Naam/Name: MEMO

Stud. Nr: ______

Toegepaste Differensiaalvergelykings
TW244 Toets 1 2019

Instruksies:

(a) 5 probleme, 54 punte (50 + 4 bonus).

(b) 2 uur, toeboek.

(c) Sakrekenaars **word toegelaat**. Selfone **nie**.

(d) Toon alle bewerkings. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(e) Daar is leë bladsye aan die agterkant van die vraestel as jou antwoorde nie inpas in die gegewe spasies nie. Dui duidelik aan as jou antwoord voortgaan op een van hierdie bladsye.

(f) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

Applied Differential Equations
TW244 Test 1 2019

Instructions:

(a) 5 problems, 54 marks (50 + 4 bonus).

(b) 2 hours, closed book.

(c) Calculators **are** allowed. Cell phones are **not**.

(d) Show all calculations. A correct answer does not earn full marks without the necessary explanation.

(e) There are blank pages at the back of the paper in case you cannot fit your answer in the space provided. Indicate clearly if your answer continues to one of these pages.

(f) The formulas below may be used without proof anywhere in the test.

Formules/Formulas:

- Fundamentele stelling van calculus/ Fundamental theorem of calculus: $$\int \frac{df}{dx}\, dx = f(x) + C$$
- Produkreël vir differensiasie/ Product rule for differentiation: $$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$
- Deelsgewyse integrasie/ Integration by parts: $$\int_a^b f\frac{dg}{dx}\, dx = [fg]_a^b - \int_a^b \frac{df}{dx} g\, dx$$
- Differensiasie van die logaritme/ Differentiation of the logarithm $$\frac{d}{dx}\ln(x-a) = \frac{1}{x-a}$$

Prob 1 (10 punte/marks)

(a) Vir elk van die volgende differensiaalvergelykings, gee die orde, dui aan of dit lineêr is of nie, en of dit 'n outonome DV is of nie.

(a) For each of the following differential equations, give the order, state whether it is linear or not, and whether it is autonomous or not.

differensiaalvergelyking *differential equation*	orde *order*	lineêr? (ja/nee) *linear? (yes/no)*	outonome DV? (ja/nee) *autonomous DE? (yes/no)*
$\left(\frac{dy}{dx}\right)^3 + y = 1$	1	✗	✓
$\dot{x} = 3x - 4y + \cos(t^2)$ $\dot{y} = 5x + 2y + \sin^2(t)$	1	✓	✗

(b) Watter van die volgende DVs is geskik om deur die integrasiefaktor metode opgelos te word.

(b) Which of the following DEs would be suitable for solving with the integrating factor method:

(A) $(\frac{dy}{dx})^3 + xy = y$ (B) $\frac{dy}{dx} + x^3y^3 = y$ (C) $\frac{dy}{dx} + xy^3 = y$ (D) $\frac{dy}{dx} + xy = y^3$ (E) niks geskik nie/ none suitable

(c) Watter van die volgende DVs is nie geskik om deur skeiding van veranderlikes opgelos te word nie.

(c) Which of the following DEs would not be suitable for solving with separation of variables:

(A) $(\frac{dy}{dx})^3 + xy = y$ (B) $\frac{dy}{dx} + x^3y^3 = y^3$ (C) $\frac{dy}{dx} + x^3y = y$ (D) alles geskik/ all suitable (E) niks geskik nie/ none suitable

(d) Toon aan m.b.v. die integrasiefaktor dat die oplossingsfamilie van die DV

(d) Show, using integrating factor, that the family of solutions to the DE

$$\frac{dy}{dt} = t - y$$

gegee word deur

is given by

$$y(t) = t - 1 + Ce^{-t}.$$

(Wenk: sien voorblad vir deelsgewyse integrasie)

(Hint: see front page for integration by parts)

