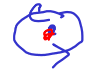
Applied differential equations

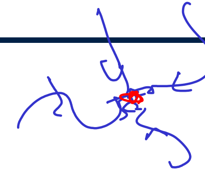
TW244 - Lecture 29

10.2 Stability of linear sytems

Prof Nick Hale - 2020

10.2: Stability of systems
Recall

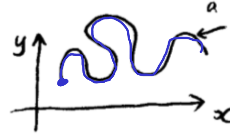


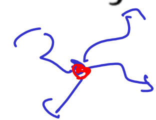


We are looking at plane autonomous systems:

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

phase plane:





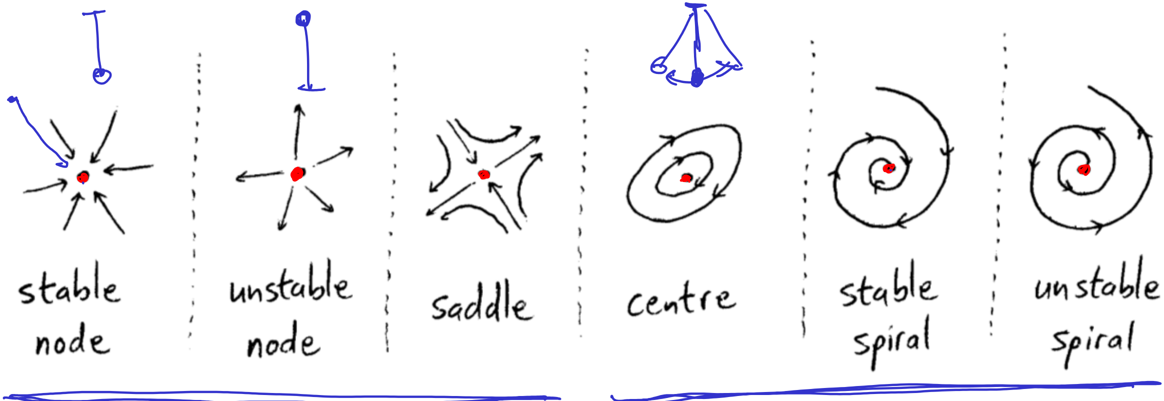
We said that a **critical solution** was an (x, y) pair for which $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

Such a solution corresponds to a point in the phase plane.

By considering how solution curves behave in a small region around the critical solution, we can classify them as one of six types...

10.2: Stability of systems
Classification of critical solutions

By considering how solution curves behave in a small region around the critical solution, we can classify them as one of six types...



Given a critical solution, we will now see how to determine which type it is.

We first consider linear autonomous systems...

10.2: Stability of systems

Classification of critical solutions

Considering the linear autonomous system of DEs

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned} \implies \underbrace{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}}_{\frac{d}{dt}\underline{x}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{A\,\underline{x}}.$$

Critical solutions have $\frac{dx}{dt} = \frac{dy}{dt} = 0$, hence

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

Hence, if A is invertible, then $(x, y) = (0, 0)$ is the unique critical solution.

So how can we determine what type of critical solution $(0, 0)$ is..?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Recall:
If A has linearly independent eigenvectors $\underline{v}_1, \underline{v}_2$ with corresponding eigenvalues λ_1, λ_2, then the solution of $\frac{d\underline{x}}{dt} = A\underline{x}$ is $\underline{x} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2$.

To find λ_1 and λ_2 observe that

$$\det(A - \lambda I) = 0 \implies \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \implies (a-\lambda)(d-\lambda) - bc = 0$$

$$\implies \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \implies \lambda^2 - \tau\lambda + \Delta = 0 \implies \lambda = \tfrac{1}{2}[\tau \pm \sqrt{\tau^2 - 4\Delta}]$$

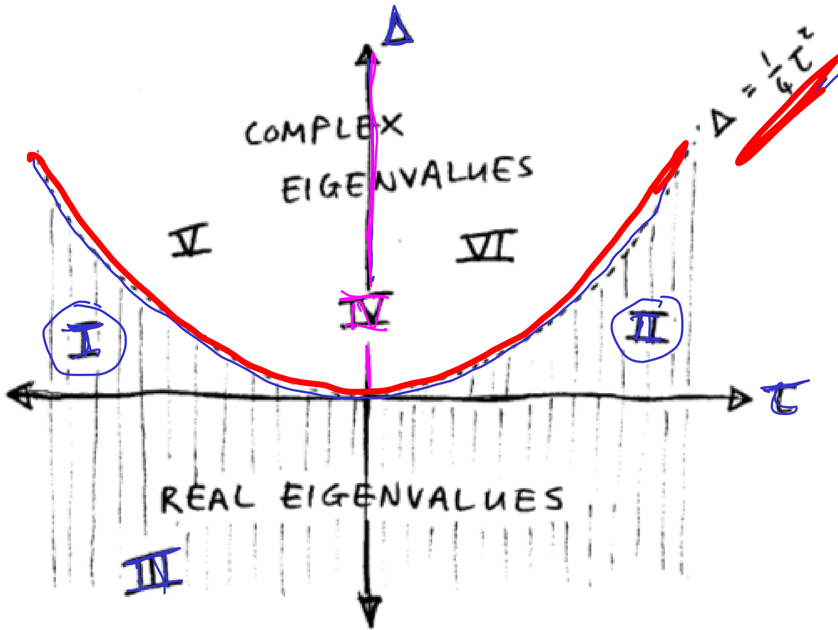
where $\Delta = \det(A) = ad - bc$ and we call $\tau = a + d$ the "trace" of A.*

Note that we have

- real eigenvalues when $\tau^2 - 4\Delta \geq 0$, i.e., $\Delta \leq \frac{1}{4}\tau^2$
- complex eigenvalues when $\tau^2 - 4\Delta < 0$, i.e., $\Delta > \frac{1}{4}\tau^2$

*More generally, the trace of a matrix is the sum of the diagonal entries.

