Naam/Name: Memo

Stud. Nr: ______

Toegepaste Differensiaalvergelykings
TW244 Toets 2 2017

Instruksies:

(a) 6 probleme, 50 + 9 bonus punte (maks = 50).

(b) 2.5 uur, toeboek.

(c) Sakrekenaars **word toegelaat**. Selfone **nie**.

(d) Toon alle bewerkings. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(e) Daar is leë bladsye aan die agterkant van die vraestel as jou antwoorde nie inpas in die gegewe spasies nie. Dui duidelik aan as jou antwoord voortgaan op een van hierdie bladsye.

(f) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

Applied Differential Equations
TW244 Test 2 2017

Instructions:

(a) 6 problems, 50 + 9 bonus marks (max = 50).

(b) 2.5 hours, closed book.

(c) Calculators **are** allowed. Cell phones are **not**.

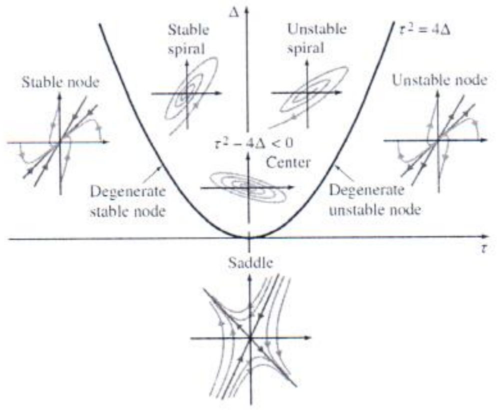
(d) Show all calculations. A correct answer does not earn full marks without the necessary explanation.

(e) There are blank pages at the back of the paper in case you cannot fit your answer in the space provided. Indicate clearly if your answer continues to one of these pages.

(f) The formulas below may be used without proof anywhere in the test.

Formules/Formulas:

- Wronskiaan/Wronskian: $$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$
- Deelsgewyse integrasie/ Integration by parts: $$\int_a^b f\frac{dg}{dx}\,dx = [fg]_a^b - \int_a^b \frac{df}{dx} g\,dx$$
- Dubbelhoek formules/ Double angle formulae: $$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$ $$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
- Laplace transform $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
- Laplace transform of derivatives/ Laplace transform of derivatives $$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
- $\tau = \text{trace}(A)$ & $\Delta = \det(A) \implies$ $$\text{eig}(A) = \tfrac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$$
- Tangent: $\tan(\pi/6) = 1/\sqrt{3}, \quad \tan(\pi/4) = 1, \quad \tan(\pi/3) = \sqrt{3}, \quad \tan(\pi/2) = \sqrt{3}.$
- Klassifikasie van kritieke punte vir **lineêre** stelsels/ Classification of critical points for **linear** systems



Prob 1 (10 + 1 punte/marks)

Beskou die DV:

Consider the differential equation:

$$y'' + 4y' + 4y = 0.$$

(a) Bevestig dat $y_1 = e^{-2x}$ en $y_2 = xe^{-2x}$ oplossings is.

2 **(a)** Verify that $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are solutions.

$y_1 = e^{-2x} \Rightarrow y_1' = -2e^{-2x},\ y_1'' = 4e^{-2x}$

$\Rightarrow y_1'' + 4y_1' + 4y_1 = e^{-2x}(4-8+4) = 0.$ 1

$y_2 = xe^{-2x} \Rightarrow y_2' = e^{-2x}(1-2x),\ y_2'' = e^{-2x}(-2+4x-2) = 4e^{-2x}(x-1)$

$\Rightarrow y_2'' + 4y_2' + 4y_2 = e^{-2x}(4x-4+4-8x+4x) = 0$ 1

(b) Wys dat y_1 & y_2 fundamentele oplossings is deur die gepaste Wronskiaan te bereken, en skryf die algemene oplossing vir die DV neer.

2 **(b)** By calculating the appropriate Wronskian, show that y_1 & y_2 are fundamental solutions and write down the general solution to the DE.

$W(x) = y_1y_2' - y_2y_1' = e^{-2x}e^{-2x}(1-2x) - xe^{-2x}(-2)e^{-2x}$

$= e^{-4x} \neq 0\ \forall x \in \mathbb{R} \Rightarrow$ fundamental 1

$y(x) = c_1y_1(x) + c_2y_2(x)$

$= c_1e^{-2x} + c_2xe^{-2x}$ 1

(c) Los die aanvangswaardeprobleem op:

1 **(c)** Solve the initial value problem:

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

$y(0) = c_1 = 1$

$y'(0) = -2c_1 + c_2 = 0 \Rightarrow c_2 = 2$

$\Rightarrow \boxed{y(x) = e^{-2x}(1+2x)}$ 1

Beskou nou die nie-homogene aanvangswaardeprobleem:

Consider now the nonhomogenous initial value problem:

$$x'' + 4x' + 4x = 25\sin(t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

(d) As die bostaande vergelyking 'n model vir 'n veer-massa stelsel was, dan was die stelsel 'n...

