



Naam/Name: _____

Stud. Nr: _____

**Toegepaste Differensiaalvergelykings
TW244 Toets 1 2019**

Instruksies:

- (a) 5 probleme, 54 punte (50 + 4 bonus).
- (b) 2 uur, toeboek.
- (c) Sakrekenaars **word toegelaat**. Selfone **nie**.
- (d) Toon alle bewerkings. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.
- (e) Daar is leë bladsye aan die agterkant van die vraestel as jou antwoorde nie inpas in die gegewe spasies nie. Dui duidelik aan as jou antwoord voortgaan op een van hierdie bladsye.
- (f) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

Formules/Formulas:

- Fundamentele stelling van calculus /
Fundamental theorem of calculus :
- Produkteël vir differensiasie /
Product rule for differentiation :
- Deelsgewyse integrasie /
Integration by parts :
- Differensiasie van die logaritme /
Differentiation of the logarithm :

**Applied Differential Equations
TW244 Test 1 2019**

Instructions:

- (a) 5 problems, 54 marks (50 + 4 bonus).
- (b) 2 hours, closed book.
- (c) Calculators **are** allowed. Cell phones are **not**.
- (d) Show all calculations. A correct answer does not earn full marks without the necessary explanation.
- (e) There are blank pages at the back of the paper in case you cannot fit your answer in the space provided. Indicate clearly if your answer continues to one of these pages.
- (f) The formulas below may be used without proof anywhere in the test.

$$\int \frac{df}{dx} dx = f(x) + C$$

$$\frac{d}{dx} (fg) = \frac{df}{dx} g + f \frac{dg}{dx}$$

$$\int_a^b f \frac{dg}{dx} dx = [fg]_a^b - \int_a^b \frac{df}{dx} g dx$$

$$\frac{d}{dx} \ln(x - a) = \frac{1}{x - a}$$

Prob 1 (10 punte/marks)

(a) Vir elk van die volgende differensiaalvergelykings, gee die orde, dui aan of dit lineêr is of nie, en of dit 'n outonome DV is of nie.

(a) For each of the following differential equations, give the order, state whether it is linear or not, and whether it is autonomous or not.

differensiaalvergelyking <i>differential equation</i>	orde <i>order</i>	lineêr? (ja/nee) <i>linear? (yes/no)</i>	outonome DV? (ja/nee) <i>autonomous DE? (yes/no)</i>
$\left(\frac{dy}{dx}\right)^3 + y = 1$			
$\dot{x} = 3x - 4y + \cos(t^2)$ $\dot{y} = 5x + 2y + \sin^2(t)$			

(b) Watter van die volgende DVs is geskik om deur die integrasiefaktor metode opgelos te word.

(b) Which of the following DEs would be suitable for solving with the integrating factor method:

(A) $\left(\frac{dy}{dx}\right)^3 + xy = y$ (B) $\frac{dy}{dx} + x^3y^3 = y$ (C) $\frac{dy}{dx} + xy^3 = y$ (D) $\frac{dy}{dx} + xy = y^3$ (E) niks geskik nie/
none suitable

(c) Watter van die volgende DVs is nie geskik om deur skeiding van veranderlikes opgelos te word nie.

(c) Which of the following DEs would not be suitable for solving with separation of variables:

(A) $\left(\frac{dy}{dx}\right)^3 + xy = y$ (B) $\frac{dy}{dx} + x^3y^3 = y^3$ (C) $\frac{dy}{dx} + x^3y = y$ (D) alles geskik/
all suitable (E) niks geskik nie/
none suitable

(d) Toon aan m.b.v. die integrasiefaktor dat die oplossingsfamilie van die DV

(d) Show, using integrating factor, that the family of solutions to the DE

$$\frac{dy}{dt} = t - y$$

gegee word deur

is given by

$$y(t) = t - 1 + Ce^{-t}.$$

(Wenk: sien voorblad vir deelsgewyse integrasie)

(Hint: see front page for integration by parts)

(e) Watter van die onderstaande rigtingvelde stem met die DV $\frac{dy}{dt} = t - y$.

