Problem 1:

(a) $\mathcal{L}\{t^n\} := \int_0^\infty e^{-st} t^n \, dt \overset{\text{parts}}{\widehat{=}} \underbrace{\left[t^n \frac{e^{-st}}{-s} \right]_{t=0}^{t=\infty}}_{=0} - \int_0^\infty \frac{e^{-st}}{-st} n t^{n-1} \, dt = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$

(b) $\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\} = \frac{n(n-1)}{s^2} \mathcal{L}\{t^{n-2}\} = \frac{n(n-1)...1}{s^n} \mathcal{L}\{t^0\} = \frac{n!}{s^n} \frac{1}{s} = \frac{n!}{s^{n+1}}$

Problem 2:

(a) $\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{-(s-a)t} f(t) \, dt = \int_0^\infty e^{-\hat{s}t} f(t) \, dt = F(\hat{s}) = F(s-a).$

(b) $\mathscr{L}\{f(at)\} = \int_0^\infty e^{-st} f(at) \, dt.$ $u = at \implies du = a \, dt$ and $\mathscr{L}\{f(at)\} = \int_0^\infty e^{-\frac{s}{a}u} f(u) \, \frac{du}{a} = \frac{1}{a} F\Big(\frac{s}{a}\Big).$

Problem 3:

(a) $\mathscr{L}\{t^2)\} = \frac{2}{s^3} = F(s) \overset{2(a)}{\widehat{\implies}} \mathscr{L}\{e^{at}t^2)\} = F(s-a) = \frac{2}{(s-a)^3}$

(b) $\mathscr{L}\{\cos(t)\} = \frac{s}{s^2+1} = F(s) \overset{2(b)}{\widehat{\implies}} \mathscr{L}\{\cos(kt)\} = \frac{1}{k} F\Big(\frac{s}{k}\Big) = \frac{1}{k} \frac{s/k}{(s/k)^2+1} = \frac{s}{s^2+k^2}$

(c) $\mathscr{L}\{\sin(t)\} = \frac{k}{s^2+1} = F(s) \overset{2(b)}{\widehat{\implies}} \mathscr{L}\{\sin(kt)\} = \frac{1}{k} F\Big(\frac{s}{k}\Big) = \frac{1}{k} \frac{1}{(s/k)^2+1} = \frac{k}{s^2+k^2}$

Problem 4:

(a) $2\mathscr{L}\{x'\} + \mathscr{L}\{x\} = \mathscr{L}\{0\} \implies 2[s\mathscr{L}\{x\} - x(0)] + \mathscr{L}\{x\} = 0 \implies (2s+1)\mathscr{L}\{x\} + 6 = 0.$

Hence $x = \mathscr{L}^{-1}\left\{\frac{-6}{2s+1}\right\} = -3\mathscr{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} = -3e^{-\frac{1}{2}t}.$

(b) $\left[s^2\mathscr{L}\{x\} - s\,x(0) - x'(0)\right] - 4[s\mathscr{L}\{x\} - x(0)] = -3\mathscr{L}\{e^{-t}\}$

$\implies (s^2 - 4s)\mathscr{L}\{x\} = s - 5 - \frac{3}{s+1} \implies \mathscr{L}\{x\} = \frac{(s-5)(s+1)-3}{(s+1)(s)(s-4)} = \frac{s^2-4s-8}{(s+1)(s)(s-4)}.$

Let $\frac{s^2-4s-8}{(s+1)(s)(s-4)} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s-4} = \frac{A(s)(s-4) + B(s+1)(s-4) + C(s+1)(s)}{(s+1)(s)(s-4)},$

so that $(A+B+C)s^2 + (-4A-3B+C)s + (-4B) = s^2 - 4s - 8 \implies A = -\frac{3}{5}, \quad B = 2, \quad C = -\frac{2}{5}.$

Hence $x = -\frac{3}{5}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} = -\frac{3}{5}e^{-t} + 2 - \frac{2}{5}e^{4t}.$

(c) $\left[s^2\mathscr{L}\{x\} - s\cdot 0 - 0\right] + 9\mathscr{L}\{x\} = \mathscr{L}\{e^t\} \implies (s^2+9)\mathscr{L}\{x\} = \frac{1}{s-1} \implies \mathscr{L}\{x\} = \frac{1}{(s-1)(s^2+9)}.$

Let $\frac{1}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9} = \frac{(s^2+9)A + (s-1)(Bs+C)}{(s-1)(s^2+9)},$

so that $(A+B)s^2 + (-B+C)s + (9A-C) \implies A = \frac{1}{10}, \quad B = -\frac{1}{10}, \quad C = -\frac{1}{10}.$

Hence $\mathscr{L}\{x\} = \frac{\frac{1}{10}}{s-1} + \frac{-\frac{1}{10}s - \frac{1}{10}}{s^2+9} = \frac{1}{10}\left(\frac{1}{s-1}\right) - \frac{1}{10}\left(\frac{s}{s^2+3^2}\right) - \frac{1}{30}\left(\frac{3}{s^2+3^2}\right).$

Consequently, $x = \frac{1}{10}e^t - \frac{1}{10}\cos(3t) - \frac{1}{30}\sin(3t)$.

