Applied differential equations

TW244 - Lecture 01

Introduction

Prof Nick Hale - 2020

Mathematical modelling

with differential equations.

Applied maths = problem solving (using maths!)

Wikipedia eloquently describes applied mathematics as "[solving] practical problems by formulating and studying mathematical models."

The following represent the steps involved in the mathematical modelling process:

- Identify / Specify a problem
- Make and state assumptions
- Mathematical model
- Solve the model
- Analysis / Verification
- Refine the model

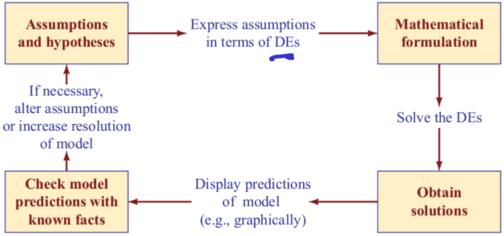


FIGURE 1.3.1 Steps in the modeling process with differential equations

Our interest is in problems that can be modelled by differential equations.

The process of mathematical modelling
Why *differential* equations?

We will soon see that there are many types of problems which can be modelled by differential equations.

Why is this? Why differential equations?

Here are at least three reasons (but there are many more!):

- Newton's 2nd law of motion, $F = ma$.
  a = acceleration = $\frac{dv}{dt} = \frac{d^2s}{dt^2}$
- Rates of change / reaction
  Often we know how fast something grows (population, etc)
- Diffusion and Brownian motion
  (Not covered in this course!)

Let's look at an example.

The process of mathematical modelling
Identify / specify a problem

A skydiver jumps out of a helicopter,

free-falls for 10 seconds,

and then opens her parachute.

How far has she fallen in these 10 seconds?

The process of mathematical modelling
Assumptions

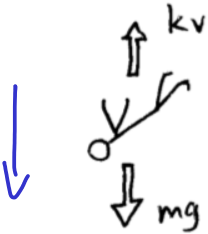
In our example, let's assume:

- The initial velocity is zero
- Motion is only vertical (no horizontal component)
- The forces are due to gravity and air resistance (drag)
- Air resistance is proportional to the instantaneous velocity

These are pretty much the simplest assumptions we could make, but by stating them clearly at least they are out in the open.

The process of mathematical modelling
Mathematical model

Newton's second law states that $ma = F$, so



$$\begin{aligned} ma &= F, \\ m\frac{dv}{dt} &= mg - kv, \\ \frac{dv}{dt} &= g - \frac{k}{m}v. \end{aligned}$$

where k is a constant

differential equation.

For simplicity, let's assume $g = 10$ and $k/m = 1$, then we have

$$\frac{dv}{dt} = 10 - v, \qquad v(0) = 0.$$

exercise.

(NB, dimensional analysis is important! What dimensions does k have? We say a constant with no dimensions is "dimensionless". More later.)

The process of mathematical modelling

Solve the model

Differential equation (DE)
Initial value problem (IVP)

We want to solve

$$\frac{dv}{dt} = 10 - v, \qquad v(0) = 0.$$

We'll talk about the details of this technique later, but for now we have

$$\int \frac{dv}{10 - v} = \int dt \implies -\ln|10 - v| = t + c_1 \implies v = 10 - c_2 e^{-t}.$$

Using the initial condition $v(0) = 0$ we see $0 = 10 - c_2 \implies c_2 = 10$, so

$$v(t) = 10(1 - e^{-t}).$$

Solution.

But we want to know how far she fell..

