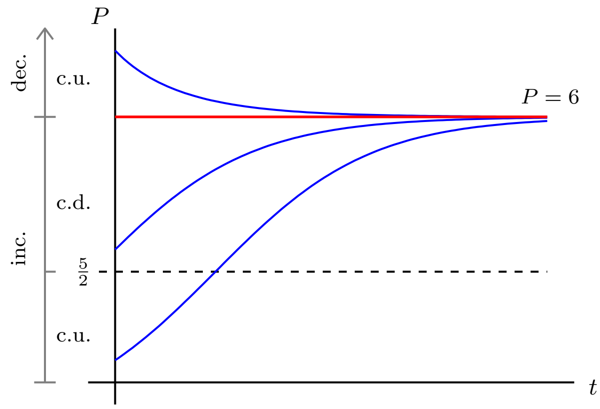
Problem 1:

(a) $\dfrac{dP}{dt} = 5P - P^2 - 4 + 10 = 5P - P^2 + 6.$

(b) Critical solutions: $\frac{dP}{dt} = -(P-6)(P+1) = 0$
$\implies P = 6 \quad$ (P cannot be negative).

P increases if $\frac{dP}{dt} > 0$, so if $0 < P < 6$,
and P decreases if $\frac{dP}{dt} < 0$, so if $P > 6$.

$\frac{d^2P}{dt^2} = (5-2P)\frac{dP}{dt} = (2P-5)(P-6)(P+1).$
P is concave-up when $0 < P < \frac{5}{2}$ or $P > 6$,
and P is concave-down when $\frac{5}{2} < P < 6$.



The fish will therefore never go extinct but will, for any initial amount, tend to 6 000.

(c) $\dfrac{dP}{dt} = -(P-6)(P+1) \implies \displaystyle\int \frac{1}{(P-6)(P+1)}\,dP = -\int dt \implies \int\left(\frac{\frac{1}{7}}{P-6} - \frac{\frac{1}{7}}{P+1}\right)dP = -\int dt$

$\implies \ln|P-6| - \ln|P+1| = -7t + c.$ Initial value $P(0) = P_0$: $c = \ln|P_0 - 6| - \ln|P_0+1|$

$\implies \ln\left|\dfrac{(P-6)(P_0+1)}{(P+1)(P_0-6)}\right| = -7t \implies \dfrac{(P-6)(P_0+1)}{(P+1)(P_0-6)} = \pm e^{-7t}$, choose the (+)-sign so that $P(0) = P_0$.

$\implies (P-6)(P_0+1) = (P+1)(P_0-6)e^{-7t}$, so that $\boxed{P(t) = \dfrac{6(P_0+1) + (P_0-6)e^{-7t}}{(P_0+1) - (P_0-6)e^{-7t}}}$

For $P(t)$ to be zero, $6(P_0+1) + (P_0-6)e^{-7t} = 0$, therefore $t = \frac{1}{7}\ln\left(\dfrac{6-P_0}{6P_0+6}\right)$.
This time t must be positive and real, which implies that $6 - P_0 > 6P_0 + 6$, that is $P_0 < 0$.
For any $P_0 > 0$ we see that $P(t)$ will never become zero, meaning that the fish will never go extinct.

Problem 2:

Let $X(t)$ be the mass of C at time t. Note: 2 g of A plus 1 g of B forms 3 g of C.
That is, the mass of A present at time t is $40 - \frac{2}{3}X$ and the mass of B present is $50 - \frac{1}{3}X$.

$\dfrac{dX}{dt} = k'(40 - \frac{2}{3}X)(50 - \frac{1}{3}X) = k(60-X)(150-X)$ with $X(0) = 0$.

Separate variables: $\displaystyle\int \frac{1}{(60-X)(150-X)}\,dX = k\int dt \implies \frac{1}{90}\int\left(\frac{1}{X-150} - \frac{1}{X-60}\right)dX = k\int dt.$

Hence $\ln\left|\frac{X-150}{X-60}\right| = 90kt + c$. Given that $X(0) = 0 \implies c = \ln(2.5)$. Thus $\ln\left|\frac{X-150}{2.5(X-60)}\right| = 90kt$.

