For examiner's use:

Naam/Name: Memo

Stud. Nr: ______________

Toegepaste Differensiaalvergelykings
TW244 Toets 2 2018

Dosent/Lecturer: Dr N Hale

Instruksies:

(a) 5 probleme (+1 bonus).

(b) 50 + 4 punte (50 maks).

(c) 2.5 uur, toeboek.

(d) Sakrekenaars **word toegelaat**. Selfone **nie**.

(e) Toon alle bewerkings. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(f) Daar is leë bladsye aan die agterkant van die vraestel as jou antwoorde nie inpas in die gegewe spasies nie. Dui duidelik aan as jou antwoord voortgaan op een van hierdie bladsye.

(g) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

Applied Differential Equations
TW244 Test 2 2018

Moderator: Dr W Brink

Instructions:

(a) 5 problems (+1 bonus).

(b) 50 + 4 marks (50 max).

(c) 2.5 hours, closed book.

(d) Calculators **are** allowed. Cell phones are **not**.

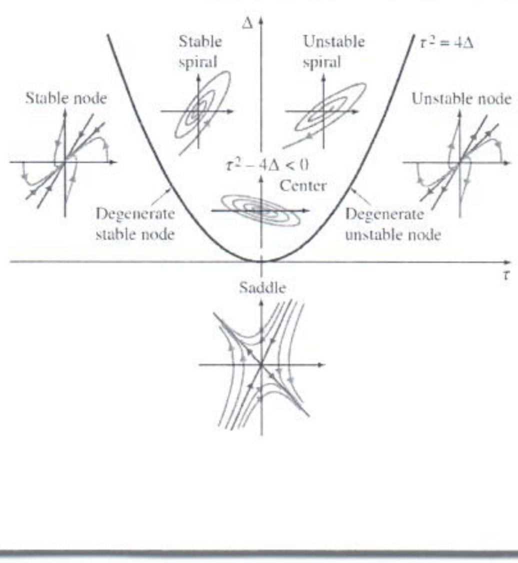
(e) Show all calculations. A correct answer does not earn full marks without the necessary explanation.

(f) There are blank pages at the back of the paper in case you cannot fit your answer in the space provided. Indicate clearly if your answer continues to one of these pages.

(g) The formulas below may be used without proof anywhere in the test.

Formules/Formulas:

- Wronskiaan/Wronskian: $$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$
- Deelsgewyse integrasie/ Integration by parts: $$\int_a^b f\frac{dg}{dx}\,dx = [fg]_a^b - \int_a^b \frac{df}{dx}g\,dx$$
- Dubbelhoek formules/ Double angle formulae: $$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$ $$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
- Laplace transform $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$
- Laplace transform van afgeleides/ Laplace transform of derivatives $$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
- Tangens/Tangent: $\tan(\pi/6) = 1/\sqrt{3}$, $\tan(\pi/4) = 1$, $\tan(\pi/3) = \sqrt{3}$, $\tan(\pi/2) = \infty$.
- $\tau = \mathrm{trace}(A)$ & $\Delta = \det(A) \implies$ $\mathrm{eig}(A) = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$
- $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \implies \begin{cases} \lambda_1 = a, & \underline{v}_1 = [1,0] \\ \lambda_2 = b, & \underline{v}_2 = [0,1] \end{cases}$
- Klassifikasie van kritieke punte vir **lineêre** stelsels/ Classification of critical points for **linear** systems



Prob 1 (14 punte/marks)

(a) Beskou die onderstaande differensiaalvergelykings en besluit watter tipe veer-massa stelsels hulle beskryf.

3 **(a)** Consider the differential equations below and decide what type of the mass-spring systems they describe.

(A) Lineêre veer, ongedemp, gedrewe / *Linear spring, undamped, driven*

(B) Nie-lineêre veer, gedemp, ongedrewe / *Nonlinear spring, damped, undriven*

(C) Lineêre veer, gedemp, gedrewe / *Linear spring, damped, driven*

(D) Lineêre veer, gedemp, ongedrewe / *Linear spring, damped, undriven*

(E) Geeneen van die bogenoemde nie / *None of the above*

(i) Die vergelyking / *The equation*

$$x'' + |x'|x' + 4x = 0$$

is tipe / *is of type* (A) (B) (C) (D) (E)

(ii) Die vergelyking / *The equation*

$$x'' + 7x = \cos(4t)$$

is tipe / *is of type* (A) (B) (C) (D) (E)

(iii) Die vergelyking / *The equation*

$$x'' + x + x^3 = 0$$

is tipe / *is of type* (A) (B) (C) (D) (E)

(b) Los die volgende outonome differensiaalvergelyking op vir $x(t)$ met enige metode van jou keuse. Gee die oplossing in **amplitude-fase** vorm.

