Applied differential equations

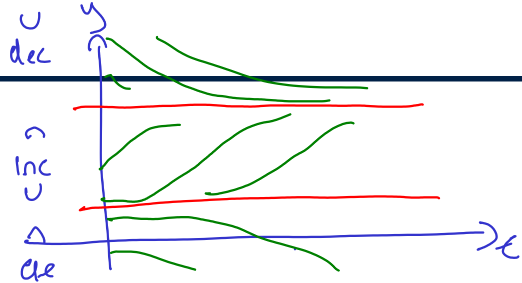
TW244 - Lecture 28

10.1 Autonomous systems

Prof Nick Hale - 2020

$y' = f(t,y)$

$y' = f(y)$



10.1: AUTONOMOUS SYSTEMS

An autonomous system of two DEs has the form*.

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y).$$

It is "autonomous" because t does not appear explicitly in the RHS.

Examples:

$$\underbrace{\begin{aligned}\frac{dx}{dt} &= x^2 + \sin y \\ \frac{dy}{dt} &= x + xy - e^y\end{aligned}}_{autonomous} \qquad \underbrace{\begin{aligned}\frac{dx}{dt} &= x + y + t \\ \frac{dy}{dt} &= x - e^{ty}\end{aligned}}_{non-autonomous}$$

*An autonomous system of two DEs is sometimes called a "*plane* autonomous system"

Example: Nonlinear pendulum

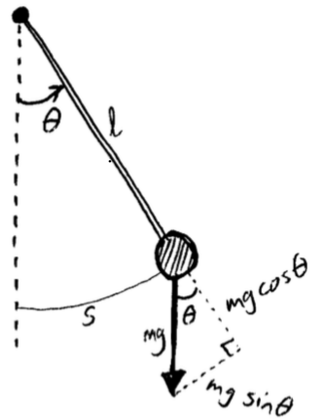
Recall the non-linear pendulum from Lecture 24:

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta.$$

Let $x = \theta$ and $y = \frac{d\theta}{dt}$ then we may write

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -\omega^2 \sin x. \end{cases}$$

i.e., $P(x,y) = y, \quad Q(x,y) = -\omega^2 \sin x.$



Linear pendulum: $dx/dt = y$

$dy/dt = -\omega^2 x$

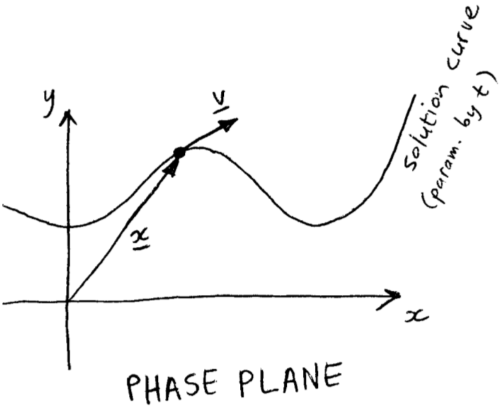
10.1: Autonomous systems
Vector field interpretation

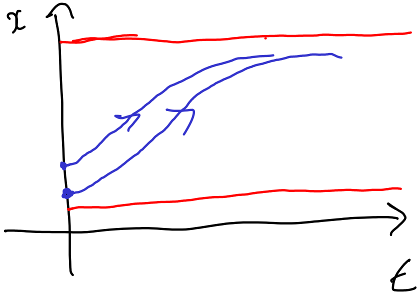
The vector-valued function $\underline{v}(x, y) = [P(x, y); Q(x, y)]$ defines a vector field in a region of the plane, and a solution to the DE $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ can be interpreted as the path of a particle as it moves through this field.

For example, consider $\underline{v}(x, y)$ as the velocity (i.e., speed and direction) of a stream at position (x, y).

A particle (e.g., a champagne cork) released from an initial position $\underline{x}_0 = (x_0, y_0)$ in this stream will trace out a solution curve $\underline{x}(t) = [x(t); y(t)]$ to the DE.