$V = \frac{ds}{dt}$

Displacement:

$$s(t) = \int v(t)\,dt = 10(t + e^{-t}) + c_3.$$

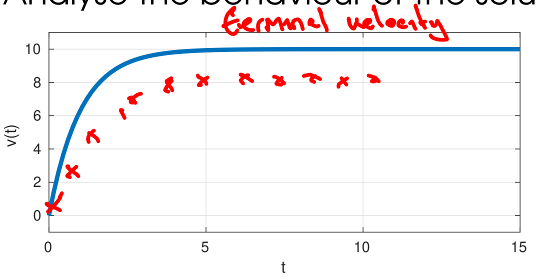
The initial displacement is zero: $s(0) = 0 \implies c_3 = -10$, so

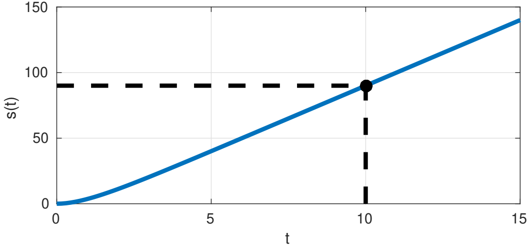
$$s(t) = 10(t + e^{-t} - 1).$$

According to our model, she falls about $\underbrace{90}_{\approx s(10)}$ metres in the first 10 seconds.

The process of mathematical modelling
Analysis / verification

Analyse the behaviour of the solution, for example, by plotting graphs:





Does it behave correctly according to the physical situation being investigated?

Does it agree with experimental data? (If available.)

The process of mathematical modelling

Refine the model / assumptions

Depending on the above, it might be necessary to refine the model.

For example,

- Is linear air resistance applicable?
- Did our assumption that $k/m = 1$ give physically meaningful results?
- ...

This step is not always necessary, but it is important to remember that your first model may not be the best approximation to the physical system.

In fact, good practice is to start with a simple model that is relatively easy to solve to get a basic understanding of the problem.

The process of mathematical modelling
Other examples

Examples of some other situations we will model (and the differential equations they give rise to) include:

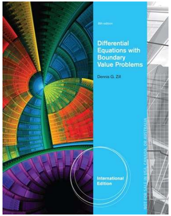
- Radioactive decay
$$\frac{dN}{dt} = -\lambda N$$
- Nonlinear pendulum
$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta$$
- Predator-Prey models
$$\begin{aligned} \frac{dr}{dt} &= \alpha r - \beta r f \\ \frac{df}{dt} &= \gamma r f - \delta f \end{aligned}$$
- Logistic population model
$$\frac{dP}{dt} = \rho P \left(1 - \frac{P}{\kappa}\right)$$
- Spring-mass systems
$$x'' + 2\gamma x' + x = f(t)$$
- Sliding bead on a wire
$$x'' = -\alpha \frac{f'(x)}{1 + [f'(x)]^2} - \beta x'$$

The process of mathematical modelling

Section 1.3 of the textbook (8th edition only...)

The course text has many examples similar to this in Chapter 1.3.

Differential Equations with Boundary-Value Problems
D.G. Zill & W.S. Wright
ISBN-10: 1111827060
ISBN-13: 978-1111827069
Brooks Cole; 8th edition (March 15, 2012), 664 pp



There are also some interesting projects in the first few pages, which will give you an idea of the very wide range of problems than differential equations are applicable to.

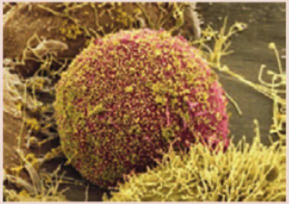
Another opportunity to apply the knowledge from this course is in the COMAP Mathematics Contest in Modelling (MCM). Typically a selection of the top students from this course are selected to compete. More details later, or see http://appliedmaths.sun.ac.za/MCM/.

The process of mathematical modelling

"Projects" of the textbook

Is AIDS an Invariably Fatal Disease?

by Ivan Kramer



Thomas Deerinck, NCMIR / Photo Researchers, Inc.

Cell infected with HIV

This essay will address and answer the question: Is the acquired immunodeficienc syndrome (AIDS), which is the end stage of the human immunodeficiency virus (HIV) infection, an invariably fatal disease?