$\frac{dy}{dt} + y = t$ IF e^t

$\frac{d}{dt}(e^t y) = t e^t$

$\int d(e^t y) = \int t e^t \, dt$

$e^t y = te^t - e^t + C$

$y = t - 1 + Ce^{-t}$

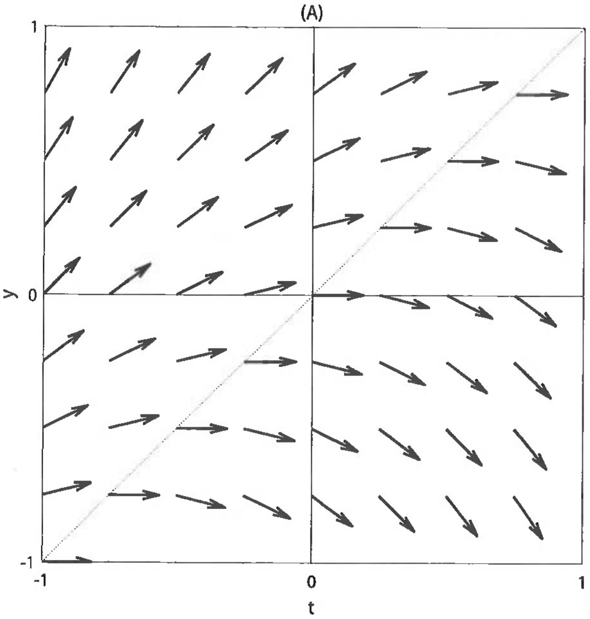
QUESTION (1e)

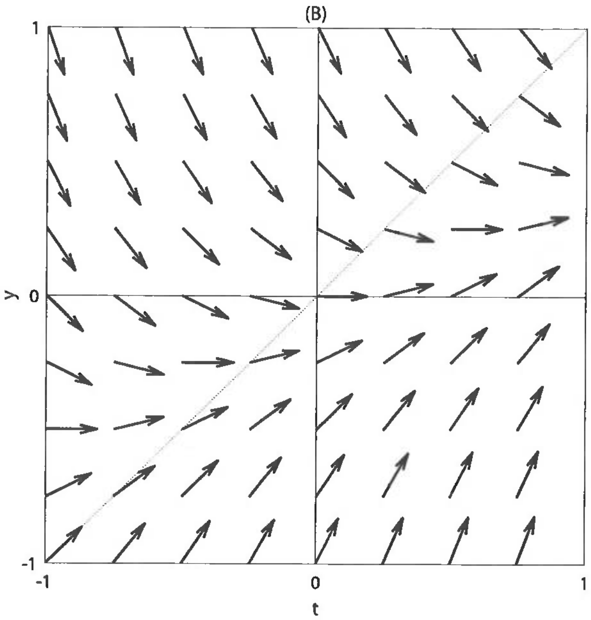
ANSWER: C

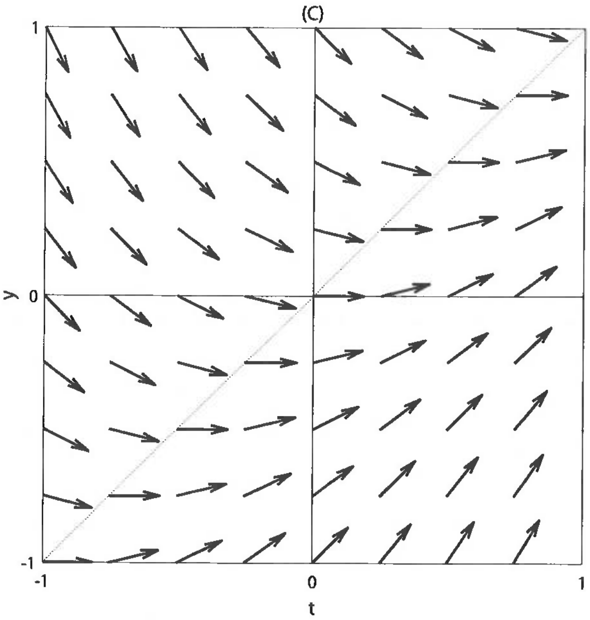
(e) Watter van die onderstaande rigtingvelde stem met die DV $\frac{dy}{dt} = t - y$.

