Applied differential equations

TW244 - Lecture 20

Example: Projectile motion

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SCIENCE
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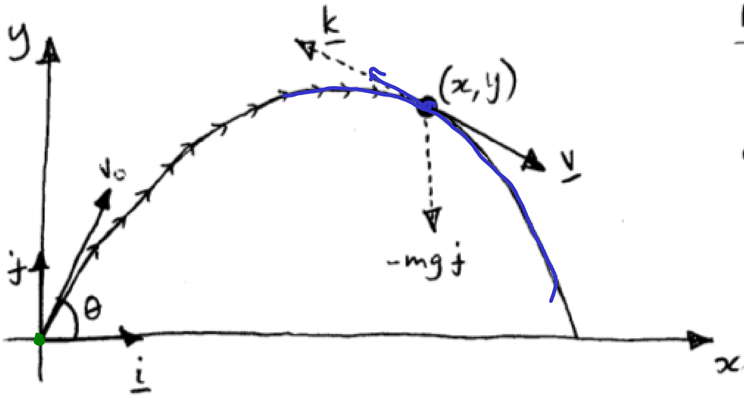
Example: Projectile motion
(See Problems 23 and 24 on p. 184 of Z&W)

A projectile with mass m is launched at an angle θ (from the horizontal) and velocity v_0. Determine a system of DEs that describes the motion of the projectile if drag (air resistance) is proportional to the instaneous velocity.



Note: the velocity $\underline{v}$ is tangent to the trajectory, and $\underline{k}$, the drag, is in a direction opposite to $\underline{v}$.

Let

$$\underline{S}(t) = x(t)\underline{i} + y(t)\underline{j} \quad \text{and} \quad \underline{v}(t) = x'(t)\underline{i} + y'(t)\underline{j}$$

be the displacement and velocity at time t, respectively.

- From the definition of velocity:

$$\frac{dx}{dt} = x' \quad \text{and} \quad \frac{dy}{dt} = y'.$$

- From Newton's second law of motion:

$M\underline{a} = \underline{F}$

$$m\frac{d\underline{v}}{dt} = -mg\underline{j} - k\underline{v}$$

$$m\left(\frac{dx'}{dt}\underline{i} + \frac{dy'}{dt}\underline{j}\right) = -mg\underline{j} - k\left(x'\underline{i} + y'\underline{j}\right)$$

$m\frac{dx'}{dt} = -kx'$

$m\frac{dy'}{dt} = -mg - ky'$

$\Rightarrow dx'/dt = \frac{-k}{m}x'$

$\frac{dy'}{dt} = -g - k/m\, y'$

Example: Projectile motion
Derivation

So we have the following system (with $r = \frac{k}{m}$):*

$$\frac{dx'}{dt} = -rx' \tag{1}$$

$$\frac{dy'}{dt} = -g - ry' \tag{2}$$

$$\frac{dx}{dt} = x' \tag{3}$$

$$\frac{dy}{dt} = y' \tag{4}$$

with initial velocity

$$x'(0) = v_0 \cos\theta \quad \text{and} \quad y'(0) = v_0 \sin\theta$$

and initial displacement

$$x(0) = 0 \quad \text{and} \quad y(0) = 0.$$

*Note the equations involving x decouple from those involving y and that the two x (and y) equations can be solved by forward substitution (like the radioactive decay example in Lecture 14). This makes this problem easy to solve and we don't need the machinery of the past few lectures.

Solution:

(1): $\frac{dx'}{dt} = -rx'$ with $x'(0) = v_0 \cos\theta \implies x'(t) = (v_0 \cos\theta)e^{-rt}$

(2): $\frac{dy'}{dt} + ry' = -g$. Use integrating factor, $f(t) = e^{\int r\,dt} = e^{rt}$.

$$\begin{aligned} \frac{d}{dt}\left[e^{rt}y'\right] &= -ge^{rt} \\ e^{rt}y' &= -\frac{g}{r}e^{rt} + C \\ e^{rt}y' &= -\frac{g}{r}e^{rt} + v_0 \sin\theta + \frac{g}{r} \\ \implies y'(t) &= \left(\frac{g}{r} + v_0 \sin\theta\right)e^{-rt} - \frac{g}{r} \end{aligned}$$

where we used the initial condition $y'(0) = v_0 \sin\theta = -\frac{g}{r} + C$ to find C.[†]

[†]Note ithat if $v_0 = 0$ then this is the same solution we found in Lecture 1!

Solution cont.:

(3): $\frac{dx}{dt} = x' = (v_0 \cos\theta) e^{-rt} \implies$

$x(0) = 0$

$$\begin{aligned} x &= -\tfrac{1}{r}(v_0\cos\theta)e^{-rt} + C_1 \\ &= -\tfrac{1}{r}(v_0\cos\theta)e^{-rt} + \tfrac{1}{r}(v_0\cos\theta) \\ \implies x(t) &= \tfrac{1}{r}v_0\cos\theta(1-e^{-rt}) \end{aligned}$$

(4): $\frac{dy}{dt} = y' = (\frac{g}{r} + v_0\sin\theta)e^{-rt} - \frac{g}{r} \implies$

$y(0) = 0$

$$\begin{aligned} y &= -\tfrac{1}{r}(\tfrac{g}{r} + v_0\sin\theta)e^{-rt} - \tfrac{g}{r}t + C_2 \\ &= -\tfrac{1}{r}(\tfrac{g}{r} + v_0\sin\theta)e^{-rt} - \tfrac{g}{r}t + \tfrac{1}{r}(\tfrac{g}{r} + v_0\sin\theta) \\ \implies y(t) &= \tfrac{1}{r}(\tfrac{g}{r} + v_0\sin\theta)(1-e^{-rt}) - \tfrac{g}{r}t \end{aligned}$$

where in both cases we used the fact that the initial position is zero.

