Applied differential equations

TW244 - Lecture 34

X.X Nondimensionalisation

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SCIENCE
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X.X: Nondimensionalisation

What is nondimensionalisation?

Many of the ODE models we have considered in this course have a number of parameters in them.

Consider, for example, the linear and quadratic models of free-fall

$$m\frac{dv}{dt} = mg - kv \qquad \text{and} \qquad m\frac{dv}{dt} = mg - kv^2,$$

and the damped spring-mass system with a forcing term:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F_0\cos(\gamma t).$$

Are all these parameters necessary, or can we make do with fewer?

Nondimensionalisation is a good way to answer this question. It does so by removing the dimensions from the equation, which often reveals what are called the characteristic units (or "length scales") of the system.

The five steps

The five steps are:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order term;
4. Choose the characteristic units for each variable so that the coefficients of as many terms as possible become 1*;
5. Rewrite the system in terms of the new dimensionless quantities.

*It is usually not possible to set all coefficients to 1, so often a choice must be made.

X.X: Nondimensionalisation

Example 1

Example 1: Let's apply idea this to the free-fall model $m\frac{dv}{dt} = mg - kv$.

1 Clearly v is the dependent variable and t is the indep. variable.

2 Introduce $\widehat{v} = v/v_c$ and $\widehat{t} = t/t_c$, where x_c and t_c are the characteristic lengths to be determined and note that

$$\left.\begin{aligned} v &= v_c\widehat{v} \\ t &= t_c\widehat{t} \end{aligned}\right\} \implies \frac{dv}{dt} = \frac{v_c}{t_c}\frac{d\widehat{v}}{d\widehat{t}}$$

and therefore

$$m\frac{v_c}{t_c}\frac{d\widehat{v}}{d\widehat{t}} = mg - kv_c\widehat{v}.$$

3 Dividing through by mv_c/t_c we have

$$1\frac{d\widehat{v}}{d\widehat{t}} = \frac{t_cg}{v_c} - \frac{kt_c}{m}\widehat{v}.$$

Notice that this equation is now dimensionless.

Example 1 (cont.)

4 We seek to make coefficients equal 1, therefore

$$\frac{t_C g}{v_C} = \frac{k t_C}{m} = 1 \implies t_C = \frac{m}{k} \;\&\; v_C = \frac{mg}{k},$$

5 and now the new (nondimensional) equation is given by

$$\boxed{\frac{d\widehat{v}}{d\widehat{t}} = 1 - \widehat{v}.}$$

What is this telling us? Well, it says that the system doesn't depend on m, k, and g, only their relations as given by t_C and v_C. Thus we have reduced the number of unknowns from three to two. Furthermore, since there are no parameters left in the equation, this tells us all solutions essentially behave in the same way.

Recall the analytical solution we found in Lecture 9 (when $x(0) = 0$):

$$\boxed{v(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) = v_C(1 - e^{-\frac{t}{t_C}}).}$$

Here we see how important the quantities t_C and v_C are in the solution.

Example 2 (Exercise): Now consider the case of nonlinear drag:

$$m\frac{dv}{dt} = mg - kv^2.$$

Show that the characteristic length scales here are now given by:

$$t_c = \sqrt{\frac{m}{gk}} \quad \text{and} \quad v_c = \sqrt{\frac{mg}{k}}$$

and the corresponding nondimensional equation by

$$\boxed{\frac{d\hat{v}}{d\hat{t}} = 1 - \hat{v}^2.}$$

Again, note the appearance of v_c & t_c in the solution from in Lecture 10:

$$\boxed{v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right) = v_c \tanh\left(\frac{t}{t_c}\right).}$$

Example 3: Consider next the the spring-mass system:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F_0\cos(\gamma t).$$

Notice here that we have five parameters (m, β, k, F_0, and γ).