(d) Let $X = \mathscr{L}\{x\}$ and $Y = \mathscr{L}\{y\}$ then

$$\left.\begin{array}{r} sX - x(0) = X - 2Y \\ sY - y(0) = 5X - Y \end{array}\right\} \implies \left.\begin{array}{r} sX + 1 = X - 2Y \\ sY - 2 = 5X - Y \end{array}\right\} \implies \begin{array}{rr} (s-1)X + 1 = -2Y & (1) \\ (s+1)Y - 2 = 5X & (2) \end{array}$$

$$\begin{array}{rcll} (s-1)\times(2) & \implies & (s^2-1)Y - 2(s-1) = 5X(s-1) & (3) \\ 5(1) - (3) & \implies & 5 = -10Y - (s^2-1)Y + 2(s-1) = -Y(s^2+9) + 2s - 2 & \end{array}$$

Hence $Y = 2\dfrac{s}{s^2+3^2} - \dfrac{7}{3}\dfrac{3}{s^2+3^2} \implies y(t) = 2\cos(3t) - \dfrac{7}{3}\sin(3t)$.
Substituting back to the original DE (details omitted) we find $x(t) = -\cos(3t) - \frac{5}{3}\sin(3t)$.

Problem 5:

$$\begin{aligned} \mathcal{L}\{\frac{d^2x}{dt^2} + 2x\} = \mathcal{L}\{\sin(t)\} \quad & \implies \quad s^2X(s) - sx(0) - x'(0) + 2X(s) = \frac{1}{s^2+1} \\ & \implies \quad (s^2+2)X(s) = \frac{1}{s^2+1} \\ & \implies \quad X(s) = \frac{1}{(s^2+1)(s^2+2)} = \frac{1}{s^2+1} - \frac{1}{s^2+2} \\ & \implies \quad x(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+1} - \frac{1}{s^2+2}\} \\ & \implies \quad \boxed{x(t) = \sin(t) - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)}. \end{aligned}$$

Problem 6:

Take the Laplace transform on both sides of the two DEs (and define $X = \mathscr{L}\{x\}$ and $Y = \mathscr{L}\{y\}$):

$$\left.\begin{array}{r} s^2X - s + 8X - 3Y = 0 \\ s^2Y + 1 - 4X + 4Y = 0 \end{array}\right\} \implies \left.\begin{array}{r} (s^2+8)X - 3Y = s \\ -4X + (s^2+4)Y = -1 \end{array}\right\} \implies \begin{array}{rr} 4(s^2+8)X & -12Y = 4s \\ -4(s^2+8)X + (s^2+4)(s^2+8)Y & = -(s^2+8) \end{array}$$

Add those last two equations together: $\big[(s^2+4)(s^2+8) - 12\big]Y = 4s - (s^2+8)$

$$\implies (s^4 + 12s^2 + 20)Y = 4s - (s^2+8) \implies Y = \frac{-s^2+4s-8}{(s^2+10)(s^2+2)}.$$

Partial fractions: $\dfrac{-s^2+4s-8}{(s^2+10)(s^2+2)} = \dfrac{-\frac{1}{2}s - \frac{1}{4}}{s^2+10} + \dfrac{\frac{1}{2}s - \frac{3}{4}}{s^2+2} = \dfrac{-\frac{1}{2}s}{s^2+10} - \dfrac{\frac{1}{4}}{s^2+10} + \dfrac{\frac{1}{2}s}{s^2+2} - \dfrac{\frac{3}{4}}{s^2+2}$.

Hence $y = \mathscr{L}^{-1}\{Y\} = -\frac{1}{2}\cos(\sqrt{10}\,t) - \frac{1}{4\sqrt{10}}\sin(\sqrt{10}\,t) + \frac{1}{2}\cos(\sqrt{2}\,t) - \frac{3}{4\sqrt{2}}\sin(\sqrt{2}\,t)$.

Probably the easiest way now to obtain x, given y, is to use the equation $y'' - 4x + 4y = 0$.

Thus $x = \frac{1}{4}\big[y'' + 4y\big]$

$$\begin{aligned} &= \tfrac{1}{4}\Big[\tfrac{10}{2}\cos(\sqrt{10}\,t) + \tfrac{\sqrt{10}}{4}\sin(\sqrt{10}\,t) - \cos(\sqrt{2}\,t) + \tfrac{3\sqrt{2}}{4}\sin(\sqrt{2}\,t)\Big] + y \\ &= \tfrac{3}{4}\cos(\sqrt{10}\,t) - \tfrac{3}{8\sqrt{10}}\sin(\sqrt{10}\,t) + \tfrac{1}{4}\cos(\sqrt{2}\,t) - \tfrac{3}{8\sqrt{2}}\sin(\sqrt{2}\,t). \end{aligned}$$

Problem 7:

$$\mathcal{L}\{t^\alpha\} := \int_0^\infty e^{-st}t^\alpha\,dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha \frac{1}{s}\,du = \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u}u^\alpha\,du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$