TW244 **Opdrag / Assignment #2** **Due 23:59 on 12 Sep 2021**

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P1: Elon Musk se SpaceX maatskappy het 'n tekort aan personeel weens die COVID-19 pandemie en het vervolgens van die wiskundige modellering aan die 2021 TW244 klas by SU uitgekontrakteer.[a] Jou eerste taak is om 'n model vir sy vuurpyllanserings te ontwikkel.

[a] Waarskynlik nie een van sy beste besigheidsidees nie...

P1: The COVID-19 pandemic has left Elon Musk's SpaceX company short-staffed and he has outsourced some of the mathematical modelling to the 2021 TW244 class at SU.[a] Your first task is to develop a model for his rocket launches.

[a] Probably not one of his most sensible business ideas...

(a) [1 punt] Wanneer die massa van 'n liggaam verander, word Newton se tweede wet van beweging $F = \frac{d}{dt}(mv)$. As ons aanneem dat 'n vuurpyl onderhewig is aan lineêre lugweerstand en 'n konstante stuwing van R ophef, aflei die volgende model vir die snelheid van die vuurpyl:

(a) [1 mark] When the mass of a body is changing, Newton's second law of motion becomes $F = \frac{d}{dt}(mv)$. Assuming that a rocket is subject to linear air resistance and provides a constant thrust R, derive the following model for the velocity of the rocket:

$$\frac{dv}{dt} + \frac{k + m'(t)}{m(t)}v = -g + \frac{R}{m(t)},$$

waar $m(t)$ die tyd-afhanklike massa van die vuurpyl is en k die sleepkoëffisiënt. Sluit 'n skets in wat die relevante kragte op die vuurpyl aandui.

where $m(t)$ is the time-dependent mass of the rocket and k is the drag coefficient. Include a sketch labelling the relevant forces acting on the rocket.

(b) [1 punt] Gestel die brandstof word gebruik teen 'n konstante tempo van λkg/s en dat die vuurpyl 'n aanvanklike totale massa m_0 het. Lei 'n DV vir $m(t)$ af. Los die DV op (met die hand) en vervang die DV in (a).

(b) [1 mark] Assuming that the fuel is used at a constant rate of λkg/s and that the rocket has an initial total mass m_0, derive a DE for $m(t)$. Solve the DE (by hand) and substitute to the DE in (a).

(c) [2 punte] Gebruik die integrasiefaktormetode om die nuwe DV vir $v(t)$ op te los as $\lambda = 1$, $m_0 = 200$kg, $R = 2000$N, $g = 9.8\text{m/s}^2$, $k = 3$kg/s, en $v(0) = 0$, om te wys dat

(c) [2 marks] Use the integrating factor method to solve the new DE for $v(t)$ if $\lambda = 1$, $m_0 = 200$kg, $R = 2000$N, $g = 9.8\text{m/s}^2$, $k = 3$kg/s, and $v(0) = 0$, to show that

$$v(t) = t(3t + 25)/125.$$

(d) [1 punt] Gebruik die formule $\frac{ds}{dt} = v$ om 'n formule vir die hoogte $s(t)$ van die vuurpyl op tyd t te bepaal.

(d) [1 mark] Use the formula $\frac{ds}{dt} = v$ to determine a formula for the height $s(t)$ of the rocket at time t.

(e) [2 punte] Gestel die vuurpyl het aanvanklik 100kg brandstof (die oorblywende 100kg is die struktuur van die vuurpyl, vragvrag, ens.). Wat is die uitbrandingstyd, t_B, waarop al die brandstof gebruik word? Wat is die snelheid van die vuurpyl op hierdie tydstip? Wat is die hoogte van die vuurpyl op hierdie tydstip? Gebruik MATLAB/Python om $v(t)$ en $s(t)$ vir $0 \le t \le t_B$ te stip.

(e) [2 marks] Suppose that the rocket initially has 100kg of fuel (the remaining 100kg is the structure of the rocket, payload, etc). What is the burnout time, t_B, at which all the fuel is used? What is the velocity of the rocket at this time? What is the height of the rocket at this time? Use MATLAB/Python to plot $v(t)$ and $s(t)$ for $0 \le t \le t_B$.

(f) [2 punte] Vorm en los 'n model op vir die snelheid en hoogte van die vuurpyl vir $t_B < t < t_E$, waar t_E die tyd is waarop die vuurpyl na die aarde terugkeer. Stip $(t, v(t))$ and $(t, s(t))$ vir $0 \le t \le t_E$. Wat is die maksimum hoogte, s_Z, wat die vuurpyl bereik? Wat is die tyd, t_E, waarop die vuurpyl na die aarde terugkeer?

