Applied differential equations

TW244 - Lecture 21

5.1: Spring-mass systems

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SCIENCE
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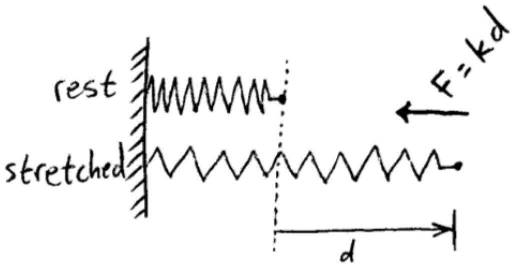
SPRING-MASS SYSTEMS

Hooke's law:

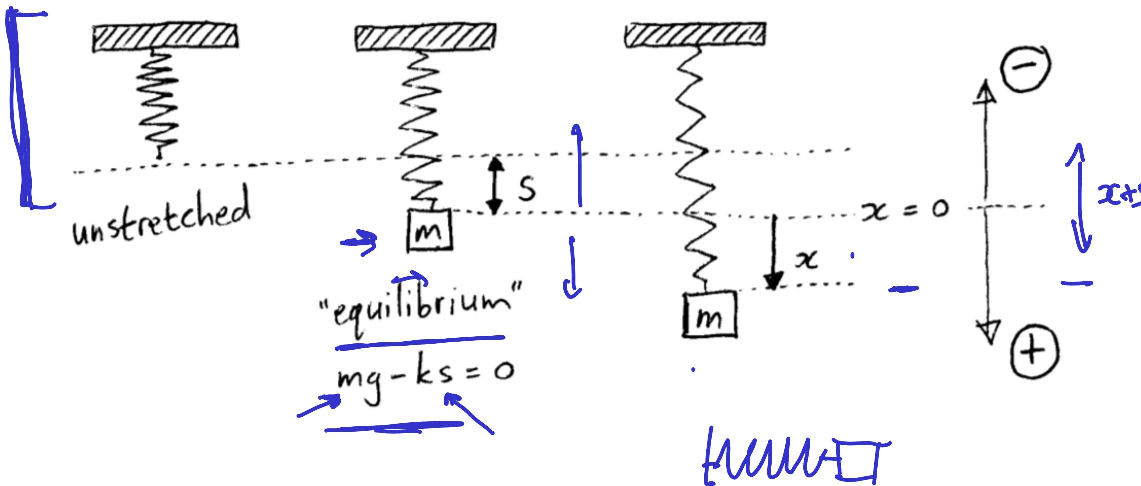
The "restoring forces" exerted by a spring is proportional to the distance by which the spring is elongated (or compressed).



Here k is the "spring constant".

Undamped motion:

Suppose an object with mass m is attached to a vertical spring:



Spring-mass systems
Hooke's law: Undamped motion

Newton's 2nd law of motion: $ma = F$

$$m\frac{d^2x}{dt^2} = mg - k(s+x) = \overbrace{mg - ks}^{=0} - kx = -kx.$$

Therefore

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0,} \quad \text{where } \omega^2 = \frac{k}{m}.$$

$x = e^{pt}$

But we've seen and solved this DE before!

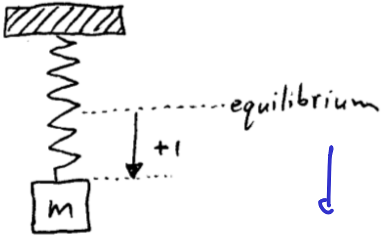
$$\boxed{x(t) = c_1\cos(\omega t) + c_2\sin(\omega t).}$$

We find c_1 & c_2 from initial position $x(0) = x_0$ and initial velocity $x'(0) = x_1$.

Spring-mass systems
Example

Example: Suppose $\omega^2 = \frac{k}{m} = 4$, $x(0) = 1, x'(0) = -2$ then $\underline{x'' + 4x} = 0$ and

$$\underline{x(t) = c_1 \cos(2t) + c_2 \sin(2t).}$$



Use the initial conditions:

$$\begin{aligned} x(0) = 1 &\implies & 1 &= c_1 \cos(0) + c_2 \sin(0) &\implies c_1 = 1 \\ x'(0) = -2 &\implies & -2 &= -2c_1 \sin(0) + 2c_2 \underset{\cos}{\cancel{\sin}}(0) &\implies c_2 = -1 \end{aligned}$$

Therefore

$$\boxed{x(t) = \cos(2t) - \sin(2t).}$$

We have the solution, but in this form it hard to visualize. So we consider...

