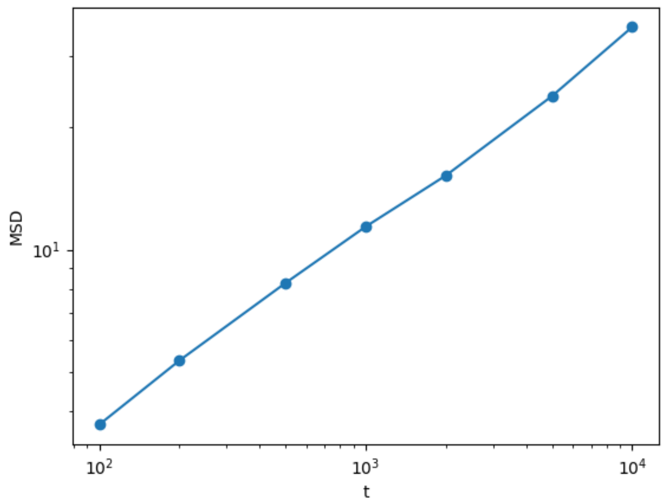

Figure 15: MSD as a function of time in the logarithmic scale

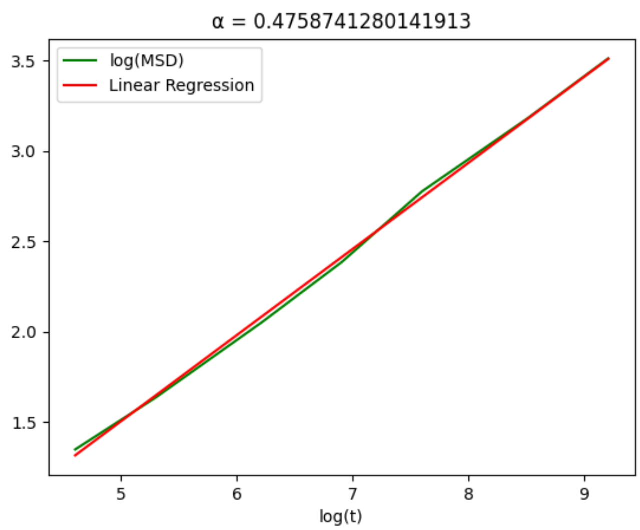

Figure 16: MSD and the corresponding regression as functions of time in the log scale

The mean square displacement represented was calculated by averaging 3000 trajectories and the value of α returned by the linear regression is nearly equal to the result we expected which is $1/2$. Thus this simulations are correct.

3.4 Generating Fractional Superdiffusion using Lévy Flights

In this part we generate CTRWs with an exponential waiting time density and a Lévy step length density with power law asymptotics:

$$\lambda_{\Delta x}(x) \sim \frac{A_\alpha}{\sigma_\alpha}|x|^{-1-\alpha}$$

where $1<\alpha<2$.
The Fourier transform of this density has the following expansion:

$$\hat{\lambda}_{\Delta x}(x) \sim 1 - \sigma^\alpha |q|^\alpha$$

As in the previous section, we will use the asymptotic expansion of the waiting time density and the master equation, and finally we get this equation:

$$u\hat{\hat{P}}(q,u) - u\hat{P}(q,0) = -\frac{\sigma^\alpha |q|^\alpha}{\tau}\hat{\hat{P}}(q,u)$$

The inverse Laplace transform leads to:

$$\frac{\partial \hat{P}(q,t)}{\partial t} = -\frac{\sigma^\alpha |q|^\alpha}{\tau}\hat{P}(q,t)$$

by using the inverse Fourier transform and some classic results from fractional calculus we get:

$$\mathcal{F}(\nabla^\alpha_{|x|}P(x,t)) = -|q|^\alpha \hat{P}(q,t)$$

where $\mathcal{F}$ is the Fourier transform operator and $\nabla^\alpha_{|x|}$ is the Riesz fractional derivative.

Using the Fourier inverse transform we get the fractional superdiffusion equation:

$$\frac{\partial P}{\partial t} = D\nabla^\alpha_{|x|}P$$

where D $= \frac{\sigma^\alpha}{\tau}$.

This equation has a solution that can be expressed using Fox H functions. The asymptotic behavior of this solution is:

$$P(x,t) \sim \frac{\sigma^\alpha t}{\tau |x|^{1+\alpha}}$$

where $1<\alpha<2$.
This probability density function has a divergent mean square displacement: $< x^2(t) > \to \infty$.

