Applied differential equations

TW244 - Lecture 31

10.3 Stability of nonlinear systems

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We cannot easily write down a general explicit solution for a nonlinear autonomous system, which makes the classification of critical solutions more complicated (but also more important!)

To proceed we first linearise the system, i.e., we want to approximate

$$\left.\begin{aligned} \frac{dx}{dt} &= \underline{P}(x,y) \\ \frac{dy}{dt} &= \underline{Q}(x,y) \end{aligned}\right\} \text{nonlinear} \quad \text{by} \quad \left.\begin{aligned} \frac{d\hat{x}}{dt} &\approx a\hat{x} + b\hat{y} \\ \frac{d\hat{y}}{dt} &\approx c\hat{x} + d\hat{y} \end{aligned}\right\} \text{linear}$$

$x - x^*$

$y - y^*$

Recall: Critical point (x_*, y_*)
$\Rightarrow P(x_*, y_*) = Q(x_*, y_*) = 0$

Consider $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$ with a critical point at $(x,y) = (x_*, y_*)$.

We expand $P(x,y)$ and $Q(x,y)$ as a Taylor series about the point (x_*, y_*):*

$$\begin{aligned}
\frac{dx}{dt} &= P(x_*, y_*) + \underbrace{(x - x_*)}_{h_x} \left.\frac{\partial P}{\partial x}\right|_{x_*,y_*} + \underbrace{(y - y_*)}_{h_y} \left.\frac{\partial P}{\partial y}\right|_{x_*,y_*} + \cdots \\
\frac{dy}{dt} &= \underbrace{Q(x_*, y_*)}_{=0} + (x - x_*) \left.\frac{\partial Q}{\partial x}\right|_{x_*,y_*} + (y - y_*) \left.\frac{\partial Q}{\partial y}\right|_{x_*,y_*} + \underbrace{\cdots}_{\text{negligible}}.
\end{aligned}$$

i.e., we have that (approximately)

$$\begin{aligned}
\frac{dx}{dt} &= (x - x_*) \left.\frac{\partial P}{\partial x}\right|_{x_*,y_*} + (y - y_*) \left.\frac{\partial P}{\partial y}\right|_{x_*,y_*} \\
\frac{dy}{dt} &= (x - x_*) \left.\frac{\partial Q}{\partial x}\right|_{x_*,y_*} + (y - y_*) \left.\frac{\partial Q}{\partial y}\right|_{x_*,y_*}.
\end{aligned}$$

$f(x+h_x, y+h_y) \to P(x,y)$
$f(x,y) \to P(x_*, y_*)$

*Recall that: $f(x+h_x, y+h_y) = f(x,y) + h_x \frac{\partial f}{\partial x}|_{x,y} + h_y \frac{\partial f}{\partial y}|_{x,y} + O(h_x^2 + h_y^2 + h_x h_y)$.

$$\frac{dx}{dt} \approx (x - x_*) \left.\frac{\partial P}{\partial x}\right|_{x_*,y_*} + (y - y_*) \left.\frac{\partial P}{\partial y}\right|_{x_*,y_*}$$
$$\frac{dy}{dt} \approx (x - x_*) \left.\frac{\partial Q}{\partial x}\right|_{x_*,y_*} + (y - y_*) \left.\frac{\partial Q}{\partial y}\right|_{x_*,y_*}.$$

Finally, if we denote $\hat{x} = x - x_*$ and $\hat{y} = y - y_*$, then we have†

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}}_{\text{evaluated at } (x_*, y_*)} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}.$$

We call this matrix the Jacobian, and note that the system above is linear!

The critical point (x_*, y_*) in the original system corresponds to $(\hat{x}, \hat{y}) = (0, 0)$.

$\therefore$ we can use our (Δ, τ) graph to classify the critical point!

†Observe that $\frac{d\hat{x}}{dt} = \frac{dx}{dt}$ and $\frac{d\hat{y}}{dt} = \frac{dy}{dt}$.

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Example

$$\left.\frac{dy}{dt}\right|_{(2+\varepsilon,3)} = 3(-2+2+\varepsilon) = 3\varepsilon > 0$$

Example: Classify the critical solutions of‡

$$\frac{dx}{dt} = 5x - x^2 - xy = x(5 - x - y)$$
$$\frac{dy}{dt} = -2y + xy = y(-2 + x).$$

Critical solutions: $\frac{dx}{dt} = \frac{dy}{dt} = 0 \implies x(5 - x - y) = 0$ and $y(x - 2) = 0$, therefore three critical solutions $(x, y) = (0, 0),\ (5, 0),\ (2, 3)$.