Spring-mass systems
Amplitude-phase form

Amplitude-phase form: In general, we may write

$$x \;=\; \sqrt{c_1^2+c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\omega t)+\frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\omega t)\right].$$

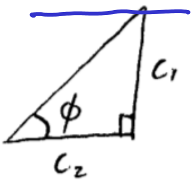
Then define then angle ϕ so that $\sin\phi=\frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\cos\phi=\frac{c_2}{\sqrt{c_1^2+c_2^2}}$ and

$$x=\sqrt{c_1^2+c_2^2}\,[\sin\phi\cos(\omega t)+\cos\phi\sin(\omega t)]=\sqrt{c_1^2+c_2^2}\,\sin(\phi+\omega t)$$

$$\Longrightarrow \boxed{x(t)=A\sin(\omega(t-\theta))} \quad : A=\sqrt{c_1^2+c_2^2},\quad \theta=\frac{2\pi-\phi}{\omega},\quad \phi=tan^{-1}\frac{c_1}{c_2}\in[0,2\pi)^\dagger$$

This is simple harmonic motion!

- amplitude: $A=\sqrt{c_1^2+c_2^2}$
- period: $T=\frac{2\pi}{\omega}$
- frequency: $f=\frac{1}{T}=\frac{\omega}{2\pi}$
- phase shift*: θ



* By convention the phase shift is always positive.

† There are two solutions for $\phi\in[0,2\pi)$. Choose the one that gives the correct sign for $\sin(\phi)$ and $\cos(\phi)$.

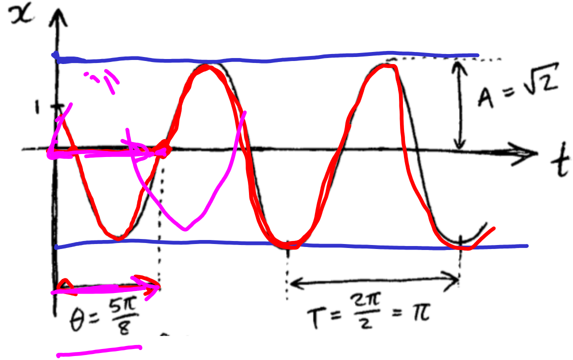
$\phi = \tan^{-1}(c_1/c_2) = \tan^{-1}(1) = \{-\pi/4, 3/4\pi$

$\operatorname{atan2}(1,-1) = 3\pi/4$

Example (cont.)

In amplitude-phase form:

$$
\begin{aligned}
x(t) &= \cos(2t) - \sin(2t) \\
&= \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos(2t) - \frac{1}{\sqrt{2}}\sin(2t)\right] \\
&= \sqrt{2}\sin\left(2t + \frac{3\pi}{4}\right) \\
&= \sqrt{2}\sin\left(2t - \frac{5\pi}{4}\right) \\
&= \sqrt{2}\sin\left[2\left(t - \frac{5\pi}{8}\right)\right]
\end{aligned}
$$

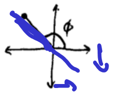


- amplitude: $A = \sqrt{2}$
- period: $T = \pi$
- frequency: $f = \frac{1}{\pi}$
- phase shift*: $\frac{5\pi}{8}$

$\sin\phi = \frac{1}{\sqrt{2}}$

$\cos\phi = -\frac{1}{\sqrt{2}}$

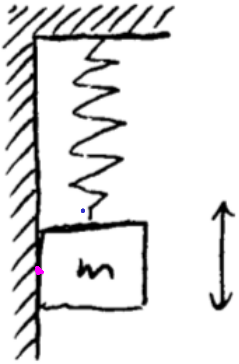
$\therefore \phi = \frac{3\pi}{4}$



* Observe the phase shift is where the curve passes zero moving upwards.

Damped motion:

Suppose now that there is also a linear damping force in the direction opposite to motion (e.g., due to air resistance or friction):



$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}.^*$$

Therefore

$$x'' + 2\gamma x' + \omega^2 x = 0 \quad \text{with} \quad \omega^2 = \frac{k}{m}, \;\; 2\gamma = \frac{\beta}{m}.$$

This is a linear homogeneous DE!

Try $x = e^{pt}$ as a solution (we use p here as m is already used for the mass).

* Convince yourself that the term in red is damping regardless of whether the spring moves up or down.

Substituting $x = e^{pt}$ in to $x'' + 2\gamma x' + \omega^2 x = 0$ gives $p^2 + 2\gamma p + \omega^2 = 0$

$$\implies p = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

We shall see that this leads to three cases:

- $\gamma^2 > \omega^2 \implies$ two real roots ("overdamped")
- $\gamma^2 = \omega^2 \implies$ one real root ("critically overdamped")
- $\gamma^2 < \omega^2 \implies$ no real roots ("underdamped")

Examples next time!