Applied differential equations

TW244 - Lecture 18

4.9: Solving linear systems: Elimination

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4.9: Solving linear systems: elimination

Here we would like to solve a system of two first-order DEs, e.g.,

$$\frac{dx}{dt} = -8x + 2y, \quad x(0) = 8$$
$$\frac{dy}{dt} = 8x - 8y, \quad y(0) = 4.$$

The method of elimination:

- Eliminate one variable to obtain a second-order DE in the other.
- Solve this second-order DE (follow the $y = y_c + y_p$ route).
- Substitute this variable and solve for the remaining variable.

$x' = -8x + 2y$

$y' = 8x - 8y$

Step 1: Eliminate one variable to obtain a second-order DE in the other.

$$\begin{aligned}
\frac{d^2x}{dt^2} &= -8\frac{dx}{dt} + 2\frac{dy}{dt} \\
&= -8\frac{dx}{dt} + 2(8x - 8y) \\
&= -8\frac{dx}{dt} + 2\left(8x - 8\left(\frac{1}{2}\frac{dx}{dt} + \frac{1}{2}8x\right)\right) \\
&= -16\frac{dx}{dt} - 48x.
\end{aligned}$$

← differentiate both sides of (1)

← subst y' from (2)

← subst y from (1).

Hence

$$\boxed{x'' + 16x' + 48x = 0}$$

(we have eliminated y!)

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Example 1

Step 2: Solve the second-order DE $\boxed{x'' + 16x' + 48x = 0}$.

This is a homogeneous second-order DE, so try $x = e^{mt}$:

$$\implies m^2 e^{mt} + 16me^{mt} + 48e^{mt} = 0$$

$$\implies m^2 + 16m + 48 = 0$$

Auxiliary eqn

$$\implies (m+4)(m+12) = 0$$

$\implies m = -4, -12$

$$\implies \boxed{x(t) = c_1 e^{-4t} + c_2 e^{-12t}.}$$

(Since this example is homogeneous we do not need to find y_p.)

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Example 1

Step 3: Substitute $x(t) = c_1 e^{-4t} + c_2 e^{-12t}$ and solve for $y(t)$.

Recall $\frac{dx}{dt} = -8x + 2y$, hence

$\Longrightarrow y = \frac{x' + 8x}{2}$

$$-4c_1 e^{-4t} - 12c_2 e^{-12t} = -8(c_1 e^{-4t} + c_2 e^{-12t}) + 2y$$

and therefore

$$\boxed{y(t) = 2c_1 e^{-4t} - 2c_2 e^{-12t}}$$

Initial conditions:

$$\left.\begin{array}{lcl} x(0) = 8 & \Longrightarrow & c_1 + c_2 = 8 \\ y(0) = 4 & \Longrightarrow & 2c_1 - 2c_2 = 4 \end{array}\right\} \Longrightarrow c_1 = 5, c_2 = 3..$$

Hence

$$\boxed{\begin{array}{rcl} x(t) & = & 5e^{-4t} + 3e^{-12t} \\ y(t) & = & 10e^{-4t} - 6e^{-12t}. \end{array}}$$

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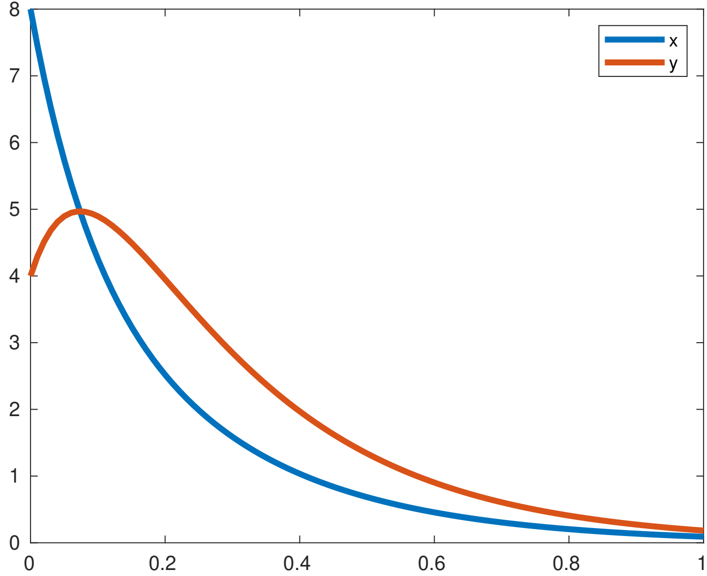
Example 1



