Applied differential equations

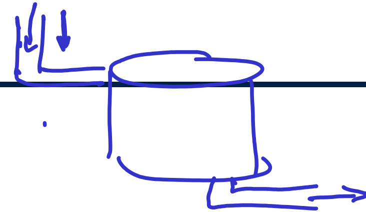
TW244 - Lecture 08

3.1: Linear Models (cont.)

Prof Nick Hale - 2020

3.1: Linear Models

Application 4: ~~Mixtures~~ Dilution



Let $m = m(t)$ be the amount (in kg) of salt in a tank at a particular time t.

The rate of change in the amount of salt in the tank is given by:

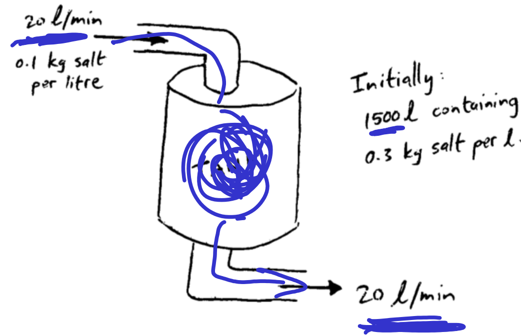
$$\boxed{\begin{array}{l}\text{rate of change in}\\ \text{amount of salt in}\\ \text{the tank}\end{array}} = \boxed{\begin{array}{l}\text{rate at which salt}\\ \text{enters the tank}\end{array}} - \boxed{\begin{array}{l}\text{rate at which salt}\\ \text{exits the tank}\end{array}}$$

M/volume

$$\frac{dm}{dt} = (\cdots kg/l)(\cdots l/min) - (\cdots kg/l)(\cdots l/min)$$

A tank initially contains 1500l of a brine solution (i.e., salt water) with concentration 0.3kg of salt per litre. In an effort to dilute the mixture, brine with a salt concentration of 0.1kg/l is pumped into the tank at 20l/min. The well-mixed solution in the tank is also pumped out at a rate of 20l/min. Determine the salt concentration in the tank at any time t.

Let $m(t)$ be the mass of salt in the tank.

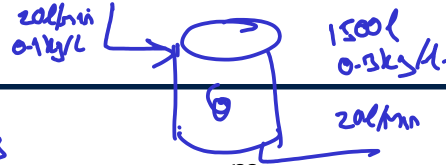


Assumptions:

- the brine is well-mixed so that all the salt dissolves
- salt is neither created nor destroyed in the system

3.1: Linear Models

Example



We have that:

rate salt enters rate salt leaves

$$\frac{dm}{dt} = (0.1kg/l)(20l/min) - \left(\frac{m(t)}{1500}kg/l\right)(20l/min) = \left(2 - \frac{m}{75}\right)(kg/min)$$

with initial condition $\quad m(0) = (0.3kg/l)(1500l) = 450kg.$

$M' + M/75 = 2$

Solve this IVP using integrating factor $e^{\int \frac{1}{75}dt} = e^{\frac{1}{75}t}$, so that

$f = e^{\int \frac{1}{75}dt} = e^{t/75}$

$$\frac{d}{dt}\left[e^{\frac{1}{75}t}m\right] = 2e^{\frac{1}{75}t} \longrightarrow e^{\frac{1}{75}t}m = \int 2e^{\frac{1}{75}t}dt = 150e^{\frac{1}{75}t} + C$$

Initial condition $m(0) = 450 \longrightarrow 450 = 150 + C \longrightarrow C = 300.$

Therefore:

$$m(t) = 150 + \underbrace{300e^{-\frac{1}{75}t}}_{\to 0} \text{ and concentration } c(t) = \frac{m(t)}{1500} = 0.1 + \underbrace{0.2e^{-\frac{1}{75}t}}_{\to 0 \text{ as } t \to \infty}.$$

What is the steady state of the tank? Compare to your physical intuition.