Solution

$$x'(t) = v_0 \cos\theta \, e^{-rt}$$

$$y'(t) = \left(g/r + v_0 \sin\theta\right) e^{-rt} + g/r$$

$$x(t) = \frac{1}{r} v_0 \cos\theta \left(1 - e^{-rt}\right)$$

$$y(t) = \frac{1}{r}\left(g/r + v_0 \sin\theta\right)\left(1 - e^{-rt}\right) - g/r \, t$$

Trajectory: Here we want y as a function of x.
Let's elimate t from $x(t)$ and $y(t)$:

$$
\begin{aligned}
x(t) &= \tfrac{1}{r} v_0 \cos\theta (1 - e^{-rt}) \\
\frac{1 - e^{-rt}}{r} &= \frac{x}{v_0 \cos\theta} \quad =: \; p \\
e^{-rt} &= 1 - rp \implies t = -\tfrac{1}{r} \ln(1 - rp)
\end{aligned}
$$

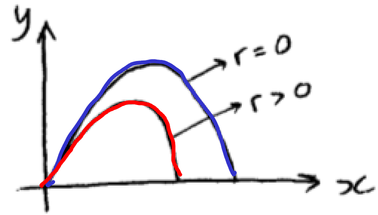
Then

$$
y(t) = \left(\tfrac{g}{r} + v_0 \sin\theta\right) \tfrac{1 - e^{-rt}}{r} - \tfrac{g}{r} t \implies
$$

$$
\boxed{y(x) = \left(\frac{g}{r} + v_0 \sin\theta\right) p + \frac{g}{r^2} \ln(1 - rp) = x \tan\theta + \frac{g}{r^2}\Big(rp + \ln(1 - rp)\Big)}
$$

where $p = \frac{x}{v_0 \cos\theta}$.

But wait ... high school physics says that if $r = 0$ (i.e., no air resistance) then we should get a parabolic arc.



What happens to our trajectory for $|r| \ll 1$?

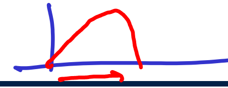
Taylor series: $\ln(1-z) = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \dots$

Substituting this with $z = \frac{rx}{v_0 \cos\theta}$ to our formula for $y(x)$, we get

$p = \frac{x}{v_0 \cos\theta} \Rightarrow p^2 = \frac{x^2}{v_0^2 \cos^2\theta}$

$$\begin{aligned}
y(x) &= x\tan\theta + \frac{g}{r^2}\Big(rp + \ln(1-rp)\Big) \\
&= x\tan\theta + \frac{g}{r^2}\Big(\cancel{rp} - \cancel{rp} - \tfrac{1}{2}(rp)^2 - \tfrac{1}{3}(rp)^3 - \tfrac{1}{4}(rp)^4 \dots\Big) \\
&= \underbrace{\tan(\theta)x - \tfrac{1}{2}\Big(\tfrac{g}{v_0^2\cos^2\theta}\Big)x^2}_{\text{parabola}} - \underbrace{\tfrac{1}{3}\cancel{grp^3} - \tfrac{1}{4}\cancel{gr^2p^4} - \cancel{\dots}}_{\text{negligible if } r \ll 1}
\end{aligned}$$

Example: Projectile motion
Range



Recall our trajectory solution $y(x) = x\tan\theta + \frac{g}{r^2}\Big(rp + \ln(1-rp)\Big),\ p = \frac{x}{v_0\cos\theta}$.

Suppose we wish to determine the range of our projectile, i.e., what value of x (other than $x = 0$) satisfies $y(x) = 0$?[†].

One way to do this is to approximate it numerically with `fzero` in MATLAB[§].

However, it turns out that an closed form expression can be obtained in terms of the Lambert W function, which is defined as the inverse of the function

$$T(w) = we^w.$$

That is, if you want to know what number w satisfies $we^w = z$, the answer is $w = W(z)$, where W is the Lambert W function. The Lambert W function is implemented as `lambertw` in the MATLAB symbolic toolbox[¶] and as `scipy.special.lambertw` in Python.

[†] Assuming we fire on a flat surface

[§] or some equivalent - see Computer Assignment 03

[¶] There's also a version on our SUNLearn page written by me.

Example: Projectile motion
Range (cont.)

Lambert W

The derivation of this result is beyond the scope of this course, but one can show that[||]

$$R(\theta) = \frac{v_0}{r}\cos\theta\left(1 - \frac{W(ue^u)}{u}\right), \quad \text{where } u = -1 - \frac{rv_0}{g}\sin\theta.$$

From this expression one can determine the optimal angle, θ_{max}, which it turns out depends only on the quantity $\alpha = \frac{rv_0}{g}$ and is given by

$$\theta_{max} = \begin{cases} \arcsin\left(\frac{\alpha W(\frac{\alpha^2-1}{e})}{\alpha^2-1-W(\frac{\alpha^2-1}{e})}\right), & \alpha \neq 1 \\ \arcsin(\frac{1}{e-1}), & \alpha = 1. \end{cases}$$

From this one can readily determine $R_{max} = R(\theta_{max})$.

[||]More details on the Lambert W function and the derivation can be found here:
E.W. Packel and D.S. Yuen, *Projectile Motion with Resistance and the Lambert W Function, vol. 35, no. 5, 2004, The College Mathematics Journal*