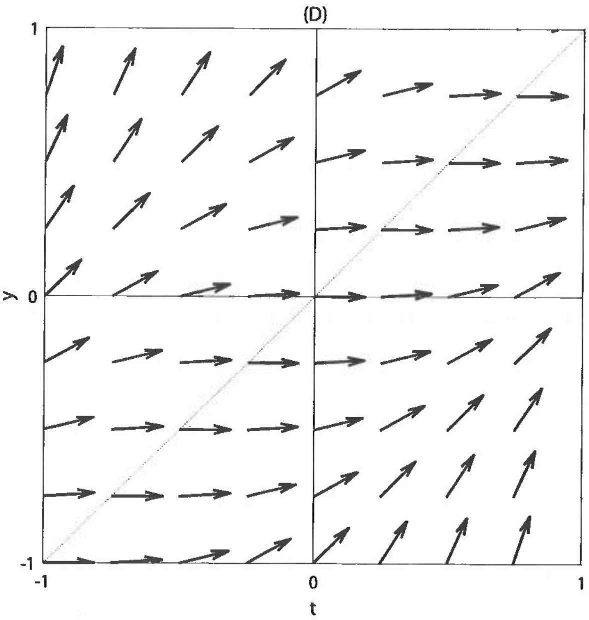
(e) Which of the direction fields below corresponds to the DE $\frac{dy}{dt} = t - y$.

(A) (B) (C) (D) (E) geen van die onderstaande / none of the below









Prob 2 (10 punte/marks)

Veronderstel ons stel belang in die gedrag van nie-negatiewe oplossings ($P \geq 0$) van die outonome DV:

Suppose we are interested in the behaviour of non-negative solutions ($P \geq 0$) of the autonomous DE:

$$\frac{dP}{dt} = P^2 - P.$$

(a) Vind die kritieke oplossings van die DV.

(a) Find the critical solutions of the DE.

$\frac{dP}{dt} = 0 \qquad P(P-1) = 0 \qquad P = 0$ or $P = 1$

(b) Vir watter waardes van P is $\frac{dP}{dt}$ positief / negatief?

(b) For what values of P is $\frac{dP}{dt}$ positive / negative?

$\frac{dP}{dt} > 0$

	<0	0 to 1	>1
P	$-$	$+$	$+$
$P-1$	$-$	$-$	$+$
	$+$	$-$	$+$

$P < 0 \vee P > 1$

$\frac{dP}{dt} < 0 \qquad 0 < P < 1$

(c) Toon aan dat $\frac{d^2P}{dt^2} = P(P-1)(2P-1)$.

(c) Show that $\frac{d^2P}{dt^2} = P(P-1)(2P-1)$.

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}P(P-1) = (P-1)\frac{dP}{dt} + P\frac{dP}{dt} = (2P-1)\frac{dP}{dt} = (2P-1)P(P-1)$$

(d) Vir watter waardes van P konkaaf op / af?

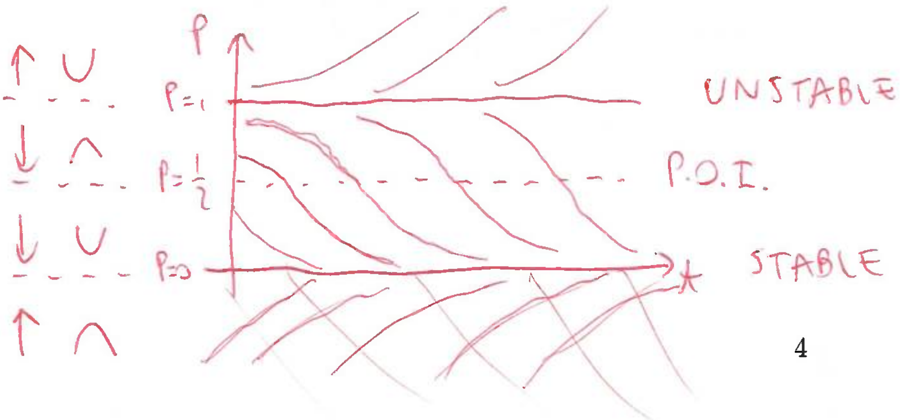
(d) For what values of P concave up / down?

