Week 7

20.8 The M/G/1/GD/∞/∞ Queuing System:

- single-server queuing system
- interarrival times are exponential
- not a birth–death process

$\frac{1}{\mu} = E(\mathbf{S})$

$\sigma^2 =$ ($\frac{1}{\mu}$)^2 = S = service time distribution

$L = L_q + \rho$ ---> $L = \frac{\lambda}{\mu - \lambda}$

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1 - \rho)}$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

π_0 , the fraction of the time that the server is idle, is 1-ρ

20.10 Exponential Queues in Series and Open Queuing Networks :

Exponential Queues in Series:

- The customer's service is not complete until the customer has been served by more than one server (**k-stage series queuing system**).

THEOREM 4

If (1) interarrival times for a series queuing system are exponential with rate λ, (2) service times for each stage i server are exponential, and (3) each stage has an infinite-capacity waiting room, then interarrival times for arrivals to each stage of the queuing system are exponential with rate λ.

- For this system we need to make sure $\lambda < s_j * \mu_j$ otherwise the system will blow up.

EXAMPLE 13 Auto Assembly

The last two things that are done to a car before its manufacture is complete are installing the engine and putting on the tires. An average of 54 cars per hour arrive requiring these two tasks. One worker is available to install the engine and can service an average of 60 cars per hour. After the engine is installed, the car goes to the tire station and waits for its tires to be attached. Three workers serve at the tire station. Each works on one car at a time and can put tires on a car in an average of 3 minutes. Both interarrival times and service times are exponential.

1 Determine the mean queue length at each work station.

2 Determine the total expected time that a car spends waiting for service.

Solution This is a series queuing system with $\lambda = 54$ cars per hour, $s_1 = 1$, $\mu_1 = 60$ cars per hour, $s_2 = 3$, and $\mu_2 = 20$ cars per hour (see Figure 25). Since $\lambda < \mu_1$ and $\lambda < 3\mu_2$, neither queue will "blow up," and Jackson's theorem is applicable. For stage 1 (engine), $\rho = \frac{54}{60} = .90$. Then (27) yields

$$L_q \text{ (for engine)} = \left(\frac{\rho^2}{1-\rho}\right) = \left[\frac{(.90)^2}{1-.90}\right] = 8.1 \text{ cars}$$

Now (32) yields

$$W_q \text{ (for engine)} = \frac{L_q}{\lambda} = \frac{8.1}{54} = 0.15 \text{ hour}$$

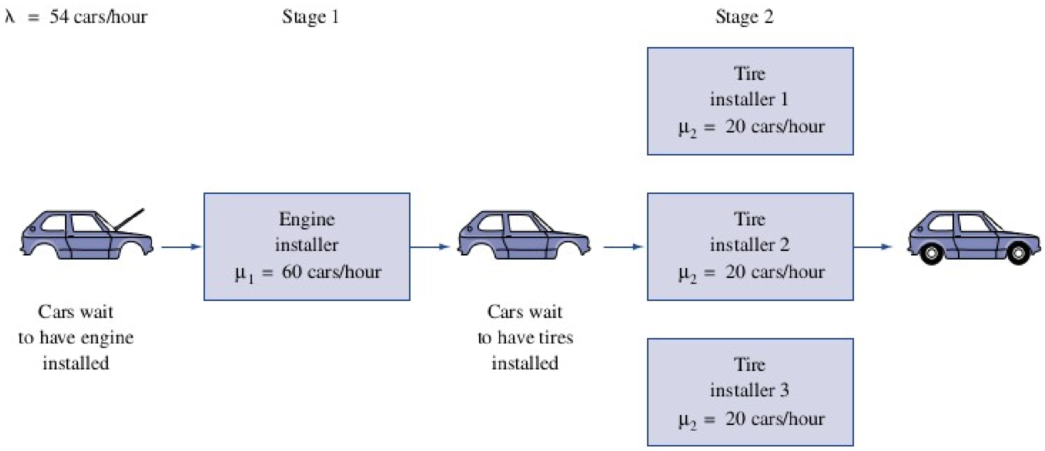


FIGURE 25
Series Queuing System for Automobile

For stage 2 (tires), $\rho = \frac{54}{3(20)} = .90$. Table 6 yields $P(j \geq 3) = .83$. Now (41) yields