$\frac{X-150}{2.5(X-60)} = \pm e^{90kt}$, and we choose the (+)-sign to satisfy $X(0) = 0$.

$X - 150 = 2.5(X-60)e^{90kt} \implies X(1 - 2.5e^{90kt}) = 150 - 150e^{90kt} \implies X(t) = \dfrac{150 - 150e^{90kt}}{1 - 2.5e^{90kt}}.$

It is given that $X5() = 10$, from which we can solve k: $k = 2.5184 \times 10^{-4}$, so that $X(20) = \boxed{29.32\,\text{g}}$

Limiting amount of C: $\lim_{t\to\infty} X(t) = \lim_{t\to\infty} \dfrac{150 - 150e^{90kt}}{1 - 2.5e^{90kt}} \times \dfrac{e^{-90kt}}{e^{-90kt}} = \lim_{t\to\infty} \dfrac{150e^{-90kt} - 150}{e^{-90kt} - 2.5} = \dfrac{-150}{-2.5} = \boxed{60\,\mathrm{g}}$

How much of A remains: $40 - \frac{2}{3}(60) = \boxed{0\,\mathrm{g}}$, and how much of B remains: $50 - \frac{1}{3}(60) = \boxed{30\,\mathrm{g}}$

We also seek t for which $X(t)$ is equal to half of the limiting amount, i.e. for which $\frac{150-150e^{90kt}}{1-2.5e^{90kt}} = 30$, that is $150 - 150e^{90kt} = 30\big(1 - 2.5e^{90kt}\big) \implies t = \frac{1}{90k} \ln\left(\frac{150-30}{150-75}\right) = \boxed{20.7\,\mathrm{min}}$

Problem 3:

(a) $m\dfrac{dv}{dt} = -mg \implies \dfrac{dv}{dt} = -g \implies v = -gt + c \implies v(t) = v_0 - gt.$

(b) Maximum height is reached when $v = 0$, that is when $v_0 - gt = 0$, therefore $t = \dfrac{v_0}{g}$.

Problem 4:

(a) As usual we form the DE by considering subtracting the amount leaving from the amount incoming:

$$\begin{aligned} & \frac{dv}{dt} = 2(\ell/min) - 3(\ell/min) = -1, \quad v(0) = 100. \\ \implies \quad & v = \int -1\,dt = C - t, \quad v(0) = 100 = C \implies C = 100. \\ \implies \quad & v(t) = 100 - t. \end{aligned}$$

(b) Since the volume of water is no longer constant (as in Lecture 8) our DE is more complicated:

$$\frac{dm}{dt} = 0.1(kg/\ell) \times 2(\ell/min) - \frac{3(\ell/min)m(kg)}{v(\ell)} = 0.2 - \frac{3m}{100 - t}$$

Solve via integrating factor:

$$\begin{aligned} \frac{dm}{dt} + \frac{3}{100 - t}m &= 0.2 \implies f(t) = e^{-2\int \frac{dt}{t-100}} = (t - 100)^{-3} \\ \implies \frac{d}{dt}\left[\frac{m}{(t-100)^3}\right] &= \frac{2}{10}\frac{1}{(t-100)^3} \\ \implies \frac{m}{(t-100)^3} &= \frac{2}{10}\int \frac{dt}{(t-100)^3} = -\frac{1}{10}\frac{dt}{(t-100)^2} + C \end{aligned}$$

$$\implies m = C(t - 100)^3 - \frac{1}{10}(t - 100).$$

Using the initial condition, we find

$$m(0) = 0.2v(0) = 20 = C(-100)^3 - \frac{1}{10}(-100) \implies C = -10^{-5}.$$

Tidying up we find

$$m(t) = (100 - t)\left[\frac{1}{10} + 10^{-5}(100 - t)^2\right].$$

(c) We seek t^* so that $c(t^*) = \frac{m(t^*)}{v(t^*)} = \frac{15}{100}$, therefore

$$\begin{aligned} \frac{15}{100} &= \frac{10}{100} + 10^{-5}(100 - t^*)^2 \\ \implies (100 - t)^2 &= 10^5 \times \frac{5}{100} = \frac{1}{2}10^4 \end{aligned}$$

$$\implies t^* = 100(1 - 1/\sqrt{2}) \approx 29.2 mins.$$

Problem 5:

(a) $y_1 = e^{4x} \implies y_1' = 4e^{4x}, y_1'' = 16e^{4x} \implies y_1'' - y_1' - 12y_1 = (16-4-12)e^{4x} = 0$ ✓
$y_2 = e^{-3x} \implies y_2' = -3e^{-3x}, y_2'' = 9e^{-3x} \implies y_2'' - y_2' - 12y_2 = (9+3-12)e^{-3x} = 0$ ✓

(b) $W(x) = y_1y_2' - y_1'y_2 = e^{4x}(-3e^{-3x}) - (4e^{4x})e^{-3x} = -7e^x < 0 \; \forall -\infty < x < \infty \implies$ fundamental.

(c) General solution: $y(x) = c_1e^{4x} + c_2e^{-3x}$.

(d) $y(0) = 1 = c_1 + c_2 \implies c_2 = 1 - c_1$.
$y'(0) = 0 = 4c_1 - 3c_2 = 4c_1 - 3(1-c_1) = 7c_1 - 3 \implies c_1 = 3/7 \implies c_2 = 1 - 3/7 = 4/7$

$$\boxed{y(x) = \frac{1}{7}(3e^{4x} + 4e^{-3x}).}$$

Problem 6:

(a) $y_p = e^{2x} \implies y_p' = 2e^{2x}, y_p'' = 4e^{2x} \implies y_p'' - y_p' - 12y_p = (4-2-12)e^{2x} = -10e^{2x}$ ✓

(b) General solution: $y(x) = c_1e^{4x} + c_2e^{-3x} + e^{2x}$. $y(0) = 3 = c_1 + c_2 + 1 \implies c_2 = 2 - c_1$.
$y'(0) = 2 = 4c_1 - 3c_2 = 4c_1 - 3(2-c_1) = 7c_1 - 6 \implies c_1 = 8/7 \implies c_2 = 2 - 8/7 = 6/7$

$$\boxed{y(x) = \frac{6}{7}e^{4x} + \frac{8}{7}e^{-3x} + e^{2x}.}$$

Problem 7:

(a) $y_1 = e^{2x} \implies y_1' = 2e^{2x}, y_1'' = 4e^{2x} \implies y_1'' - 4y_1' + 4y_1 = (4-8+4)e^{2x} = 0$ ✓
$y_2 = xe^{2x} \implies y_2' = (1+2x)e^{2x}, y_2'' = 4(1+x)e^{2x} \implies y_2'' - 4y_2' + 4y_2 = (4+4x-4-8x+4x)e^{2x} = 0$ ✓
$W(x) = y_1y_2' - y_1'y_2 = e^{2x}(1+2x)e^{2x}) - (2e^{2x})xe^{2x} = e^{2x} > 0 \; \forall -\infty < x < \infty \implies$ fundamental ✓.

(b) $y_p = e^x \implies y_p' = y_p'' = e^x \implies y_p'' - y_p' - 12y_p = (1-4+4)e^x = e^x$ ✓

(c) General solution: $y(x) = c_1e^{2x} + c_2xe^{2x} + e^x$.
$y(0) = 0 = c_1 + 1 \implies c_1 = -1$.
$y'(0) = 1 = 2c_1 + c2 + 1 = c2 - 1 \implies c_2 = 2$

$$\boxed{y(x) = (2x-1)e^{2x} + e^x.}$$