$\frac{d^2P}{dt^2} > 0$

	<0	0 to $1/2$	$1/2$ to 1	>1
P	$-$	$+$	$+$	$+$
$P-1$	$-$	$-$	$-$	$+$
$2P-1$	$-$	$-$	$+$	$+$
	$-$	$+$	$-$	$+$

(e) Gebruik die informasie vanuit dele (a)-(d) om die 'familieportret' van oplossings vir die DV te teken op die oorkantste bladsy. Wys die gebiede waarin oplossings konkaaf na bo of konkaaf na onder is, en merk die infleksiepunt indien daar een is. Dui verteenwoordigende oplossingskurwes vir 'n paar aanvangsvoorwaardes aan, en klassifiseer die kritieke oplossings as stabiel, semi-stabiel, of onstabiel.

(e) Use the information from parts (a)-(d) to draw the 'family portrait' of solutions for the DE on the page opposite. Show the regions in which solutions are concave up or concave down, and mark the point of inflection if there is one. Include representative solution curves for some initial conditions, and classify the critical solutions as stable, semi-stable, or unstable.



Beskou die aanvangswaardeprobleem

Consider the initial value problem

$$\frac{dP}{dt} = k(P^2 - P), \qquad k > 0, \qquad 0 < P(0) = P_0 < 1.$$

(f) Bereken die waardes van A en B waarvoor

(f) Compute the values of A and B for which

$$\frac{1}{P(P-1)} = \frac{A}{P} + \frac{B}{P-1} = \frac{A(P-1)+BP}{P(P-1)}$$

$$P(A+B) - A = 1$$

$$A+B=0 \qquad A=-B \qquad A=-1$$
$$B=1$$

(g) Gebruik skeiding van veranderlikes om dié DV op te los en wys dat die oplossing geskryf kan word as

(f) Use separation of variables to solve the DE and show that the solution can be written as

$$P(t) = \frac{-P_0 e^{-kt}}{P_0(1 \cancel{+} - e^{-kt}) - 1}.$$

(SORRY!!! MARKED ACCORDINGLY

$$\int \frac{dP}{P(P-1)} = \int k\,dt = kt + C$$

$$\int \frac{1}{P-1} - \frac{1}{P}\,dP = \ln\left|\frac{P-1}{P}\right| \qquad \frac{P-1}{P} = \pm C e^{kt}$$

$$0 < P_0 < 1 \qquad \boxed{\frac{P_0-1}{P_0}} = C, \quad \frac{P_0-1}{P_0} < 0$$

$$\boxed{\frac{P-1}{P}} = \boxed{\frac{P_0-1}{P_0}}\boxed{e^{kt}}$$

$\frac{P-1}{P} < 0$ (see 2.e), $\frac{P_0-1}{P_0} < 0$, $e^{kt} > 0$

$$P\left(1 - \frac{P_0-1}{P_0}e^{kt}\right) = 1$$

$$P = \frac{1}{\frac{P_0 - (P_0-1)e^{kt}}{P_0}} = \frac{P_0}{P_0(1-e^{kt}) + e^{kt}}$$

$$= \frac{-P_0 e^{-kt}}{P_0(1-e^{-kt}) - 1}$$

OR

$$= \frac{P_0 e^{-kt}}{P_0(e^{-kt}-1) + 1}$$

Prob 3 (10 punte/marks)

'n Moordslagoffer is dood gevind in 'n vleiskamer met 'n konstante kamertemperatuur $T_m = 4°C$. Jy, die hoof van CSI Stellenbosch, moet Newton se Wet van Verkoeling gebruik om die tyd van sy dood te bepaal:

A murder victim is found dead in a meat locker kept at a constant ambient temperature of $T_m = 4°C$. You, the head of CSI Stellenbosch, must use Newton's Law of Cooling to establish the time of death:

$$\frac{dT}{dt} = k(T_m - T), \quad T(0) = T_0.$$

(a) Jou kollega stry dat as $T_m > T_0$ dan moet k positief wees, en as $T_m < T_0$ dan moet k negatief wees. Verduidelik kortliks waarom hy/sy verkeerd is.