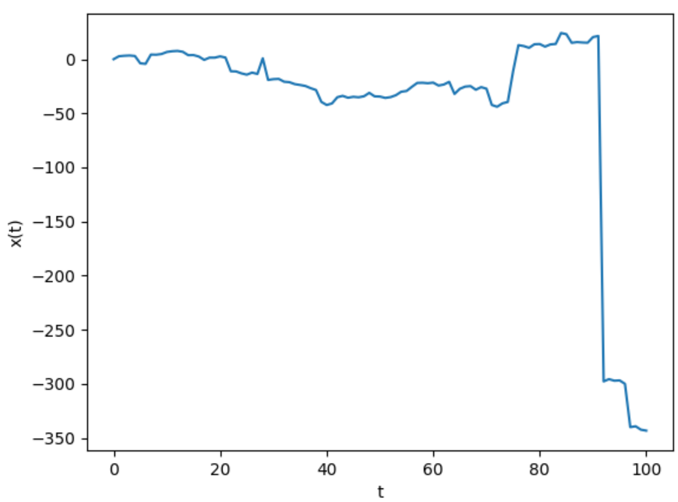

Figure 17: Position as a function of time

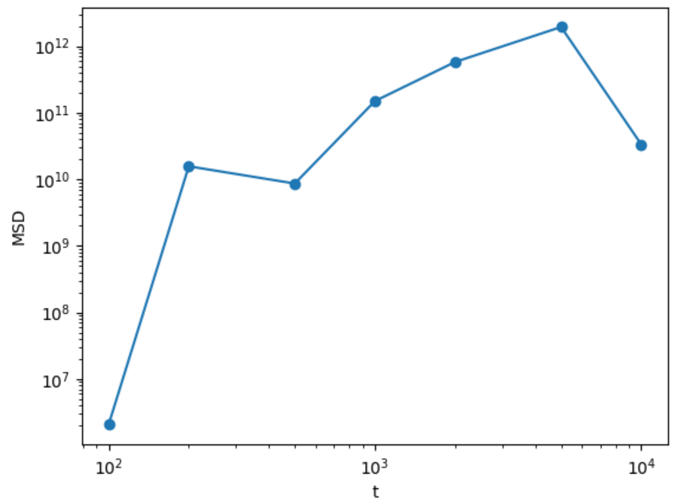

Figure 18: MSD as a function of time in the logarithmic scale

4 Conclusions and perspectives

Brownian motion is a standard diffusion process that can be modeled in different ways using physical or mathematical models, and like any other standard diffusion process the mean square displacement of the particles is a linear function of time. However, many other diffusion phenomenons have a mean square displacement that is a non-linear function of time. They are called anomalous diffusion, and they be modeled using continuous time random walks and fractional diffusion equations.
This work can be further developed and extended to show concrete applications of anomalous diffusion in physics. These applications are very recent, for instance the first application of CTRWs to plasma physics problems was done by R.Balescu in 1995 [18], and consequently they can be very useful to solve many problems in physics.

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6 Appendix

6.1 Fractional Browian motion

A fractional Brownian motion (fBm) is a continuous-time Gaussian process $B_H(t)$ on a segment [0,T] such that:

$$\begin{cases} B_H(0) = 0 \\ E(B_H(t)) = 0 \\ E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \end{cases}$$

$H \in [0, 1]$ is called Hurst index.

A Brownian motion is a special case of fBms where H $= \frac{1}{2}$.

6.2 Riemann-Liouville fractional integral

There are two different Riemann-Liouville integrals, the left-sided fractional integral and the right-sided fractional integral.
The right-sided Riemann-Liouville integral of a function f is defined as follows:

$$_aD_x^{-q} f(x) = \frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_a^x \frac{f(y)}{(x-y)^{-q+1}} \, \mathrm{d}y$$

with: $q \in \mathbb{R}^2$ and $a, x \in \mathbb{R}$.

The left-sided Riemann-Liouville integral of a function f is defined as follows:

$$_xD_a^{-q} f(x) = \frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_x^a \frac{f(y)}{(x-y)^{-q+1}} \, \mathrm{d}y$$

with: $q \in \mathbb{R}^2$ and $a, x \in \mathbb{R}$.

6.3 Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative of a function f is defined as follows:

$$D_x^q f(x) = \frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)} f(x)}{dx^{-(n-q)}}\right) \; ; \; q \in \mathbb{R}^2; \; n = \lfloor q \rfloor + 1$$

with: $q \in \mathbb{R}^2$ and $n = \lfloor q \rfloor + 1$

The derivative of negative order that appears in the expression of this derivative is the right-sided Riemann-Liouville integral.

6.4 Riesz fractional derivative

The Riesz fractional derivative of a function f is:

$$\nabla^\alpha_{|x|} f(x) = -\frac{1}{2cos(\pi\alpha/2)} ({}_{-\infty}D^x_\alpha +_x D^\alpha_\infty) f(x)$$

with: $1 < \alpha < 2$.