Applied differential equations

TW244 - Lecture 06

3.1: Linear Models

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SCIENCE
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3.1: Linear models

- Population growth
- Radioactive decay
- Newton's law of cooling
- Mixtures
- Compound interest
- Series circuits
- Free-fall with drag
- Many more in Z&W 3.1

Consider the initial value problem, I:

$$\frac{dx}{dt} = kx, \qquad x(0) = x_0.$$

For the next few lectures we will be looking at some simple linear models of this form.

But first, let's quickly recall how to solve this linear DE using

1. Separation of variables
2. Integrating factor method

We have:

$$\frac{dx}{dt} = kx \implies \frac{1}{x}\,dx = k\,dt \implies \int \frac{1}{x}\,dx = \int k\,dt \implies \ln|x| = kt + c.$$

Initial condition:

$$x(t=0) = x_0 \implies c = \ln|x_0|.$$

Therefore

$$\ln|x| = kt + \ln|x_0| \implies \ln\left|\frac{x}{x_0}\right| = kt \implies \left|\frac{x}{x_0}\right| = e^{kt} \implies x = \pm x_0 e^{kt}.$$

We choose the $+$ solution, so that the initial condition $x(0) = x_0$ is satisfied, i.e.,

$$\boxed{x(t) = x_0 e^{kt}.}$$

$x' + p(t)x = q(t)$

$f(t) = e^{\int p(t)\,dt}$

We have:

$$\frac{dx}{dt} = kx \implies \frac{dx}{dt} - kx = 0.$$

Integrating factor: $e^{\int(-k)\,dt} = e^{-kt}$ ✓

$$e^{-kt}\frac{dx}{dt} - ke^{-kt}x = \frac{d}{dt}[e^{-kt}x] = 0 \implies e^{-kt}x = \int 0\,dt + c = c$$

$$\implies x = ce^{kt}$$

✓

Initial condition:

$$x(t=0) = x_0 \implies c = x_0,$$

i.e.,

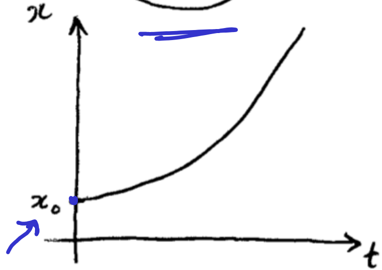
$$\boxed{x(t) = x_0e^{kt}.}$$

Growth or decay?

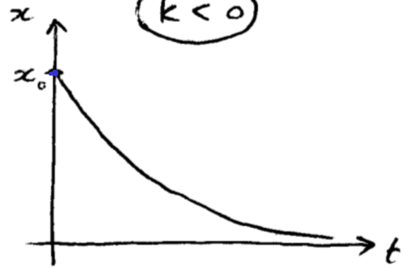
$x(t) = x_0 e^{kt}$

Note :

$k > 0$



$k < 0$



Exponential growth

Exponential decay

3.1: Linear Models
Application 1: Population growth

Let $P = P(t)$ be the size of a population at time t.

Assumptions:

- We can approximate $P(t)$ with a smooth continuous function.
- The rate of growth at any time t is proportional to the population size at that time. (The Malthus model.)

So we have

$$\frac{dP}{dt} = kP$$

with (constant of proportionality or "growth rate")

$$k > 0,$$

and (initial population size)

$$P(0) = P_0.$$

Therefore

$$\boxed{P(t) = P_0 e^{kt}.}$$

3.1: Linear Models

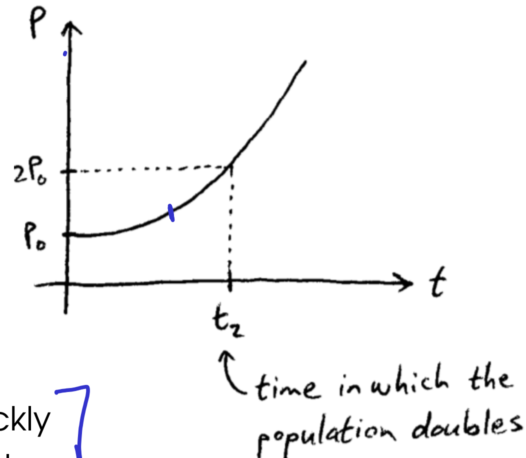
Application 1: Population growth: Doubling time

$\boxed{P(t) = P_0 e^{kt}}$

We could ask "how long does it take the population to double in size?"

i.e., we want the time t_2 such that

$$
\begin{aligned}
P(t_2) &= 2P_0 \\
P_0 e^{kt_2} &= 2P_0 \\
e^{kt_2} &= 2 \\
kt_2 &= \ln 2 \\
t_2 &= \frac{1}{k}\ln 2
\end{aligned}
$$



Therefore

- k large means pop. doubles quickly
- k small means pop. doubles slowly

Example 1

$dP/dt = kP$, $P(0) = P_0$

What if we don't know the growth rate, k?

$\to P(t) = P_0 e^{kt}$

Suppose a culture initially has P_0 number of bacteria and at $t = 1$ (hour) the number of bacteria is measured to be $\frac{3}{2}P_0$.

If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the necessary time for the bacteria to triple.

Solution:

$$\frac{dP}{dt} = kP \text{ with } P(0) = P_0 \implies P = P_0 e^{kt}. \quad (\leftarrow t \text{ measured in hours})$$

At $t = 1$ we have $\frac{3}{2}P_0 = P_0 e^{k \cdot 1} \implies \frac{3}{2} = e^k \implies k = \ln \frac{3}{2}$.

$P(t) = P_0 e^{\ln\frac{3}{2} t}$

So when is $P = 3P_0$?

$$3P_0 = P_0 e^{\ln(\frac{3}{2})t} \implies \ln 3 = \ln(\tfrac{3}{2})t \implies t = \frac{\ln 3}{\ln \frac{3}{2}} \approx 2.71 \text{ hours.}$$