(a) Your colleague argues that if $T_m > T_0$ then k should be positive and if $T_m < T_0$ then k should be negative. Explain briefly why he/she is wrong.

$$\left.\begin{array}{l} \text{if } T_m > T, \ T \text{ should decrease}, \ \frac{dT}{dt} < 0 \\ \text{if } T < T_m, \ T \text{ should increase}, \ \frac{dT}{dt} > 0 \end{array}\right\} \Rightarrow k > 0$$

(b) Gebruik die integrasiefaktor om die AWP op te los en wys dat

(b) Use integrating factor to solve the IVP and show that

$$T(t) = T_m + (T_0 - T_m)e^{-kt}.$$

$$\frac{dT}{dt} + kT = kT_m \qquad \text{I.F. } e^{kt}$$

$$\frac{d}{dt}\left(e^{kT}T\right) = e^{kt} k T_m$$

$$e^{kt}T = \int e^{kt} k T_m \, dt = e^{kt} T_m + C$$

$$T = T_m + Ce^{-kt}$$

$$T(0) = T_0 = T_m + C \qquad C = T_0 - T_m$$

$$T = T_m + (T_0 - T_m)e^{-kt}$$

(c) Toe jy 5nm gearriveer het, het jy die kerntemperatuur van die slagoffer se liggaam as 20°C gemeet. 'n Uur later, teen 6nm, meet jy weer en vind jy dat die liggaam se kerntemperatuur 15°C is. Gebruik hierdie metings om k te bereken.

(b) When you arrived at 5PM you measured the victim's core body temperature as 20°C. Now, an hour later at 6PM, you measure again and find the core body temperature is 15°C. Use these measurements to determine k.

$T_m = 4$

$T_0 = 20 \qquad T(1) = 15$

$t = 0 = 5$ PM

$$15 = 4 + (20-4)e^{-k} \qquad e^{-k} = \frac{11}{16} \qquad k = \ln\frac{16}{11} \simeq 0{,}37$$

$k \simeq$ 0,37

(d) Laastens, gebruik hierdie waarde van k saam met die feit dat die kerntemperatuur van 'n lewendige persoon om en by 37°C is om die tyd van die persoon se dood te bepaal.

(d) Finally, use this value of k and fact that the core temperature of a living human is around 37°C to estimate the time of death.

$$T(t_D) = 4 + (20-4)e^{-k \cdot t_D} = 37 \qquad e^{-k \cdot t_D} = \frac{33}{16} \qquad t_D = \frac{\ln\frac{16}{33}}{\ln\frac{16}{11}} \simeq -2 \text{ hrs.}$$

Tyd/Time $\simeq$ 3 PM

Prob 4 (10 punte/marks)

Twee chemikalieë A en B word gekombineer om 'n chemikalie C te vorm. Die reaksietempo is direkeweredig aan die produk van die hoeveelhede van A en B wat nog nie na C omgeskakel is nie. Aanvanklik is daar 40 gram van A en 30 gram van B. Om 5 gram van C te maak word 3 gram van B en 2 gram van A benodig.

Two chemicals A and B are combined to form a chemical C. The rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to C. Initially, there are 40 grams of A and 30 grams of B. To make 5 grams of C requires 3 grams of B and 2 grams of A.

(a) Skryf 'n AWP neer wat hierdie stelsel beskryf (Geen bewerkings is nodig nie).

(a) Write down an IVP describing this system. (No working necessary).

$A(t) = 40 - \frac{2}{5}C(t)$

$B(t) = 30 - \frac{3}{5}C(t)$

$\therefore \quad \frac{dC}{dt} = k(C-100)(C-50), \qquad C(0) = 0$

(b) Los die AWP op met skeiding van veranderlikes en wys dat die oplossing gegee word deur

(b) Solve the IVP using separation of variables and show that the solution is given by

$$C(t) = \frac{100(1 - e^{-50kt})}{2 - e^{-50kt}}.$$

$$\int \frac{dC}{(C-100)(C-50)} = \int k\, dt$$

$$\frac{1}{(C-100)(C-50)} = \frac{\alpha}{C-100} - \frac{\alpha}{C-50}$$

$$1 = \alpha(C-50) - \alpha(C-100) = -50\alpha + 100\alpha$$

$$\alpha = 1/50$$

$$\int \frac{1}{C-100} - \frac{1}{C-50}\, dC = 50kt + c$$

$$\ln\left|\frac{C-100}{C-50}\right| = 50kt + c \qquad C(0) = 0 \qquad c = \ln\left|\frac{0-100}{0-50}\right|$$