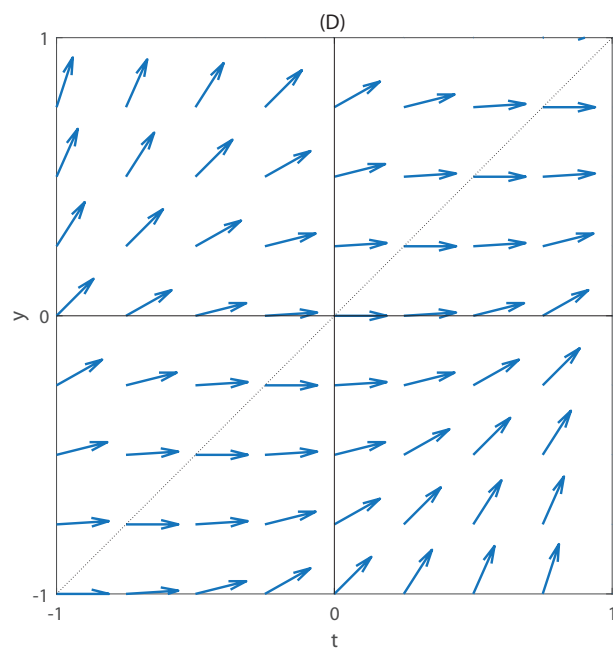
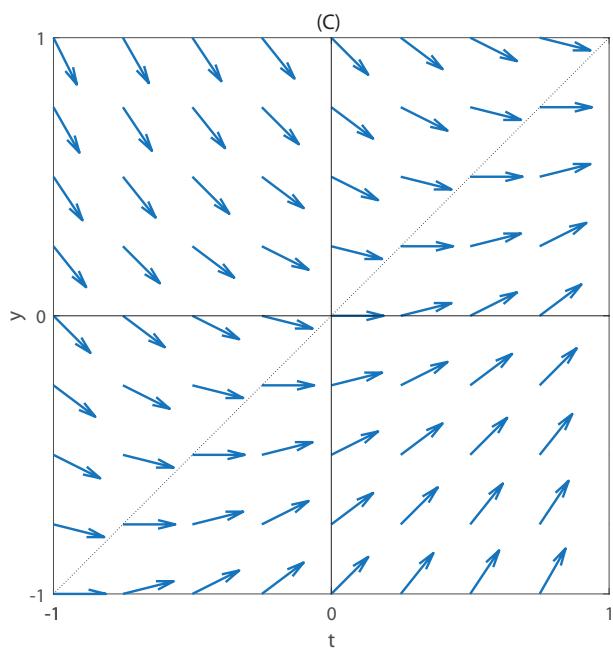
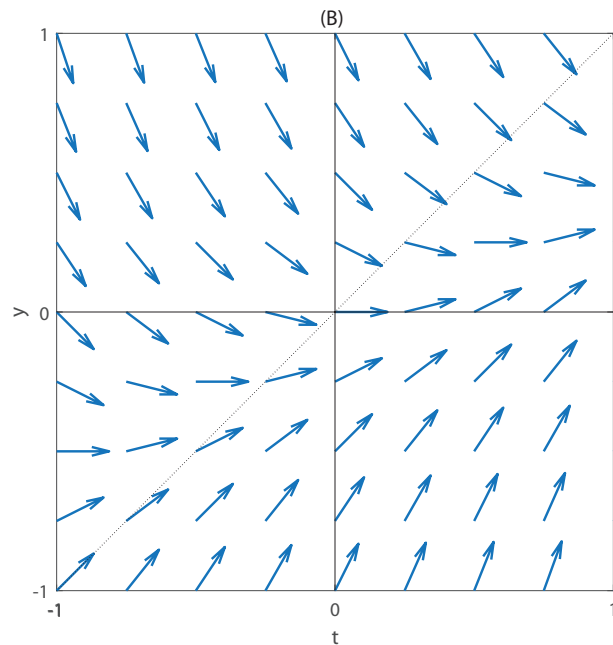
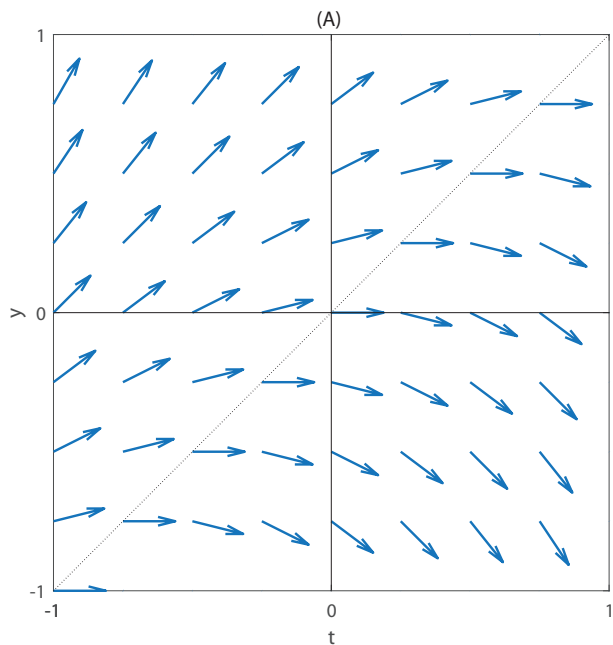
(A)

