Hierdie opdrag moet as 'n enkele PDF-lêer via SUNLearn ingehandig word voor die sperdatum hierbo. Laat inhandigings sal gepenaliseer word. Kommunikasie tussen studente rakende werksopdragte is streng verbode en plagiaat sal tot ernstige gevolge lei. Jou inhandiging moet 'n getekende verklaring bevat dat dit jou eie werk is. Raadpleeg die TW244 SUNLearn-blad vir verdere instruksies.

This assignment must be submitted as a single PDF file via SUNLearn before the due date above. Late submissions will be penalized. Communication between students regarding assignments is strictly prohibited and plagiarism will have severe consequences. Your submission should contain a signed declaration that it is your own work. See TW244 SUNLearn page for further instructions.

P1: Beskou die volgende outonome 1ste-orde DV:

P1: Consider the following autonomous 1st-order DE:

$$\frac{dP}{dt} = -P^4 + 4P^3 - 5P^2 + 2P.$$

(a) Vind al die kritieke oplossings van hierdie DV. (Wenk: dit behoort deur inspeksie moontlik te wees, maar indien nie, oorweeg dit om die `roots`-funksie in MATLAB of NumPy te gebruik.)

(a) Find all the critical solutions of this DE. (Hint: This should be possible by inspection, but if not, consider using the `roots` function in MATLAB or NumPy.)

(b) Bepaal gebiede waar oplossings stygend/dalend is en ook waar dit konkaaf-op/konkaaf-af is, sonder om die DV op te los.

(b) Without solving the DE, determine regions where solutions increase/decrease and also where they are concave-up/concave-down.

(c) Skets met die hand 'n paar tipiese oplossingskrommes in die tP-vlak.

(c) Sketch by hand a few typical solution curves in the tP-plane.

(d) Gebruik dan jou skets om elke kritieke oplossing as stabiel, onstabiel of semi-stabiel te klassifiseer.

(d) Use your sketch to classify every critical point as stable, unstable, or semi-stable.

(e) As $P(0) = 0.25$, wat is $P(t \to \infty)$? Wat as $P(0) = 1.4$?

(e) If $P(0) = 0.25$, what is $P(t \to \infty)$? What if $P(0) = 1.4$?

(f) Verifieer jou handskets van deel (c) met behulp van `dfield8`. Stip vir $0 \le t \le (10 + \omega)$, waar ω die finale syfer van jou studentenommer is, en kies verstandige P limiete.

(f) Verify your hand sketch from part (c) with the aid of `dfield8`. Plot for $0 \le t \le (20 - \omega)$, where ω is the final digit of your student number, and choose sensible P limits.

P2: Die differensiaalvergelyking hieronder modelleer die snelheid van 'n vryvallende valskermspringer met massa m onderhewig aan lineêre sleur met sleurkoëffisiënt k en aanvanklike snelheid v_0 (sien Lesing 07):

P2: The differential equation below models the velocity of a free-falling skydiver of mass m subject to linear drag with drag coefficient k and initial velocity v_0 (see Lecture 07):

$$m\frac{dv}{dt} = mg - kv$$

(a) Gebruik skeiding van veranderlikes om te wys dat die oplossing vir hierdie DV gegee word deur

(a) Use separation of variables to show that the solution to this DE is given by

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) + \frac{m}{k}e^{-\frac{k}{m}t}v_0, \qquad v(t_0) = v_0.$$

(b) Toon aan dat die afstand wat die valskermspringer afgelê het na 'n tyd t gegee word deur

(b) Show that the distance travelled by the skydiver after time t is given by

$$s(t) = \ldots$$

(b) Die terminale snelheid van 'n vryvallende mens is ongeveer 56m/s. Gebruik hierdie inligting om die relatiewe sleurkoëffisiënt, $\frac{k}{m}$, te skat. (Neem $g = 9.81$.) Wat is die snelheid van die valskermspringer na 60 sekondes? Hoe ver het sy geval?

(b) The terminal velocity of a free-falling human is around 56m/s. Use this information to estimate the relative drag coefficient, $\frac{k}{m}$. (Take $g = 9.81$.) What is the velocity of the skydiver after 60 seconds? How far has she fallen?