3 **(b)** Solve the following autonomous differential equation for $x(t)$ by any means you like. Express the solution in **amplitude-phase** form.

$$x'' + 2x' + 401x = 0, \qquad x(0) = 1,\ x'(0) = 19.$$

Try $x = e^{mt} \Rightarrow m^2 + 2m + 401 = 0 \Rightarrow m = \dfrac{-2 \pm \sqrt{4 - 4\times 401}}{2} = -1 \pm 20i$

$\Rightarrow x(t) = e^{-t}(c_1\cos(20t) + c_2\sin(20t))$

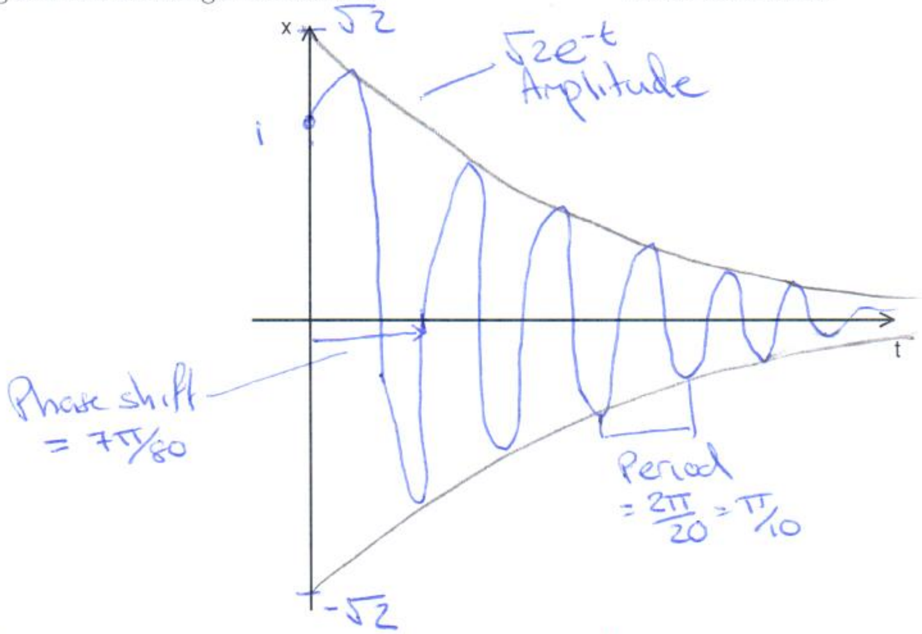
$\Rightarrow x'(t) = -e^{-t}(c_1\cos(20t) + c_2\sin(20t))$
$\quad + e^{-t}(-20c_1\sin(20t) + 20c_2\cos(20t))$

$x(0) = 1 = c_1$
$x'(0) = 19 = -1 + 20c_2 \Rightarrow c_2 = 1$

$\Rightarrow x(t) = e^{-t}(\cos(20t) + \sin(20t))$
$= \sqrt{2}e^{-t}\left(\frac{1}{\sqrt{2}}\cos(20t) + \frac{1}{\sqrt{2}}\sin(20t)\right)$
$= \sqrt{2}e^{-t}(\sin\pi/4\cos 20t + \cos\pi/4 \sin 20t)$
$= \sqrt{2}e^{-t}\sin(20t + \pi/4)$
$= \sqrt{2}e^{-t}\sin(20t - 7\pi/4)$
$= \sqrt{2}e^{-t}\sin(20(t - 7\pi/80))$

(c) Skets 'n grafiek van jou oplossing van hierbo as 'n funksie van t. Toon die amplitude, periode en faseverskuiwing duidelik aan en gee in besonder aandag aan die aanvangsvoorwaardes.

(c) Sketch a graph of your solution above as a function of t. Indicate clearly the amplitude, period, and phase shift and pay special attention to the initial conditions.



(d) Ervaar hierdie veerstelsel (omkring alles wat van toepassing is)

(d) Is this spring system (circle all that apply)

(A) swaar demping / *overdamped* (B) kritieke demping / *critically damped* (C) ligte demping / *underdamped* (D) resonansie / *resonant* (E) swewinge / *beating*

Beskou nou die volgende aanvangswaardeprobleem wat 'n ander veer-massa stelsel beskryf:

Consider now the following initial value problem, which describes a different spring-mass system:

$$\frac{d^2x}{dt^2} + 4x = f(t), \qquad x(0) = 1, \qquad x'(0) = 0.$$

(e) Doen 'n geskikte funksie f aan die hand sodat die ooreenstemmende oplossing die "swewinge" ("beats") verskynsel sal vertoon. Teken 'n rowwe skets van sodanige oplossing in Figuur 1(e) onder.