Proceed as before:

1 Clearly x is the dependent variable and t is the indep. variable.

2 Introduce $\widehat{x} = x/x_c$ and $\widehat{t} = t/t_c$ so that (note $\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\widehat{x}}{d\widehat{t}^2}$)

$$m\underbrace{\frac{x_c}{t_c^2}}\frac{d^2\widehat{x}}{d\widehat{t}^2} + \beta\frac{x_c}{t_c}\frac{d\widehat{x}}{d\widehat{t}} + kx_c\widehat{x} = F_0\cos(\gamma t_c\widehat{t}).$$

3 Dividing through by mx_c/t_c^2 we have

$$\frac{d^2\widehat{x}}{d\widehat{t}^2} + \frac{\beta t_c}{m}\frac{d\widehat{x}}{d\widehat{t}} + \frac{kt_c^2}{m}\widehat{x} = \frac{F_0 t_c^2}{x_c m}\cos(\gamma t_c\widehat{t}).$$

$x = x_c\hat{x}, \quad t = t_c\hat{t}$

$\frac{dx}{dt} = \frac{x_c}{t_c}\frac{d\hat{x}}{d\hat{t}}$

$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt}$

$= \frac{1}{t_c}\frac{d}{d\hat{t}}\frac{x_c}{t_c}\frac{d\hat{x}}{d\hat{t}}$

$= \frac{1}{t_c}\frac{x_c}{t_c}\frac{d^2\hat{x}}{d\hat{t}^2}$

$= \frac{x_c}{t_c^2}\frac{d^2\hat{x}}{d\hat{t}^2}$

Example 3 (cont.)

$$\frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\beta t_c}{m}\frac{d\hat{x}}{d\hat{t}} + \frac{k t_c^2}{m}\hat{x} = \frac{F_0 t_c^2}{x_c m}\cos(\gamma t_c \hat{t}).$$

4 We cannot set all the terms to unity. We choose the 0th-order term so

$$\frac{k t_c^2}{m} = 1 \implies t_c = \sqrt{\frac{m}{k}} \quad \text{and} \quad \frac{F_0 t_c^2}{x_c m} = 1 \implies x_c = \frac{F_0}{k}.$$

5 Hence we arrive at the nondimensional equation

$$\boxed{\frac{d^2\hat{x}}{d\hat{t}^2} + 2\xi\frac{d\hat{x}}{d\hat{t}} + \hat{x} = \cos(\phi\hat{t})} \quad : \quad 2\xi = \beta/\sqrt{mk} \text{ and } \phi = \gamma t_c.$$

Thus the behaviour of such spring-mass systems can be characterised by just two parameters, rather than the initial five we started with! Notice over- and under-damping, as described in Lecture 22, correspond to $\xi < 1$ and $\xi > 1$, respectively. t_c is the period of the undamped motion.

Example 4: Recall the Lotka–Volterra predator-prey model:

$$\frac{dx}{dt} = x(-a + by), \quad \frac{dy}{dt} = y(c - dx).$$

Here we have four parameters; the birth and death rates of each species. But can we characterise the behaviour of the solution with fewer?

1. Here x and y are dependent variables and t is the indep. variable.
2. Introduce $\widehat{x} = x/x_C$, $\widehat{y} = y/y_C$, and $\widehat{t} = t/t_C$ so that

$$\frac{x_C}{t_C}\frac{d\widehat{x}}{d\widehat{t}} = x_C\widehat{x}(-a + by_C\widehat{y}), \quad \frac{y_C}{t_C}\frac{d\widehat{y}}{d\widehat{t}} = y_C\widehat{y}(c - dx_C\widehat{x}).$$

3. Dividing through by x_C/t_C and y_C/t_C and expanding out we have

$$\frac{d\widehat{x}}{d\widehat{t}} = (-at_C)\widehat{x} + (bt_Cy_C)\widehat{x}\widehat{y}, \quad \frac{d\widehat{y}}{d\widehat{t}} = (ct_C)\widehat{y} - (dt_Cx_C)\widehat{x}\widehat{y}.$$

Example 4 (cont.)

$$\frac{d\widehat{x}}{d\widehat{t}} = \underline{(-at_c)}\widehat{x} + \underline{(bt_cy_c)}\widehat{x}\widehat{y}, \quad \frac{d\widehat{y}}{d\widehat{t}} = \underline{(ct_c)}\widehat{y} - \underline{(dt_cx_c)}\widehat{x}\widehat{y}.$$

