Applied differential equations

TW244 - Lecture 02

Introduction:
Definitions & terminology, IVPs

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1.1: Definitions and terminology

$y(x)$ ← indep.
dep.

Differential equation (DE): An equation containing the derivatives of one or more unknown functions with respect to one or more independent variables. Examples:

$$\frac{dy}{dx} = x\sqrt{y}, \quad y'' - 2y' + y = 0, \quad \ddot{x} + \sin(\pi x) = \cos(t).$$

Classification of DEs

Type: Ordinary (ODE): 1 independent variable, e.g., $\frac{dy}{dx} = x - y$.
Partial (PDE): 2 or more independent variables, e.g., $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x - y$.

$u(x,y)$

Order: The order of a DE is the order of the highest appearing derivative.

Linearity: A DE is linear if the dep. variable and its derivatives are linear,

$$\text{e.g., } \frac{dy}{dx} = x - y, \quad \frac{dy}{dx} = x^2 - y, \quad \text{or} \quad \frac{d^2y}{dx^2} + xy = 0.$$

Otherwise it is nonlinear,

$$\text{e.g., } \frac{dy}{dx} = x - y^2, \quad \text{or} \quad (1 - y)\frac{dy}{dx} + 2y = e^x + x.$$

Solution of a DE: Any function with *n* continuous derivatives that satisfies an *n*th-order DE on some interval *I* is said to be "a solution of the DE on *I*".

Example:

DE: $\frac{dy}{dx} = x - y$

solution: $y = x - 1 + ce^{-x}$, $\quad I = (-\infty, \infty)$ and $c \in \mathbb{R}$

verify: $\frac{dy}{dx} = 1 - ce^{-x} = x - (x - 1 + ce^{-x}) = x - y$ ✓

Example:

DE: $x\frac{dy}{dx} + y = 0$

solution: $y = \frac{c}{x}$, $\quad I = (-\infty, 0) \cup (0, \infty)$ (NB, y is not cts at $x = 0$)

verify: $x\frac{dy}{dx} + y = x(-\frac{c}{x^2}) + \frac{c}{x} = -\frac{c}{x} + \frac{c}{x} = 0.$ ✓

Solution curve: The graph of a solution over the interval *I*. ✓

Explicit solutions: Dep. variable is expressed in terms of indep. variable. For example,

$$\text{DE}: \quad \frac{dy}{dx} = x - y$$
$$\text{solution:} \quad y = x - 1 + ce^{-x}$$

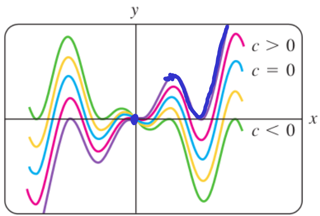
Implicit solutions: Otherwise. For example,

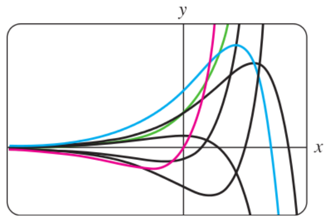
$$\text{DE}: \quad \frac{dy}{dx} = -\frac{x}{y}$$
$$\text{solution:} \quad x^2 + y^2 = c^2 \text{ (implicit) or } y = \pm\sqrt{c^2 - x^2} \text{ (explicit)}$$
$$\text{verify:} \quad \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(c) \implies 2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

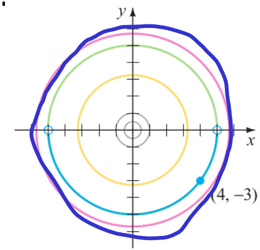
Note that in example 2 we require $x \in [-c, c]$ for the solution to be real. However, $y \neq 0$ (look at the original DE), so $x \neq \pm c$, and hence $I = (-c, c)$.

1.1 Definitions and terminology
Family of solutions

Family of solutions: Set of solutions parameterized by integration constant(s). For example, $x^2 + y^2 = c^2$ forms one-parameter family of solutions for $\frac{dy}{dx} = -\frac{x}{y}$. Here are some other examples:







Exercise: Verify that the one-parameter family of solutions to $\frac{dy}{dx} = x\sqrt{y}$ is given by $y = (\frac{1}{4}x^2 + c)^2$.

Special cases:

Particular solution : A solution free from arbitrary parameters

Singular solution : Not a member of any family, e.g., $y = 0$ for $\frac{dy}{dx} = x\sqrt{y}$

1.2: Initial value problems (IVPs)

1.2 Initial value problems

IVPs and BVPs


Typically, additional information is given, from which the integration constant can be determined. (E.g., our skydiver had velocity $v(0) = 0$.)