(B)

(C)

(D)

(E) geen van die onderstaande /
none of the below



Prob 2 (10 punte/marks)

Veronderstel ons stel belang in die gedrag van nie-negatiewe oplossings ($P \geq 0$) van die outonome DV:

Suppose we are interested in the behaviour of non-negative solutions ($P \geq 0$) of the autonomous DE:

$$\frac{dP}{dt} = P^2 - P.$$

(a) Vind die kritieke oplossings van die DV.

(a) Find the critical solutions of the DE.

(b) Vir watter waardes van P is $\frac{dP}{dt}$ positief / negatief?

(b) For what values of P is $\frac{dP}{dt}$ positive / negative?

(c) Toon aan dat $\frac{d^2P}{dt^2} = P(P-1)(2P-1)$.

(c) Show that $\frac{d^2P}{dt^2} = P(P-1)(2P-1)$.

(d) Vir watter waardes van P konkav op / af?

(d) For what values of P concave up / down?

(e) Gebruik die informasie vanuit dele (a)-(d) om die ‘familieportret’ van oplossings vir die DV te teken op die oorkantste bladsy. Wys die gebiede waarin oplossings konkav na bo of konkav na onder is, en merk die infleksiepunt indien daar een is. Dui verteenwoordigende oplossingskurwes vir ‘n paar aanvangsvoorwaardes aan, en klassifiseer die kritieke oplossings as stabiel, semi-stabiel, of onstabiel.

(e) Use the information from parts (a)-(d) to draw the ‘family portrait’ of solutions for the DE on the page opposite. Show the regions in which solutions are concave up or concave down, and mark the point of inflection if there is one. Include representative solution curves for some initial conditions, and classify the critical solutions as stable, semi-stable, or unstable.

Beskou die aanvangswaardeprobleem

Consider the initial value problem

$$\frac{dP}{dt} = k(P^2 - P), \quad k > 0, \quad 0 < P(0) = P_0 < 1.$$

(f) Bereken die waardes van A en B waarvoor

(f) Compute the values of A and B for which

$$\frac{1}{P(P-1)} = \frac{A}{P} + \frac{B}{P-1}$$

(g) Gebruik skeiding van veranderlikes om dié DV
op te los en wys dat die oplossing geskryf kan word
as

(g) Use separation of variables to solve the DE and
show that the solution can be written as

$$P(t) = \frac{P_0 e^{-kt}}{P_0(e^{-kt} - 1) + 1}.$$

Prob 3 (10 punte/marks)