$$L_q \text{ (for tires)} = \frac{.83(.90)}{1 - .90} = 7.47 \text{ cars}$$

Then

$$W_q \text{ (for tires)} = \frac{L_q}{\lambda} = \frac{7.47}{54} = 0.138 \text{ hour}$$

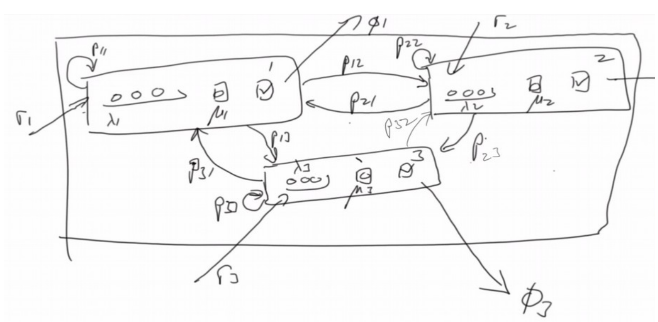
Thus, the total expected time a car spends waiting for engine installation and tires is $0.15 + 0.138 = 0.288$ hour.

Open Queuing Networks:

- a generalization of queues in series.(has more than one station)
- If $s_j * \mu_j > \lambda_j$ holds then system can be treated as M/M/1/GD/∞/∞ (j is the station).

Each station has :

- λ_j : the rate at which customers arrive at station j.
- μ_j : operating at rate at station j.
- r_j : arrival rate at station j from outside the queuing system.
- p_{ij} : probability of arrival at station j from i.
- θ_i : probability that consumer completes service at i



Formulas :

$$\lambda_j = r_j + \sum_{i=1}^{k} p_{ij}\lambda_i.$$

$$\phi_i = 1 - \sum_{j=1}^{k} p_{ij}$$

Since if $s_j * \mu_j > \lambda_j$ holds then system can be treated as M/M/1/GD/∞/∞ (j is the station). So we have the formulas of :

$$L = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

$$L_q = \frac{}{1-\rho} - \rho = \frac{}{1-\rho} = \frac{}{\mu(\mu-\lambda)}$$

$$L_s = 0\pi_0 + 1(\pi_1 + \pi_2 + \cdots) = 1 - \pi_0 = 1 - (1-\rho) = \rho$$

$\lambda = r_1 + r_2 + \cdots + r_k,$ Where k is the # of entries into the system

THEOREM 3

For *any* queuing system in which a steady-state distribution exists, the following relations hold:

$$L = \lambda W \quad (28)$$

$$L_q = \lambda W_q \quad (29)$$

$$L_s = \lambda W_s \quad (30)$$

EXAMPLE 14 Open Queuing Network Example

Consider two servers. An average of 8 customers per hour arrive from outside at server 1, and an average of 17 customers per hour arrive from outside at server 2. Interarrival times are exponential. Server 1 can serve at an exponential rate of 20 customers per hour, and server 2 can serve at an exponential rate of 30 customers per hour. After completing service at server 1, half of the customers leave the system, and half go to server 2. After completing service at server 2, $\frac{3}{4}$ of the customers complete service, and $\frac{1}{4}$ return to server 1.