(A) Lineêre veer, ongedemp, gedrewe

(B) Nie-lineêre veer, gedemp, ongedrewe

(C) Lineêre veer, gedemp, ongedrewe

(D) Lineêre veer, gedemp, gedrewe

(E) Nie-lineêre veer, ongedemp, gedrewe

1 **(d)** If the equation above were a model for a spring-mass system, the system would be ...

(A) Linear spring, undamped, driven

(B) Nonlinear spring, damped, undriven

(C) Linear spring, damped, undriven

(D) Linear spring, damped, driven

(E) Nonlinear spring, undamped, driven

(e) Gebruik die metode van onbepaalde koëffisiënte om die aanvangswaardeprobleem op te los.

4 **(e)** Use the method of undetermined coefficients to solve the initial value problem.

Let $y_p = A\cos(t) + B\sin(t)$ 1

$\Rightarrow y_p' = -A\sin(t) + B\cos(t), \quad y_p'' = -A\cos(t) - B\sin(t)$

$\Rightarrow y_p'' + 4y_p' + 4y = (-A+4B+4A)\cos(t) + (-B-4A+4B)\sin(t)$

$= 25\sin(t)$

$\Rightarrow \begin{cases} 3A+4B=0 \\ -4A+3B=25 \end{cases}$ 1 $\Rightarrow \begin{cases} 12A+16B=0 \\ -12A+9B=75 \end{cases} \Rightarrow 25B = 75$

$\Rightarrow B = 3$

$\Rightarrow \boxed{y_p(x) = -4\cos(t) + 3\sin(t)}$ 1 $\quad \Rightarrow A = -4$

$\Rightarrow x(t) = c_1e^{-2t} + c_2e^{-2t}t - 4\cos(t) + 3\sin(t)$

$0 = x(0) = c_1 - 4 \Rightarrow c_1 = 4$

$0 = x'(0) = -2c_1 + c_2 + 3 \Rightarrow c_2 = 2c_1 - 3 = 5$

$\Rightarrow \boxed{x(t) = e^{-2t}(4+5t) - 4\cos(t) + 3\sin(t)}$ 1

(bonus) Ervaar die bostaande veer-massa stelsel ... 1 **(bonus)** Is the spring-mass system above ...

(A) ligte demping/ underdamped

(B) kritieke demping/ critically damped

(C) swaar demping/ overdamped

('n Verkeerde antwoord word negatief gemerk!)

(A negative mark will be awarded for an incorrect answer!)

Prob 2 (5 punte/marks)

Beskou die volgende aanvangswaardeprobleem wat 'n veer-massa stelsel beskryf:

Consider the following initial value problem, which describes a spring-mass system:

$$\frac{d^2x}{dt^2} + 3x = 0, \quad x(0) = 1, \quad x'(0) = \cancel{3}.\ 1$$

(a) Los op die aanvangswaardeprobleem (op enige manier wat jy wil) en druk die oplossing in amplitude-fase vorm uit.

(3) **(a)** Solve the initial value problem (by any means you like) and express the solution in amplitude-phase form.

(b) Skets 'n grafiek van die oplossing. Toon die amplitude, periode en faseverskuiwing duidelik aan en gee in besonder aandag aan die aanvangsvoorwaardes.

(2) **(b)** Sketch a curve of the solution. Indicate clearly the amplitude, period, and phase shift. Pay special attention to the initial conditions.

$x(t) = c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t)$. (1)

