Problem 1:

(a) Complementary solution: try $y = e^{mx}$ as a solution for $y'' - 8y' + 16y = 0$.

Then $m^2 e^{mx} - 8me^{mx} + 16e^{mx} = 0 \implies m^2 - 8m + 16 = 0 \implies (m-4)^2 = 0 \implies m = 4$.

The auxiliary equation has one real root, therefore $y_c = c_1 e^{4x} + c_2 x e^{4x}$.

Particular solution: the right-hand side of the DE is $-10\cos(3x)$, so try $y_p = A\cos(3x) + B\sin(3x)$.

The left-hand side of the DE then becomes:

$$[-9A\cos(3x) - 9B\sin(3x)] - 8[-3A\sin(3x) + 3B\cos(3x)] + 16[A\cos(3x) + B\sin(3x)]$$
$$= [7A - 24B]\cos(3x) + [24A + 7B]\sin(3x)$$

We want it to be equal to $-10\cos(3x)$, therefore let $7A - 24B = -10$ and $24A + 7B = 0$, so that $A = -\frac{14}{125}$ and $B = \frac{48}{125}$. Hence $y_p = -\frac{14}{125}\cos(3x) + \frac{48}{125}\sin(3x)$.

General solution: $\boxed{y = c_1 e^{4x} + c_2 x e^{4x} - \frac{14}{125}\cos(3x) + \frac{48}{125}\sin(3x)}$

(b) Complementary solution: try $y = e^{mx}$, so that $m^2 + 6m + 13 = 0 \implies m = \frac{-6\pm\sqrt{-16}}{2} = -3 \pm 2i$.

The auxiliary equation has two complex roots, therefore $y_c = e^{-3x}[c_1\cos(2x) + c_2\sin(2x)]$.

Particular solution: the right-hand side of the DE is $e^{-x}(1-x)$, so try $y_p = e^{-x}(Ax + B)$.

The left-hand side of the DE then becomes:

$$[e^{-x}(Ax + B - 2A)] + 6[-e^{-x}(Ax + B - A)] + 13[e^{-x}(Ax + B)] = (8A)xe^{-x} + (4A + 8B)e^{-x}$$

We want it to be equal to $e^{-x}(1-x) = -xe^{-x} + e^{-x}$, therefore let $8A = -1$ and $4A + 8B = 1$, so that $A = -\frac{1}{8}$ and $B = \frac{3}{16}$. Hence $y_p = e^{-x}\left(\frac{3}{16} - \frac{1}{8}x\right)$.

General solution: $\boxed{y = e^{-3x}[c_1\cos(2x) + c_2\sin(2x)] + e^{-x}\left(\frac{3}{16} - \frac{1}{8}x\right)}$

Problem 2:

Complementary solution: $2m^2 + 4m - 16 = 0 \implies m = -4$ of $m = 2$. Hence $y_c = c_1 e^{-4x} + c_2 e^{2x}$.

Particular solution: the right-hand side of the DE is $x - e^x$, so try $y_p = Ax + B + Ce^x$.

The left-hand side of the DE then becomes:

$$2Ce^x + 4(A + Ce^x) - 16(Ax + B + Ce^x) = (-16A)x + (4A - 16B) + (-10C)e^x.$$

We want it to be equal to $x - e^x$, so $A = -\frac{1}{16}$, $B = -\frac{1}{64}$ and $C = \frac{1}{10}$.

The general solution is thus $y = c_1 e^{-4x} + c_2 e^{2x} - \frac{1}{16}x - \frac{1}{64} + \frac{1}{10}e^x$.

Now we plug in the initial conditions:

$y(0) = 1 \implies c_1 + c_2 - \frac{1}{64} + \frac{1}{10} = 1$ and $y'(0) = -\frac{1}{4} \implies -4c_1 + 2c_2 - \frac{1}{16} + \frac{1}{10} = -\frac{1}{4}$.

These are two linear equations in the unknowns c_1 and c_2, that can be solved: $c_1 = \frac{113}{320}$ and $c_2 = \frac{9}{16}$.

The solution of the given initial value problem is thence $\boxed{y = \frac{113}{320}e^{-4x} + \frac{9}{16}e^{2x} - \frac{1}{16}x - \frac{1}{64} + \frac{1}{10}e^x}$

Problem 3: Complementary solution: $m^2 + 2m + 1 = 0 \implies m = -1$. Hence $y_c = c_1 e^{-x} + c_2 x e^{-x}$.

Wronskian: $W(x) = y1y2' - y1'y2 = e^{-2x}$.

Particular solution 1: $u_1 = -\int \frac{xe^{-x}e^{-x}\log(x)}{e^{-2x}}\,dx = -\int x\log(x)\,dx = x^2(\frac{1}{4} - \frac{1}{2}\log(x))$ (integration by parts)

Particular solution 2: $u_2 = \int \frac{e^{-x}e^{-x}\log(x)}{e^{-2x}}\,dx = \int \log(x)\,dx = x(\log(x) - 1)$ (integration by parts)

Particular solution: $y_p = u_1 y_1 + u_2 y_2 = x^2 e^{-x}(\frac{1}{2}\log(x) - \frac{3}{4})$

General solution: $y = c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x}(\frac{1}{2}\log(x) - \frac{3}{4})$

Initial conditions: $y(0) = c_1 = 0, y'(0) = c_1 + c_2 = 0 \to c_1 = c_2 = 0$

Solution: $y(x) = x^2 e^{-x}(\frac{1}{2}\log(x) - \frac{3}{4})$

Problem 4:

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} = \frac{e^{-t}xe^{-x} - e^{-x}te^{-t}}{e^{-2t}} = e^{2t}e^{-t-x}(x-t) = e^{-(x-t)}(x-t).$$

$$\begin{aligned} y_p(x) &= \int^x G(x,t)f(t)\,dt = \int^x e^{-(x-t)}(x-t)e^{-t}\log(t)\,dt = e^{-x}\int^x (x-t)\log(t)\,dt \\ &= e^{-x}\left(x\int^x \log(t)\,dt - \int^x t\log(t)\,dt\right) \\ &= e^{-x}\left(x(-1+\log(x)) - (x^2(-1+2\log(x)))/4\right) \\ &= x^2e^{-x}(\frac{1}{2}\log(x) - \frac{3}{4}) \end{aligned}$$

Problem 5:

The system in D-notation: $(D-1)x + y = 0,$
$x + (D-1)y = 0.$

Multiply the first equation with $(D-1)$: $(D-1)^2x + (D-1)y = 0,$
and multiply the second eqn with (-1): $-x - (D-1)y = 0.$

Add these two equations together: $(D-1)^2x - x = 0 \implies D^2x - 2Dx + x - x = 0 \implies x'' - 2x' = 0.$

Solve for x: $x = e^{mt} \implies m^2 - 2m = 0 \implies m = 0$ or $m = 2$. Hence $x(t) = c_1 + c_2e^{2t}$.

From the given DE $dx/dt = x - y$ follows that $y = x - dx/dt = c_1 + c_2e^{2t} - 2c_2e^{2t}$. Hence $y(t) = c_1 - c_2e^{2t}$.

Initial values: $x(0) = 4$ and $y(0) = -2 \implies c_1 + c_2 = 4$ and $c_1 - c_2 = -2 \implies c_1 = 1$ and $c_2 = 3$.

Consequently, $\boxed{x(t) = 1 + 3e^{2t} \text{ and } y(t) = 1 - 3e^{2t}}$

Problem 6:

$$\begin{array}{r} (D-1)x-5y=-t^2 \\ 2x+(D+1)y=\frac{3}{5}t \end{array} \implies \begin{array}{r} (D-1)(D+1)x-5(D+1)y=-(D+1)t^2 \\ 10x+5(D+1)y=3t \end{array} \implies (D^2-1)x+10x=-2t-t^2+3t$$

Hence $x''+9x=t-t^2$. $\left[\text{remember: } -(D+1)t^2=-Dt^2-t^2=-\frac{d}{dt}(t^2)-t^2=-2t-t^2\right]$

Complementary solution: $x_c''+9x_c=0 \implies x_c''=-(3)^2x_c \implies x_c=c_1\cos(3t)+c_2\sin(3t)$.

Particular solution: the right-hand side of the DE is $t-t^2$, so try $x_p=At^2+Bt+C$:

$2A+9(At^2+Bt+C)=t-t^2 \implies 2A+9C=0,\ 9B=1,\ 9A=-1 \implies A=-\frac{1}{9},\ B=\frac{1}{9},\ C=\frac{2}{81}$

Therefore $\boxed{x(t)=c_1\cos(3t)+c_2\sin(3t)-\frac{1}{9}t^2+\frac{1}{9}t+\frac{2}{81}}$

From the given DE $dx/dt=x+5y-t^2$ follows that

$$\begin{aligned} y &= \tfrac{1}{5}\left(dx/dt-x+t^2\right) \\ &= \tfrac{1}{5}\left[-3c_1\sin(3t)+3c_2\cos(3t)-\tfrac{2}{9}t+\tfrac{1}{9}-c_1\cos(3t)-c_2\sin(3t)+\tfrac{1}{9}t^2-\tfrac{1}{9}t-\tfrac{2}{81}+t^2\right] \end{aligned}$$

Therefore $\boxed{y(t)=\frac{1}{5}\left[(3c_2-c_1)\cos(3t)-(3c_1+c_2)\sin(3t)+\frac{10}{9}t^2-\frac{1}{3}t+\frac{7}{81}\right]}$

Problem 7:

See page 183 of Z&W:

EXAMPLE 3 A Mixture Problem Revisited

In (3) of Section 3.3 we saw that the system of linear first-order differential equations

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$
$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

is a model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ in brine mixtures in tanks A and B, respectively, shown in Figure 3.3.1. At that time we were not able to solve the system. But now, in terms of differential operators, the foregoing system can be written as

$$\left(D + \frac{2}{25}\right)x_1 - \frac{1}{50}x_2 = 0$$
$$-\frac{2}{25}x_1 + \left(D + \frac{2}{25}\right)x_2 = 0.$$

Operating on the first equation by $D + \frac{2}{25}$, multiplying the second equation by $\frac{1}{50}$, adding, and then simplifying gives $(625D^2 + 100D + 3)x_1 = 0$. From the auxiliary equation

$$625m^2 + 100m + 3 = (25m + 1)(25m + 3) = 0$$

we see immediately that $x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}$. We can now obtain $x_2(t)$ by using the first DE of the system in the form $x_2 = 50(D + \frac{2}{25})x_1$. In this manner we find the solution of the system to b

$$x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}, \qquad x_2(t) = 2c_1e^{-t/25} - 2c_2e^{-3t/25}.$$

In the original discussion on page 108 we assumed that the initial conditions were $x_1(0) = 25$ and $x_2(0) = 0$. Applying these conditions to the solution yields $c_1 + c_2 = 25$ and $2c_1 - 2c_2 = 0$. Solving these equations simultaneously gives $c_1 = c_2 = \frac{25}{2}$. Finally, a solution of the initial-value problem is

$$x_1(t) = \frac{25}{2}e^{-t/25} + \frac{25}{2}e^{-3t/25}, \qquad x_2(t) = 25e^{-t/25} - 25e^{-3t/25}.$$