$$\left(\frac{50}{100}\right)\left(\frac{C-100}{C-50}\right) = e^{50kt}$$

$$100(C-50) = 50(C-100)e^{-50kt}$$

$$\cancel{50}C(2 - e^{-50kt}) = \cancel{50}\cdot 100 - 100\cdot\cancel{50}\cdot e^{-50kt}$$

$$C = \frac{100(1 - e^{-50kt})}{2 - e^{-50kt}}$$

(c) Dit word waargeneem dat 5 gram van C in 2 minute gevorm word. Hoeveel word gevorm in 10 minute?

(c) It is observed that 5 grams of C is formed in 2 minutes. How much is formed in 10 minutes?

$$C(2) = 5 \qquad 5 = \frac{100(1 - e^{-100k})}{2 - e^{-100k}}$$

$$10 - 5e^{-100k} = 100 - 100e^{-100k} \qquad e^{-100k} = \frac{18}{19} \qquad k = \frac{\ln\frac{19}{18}}{100} \approx 5{,}4\cdot 10^{-4}$$

$$C(10) = \frac{100(1 - e^{-500k})}{2 - e^{-500k}} \approx 19\ g$$

(d) Wat is die beperkte aantal van C na 'n lang tydperk?

(d) What is the limiting amount of C after a long time?

$$C(t\to\infty) = 50\ g$$

(e) Hoeveel van chemikalieë A en B bly oor na 'n lang tydperk?

(e) How much of the chemicals A and B remain after a long time?

$$A(t\to\infty) = 40 - \frac{2}{5}\cdot 50 = 20\ g$$

$$B(t\to\infty) = 30 - \frac{3}{5}\cdot 50 = 0\ g$$

Prob 5 (14 punte/marks)

Toe Skoonlief en die Ondier mekaar se foto's op hul "dating app" sien, raak die Ondier verlief op Skoonlief weens haar aangename voorkoms ($A_B > 0$), terwyl Skoonlief nie van die Ondier hou nie vanweë sy skrikwekkende voorkoms ($A_T < 0$). Beide van hul vergeet van hul gevoelens teen 'n konstante eenheidstempo, ongeag of hul geliefd is (positief) of nie (negatief). Die gevoelens van die Ondier teenoor Skoonlief word daarom beskryf deur

When Beauty and The Beast are shown each other's picture on their dating app, The Beast falls in love with Beauty because of her pleasant appearance ($A_B > 0$), while Beauty dislikes The Beast because of his terrifying appearance ($A_T < 0$). Also, they both forget their feelings with a constant unit rate, no matter if they are loved (positive) or disliked (negative). Thus, the feeling of The Beast towards Beauty is described by

$$\dot{T} = -T + A_B$$

terwyl die gevoelens van Skoonlief teenoor die Ondier beskryf word deur

while the feeling of Beauty towards The Beast is described by

$$\dot{B} = -B + A_T$$

(a) Bereken die ewewig vir albei se gevoelens.

(a) Compute the equilibrium for both feelings.

$\dot{T} = 0 \quad T = A_B$

$\dot{B} = 0 \quad B = A_T$

(b) Los beide DVs op (deur gebruik te maak van die skeiding van veranderlikes of die integrasiefaktor) om $T(t)$ en $B(t)$ te bepaal, as dit gegee word dat hul aanvanklike gevoelens onverskillig is ($T(0) = B(0) = 0$) aangesien hul mekaar nie vantevore geken het nie.

(b) Solve both DEs (using either separation of variables or integrating factor) to obtain $T(t)$ and $B(t)$, given that, since they did not know each other before, their initial feelings are indifferent ($T(0) = B(0) = 0$).

$\dot{T} + T = A_B \quad$ I.F. e^t

$Te^t = A_B e^t + C \quad T(0) = 0$

$T = A_B + Ce^{-t} \quad C = -A_B$

$T(t) = A_B(1 - e^{-t})$

$\dot{B} + B = A_T$

SAME HERE

(c) Verifieer die stabiliteit van die ewewigsgevoelens wat in vraag (a) bereken is deur gebruik te maak van die oplossings uit vraag (b) en lewer kommentaar op die resultaat (onthou $A_T < 0$).

