Applied differential equations

TW244 - Lecture 33

10.4 Nonlinear models

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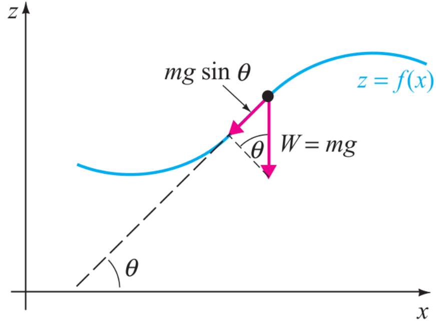
Sliding bead on a wire

Consider the problem of a bead of mass m sliding along a thin, frictionless wire whose shape is described by the function $z = f(x)$.

- The tangential force is $mg\sin\theta$
- In the x direction:

$$F_x = -mg\sin\theta\cos\theta = -mg\frac{\sin\theta}{\cos\theta}\cos^2\theta$$
$$= -mg\frac{\tan\theta}{\sec^2\theta} = -mg\frac{\tan\theta}{1+\tan^2\theta}$$
$$= -mg\frac{f'(x)}{1+[f'(x)]^2}$$



- Damping force proportional to velocity:
  $D_x = -\beta x'$
- Newton's second law ($ma = F$):

$$mx'' = -mg\frac{f'(x)}{1+[f'(x)]^2} - \beta x' \implies \begin{cases} x' = y \\ y' = -g\dfrac{f'(x)}{1+[f'(x)]^2} - \dfrac{\beta}{m}y \end{cases}$$

10.4: Nonlinear models
Sliding bead on a wire

$$\frac{\partial Q}{\partial x} = \frac{-g f''(x)}{1+[\cancel{f'(x)}]^2} + \frac{2g[\cancel{f'(x)}]^2 f''(x)}{[1+[\cancel{f'(x)}]^2]^2}$$

Let's analyse the stability of the system $x' = y, y' = -g\dfrac{f'(x)}{1+[f'(x)]^2} - \dfrac{\beta}{m}y$.
(For now we leave f general, we will pick an example later.)

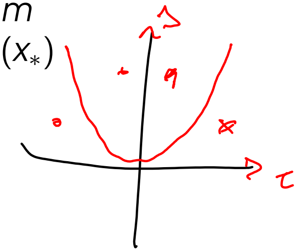
Notice that all critical points require $y = 0$ and hence $f'(x) = 0$. This corresponds to a point where the wire is horizontal - which makes sense!

Consider a critical point $(x_*, 0)$, then we have a Jacobian* when $f'(x^*) = 0$.

$$J(x_*, 0) = \begin{bmatrix} 0 & 1 \\ -gf''(x_*) & -\beta/m \end{bmatrix} \Longrightarrow \begin{cases} \tau = -\beta/m \\ \Delta = gf''(x_*) \end{cases}$$

and

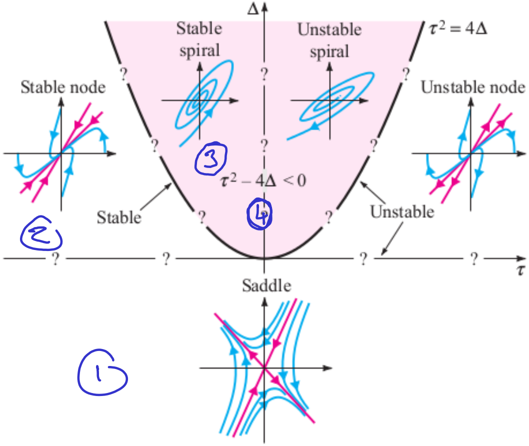
$$\tau^2 - 4\Delta = (\beta/m)^2 - 4gf''(x_*).$$



*Exercise: Show $\frac{d}{dx} f'(x)/(1+[f'(x)]^2)\Big|^{x_*} = f''(x_*)$ when $f'(x_*) = 0$.

We have $\tau = -\beta/m$, $\quad \Delta = gf''(x_*)$, $\quad \tau^2 - 4\Delta = (\beta/m)^2 - 4gf''(x_*)$.

