Applied differential equations

TW244 - Lecture 30

10.2 Stability of linear systems

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SCIENCE
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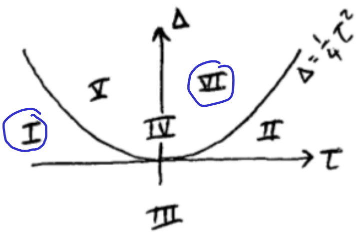
10.2: Stability of systems
Recall

We are trying to classify the crictical solution $(x, y) = (0, 0)$ of the linear plane autonomous system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$

We saw all we needed to do was to

- Calculate the trace $\tau = \text{trace}(A)$ and determinant $\Delta = \det(A)$
- See where the point (τ, Δ) is located on the following graph:



I : stable node
II : unstable node
III : saddle

IV : centre
V : stable spiral
VI : unstable spiral

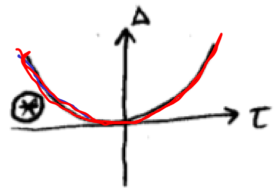
10.2: Stability of systems
Example 1

Classify the critical solution $(x, y) = (0, 0)$ of the system

$$\frac{dx}{dt} = -x + y,$$
$$\frac{dy}{dt} = \frac{1}{4}x - y.$$

In this case $A = \begin{bmatrix} -1 & 1 \\ \frac{1}{4} & -1 \end{bmatrix} \Longrightarrow \begin{cases} \tau = -1 - 1 = -2, \\ \Delta = (-1)(-1) - (1)(\frac{1}{4}) = \frac{3}{4}. \end{cases}$

$\frac{1}{4}\tau^2 = 1 \Rightarrow \Delta < \frac{1}{4}\tau^2$



Therefore $(x, y) = (0, 0)$ is a **stable node**.

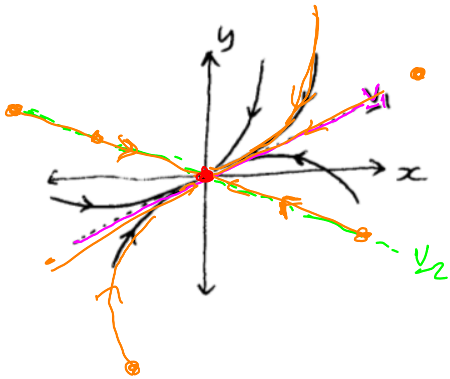
10.2: Stability of systems
Example 1

How can we get a better idea of how the solution curves will behave in the phase plane?

Well, $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2.$

In this example $\begin{matrix} \lambda_1 = -\frac{1}{2}, & \underline{v}_1 = [+2, 1]^\top \\ \lambda_2 = -\frac{3}{2}, & \underline{v}_2 = [-2, 1]^\top \end{matrix}$.

Note that $|\lambda_1| < |\lambda_2|$, so $e^{\lambda_2} \to 0$ faster than $e^{\lambda_1 t} \to$ as $t \to \infty$, hence the direction of $\underline{v}_1$ dominates for large values of t.

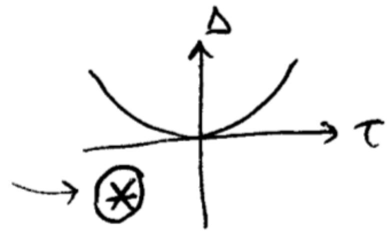


Classify the critical solution $(x, y) = (0, 0)$ of the system

$$\frac{dx}{dt} = -x + y,$$
$$\frac{dy}{dt} = 4x - y.$$

In this case $A = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$

$$\Longrightarrow \begin{cases} \tau = -1 - 1 = -2, \\ \Delta = (-1)(-1) - (1)(4) = -3. \end{cases}$$



Since τ and Δ are negative and therefore $(x, y) = (0, 0)$ is a **saddle**.

How can get a better idea of how the solution curves will behave in the phase plane?

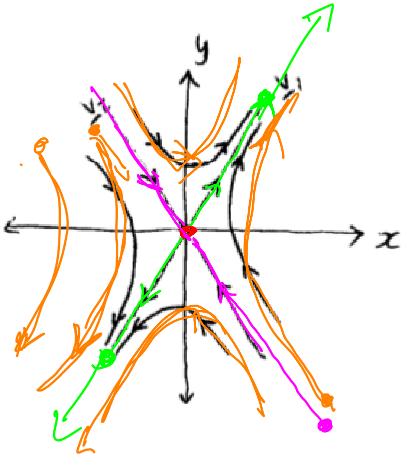
In this example

$$\lambda_1 = +1,\ \underline{v}_1 = [+1, 2]^\top$$
$$\lambda_2 = -3,\ \underline{v}_2 = [-1, 2]^\top$$

and we have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2$$

where $e^{\lambda_1 t} \to \infty$ and $e^{\lambda_2 t} \to 0$ as $t \to \infty$.