(c) Verify the stability of the equilibrium feelings computed in question (a) using the solutions from question (b) and comment on the result (remember $A_T < 0$).

$T(t \to \infty) = A_B$

$B(t \to \infty) = A_T$

STABLE, THE BEAST LOVES BEAUTY

BUT SHE DISLIKES HIM

As hulle egter persoonlik ontmoet, kan hul gevoelens mekaar beïnvloed met koëffisiënte k_T en k_B ($0 < k_T, k_B < 1$). Dít kan gemodelleer word deur die stelsel

However, if they instead meet in person, their feelings can affect each other with coefficients k_T and k_B ($0 < k_T, k_B < 1$). This can be modelled by the system

$$\dot{T} = -T + A_B + k_T B$$

$$\dot{B} = -B + A_T + k_B T$$

(d) Wat is die betekenis van k_T en k_B, en die feit dat hul groter as 0 is?

(d) What is the meaning of k_T and k_B, and their being greater than 0?

k_T AND k_B ARE REACTION TO THE PARTNER'S FEELINGS: IF YOUR PARTNER LOVES YOU (POSITIVE FEELINGS RECEIVED), YOU TEND TO LOVE HER/HIM BACK (POSITIVE EFFECT ON YOUR FEELINGS IN HER/HIS REGARD). IF YOU ARE DISLIKED THAT WILL HAVE A NEGATIVE EFFECT ON YOUR FEELINGS

(e) Aangesien hulle aanvanklik onverskillig teenoor mekaar voel ($T(0) = B(0) = 0$), wat is die aanvanklike rigting van Skoonlief se gevoelens teenoor die Ondier?

(e) Given that they are initially indifferent to each other ($T(0) = B(0) = 0$), what is the initial direction of Beauty's feelings towards The Beast?

THE BEAST $\dot{T}(0) = -\cancel{T(0)} + A_B + k_T \cancel{B(0)}$ POSITIVE $(A_B > 0)$

BEATY $\dot{B}(0) = A_T < 0$ NEGATIVE

(f) Bereken die ewewig van die stelsel van DVs en neem aan dat dit stabiel is. Toon aan of die Ondier uiteindelik deur Skoonlief liefgehê kan word.

(f) Compute the equilibrium of the system of DEs and, assuming it is stable, show if The Beast can eventually be loved by Beauty.

$\dot{T} = 0$, $\dot{B} = 0$ $\begin{cases} T = A_B + k_T B \\ B = A_T + k_B T \end{cases}$

$B = A_T + k_B(A_B + k_T B)$ $B(1 - k_T k_B) = A_T + k_B A_B$

$B = \dfrac{A_T + k_B A_B}{1 - k_T k_B}$, $1 - k_T k_B > 0$

(SIMILAR FOR THE BEAST)

$T = \dfrac{A_B + k_T A_T}{1 - k_T k_B}$

BEAUTY WILL LOVE THE BEAST if $B > 0$ $A_T + k_B A_B > 0$ $k_B A_B > |A_T|$

if HER REACTION (BEAUTY APPEAL) LARGER THAN THE BEAST'S TERRIFYING APPEARANCE

(g) Byvoorbeeld, wat is die waarde van die ewewig in vraag (f) as $A_B = 1, A_T = -1/3, k_T = k_B = 1/2$?

(g) For example, what is the value of the equilibrium in question (f) if $A_B = 1, A_T = -1/3, k_T = k_B = 1/2$?

$$B = \frac{-\frac{1}{3} + \frac{1}{2} \cdot 1}{1 - \frac{1}{2} \cdot \frac{1}{2}} = \frac{-\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{4}} = \frac{\frac{-2+3}{6}}{\frac{3}{4}} = \frac{1}{\cancel{6}} \cdot \frac{\cancel{4}^{2}}{3} = \frac{2}{9}$$

> 0 !

in fact $k_B \cdot A_B > |A_T|$

$\frac{1}{2} \cdot 1 > \frac{1}{3}$

Hierdie bladsy is doelbewus leeg gelaat. Jy kan dit gebruik vir rofwerk of vir ekstra spasie, indien nodig.

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