(e) Suggest a suitable function f so that the resulting solution will exhibit the "beats" phenomenon. Draw a rough sketch of such a solution in Figure 1(e) below.

$f = \ldots\cos(2.1t)\ldots$

(f) Doen 'n geskikte funksie f aan die hand sodat die ooreenstemmende oplossing die "resonansie" verskynsel sal vertoon. Teken 'n rowwe skets van sodanige oplossing in Figuur 1(f) onder.

(f) Suggest a suitable function f so that the resulting solution will exhibit the "resonance" phenomenon. Draw a rough sketch of such a solution in Figure 1(f) below.

$f = \ldots\cos(2t)\ldots$

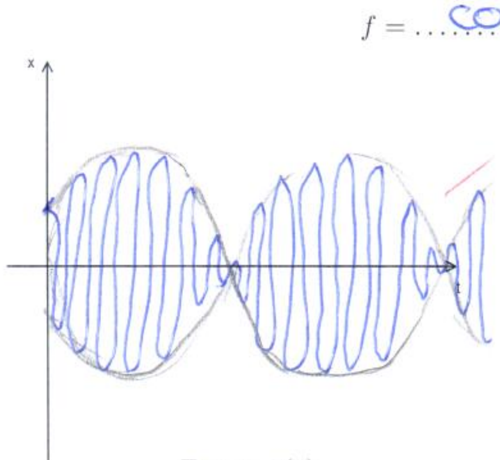


Figure 1(e)

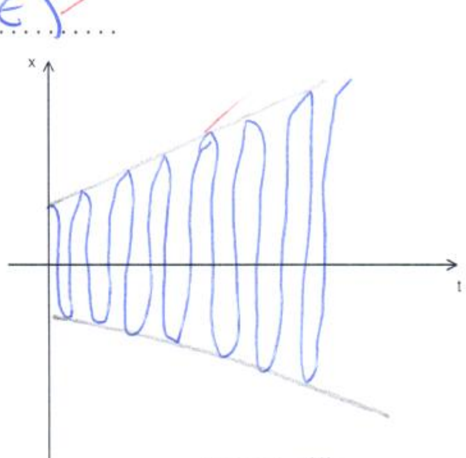


Figure 1(f)

Prob 2 (13 punte/marks)

(a) Toon aan dat

3 (a) Show that

(i) $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0,$ en/*and* (ii) $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}, \quad s > 0.$

i) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt$
$= \left[-\frac{1}{s}e^{-st}\right]_0^\infty$
$= 0 + \frac{1}{s}, s>0$
$= 1/s$

ii) $\mathcal{L}\{\sin(t)\} = \int_0^\infty e^{-st}\sin(t)dt$
$= \left[-\frac{1}{s}e^{-st}\sin(t)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\cos(t)dt$
$= \frac{1}{s}\mathcal{L}\{\cos(t)\}$
$\mathcal{L}\{\cos(t)\} = \int_0^\infty e^{-st}\cos(t)dt$
$= \left[-\frac{1}{s}e^{-st}\cos(t)\right]_0^\infty - \frac{1}{s}\int_0^\infty e^{-st}\sin(t)dt$
$= \frac{1}{s} - 1/s\,\mathcal{L}\{\sin(t)\}$
$\Rightarrow \mathcal{L}\{\sin(t)\} = \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$

(b) Wys dat as die Laplace transform van 'n funksie $f(t)$ gegee word deur $F(s)$ vir $s > s_0$, dan

2 (b) Show that if the Laplace transform of a function $f(t)$ is given by $F(s)$ for $s > s_0$, then

(i) $\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > s_0 + a,$ en/*and* (ii) $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad s > as_0.$

i) $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt$
$= \int_0^\infty e^{-(s-a)t}f(t)dt$
$\hat{s} = s-a \Rightarrow \quad = \int_0^\infty e^{-\hat{s}t}f(t)dt$
$= F(\hat{s}), \ \hat{s} > s_0$
$= F(s-a), \ s > s_0 + a$

ii) $\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st}f(at)dt \quad u=at, \ du = a\,dt$
$= \frac{1}{a}\int_0^\infty e^{-s(u/a)}f(u)du$
$= \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})u}f(u)du$
$\hat{s} = s/a \quad = \frac{1}{a}\int_0^\infty e^{-\hat{s}u}f(u)du$
$= \frac{1}{a}F(\hat{s}), \ \hat{s} > s_0$
$= \frac{1}{a}F(s/a), \ s > as_0$

