TW244 **Opdrag / Assignment #3** **Due 23:59PM on 10 Oct 2021**

Hierdie opdrag moet as 'n enkele PDF-lêer via SUNLearn ingehandig word voor die sperdatum hierbo. Laat inhandigings sal gepenaliseer word. Kommunikasie tussen studente rakende werksopdragte is streng verbode en plagiaat sal tot ernstige gevolge lei. Jou inhandiging moet 'n getekende verklaring bevat dat dit jou eie werk is. Raadpleeg die TW244 SUNLearn-blad vir verdere instruksies.

This assignment must be submitted as a single PDF file via SUNLearn before the due date above. Late submissions will be penalized. Communication between students regarding assignments is strictly prohibited and plagiarism will have severe consequences. Your submission should contain a signed declaration that it is your own work. See TW244 SUNLearn page for further instructions.

P1: Beskou die tweede-orde aanvangswaardeprobleem,

P1: Consider the second order initial value problem,

$$y'' + \tfrac{2}{x}y' + y = \tfrac{1}{x^2}, \quad y(\pi) = 0, \quad y'(\pi) = 0.$$

(a) [2 punte] Deur 'n nuwe veranderlike $z = y'$ in te stel, skryf die DV hierbo as 'n stelsel van eerste-orde vergelykings en gebruik `ode45` om 'n numeriese benadering vir $y(x)$ te verkry met $\pi \leq x \leq 20$. Stip die berekende oplossing (slegs y, nie y' nie) as 'n soliede lyn en gee die waarde van $y(20)$ in die titel (asook geskikte name vir die asse, ens.)

(a) [2 marks] By introducing a new variable, $z = y'$, write the DE above as a system of first order equations and hence use `ode45` to obtain a numerical approximation to $y(x)$ for $\pi \leq x \leq 20$. Plot the computed solution (y only, not y') as a solid line, giving the value of $y(20)$ in the title (as well as suitable axes labels, etc).

(b) [2 punte] (i) Verifieer dat $y_1 = \sin(x)/x$ en $y_2 = \cos(x)/x$ oplossings vir die homogene DV is.[a] (ii) Wys dat y_1 en y_2 fundamentele oplossings is.

[a] y_1 word die *sinc funksie* genoem, $\text{sinc}(x) = \sin(x)/x$.

(b) [2 marks] (i) Verify that $y_1 = \sin(x)/x$ and $y_2 = \cos(x)/x$ are solutions to the homogeneous DE.[a] (ii) Show that y_1 and y_2 are fundamental solutions.

[a] y_1 is called the *sinc function*, $\text{sinc}(x) = \sin(x)/x$.

(c) [1 punt] Bereken Green se funksie vir die DV en genereer 'n 3D-grafiek daarvan (m.b.v. bv. `surf` or `mesh` in MATLAB) vir $\pi \leq t, x \leq 20$.[a]

[a] Daar is niks besonders interessant aan hierdie figuur nie; dit is slegs 'n oefening om (i) Green se funksie te vorm, (ii) 'n funksie van twee veranderlikes te definieer, en (iii) 'n 3D-grafiek te maak.

(c) [1 mark] Calculate the Green's function for the DE and generate a 3D plot of it (e.g., using `surf` or `mesh` in MATLAB) for $\pi \leq t, x \leq 20$.[a]

[a] There's nothing particularly interesting about this figure; it's just an exercise in (i) forming the Green's function, (ii) defining a function of two variables, and (iii) making a 3D plot.)

(d) [2 punte] Gebruik die metode van variasie van parameters (of die Green's funksie)) om te wys dat die oplossing vir die nie-homogene DV gegee word deur

(d) [2 marks] Use the method of variation of parameters (or the Green's function) to show that the solution to the non-homogenous DE is given by

$$y(x) = \frac{\sin(x)}{x}\Big(c_1 + Ci(x)\Big) + \frac{\cos(x)}{x}\Big(c_2 - Si(x)\Big), \qquad \text{waar/where} \quad Si(x) = \int_0^x \frac{\sin(t)}{t}, \quad Ci(x) = -\int_x^\infty \frac{\cos(t)}{t}\,dt$$

die *sinus integraal* en *cosinus integraal funksies* is.

are the *sine integral* and *cosine integral functions*.

