Applied differential equations

TW244 - Lecture 16

4.4: Non-homogeneous linear DEs

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Consider the DE

$$ay'' + by' + cy = g(x)$$

with $g(x) \not\equiv 0$ and a, b, c constant.

Recall (from Lecture 14), we may write the solution as

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

where

- $y_c(x)$ is the general solution of the corresponding homogeneous DE $ay'' + by' + cy = 0$.
- $y_p(x)$ is any particular solution of $ay'' + by' + cy = g(x)$.

Last time we saw how to find $y_c(x)$ (try $y = e^{mx}$, auxiliary equation, etc), but how do we find $y_p(x)$?

$ay'' + by' + cy = g(x)$

The method of undetermined coefficients:

particular soln

We make an "educated guess" for the form of y_p by looking at g!

Constants, polynomials, exponentials, sines, and/or cosines all have the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials, sines, and/or cosines.

Since the linear combination of derivatives $ay'' + by' + cy$ must be equal to $g(x)$, it is reasonable to assume that y_p is similar in form to $g(x)$.

TABLE 4.4.1 Trial Particular Solutions

$ay'' + by' + cy = g(x)$

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Find the general solution of $y'' + 6y' + 5y = e^{2x}$. , $y(0)=1$, $y'(0)=2$

($\mathbf{y_c}$): $y'' + 6y' + 5y = 0$.
Try $y = e^{mx} \implies m^2 + 6m + 5 = 0 \implies (m+1)(m+5) = 0$ $\Rightarrow m=-1,-5$

$$\implies \boxed{y_c = c_1 e^{-x} + c_2 e^{-5x}}$$

($\mathbf{y_p}$): $g(x) = e^{2x}$.
Try $y_p = Ae^{2x} \implies y_p' = 2Ae^{2x}, y_p'' = 4Ae^{2x}$ and substituting this to the DE gives:

$$4Ae^{2x} + 12Ae^{2x} + 5Ae^{2x} = \underbrace{e^{2x}}_{=g(x)} \implies 21A = 1 \implies y_p = \frac{1}{21}e^{2x}$$

Exercise: verify.

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-5x} + \frac{1}{21}e^{2x}}.$$

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Example 2

Find the general solution of $y'' + 6y' + 5y = e^{-x}$.

($\mathbf{y_c}$): $y'' + 6y' + 5y = 0 \implies y_c = c_1e^{-x} + c_2e^{-5x}$ (same as previous!)

($\mathbf{y_p}$): $g(x) = e^{-x}$.
Try $y_p = Ae^{-x} \implies y_p' = -Ae^{-x}, y_p'' = Ae^{-x}$ and substituting this to the DE gives:

$$Ae^{-x} - 6Ae^{-x} + 5Ae^{-x} = e^{-x} \implies 0 = 1 \implies \text{contradiction!}$$

We should obviously get zero here because c_1e^{-x} is a solution of the homogeneous DE!

Instead, we try $y_p = Axe^{-x} \implies y_p' = Ae^{-x}(1-x), y_p'' = Ae^{-x}(x-2)$.

Example 2 (cont.)

$y'' + 6y' + 5y = e^{-x}$

Instead, we try $y_p = Axe^{-x} \implies y_p' = Ae^{-x}(1-x), y_p'' = Ae^{-x}(x-2)$.

Substituting this to the DE we find

$$\begin{aligned}
Ae^{-x}(x-2) + 6Ae^{-x}(1-x) + 5Axe^{-x} &= e^{-x} \\
A(x-2) + 6A(1-x) + 5Ax &= 1 \\
Ax - 2A + 6A - 6Ax + 5Ax &= 1 \\
4A &= 1 \implies A = \tfrac{1}{4}.
\end{aligned}$$

Therefore

$$\boxed{y_p = \tfrac{1}{4}xe^{-x}}$$

and

$$\boxed{y(x) = c_1e^{-x} + c_2e^{-5x} + \tfrac{1}{4}xe^{-x}.}$$

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Example 3

Find the general solution of $y'' - 9y' + 14y = 3x^2 - 5\sin(2x)$.

$(\mathbf{y_c})$: $y'' - 9y' + 14y = 0$.

Try $y = e^{mx} \implies m^2 - 9m + 14 = 0 \implies (m-7)(m-2) = 0$ => $m_1 = 7$, $m_2 = 2$

$$\implies \boxed{y_c = c_1 e^{7x} + c_2 e^{2x}}$$

$(\mathbf{y_p})$: $g(x) = 3x^2 - 5\sin(2x)$.

