Applied differential equations

TW244 - Lecture 25

7.1: Laplace transforms

Prof Nick Hale - 2020

6: SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Although Prof Hale really likes special functions,
we don't have time to cover them in this course.
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Fortunately, Prof Hale also likes...

7.1: LAPLACE TRANSFORMS

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Definition:
Suppose $f(t)$ is defined for all $t \geq 0$.

The Laplace transform of $f(t)$ is defined as:

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

for all values of s for which the integral exists.

Linearity:
It follows from the definition that the Laplace transform is a linear operator.
That is, if $f(t)$ and $g(t)$ are continuous functions on $[0, \infty)$, then

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}.$$

Let $f(t) = 1$ then

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty} \left[-\tfrac{1}{s}e^{-st}\right]_{t=0}^{t=b} \\
&= \lim_{b\to\infty} \left[-\tfrac{1}{s}e^{-sb} + \tfrac{1}{s}\right] \\
&= \boxed{\frac{1}{s}} \text{ for } s > 0.
\end{aligned}$$

Note we require $s > 0$ as the integral doesn't converge for $s \leq 0$!

We will usually skip a few of these steps and just write:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

7.1: Laplace transforms
Example 2

Recall integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Let $f(t) = t$ then

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty \underbrace{e^{-st}}_{g'}\underbrace{t}_{f}\,dt \\
&= \Big[-\tfrac{t}{s}e^{-st}\Big]_0^\infty - \int_0^\infty (-\tfrac{1}{s})e^{-st}\,dt \\
&= -\tfrac{1}{s}\Big[te^{-st}\Big]_0^\infty + \tfrac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= -\tfrac{1}{s}\Big[0-0\Big] + \tfrac{1}{s}\Big[-\tfrac{1}{s}e^{-st}\Big]_0^\infty \\
&= \boxed{\frac{1}{s^2}} \text{ for } s > 0.
\end{aligned}$$

Exercise: Show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 1, 2, 3, \ldots$

7.1: Laplace transforms

Example 3

Let $f(t) = e^{at}$, $a \in \mathbb{R}$ then

$$
\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st} e^{at}\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \\
&= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty \\
&= \frac{1}{a-s}\left[0 - 1\right] \text{ for } s > a \\
&= \boxed{\frac{1}{s-a}} \text{ for } s > a.
\end{aligned}
$$

Note how Example 1 is a special case of Example 3 with $a = 0$!
Exercise: What changes if a is a complex number?

Example 4

Let $f(t) = \sin(t)$ then (again using integration by parts)

$$\begin{aligned}
\mathcal{L}\{\sin(t)\} &= \int_0^\infty e^{-st}\sin(t)\,dt = \left[-\tfrac{1}{s}e^{-st}\sin(t)\right]_0^\infty + \tfrac{1}{s}\int_0^\infty e^{-st}\cos(t)\,dt \\
&= 0 + \tfrac{1}{s}\mathcal{L}\{\cos(t)\}, \quad s > 0. \qquad (1) \\
\mathcal{L}\{\cos(t)\} &= \int_0^\infty e^{-st}\cos(t)\,dt = \left[-\tfrac{1}{s}e^{-st}\cos(t)\right]_0^\infty - \tfrac{1}{s}\int_0^\infty e^{-st}\sin(t)\,dt \\
&= \tfrac{1}{s} - \tfrac{1}{s}\mathcal{L}\{\sin(t)\}, \quad s > 0. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
(1)\,\&\,(2) \implies\ & \mathcal{L}\{\sin(t)\} = \tfrac{1}{s^2}(1 - \mathcal{L}\{\sin(t)\}) \\
\implies\ & (1 - \tfrac{1}{s^2})\mathcal{L}\{\sin(t)\} = \tfrac{1}{s^2} \\
\implies\ & \boxed{\mathcal{L}\{\sin(t)\} = \tfrac{1}{s^2+1}}, \quad s > 0. \qquad (3)
\end{aligned}$$

Exercise: Rederive (3) using example 3, linearity of $\mathcal{L}\{\}$, and the fact $\sin t = (e^{it} - e^{-it})/2i$.
Exercise: Show that $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$ and $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$.

Table of Laplace transforms $(n = 1, 2, 3, \ldots)$

$$
\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s} \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}, \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \\
\mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2} \\
\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2} \\
\mathcal{L}\{\sinh(kt)\} &= \frac{k}{s^2-k^2} \\
\mathcal{L}\{\cosh(kt)\} &= \frac{s}{s^2-k^2}
\end{aligned}
$$

Exercise: Compute the Laplace transform of $f(t) = 2t + \sin(3t)$.
Note: You do not have to remember this table, but you may need to derive it!

The Laplace transform has several other useful properties.

In particular if $F(s) = \mathcal{L}\{f(t)\}$ then

- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$
- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

The final one will be particularly useful to us, and we will prove this result next time. For the others, see Section 7.3 of the textbook or Tutorial 05.

Definition

If $F(s)$ is the Laplace transform of a function $f(t)$, i.e., $F(s) = \mathcal{L}\{f(t)\}$ then we say $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Examples: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1, \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}, \mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos(kt)$, etc.

Exercise: Show that the inverse Laplace transform is also linear.

Example 1: Find the inverse Laplace transform of $F(s) = \frac{s+1}{s^2-4s}$.

This is not in our table! Let's use partial fractions to write it in an easier form:

$$\frac{s+1}{s^2-4s} = \frac{A}{s} + \frac{B}{s-4} \overbrace{\Longrightarrow}^{\text{partial fractions}} A = -\frac{1}{4}, B = \frac{5}{4} \Longrightarrow \frac{s+1}{s^2-4s} = -\frac{1}{4}\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}$$

$$\Longrightarrow \mathcal{L}^{-1}\{\tfrac{s+1}{s^2-4s}\} = -\tfrac{1}{4}\mathcal{L}^{-1}\{\tfrac{1}{s}\} + \tfrac{5}{4}\mathcal{L}^{-1}\{\tfrac{1}{s-4}\} = -\tfrac{1}{4} \cdot 1 + \tfrac{5}{4}e^{4t}.$$

Example 2: Exercise: Find the inverse Laplace transform of $F(s) = \frac{s^5+s^2+7}{s^7+7s^5}$.*

*Hint: Use partial fractions, rectify constants, then use the table from previous page. Solution: $f(t) = t^4/24 + \sin(\sqrt{7}t)/\sqrt{7}$.