We have that:

$$\frac{dm}{dt} = (0.1kg/l)(20l/min) - \left(\frac{m}{1500}kg/l\right)(20l/min) = \left(2 - \frac{m}{75}\right)(kg/min)$$

with initial condition $\quad m(0) = (0.3kg/l)(1500l) = 450kg.$

Solve this IVP using integrating factor $e^{\int \frac{1}{75}dt} = e^{\frac{1}{75}t}$, so that

$$\frac{d}{dt}\left[e^{\frac{1}{75}t}m\right] = 2e^{\frac{1}{75}t} \implies e^{\frac{1}{75}t}m = \int 2e^{\frac{1}{75}t}dt + C = 150e^{\frac{1}{75}t} + C$$

Initial condition $m(0) = 450 \implies 450 = 150 + C \implies C = 300.$

Therefore:

$$m(t) = 150 + 300e^{-\frac{1}{75}t} \text{ and concentration } c(t) = \frac{m}{1500} = 0.1 + 0.2e^{-\frac{1}{75}t}.$$

What is the steady state of the tank? Compare to your physical intuition.

Suppose we invest an amount S_0 at a yearly interest rate r.

How much is the investment worth after t years if the interest is compounded:

- yearly
- 6-monthly
- quarterly
- daily
- continuously?

Let $S = S(t)$ be the value of the interest after t years.

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Yearly:

$$\begin{aligned} S(1) &= S_0 + rS_0 = S_0(1+r) \\ S(2) &= S(1) \times (1+r) = S_0(1+r)^2 \\ &\vdots \\ S(t) &= S_0(1+r)^t. \end{aligned}$$

6-monthly:

$$\begin{aligned} S(\tfrac{1}{2}) &= S_0 + \tfrac{r}{2}S_0 = S_0(1+\tfrac{r}{2}) \\ S(1) &= S_0(1+\tfrac{r}{2})^2 \\ &\vdots \\ S(t) &= S_0(1+\tfrac{r}{2})^{2t}. \end{aligned}$$

Quarterly:

$$S(t) = S_0(1+\tfrac{r}{4})^{4t}.$$

Daily:

$$S(t) = S_0(1+\tfrac{r}{365})^{365t}.$$

What if the interest is compounded continuously?

Interest compounded continuously:

Take $S(t) = S_0(1+\frac{r}{m})^{mt}$ and let $m \to \infty$ then

$$
\begin{aligned}
\lim_{m\to\infty}(1+\tfrac{r}{m})^{mt} &= \lim_{m\to\infty}(1+\tfrac{r}{m})^{mt} \\
&= \lim_{h\to 0}(1+h)^{\frac{r}{h}t} \qquad \text{where} \quad h = \tfrac{r}{m} \\
&= \left[\lim_{h\to 0}(1+h)^{\frac{1}{h}}\right]^{rt} \\
&= \left[\lim_{h\to 0} e^{\frac{\ln(1+h)}{h}}\right]^{rt} \overbrace{=}^{l'Hospital} \left[\lim_{h\to 0} e^{\frac{1}{1+h}}\right]^{rt} \\
&= e^{rt}
\end{aligned}
$$

Hence $S(t) = S_0 e^{rt}$ and continuous compound of interest is therefore equivalent to the Malthus model of population growth!

$$\frac{dS}{dt} = rS \quad \text{with} \quad S(0) = S_0.$$

3.1: Linear Models

Application 6: Series circuits (warning - this slide may contain physics!)

For a series circuit containing a resistor and an inductor, Kirchoff's first law states "*the sum of the voltage drop across the inductor and the voltage drop across the resistor equals the impressed voltage on the circuit.*".

Mathematically this can be expressed as

$$L\frac{di}{dt} + Ri = E(t),$$

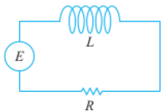


FIGURE 3.1.7 *LR*-series circuit

where L and R are known as the inductance and resistance, respectively. The current $i(t)$ is called the *response* of the circuit.