(f) [2 marks] Form and solve a model for the velocity and height of the rocket for $t_B < t < t_E$, where t_E is the time at which the rocket returns to earth. Plot $(t, v(t))$ and $(t, s(t))$ for $0 \le t \le t_E$. What is the maxium height, s_Z, achieved by the rocket? What is the time, t_E, at which the rocket returns to Earth?

P2: Nadat SpaceX 'n aantal vuurpyle suksesvol gelanseer het, is SpaceX se terraformering van Mars nou aan die gang. Sir Elon het jou gevra om 'n model te ontwikkel vir die groei van 'n geneties gemodifiseerde plantsoort wat gebruik word om O_2 te genereer. Die data in die onderstaande tabel toon die groei van die plante in die toetsarea vir die eerste 12 maande.

P2: Having now successfully launched a number of rockets, SpaceX's terraforming of Mars is now underway. Sir Elon has asked you to develop a model for the growth of a genetically-modified plant species being used to generate O_2. The data in the table below shows the growth of the plants in testing area for the first 12 months.

t (months)	0	1	2	3	4	5	6	7	8	9	10	11	12
P (hectares)	4.2	5.7	8.4	11.3	15.4	20	25.9	32.3	38.1	47.3	54.3	62.9	77

(a) [1 punt] Gebruik eers die Malthus-model om die plantbevolking te modelleer, m.a.w., $P(t) = P_0 e^{kt}$, waar t in maande gemeet word. Om die groeitempo k te vind, pas die oplossing sodat $P(1) = 5.7$. Stip die data vanaf die tabel (as sirkels of kolletjies) en stip die Malthus model oplossing op dieselfde figuur (as 'n soliede lyn) met jou gekose k-waarde. Gee die model 'n goeie passing?

(b) [1 punt] 'n Beter manier om k te pas is om daarop te let dat as $P(t) = P_0 e^{kt}$ dan $\ln(P(t)) = \ln(P_0) + kt$. Stip die log van ons data om jouself te oortuig dat hierdie (ongeveer) lineêr in t is. Ons kan nou k vind deur 'n reguit lyn op $(t, \ln(P))$ te pas, wat bekend staan as *lineêre-log regressie*. Gebruik `polyfit` in MATLAB (of `numpy.polyfit`) om 'n reguit lyn op $(t, \ln(P))$ te pas, en gebruik die helling om k te vind. Stip weereens die data $(t, P(t))$ as kolletjies asook jou nuwe passing met hierdie waarde van k. Is hierdie passing beter?

(c) [1 punt] Een van die redes waarom die Malthus-model nie die data goed pas nie, is oor dit onbeperkte groei van die plantbevolking toelaat. Kom ons beskou eerder die Logistieke model, $\frac{dP}{dt} = P(a - bP)$ met $P(0) = P_0$. Ons het nou twee parameters om te pas. Alhoewel daar meer gesofistikeerde tegnieke is, kan ons begin deur te kyk na $b = 0$ and a soos in **(b)** en van die probeer en fouteer metode gebruik maak totdat ons 'n goeie passing kry. Deur dit te doen, kry ek $a = 0.325$ en $b = 0.003$. **(i)** Stip die oplossing vir die Logistieke model met hierdie parameters en wys dat dit die data goed pas. **(ii)** Gebruik die model om die plantbevolking na 18 en 24 maande te voorspel, asook die drakrag van die toetsarea.

(d) [1 punt] Ses maande later kry jy 'n oproep van Elon wat sê dat jou model verkeerd is en dat die plante vinniger groei as verwag! Die opnames vir die afgelope ses maande word hieronder getoon. Voeg dit by jou grafiek en vergelyk met jou oplossingskurwe van **(c)** wat tot 18 maande verleng is.

(a) [1 mark] First use the Malthus model to model the plant population, i.e., $P(t) = P_0 e^{kt}$, where t is measured in months. To find the growth rate k, fit the solution so that $P(1) = 5.7$. Plot the data from the table (as circles or dots) and on the same figure plot the Malthus model solution (as a solid line) with your chosen value of k. Does the model give a good fit?

(b) [1 mark] A better way to fit k is to observe that if $P(t) = P_0 e^{kt}$ then $\ln(P(t)) = \ln(P_0) + kt$. Plot the log of our data to convince yourself that this is (approximately) linear in t. We can now find k by fitting a straight line to $(t, \ln(P))$, which is known as *linear-log regression*. Use `polyfit` in MATLAB (or `numpy.polyfit`) to fit a straight line to $(t, \ln(P))$, and use the slope find k. Plot again the data $(t, P(t))$ as circles/dots and your new fit with this value of k. Is the fit any better?

