Applied differential equations

TW244 - Lecture 32

10.4 Nonlinear models

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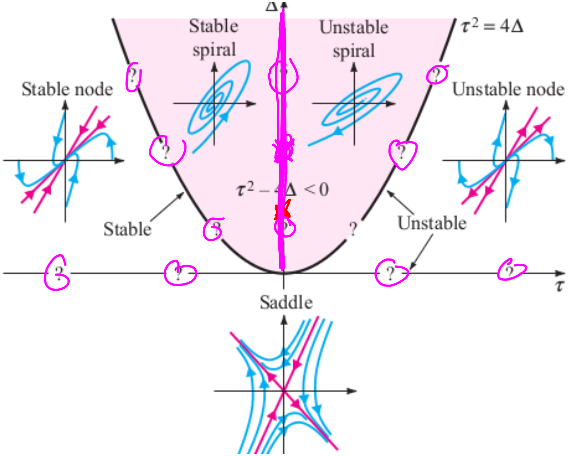
To sketch phase plane of nonlinear planar autonomous DES:

1. Locate critical points
2. Compute Jacobian matrix, $J = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix}$
3. Classify critical points by substituting to J and checking (τ, Δ) diagram – see next page.
4. Sketch solutions at critical points. (using evals and evecs)
5. Sketch solutions between critical points (using nullclines to help)

$$dx/dt = P(x,y)$$
$$dy/dt = Q(x,y)$$

Borderline cases

Important to note: For nonlinear systems the method of linearisation cannot concretely classify criticial solutions in the borderline cases! Compare the figure above with the one we saw in Lecture 30.



Example: Consider the nonlinear spring:

$$x'' = -2x - 3x^3.$$

We may rewrite this as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x(2 + 3x^2).$$

We see immediately that the only critical point is at $(0, 0)$ and

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ -2 - 9x^2 & 0 \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \implies \Delta = 2, \tau = 0.$$

This is a borderline case and the solution may be a stable spiral, an unstable spiral, or a centre! How can we decide?

Phase-plane method

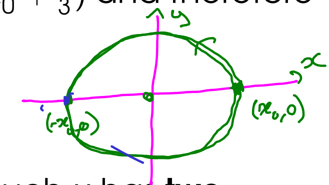
Use for borderline cases where may be a centre.

The phase-plane method is based on the fact that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)}$.
Solving for $y(x)$ sometimes tells us the nature of borderline critical points.
Returning to our example, we find

$$\frac{dy}{dx} = -\frac{x(2+3x^2)}{y} \quad \overbrace{\Longrightarrow}^{\text{tutorial 7}} \quad y^2 = -x^2\tfrac{3}{2}(x^2+\tfrac{4}{3}) + c^2$$

Choosing an initial condition $(x_0, 0)$ we find $c^2 = \frac{3}{2}x_0^2(x_0^2 + \frac{4}{3})$ and therefore*

$$y^2 = \tfrac{3}{2}(\tfrac{4}{3} + x_0^2 + x^2)(x_0^2 - x^2)$$



Noting that

- $y(-x_0) = y(x_0) = 0$
- For any $|x| < |x_0|$ the RHS is positive, hence each such x has **two** corresponding values of y

it follows the solution forms a closed curve and (0,0) is therefore a **centre**.

*complete the square

$$y^2 = -x^2\tfrac{3}{2}(x^2+\tfrac{4}{3}) + c^2,\ c^2 = \tfrac{3}{2}x_0^2(x_0^2+\tfrac{4}{3})$$