$x(0) = c_1 = 1,\ x'(0) = \sqrt{3}c_2 = 1 \Rightarrow c_2 = 1/\sqrt{3}$

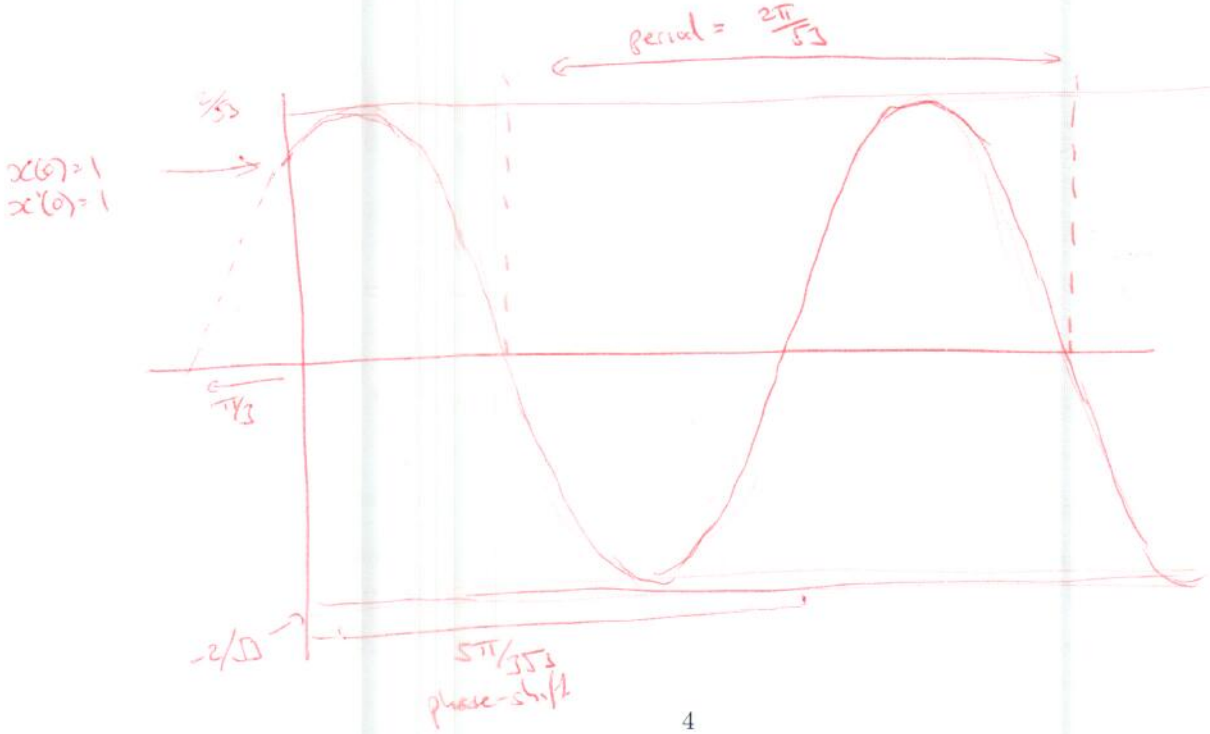
$\Rightarrow \boxed{x(t) = \cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)}$ (1)

$= A(\sin\phi\cos\sqrt{3}t + \cos\phi\sin\sqrt{3}t)$; $A = \sqrt{1^2 + 1/3} = 2/\sqrt{3}$

$= A\sin(\sqrt{3}t + \phi)$; $\phi = \tan^{-1}\left(\frac{1}{(1/\sqrt{3})}\right) = \pi/3 = -5\pi/3$

$\boxed{= \underbrace{2/\sqrt{3}}_{\text{Amplitude}}\sin(\underbrace{\sqrt{3}}_{\text{frequency}}(t \underbrace{- \tfrac{5\pi}{3\sqrt{3}}}_{\text{phase-shift}}))}$ (1)

(2)



Prob 3 (5 punte/marks)

Beskou die veer-massa stelsel wat beskryf word deur:

Consider the spring-mass system described by:

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \qquad x(0) = x'(0) = 0, \qquad 0 < \gamma \neq \omega,$$

met oplossing

with solution

$$x(t) = \frac{2F_0}{\omega^2 - \gamma^2} \sin\left[\tfrac{1}{2}(\omega + \gamma)t\right] \sin\left[\tfrac{1}{2}(\omega - \gamma)t\right].$$

(a) Beskou 'n drywingsfrekwensie naby die natuurlike frekwensie van die veer (m.a.w., $\gamma = \omega + 2\varepsilon$ met $0 < \varepsilon \ll 1$), en lei 'n vergelyking af wat *swewinge* demonstreer. Sluit 'n skematiese diagram van die effek in.

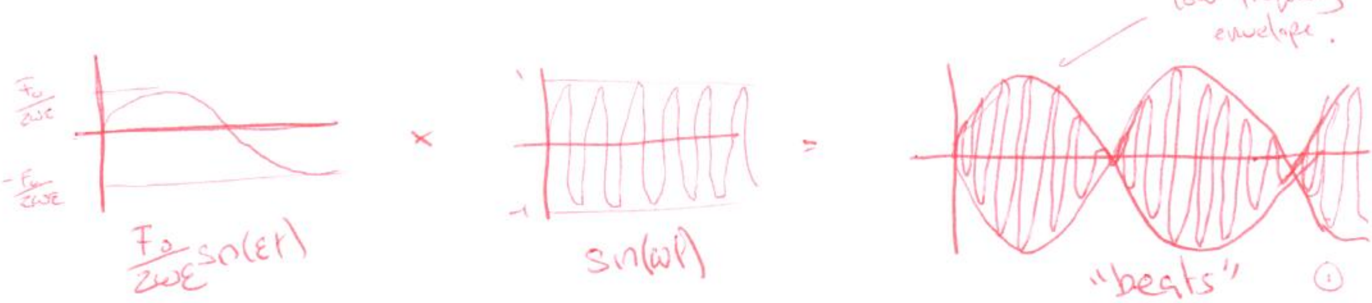
3 **(a)** By considering a forcing frequency near the natural frequency of the spring (i.e., $\gamma = \omega + 2\varepsilon$ with $0 < \varepsilon \ll 1$) derive an equation which demonstrates the *beats* phenomenon. Include a schematic diagram of the effect.