(c) Toon aan dat

(c) Show that

(i) $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \quad s > a,$ en/*and* 2 (ii) $\mathcal{L}\{e^{mt}\sin(t)\} = \frac{1}{(s-m)^2+1}, \quad s > 0.$

i) $f(t) = \sin(t) \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} = F(s)$
b(ii) $\Rightarrow \mathcal{L}\{\sin(kt)\} = \mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right) = \frac{1}{k}\frac{1}{s^2/k^2+1} = \frac{k}{s^2+k^2}$

ii) $f(t) = \sin(t) \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} = F(s)$
b(i) $\Rightarrow \mathcal{L}\{e^{mt}\sin(t)\} = \mathcal{L}\{e^{mt}f(t)\} = F(s-m) = \frac{1}{(s-m)^2+1}$

(d) Herlei die volgende formules

2(d) Derive the following formulas

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \qquad \text{en}/\textit{and} \qquad \frac{1}{(s^2+1)(s^2+2)} = \frac{1}{s^2+1} - \frac{1}{s^2+2}.$$

i) $\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$
(Given on formula sheet).

$= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)$

$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$$\frac{1}{(s^2+1)(s^2+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$$

$\Rightarrow 1 = As^3 + 2As + Bs^2 + 2B + Cs^3 + Cs + Ds^2 + D$

$1 = \underset{\overset{a}{=}0}{(A+C)}s^3 + \underset{\overset{b}{=}0}{(B+D)}s^2 + \underset{\overset{c}{=}0}{(2A+C)}s + \underset{\overset{d}{=}1}{(2B+D)}$

$a+c \Rightarrow A = C = 0$

$b+d \Rightarrow B = 1,\ D = -1$

$$\Rightarrow \frac{1}{(s^2+1)(s^2+2)} = \frac{1}{s^2+1} - \frac{1}{s^2+2}.$$

(e) Gebruik die metode van Laplace transforms om die volgende aanvangswaardeprobleem op te los

2(e) Use the method of Laplace transforms to solve the following initial value problem

$$y'' + y = \sin(\sqrt{2}t), \qquad y(0) = 0,\ y'(0) = 1.$$

Denote $Y = \mathcal{L}\{y(t)\}$

Nota: Jy moet die metode van Laplace transforms gebruik.

Note: You must use the method of Laplace transforms.

$y'' + y = \sin(\sqrt{2}t) \Rightarrow \mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(\sqrt{2}t)\}$

$\Rightarrow s^2Y - sy(0) - y'(0) + Y = \frac{\sqrt{2}}{s^2+2}$

$\Rightarrow (s^2+1)Y = \frac{\sqrt{2}}{s^2+2} + 1$

$\Rightarrow Y = \frac{\sqrt{2}}{(s^2+1)(s^2+2)} + \frac{1}{s^2+1} \underset{d(ii)}{=} \frac{\sqrt{2}+1}{s^2+1} - \frac{\sqrt{2}}{s^2+2}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}+1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} = (\sqrt{2}+1)\sin(t) - \sin(\sqrt{2}t).$

(f) Die eenheid-stapfunksie $\mathcal{U}(t-a)$ word gedefinieer deur

2(f) The unit step function $\mathcal{U}(t-a)$ is defined by

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t \le a, \\ 1, & t \ge a. \end{cases}$$

Wys dat vir $a > 0$

Show that for $a > 0$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}.$$

$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\mathcal{U}(t-a)\,dt = \int_0^a e^{-st}\cancelto{=0}{\mathcal{U}(t-a)}\,dt + \int_a^\infty e^{-st}\underbrace{\mathcal{U}(t-a)}_{=1}\,dt$

$= \int_a^\infty e^{-st}\,dt$

$= \left[-\frac{1}{s}e^{-st}\right]_a^\infty = 0 + \frac{e^{-as}}{s}$

$= \frac{e^{-as}}{s}$

Prob 3 (15 punte/marks)