- $\mathbf{f''(x_*) < 0}$
  $\implies \Delta < 0 \to$ saddle
  (A local maximum in the wire)
- $\mathbf{0 < f''(x_*) < \beta^2/(4gm^2)}$
  $\implies \tau^2 - 4\Delta > 0 \to$ stable node
  (overdamped)
- $\mathbf{0 < \beta^2/(4gm^2) < f''(x_*)}$
  $\implies \tau^2 - 4\Delta < 0 \to$ stable spiral
  (underdamped)
- $\mathbf{0 = \beta < f''(x_*)}$
  $\implies \tau^2 - 4\Delta < 0 \to$ centre*
  (no damping!)



* Actually this is borderline and uncertain, but it can be shown by the phase-plane method to be a centre.

Example: Let's consider an example where the wire is described by the function $z = f(x) = \sin(x)$. Let's assume the mass of the bead is $m = 0.01$, and (to keep the numbers nicer) $g = 10$.

The motion of the bead is then described by the autonomous system

$$x' = y, \qquad y' = -10\cos(x)/(1+\cos^2(x)) - 100\beta y.$$

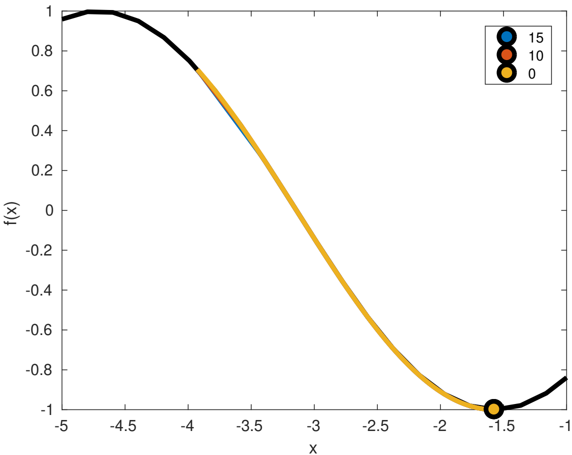
The critical points of this system occur when $f'(x) = \cos(x) = 0$ and $y = 0$, i.e., $(x, y) = ([k + \frac{1}{2}]\pi, 0),\ k \in \mathbb{Z}$.

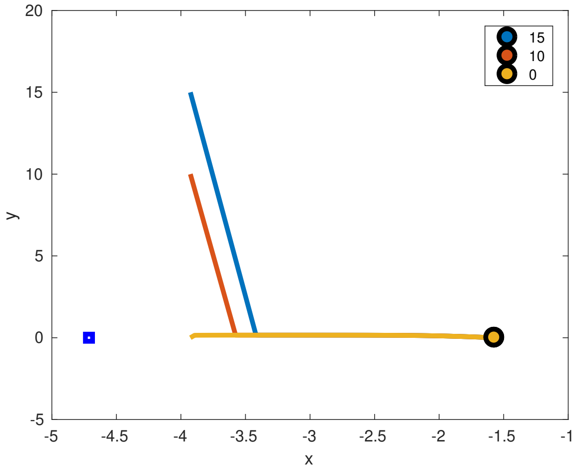
Rather than rederive the the nature of these critical points from scratch, let's use the general formula from the previous slide.

When k is odd then $f''([k + \frac{1}{2}]\pi) = -1$ and so the critical point is a saddle. When k is even the critical value of $\beta \neq 0$ is when $0 < f''([k + \frac{1}{2}]\pi) = \beta^2/(4gm^2)$, i.e., $\beta^* = 0.0632$. Let's consider three cases...

Case 1:

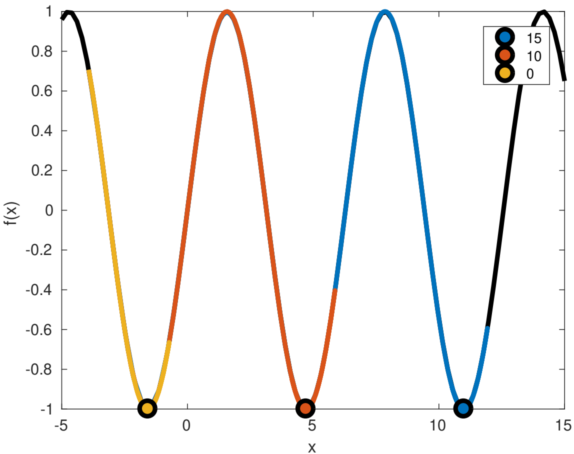
$\beta = 0.07 > \beta^* \implies$ stable point $\implies$ overdamped.