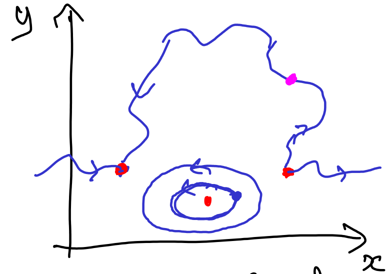
These are similar to (but not the same as) the direction fields we saw in Lecture 3 (as here the curve is parameterised by t).





Direction field

$$dx/dt = P(x)$$



Vector field

$$dx/dt = P(x,y)$$

$$dy/dt = Q(x,y)$$

Phase plane.

10.1: Autonomous systems
Types of solutions

There are three basic types of solutions to planar autonomous systems:

(1): Equilibrium solutions (aka "stationary" or "critical" solutions) at the same time

- An equilibrium solution is an (x, y) pair for which $\frac{dx}{dt} = \frac{dy}{dt} = 0$.
- When a solution curve reaches a stationary point it remains there indefinitely.
- The solution is said the be "in equilibrium" (both x and y are constant).

Example: Consider the predatory-prey model $\frac{dx}{dt} = 5x - x^2 - xy$; $\frac{dy}{dt} = -2y + xy$.
For equilibrium we require

$$\begin{cases} \frac{dx}{dt} = 0 \quad \Longrightarrow \quad x(5 - x - y) = 0 \\ \frac{dy}{dt} = 0 \quad \Longrightarrow \quad y(x - 2) = 0 \end{cases}$$

and therefore the equilibrium solutions are $(x, y) = (0, 0); (5, 0); (2, 3)$.

(2): Arcs

- A solution $(x(t), y(t))$ defining a plane curve that does not intersect itself is an "arc".

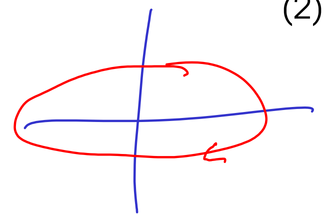
10.1: Autonomous systems
Types of solutions (cont.)

(3): Periodic solution (or "cycle")

- A solution $(x(t), y(t))$ forming a closed curve is a periodic solution or "cycle".

Example: Consider the linear pendulum $\frac{d^2\theta}{dt^2} = -\omega^2\theta$.
Let $x = \theta, y = \frac{d\theta}{dt}$ then

$$\frac{dx}{dt} = y \qquad (1)$$
$$\frac{dy}{dt} = -\omega^2 x \qquad (2)$$



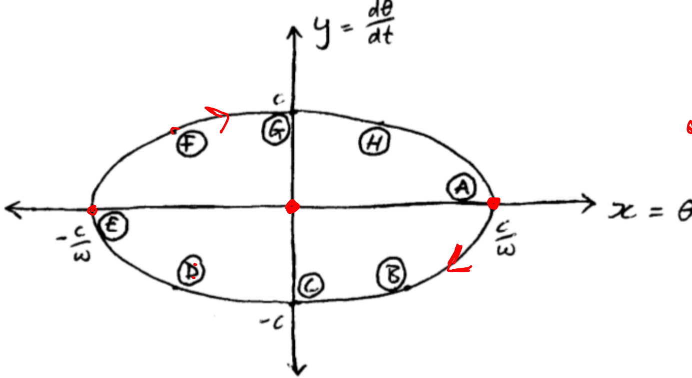
Adding $x\omega^2 \times (1)$ to $y \times (2)$ gives

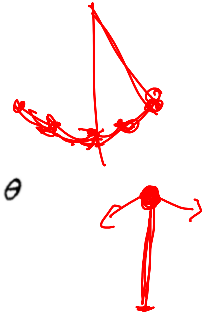
$$\omega^2 x \frac{dx}{dt} + y\frac{dy}{dt} = 0$$

