Applied differential equations

TW244 - Lecture 07

3.1: Linear Models (cont.)

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SCIENCE
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Recall:

$$\underline{\frac{dx}{dt} = kx,} \quad x(0) = x_0 \implies \underline{x(t) = x_0 e^{kt}}.$$

Let's look at some more applications of linear models.

3.1: Linear Models
Application 2: Radioactive decay

Let $N = N(t)$ be the number of radioactive atoms at time t.

Assumptions:

- We can approximate $N(t)$ with a smooth continuous function.
- At any time instance the rate of decay is proportional to the number of radioactive atoms present.

So we have

$$\frac{dN}{dt} = -\lambda N \quad \text{with} \quad N(0) = N_0, \quad \lambda > 0$$

($-\lambda$ emphasizes that there is a decrease) and therefore

$$\boxed{N(t) = N_0 e^{-\lambda t}.}$$

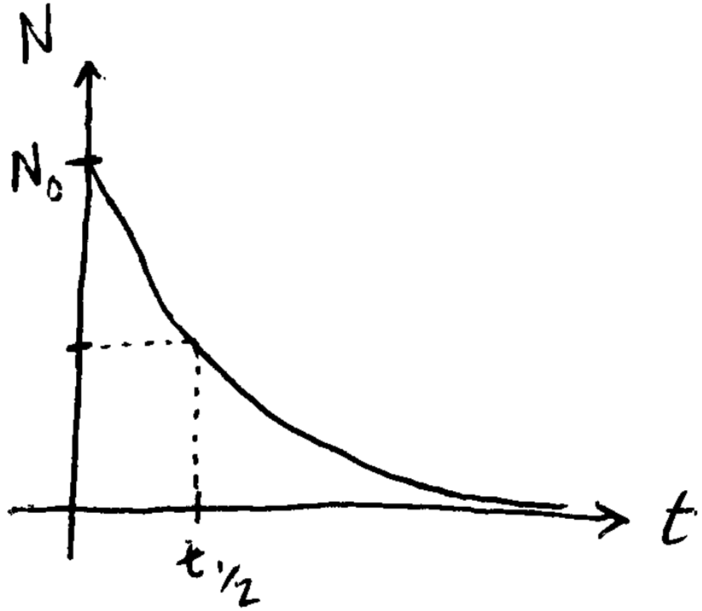
3.1: Linear Models

Application 2: Radioactive decay: Half-life

$N(t) = N_0 e^{-\lambda t}$

Half-life: The time is takes for half of the atoms to decay* ✗

$$
\begin{aligned}
\frac{1}{2}N_0 &= N_0 e^{-\lambda t_{0.5}} = N(t_{0.5}) \\
\ln\frac{1}{2} &= -\lambda t_{0.5} \\
t_{0.5} &= \frac{1}{\lambda}\ln 2
\end{aligned}
$$



Therefore

- λ large means short half life
- λ small means long half life

*Also an excellent late-90s video game! ✗

3.1: Linear Models

Application 2: Radioactive decay: Half-life (cont.)

We can write $N(t) = N_0 e^{-\lambda t}$ in terms of the half-life as follows:

Since $t_{0.5} = \frac{1}{\lambda} \ln 2$, we have $\lambda = \frac{1}{t_{0.5}} \ln 2$

Substituting this to $N(t)$ we have

$$N(t) = N_0 [e^{-\ln 2}]^{t/t_{0.5}} = \left(\frac{1}{2}\right)^{t/t_{0.5}} N_0.$$

Note:

- if $t = t_{0.5}$ then $N = \frac{1}{2} N_0$
- if $t = 2t_{0.5}$ then $N = \frac{1}{4} N_0$

etc.

3.1: Linear Models

Example: "Carbon dating"

$N(t) = \left(\frac{1}{2}\right)^{t/t_{0.5}} N_0$

A piece of fossilized wood contains 63% as much C-14 (a radioactive carbon isotope) as a living piece of wood with the same mass.

$\frac{N(t)}{N_0} = 0.63$

If the half-life of C-14 is roughly 5730 years, how old is the fossil?

Solution:

$$N(t) = N_0 e^{-\lambda t} = \left(\frac{1}{2}\right)^{t/5730} N_0 \qquad (\leftarrow t \text{ in years})$$

We're given that $\frac{N}{N_0} = \frac{\text{current \# of C-14 isotopes}}{\text{\# of isotopes when alive}} = 0.63.$

Therefore

$$0.63 = \left(\frac{1}{2}\right)^{t/5730} \implies \ln(0.63) = \frac{t}{5730}\ln\frac{1}{2} \implies t = 5730\frac{\ln(0.63)}{\ln(0.5)} \approx 3819 \text{ years old.}$$

Exercise: Exercise 3.1.12 on p. 90 of Z&W.
Note: See discussion on p. 85-86 for more details of carbon dating in practice.