'n Moordslagoffer is dood gevind in 'n vleiskamer met 'n konstante kamertemperatuur $T_m = 4^\circ\text{C}$. Jy, die hoof van CSI Stellenbosch, moet Newton se Wet van Verkoeling gebruik om die tyd van sy dood te bepaal:

A murder victim is found dead in a meat locker kept at a constant ambient temperature of $T_m = 4^\circ\text{C}$. You, the head of CSI Stellenbosch, must use Newton's Law of Cooling to establish the time of death:

$$\frac{dT}{dt} = k(T_m - T), \quad T(0) = T_0.$$

(a) Jou kollega stry dat as $T_m > T_0$ dan moet k positief wees, en as $T_m < T_0$ dan moet k negatief wees. Verduidelik kortliks waarom hy/sy verkeerd is.

(a) Your colleague argues that if $T_m > T_0$ then k should be positive and if $T_m < T_0$ then k should be negative. Explain briefly why he/she is wrong.

(b) Gebruik die integrasiefaktor om die AWP op te los en wys dat

(b) Use integrating factor to solve the IVP and show that

$$T(t) = T_m + (T_0 - T_m)e^{-kt}.$$

(c) Toe jy 5nm gearriveer het, het jy die kerntemperatuur van die slagoffer se liggaam as 20°C gemeet. 'n Uur later, teen 6nm, meet jy weer en vind jy dat die liggaam se kerntemperatuur 15°C is. Gebruik hierdie metings om k te bereken.

(c) When you arrived at 5PM you measured the victim's core body temperature as 20°C . Now, an hour later at 6PM, you measure again and find the core body temperature is 15°C . Use these measurements to determine k .

$$k = \dots\dots\dots$$

(d) Laastens, gebruik hierdie waarde van k saam met die feit dat die kerntemperatuur van 'n lewendige persoon om en by 37°C is om die tyd van die persoon se dood te bepaal.

(d) Finally, use this value of k and fact that the core temperature of a living human is around 37°C to estimate the time of death.

$$Tyd/Time = \dots\dots\dots$$

Prob 4 (10 punte/marks)

Twee chemikalieë A en B word gekombineer om 'n chemikalie C te vorm. Die reaksietempo is direk eweredig aan die produk van die hoeveelhede van A en B wat nog nie na C omgeskakel is nie. Aanvanklik is daar 40 gram van A en 30 gram van B . Om 5 gram van C te maak word 3 gram van B en 2 gram van A benodig.

(a) Skryf 'n AWP neer wat hierdie stelsel beskryf (Geen bewerkings is nodig nie).

$$\begin{array}{l} A(t) = \dots\dots\dots \\ B(t) = \dots\dots\dots \end{array} \quad \therefore \quad \frac{dC}{dt} = \dots\dots\dots, \quad C(0) = \dots\dots\dots$$

(b) Los die AWP op met skeiding van veranderlikes en wys dat die oplossing gegee word deur

$$C(t) = \frac{100(1 - e^{-50kt})}{2 - e^{-50kt}}.$$

(c) Dit word waargeneem dat 5 gram van C in 2 minute gevorm word. Hoeveel word gevorm in 10 minute?

(d) Wat is die beperkte aantal van C na 'n lang tydperk?

(e) Hoeveel van chemikalieë A en B bly oor na 'n lang tydperk?

Two chemicals A and B are combined to form a chemical C . The rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to C . Initially, there are 40 grams of A and 30 grams of B . To make 5 grams of C requires 3 grams of B and 2 grams of A .

(a) Write down an IVP describing this system. (No working necessary).

(b) Solve the IVP using separation of variables and show that the solution is given by

(c) It is observed that 5 grams of C is formed in 2 minutes. How much is formed in 10 minutes?

(d) What is the limiting amount of C after a long time?

(e) How much of the chemicals A and B remain after a long time?