(c) [1 mark] One reason why the Malthus model is not giving a good fit to the data is that it allows unbounded growth of the plant species. Let us instead consider the Logistic model, $\frac{dP}{dt} = P(a - bP)$ with $P(0) = P_0$. We now have two parameters to fit. Although there are more sophisticated techniques, one thing we could do is start with $b = 0$ and a as in **(b)** and use trial and error until we get a good fit. Doing so, I find $a = 0.325$ and $b = 0.003$. **(i)** Plot the solution to the Logistic model with these parameters and show it gives a good fit to the data. **(ii)** Use the model to predict the plant population after 18 and 24 months, as well as the carrying capacity of the test area.

(d) [1 mark] Six months later you get a phone call from Elon telling you that your model is off and the plants are growing faster than expected! The counts for the past six months are shown below. Add them to you plot and compare to your solution curve from **(c)** extended to 18 months.

t	13	14	15	16	17	18
P	86.7	98.1	113.6	123.6	136.4	151.9

(e) [1 punt] Tyd om ons aannames en ons model te hersien! Dit lyk asof die Logistieke model die beperkende effek van die omgewing op die bevolking oorskat. Laat ons eerder 'n model van die volgende vorm gebruik:

(e) [1 mark] Time to revise our assumptions and our model! It looks like the Logistic model is overestimating the limiting effect of the environment on the population. Let us instead use a model of the form

$$\frac{dP}{dt} = P(a - b \log P), \qquad P(0) = P_0,$$

wat bekend staan as die Gompertz model. (Die idee hier is dat die tempo bP^2 waarteen die visse sterf te groot is in die Logistieke model, so ons beskou 'n laer tempo van $bP \log P$.) Gebruik skeiding van veranderlikes om die AWP hierbo op te los, en toon aan dat die oplossing die vorm het

known as the Gompertz model. (The idea here is that the 'death rate', bP^2, in the Logistic model was too strong, so we consider a weaker rate $bP \log P$.) Use the method of separation of variables to solve the IVP above, to show the solution is of the form

$$P(t) = e^{a/b + Ce^{-bt}},$$

waar u die konstante C moet bepaal. (Die verandering van veranderlikes $P = e^u$ sal nuttig wees.)

where you must determine the constant C. (The change of variables $P = e^u$ will be useful.)

(f) [1 punt] Weereens het ons twee parameters om te pas. Deur die probeer en fouteer metode vind ek dat $a = 0.486$ en $b = 0.079$ 'n goeie passing gee. Voeg hierdie oplossing by jou grafiek van **(d)** en wys dat die model die data goed pas vir al 18 maande. Bepaal die bevolking wat die model na 24 maande voorspel, asook die drakrag van die plaas soos voorspel deur die model.

(f) [1 mark] Again, we have two parameters to fit in the model. By trial and error I find that $a = 0.486$ and $b = 0.079$ give a good fit. Add this solution to your plot from **(d)** and show the model gives a good fit to the data for all 18 months. Determine the population after 24 months and the carrying capacity of the farm predicted by the model.

P3: 'n Paar maande gaan verby voordat 'n interessante ontdekking gemaak word: dit lyk asof 'n soort uitheemse organisme die plante vreet! Dit lyk nie asof die organisme enige voedingswaarde uit die plante kry nie, maar dit blyk dat dit seisoenale gedrag toon. Ons kan die organisme-bevolking, $O(t)$, modelleer as

P3: A few months pass before an interesting discovery is made: some kind of alien organism appears to be eating the plants! The organism doesn't seem to gain any nourishment from the plants, but does seem to exhibit seasonal behaviour. We can model the organism population, $O(t)$, as

$$\frac{dO}{dt} = \gamma \sin\left(\frac{2\pi}{23}t\right) O(t),$$

waar die $2\pi/23$ voortspruit uit die feit dat 'n jaar op Mars ongeveer 23 maande is.

where the $2\pi/23$ arises from the fact that a year on Mars is approximately 23 months.

(a) Dit is moontlik dat hierdie organisme die beperking in ons plantebevolking veroorsaak het. Kom ons ondersoek dit deur die model wat vanuit die DV hierbo gevorm is, te kombineer met

(a) It is likely that this organism was the cause of limiting our plant population. Let's test this by investigating the model formed of the DE above combined with

$$\frac{dP}{dt} = P\big(a - bO(t)\log(P(t))\big)$$

Uit waarnemings/passings soortgelyk aan die vorige vrae, lei ons $a = 0.5$, $b = 0.05$, $\gamma = 0.01$, $P(0) = 4.2$, en $O(0) = 2$ af.

Gebruik `ode45` in MATLAB (of `scipy.integrate.solve_ivp` in Python) om die gekoppelde DV-stelsel vir $P(t)$ en $O(t)$ op te los vir die eerste 4 jaar (d.w.s., 92 maande). Stel 'n figuur van $(t, P(t))$ en $(t, O(t))$ op (nie op dieselfde asse nie, aangesien die skale so verskillend is[a]).

