
Problem 1:

$x'' + 9x = 0 \implies x = c_1\cos(3t) + c_2\sin(3t)$. $x(0) = 1$ and $x'(0) = 3\sqrt{3} \implies c_1 = 1$ and $c_2 = \sqrt{3}$

$$\begin{aligned}\text{Hence}\quad x &= \cos(3t) + \sqrt{3}\sin(3t)\\ &= 2\left[\tfrac{1}{2}\cos(3t) + \tfrac{\sqrt{3}}{2}\sin(3t)\right]\\ &= 2\left[\sin\!\left(\tfrac{\pi}{6}\right)\cos(3t) + \cos\!\left(\tfrac{\pi}{6}\right)\sin(3t)\right] = 2\sin\!\left(3t + \tfrac{\pi}{6}\right) = 2\sin\!\left(3t - \tfrac{11\pi}{6}\right) = 2\sin\left[3\!\left(t - \tfrac{11\pi}{18}\right)\right].\end{aligned}$$

Problem 2:

(a) Here $\gamma = 1$ and $\omega = \sqrt{5}$ so that $\gamma^2 - \omega^2 < 0 \implies \boxed{\text{under-damped}}$

$x'' + 2x' + 5x = 0$. Try $x = e^{pt}$: $p^2 + 2p + 5 = 0 \implies p = -1 \pm 2i$

$x = e^{-t}\left[c_1\cos(2t) + c_2\sin(2t)\right]$ and $x' = -e^{-t}\left[c_1\cos(2t) + c_2\sin(2t)\right] + e^{-t}\left[-2c_1\sin(2t) + 2c_2\cos(2t)\right]$

Initial conditions: $x(0) = 1$: $1 = c_1\cos(0) + c_2\sin(0) \implies c_1 = 1$

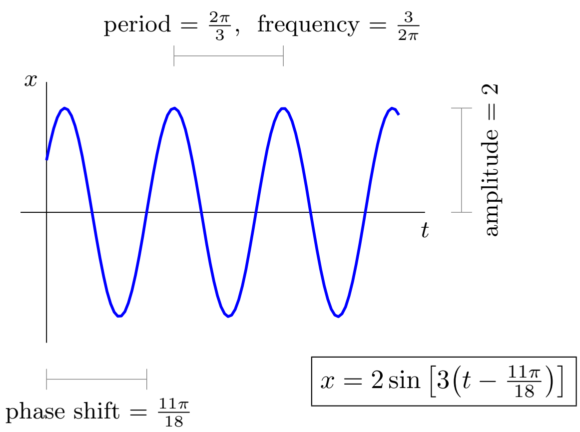
$x'(0) = -3$: $-3 = -c_1\cos(0) - c_2\sin(0) - 2c_1\sin(0) + 2c_2\cos(0) \implies c_2 = -1$

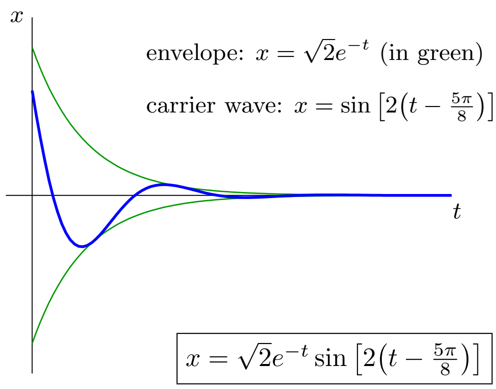
$$\begin{aligned}\text{Hence}\quad x &= e^{-t}\left[\cos(2t) - \sin(2t)\right]\\ &= \sqrt{2}e^{-t}\left[\tfrac{1}{\sqrt{2}}\cos(2t) - \tfrac{1}{\sqrt{2}}\sin(2t)\right]\\ &= \sqrt{2}e^{-t}\left[\sin\!\left(\tfrac{3\pi}{4}\right)\cos(2t) + \cos\!\left(\tfrac{3\pi}{4}\right)\sin(2t)\right] = \sqrt{2}e^{-t}\sin\!\left(2t + \tfrac{3\pi}{4}\right) = \sqrt{2}e^{-t}\sin\left[2\!\left(t - \tfrac{5\pi}{8}\right)\right].\end{aligned}$$

(b) Here $\gamma = \frac{3}{2}$ and $\omega = \sqrt{2}$ so that $\gamma^2 - \omega^2 > 0 \implies \boxed{\text{over-damped}}$

$x'' + 3x' + 2x = 0$. Try $x = e^{pt}$: $p^2 + 3p + 2 = 0 \implies p = -1$ of $p = -2$

$x = c_1e^{-t} + c_2e^{-2t}$. From $x(0) = -1$ and $x'(0) = 4$ we get $c_1 = 2$ en $c_2 = -3 \implies x = 2e^{-t} - 3e^{-2t}$.





Problem 3:

(a) The system is linear, undamped, and driven.
The natural frequency is $\frac{1}{\sqrt{2}\pi}$ and the forcing frequency is $\frac{1}{2\pi}$.

(b) (This problem can be solved with the Method of Undetermined Coefficients or the Method of Variation of Parameters. Here we instead use Laplace Transforms. See Lectures 27/28.)

$$\begin{aligned}
\mathcal{L}\{\frac{d^2x}{dt^2}+2x\}=\mathcal{L}\{\sin(t)\} \quad &\Longrightarrow \quad s^2X(s)-sx(0)-x'(0)+2X(s)=\frac{1}{s^2+1}\\
&\Longrightarrow \quad (s^2+2)X(s)=\frac{1}{s^2+1}\\
&\Longrightarrow \quad X(s)=\frac{1}{(s^2+1)(s^2+2)}=\frac{1}{s^2+1}-\frac{1}{s^2+2}\\
&\Longrightarrow \quad x(t)=\mathcal{L}^{-1}\{\frac{1}{s^2+1}-\frac{1}{s^2+2}\}\\
&\Longrightarrow \quad \boxed{x(t)=\sin(t)-\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)}.
\end{aligned}$$

Problem 4:

- 2a: The forcing term
- 2b: Easily verified
- 2c: $x(t)=20\sin(2.05t)\sin(0.05t)$
- 2d: opposite