Monte Carlo sampling: → procedure of **generating** service and interarrival times from the given probability distributions.

Steps of Monte Carlo sampling:

1. Establish probability distributions for important input variables.
   A. Service times
   B. Interarrival times
2. Build a cumulative probability distribution for each variable in 1 (if it does not exist).
3. Generate an interval to translate random numbers to random variable outcomes.
   A. Continuous variables: use inverse transformation method.
4. Generate random numbers.
5. Generate random variable outcomes.
6. Simulate a series of trials.

Random number generators

▶ Important characteristics:

1. Computationally fast.
2. Requires little computer memory.
3. Sufficiently spread out.
4. Identically replicable.
5. Has a long cycle (before repeating).

Example: Linear Congruential Random Number Generator

$$\boxed{R_i = \frac{x_i}{m}} \quad \text{where} \quad \boxed{x_i = (a x_{i-1} + c) \% m}$$

and : ▶ x_0 : seed number (given/specified)
▶ a : constant multiplier ($a = 7^5$)
▶ c : increment (given/specified)
▶ m : modulos ($m = 2^{31} - 1$)

$m = 2^{31} - 1$ = prime number
— ensures a cycle length
= magnitude of the modulus

m = non-prime number
— ensure a cycle length
< magnitude of the modulus

Inverse transformation method:

1. Find the cumulative density function:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt,$$

2. Create a random number R
3. Set F(x)=R and solve for x with:

$$x = F^{-1}(R)$$

4. $x_i = F^{-1}(R_i)$ = random variable generator.

① Examples: uniform distribution

pdf : $f(t) = \frac{1}{b-a}, \quad a \le t \le b$

cdf : $F(t) = \int_a^t \frac{1}{b-a} \cdot dt$

$$= \begin{cases} 0 & ; t < a \\ \frac{t-a}{b-a} & ; a \le t \le b \\ 1 & ; t > b \end{cases}$$

inverse transform : $\frac{t_i - a}{b-a} = R_i$

$\Rightarrow t_i = R_i(b-a) + a$

② Examples: exponential distribution

pdf : $f(t) = \lambda e^{-\lambda t} \; ; \; t \ge 0, \lambda > 0$

cdf : $F(t) = \int_0^t \lambda e^{-\lambda t} \cdot dt$

$$= \begin{cases} 0 & ; t < 0 \\ 1 - e^{-\lambda t} & ; t \ge 0, \lambda > 0 \end{cases}$$

inverse transform : $1 - e^{-\lambda t} = R_i$

$t_i = -\frac{1}{\lambda} \cdot \ln(1 - R_i)$

③ Examples: normal distribution

inverse transform (in R):

$\underline{t}$ = qnorm ($\underline{R}$, mu , sigma)

Acceptance Rejection Method:

1. Select a constant M such that: $M = \max\{ f(t) \mid t \in [a,b] \}$
2. While i < n:
   A. Generate two random numbers r1 and r2.
   B. Compute $t^* = a + (b - a)*r1$
   C. Evaluate $f(t^*)$
      a. If $r2 < f(t^*)/M$ —> $t\_i = t^*$ and $i = i+1$
   D. Return to step 2

When do we use ARM?

- f(t) defined over a finite interval ∴ [a, b]
- distributions where CDF don't exist in closed form

Direct Method:

1. Generate two random numbers r1 and r2.
2. Transform r1 and r2 into two normal random variates, (each with mean 0 and variance 1), using the direct transformations:

$$\left.\begin{aligned} Z_1 &= (-2 \ln r_1)^{1/2} \sin 2\pi r_2 \\ Z_2 &= (-2 \ln r_1)^{1/2} \cos 2\pi r_2 \end{aligned}\right\} \sim N(0,1) \quad \text{choose one}$$

3. Transform these standardized normal variates into normal variates from the distribution with mean μ and variance σ^2, using the equations:

$$\begin{aligned} X_1 &= \mu + \sigma Z_1 \\ X_2 &= \mu + \sigma Z_2 \end{aligned} \sim N(\mu, \sigma^2)$$

When do we use DM?

- distributions where CDF don't exist in closed form
- F(t) not defined over a finite interval [a, b]

→ normal distribution