Applied differential equations

TW244 - Lecture 19

8.2 Homogeneous linear systems and the method of eigenvalues

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solving

8.2: HOMOGENEOUS LINEAR SYSTEMS

using eigenvalues

Note: This lecture is a recap of the essentials from TW214.

Consider the following systems of n homogeneous linear DEs:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

In this lecture we will remind ourselves how to solve such systems using eigenvalues and eigenvectors*

To proceed, we write the system above in matrix form as†

$$\frac{d}{dt}\underline{x} = A\underline{x}.$$

*If your memory of this topic is rusty, I suggest you read through Appendix II in Z&W.

†Recall Lecture 12.

$\frac{dx}{dt} = Ax$, $x(0) = x_0$

Now suppose the $n \times n$ matrix A has n linearly independent eigenvectors, $\underline{v}_1, \ldots, \underline{v}_n$, and n eigenvalues, $\lambda_1, \ldots, \lambda_n$, such that $A\underline{v}_k = \lambda_k \underline{v}_k$, $k = 1, \ldots, n$.

evec

eval

It follows that

$$\overset{n\times n}{[\ A\underline{v}_1 \mid A\underline{v}_2 \mid \cdots \mid A\underline{v}_n\ ]} = \overset{n\times n}{[\ \lambda_1\underline{v}_1 \mid \lambda_2\underline{v}_2 \mid \cdots \mid \lambda_n\underline{v}_n\ ]}$$

which we may then write as

$$A\overset{V}{[\ \underline{v}_1 \mid \underline{v}_2 \mid \cdots \mid \underline{v}_n\ ]} = \overset{V}{[\ \underline{v}_1 \mid \underline{v}_2 \mid \cdots \mid \underline{v}_n\ ]}\overset{\Lambda}{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}}.$$

Hence $\boxed{AV = V\Lambda \implies A = V\Lambda V^{-1}}$; we have "diagonalised" A.

eval decomposition of A

$$A = V\Lambda V^{-1}$$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \lambda_1 w_1 \\ \lambda_2 w_2 \\ \vdots \\ \lambda_n w_n \end{bmatrix}$$

We now have that

$$\begin{aligned} \frac{d}{dt}\underline{x} = A\underline{x} \quad \Longrightarrow \quad \frac{d}{dt}\underline{x} &= V\Lambda V^{-1}\underline{x} \\ \Longrightarrow \quad V^{-1}\frac{d}{dt}\underline{x} &= \Lambda V^{-1}\underline{x} \\ \Longrightarrow \quad \frac{d}{dt}(V^{-1}\underline{x}) &= \Lambda(V^{-1}\underline{x}) \\ \Longrightarrow \quad \frac{d}{dt}\underline{w} &= \Lambda\underline{w} \qquad : \underline{w} = V^{-1}\underline{x} \Longrightarrow \underline{x} = V\underline{w} \end{aligned}$$

But these DEs are trivial to solve!

$$\begin{aligned} \frac{dw_1}{dt} &= \lambda_1 w_1 \quad \Longrightarrow \quad w_1 = c_1 e^{\lambda_1 t} \\ \frac{dw_2}{dt} &= \lambda_2 w_2 \quad \Longrightarrow \quad w_2 = c_2 e^{\lambda_2 t} \\ &\vdots \\ \frac{dw_n}{dt} &= \lambda_n w_n \quad \Longrightarrow \quad w_n = c_n e^{\lambda_n t} \end{aligned}$$

$$\begin{bmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \\ \vdots \\ \frac{dw_n}{dt} \end{bmatrix} = \begin{bmatrix} \lambda_1 w_1 \\ \lambda_2 w_2 \\ \vdots \\ \lambda_n w_n \end{bmatrix}$$

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Diagonalisation

$\underline{w} = V^{-1}\underline{x}$

$A\underline{v}_k = \lambda_k \underline{v}_k$

To recover the solution in x we want

$$\underline{x} = V\underline{w} = \left[\ \underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \cdots \,\middle|\, \underline{v}_n\ \right]\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{bmatrix}$$

and therefore

$$\boxed{\underline{x} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2 + \cdots + c_n e^{\lambda_n t}\underline{v}_n.}$$

evls of A

evecs of A.

Summary: To solve the system $\frac{d}{dt}\underline{x} = A\underline{x}$

1. Determine eigenvalues, $\lambda_1, \ldots, \lambda_n$, and eigenvectors, $v_1, \ldots, v_n$ of A.
2. General solution is given by $\underline{x} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2 + \cdots + c_n e^{\lambda_n t}\underline{v}_n$.

As usual, the constants $c_1, \ldots, c_n$ are determined from initial conditions.
Notice the linear system we have to solve here is $V\underline{c} = \underline{x}_0$.