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D notation

'D notation' makes it easier to see how one variable can be eliminated:

$$
\begin{aligned}
Dx &= \frac{dx}{dt} \\
D^2x &= \frac{d^2x}{dt^2} \\
&\vdots \\
D^mx &= \frac{d^mx}{dt^m}
\end{aligned}
$$

Note that D is a linear operator, i.e.,

$$D(\alpha x + \beta y) = \alpha Dx + \beta Dy.$$

Let's see another example...

Find the general solution of

$$(1) : \frac{dx}{dt} = -y + t, \qquad (2) : \frac{dy}{dt} = x - t$$

or in D notation

$$(1) : Dx + y = t, \qquad (2) : -x + Dy = -t$$

Step 1: Eliminate y:

$$\begin{aligned} D(1) &\implies D^2x + Dy = Dt = 1, \\ (2) &\implies x - Dy = t \end{aligned}$$

Add these two equations:

$$D^2x + x = 1 + t,$$

i.e.,

$$\boxed{x'' + x = 1 + t.}$$

$x'' + k^2 x = 0$

$\Rightarrow x = c_1 \cos kt + c_2 \sin kt.$

Step 2: Solve $\boxed{x'' + x = 1 + t}$ for $x(t)$:

$(x_c) : x_c'' + x_c = 0 \implies \boxed{x_c = c_1 \cos(t) + c_2 \sin(t)}$*

$x = x_c + x_p$

$(x_p) : g(t) = 1 + t$, so try

$x_p = At + B \implies 0 + At + B = 1 + t \implies A = B = 1 \implies \boxed{x_p = 1 + t}.$

Hence

$$\boxed{x(t) = c_1 \cos(t) + c_2 \sin(t) + 1 + t.}$$

$x' = -c_1 \sin t + c_2 \cos t + 1$

Step 3: Substitute and solve for y:

From (1) we have $Dx = -y + t \implies y = t - Dx$ hence

$$\boxed{y(t) = c_1 \sin(t) - c_2 \cos(t) - 1 + t.}$$

*See Lecture 15.

$$x' = -y + t, \quad y' = x - t, \quad \underline{x(0) = 1,\ y(0) = 0}$$

$$\begin{cases} x(t) = c_1 \cos t + c_2 \sin t + 1 + t \\ y(t) = c_1 \sin t - c_2 \cos t - 1 + t \end{cases}$$

$$1 = c_1 \cdot \cos 0 + c_2 \sin 0 + 1 + 0 = c_1 + 1 \Rightarrow c_1 = 0$$

$$0 = c_1 \sin 0 - c_2 \cos 0 - 1 + 0 = 0 - c_2 - 1 \Rightarrow c_2 = -1$$

$$\boxed{\begin{aligned} x(t) &= -\sin t + 1 + t \\ y(t) &= \cos t - 1 + t \end{aligned}}$$

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Example 3

Solve the DE system $\frac{dx}{dt} = 4x + 7y$, $\frac{dy}{dt} = x - 2y$ with $x(0) = 6, y(0) = 2$.

In D notation:

$$
\begin{aligned}
(D-4)x - 7y &= 0, && (1)\\
-x + (D+2)y &= 0. && (2)
\end{aligned}
$$

Step 1:

$$
\begin{aligned}
\tfrac{1}{7}(D+2)\times(1):\quad & \tfrac{1}{7}(D+2)(D-4)x - (D+2)y = 0 && (3)\\
(3)+(2):\quad & \tfrac{1}{7}(D+2)(D-4)x - x = 0\\
& (D^2 - 2D - 8)x - 7x = 0\\
& D^2x - 2Dx - 15x = 0.
\end{aligned}
$$