(e) [1 punt] Wys dat die oplossing vir die aanvangswaardeprobleem gegee word deur

(e) [1 mark] Show that the solution to the initial value problem is given by

$$y(x) = \frac{\sin(x)}{x}\Big(Ci(x) - Ci(\pi)\Big) - \frac{\cos(x)}{x}\Big(Si(x) - Si(\pi)\Big).$$

(f) [1 punt] Stip die analitiese oplossing hierbo (gebruik nou 'n stippellyn) op dieselfde figuur as jou numeriese oplossing van **(a)**. Benoem die asse en gee 'n legende. (In Python kan jy `scipy.special.sici` gebruik. As jy die MATLAB Symbolc Toolbox het dan kan jy `sinint` en `cosint` gebruik, andersins kan jy `sici.m` vanaf die kursuswebwerf gebruik.)

(f) [1 mark] Plot the analytical solution above (now using a dashed line) on the same figure as your numerical solution from **(a)**. Include a legend and suitable axes labels. (In Python you can use `scipy.special.sici`. If you have the MATLAB Symbolic Toolbox then you can use `sinint` and `cosint`, otherwise use `sici.m` from the course website.)

P2: Hier leer ons meer oor MATLAB se `fzero` – 'n numeriese metode om vergelykings van die vorm $f(x) = 0$ op te los. Tik >> `help fzero` om te leer hoe om dit te gebruik. (`scipy.optimize.fsolve` is die Python-alternatief.)

P2: Here we learn about MATLAB's `fzero` – a numerical method for solving equations of the form $f(x) = 0$. Type >> `help fzero` to learn how to use it. (`scipy.optimize.fsolve` is the Python alternative.)

(a) [1 punt] Gebruik `fzero` of 'n soortgelyke funksie om die eerste waarde van $x > 0$ te vind sodat $y(x) = 0$. ('n Aanvanklike raaiskoot kan verkry word vanuit die figuur in **(f)**. Verkry die waarde tot ten minste vier beduidende syfers.)

(a) [1 mark] Use `fzero` or some equivalent to find the first value of $x > 0$ such that $y(x) = 0$. (An initial guess can be obtained from the figure in **(f)**. Obtain the value to at least four significant figures.)

(b) [1 punt] Deur te differensieer en $y'(x) = 0$ te stel, bepaal waar $y(x)$ sy maksimum en minimum bereik. Vertoon hierdie waardes en voeg sirkels by jou grafiek van **(f)** by (x_{max}, y_{max}) en (x_{min}, y_{min}). (As alternatief vir `fzero` hier kan jy `fminbnd`/`scipy.optimize.fminbnd` gebruik om die maksimum/minimum te vind.)

(b) [1 mark] By differentiating and setting $y'(x) = 0$ determine where $y(x)$ obtains its maximum and its minimum. Display these values and add circles to your plot from **(f)** at (x_{max}, y_{max}) and (x_{min}, y_{min}). (As an alternative to `fzero` here, you may use `fminbnd`/`scipy.optimize.fminbnd` to find the maximum/minimum.)

P3: Die situasie waar 'n teiken T langs 'n bekende pad $p(t)\underline{i} + q(t)\underline{j}$ beweeg en agternagesit word deur nastrewer P wat op 'n gegewe tydstip direk na die teiken beweeg, staan bekend as 'n **nastrewingskromme**. Die pad $x(t)\underline{i} + y(t)\underline{j}$ wat P volg kan beskryf word deur die gekoppelde stelsel van differensiaalvergelykings:

P3: The situation where a target T moving along a known path $p(t)\underline{i} + q(t)\underline{j}$ is chased by a pursuer P who at any given time heads directly towards the target is known as a **pursuit curve**. The path $x(t)\underline{i} + y(t)\underline{j}$ taken by P can be described by the coupled system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= k\sqrt{\left(\frac{dp}{dt}\right)^2 + \left(\frac{dq}{dt}\right)^2}\frac{p(t)-x}{\sqrt{(p(t)-x)^2 + (q(t)-y)^2}}\\ \frac{dy}{dt} &= k\sqrt{\left(\frac{dp}{dt}\right)^2 + \left(\frac{dq}{dt}\right)^2}\frac{q(t)-y}{\sqrt{(p(t)-x)^2 + (q(t)-y)^2}}\end{aligned}$$

waar k die relatiewe spoed van die nastrewer is.

where k is the relative speed of the pursuer.

(a) Terwyl jy op soek is na voorraad op die onlangs-gekoloniseerde Mars, sien jy 'n vyandige ruimtewese ("alien") 100m na die weste. Jou ruimtevaart staan 200m noord-oos van jou af. Ongelukkig is die ruimtewese vinnig, en kan dit 33% vinniger as jy hardloop. Gelukkig is dit nie intelligent nie, en sal dit direk na jou toe hardloop (d.w.s. nastreef) eerder as om jou te probeer afsny. Gebruik `ode45` of 'n soortgelyke funksie om die vergelykings hierbo op te los om die nastrewingskromme $P(t)$ van die ruimtewese te vind as jy direk na jou ruimtevaart toe hardloop. Stip jou pad en die nastrewingskromme op dieselfde figuur. Voeg 'n tweede figuur by om die afstand-na-ruimtewese (y-as) vs jou-afstand-gereis (x-as) te wys. Kom jy betyds by die ruimtevaart uit?