1 What fraction of the time is server 1 idle?

2 Find the expected number of customers at each server.

3 Find the average time a customer spends in the system.

4 How would the answers to parts (1)–(3) change if server 2 could serve only an average of 20 customers per hour?

Solution We have an open queuing network with $r_1 = 8$ customers/hour and $r_2 = 17$ customers/hour. Also, $p_{12} = .5$, $p_{21} = .25$, and $p_{11} = p_{22} = 0$. We can find λ_1 and λ_2 by solving $\lambda_1 = 8 + .25\lambda_2$ and $\lambda_2 = 17 + .5\lambda_1$. This yields $\lambda_1 = 14$ customers/hour and $\lambda_2 = 24$ customers/hour.

1 Server 1 may be treated as an $M/M/1/GD/\infty/\infty$ system with $\lambda = 14$ customers/hour and $\mu = 20$ customers/hour. Then $\pi_0 = 1 - \rho = 1 - .7 = .3$. Thus, server 1 is idle 30% of the time.

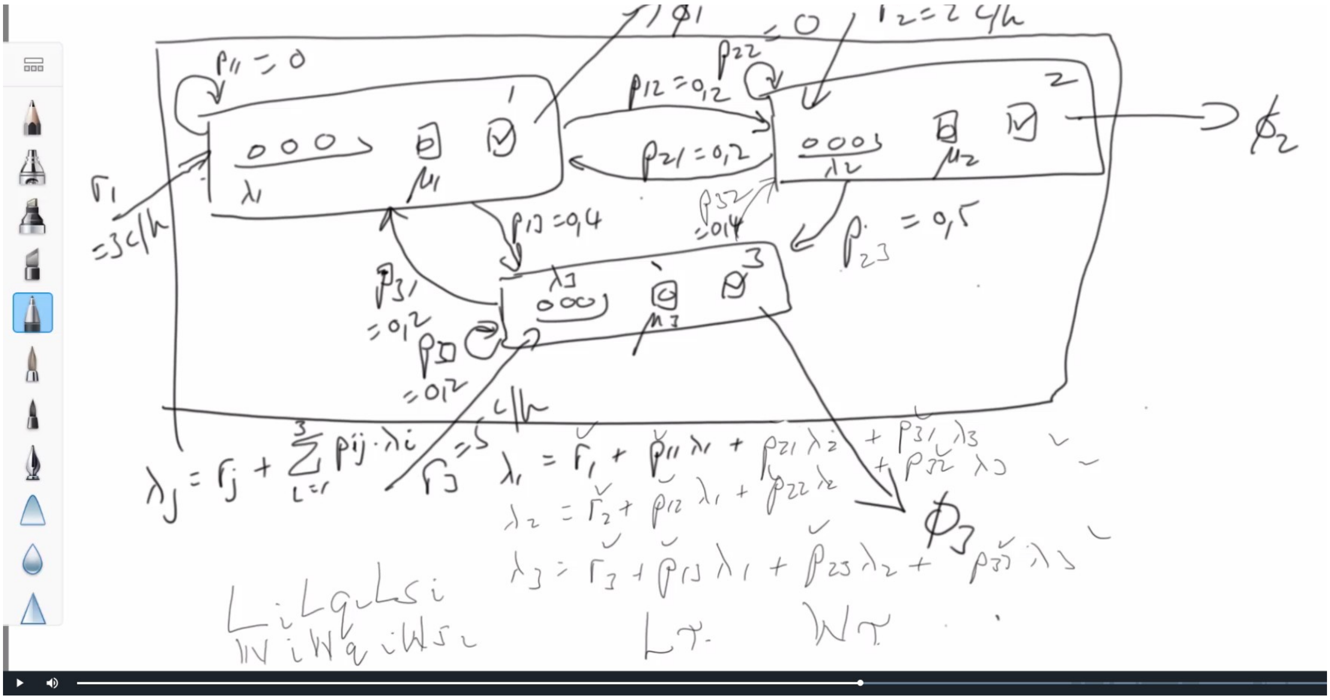
2 From (26), we find L at server $1 = \frac{14}{20-14} = \frac{7}{3}$ and L at server $2 = \frac{24}{30-24} = 4$. Thus, an average of $4 + \frac{7}{3} = \frac{19}{3}$ customers will be present in the system.