Prob 5 (14 punte/marks)

Toe Skoonlief en die Ondier mekaar se foto's op hul "dating app" sien, raak die Ondier verlief op Skoonlief weens haar aangename voorkoms ($A_B > 0$), terwyl Skoonlief nie van die Ondier hou nie vanweë sy skrikwekkende voorkoms ($A_T < 0$). Beide van hul vergeet van hul gevoelens teen 'n konstante eenheidstempo, ongeag of hul geliefd is (positief) of nie (negatief). Die gevoelens van die Ondier teenoor Skoonlief word daarom beskryf deur

$$\dot{T} = -T + A_B$$

terwyl die gevoelens van Skoonlief teenoor die Ondier beskryf word deur

$$\dot{B} = -B + A_T$$

(a) Bereken die ewewig vir albei se gevoelens.

(b) Los beide DVs op (deur gebruik te maak van die skeiding van veranderlikes of die integrasiefaktor) om $T(t)$ en $B(t)$ te bepaal, as dit gegee word dat hul aanvanklike gevoelens onverskillig is ($T(0) = B(0) = 0$) aangesien hul mekaar nie vantevore geken het nie.

(c) Verifieer die stabiliteit van die ewewigsgevoelens wat in vraag (a) bereken is deur gebruik te maak van die oplossings uit vraag (b) en lewer kommentaar op die resultaat (onthou $A_T < 0$).

When Beauty and The Beast are shown each other's picture on their dating app, The Beast falls in love with Beauty because of her pleasant appearance ($A_B > 0$), while Beauty dislikes The Beast because of his terrifying appearance ($A_T < 0$). Also, they both forget their feelings with a constant unit rate, no matter if they are loved (positive) or disliked (negative). Thus, the feeling of The Beast towards Beauty is described by

while the feeling of Beauty towards The Beast is described by

(a) Compute the equilibrium for both feelings.

(b) Solve both DEs (using either separation of variables or integrating factor) to obtain $T(t)$ and $B(t)$, given that, since they did not know each other before, their initial feelings are indifferent ($T(0) = B(0) = 0$).

(c) Verify the stability of the equilibrium feelings computed in question (a) using the solutions from question (b) and comment on the result (remember $A_T < 0$).

As hulle egter persoonlik ontmoet, kan hul gevoelens mekaar beïnvloed met koëffisiënte k_T en k_B ($0 < k_T, k_B < 1$). Dit kan gemodelleer word deur die stelsel

$$\dot{T} = -T + A_B + k_TB$$

$$\dot{B} = -B + A_T + k_BT$$

(d) Wat is die betekenis van k_T en k_B , en die feit dat hul groter as 0 is?

However, if they instead meet in person, their feelings can affect each other with coefficients k_T and k_B ($0 < k_T, k_B < 1$). This can be modelled by the system

(d) What is the meaning of k_T and k_B , and their being greater than 0?

(e) Aangesien hulle aanvanklik onverskillig teenoor mekaar voel ($T(0) = B(0) = 0$), wat is die aanvanklike rigting van Skoonlief se gevoelens teenoor die Ondier?

(e) Given that they are initially indifferent to each other ($T(0) = B(0) = 0$), what is the initial direction of Beauty's feelings towards The Beast?

(f) Bereken die ewewig van die stelsel van DVs en neem aan dat dit stabiel is. Toon aan of die Ondier uiteindelik deur Skoonlief liefgehê kan word.

(f) Compute the equilibrium of the system of DEs and, assuming it is stable, show if The Beast can eventually be loved by Beauty.

(g) Byvoorbeeld, wat is die waarde van die ewewig in vraag (f) as $A_B = 1, A_T = -1/3, k_T = k_B = 1/2$?

(g) For example, what is the value of the equilibrium in question (f) if $A_B = 1, A_T = -1/3, k_T = k_B = 1/2$?

Hierdie bladsy is doelbewus leeg gelaat. Jy kan dit gebruik vir rofwerk of vir ekstra spasie, indien nodig.

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