Wenk: Slegs die relatiewe spoed is belangrik, so dit maak nie eintlik saak hoe vinnig jy kan hardloop nie. As dit jou egter beter laat voel, neem aan dat jy in jou ruimtepak teen 5m/s op Mars kan hardloop.

(a) Whilst foraging for supplies on recently colonised Mars, you notice a hostile alien life form 100m to the west. Your rover is parked 200m to your north-east. Unfortunately, the life form is fast, and can run 33% faster than you can. Fortunately, it is not intelligent, and will run directly towards you (i.e., pursue) rather than try to cut you off. Use `ode45` or some alternative to solve the equations above to find the pursuit curve $P(t)$ of the alien if you run directly towards your rover. Plot your path and the pursuit curve of the alien on the same figure. Include a second figure showing distance-you-traveled (x-axis) vs distance-to-alien (y-axis). Do you make it to safety in time?

Hint: Only the relative speed is important, so it doesn't actually matter how fast you can run. However, if it makes you feel better, assume that you can run at 5m/s on Mars in your spacesuit.

(b) Jy het nie die tweede ruimtewese 250m oos van jou raakgesien nie. Herhaal P3(a) om die nastrewingskurwe van die tweede ruimtewese te bepaal.

(b) You failed to notice a second alien 250m to your east. ("Clever girl"!) Repeat P3(a) to determine the pursuit path of the second alien.

(c) Gelukkig slaag jou maat, Ripley, daarin om die tweede ruimtewese met haar plasma-geweer uit te haal. Jy is so vreesbevange weens die eerste ruimtewese wat jou jaag en jou instink is om daarvan af te wyk, so in plaas daarvan om direk na jou ruimtevaart te hardloop, hardloop jy op 'n sirkelboog (met radius $\sqrt{2} \times 100$, wat begin by die 6-uur posisie en anti-kloksgewys). Herhaal P3(a) met hierdie aanname. Is dit 'n beter strategie as om direk na die ruimtevaart toe te hardloop?

(c) Fortunately your partner, Ripley, manages to take out the second alien with her plasma rifle. However, terrified by the first alien chasing you, your instinct is to veer away from it, and instead of running directly to your rover you end up running on a circular arc (of radius $\sqrt{2}\times 100$, starting at the 6 o'clock position and anti-clockwise). Repeat P3(a) under this assumption. Is this a better strategy than running directly toward the rover?