Example 3

Classify the critical solution $(x, y) = (0, 0)$ of the system

$$\frac{dx}{dt} = -y,$$
$$\frac{dy}{dt} = x - \sqrt{2}y.$$

In this case $\underbrace{A = \begin{bmatrix} 0 & -1 \\ 1 & -\sqrt{2} \end{bmatrix}}$ $\Longrightarrow \begin{cases} \tau = -\sqrt{2}, \\ \Delta = (0)(-\sqrt{2}) - (-1)(1) = \underline{1}. \end{cases}$

$\Delta, \ \tau^2/4 = 1/2$

We have $\tau^2 - 4\Delta = 2 - 4 = -2$ and since $\tau < 0$ the solution is a <u>**stable spiral**</u>.

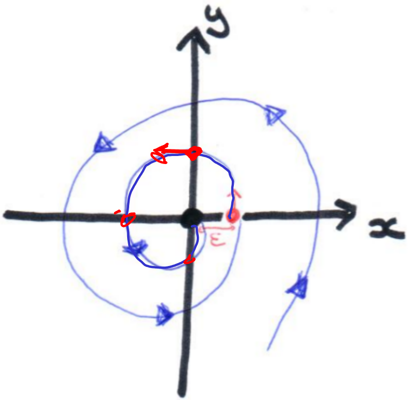
Example 3

How can get a better idea of how the solution curves will behave in the phase plane?

$dx/dt = -y$

$dy/dt = x - \sqrt{2}y$

One can use the eigenvectors to determine the direction of orientation, but instead consider this simple trick:



Consider a 'particle' at the position $(\varepsilon, 0)$, where $0 < \varepsilon \ll 1$ (i.e., slightly to the right of the origin).* At this point, for this example, the particle will move according to $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = \varepsilon$. Since $\varepsilon > 0$, this tells us the particle must be move upwards, from which we can conclude that our stable spiral is anti-clockwise.

* Alternatively one can consider a point at $(0, \varepsilon)$ slightly above the origin.

10.X NULLCLINES

(Note that the topic of nullclines is not covered in the textbook.)

$y = -\frac{a}{b}x$

$ax + by = 0$
$cx + dy = 0$

Consider again our linear planar DE of the form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Recall that a critical point satisfies both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

If only one of these is true (i.e, $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$) we call the resulting curve in the phase plane a nullcline.

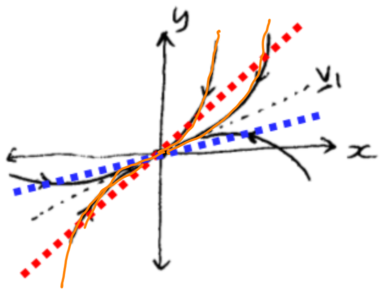
In particular, when $y = -\frac{a}{b}x$ then $\frac{dx}{dt} = 0$, and so curves passing through this point can only be vertical. Similarly, curves must be flat when $y = -\frac{c}{d}$, as here $\frac{dy}{dt} = 0$. Let's consider examples.

* Notice that critical points can only occur at the intersection of nullclines!

10.2: Stability of systems
Example 1 revisited

Consider again the DE

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = x - 4y$$



We can immediately see that along the line $y = x$, $\frac{dx}{dt} = 0$ and so curves must be vertical.

Along the line $y = \frac{x}{4}$ we have $\frac{dy}{dt} = 0$ and so curves are flat.

10.2: Stability of systems
Example 2 revisited

Consider again the DE

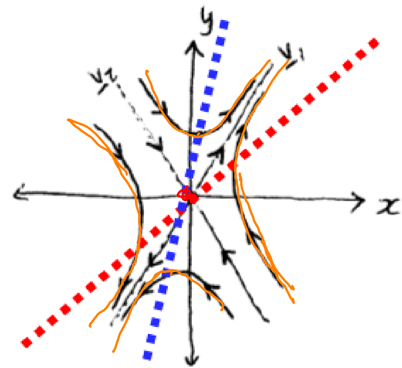
$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = 4x - y$$

In this case the nullclines are

$$y = x \text{ and } y = 4x.$$

$\frac{dx}{dt} = 0$ vertical

$\frac{dy}{dt} = 0$ flat



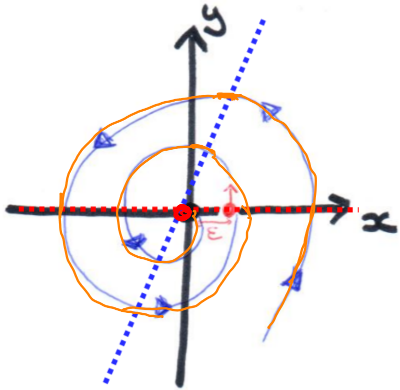
* Exercise: One can sometimes gain further information by checking whether $\frac{dx}{dt}$ and/or $\frac{dx}{dt}$ is positive/negative on each side of the nullcline. Investigate this here.

Consider again the DE

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - \sqrt{2}y$$

In this case the nullclines are

$$y = 0 \text{ and } y = \frac{1}{\sqrt{2}}x.$$



For linear problems, the nullclines are straight lines, but we will see later that for nonlinear problems, they may be more general curves.