Beskou die nie-lineêre stelsel van DVs gegee deur

Consider the nonlinear system of DEs given by

$$\frac{dx}{dt} = x + xy - 3x^2, \qquad \frac{dy}{dt} = 4y - 2xy - y^2.$$

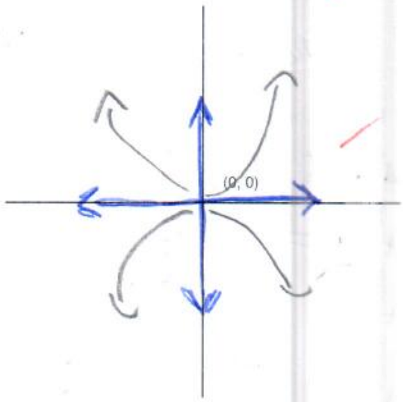
(a) Drie van die vier kritieke punte van die stelsel is $(0,0)$, $(1/3,0)$, en $(1,2)$, en gee aanleiding tot die Jacobiane onder. Gebruik hierdie inligting om die kritieke punte te klasssifiseer en teken die gedrag van die oplossings in die omgewing van die punte. Gebruik die spasie hieronder vir jou berekenings.

(a) Three of the four critical points of the system are $(0,0)$, $(1/3,0)$, and $(1,2)$, and give rise to the Jacobians below. Use this information to classify the critical points and sketch the behaviour of the solutions in their neighbourhoods. Use the space below to show your working.

$\tau = 5,\ \Delta = 4,\ \frac{1}{4}\tau^2 = \frac{25}{4} > 4 = \Delta$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

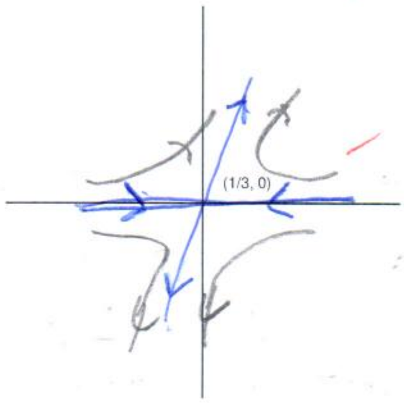
tipe/type = unstable node



$\tau = \frac{7}{3},\ \Delta = -\frac{30}{3} = -10$

$$J(1/3,0) = \frac{1}{3}\begin{bmatrix} -3 & 1 \\ 0 & 10 \end{bmatrix}$$

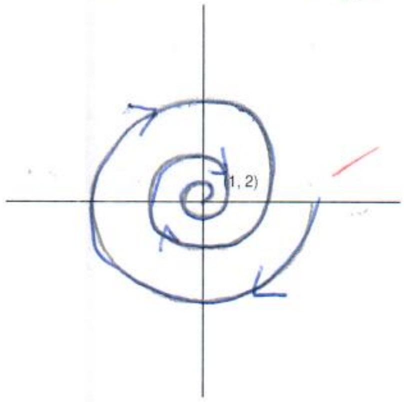
tipe/type = saddle



$\tau = -5,\ \Delta = 10,\ \frac{1}{4}\tau^2 = \frac{25}{4} < 10 = \Delta$

$$J(1,2) = \begin{bmatrix} -3 & 1 \\ -4 & -2 \end{bmatrix}$$

tipe/type = stable spiral



Wenke: Een van die matrikse het die eiewaardes en eievektore: $\lambda_1 = 10/3$, $\lambda_2 = -1$, $v_1 = [1,13]$, $v_2 = [1,0]$, en by tenminste een van die ander is daar 'n spiraal of senter. In lg geval, bepaal die rigting van rotasie en motiveer jou keuse.

Hints: One of the matrices has the eigenvalues and eigenvectors: $\lambda_1 = 10/3$, $\lambda_2 = -1$, $v_1 = [1,13]$, $v_2 = [1,0]$ and at least one of the others is a spiral or centre. In the latter case determine the direction of rotation and justify your choice.

Diagonal matrix

$\Rightarrow \lambda_1 = 1,\ \underline{v}_1 = [1,0]^T$

$\lambda_2 = 4,\ \underline{v}_2 = [0,1]^T$

$\underline{x} = c_1 e^{t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}0\\1\end{bmatrix}$

$t \to \infty \Rightarrow x \to [0,1]^T$

From hint:

$\lambda_1 = 10/3,\ \underline{v}_1 = [1,13]^T$

$\lambda_2 = -1,\ \underline{v}_2 = [1,0]$

$\left.\frac{dy}{dt}\right|_{1+\varepsilon,2} = 4\cdot 2 - 2(1+\varepsilon)2 - 2^2$

$= 8 - 4 - 4 - 4\varepsilon$

$= -4\varepsilon$

$\varepsilon > 0 \Rightarrow \frac{dy}{dt} < 0$

(1,2) (1+ε,2)