4 We have four terms, but only three choices. One option is

$$\underline{t_cx_cd = bt_cy_c = t_cc = 1} \implies t_c = \frac{1}{c}, x_c = \frac{c}{d}, y_c = \frac{c}{b}$$

5 Hence we arrive at the nondimensional equations

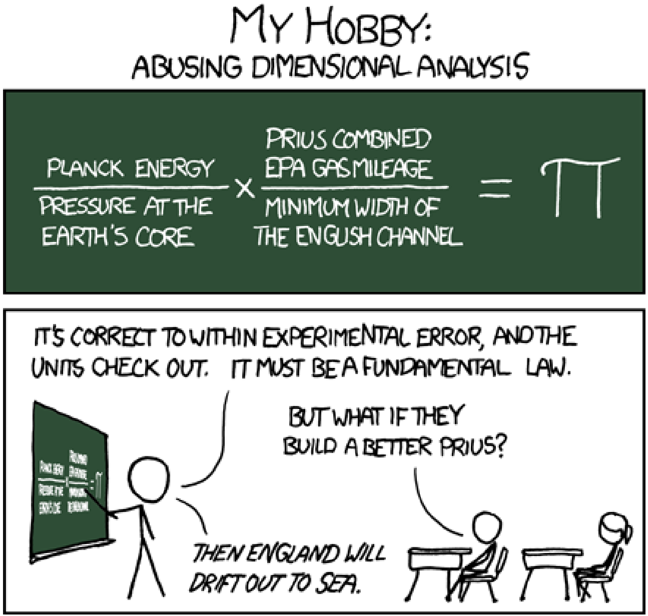
$$\frac{d\widehat{x}}{d\widehat{t}} = \widehat{x}(-\alpha + \underline{\widehat{y}}), \quad \frac{d\widehat{y}}{d\widehat{t}} = \underline{\widehat{y}}(1 - \underline{\widehat{x}}) \quad : \quad \alpha = \frac{a}{c}.$$

In this case see there is only one quantity which fundamentally changes the nature of the system; the ratio of the birth rates of the two species.

Summary

- In summary, nondimensional analysis potentially allows a reduction in the number of parameters dictating the behavior of a system.
- This can be useful both in analytical solutions of ODEs as well as numerical computations.
- Closely related is the idea of 'Dimensional Analysis' which is used often in areas like fluid dynamics.
- Like the Direction Fields and Phase Plane diagrams we have considered in this course, this is yet another way to get qualitative information about the solution to a DE, even if it cannot be solved.

Final Exercise: Revisit some of the other models we have considered in this course (e.g., the Logistic model) and see if nondimensional analysis reveals any further insights to these systems.


MY HOBBY:
ABUSING DIMENSIONAL ANALYSIS
PLANCK ENERGY
PRESSURE AT THE EARTH'S CORE
PRIUS COMBINED EPA GAS MILEAGE
MINIMUM WIDTH OF THE ENGLISH CHANNEL
= π
IT'S CORRECT TO WITHIN EXPERIMENTAL ERROR, AND THE UNITS CHECK OUT. IT MUST BE A FUNDAMENTAL LAW.
BUT WHAT IF THEY BUILD A BETTER PRIUS?
THEN ENGLAND WILL DRIFT OUT TO SEA.

What next?
Further reading and follow-on courses

Further reading: Prof Trefethen (my PhD supervisor) recently released a free book on Exploring ODEs using MATLAB:
`https://people.maths.ox.ac.uk/trefethen/ExplODE/`

SU Applied Math modules:

- Numerical methods (20710-324)
- Applied Fourier analysis (20710-364)
- Flow Modelling (20710-354)

SU Math Bio modules:

- Mathematical Applications in Biology and Medicine (21539-214)
- Introduction to Biological Modelling I (21539-314)
- Introduction to Biological Modelling II (21539-344)

The End