$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Example: $\frac{dx_1}{dt} = -\frac{4}{25}x_1 + \frac{1}{25}x_2, \quad \frac{dx_2}{dt} = \frac{4}{25}x_1 - \frac{4}{25}x_2$ with $x_1(0) = 4, x_2(0) = 0.^\ddagger$

In matrix form, this system becomes

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} -\frac{4}{25} & \frac{1}{25} \\ \frac{4}{25} & -\frac{4}{25} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}}.$$

We want the eigenvalues and eigenvectors of this matrix, i.e.,

$$A\underline{v} = \lambda\underline{v} \implies (A - \lambda I)\underline{v} = \underline{0} \implies \det(A - \lambda I) = 0 \implies$$

$$\begin{vmatrix} -\frac{4}{25} - \lambda & \frac{1}{25} \\ \frac{4}{25} & -\frac{4}{25} - \lambda \end{vmatrix} = (\tfrac{4}{25} + \lambda)^2 - (\tfrac{2}{25})^2 = (\tfrac{4}{25} + \lambda - \tfrac{2}{25})(\tfrac{4}{25} + \lambda + \tfrac{2}{25}) = 0$$

$$\implies \lambda_1 = -\underline{\frac{2}{25}} \text{ and } \lambda_2 = -\underline{\frac{6}{25}}.$$

‡This DE system comes from a mixing problem of similar type to those we saw before. Can you figure out the tank set-up?

8.2: Homogeneous linear systems
Example

$A\underline{v}_k = \lambda_k v_k$

$\Rightarrow (A-\lambda_k I)\underline{v}_k = \underline{0}$

$$\Longrightarrow \lambda_1 = -\frac{2}{25} \text{ and } \lambda_2 = -\frac{6}{25}.$$

We have the eigenvalues, now compute the corresponding eigenvectors:

(1):

$x_1 = 1 \Rightarrow -2/ \;+\; x_2 = 0 \Rightarrow x_2 = 2$

$$(A + \tfrac{2}{25}I)\underline{v}_1 = 0 \Longrightarrow \begin{bmatrix} -\frac{2}{25} & \frac{1}{25} \\ \frac{4}{25} & -\frac{2}{25} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \checkmark$$

(2):

$2/ \;+\; x_2 = 0 \Rightarrow x_2 = -2$

$$(A + \tfrac{6}{25}I)\underline{v}_2 = 0 \Longrightarrow \begin{bmatrix} \frac{2}{25} & \frac{1}{25} \\ \frac{4}{25} & \frac{2}{25} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \checkmark$$

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Example

Hence we have that

$x_1 = c_1 e^{-2t/25} + c_2 e^{-6t/25}$

$x_2 = 2c_1 e^{-2t/25} - 2c_2 e^{-6t/25}$

$$
\begin{aligned}
\underline{x} &= c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= c_1 e^{-2t/25}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{6t/25}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\end{aligned}
$$

From the initial conditions

$$
\left.\begin{aligned}
x_1(0) = 4 \quad &\Longrightarrow \quad c_1 + c_2 = 4 \\
x_2(0) = 0 \quad &\Longrightarrow \quad 2c_1 + 2c_2 = 0
\end{aligned}\right\} \Longrightarrow c_1 = c_2 = 2.
$$

Therefore

$$
\boxed{\begin{aligned}
x_1(t) &= 2e^{-2t/25} + 2e^{-6t/25} \\
x_2(t) &= 4e^{-2t/25} - 4e^{-6t/25}
\end{aligned}}.
$$

The hardest part of this method is to calculate the eigenvalues and eigenvectors, but that's very easy in MATLAB! (`[V, D] = eig(A);`)

8.2: Homogeneous linear systems
Additional comments (non-examinable)

Repeated eigenvalues and linearly dependent eigenvectors:
Above we made the assumption that the eigenvalues of A were linearly independent. If that is not the case, our solution may have terms like $te^{\lambda_i t}$ appearing, much like we same for the case of repeated roots of the auxilary equation in Lecture 15. See Section 8.2.2 in Z&W.

Complex eigenvalues:
The example above had real eigenvalues, but very little changes if we have complex eigenvalues. As usual, this will lead to solutions involving $\sin$ and $\cos$ terms. See Section 8.2.3 in Z&W.

Nonhomogeneous problems:
The eigenvalue technique can be extended to solve nonhomogeneous problems of the form $\frac{d}{dt}\underline{x} = A\underline{x} + \underline{f}(t)$. See Section 8.3 in Z&W.

Matrix exponential:

$e^{At}\underline{x}_0$

If $\frac{d\underline{x}}{dt} = A\underline{x}, \underline{x} = \underline{x}_0$, does $\underline{x}(t) = \cancel{\underline{x}_0 e^{At}}$? – Yes! (If we're careful about how we define the "matrix exponential", e^A. See Section 8.4 in Z&W or MATLAB's `expm` command.