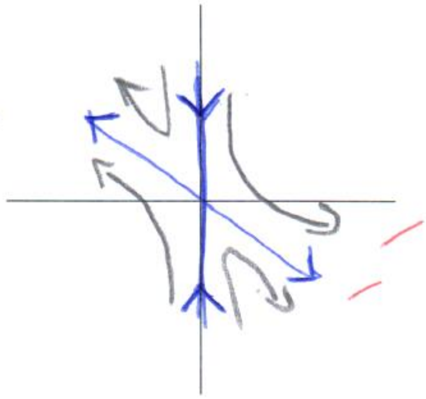
$\Rightarrow$ clockwise

(b) Lokaliseer en klassifiseer die oorblywende kritiese punt van die stelsel hierbo, en skets die oplossing in die omgewing van hierdie punt op die diagram onder.

3 (b) Locate and classify the remaining critical point of the system above and sketch the solution in the neighbourhood of this point on the diagram below.

$\frac{dx}{dt} = x(1+y-3x)$, $\quad x=0 \Rightarrow \begin{cases} \frac{dx}{dt}=0 \\ \frac{dy}{dt} = y(4-y) \end{cases}$

$\frac{dy}{dt} = y(4-2x-y)$ $\quad \Rightarrow (0,4)$ is a critical point.



$J(x,y) = \begin{bmatrix} 1+y-6x & x \\ -2y & 4-2x-2y \end{bmatrix}$, $J(0,4) = \begin{bmatrix} 5 & 0 \\ -8 & -4 \end{bmatrix}$

$\tau = 1$, $\Delta = -20 \Rightarrow$ saddle

Lower tri $\Rightarrow$ eigenvalues $\lambda_1 = 5$, $\lambda_2 = -4$

$\underline{v}_1$: $\begin{bmatrix} 0 & 0 \\ -8 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 9 \\ -8 \end{bmatrix}$

$\underline{v}_2$: $\begin{bmatrix} 9 & 0 \\ -8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 2

(c) Skets die fasediagram van die stelsel in die eerste kwadrant (m.a.w., $x, y \geq 0$). Sluit 'nulkliene' en die 4de kritiese punt in jou skets in.

3 (c) Sketch the phase diagram of the system in the first quadrant (i.e., $x, y \geq 0$). Include nullclines and the 4th critical point in your sketch.