Like other viruses, HIV has no metabolism and cannot reproduce itself outside of a living cell. The genetic information of the virus is contained in two identical strands of RNA. To reproduce, HIV must use the reproductive apparatus of the cell it invades and infects to produce exact copies of the viral RNA. Once it penetrates a cell, HIV transcribes its RNA into DNA using an enzyme (reverse transcriptase) contained in the virus. The double-stranded viral DNA migrates into the nucleus of the invaded cell and is inserted into the cell's genome with the aid of another viral enzyme (integrase). The viral DNA and the invaded cell's DNA are then integrated, and the cell is infected. When the infected cell is stimulated to reproduce, the proviral DNA is transcribed into viral DNA, and new viral particles are synthesized. Since anti-retroviral drugs like zidovudine inhibit the HIV enzyme reverse transcriptase and stop proviral DNA chain synthesis in the laboratory, these drugs, usually administered in combination, slow down the progression to AIDS in those that are infected with HIV (hosts).

The Allee Effect

by Jo Gascoigne

Courtesy of Jo Gascoigne

Dr Jo with Queenie; Queenie is on the left

The top five most famous Belgians apparently include a cyclist, a punk singer, the inventor of the saxophone, the creator of Tintin, and Audrey Hepburn. Pierre François Verhulst is not on the list, although he should be. He had a fairly short life, dying at the age of 45, but did manage to include some excitement—he was deported from Rome for trying to persuade the Pope that the Papal States needed a written constitution. Perhaps the Pope knew better even then than to take lectures in good governance from a Belgian. . . .

Aside from this episode, **Pierre Verhulst** (1804–1849) was a mathematician who concerned himself, among other things, with the dynamics of natural populations—fish, rabbits, buttercups, bacteria, or whatever. (I am prejudiced in favour of fish, so we will be thinking fish from now on.) Theorizing on the growth of natural populations had up to this point been relatively limited, although scientists had reached the obvious conclusion that the growth rate of a population (dN/dt, where $N(t)$ is the population size at time t) depended on (i) the birth rate b and (ii) the mortality rate m, both of which would vary in direct proportion to the size of the population N:

$$\frac{dN}{dt} = bN - mN. \qquad (1)$$

The process of mathematical modelling
“Projects” of the textbook

Wolf Population Dynamics

by C. J. Knickerbocker

A gray wolf in the wild

Early in 1995, after much controversy, public debate, and a 70-year absence, gray wolves were re introduced into Yellowstone National Park and Central Idaho. During this 70-year absence, significant changes were recorded in the populations of other predator and prey animals residing in the park. For instance, the elk and coyote populations had risen in the absence of influence from the larger gray wolf. With the reintroduction of the wolf in 1995, we anticipated changes in both the predator and prey animal populations in the Yellowstone Park ecosystem as the success of the wolf population is dependent upon how it influences and is influenced by the other species in the ecosystem.

For this study, we will examine how the elk (prey) population has been influ enced by the wolves (predator). Recent studies have shown that the elk population has been negatively impacted by the reintroduction of the wolves. The elk population fell from approximately 18,000 in 1995 to approximately 7,000 in 2009. This article asks the question of whether the wolves could have such an effect and, if so, could the elk population disappear?

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Bungee Jumping

by Kevin Cooper

Sean Nel/Shutterstock.com

Bungee jumping from a bridge

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River canyon. Suppose that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 174 feet below. You have brought several different cords with which to affix your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of the cord so as to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that “natural length” as $x(t)$, where x increases as you go down and is a function of time t. See Figure 1. Then, at the time you jump, $x(0) = -100$, while if your six-foot frame hits the water head first, at that time $x(t) = 174 - 100 - 6 = 68$. Notice that distance increases as you fall, and so your velocity is positive as you fall and negative when you bounce back up. Note also that you plan to dive so your head will be six feet below the end of the chord when it stops you.