3.1: Linear Models
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3.1: Linear Models

Application 3: Newton's Law of cooling/warming

Let $T = T(t)$ be the temperature of an object at time t.

Let T_m be the temperature of the environment ("ambient temperature").

Assumptions

- The environment is large, so we may assume T_m is constant.
- Rate of change in temperature of the object is proportional to the difference between the temp. of the object and the ambient temp. i.e.,
  - Large difference in temp $\rightarrow$ T changes quickly
  - Small difference in temp $\rightarrow$ T changes slowly

So we have

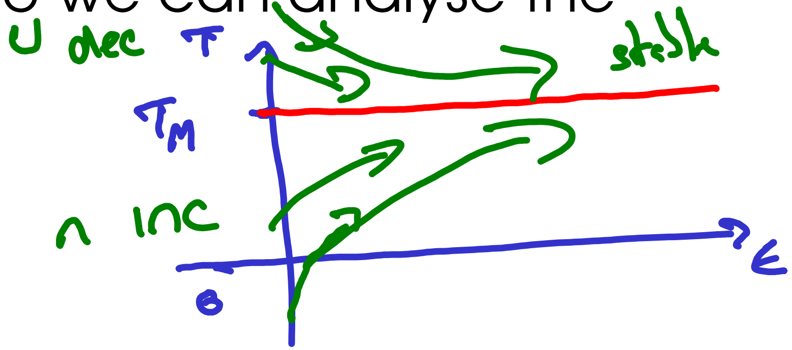
$$\frac{dT}{dt} = k(T_m - T) \quad \text{with} \quad T(0) = T_0.$$

Exercise: Why is k positive (irrespective of T_m and T_0)?

3.1: Linear Models

Application 3: Newton's Law of cooling/warming

Note that the DE $\frac{dT}{dt} = k(T_m - T)$ is autonomous, so we can analyse the behaviour of solutions as we did in Lecture 3.



Critical points:

$\frac{dT}{dt} = 0 \implies k(T_m - T) = 0 \implies \underline{T = T_m}$

Increase/decrease:

T increases when $\frac{dT}{dt} > 0 \implies k(T_m - T) > 0 \implies T < T_m$.

T decreases when $\frac{dT}{dt} < 0 \implies k(T_m - T) < 0 \implies T > T_m$.

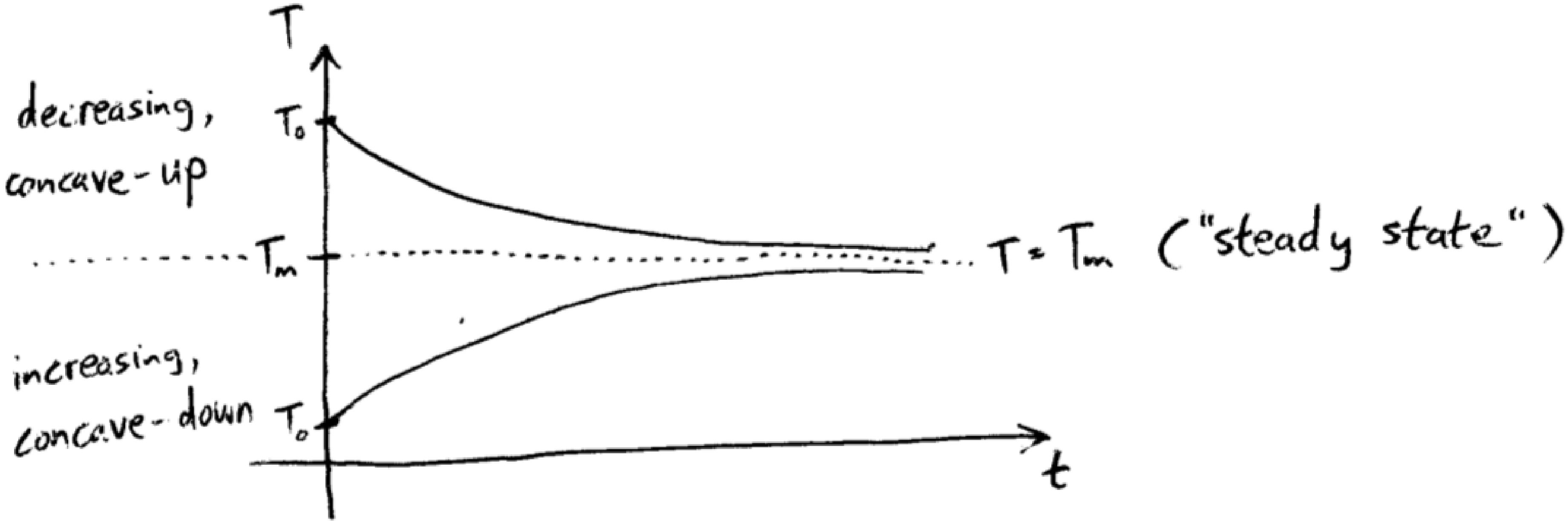
Concavity:

$\frac{d^2T}{dt^2} = \frac{d}{dt}[k(T_m - T)] = -k\frac{dT}{dt} = -k^2(T_m - T)$

$\therefore T$ is concave-up when $T > T_m$ and concave-down when $T < T_m$.

3.1: Linear Models
Application 3: Newton's Law of cooling/warming

We may combine all this information to produce the following figure:



Note: $T \to T_m$ as $t \to \infty$.
$T = T_m$ is an asymptotically stable (or "steady state") solution.

3.1: Linear Models

Application 3: Newton's Law of cooling/warming

Consider the initial value problem, II:

$$\frac{dT}{dt} = k(T_m - T), \qquad T(0) = T_0.$$

Solution by separation of variables:

$$\text{II} \implies \frac{1}{T_m - T}\,dT = k\,dt \implies -\int \frac{1}{-T_m + T}\,dT = \int k\,dt \implies -\ln|T_m - T| = kt + c.$$

Initial condition: $T(t=0) = T_0 \implies c = -\ln|T_m - T_0|$.

Therefore

$$\boxed{T(t) = T_m + (T_0 - T_m)e^{-kt}.}$$

Exercise: Solve by integrating factor method.

$$* \quad -\ln|T_M - T| = kt - \ln|T_M - T_0|$$

$$\ln|T_M - T_0| - \ln|T_M - T| = kt$$

$$\ln\left(\frac{T_M - T_0}{T_M - T}\right) = kt$$

$$\frac{T_M - T_0}{T_M - T} = e^{kt}$$

$$T_M - T_0 = (T_M - T)e^{kt}$$

$$T_M - T = (T_M - T_0)e^{-kt}$$

$$T = T_M + (T_0 - T_M)e^{-kt}$$

3.1: Linear Models

$T(t) = T_M + (T_0 - T_M)e^{-kt}$

Example (taken from the non-metric version of the textbook)

A cake is removed from an oven and its temperature is measured at 300°F. Three minutes later ~~someone hits you on the head with a pan for using °F and~~ its temperature is 200°F. How long will it take for the cake to cool off to the room temperature at 70°F?

T_M

?

$$T(t) = 70 + 230\,e^{-kt}$$ where t in mins

$$T(3) = 200 = 70 + 230e^{-3k}$$

$$\Rightarrow e^{-3k} = \frac{130}{230} \Rightarrow k = \frac{1}{3}\ln\frac{23}{13} \approx 0.19$$

$$T(t) = 70 + 230e^{-0.19t}$$

3.1: Linear Models

Example (taken from the non-metric version of the textbook)

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Newton's law of cooling (recall Lecture 7):

$$\frac{dT}{dt} = k(70 - T) \text{ with } T(0) = 300 \implies T(t) = 70 + (300 - 20)e^{-kt}.$$

We're given that $T(3) = 200$, therefore:

$$200 = 70 + 230e^{-3k} \implies e^{-3k} = \frac{130}{230} \implies k = \frac{1}{3}\ln\frac{23}{13} \approx 0.19018$$

and

$$T(t) = 70 + 230e^{-0.1918t}.$$

3.1: Linear Models

Example (taken from the non-metric version of the textbook)

But when will T reach 70°F?
Well... when $t \to \infty$!!!

According to our model and our assumptions, the cake will not reach a temperature of 70°F in finite time (which is consistent with the first law of thermodynamics).

However, as we see in the table opposite, it gets pretty close to 70°F after about 30 minutes.

Exercise: How long until T reaches 80°F?

$T(t)$	t (min)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

(b)

FIGURE 3.1.4 Temperature of cooling cake in Example 4