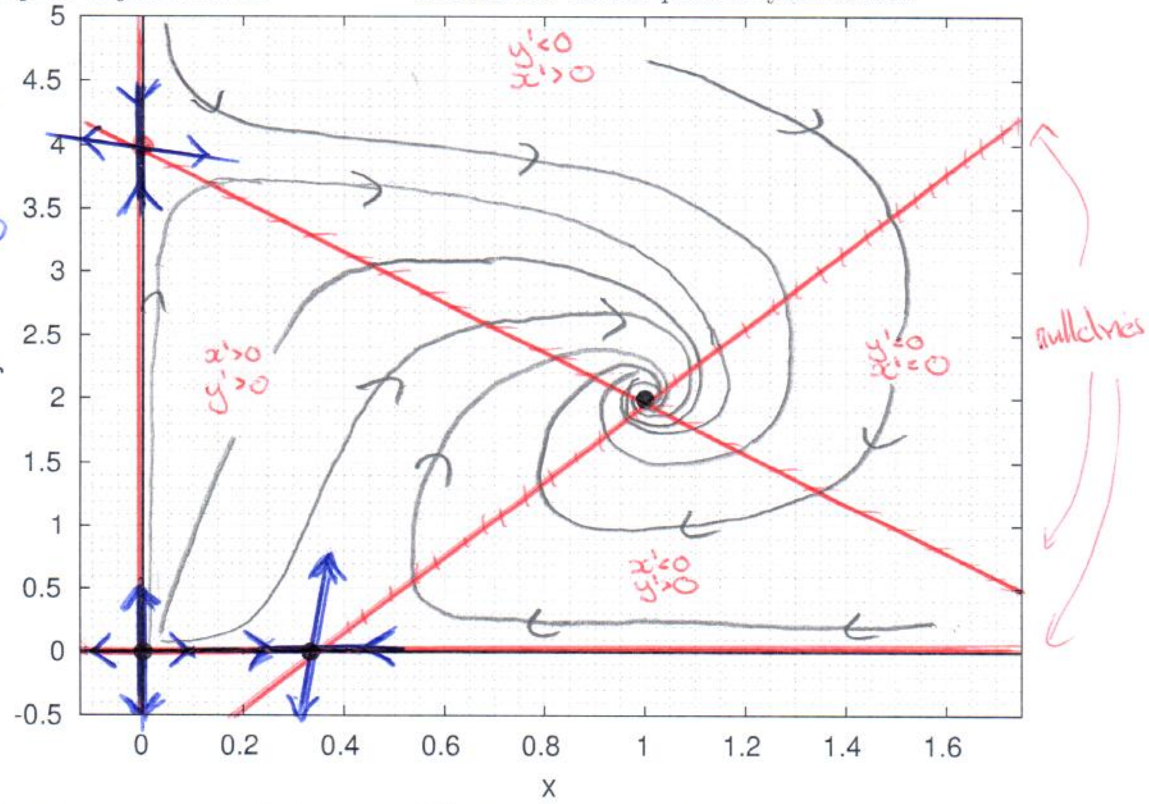
Nullclines

$x' = 0 \Rightarrow x(1+y-3x) = 0$

$\Rightarrow \begin{cases} x = 0 \\ y = 3x-1 \end{cases}$

$y' = 0 \Rightarrow y(4-2x-y) = 0$

$\Rightarrow \begin{cases} y = 0 \\ y = 4-2x \end{cases}$



(d) Vir elk van die volgende aanvangspopulasies, gee die bevolkingslimiet as $t \to \infty$.

(d) For each of the following initial populations give the limiting population as $t \to \infty$.

$(x(0), y(0)) = (0,0) \implies (x(t), y(t)) \xrightarrow{t\to\infty} (\ldots 0 \ldots, \ldots 0 \ldots)$

$(x(0), y(0)) = (0,1) \implies (x(t), y(t)) \xrightarrow{t\to\infty} (\ldots 0 \ldots, \ldots 4 \ldots)$

$(x(0), y(0)) = (1,0) \implies (x(t), y(t)) \xrightarrow{t\to\infty} (\ldots 1/3 \ldots, \ldots 0 \ldots)$

$(x(0), y(0)) = (1,1) \implies (x(t), y(t)) \xrightarrow{t\to\infty} (\ldots 1 \ldots, \ldots 2 \ldots)$

Prob 4 (4 punte/marks)

Beskou die veer-massa stelsel:

Consider the spring-mass system:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F_0\cos(\gamma t).$$

Nie-dimensionaliseer hierdie stelsel om die karakteristieke lengtes t_c en x_c te kry (in terme van m, β, k, F_0 en γ), en wys dat die stelsel in die volgende vorm geskryf kan word:

Nondimensionalise this system to obtain the characteristic lengths t_c and x_c (in terms of m, β, k, F_0, and γ), and show that the system may be written in the form

$$\frac{d^2\widehat{x}}{d\widehat{t}^2} + 2\xi\frac{d\widehat{x}}{d\widehat{t}} + \widehat{x} = \cos(\phi\widehat{t}).$$

- t and x are the variables. Let $x = x_c\hat{x}$, $t = t_c\hat{t}$ $\Rightarrow$ $\frac{dx}{dt} = \frac{x_c}{t_c}\frac{d\hat{x}}{d\hat{t}}$, $\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\hat{x}}{d\hat{t}^2}$
- Therefore $\frac{m x_c}{t_c^2}\frac{d^2\hat{x}}{d\hat{t}^2} + \beta\frac{x_c}{t_c}\frac{d\hat{x}}{d\hat{t}} + k x_c\hat{x} = F_0\cos(\gamma t_c\hat{t})$
- Divide by $\frac{m x_c}{t_c^2}$ $\Rightarrow$ $\frac{d^2\hat{x}}{d\hat{t}^2} + \underbrace{\frac{\beta t_c}{m}}_{=2\xi}\frac{d\hat{x}}{d\hat{t}} + \underbrace{\frac{k t_c^2}{m}}_{=1}\hat{x} = \underbrace{\frac{t_c^2 F_0}{m x_c}}_{=1}\cos(\underbrace{\gamma t_c}_{=\phi}\hat{t})$
- $k t_c^2/m = 1 \Rightarrow t_c = \sqrt{m/k}$, $\frac{t_c^2F_0}{m x_c} = 1 \Rightarrow x_c = t_c^2F_0/m = F_0/k$
  $\beta t_c/m = 2\xi \Rightarrow \xi = \frac{\beta}{2}\sqrt{1/mk}$, $\gamma t_c = \phi \Rightarrow \phi = \gamma\sqrt{m/k}$