This is how we choose a particular solution from the family.

These additional constraints usually come in two varieties

$$\text{initial conditions}: \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \quad , \dots$$

$$\text{boundary conditions}: \quad y(x_0) = y_0, \quad y(x_1) = y_1, \quad , \dots$$

although it is possible to have more exotic constraints

$$\text{other conditions}: \quad \int_{x_0}^{x_1} y(x)\, dx = 0, \dots$$

Let's look at some examples.

$y' + 2xy^2 = 0, \; y(0) = -1$

IVP.

Consider the following DE:

DE : $\frac{dy}{dx} + 2xy^2 = 0$

solution : $y(x) = \frac{1}{x^2+c}$

verify : $\frac{dy}{dx} + 2xy^2 = -2x/(x^2+c)^2 + 2x/(x^2+c)^2 = 0$

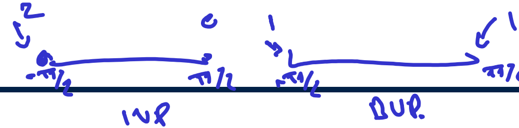
Exercise: Verify that this DE also has a singular solution $y = 0$.

Given the initial condition $y(0) = -1$, determine the particular solution.

$$y(0) = -1 \implies -1 = \frac{1}{0+c} \implies c = -1 \implies \boxed{y = \frac{1}{x^2-1}}$$

Exercise: What is the solution interval for $y = 1/(x^2 - 1)$?

Example 2



Consider the following DE:

$$\begin{aligned} \text{DE}: &\quad \frac{d^2y}{dx^2} + 16y = 0 \\ \text{solution}: &\quad y = c_1\cos(4x) + c_2\sin(4x) \qquad \leftarrow \text{(two-parameter family)} \\ \text{verify}: &\quad \text{exercise} \end{aligned}$$

Suppose we are given initial conditions $y(\frac{\pi}{2}) = -2$ and $y'(\frac{\pi}{2}) = 1$, then

$$\left.\begin{aligned} y(\tfrac{\pi}{2}) = -2 &\implies c_1.1 + c_2.0 = -2 \\ y'(\tfrac{\pi}{2}) = -1 &\implies 4c_1.0 + 4c_2.1 = 1 \end{aligned}\right\} \implies \begin{aligned} c_1 &= -2, \\ c_2 &= \tfrac{1}{4}, \end{aligned}$$

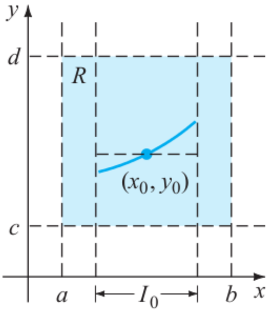
and the particular solution is given by $\boxed{y = -2\cos(4x) + \frac{1}{4}\sin(4x).}$

Exercise: Find the particular solution given boundary conditions

$$y(\tfrac{-\pi}{2}) = -2 \ \text{ and } \ y'(\tfrac{\pi}{2}) = 1.$$

Consider the first-order IVP $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.

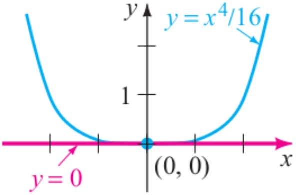
Theorem 1.2.1: Existence of a unique solution
Suppose R is a rectangular region in the xy-plane defined by $a \leq x \leq b$ and $c \leq y \leq d$ and containing the point (x_0, y_0). Suppose further that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in R. Then there exists an interval $I_0 = (x_0 - h, x_0 + h)$ in $[a, b]$ over which the solution $y(x)$ is unique.



$y(x) = (\frac{1}{4}x^2 + c)^2$

$y(x) = 0.$

Let's look at an example.



Consider the first-order IVP $\frac{dy}{dx} = x\sqrt{y}$ with $y(x_0) = y_0$.

$f(x, y) = x\sqrt{y}$ and $\frac{\partial f}{\partial y} = x\frac{1}{2\sqrt{y}}$ are continuous for $y > 0$.

By the theorem above, solution through (x_0, y_0) is unique as long as $y_0 > 0$:

$y(0) = 0 \implies \quad y = 0$ or $y = \frac{1}{16}x^4 \quad \implies$ solution is not unique

$y(2) = 1 \implies \quad y = \frac{1}{16}x^4$ and solution is unique

Remark: The conditions of Theorem 1.2.1 are sufficient but not necessary.
(See p17 of Z&W for details.)