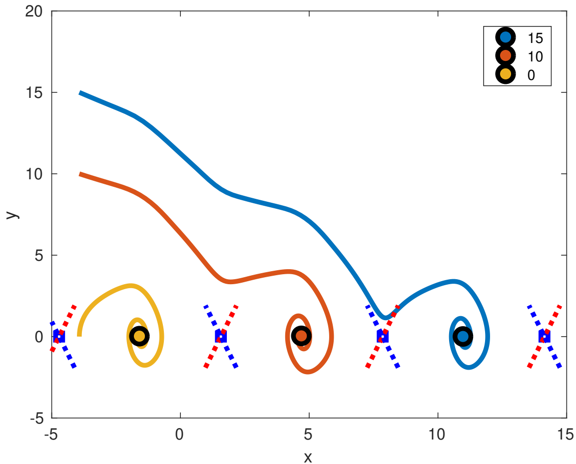




Case 2:

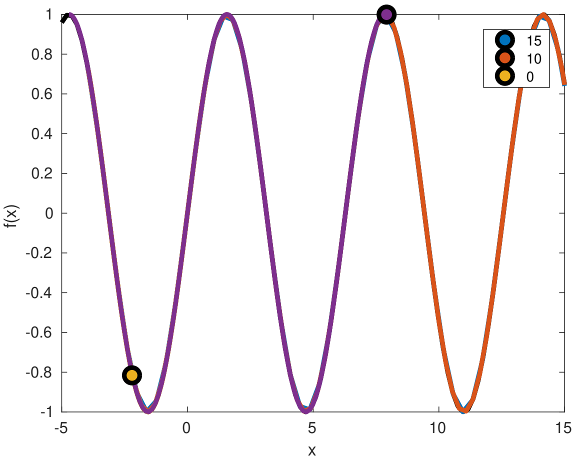
$\beta = 0.01 < \beta^* \implies$ stable spiral $\implies$ underdamped.

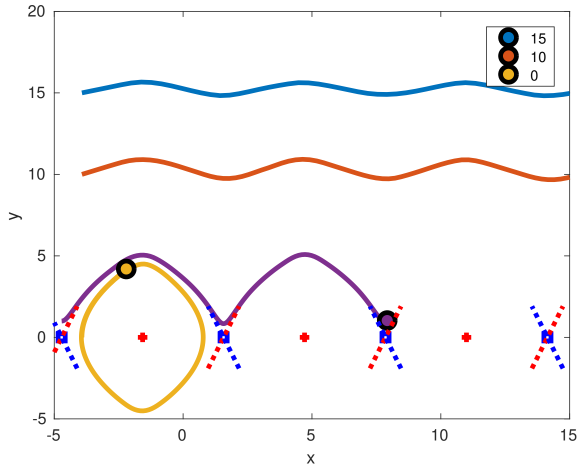




Case 3:

$\beta = 0 \implies$ no damping $\implies$ periodic solutions.

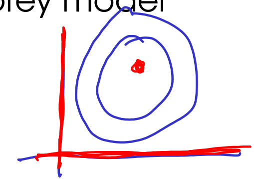




10.4: Nonlinear models
Predator-Prey and competition models

Lotka–Volterra Predator-Prey model: Recall the predator-prey model

$$\frac{dx}{dt} = x(-a + by) \qquad \frac{dy}{dt} = y(c - dx).$$



It is clear that the two critical points are $(0,0)$ and $(\frac{c}{d}, \frac{a}{b})$.

Exercise: Show that $(0,0)$ is a saddle and that $(\frac{c}{d}, \frac{a}{b})$ is a borderline case.

One can actually use the Phase Plane method to show that $(\frac{c}{d}, \frac{a}{b})$ is always a centre, but the proof is a little tricky, and we omit it here.

Self-study: Lotka–Volterra competition models: Incorporating logistic growth in the model above or considering 'competition models' (see p. 419 of Z&W) can introduce more complex behaviour, and phase diagrams are an excellent means of analysing this. See p. 419-420 in Z&W and/or play around with some models using the `pplane` software.