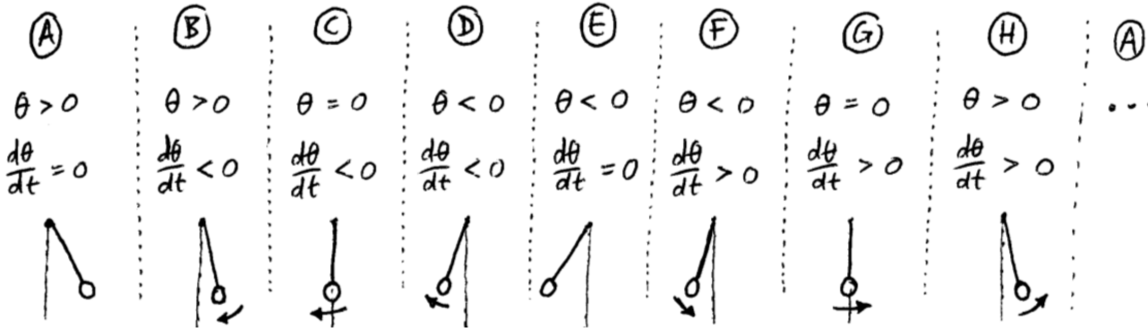
and therefore

$$2\omega^2 x \frac{dx}{dt} + 2y\frac{dy}{dt} = \omega^2 \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(\omega^2 x^2 + y^2) = 0 \implies \omega^2 x^2 + y^2 \text{ is constant.}$$

Hence $\omega^2 x^2 + y^2 = c^2$, which defines an ellipse, i.e., a closed curve!


$y = \frac{d\theta}{dt}$
$x = \theta$
c
$-c$
$\frac{c}{\omega}$
$-\frac{c}{\omega}$
A
B
C
D
E
F
G
H

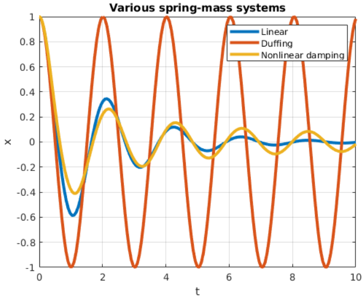


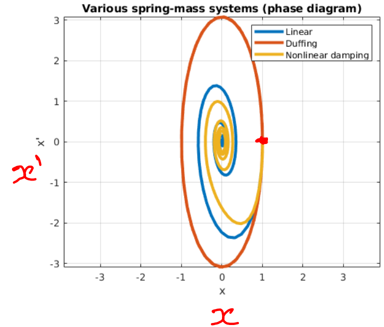

A
$\theta > 0$
$\frac{d\theta}{dt} = 0$
B
$\theta > 0$
$\frac{d\theta}{dt} < 0$
C
$\theta = 0$
$\frac{d\theta}{dt} < 0$
D
$\theta < 0$
$\frac{d\theta}{dt} < 0$
E
$\theta < 0$
$\frac{d\theta}{dt} = 0$
F
$\theta < 0$
$\frac{d\theta}{dt} > 0$
G
$\theta = 0$
$\frac{d\theta}{dt} > 0$
H
$\theta > 0$
$\frac{d\theta}{dt} > 0$
A
...

10.1: Autonomous systems

Exercises:

- Relate the above to the (x, x') graphs we drew in Assignment 04.





- Find the critical points of the following

$$\begin{aligned} x' &= y \\ y' &= -9x - y \end{aligned} \qquad \begin{aligned} x' &= x^2 + y^2 - 6 \\ y' &= x^2 - y \end{aligned} \qquad \begin{aligned} x' &= 0.01x(100 - x - y) \\ y' &= 0.05y(60 - y - 0.2x) \end{aligned}$$

When a plane autonomous system is linear we can use the method of eigenvalues to investigate solutions. (We'll do this next time!)