3 $W = \frac{L}{\lambda}$, where $\lambda = 8 + 17 = 25$ customers/hour. Thus,

$$W = \frac{\left(\frac{19}{3}\right)}{25} = \frac{19}{75} \text{ hour}$$

4 In this case, $s_2\mu_2 = 20 < \lambda_2$, so no steady state exists.

LINGO



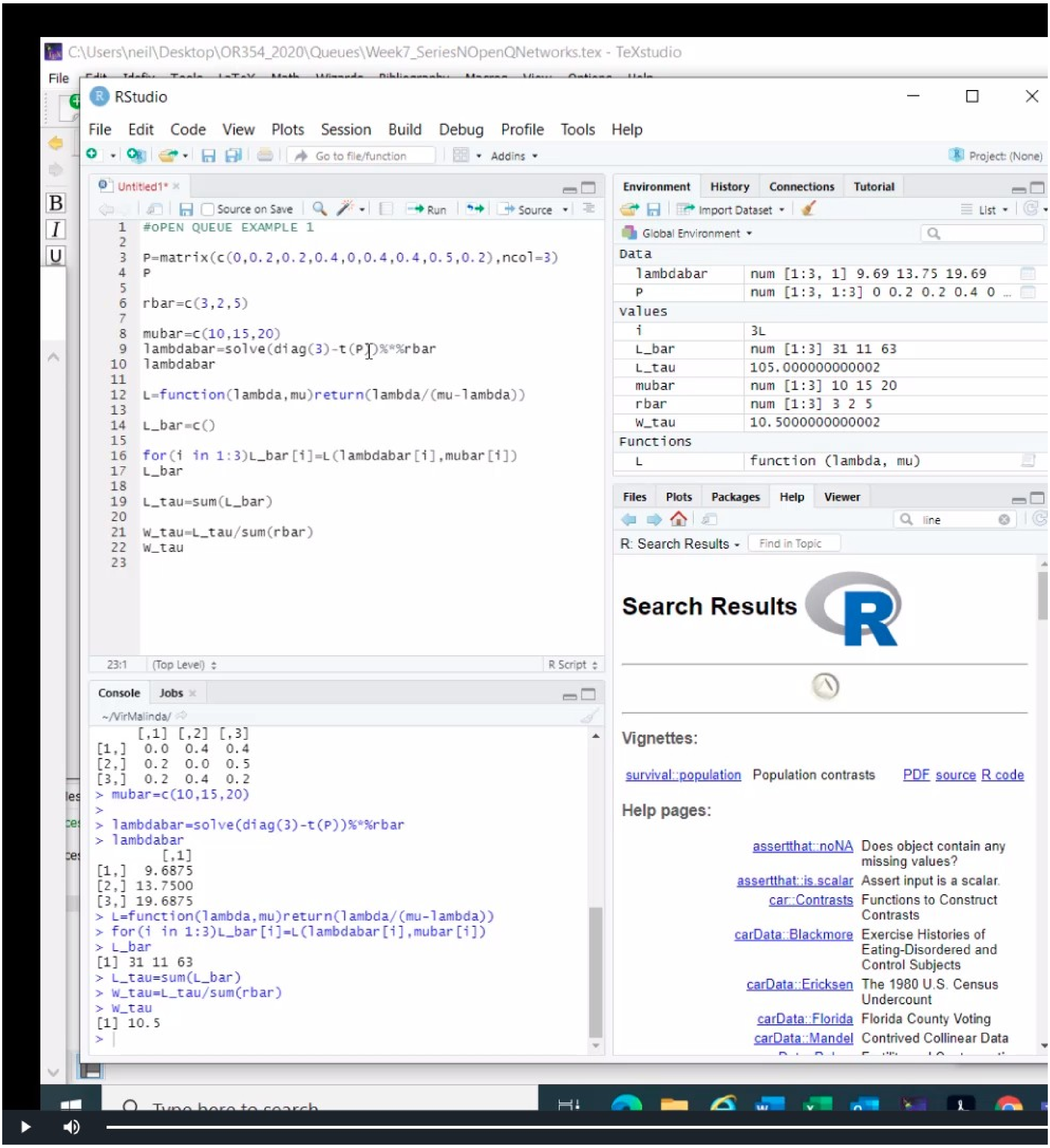
For the above example he uses the following way of calculation in Lingo (it is not needed for calculating using the formulas, but is faster when using lingo

