Applied differential equations

TW244 - Lecture 14

4.1: Intro to higher-order DEs

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4.1: Introduction to higher-order DEs

Introduction

The most general form of an n^{th}-order linear initial value problem:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = \overset{0}{\cancel{g(x)}}$$

subject to

$$y(x_0) = y_0\ ;\ y'(x_0) = y_1\ ;\ \ldots\ ;\ y^{(n-1)}(x_0) = y_{n-1}.$$

n conditions

If $g(x) \equiv 0$ the DE "homogeneous".* Otherwise it is "non-homogeneous".

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = 0.$$

If the a_i are constants, we call the DE a "constant coefficient problem". (Otherwise we say it has "variable coefficients".)

*Pronounced "homo-genius"

4.1a Homogeneous DEs

Priniciple of superposition:
Suppose $y = y_1(x)$ and $y = y_2(x)$ are solutions of a linear homogeous DE, then any linear combination $y = c_1y_1(x) + c_2y_2(x)$ is also a solution.

Proof:

$$a_n(x)\frac{d^n}{dx^n}[c_1y_1 + c_2y_2] + \cdots + a_1(x)\frac{d}{dx}[c_1y_1 + c_2y_2] + a_0(x)[c_1y_1 + c_2y_2]$$

$$= c_1\left[a_n(x)\frac{d^ny_1}{dx^n} + \cdots + a_1(x)\frac{dy_1}{dx} + a_0(x)y_1\right] + c_2\left[a_n(x)\frac{d^ny_2}{dx^n} + \cdots + a_1(x)\frac{dy_2}{dx} + a_0(x)y_2\right]$$

$$= c_1 \cdot [0] + c_2 \cdot [0] = 0.$$

Exercise: Verify that $y_1 = \sin(4x)$ and $y_2 = \cos(4x)$ are solutions to $y'' + 16y = 0$, and that so is, for example, $y(x) = \sqrt{2}y_1 + \pi y_2$.

Suppose y_1 and y_2 are solutions of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy(x)=0$$

on an interval I.

$a = a(x)$
$b = b(x)$
$c = c(x)$

$y_1 = 2y_2$
$W(x) = \begin{vmatrix} y_1 & 2y_1 \\ y_1' & 2y_1' \end{vmatrix} = 2y_1y_1' - 2y_1y_1' = 0$

The Wronskian is defined as:

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x)-y_2(x)y_1'(x)$$

and it turns out that exactly one of the following always holds:

- $W(x)=0$ for all $x\in I$ (i.e., everywhere zero)
- $W(x)\neq 0$ for all $x\in I$ (i.e., nowhere zero)

If $W(x)\neq 0$ (i.e., case 2) then y_1 and y_2 are called fundamental solutions on the interval I and we say that they are linearly independent.

The general solution

If $y_1(x)$ and $y_2(x)$ are fundamental solutions of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy(x) = 0,$$

then any and every solution of this DE may be written uniquely as a linear combination of the two fundamental solutions!

This allows us to form the general family of solutions as

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

The constants c_1 and c_2 are determined by the initial conditions.

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Example (#7 & #8, p. 124)

Consider the DE $y'' - 9y = 0$ and the functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$. Let us first verify that y_1 and y_2 are solutions:

$$y_1' = 3e^{3x}, \quad y_1'' = 9e^{3x} \implies y_1'' - 9y_1 = 0.$$
$$y_2' = -3e^{-3x}, \quad y_2'' = 9e^{-3x} \implies y_2'' - 9y_2 = 0.$$

Therefore, both are solutions... but are they fundamental?

Wronskian:

$$W(x) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3 - 3 = -6 \neq 0 \;\forall\; -\infty < x < \infty$$

therefore y_1 and y_2 are fundamental solutions and

$$y(x) = c_1 e^{3x} + c_2 e^{-3x}$$

is the general solution to the DE $y'' - 9y = 0$.

Note: The Wronskian need not be constant!

Consider the DE $y'' + 4y' + 3y = 0$ and $y_1(x) = e^{-3x}$ and $y_2(x) = e^{-x}$.

Let us first verify that y_1 and y_2 are solutions:

Exercise/
Oefening

✓

Therefore, both are solutions... but are they fundamental?

Wronskian:

$$W(x) = \begin{vmatrix} e^{-3x} & e^{-x} \\ -3e^{-3x} & -e^{-x} \end{vmatrix} = -e^{-3x}e^{-x} + 3e^{-x}e^{-3x} = 2e^{-4x} \neq 0 \, \forall -\infty < x < \infty$$

$(-\infty, \infty)$

therefore y_1 and y_2 are fundamental solutions and

$$y(x) = c_1 e^{-3x} + c_2 e^{-x}$$

is the general solution to the DE $y'' + 4y' + 3 = 0$.

Example (cont.)

Suppose now we wish to solve the IVP

$$y'' + 4y' + 3y = 0, \quad y(0) = 0, \;\; y'(0) = 1.$$

We know from the previous slide that the general solution to the DE is

$$\begin{aligned} y(x) &= c_1 e^{-3x} + c_2 e^{-x}, \\ \Longrightarrow \; y'(x) &= -3c_1 e^{-3x} - c_2 e^{-x}. \end{aligned}$$

From the initial conditions we have

$$\left.\begin{aligned} y(0) &= 0 &= c_1 + c_2, \\ y'(0) &= 1 &= -3c_1 - c_2 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c_2 &= -\tfrac{1}{2}, \\ c_1 &= \tfrac{1}{2}. \end{aligned}$$

Therfore we have

$$\boxed{y(x) = \frac{1}{2}\left(e^{-x} - e^{-3x}\right)}$$

4.1b Non-homogeneous DEs

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Non-homogeneous DEs

Recall the homogeneous and non-homogeneous forms of our linear DE:

$$ay'' + by' + cy = 0, \quad (1)$$
$$ay'' + by' + cy = g(x). \quad (2)$$

The general solution of the inhomogeneous problem (2) can be written as

$$y(x) = y_c(x) + y_p(x),$$

where

- $y_c(x)$ is the general solution to the homogeneous problem (1)
- $y_p(x)$ is <u>any</u> particular solution† of (2).

The function $y_c(x)$ is often called the "complementary solution".‡

†i.e., free from parameters.

‡Not to be confused with "complimentary"!

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Non-homogeneous DEs cont.

Hence, the procedure to solve $ay'' + by' + cy = g(x)$ is

- Find fundamental solutions y_1 and y_2 of $ay'' + by' + cy = 0$;
- Find any particular solution y_p of $ay'' + by' + cy = g(x)$;
- Form the general solution $y(x) = c_1 y_1(x) + c_2 y_2 + y_p(x)$;

OK, so step 3 is easy... but how exactly do we do steps 1 and 2?!

See the next two lectures!

Example: $y'' - 9y = x$.

We saw that e^{3x} and e^{-3x} are fundamental solutions of $y'' - 9y = 0$.

We can easily verify that $y(x) = -\frac{x}{9}$ is a solution to $y'' - 9y = x$, therefore

$$y(x) = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9}$$

is the general solution to $y'' - 9y = x$.