Applied differential equations

TW244 - Lecture 03

First-order DEs: Direction fields

Prof Nick Hale - 2020

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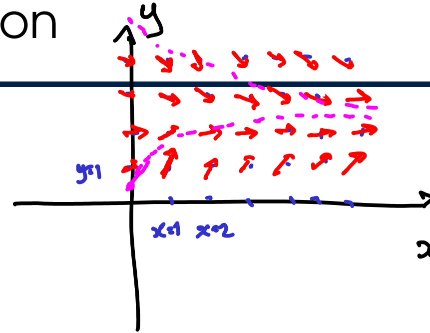
2.1 Solution curves without a solution

Qualitative analysis of the behaviour to some DEs, without solving the DE!

2.1 Solution curves without a solution

Direction fields



Consider the first-order DE, $\boxed{\frac{dy}{dx} = f(x, y).}$

- Evaluate $f(x, y)$ at a bunch of (x, y) points
- Draw a little arrow at each (x, y) to represent the direction $\frac{dy}{dx}$
- Collection of arrows forms the DE's direction field
- Gives some indication of the behaviour of the family of solutions curves in the xy-plane without solving the DE.

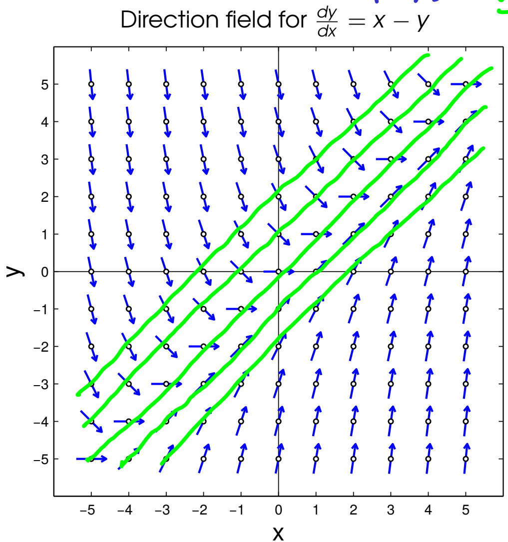
2.1 Solution curves without a solution

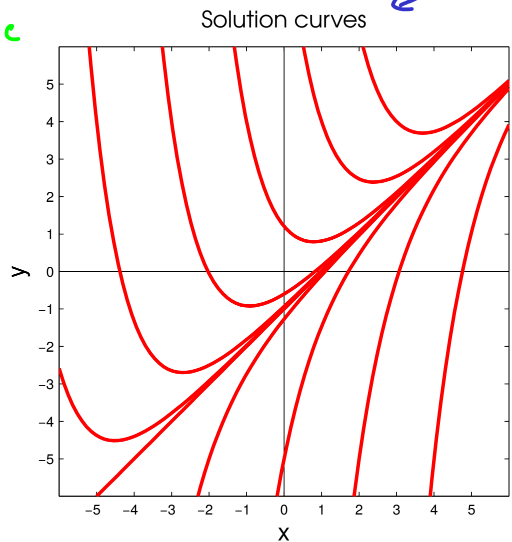
Direction fields: Example $dy/dx = x - y$