The Jacobian of this system is

$$J = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} 5 - 2x - y & -x \\ y & x - 2 \end{bmatrix}$$

Now we evaluate the Jacobian at each critical point to classify it...

‡Note: This DE system arises from a logistic predator-prey model.

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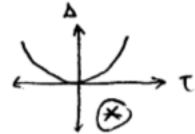
Example

$$J = \begin{bmatrix} 5-2x-y & -x \\ y & x-2 \end{bmatrix}$$

$\boxed{(x, y) = (0, 0)}$

$$J = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \implies \begin{cases} \tau = \text{trace}(J) = 3 \\ \Delta = \det(J) = -10 \end{cases}$$

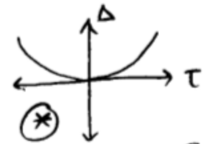


∴ SADDLE

$\lambda_1 = 5, \underline{v}_1 = [1, 0]^\top, \lambda_2 = -2, \underline{v}_2 = [0, 1]^\top$

$\boxed{(x, y) = (5, 0)}$

$$J = \begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix} \implies \begin{cases} \tau = \text{trace}(J) = -2 \\ \Delta = \det(J) = -15 \end{cases}$$

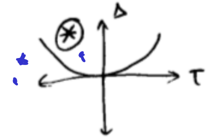


∴ SADDLE

$\lambda_1 = -5, \underline{v}_1 = [1, 0]^\top, \lambda_2 = 3, \underline{v}_2 = [5, -8]^\top$

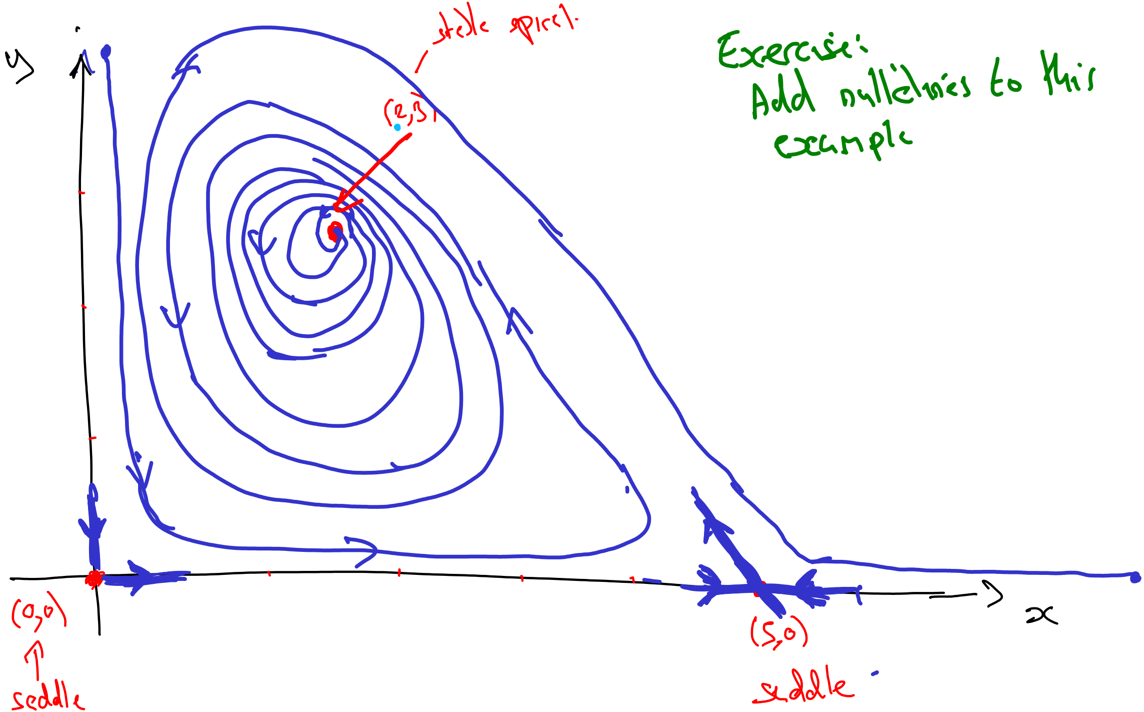
$\boxed{(x, y) = (2, 3)}$

$$J = \begin{bmatrix} -2 & -2 \\ 3 & 0 \end{bmatrix} \implies \begin{cases} \tau = \text{trace}(J) = -2 \\ \Delta = \det(J) = 6 \end{cases}$$



∴ STABLE
SPIRAL

Exercise:
Add nullclines to this example



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Direction of rotation

In the example above we might infer that the stable spiral rotated anticlockwise by looking at the direction of the saddle points.