$\gamma = \omega + 2\varepsilon \Rightarrow \frac{1}{2}(\omega + \gamma) = \frac{1}{2}(\omega + \omega + 2\varepsilon) = \omega + \varepsilon \approx \omega$

$\frac{1}{2}(\omega - \gamma) = \frac{1}{2}(\omega - \omega - 2\varepsilon) = -\varepsilon$ (1)

$\omega^2 - \gamma^2 = (\omega - \gamma)(\omega + \gamma) \approx -4\omega\varepsilon.$

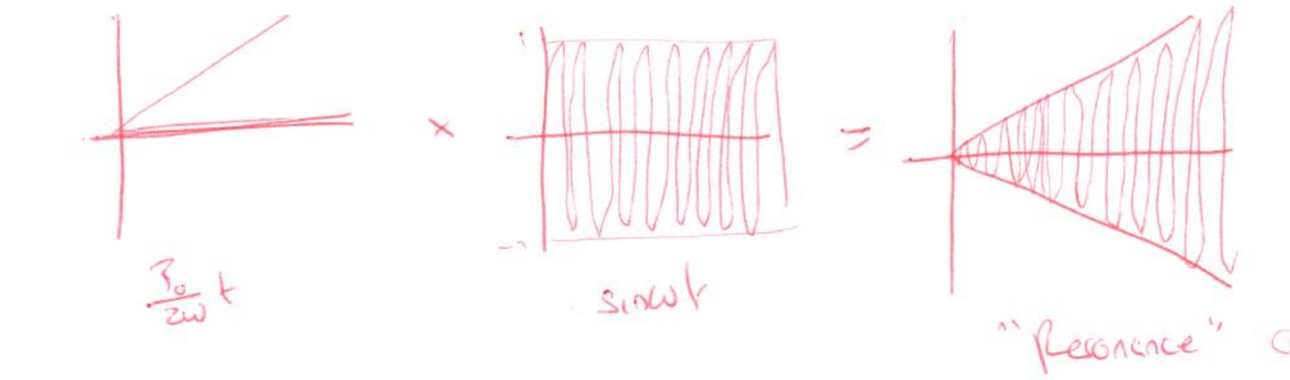
$\Rightarrow x(t) \approx \frac{2F_0}{-4\omega\varepsilon} \sin(\omega t)\sin(-\varepsilon t) = \frac{F_0}{2\omega\varepsilon}\sin(\varepsilon t) \cdot \sin(\omega t)$ (1)



(b) Beskou die limiet $\varepsilon \to 0$ in deel **(a)** en wys dat dit na *resonansie* lei. Sluit 'n gepaste diagram in.

2 **(b)** Consider the limit $\varepsilon \to 0$ in part **(a)** and show this leads to *resonance*. Include a suitable diagram.

$\varepsilon \to 0 \Rightarrow x(t) \approx \lim_{\varepsilon \to 0} \frac{F_0}{2\omega\varepsilon} \sin\varepsilon t \sin\omega t = \frac{F_0}{2\omega} t \sin\omega t$ (1)



Prob 4 (9 punte/marks)

Neem aan dat $f(t)$ glad genoeg is vir sy Laplace transform om te bestaan, asook die van sy vereiste afgeleides.

In the following assume $f(t)$ is sufficiently smooth so that its Laplace transform exists, as does that of its required derivatives.

(a) Wys vanaf die definisie van die Laplace transform dat

(1) **(a)** Show from the definition of the Laplace transform that

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \left[-\tfrac{1}{s}e^{-st}\right]_0^\infty = 0 - \left(-\tfrac{1}{s}\right)e^{-0} \quad \text{if } s > 0$$

$$= \tfrac{1}{s}$$

(b) Wys vanaf die definisie van die Laplace transform dat

(1) **(b)** Show from the definition of the Laplace transform that

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > 0.$$

(Wenk: Jy mag die resultaat van deel **(a)** gebruik indien nodig.)

(Hint: You may use the result of part **(a)** if required.)

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt \overset{\text{IP}}{=} \underbrace{\cancel{\left[\left(-\tfrac{1}{s}\right)e^{-st}\cdot t\right]_0^\infty}}_{=0} - \int_0^\infty \left(-\tfrac{1}{s}e^{-st}\cdot 1\right)dt$$

$$= \tfrac{1}{s}\int_0^\infty e^{-st}\cdot 1\,dt = \tfrac{1}{s}\mathcal{L}\{1\} = \tfrac{1}{s^2}$$

(c) Wys dat as die Laplace transform van 'n funksie $f(t)$ gegee word deur $F(s)$ vir $s > s_0$, dan

(2) **(c)** Show that if the Laplace transform of a function $f(t)$ is given by $F(s)$ for $s > s_0$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \qquad s > s_0 + a.$$

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t) = \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

let $\hat{s} = s-a$

$$= \int_0^\infty e^{-\hat{s}t}f(t)\,dt = F(\hat{s}) = F(s-a)$$

if $\hat{s} > s_0$

$\Rightarrow s-a > s_0$

$\Rightarrow s > s_0 + a.$

Prob 5 (8 punte/marks)

Beskou die lineêre outonome stelsel in die vlak:

Consider the linear plane autonomous system:

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = -2x - y.$$

(a) Skryf die stelsel in matriks vorm: $\frac{d}{dt}\underline{x} = A\underline{x}$ en bereken dan die spoor en die determinant van A.