$f(x,y)$

$y = x + c$

$\frac{dy}{dx} = c$

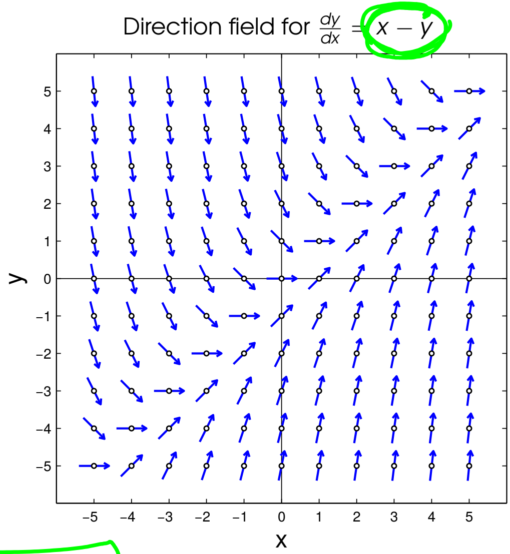


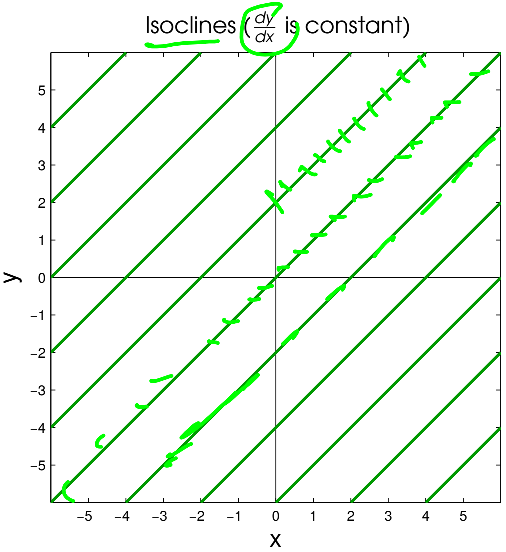


The solution curves follow the arrows!

2.1 Solution curves without a solution

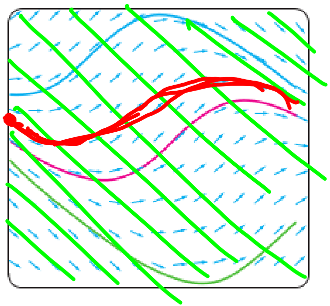
Direction fields: Example $dy/dx = x - y$



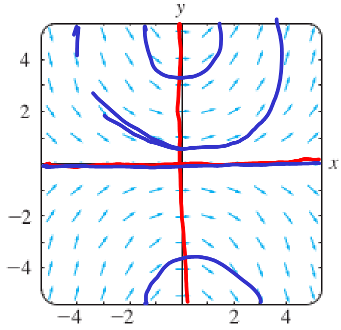


Isoclines can help you plot the direction field.

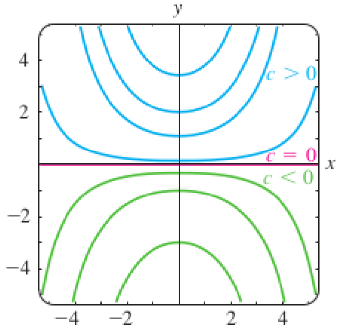
Some more examples (from the textbook):



Direction field for $\frac{dy}{dx} = \sin(x+y)$.



(a) direction field for $dy/dx = 0.2xy$



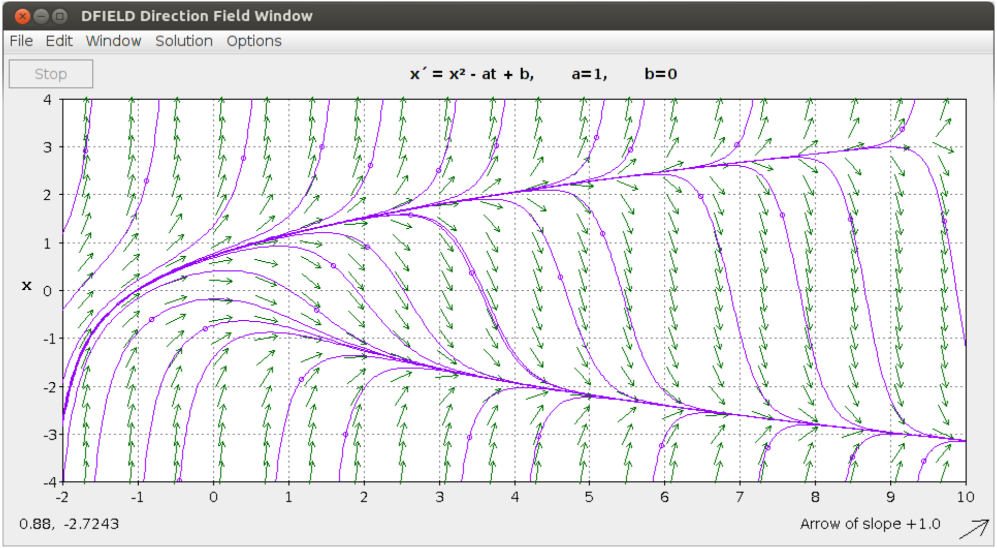
(b) some solution curves in the family $y = ce^{0.1x^2}$