When the direction of rotation is unclear, one can use the same trick as we saw in the linear case (although here it is important that that we take ε as small):

- Consider a small perturbation in the x direction of the critical point corresponding to a stable/unstable spiral or centre, i.e., $(x_1 + \varepsilon, y_1)$.
- Substitute this point in to the expression for $\frac{dy}{dt}$.
- If $\left.\frac{dy}{dt}\right|_{x_1+\varepsilon,y_1}$ is positive (negative), the rotation is anticlockwise (clockwise)

Example: Consider $(x, y) = (2 + \varepsilon, 3)$ and $\frac{dy}{dx} = -2y + xy$.

- Then $\left.\frac{dy}{dx}\right|_{2+\varepsilon,3} = -2(3) + (2 + \varepsilon)3 = 3\varepsilon > 0$
- Since this is positive, the spiral rotates anticlockwise.

(Alternatively one can substitute $(x_1, y_1 + \varepsilon)$ into $\frac{dx}{dt}$. Exercise: Try this on the example above.)

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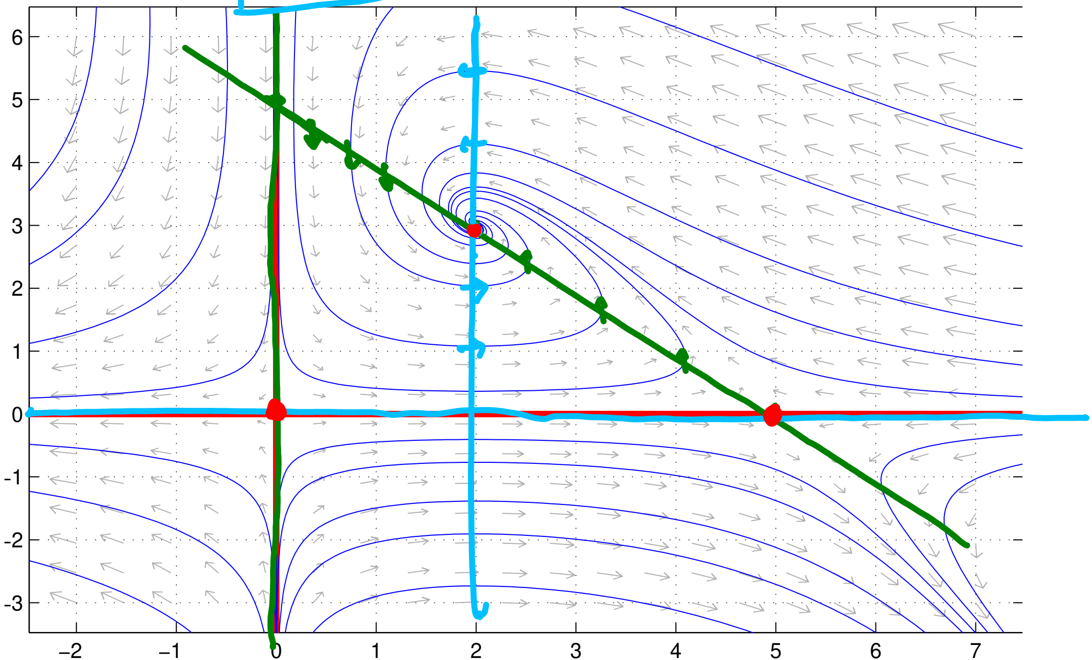
Example: Plot using pplane8



As in the case of linear planar DEs, considering nullclines (i.e., curves where $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$) can tell us more about the behaviour of solutions.

In particular, it they help us connect the local linear approximations about critical points we made above (recall that critical points are the intersections of two or more nullclines!)

In our example above we had $\frac{dx}{dt} = x(5 - x - y)$ and $\frac{dy}{dt} = y(-2 + x)$, hence the nullclines are

$$\frac{dx}{dt} = 0 \quad \text{when} \begin{cases} x = 0 \\ y = 5 - x \end{cases} \implies \quad \text{curve is vertical}$$

and

$$\frac{dy}{dt} = 0 \quad \text{when} \begin{cases} y = 0 \\ x = 2 \end{cases} \implies \quad \text{curve is flat}$$

Let's add this information to our phase diagram...

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Example: Plot using `pplane8`