(2) **(a)** Write the system in matrix form: $\frac{d}{dt}\underline{x} = A\underline{x}$ and compute the trace and the determinant of A.

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

(1)

$\tau = 0$

$\Delta = -1$ (1)

$\rightarrow \lambda = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) = \frac{1}{2}(\pm 2) = \pm 1$

(b) Vind en klassifiseer die enigste kritieke punt van hierdie stelsel. (Geen motivering nodig nie.)

(1) **(b)** Locate and classify the only critical point of this system. (No justification needed.)

(0,0). Saddle

(c) As die eievektore van A gegee word as $[0,1]^\top$ en $[1,-1]^\top$, bereken die ooreenstemmende eiewaardes en gee die oplossing van die stelsel.

(2) **(c)** Given that the eigenvectors of A are $[0,1]^\top$ and $[1,-1]^\top$, compute the corresponding eigenvalues and hence write down the general solution of the system.

$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = -1\begin{bmatrix}0\\1\end{bmatrix}$ (1)

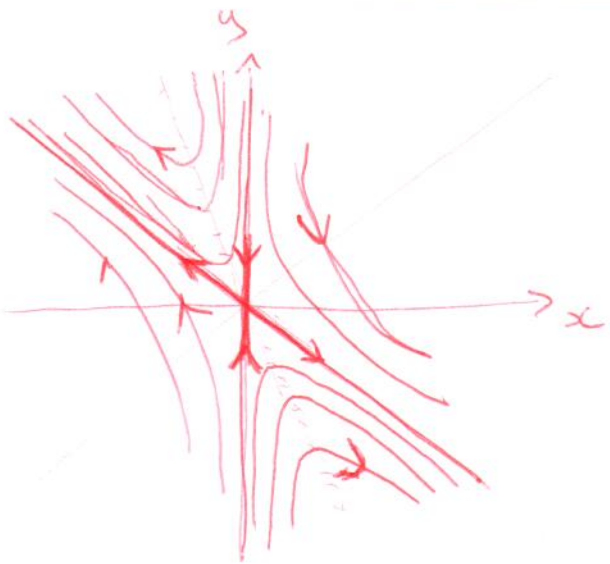
$A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = c_1\begin{pmatrix}0\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$$

(1)

(d) Skets die gedrag van die oplossings vir die bostaande stelsel in die omgewing van die kritieke punt.

(3) **(d)** Sketch the behaviour of solutions to the system above in the neighbourhood of the critical point.



(d) Wys dat

(1) **(d)** Show that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \text{en/and} \qquad \mathcal{L}\{e^{at}t\} = \frac{1}{(s-a)^2}, \qquad s > a.$$

These follow immediately from (a)->(c). //

(e) Gebruik die metode van Laplace transforms om die volgende stelsel van DVs vir $x(t)$ en $y(t)$ op te los:

(4) **(e)** Use the method of Laplace transforms to solve the following system of DEs for $x(t)$ and $y(t)$:

$$\begin{aligned} \frac{dx}{dt} &= x - y, & x(0) = 0, \\ \frac{dy}{dt} &= x + 3y + e^t, & y(0) = 0. \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$

$\Rightarrow \begin{cases} sX - 0 = X - Y \\ sY - 0 = X + 3Y + \frac{1}{s-1} \end{cases}$ (1)

$\Rightarrow \begin{cases} (s-1)X = -Y \\ (s-3)Y = X + \frac{1}{s-1} \Rightarrow (s-1)(s-3)Y = (s-1)X + 1 \end{cases}$

$= -Y + 1$

$\Rightarrow (s^2-4s+3+1)Y = (s-2)^2 Y = 1$

$\Rightarrow Y(s) = \frac{1}{(s-2)^2}$ (1)

$\Rightarrow \boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = te^{+2t}}$ (1)

$x(t) = y' - 3y - e^t$

$= e^{+2t}(1+2t) - 3te^{+2t} - e^t$

$\Rightarrow \boxed{x(t) = e^{+2t}(1-t) - e^t}$ (1)

Prob 6 (13 punte/marks)

Beskou die aangepaste Lotka–Volterra roofdier-prooi model hieronder, waar 'n logistiese groei model gebruik word vir die prooi in die afwesigheid van roofdiere:

Consider the modified Lotka–Volterra predator-prey model below, where a logistic growth model has been used for the prey in absence of predators:

$$* \quad \frac{dx}{dt} = -ax + bxy, \qquad x(0) = 0,$$
$$\frac{dy}{dt} = cy - dxy - ey^2. \qquad y(0) = 0.$$

(a) Nie-dimensionaliseer hierdie stelsel om die karakteristieke lengtes t_c, x_c, en y_c (in terme van a–e) te bepaal, en wys dat die stelsel op die volgende manier geskryf kan word:

(4½) **(a)** Nondimensionalise this system to obtain the characteristic lengths t_c, x_c, and y_c (in terms of a–e), and show that the system may be written in the form:

Let $x_c\hat{x} = x$, $y_c\hat{y} = y$, $t_c\hat{t} = t$ (1/2)

$$\frac{d\widehat{x}}{d\widehat{t}} = \widehat{x}(-\alpha + \widehat{y}), \qquad \widehat{x}(0) = 0,$$
$$\frac{d\widehat{y}}{d\widehat{t}} = \widehat{y}(1 - \widehat{x} - \beta\widehat{y}). \qquad \widehat{y}(0) = 0.$$

$* \Rightarrow \frac{x_c}{t_c}\frac{d\hat{x}}{d\hat{t}} = -ax_c\hat{x} + bx_cy_c\hat{x}\hat{y}, \quad \frac{y_c}{t_c}\frac{d\hat{y}}{d\hat{t}} = cy_c\hat{y} - dx_cy_c\hat{x}\hat{y} - ey_c^2\hat{y}^2$ (1)

$\Rightarrow \frac{d\hat{x}}{d\hat{t}} = \underbrace{-at_c}_{=\alpha}\hat{x} + \underbrace{bt_cy_c}_{=1}\hat{x}\hat{y}, \quad \frac{d\hat{y}}{d\hat{t}} = \underbrace{ct_c}_{=1}\hat{y} - \underbrace{dt_cx_c}_{=1}\hat{x}\hat{y} - \underbrace{et_cy_c}_{=1}\hat{y}^2$ (1)

(1) $ct_c = 1 \Rightarrow t_c = 1/c$. $bt_cy_c = 1 \Rightarrow y_c = \frac{1}{bt_c} = c/b$. $\alpha = at_c \Rightarrow \alpha = a/c$

$dt_cx_c = 1 \Rightarrow x_c = \frac{1}{dt_c} = c/d$ $\qquad \beta = et_cy_c = \frac{e}{c}\cdot\frac{c}{b} = \frac{e}{b}$

$\Rightarrow \frac{d\hat{x}}{d\hat{t}} = -\alpha\hat{x} + \hat{x}\hat{y} = \hat{x}(-\alpha + \hat{y})$ (1)

$\frac{d\hat{y}}{d\hat{t}} = \hat{y} - \hat{x}\hat{y} - \beta\hat{y}^2 = \hat{y}(1 - \hat{x} - \beta\hat{y})$

$t_c = \frac{1}{c}, \qquad x_c = c/d, \qquad y_c = c/b, \qquad \alpha = a/c, \qquad \beta = e/b.$

(b) Vind die drie kritieke punte van die nie-gedimensionaliseerde stelsel van deel (a).

(2) **(b)** Find the three critical points of the nondimensionalised system from part (a).

$(0,0), \ (0, 1/\beta), \ (1-\alpha\beta, \alpha)$

(1) (1)

$\hat{x}' = \hat{x}(-1+\hat{y})$ cp: $(0,0)$, $(0,2)$
$\hat{y}' = \hat{y}(1-\hat{x}-\frac{1}{2}\hat{y})$ $(\frac{1}{2},1)$

(c) Beskou nou die geval waar $\alpha = 1, \beta = \frac{1}{2}$. Klassifiseer die kritieke punte wat in deel (b) bepaal is. Skets die fasediagram van die stelsel in die eerste kwadrant (m.a.w., $x, y \geq 0$), en skenk spesiale aandag aan die rigting van enige saalpunte of stabiele/onstabiele nodusse, asook die rigting van rotasie van enige spirale/senters.

5 **(c)** Consider now the case of $\alpha = 1, \beta = \frac{1}{2}$. Classify the critical points obtained in part (b). Sketch the phase diagram of the system in the first quadrant (i.e., $x, y \geq 0$), paying special attention to the direction of any saddle points or stable/unstable nodes and the direction of rotation of any spirals or centres.

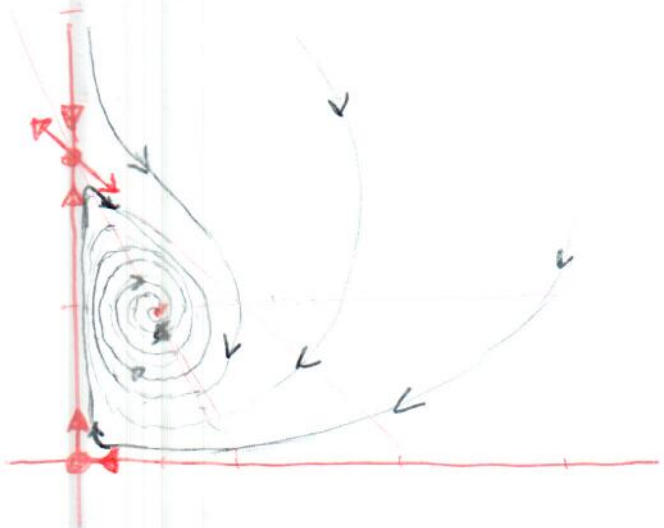
$J(x,y) = \begin{pmatrix} -1+\hat{y} & \hat{x} \\ -\hat{y} & 1-\hat{x}-\hat{y} \end{pmatrix}$

