Applied differential equations

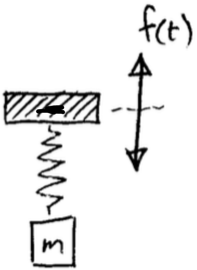
TW244 - Lecture 23

5.1: Spring-mass systems (cont.)

Prof Nick Hale - 2020

Driven motion:
Suppose now that there is also an external force acting on our spring-mass system:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$



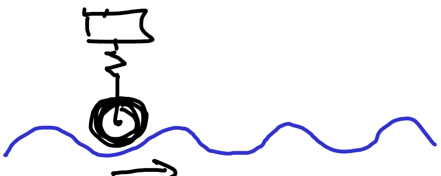
then

$$x'' + 2\gamma x' + \omega^2 x = g(t)$$

where $\omega^2 = k/m, 2\gamma = \beta/m$, and $g(t) = f(t)/m$.

This is a non-homogeneous DE! We seeks a solution of the form

$$x = x_c + x_p.$$



Example: Consider the DE

$$x'' + 2x' + 4x = 13\cos t.$$

($\mathbf{x_c}$): In the previous lecture we already worked out that

$$x_c = e^{-t}\left(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t)\right) = Ae^{-t}\sin(\sqrt{3}(t-\phi)).$$

($\mathbf{x_p}$): For x_p let's try $x_p = A\cos t + B\sin t \implies$

$$\begin{aligned}(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + 4(\overset{A}{\cos} t + \overset{B}{\sin} t) &= 13\cos t\\ (-A + B + 4A)\cos t + (-B - 2A + 4B)\sin t &= 13\cos t\end{aligned}$$

We want $\begin{cases} 3A + 2B = 13 \\ -2A + 3B = 0 \end{cases} \implies A = 3, B = 2.$

$x_p = 3\cos t + 2\sin t.$

$x = x_c + x_p$

Spring-mass systems (cont.)
Example

Therefore

$$\begin{aligned} x_p &= 3\cos t + 2\sin t \\ &= \sqrt{13}\sin(t+\theta) \quad : \quad \theta = \arctan(3/2) \approx 0.98 \end{aligned}$$

exercise

and therefore

$$\boxed{x(t) = \underbrace{Ae^{-t}\sin(\sqrt{3}(t-\phi))} + \underbrace{\sqrt{13}\sin(t-\theta)}}$$

Internal oscillation

- depends on initial conditions
- vanishes as $t \to \infty$
- 'transient term"

Forced oscillation

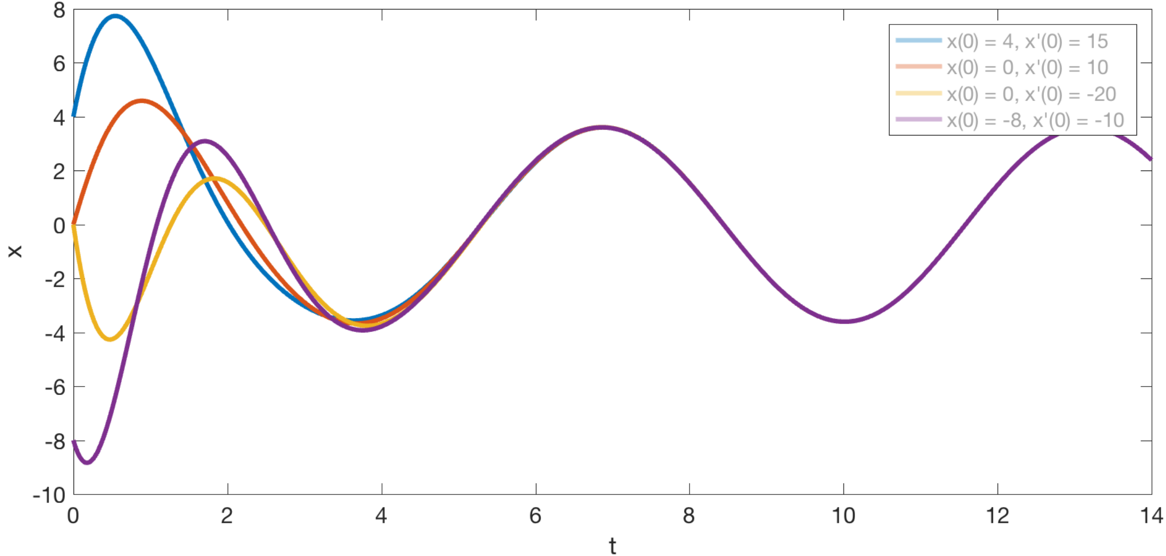
- indep. of initial conditions
- dominates as $t \to \infty$
- 'steady state term"

Long term behaviour of this system is indep. of the initial conditions!

Spring-mass systems (cont.)

Example

Long term behaviour of this system is indep. of the initial conditions!!



Spring-mass systems (cont.)

Driven motion without damping

Consider the undamped system*

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \quad x(0) = x'(0) = 0, \quad 0 < \gamma \neq \omega.$$

Solution:

$(\mathbf{x_c})$: $x'' + \omega^2 x = 0 \implies x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t)$.

$(\mathbf{x_p})$: Try $x_p = \alpha \cos(\gamma t) + \beta \sin(\gamma t)$ then

$$\begin{aligned} x'' + \omega^2 x &= -\gamma^2 \alpha \cos(\gamma t) - \gamma^2 \beta \sin(\gamma t) + \omega^2 \alpha \cos(\gamma t) + \omega^2 \beta \sin(\gamma t) \\ &= (\omega^2 - \gamma^2)\alpha \cos(\gamma t) + (\omega^2 - \gamma^2)\beta \sin(\gamma t) = F_0 \cos \gamma t \end{aligned}$$

Then $(\omega^2 - \gamma^2)\alpha = F_0 \implies \alpha = F_0/(\omega^2 - \gamma^2)$ and $(\omega^2 - \gamma^2)\beta = 0 \implies \beta = 0$ and

$$\begin{aligned} x &= c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \cos(\gamma t) \\ x' &= -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t) - \frac{\gamma F_0}{\omega^2 - \gamma^2} \sin(\gamma t) \end{aligned}$$

*Note: γ here is not the same as slide 1!

Spring-mass systems (cont.)
Driven motion without damping

Initial conditions:

$$\begin{cases} x(0) = 0 \implies & 0 = c_1 + F_0/(\omega^2 - \gamma^2) & \implies c_1 = -F_0/(\omega^2 - \gamma^2) \\ x'(0) = 0 \implies & 0 = \omega c_2 & \implies c_2 = 0 \end{cases}.$$

Therefore

$$\boxed{x(t) = \frac{F_0}{\omega^2 - \gamma^2}\big(\cos(\gamma t) - \cos(\omega t)\big).}$$

Note that we obviously require $\gamma \neq \omega$, but what if

- $\gamma \approx \omega$?
- $\gamma \to \omega$?

Find out next time!