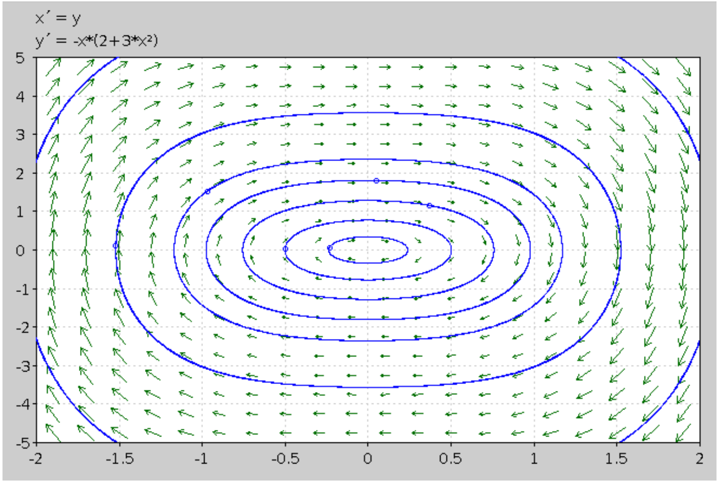
$$y^2 = -x^2\, 3/2\,(x^2+4/3) + 3/2\, x_0^2(x_0^2+4/3)$$

$$= 3/2\left[x_0^2(x_0^2+4/3) - x^2(x^2+4/3)\right] = 3/2\left[x_0^4 + 4/3\, x_0^2 - 4/3\, x^2 - x^4\right]$$

$$= 3/2\left[(x_0^2-x^2)(x_0^2+x_0^2+4/3)\right]$$

Phase-plane method

Confirm with `pplane`:



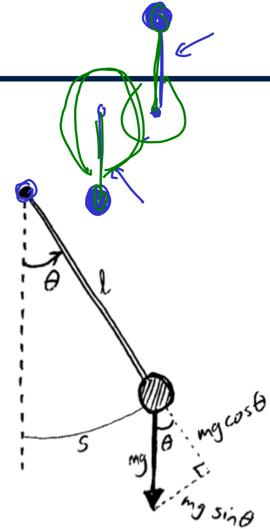
For more examples of using the Phase Plane method see p. 411-412 in Z&W.

NONLINEAR PENDULUM

10.4: Nonlinear models
Nonlinear pendulum

Recall the nonlinear pendulum from Lecture 26:

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta \quad : \quad \omega^2 = \frac{g}{\ell}$$



Let $x = \theta$ and $y = \frac{d\theta}{dt}$ then we may model the nonlinear pendulum as the plane autonomous system

$$\boxed{\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\omega^2 \sin x. \end{aligned}}$$

Critical solutions: $\frac{dx}{dt} = \frac{dy}{dt} = 0$

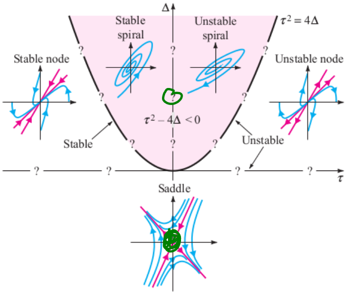
$\implies y = 0 \;\&\; \sin(x) = 0 \implies x = 0, \pm\pi, \pm 2\pi, \ldots \implies (x, y) = (k\pi, 0) \;\forall\, k \in \mathbb{Z}.$

(1) What positions do these correspond to? (2) Can we classify them?

The system is nonlinear, so we need the Jacobian:

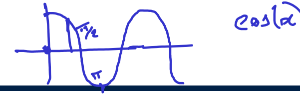
$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos(x) & 0 \end{bmatrix} \text{ and we have } J(k\pi,0) = \begin{bmatrix} 0 & 1 \\ \omega^2(-1)^{k+1} & 0 \end{bmatrix} \text{ with}$$

- $\tau = \text{trace}(J) = 0$
- $\Delta = \det(J) = \begin{cases} -\omega^2 & : k \text{ odd} \implies \text{saddle} \\ \omega^2 & : k \text{ even} \implies \text{centre?} \end{cases}$
- $\lambda = \text{eig}(J) = \begin{cases} \pm\omega & : k \text{ odd} \\ \pm i\omega & : k \text{ even} \end{cases}$

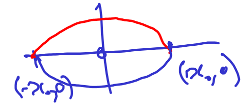


When k is odd we get saddle solutions, but k even gives borderline cases.

