Applied differential equations

TW244 - Lecture 22

5.1: Spring-mass systems (cont.)

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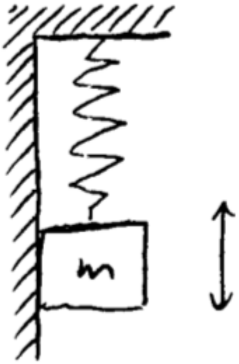
Damped motion:

Suppose now that there is also a linear damping force in the direction opposite to motion (e.g., due to air resistance or friction):

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}.^*$$



Therefore

$$x'' + 2\gamma x' + \omega^2 x = 0 \quad \text{with} \quad \omega^2 = \frac{k}{m}, \;\; 2\gamma = \frac{\beta}{m}.$$

This is a linear homogeneous DE!

Try $x = e^{pt}$ as a solution (we use p here as m is already used for the mass).

* Convince yourself that the term in red is damping regardless of whether the spring moves up or down.

Spring-mass systems (cont.)
Damped motion (reminder)

Recall that we derived an equation for damped motion as

$$x'' + 2\gamma x' + \omega^2 x = 0,$$

where $2\gamma = \beta/m$ and $\omega^2 = k/m$.* Substituting $x = e^{pt}$ gives

$$p^2 + 2\gamma p + \omega^2 = 0,$$

$x = e^{pt}$

and therefore

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

This leads to three cases:

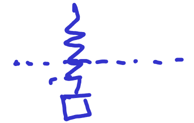
- $\gamma^2 > \omega^2 \implies$ two real roots ("overdamped")
- $\gamma^2 = \omega^2 \implies$ one real root ("critically overdamped")
- $\gamma^2 < \omega^2 \implies$ no real roots ("underdamped")

*Recall that β is the damping constant, k is the spring constant, and m is the mass.

Spring-mass systems (cont.)

Damped motion (cont.)

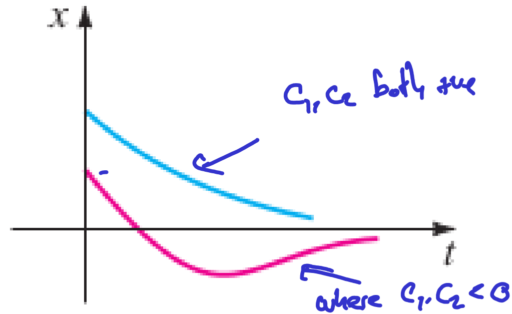
Case 1: $\gamma^2 > \omega^2 \implies$ two real roots ("overdamped")



$$\gamma^2 > \omega^2 \implies \frac{\beta}{4m^2} > \frac{k}{m} \implies \beta^2 > 4km.$$

Then $\quad p_1 = -\gamma + \sqrt{\gamma^2 - \omega^2} < 0 \quad$ and $\quad p_2 = -\gamma - \sqrt{\gamma^2 - \omega^2} < 0$ and

$$\begin{aligned} x(t) &= c_1 e^{p_1 t} + c_2 e^{p_2 t} \\ &= e^{-\gamma t}\left(c_1 e^{+\sqrt{\gamma^2-\omega^2}t} + c_2 e^{-\sqrt{\gamma^2-\omega^2}t}\right) \\ &= c_1 e^{p_2 t}\left(e^{2\sqrt{\gamma^2-\omega^2}t} + \frac{c_2}{c_1}\right) \end{aligned}$$



Note that $x \to 0$ as $t \to \infty$,
i.e., it returns to the equilibrium position.
Notice there is at most one 'oscillation' (if c_1 and c_2 are of different signs).