(c) [Bonus - moeilik] Na 60 sekondes maak die valskermspringer haar valskerm oop, wat haar terminale snelheid met ongeveer 90% verminder. As sy aanvanklik van 'n hoogte van 4000m gespring het, hoe lank in totaal voordat sy land? [Wenk: Jy sal dit nie analities kan oplos nie. Jy moet óf 'n nulpuntsoekfunksie moet gebruik (soos `fzero`, wat ons in CA02 sal bespreek), óf 'n benaderde oplossing moet vind deur die hoogte in (d) te stip en in te "zoom" op die figuur om uit te vind waar dit nul bereik.]

(c) [Bonus - hard] After 60 seconds the skydiver opens her parachute, which decreases her terminal velocity by around 90%. If the she initially jumped from a height of 4000m, how long in total before she lands? [Hint: You will not be able to solve this analytically. You will either need to use a zero-finding function (such as `fzero`, which we will discuss in CA02) or to find an approximate solution by plotting the altitude in (d) and zooming in on the figure to find out where it reaches zero.]

(d) [Opsioneel] Gebruik MATLAB/Python om die snelheid en hoogte van die valskermspringer te stip as 'n funksie van tyd. Bespreek enige beperkings/twyfelagtige aannames in die model.

(d) [Optional] Use MATLAB/Python to plot the velocity and altitude of the skydiver as a function of time. Discuss any limitations/questionable assumptions in the model.

P3: Beskou die aanvangswaardeprobleem

P3: Consider the initial value problem

$$\frac{dy}{dt} = -\cos(t)y^2, \quad y(0) = 1,$$

(a) Verifieer dat die oplossing gegee word deur

(a) Verify that the solution is given by

$$y(t) = \frac{1}{1+\sin(t)},$$

en stip die oplossing vir $0 \le t \le 1$.

and plot the solution for $0 \le t \le 1$.

Veronderstel nou dat die eksakte oplossing onbekend is en dat ons die probleem numeries wil oplos.

Now suppose that the exact solution is not known and that we wish to solve the problem numerically.

(b) Gebruik Euler se metode in MATLAB of Python met staplengte $h = 0.25$ om benaderde waardes vir $y(0.25)$, $y(0.5)$, $y(0.75)$, $y(1)$ te bereken. Vergelyk jou resultate met die eksakte funksiewaardes (soos bereken met die gegewe oplossing) in beide 'n grafiek an 'n tabel van die vorm:

(b) Use Euler's method in MATLAB or Python with step size $h = 0.25$ to calculate approximate values for $y(0.25)$, $y(0.5)$, $y(0.75)$, $y(1)$. Compare your results to the exact function values (as calculated with the given solution) in both a graph and a table of the form:

t	y(t) exact	y(t) approx	error
0.2	0.8343	0.8000	0.03426
0.4	⋮	⋮	⋮
0.6	⋮	⋮	⋮
0.8	⋮	⋮	⋮
1.0	⋮	⋮	⋮

(c) Gebruik die gewysigde Euler metode om dieselfde waardes as hierbo te bereken. Vergelyk weereens met die eksakte waardes. Is hierdie benaderings beter of slegter as dié van deel (b)?

(c) Use the modified Euler method to approximate the same values as above. Again, compare to the exact values. Are these approximations better or worse than those of part (b)?

t	y(t) exact	y(t) approx	error
0.2	0.8343	0.8373	0.003017
0.4	⋮	⋮	⋮
0.6	⋮	⋮	⋮
0.8	⋮	⋮	⋮
1.0	⋮	⋮	⋮

(d) Herhaal (b) met $h = 0.1$, 0.05, en 0.025. Stel 'n tabel op om aan te toon dat die fout by $t = 1$ ongeveer halveer namate ons h halveer. Doen dieselfde vir (c) en toon aan dat die fout met ongeveer 'n faktor van 4 verminder wanneer h gehalveer word. Kom tot die gevolgtrekking dat gewysigde Euler nie net meer akkuraat is nie, maar ook vinniger *konvergeer*.

(d) Repeat (b) using $h = 0.1$, 0.05, and 0.025. Produce a table to show that the error at $t = 1$ roughly halves as we halve h. Do the same for (c) and show that the error decreases by roughly a factor of 4 when h is halved. Conclude that not only is modified Euler more accurate, but it *converges* faster.

h	Euler error	Mod Euler error
0.2	0.04952	0.003117
0.1	⋮	⋮
0.05	⋮	⋮