Try

$$\begin{aligned} y_p &= Ax^2 + Bx + C + D\cos(2x) + E\sin(2x) \\ y_p' &= 2Ax + B - 2D\sin(2x) + 2E\cos(2x) \\ y_p'' &= 2A - 4D\cos(2x) - 4E\sin(2x) \end{aligned}$$

Substitute to the DE to find (exercise)

$A = \frac{3}{14}, B = \frac{27}{98}, C = \frac{201}{1372}, D = -\frac{45}{212}, E = -\frac{25}{212}$ and hence

$$\boxed{y(x) = c_1 e^{7x} + c_2 e^{2x} + \frac{3}{14}x^2 + \frac{27}{98}x + \frac{201}{1372} - \frac{45}{212}\cos(2x) - \frac{25}{212}\sin(2x).}$$

$$y = Ax^2 + Bx + C + D\cos 2x + E\sin 2x$$
$$y' = 2Ax + B - 2D\sin 2x + 2E\cos 2x$$
$$y'' = 2A - 4D\cos 2x - 4E\sin 2x$$

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x$$

$$2A - 4D\cos 2x - 4E\sin 2x - 9\left[2Ax + B - 2D\sin 2x + 2E\cos 2x\right]$$
$$+ 14\left[Ax^2 + Bx + C + D\cos 2x + E\sin 2x\right] = 3x^2 - 5\sin 2x$$

$$(2A - 9B + 14C) + x\left[-18A + 14B\right] + x^2\left[14A\right] + \cos 2x\left[-4D - 18E + 14D\right]$$
$$+ \sin 2x\left[-4E + 18D + 14E\right] = 3x^2 - 5\sin 2x$$

$$\left.\begin{aligned} 2A - 9B + 14C &= 0 \\ -18A + 14B &= 0 \\ 14A &= 3 \\ 10D - 18E &= 0 \\ 18D - 10E &= -5 \end{aligned}\right\}$$

$$A = 3/14$$
$$B = \frac{1}{14} 18A = \frac{1}{14} 18 \cdot 3/14 = \frac{3 \cdot 18}{14 \cdot 14} = \frac{27}{98}$$
$$C = 1/14\,(9B - 2A) = \underline{\qquad\qquad}$$

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Example 4

Solve $y'' + 4y = -2$ subject to initial conditions $y(\pi/8) = 1/2, y'(\pi/8) = 2$.

$(\mathbf{y_c})$: From Lecture 15 (final slide) we have $\boxed{y_c = c_1 \sin(2x) + c_2 \cos(2x)}$.

$y = e^{mx}$

$y_p(x) = -1/2$

$(\mathbf{y_p})$: We try $y_p = A$ so that $y_p'' = 0$ and $y_p'' + 4y_p = 4A = -2 \implies A = -\frac{1}{2}$.

Hence we have the general solution $\boxed{y(x) = c_1 \sin(2x) + c_2 \cos(2x) - \frac{1}{2}}$.

We now use the initial conditions to solve for c_1 and c_2:*

$$\left.\begin{array}{rclcl} y(\frac{\pi}{8}) & = & c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2} - \frac{1}{2} & = \frac{1}{2} & \implies c_1 + c_2 = \frac{2}{\sqrt{2}} \\ y'(\frac{\pi}{8}) & = & 2c_1\frac{\sqrt{2}}{2} - 2c_2\frac{\sqrt{2}}{2} & = 2 & \implies c_1 - c_2 = \frac{2}{\sqrt{2}} \end{array}\right\} \implies \begin{array}{rcl} c_1 & = & \sqrt{2}, \\ c_2 & = & 0. \end{array}$$

Therefore, the solution to the IVP is given by

$$\boxed{y(x) = \sqrt{2}\sin(2x) - \frac{1}{2}}.$$

*Note that we apply the boundary conditions to the general solution, $y(x)$, and not just the complimentary solution, y_c. This is a common mistake!

There are **lots** of examples in the textbook. I suggest you try some!

Exercise, solve the following:

$$\begin{aligned} y'' + 2y' + y &= e^{-x}, \qquad y(0) = 1, \quad y'(0) = 0. \\ 5y'' + y' &= -6x, \qquad y(0) = 0, \quad y'(0) = -10. \end{aligned}$$