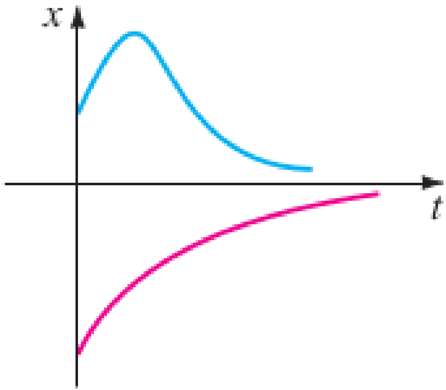
Case 2: $\gamma^2 = \omega^2 \implies$ one real root ("critically damped")

$$\gamma^2 = \omega^2 \implies \beta^2 = 4km \quad \text{and} \quad p = -\gamma.$$

Then we have that (recall Lecture 15)

$$\begin{aligned} x(t) &= c_1 e^{pt} + c_2 t e^{pt} \\ &= c_2 e^{-\gamma t}\left(t + \frac{c_1}{c_2}\right) \end{aligned}$$

and again note that $x \to 0$ as $t \to \infty$ and that there is at most one oscillation.



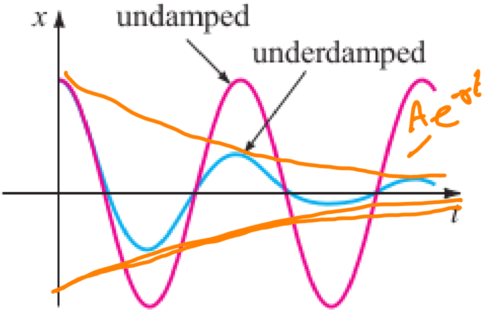
Spring-mass systems (cont.)
Damped motion (cont.)

Case 3: $\gamma^2 < \omega^2 \implies$ no real roots ("underdamped")

$$\gamma^2 < \omega^2 \implies \frac{\beta}{4m^2} > \frac{k}{m} \implies \beta^2 < 4km.$$

Then $\quad p_1 = -\gamma + i\sqrt{\omega^2 - \gamma^2} < 0 \quad$ and $\quad p_2 = -\gamma - i\sqrt{\omega^2 - \gamma^2} < 0$ and†

$$\begin{aligned} x(t) &= c_1 e^{p_1 t} + c_2 e^{p_2 t} \\ &= e^{-\gamma t}\left(d_1 \cos(\sqrt{\omega^2 - \gamma^2}t) + d_2 \sin(\sqrt{\omega^2 - \gamma^2}t)\right) \\ &= \underbrace{Ae^{-\gamma t}} \sin(\sqrt{\omega^2 - \gamma^2}t + \phi) \end{aligned}$$



Note that the "amplitude" decays exponentially.

We call this a "damped oscillator".

†Recall Lecture 15.

$x'' + 2\gamma x' + \omega^2 x = 0$

Example 1: Consider the spring-mass system described by

$$x'' + 4x' + 4x = 0 \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x'(0) = -2.$$

(Note $\omega^2 = 4$ and $\gamma = 2$.) If we try $x = e^{pt}$ we have

$$p^2 + 4p + 4 = (p+2)^2 = 0 \implies p = -2,$$

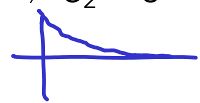
i.e., one real root $\implies$ critical damping and therefore

$$\begin{aligned} x &= c_1 e^{-2t} + c_2 t e^{-2t} \\ x' &= -2c_1 e^{-2t} + c_2(1-2t)e^{-2t}. \end{aligned}$$

Initial conditions $\begin{cases} x(0) = 1 \implies & 1 = c_1 + 0 & \implies c_1 = 1 \\ x'(0) = -2 \implies & -2 = -2c_1 + c_2 & \implies c_2 = 0 \end{cases}.$

Hence,

$$\boxed{x(t) = e^{-2t}.}$$



Example 2: Consider the spring-mass system described by

$$x'' + 2x' + 4x = 0 \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x'(0) = -2.$$