For a series circuit containing a capacitor and a resistor, we instead have

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

where $q(t)$ is the charge on the capacitor.

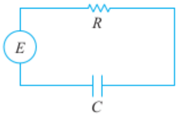


FIGURE 3.1.8 *RC*-series circuit

Exercises:

- Show that if the voltage $E(t)$ is kept constant (i.e., $E(t) = E_0$) then the solution to equation for the current $i(t)$ in the first equation is given by

$$i(t) = \frac{E_0}{R} + ce^{(-R/L)t}.$$

- With reference to the above, explain why for large times the system is simply governed by Ohm's law, i.e.,

$$E = iR.$$

- Suppose a 12-volt battery is connected to such a circuit in which the induction is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current $i(t)$ if the initial current is zero.*
- Find the solution to the DE describing i when the $E(t)$ is not constant.**

*Hint: answer = $i(t) = \frac{6}{5}(1 - e^{-20t})$. ** Hint: answer = $i(t) = \frac{e^{-(R/L)t}}{L} \int e^{(R/L)t} E(t)\, dt + ce^{-(R/L)t}$.

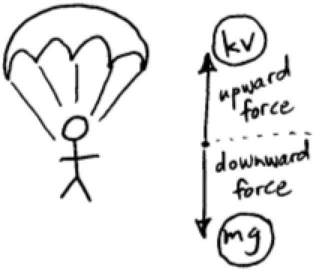
Application 7: Free-fall with linear drag

Consider the following variation of the free fall problem we saw in Lecture 1:

Jane Bond falls out of a helicopter. Determine her velocity $v(t)$ and displacement $s(t)$ at time t if drag (air resistance) is proportional to her instantaneous velocity and $v(0) = s(0) = 0$ (with "coefficient of drag" k).

Newton's second law of motion:

$$\begin{aligned} & ma && = && F \\ \therefore\; & m\frac{dv}{dt} && = && mg \underline{- kv} \\ \textit{units}\; & [kg][m/s^2] && = && [kg][m/s^2] - [kg/s][m/s] \end{aligned}$$



Application 7: Free-fall with linear drag

Solve with integrating factor $e^{\int \frac{k}{m} dt} = e^{\frac{k}{m}t}$:

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

$$e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = \frac{d}{dt}\left[e^{\frac{k}{m}t}v\right] = ge^{\frac{k}{m}t}$$

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C$$

$= \int g e^{k/m\, t} dt$

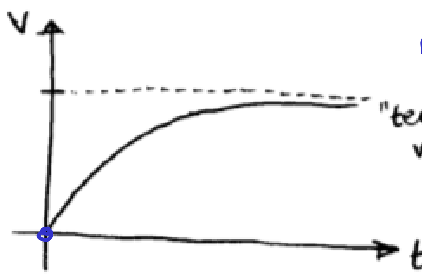
Exercise: Terminal 50m/s
Sam 60kg
Determine k.

Using the initial condition we have

$$v(0) = 0 \implies 0 = \frac{mg}{k} + C \implies C = -\frac{mg}{k}$$

therefore

$$\boxed{v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right).}$$



What about displacement? Well, we know $\frac{ds}{dt} = v$, so

$v(t)$

$$
\begin{aligned}
\frac{ds}{dt} &= \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) \implies \\
s &= \frac{mg}{k}\int(1 - e^{-\frac{k}{m}t})\,dt + C \;=\; \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right) + C
\end{aligned}
$$

Using the initial condition $s(0) = 0$ we have

$$s(0) = 0 \implies 0 = \frac{mg}{k}\frac{m}{k} + C \implies C = -\frac{m^2 g}{k^2}$$

and therefore

$$\boxed{s(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right).}$$

Exercise!

Sene jumps from 4500m how long before she is 500m above the parachute?

Next time we will consider the effect of nonlinear drag.