Let's use the phase-plane method to determine if $(\pm 2k\pi, 0)$ are centres.



Recall: The phase-plane method attempts to relate x and y by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$ and eliminating t. In this case



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\omega^2 \frac{\sin(x)}{y}.$$

We can solve via separation of variables to find $y^2 = 2\omega^2 \cos(x) + c$ and if we choose an initial point on the x axis, i.e., $(x_0, 0)$, then we have

$$y^2 = 2\omega^2[\cos(x) - \underline{\cos(x_0)}].$$

Similar to nonlinear spring in the previous lecture, we can now show that

- $y(-x_0) = y(x_0) = 0$ ✓
- For any $|x| < |x_0| < \pi$ the RHS is positive, hence each such x has **two** corresponding values of y

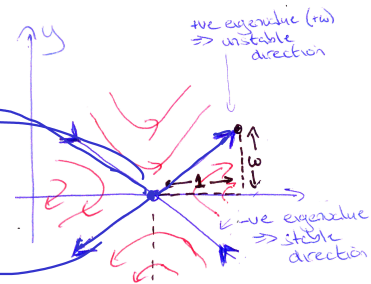
it follows the solution forms a closed curve and (0,0) is therefore a **centre**.

We can apply this same idea to each critical point $(\pm 2k\pi, 0), k = 1, 2, \ldots$.

$J = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix}$

We are almost ready to sketch our phase diagram, but let's first find the direction of the saddles at $(\pm[2k+1]\pi, 0)$. The eigenvalues of $+\omega$ and $-\omega$ have corresponding eigenvectors $[1, \omega]^\top$ and $[1, -\omega]^\top$, respectively. The figure on the right therefore shows the behaviour of solutions in the neighbourhood of these critical points.

$-\pi$



Recalling that for initial condition $(x_0, 0)$ we had $y^2 = 2\omega^2[\cos(x) - \cos(x_0)]$. We can substitute $x_0 = \pi$ to find the equation

$$y^2 = 2\omega^2(\cos(x) - (-1)) = 4\omega^2\cos^2(x/2) \implies y = \pm 2\omega\cos(x/2)$$

which connects each of the saddle points.

Let's combine this with the centres at $(k\pi, 0)$ to sketch the full figure.

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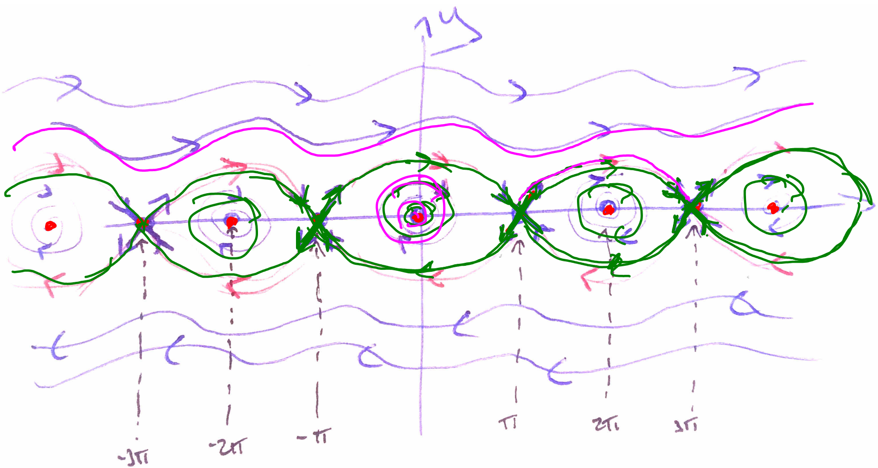

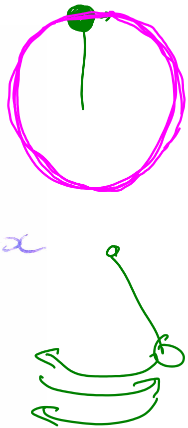

We can observe two kinds of motion, determined by the initial conditions.

- periodic oscillation 

- whirling about the pivot 

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