If we try $x = e^{pt}$ we have

$$p^2 + 2p + 4 = 0 \implies p = -\tfrac{1}{2}(-2 \pm \sqrt{4 - 16}) = -1 \pm i\sqrt{3}$$

i.e., no real roots $\implies$ underdamped and therefore

$$\begin{aligned}
x &= c_1 e^{(-1+i\sqrt{3})t} + c_2 e^{(-1-i\sqrt{3})t} \\
&= e^{-t}(c_1 e^{i\sqrt{3}t} + c_2 e^{-i\sqrt{3}t}) \\
&= e^{-t}(d_1 \cos(\sqrt{3}t) + d_2 \sin(\sqrt{3}t)) \quad \longleftarrow \text{lecture 15} \\
x' &= -e^{-t}(d_1 \cos(\sqrt{3}t) + d_2 \sin(\sqrt{3}t)) \\
&\quad + e^{-t}(-\sqrt{3}d_1 \sin(\sqrt{3}t) + \sqrt{3}d_2 \cos(\sqrt{3}t)) \\
&= e^{-t}((-d_1 + \sqrt{3}d_2)\cos(\sqrt{3}t) + (-d_2 - \sqrt{3}d_1)\sin(\sqrt{3}t))
\end{aligned}$$

Spring-mass systems (cont.)

Example 2 (cont.)

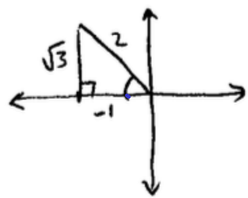
Initial conditions:

$$\begin{cases} x(0) = 1 \implies & 1 = d_1 + 0 & \implies d_1 = 1 \\ x'(0) = -2 \implies & -2 = -d_1 + \sqrt{3}d_2 & \implies d_2 = -1/\sqrt{3} \end{cases}$$

Hence,

$$\begin{aligned} x(t) &= e^{-t}\left(\cos(\sqrt{3}t) - \tfrac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right) \\ &= e^{-t}\tfrac{2}{\sqrt{3}}\left(\tfrac{\sqrt{3}}{2}\cos(\sqrt{3}t) - \tfrac{1}{2}\sin(\sqrt{3}t)\right). \end{aligned}$$

Let $\sin\phi = \frac{\sqrt{3}}{2}$ and $\cos\phi = -\frac{1}{2}$ then $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and



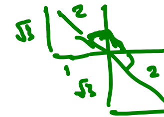
$$\boxed{x(t) = \frac{2}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}\left(t - \frac{4\pi}{3\sqrt{3}}\right)\right).}$$

Task: Write $x(t) = e^{-t}\left(\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)$ in Amplitude-Phase form.

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{1 + 1/3} = 2/\sqrt{3}$$

$$\Rightarrow x(t) = 2/\sqrt{3}\, e^{-t}\left(\underbrace{\frac{\sqrt{3}}{2}}_{\sin\phi}\cos\sqrt{3}t - \underbrace{\frac{1}{2}}_{\cos\phi}\sin\sqrt{3}t\right)$$

$$\Rightarrow \sin\phi = \sqrt{3}/2$$
$$\cos\phi = -1/2$$
$$\tan\phi = -\sqrt{3}$$



$$\Rightarrow \phi = -\pi/3 + \pi$$
$$= 2\pi/3$$

$$\Rightarrow x(t) = 2/\sqrt{3}\, e^{-t}\sin\left(\sqrt{3}t + 2\pi/3\right)$$
$$= 2/\sqrt{3}\, e^{-t}\sin\left(\sqrt{3}t + 2\pi/3 - 2\pi\right)$$
$$= 2/\sqrt{3}\, e^{-t}\sin\left(\sqrt{3}t - 4\pi/3\right)$$
$$= \underline{2/\sqrt{3}}\, e^{-t}\sin\left(\sqrt{3}\left(t - \underline{\frac{4\pi}{3\sqrt{3}}}\right)\right)$$

Spring-mass systems (cont.)
Example 2 (cont.)

$$\boxed{x(t) = \frac{2}{\sqrt{3}} e^{-t} \sin\left(\sqrt{3}\left(t - \frac{4\pi}{3\sqrt{3}}\right)\right).}$$