$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} \Delta = -1 \\ \tau = 0 \end{matrix} \Rightarrow$ Saddle $\lambda_1 = -1, \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 1
$\lambda_2 = 1, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \rightarrow \begin{matrix} \Delta = -1 \\ \tau = 0 \end{matrix} \Rightarrow$ Saddle (See Prob 5) 1

$J(\frac{1}{2},1) = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \Rightarrow \left.\begin{matrix} \Delta = \frac{1}{2} \\ \tau = -\frac{1}{2} \\ \tau^2/4 = 1/8 < \Delta \end{matrix}\right\} \Rightarrow$ stable spiral 1

$\left.\frac{dy}{dt}\right|_{\frac{1}{2}+\varepsilon,1} = 1 - \frac{1}{2} - \varepsilon - \frac{1}{2} = -\varepsilon < 0 \Rightarrow$ clockwise



2

(d) Vir elk van die volgende aanvangspopulasies, gee die bevolkingslimiet as $t \to \infty$. (Geen motivering nodig nie.)

2 **(d)** For each of the following initial populations give the limiting population as $t \to \infty$. (No justification required.)

$(\widehat{x}(0), \widehat{y}(0)) = (0,0) \implies (\widehat{x}(t), \widehat{y}(t)) \xrightarrow{t\to\infty} (\ldots 0 \ldots, \ldots 0 \ldots)$

$(\widehat{x}(0), \widehat{y}(0)) = (0,1) \implies (\widehat{x}(t), \widehat{y}(t)) \xrightarrow{t\to\infty} (\ldots 0 \ldots, \ldots 2 \ldots)$

$(\widehat{x}(0), \widehat{y}(0)) = (1,0) \implies (\widehat{x}(t), \widehat{y}(t)) \xrightarrow{t\to\infty} (\ldots 0 \ldots, \ldots 0 \ldots)$

$(\widehat{x}(0), \widehat{y}(0)) = (1,1) \implies (\widehat{x}(t), \widehat{y}(t)) \xrightarrow{t\to\infty} (\ldots \frac{1}{2} \ldots, \ldots 1 \ldots)$

Bonus (4 punte/marks)

Kyk weer na die roofdier-prooi stelsel van Prob 6:

Recall the predator-prey system from Prob 6:

$$\begin{aligned}\frac{dx}{dt} &= x(-\alpha + y),\\ \frac{dy}{dt} &= y(1 - x - \beta y).\end{aligned}$$

(a) Wys dat as $\alpha\beta \geq 1$ dan lê slegs twee kritieke punte in die eerste kwadrant, en dat een van hulle 'n stabiele nodus is.

(b) Skets die fasediagram van die stelsel in die eerste kwadrant en beskryf wat met die roofdier populasie gebeur wanneer $t \to \infty$.

(3) **(a)** Show that if $\alpha\beta \geq 1$ then only two critical points lie in the first quadrant, and that one of them is a stable node.

(1) **(b)** Sketch the phase diagram of the system in the first quadrant and describe what happens to the predator population as $t \to \infty$.

(a) i Critical points: $(0,0)$ ✓, $(0,1/\beta)$ ✓, $(1-\alpha\beta, \alpha)$?

$\alpha\beta > 1 \Rightarrow 1-\alpha\beta < 0 \Rightarrow$ second quadrant

$\alpha\beta = 1 \Rightarrow 1-\alpha\beta = 0,\ 1/\beta = \alpha \Rightarrow (1-\alpha\beta, \alpha) = (0, 1/\beta)$. (1)

$\therefore$ Only two critical points in first quadrant: $(0,0), (0,1/\beta)$

ii $J(x,y) = \begin{pmatrix} -\alpha+y & x \\ -y & 1-x-2\beta y \end{pmatrix}$

$\Rightarrow J(0,0) = \begin{pmatrix} -\alpha & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \Delta = -\alpha,\ \tau = 1-\alpha \Rightarrow$ saddle. $\lambda_1 = -\alpha,\ \underline{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$; $\lambda_2 = 1,\ \underline{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

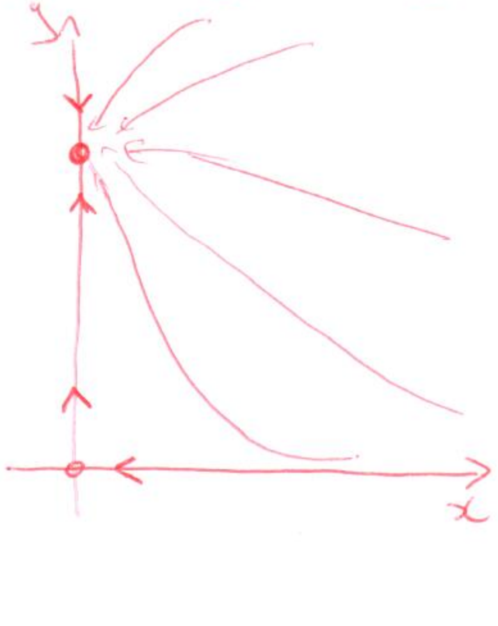
$J(0,1/\beta) = \begin{pmatrix} -\alpha+1/\beta, & 0 \\ -1/\beta, & 1-\frac{1}{\beta}2\beta \end{pmatrix} = \begin{pmatrix} -1/\beta(\alpha\beta-1) & 0 \\ -1/\beta & -1 \end{pmatrix} \Rightarrow \Delta = 1/\beta(\alpha\beta-1) > 0$, $\tau = -1 - 1/\beta(\alpha\beta-1) < 0$ (1)