You know that the acceleration due to gravity is a constant, called g, so that the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in

The Collapse of the Tacoma Narrows Suspension Bridge

by Gilbert N. Lewis

AP Photo

Collapse of the Tacoma Narrows Bridge

In the summer of 1940, the Tacoma Narrows Suspension Bridge in the State of Washington was completed and opened to traffic. Almost immediately, observers noticed that the wind blowing across the roadway would sometimes set up large vertical vibrations in the roadbed. The bridge became a tourist attraction as people came to watch, and perhaps ride, the undulating bridge. Finally, on November 7, 1940, during a powerful storm, the oscillations increased beyond any previously observed, and the bridge was evacuated. Soon, the vertical oscillations became rotational, as observed by looking down the roadway. The entire span was eventually shaken apart by the large vibrations, and the bridge collapsed. Figure 1 shows a picture of the bridge during the collapse. See [**1**] and [**2**] for interesting and sometimes humorous anecdotes associated with the bridge. Or, do an Internet search with the key words “Tacoma Bridge Disaster” in order to find and view some interesting videos of the collapse of the bridge.

The process of mathematical modelling

"Projects" of the textbook

Murder at the Mayfair Diner

by Tom LoFaro

© Ronald C. Saari

The Mayfair diner in Philadelphia, PA

Dawn at the Mayfair Diner. The amber glow of streetlights mixed with the violent red flash of police cruisers begins to fade with the rising of a furnace orange sun. Detective Daphne Marlow exits the diner holding a steaming cup of hot joe in one hand and a summary of the crime scene evidence in the other. Taking a seat on the bumper of her tan LTD, Detective Marlow begins to review the evidence.

At 5:30 a.m. the body of one Joe D. Wood was found in the walk in refrigerator in the diner's basement. At 6:00 a.m. the coroner arrived and determined that the core body temperature of the corpse was 85 degrees Fahrenheit. Thirty minutes later the coroner again measured the core body temperature. This time the reading was 84 degrees Fahrenheit. The thermostat inside the refrigerator reads 50 degrees Fahrenheit.

Daphne takes out a fading yellow legal pad and ketchup-stained calculator from the front seat of her cruiser and begins to compute. She knows that Newton's Law of Cooling says that the rate at which an object cools is proportional to the difference between the temperature T of the body at time t and the temperature T_m of the environment surrounding the body. She jots down the equation

$$\frac{dT}{dt} = k(T - T_m), \quad t > 0, \tag{1}$$

Earthquake Shaking of Multistory Buildings

by Gilbert N. Lewis

s76/ZUMA Press/Newscom

Collapsed apartment building in San Francisco, October 18, 1989, the day after the massive Loma Prieta earthquake

Large earthquakes typically have a devastating effect on buildings. For example, the famous 1906 San Francisco earthquake destroyed much of that city. More recently, that area was hit by the Loma Prieta earthquake that many people in the United States and elsewhere experienced second-hand while watching on television the Major League Baseball World Series game that was taking place in San Francisco in 1989.

In this project, we attempt to model the effect of an earthquake on a multi-story building and then solve and interpret the mathematics. Let x_i represent the horizontal displacement of the ith floor from equlibrium. Here, the equilibrium position will be a fixed point on the ground, so that $x_0 = 0$. During an earthquake, the ground moves horizontally so that each floor is considered to be displaced relative to the ground. We assume that the ith floor of the building has a mass m_i, and that successive floor are connected by an elastic connector whose effect resembles that of a spring.

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Modeling Arms Races

by Michael Olinick

NICOLAS ASFOURI/AFP/Getty Images/Newscom

Weapons and ammunition recovered during military operations against Taliban militants in South Waziristan in October 2009

The last hundred years have seen numerous dangerous, destabilizing, and expensive arms races. The outbreak of World War I climaxed a rapid buildup of armaments among rival European powers. There was a similar mutual accumulation of conventional arms just prior to World War II. The United States and the Soviet Union engaged in a costly nuclear arms race during the forty years of the Cold War. Stockpiling of ever-more deadly weapons is common today in many parts of the world, including the Middle East, the Indian subcontinent, and the Korean peninsula.

British meteorologist and educator Lewis F. Richardson (1881–1953) developed several mathematical models to analyze the dynamics of arms races, the evolution over time of the process of interaction between countries in their acquisition of weapons. Arms race models generally assume that each nation adjusts its accumulation of weapons in some manner dependent on the size of its own stockpile and the armament levels of the other nations.