$$\lambda_1 = r_1 + p_{11}\lambda_1 + p_{21}\lambda_2 + p_{31}\lambda_3$$
$$\lambda_2 = r_2 + p_{12}\lambda_1 + p_{22}\lambda_2 + p_{32}\lambda_3$$
$$\lambda_3 = r_3 + p_{13}\lambda_1 + p_{23}\lambda_2 + p_{33}\lambda_3$$

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

$$\bar{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \quad \bar{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\bar{\lambda} = \bar{r} + P^T \bar{\lambda}$$
$$\bar{\lambda} - P^T \bar{\lambda} = \bar{r}$$
$$[I - P^T]\,\bar{\lambda} = \bar{r}$$
$$\bar{\lambda} = [I - P^T]^{-1} \cdot \bar{r}$$


C:\Users\neil\Desktop\OR354_2020\Queues\Week7_SeriesNOpenQNetworks.tex - TeXstudio
RStudio
File Edit Code View Plots Session Build Debug Profile Tools Help
Untitled1*
Source on Save
Run
Source
#OPEN QUEUE EXAMPLE 1
P=matrix(c(0,0.2,0.2,0.4,0,0.4,0.4,0.5,0.2),ncol=3)
P
rbar=c(3,2,5)
mubar=c(10,15,20)
lambdabar=solve(diag(3)-t(P))%*%rbar
lambdabar
L=function(lambda,mu)return(lambda/(mu-lambda))
L_bar=c()
for(i in 1:3)L_bar[i]=L(lambdabar[i],mubar[i])
L_bar
L_tau=sum(L_bar)
W_tau=L_tau/sum(rbar)
W_tau
Console Jobs
~/VirMalinda/
[,1] [,2] [,3]
[1,] 0.0 0.4 0.4
[2,] 0.2 0.0 0.5
[3,] 0.2 0.4 0.2
> mubar=c(10,15,20)
>
> lambdabar=solve(diag(3)-t(P))%*%rbar
> lambdabar
[,1]
[1,] 9.6875
[2,] 13.7500
[3,] 19.6875
> L=function(lambda,mu)return(lambda/(mu-lambda))
> for(i in 1:3)L_bar[i]=L(lambdabar[i],mubar[i])
> L_bar
[1] 31 11 63
> L_tau=sum(L_bar)
> W_tau=L_tau/sum(rbar)
> W_tau
[1] 10.5
Environment History Connections Tutorial
Import Dataset
Global Environment
Data
lambdabar num [1:3, 1] 9.69 13.75 19.69
P num [1:3, 1:3] 0 0.2 0.2 0.4 0 ...
Values
i 3L
L_bar num [1:3] 31 11 63
L_tau 105.000000000002
mubar num [1:3] 10 15 20
rbar num [1:3] 3 2 5
W_tau 10.5000000000002
Functions
L function (lambda, mu)
Files Plots Packages Help Viewer
R: Search Results
Search Results
Vignettes:
survival::population Population contrasts PDF source R code
Help pages:
assertthat::noNA Does object contain any missing values?
assertthat::is.scalar Assert input is a scalar.
car::Contrasts Functions to Construct Contrasts
carData::Blackmore Exercise Histories of Eating-Disordered and Control Subjects
carData::Ericksen The 1980 U.S. Census Undercount
carData::Florida Florida County Voting
carData::Mandel Contrived Collinear Data