$\tau^2 - 4\Delta = 1 + 2/\beta(\alpha\beta-1) + \frac{1}{\beta^2}(\alpha\beta-1)^2 - \frac{4}{\beta}(\alpha\beta-1)$

$= 1 - 2/\beta(\alpha\beta-1) + 1/\beta^2(\alpha\beta-1) = (1-\frac{1}{\beta}(\alpha\beta-1))^2 > 0$ (1)

$\therefore$ real negative eigenvalues $\Rightarrow$ stable node. //

(b)



The predator population (x) always dies out.

(1)

Bonus (4 punte/marks)

Beskou 'n ongedempte veer-massa stelsel met 'n nie-lineêre veer wat beskryf word deur die AWP

Consider an undamped spring-mass system with a nonlinear spring described by the IVP

$$x'' + x - x^3 = 0, \qquad x(0) = x_0, \quad x'(0) = x_0'.$$

(a) Gebruik die fasevlak metode om te wys dat die stelsel periodiese ossilasies vertoon vir aanvangsverplasings en/of -snelhede wat klein genoeg is.

3 **(a)** Use the phase plane method to show that for sufficiently small initial displacements and/or velocities, the system exhibits period oscillations.

(b) As $x_0 = 0$, vind die maksimum grootte van die aanvangsnelheid wat sulke periodiese ossilasies toelaat.

1 **(b)** If $x_0 = 0$ find the maximum magnitude of the initial velocity which allows such periodic oscillations.

(a) Let $x' = y$, $y' = -x + x^3$

$\Rightarrow \frac{dy}{dx} = \frac{-x+x^3}{y} \Rightarrow \frac{1}{2}y^2 = \int (-x+x^3)\,dx = -x^2/2 + x^4/4 + c_1$

$\Rightarrow y^2 = -x^2 + \frac{1}{2}x^4 + c$ (1)

Consider an initial value pair $(x_0, 0)$ so that $y(x_0) = y^2(x_0) = 0$.

$\Rightarrow y^2(x_0) = 0 = -x_0^2 + \frac{1}{2}x_0^4 + c \Rightarrow c = x_0^2 - \frac{1}{2}x_0^4$

$\Rightarrow y^2(x) = -x^2 + \frac{1}{2}x^4 + x_0^2 - \frac{1}{2}x_0^4 = \frac{1}{2}(x^4 - 2x^2 - x_0^4 - 2x_0^2)$

$= \frac{1}{2}(x^2 - x_0^2)(x^2 + x_0^2 - 2) = \frac{1}{2}(x_0^2 - x^2)(2 - x_0^2 - x^2)$ (1)

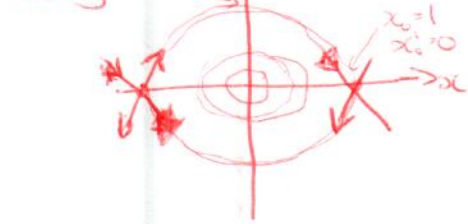
Let $|x| < |x_0| < 1$ then

- $y(-x_0) = y(x_0) = 0$
- $x_0^2 - x^2 > 0$, $x_0^2 + x^2 < 2$ $\Rightarrow y^2(x) > 0 \;\; \forall\, |x| < |x_0| < 1$

$\Rightarrow y$ has +ve and -ve solution $\forall\, |x| < |x_0| < 1$ (1)

Therefore $y(x)$ forms a closed orbit in the phase plane $\Rightarrow$ periodic sol^n

(b) Max velocity on curve occurs when $|x_0| = 1$ and $x_0' = 0$.



$\Rightarrow y^2(x) = \frac{1}{2}(\cancel{1-x^2})(2-1-x^2)$ (1)

Max velocity at $x = 0$

$y_{max}^2 = y^2(0) = \frac{1}{2}(1-0)(2-1-0) = \frac{1}{2}$

$\Rightarrow y_{max} = 1/\sqrt{2}$ ← max velocity

[Opsionele vraag. Waarde zero punte]: Verduidelik waarom die veer-massa sisteem hierbo nie fisies moontlik is, wanneer $|x(t)|$ groter as 1 word, nie.

0 **[Optional question. Worth zero marks]**: Explain why the spring-mass system above is nonphysical if $|x(t)|$ becomes greater than 1.

Because the "restoring" force of the spring becomes negative and pushes the mass away from equilibrium!