[a]Of jy kan probeer om die `plotyy`-funksie in MATLAB te gebruik.

From observations/fittings similar to the previous questions, we deduce $a = 0.5$, $b = 0.05$, $\gamma = 0.01$, $P(0) = 4.2$, and $O(0) = 2$.

Use `ode45` in MATLAB (or `scipy.integrate.solve_ivp` in Python) to solve the coupled DE system for $P(t)$ and $O(t)$ for first 4 years (i.e., 92 months). Produce a figure of $(t, P(t))$ and $(t, O(t))$ (not on the same axes, since the scales are so different[a]).

[a]Or you can try using the `plotyy` function in MATLAB.

(b) [Bonus] Verbeter die model van P3a om ook die suurstof wat deur die plante geproduseer word, in te sluit. Aanvaar dat die verandering in O_2 van die vorm $d \times P(100 - O_2)$ is en dat die verhoogde O_2 plantgroei beperk deur 'n bykomende term $e \times P\log(P)O_2$ by ons model vir $dP(t)/dt$ te voeg. Los die nuwe stelsel op m.b.v. `ode45` of `solve_ivp`, en neem $d = 0.0001$, $e = 0.000005$, en $O_2(0) = 0\%$. Stip die oplossing soos in **(a)**, maar sluit ook $(t, O_2(t))$ in. Hoeveel maande voordat die O_2-vlak 'n bewoonbare 20% bereik?

(b) [Bonus] Improve the model from P3a to also include the Oxygen produced by the plants. Assume that the change in O_2 is of the form $d \times P(100 - O_2)$ and that the increased O_2 limits plant growth by introducing an additional term $e \times P\log(P)O_2$ in our model for $dP(t)/dt$. Solve the new system using `ode45` or `solve_ivp`, taking $d = 0.0001$, $e = 0.000005$, and $O_2(0) = 0\%$. Plot the solution as in **(a)** but also including $(t, O_2(t))$. How many months before the O_2 level reaches a habitable 20%?

Neem kennis: Die biologie in hierdie vraag is 'n bietjie wispelturig (d.w.s., feitlik heeltemal opgemaak). Die bedoeling was om intuïsie te gee oor hoe ons so 'n stelsel met DV's kan modelleer.

Disclaimer: The biology in this question is a little bit wonky (i.e., almost entirely made up). The intention was give some intuition how we might model such a system with DEs.

P4: [Opsioneel] Ondersoek die oplossing vir **P1(c)** wanneer (i) λ/k nie 'n heelgetal is nie, (ii) $\lambda = k$, (iii) $\lambda = k/2$.

P4: [Optional] Investigate the solution to **P1(c)** when (i) λ/k is not an integer, (ii) $\lambda = k$, (iii) $\lambda = k/2$.

P5: [Opsioneel] Herhaal **P1(a-e)** met die aanname van 'n kwadratiese sleepkrag. (Jy sal waarskynlik die resulterende DV numeries moet oplos, bv. met `ode45`.)

P5: [Optional] Repeat **P1(a-e)** with the assumption of a quadratic drag force. (You will likely need to solve the resulting DE numerically, e.g., with `ode45`.)

P6: [Opsioneel] In die praktyk word meerstapige vuurpyle gereeld gebruik. (Trouens, volgens Wikipedia: *"A multistage rocket is required to reach orbital speed. Single-stage-to-orbit designs are sought, but have not yet been demonstrated."*) Die grootste voordeel is dat sodra die brandstof van die een fase opgebruik is, kan daardie gedeelte van die vuurpyl afgegooi word, en die effektiewe gewig van die vuurpyl verminder word. Herhaal P1 onder die aanname van 'n tweestapige vuurpyl, met elke fase wat 25kg weeg en 50kg brandstof bevat. Vergelyk die maksimum hoogte wat so 'n vuurpyl sal bereik met dié van **P1(f)**.

P6: [Optional] In practice, multistage rockets are often used. (In fact, according to Wikipedia: *"A multistage rocket is required to reach orbital speed. Single-stage-to-orbit designs are sought, but have not yet been demonstrated."*) The main advantage is that once the fuel from one stage has been expended, that part of the rocket may be jettisoned, and the effective weight of the rocket reduced. Repeat P1 under the assumption of a two stage rocket, with each stage weighing 25kg and containing 50kg of fuel. Compare the maximum height obtainable with such a rocket to that of **P1(f)**.

P7: [Opsioneel] Gebruik `lsqnonlin` of `scipy.optimize.least_squares` om die a en b in P2(c) en P2(f) te vind.

P7: [Optional] Use `lsqnonlin` or `scipy.optimize.least_squares` to find the coefficients a and b in P2(c) and P2(f).