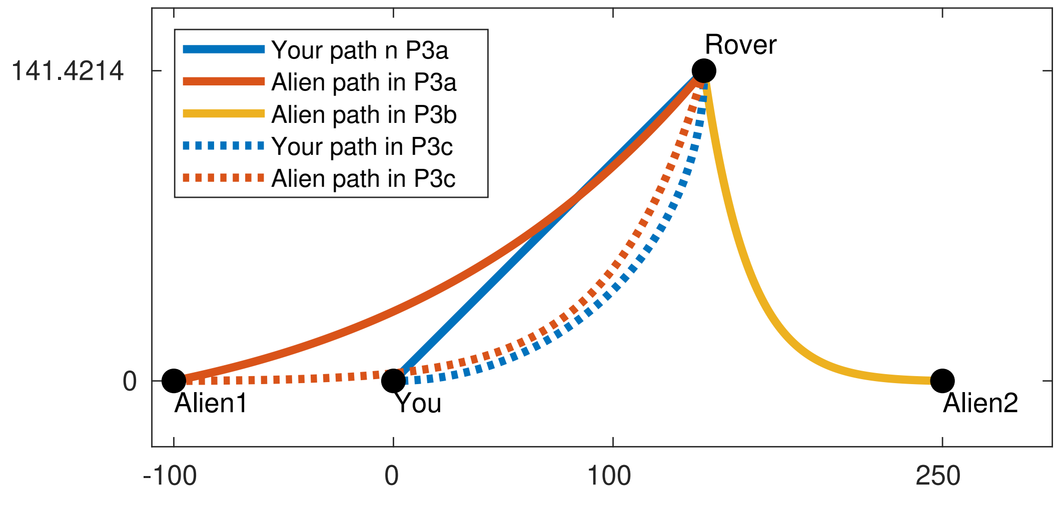

Figure 1: *Skets van die krommes* | Sketch of the curves

Groepsprobleem! Ons gaan iets ongewoons probeer. Die volgende vraag is vier bonuspunte werd, maar die *hele klas* moet dieselfde antwoord ingee (wat ons sal bereik m.b.v. 'n meningspeiling op SUNLearn). Jy mag of alleen of in 'n groep van jou keuse aan die probleem werk (jy mag egter nie hulp ontvang van iemand wat nie in die klas is nie). Op die sperdatum moet jy/jou groep jul oplossing (en bewerkings) op SUNLearn indien. Ek sal dan alle inhandigings oor hierdie probleem aan die hele klas beskikbaar maak, en almal kan stem vir wat hul dink die regte antwoord is. As die meerderheid stemme vir 'n inhandiging met die regte antwoord is, kry elkeen wat stem (al was dit nie noodwendig die korrekte antwoord nie) die punte. As die regte antwoord nie gekies word nie, kry niemand punte nie (dus is daar motivering om nie net die regte antwoord te kry nie, maar ook om jou oplossing so goed aan te bied dat jou mede-studente glo dit is korrek).

Neem kennis: Jy hoef nie iets in te handig om te mag stem nie. Jy moet egter stem om in aanmerking te kom vir punte. Net een inhandiging per groep is nodig. Inhandigings van verskillende groepe met dieselfde antwoord sal saamgevoeg word. Inhandigings sal anoniem geplaas word. As slegs een persoon inhandig, is dit nie nodig om te stem nie. As niemand iets inhandig nie, kry niemand die punte nie (en ek sal nooit weer iets pret probeer doen nie..)

P4 [**4 bonus punte**] Beskou 'n vier-fase vuurpyl, soortgelyk aan die een wat in P1 en P6 van RO02 beskryf is. Die vuurpyl bestaan uit drie 50kg boosters wat elk 25kg brandstof bevat, en een 50kg loonvrag. Die boosters word opeenvolgend gebruik. Wanneer die brandstof in 'n booster opgebruik word, word dit uitgestoot en die volgende booster neem oor. Elke booster bied 'n konstante dryfkrag van 2000N en verbrand brandstof teen 'n tempo van 1kg/s. Wat is die maksimum hoogte wat hierdie vuurpyl kan bereik?

Aannames: Die vuurpyl word vanuit rus vanaf die aardoppervlak gelanseer. Neem g konstant teen 9.81. Aanvaar dat die boosters onmiddellik van mekaar skei en aansteek by maksimum dryfkrag. Aanvaar 'n *kwadratiese* sleepkrag met konstante sleepkoëffisiënt $kuntitled = 0.3$m/kg. (Ek glo dat dit al die vereiste aannames is. Enige ander aannames moet eers met my bespreek word voor jy jou antwoord indien.)

Group Problem! We are going to try something unusual. The following question is worth four bonus marks, however, the *entire class* must submit the same answer (which we will achieve via a poll on SUNLearn). You may work on the problem alone, or in a group of your choosing (however, you may not seek assistance on the problem from anyone outside the class). On the due date, you/your group should submit their solution (and working) on SUNLearn. I will then make all submissions to this problem available to the whole class, and everyone gets to vote on what they think is the correct answer. If the majority of votes are for a submission with the correct answer, then everyone who votes (not necessarily for the correct answer) will get the marks. If the correct answer is not chosen, then nobody gets marks (so there is motivation not only to get the right answer, but also present/-explain your solution clearly so that people believe it is correct).

Points of clarification: You do not need to submit in order to vote. You need to vote in order to be eligible for marks. Only one submission is required per group. Submissions from multiple groups with the same answer will be collated together. Submissions will be posted anonamously. If only one person submits, then there will be no need for a vote. If nobody submits, then nobody will get marks (and I will never try to do anything fun ever again..).

P4 [**4 bonus marks**] Consider a four-stage rocket, similar to that described in P1 & P6 of CA02. The rocket is formed of three 50kg boosters, each containing 25kg of fuel, and one 50kg payload. The boosters are used sequentially. When the fuel in a booster is expended it is ejected and the next booster in the sequence takes over. Each booster provides a constant thrust of 2000N and burns fuel at a rate of 1kg/s. What is the maximum height achievable by this rocket?

Assumptions: The rocket is launched from rest from the Earth's surface. Take g constant at 9.81. Assume boosters separate instantaneously and ignite at maximum thrust. Assume a *quadratic* drag force with constant drag coefficient $kuntitled = 0.3$m/kg. (I believe those are all the required assumptions. Any others should be cleared with me before submitting.)