i.e.,

$$\boxed{x'' - 2x' - 15x = 0.}$$

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Example 3

Step 2: Solve $\boxed{x'' - 2x' - 15x = 0.}$

Try $x = e^{mt} \implies m^2 - 2m - 15 = (m-5)(m+3) = 0 \implies m = -3, 5$

$$\therefore \boxed{x(t) = c_1 e^{-3t} + c_2 e^{5t}.}$$

$x' = 4x + 7y$
$y' = x - 2y$

Step 3: Substitute $x(t)$ to (1) and solve for $y(t)$:

$$y(t) = \tfrac{1}{7}(D-4)x = \tfrac{1}{7}(-3c_1 e^{-3t} + 5c_2 e^{5t}) - \tfrac{4}{7}(c_1 e^{-3t} + c_2 e^{5t}) = -c_1 e^{-3t} + \tfrac{1}{7}c_2 e^{5t}.$$

Initial conditions:

$$\left.\begin{array}{lll} x(0) = 6 & \implies & c_1 + c_2 = 6 \\ y(0) = 2 & \implies & -c_1 + \frac{1}{7}c_2 = 2 \end{array}\right\} \implies c_1 = -1, c_2 = 7.$$

Therefore

$$\boxed{\begin{aligned} x(t) &= -e^{-3t} + 7e^{5t} \\ y(t) &= e^{-3t} + e^{5t}. \end{aligned}}$$

Exercise: Repeat step (1) and (2) but solve first for y rather than x. (You will find that the algebra is easier.)

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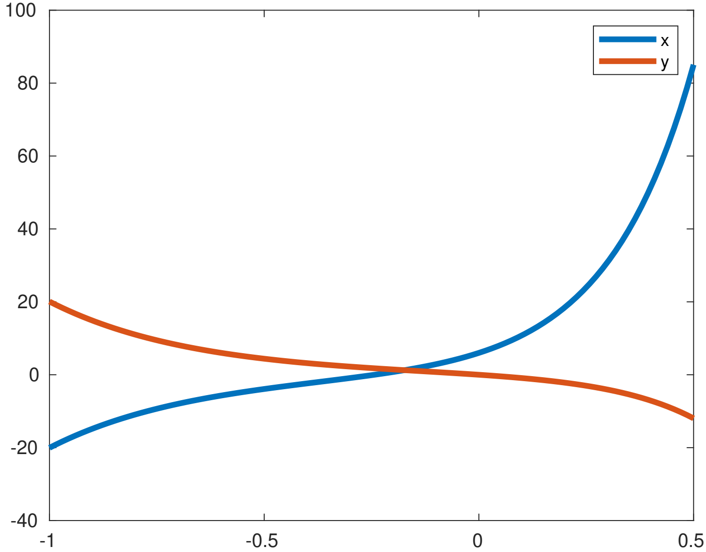
Example 3



Exercise: Solve the DE system $\frac{dx}{dt} = 4x + 7y + 6\sin(x)$, $\frac{dy}{dt} = x - 2y + 4\cos(x)$ with $x(0) = 6, y(0) = 1$ by eliminating y. (Hints below.)

Step 1 Eliminate y to find $x'' - 2x' - 15x = 34\cos(t) + 12\sin(t)$.

Step 2a Solve $x'' - 2x' - 15x = 0$ as in Example 3 to find $x_C(t) = c_1e^{-3t} + c_2e^{5t}$.

Step 2b Solve $x_p'' - 2x_p' - 15x_p = 34\cos(t) + 12\sin(t)$ using the method of undetermined coefficients with the guess $x_p = A\sin(t) + B\cos(t)$ to find

$$x_p(t) = -\sin(t) - 2\cos(t)$$

and hence

$$x(t) = c_1e^{-3t} + c_2e^{5t} - \sin(t) - 2\cos(t)$$

.

Step 3a Substitute $x(t)$ to (1) and solve for $y(t)$ to find $y(t) = -c_1e^{-3t} + \frac{1}{7}c_2e^{5t} + \cos(t)$

Step 3b Use the initial conditions to find $c_1 = 1$ and $c_2 = 7$ and hence that

$$x(t) = e^{-3t} + 7e^{5y} - \sin(t) - 2\cos(t), \qquad y(t) = -e^{-3t} + e^{5t} + \cos(t).$$

Exercise: Repeat the above, but eliminate x rather than y.

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Example 4