Exercise: Verify that $y = ce^{0.1x^2}$ is a family of solutions to $\frac{dy}{dx} = 0.2xy$.

2.1 Solution curves without a solution

dfield

Direction fields: Example



For more complicated examples, software can be used (see CA01).

Autonomous first-order DEs

2.1 Solution curves without a solution

Autonomous first-order DEs

$dy/dx = f(x,y)$

An autonomous equation is one in which the independent variable does not appear explicitly, i.e., a DE of the form

$$\frac{dy}{dx} = f(y)$$

(Remember, we're interested in functions $y = y(x)$ that satisfy this DE.)

Examples:

$$\frac{dy}{dx} = 1 + y^2 \quad \leftarrow \text{autonomous, } \frac{dy}{dx} = f(y)$$

$$\frac{dy}{dx} = 0.2xy \quad \leftarrow \text{not autonomous, } \frac{dy}{dx} = f(x, y)$$

$(y')^2 = xy$

order = 1

2.1 Solution curves without a solution

$dy/dx = f(y)$

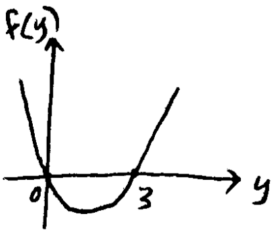
Critical points (also known as "stationary" or "equilibrium" points)

An critical point of an autonomous DE is any $c \in \mathbb{R}$ for which $f(c) = 0$.

If $y(x) = c$ then $\frac{dy}{dx} = 0 = f(y)$, so $y = c$ satisfies the DE.

We call this an equilibrium solution. (Why?)

Example:

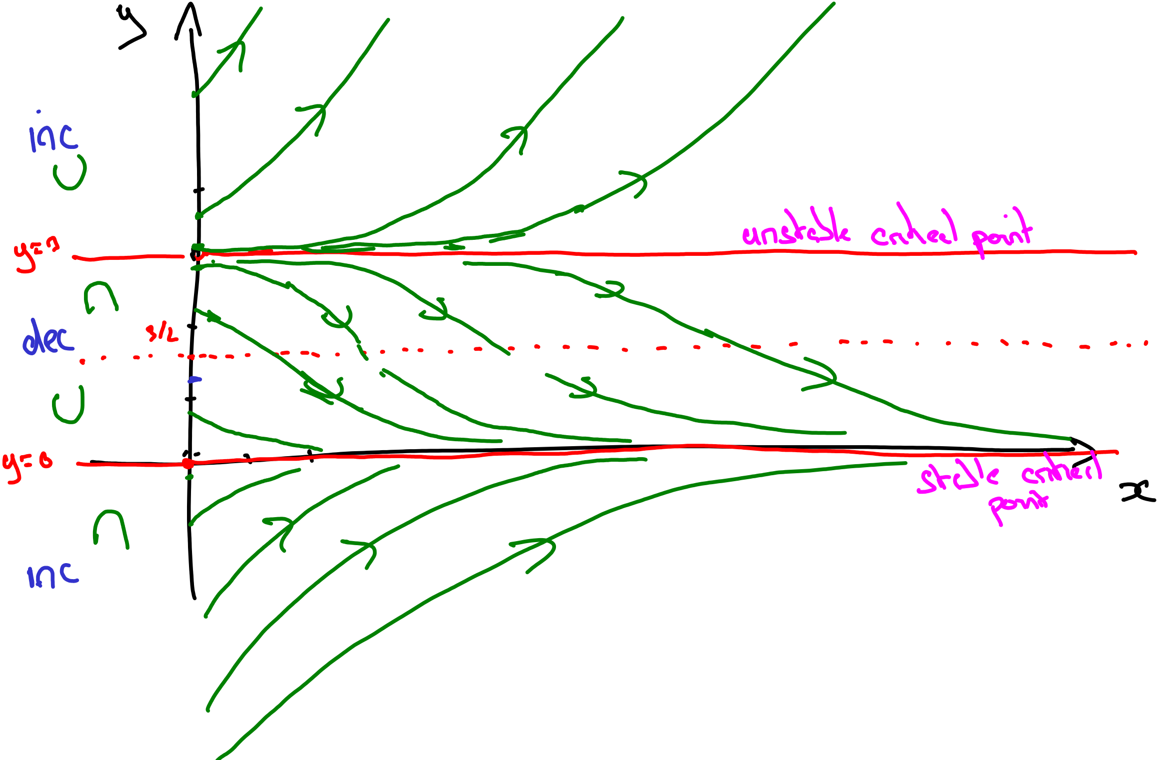


DE $\qquad \frac{dy}{dx} = \overbrace{y^2 - 3y}^{f(y)}$ $\leftarrow$ autonomous

Critical points:

$$y^2 - 3y = 0$$
$$y(y-3) = 0$$
$$\therefore y = 0 \text{ or } y = 3$$

We can now continue our qualitative analysis..



y
x
inc
dec
inc
y=1
3/2
y=0
unstable critical point
stable critical point

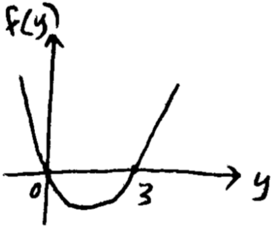
Increasing or decreasing?

$\frac{dy}{dx} = f(y)$

We can tell when a non-constant solution $y = y(x)$ is increasing or decreasing by looking at the sign of f. In particular

- $f(y) > 0 \implies \frac{dy}{dx} > 0 \implies y(x)$ is strictly increasing
- $f(y) < 0 \implies \frac{dy}{dx} < 0 \implies y(x)$ is strictly decreasing

For the example $f(y) = y^2 - 3y$ we have



$\therefore$ y increases when $y < 0$ or $y > 3$ (critical points!)
y decreases when $0 < y < 3$

2.1 Solution curves without a solution
Concavity

$\frac{dy}{dx} = f(y)$

$\frac{d^2y}{dx^2} = \frac{df(y)}{dx}$

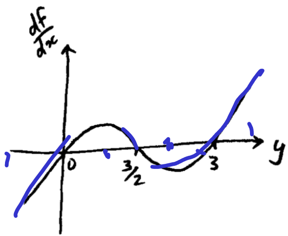
We just looked at the sign of $f(y)$. Now let's look at the sign of $\frac{df}{dx}$.

- $\frac{df}{dx} > 0 \implies \frac{d^2y}{dx^2} > 0 \implies y$ is concave up
- $\frac{df}{dx} < 0 \implies \frac{d^2y}{dx^2} < 0 \implies y$ is concave down

Chain rule:

$\frac{d}{dx} y^2 = 2y\frac{dy}{dx}$

For the example $f(y) = y^2 - 3y$ we have

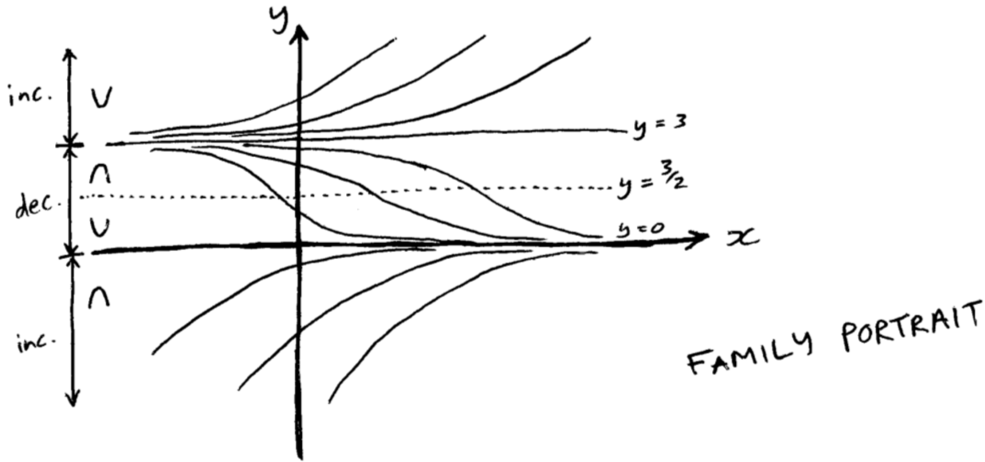


$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}(y^2 - 3y) &&= 2y\frac{dy}{dx} - 3\frac{dy}{dx} \\
&= 2yf(y) - 3f(y) &&= (2y-3)f(y) \\
&= (2y-3)(y^2-3y) &&= y(2y-3)(y-3)
\end{aligned}$$

$\therefore$ y is concave up when $0 < y < \frac{3}{2}$ or $y > 3$

y is concave down when $y < 0$ or $\frac{3}{2} < y < 3$

For the DE $\frac{dy}{dx} = y^2 - 3y$, we can put all we've established in one picture:



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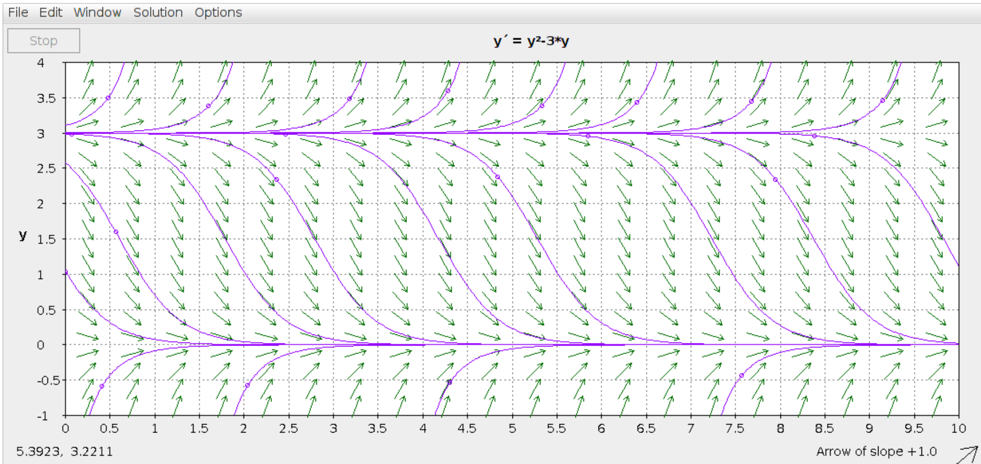
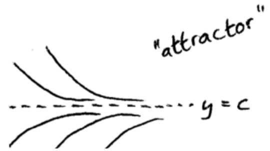
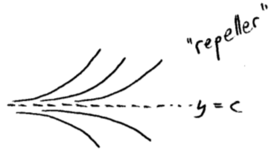


2.1 Solution curves without a solution
Classifying critical solutions

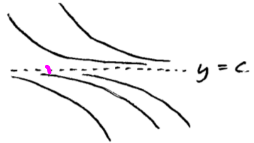
Asymptotically stable: Solution curves tend to $y = c$ on both sides as $x \to \infty$.



Asymptotically unstable: Solution curves tend away $y = c$ on both sides as $x \to \infty$.



Asymptotically semi-stable: Tend to $y = c$ on one side and away on the other side.



Exercise: Classify the equilibrium solutions of the DE $\frac{dy}{dx} = y^2 - 3y$.