$x_c = F_0/k$, $t_c = \sqrt{m/k}$, $\xi = \frac{\beta}{2m}\sqrt{m/k}$, $\phi = \gamma\sqrt{m/k}$

Prob 5 (4 punte/marks)

Beskou die nie-lineêre veer-massa stelsel:

Consider the nonlinear spring-mass system:

$$x'' = -x - x^3$$

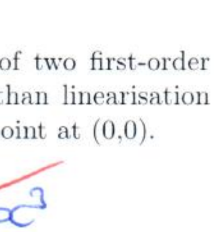
Herskryf die bostaande as 'n stelsel van twee eerste-orde differensiaalvergelykings en wys dat dit nie moontlik is om die kritieke punt by (0,0) te klassifiseer nie.

Rewrite the above as a system of two first-order differential equations and show than linearisation is unable to classify the critical point at (0,0).

Let $x' = y$ $\Rightarrow$ $y' = x'' = -x - x^3$, ie $\begin{cases} x' = y \\ y' = -x - x^3 \end{cases}$

Linearisation $\Rightarrow$ $J(x,y) = \begin{bmatrix} 0 & 1 \\ -1-3x^2 & 0 \end{bmatrix}$ $\Rightarrow$ $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow \tau = 0, \Delta = 1$



Since the system is nonlinear this is a borderline case and we cannot classify the nature of the CP at (0,0) with this method.

Bonus Prob (4 bonus punte/marks)

(a) Gebruik die fasevlak metode om te wys dat die kritieke punt by (0,0) in die stelsel hierbo 'n senter is. **(b)** Bepaal en klassifiseer enige ander kritiese punte van die stelsel en teken die fasediagram.

(a) Use the phase-plane method to show that critical point at (0,0) in the system above is a centre. **(b)** Locate and classify any other critical points of the system and hence plot the phase diagram.

a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x-x^3}{y}$

Seperation of variables $\Rightarrow \int y\,dy = \int(-x-x^3)\,dx$

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 - \frac{1}{4}x^4 + c'$

$\Rightarrow y^2 = -x^2 - \frac{1}{2}x^4 + c$

Initial condition $(x_0, 0) \Rightarrow 0 = -x_0^2 - \frac{1}{2}x_0^4 + c$

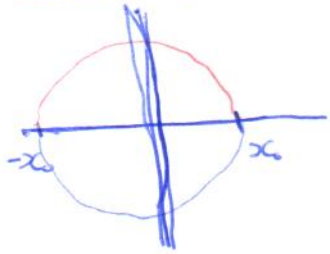
$\Rightarrow c = x_0^2 + \frac{1}{2}x_0^4$

$\Rightarrow y^2 = \underbrace{(x_0^2 - x^2)}_{>0} + \frac{1}{2}\underbrace{(x_0^4 - x^4)}_{>0}$

Notice: • $y^2(-x_0) = y^2(x_0) = 0$

• For $0 < |x| < |x_0|$ we have $y^2 > 0$ and hence y has +ve and -ve solutions

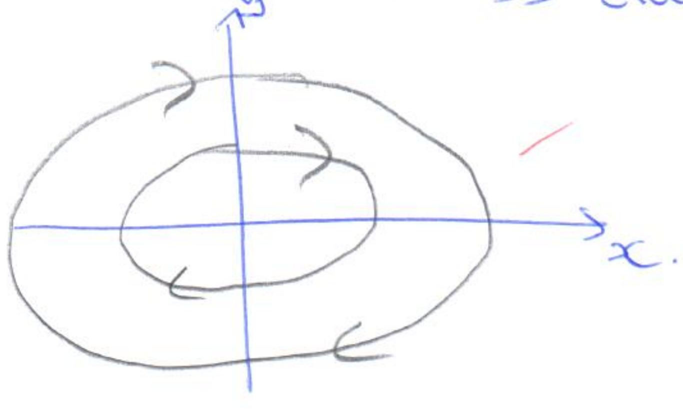
Therefore $y(x)$ forms a closed curve in the phase plane (for any $x_0 \neq 0$) and hence the critical point is a centre.



b) There are no other critical points

Direction of rotation: $\left.\frac{dy}{dt}\right|_{0,\varepsilon} = -\varepsilon - \varepsilon^3 < 0$ for $\varepsilon > 0$

$\Rightarrow$ clockwise.



Hierdie bladsy is doelbewus leeg gelaat. Jy kan dit gebruik vir rofwerk of vir ekstra spasie, indien nodig.

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