10.2: Stability of systems
Classification of critical solutions



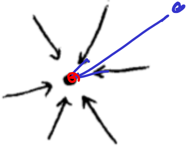
10.2: Stability of systems

$\Delta < \frac{1}{4}\tau^2 \implies$ real eigenvalues

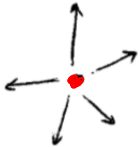
$$\lambda = \tfrac{1}{2}[\tau \pm \sqrt{\tau^2 - 4\Delta}]$$

$$\underline{x} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2.$$

I $\Delta > 0$ and $\tau < 0 \implies$ both eigenvalues are negative
$\therefore e^{\lambda_1 t} \to 0$ and $e^{\lambda_2 t} \to 0$ as $t \to \infty \implies$ **stable node**

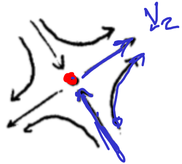


II $\Delta > 0$ and $\tau > 0 \implies$ both eigenvalues are positive
$\therefore e^{\lambda_1 t} \to \infty$ and $e^{\lambda_2 t} \to \infty$ as $t \to \infty \implies$ **unstable node**



III $\Delta < 0 \implies$ eigenvalues differ in sign
$\therefore$, e.g., $e^{\lambda_1 t} \to 0$ and $e^{\lambda_2 t} \to \infty$ as $t \to \infty \implies$ **saddle**

$\lambda_1 < 0, \lambda_2 > 0$



10.2: Stability of systems

$\Delta \geq \frac{1}{4}\tau^2 \implies$ complex eigenvalues

$$\lambda = \tfrac{1}{2}[\tau \pm \sqrt{\tau^2 - 4\Delta}]$$

$$\underline{x} = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2.$$

IV $\tau = 0 \implies \lambda = \pm\frac{1}{2}\sqrt{-4\Delta} = \pm i\sqrt{\Delta}$

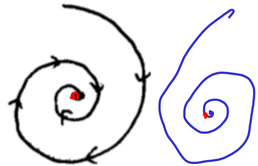
$$\therefore \left.\begin{aligned} x(t) &= c_1 \sin(\sqrt{\Delta}(t-\theta_1)) \\ y(t) &= c_2 \sin(\sqrt{\Delta}(t-\theta_2)) \end{aligned}\right\} \implies$$

centre

V $\tau < 0 \implies \lambda = \frac{\tau}{2} \pm \frac{1}{2} i\sqrt{4\Delta - \tau^2}$

$$\therefore e^{\lambda_\pm t} \to 0 \text{ and } \underbrace{e^{\tau t/2}}_{\to 0}\underbrace{e^{\pm\frac{1}{2}i\sqrt{4\Delta-\tau^2}t}}_{\textit{oscillation}} \implies$$

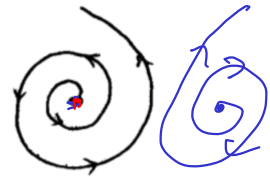
stable spiral



VI $\tau > 0 \implies \lambda = \frac{\tau}{2} \pm \frac{1}{2} i\sqrt{4\Delta - \tau^2}$

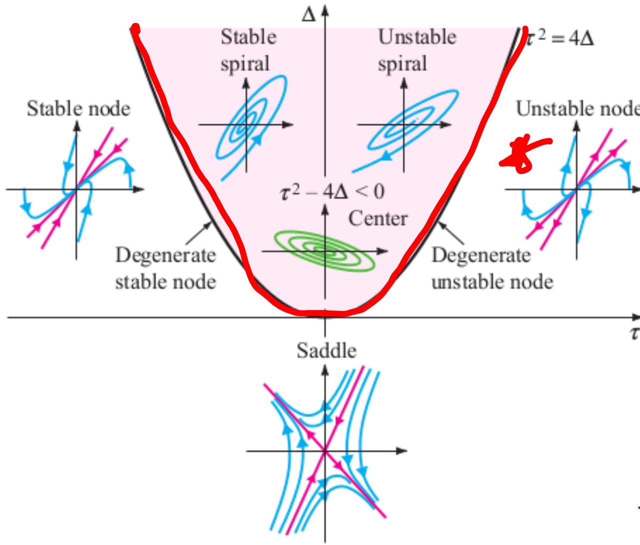
$$\therefore e^{\lambda_\pm t} \to \infty \text{ and } \underbrace{e^{\tau t/2}}_{\to \infty}\underbrace{e^{\pm\frac{1}{2}i\sqrt{4\Delta-\tau^2}t}}_{\textit{oscillation}} \implies$$

unstable spiral



10.2: Stability of systems

Perhaps this is better illustrated in the textbook (or perhaps not...)



$$\frac{dx}{dt} = Ax$$

$$\tau = \text{trace}(A)$$
$$\Delta = \det(A)$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 4x - 2y \\ \frac{dy}{dt} &= -x + 3y \end{aligned}}$$

$$\tau = 4+3 = 7$$
$$\Delta = 12 - 2 = 10$$
$$\tau^2/4 = \frac{49}{4} \approx 12$$

Note: You will be given this diagram in the test if it is needed, but you will be expected to interpret it and possibly explain why, for example, $\tau < 0$ corresponds to a saddle.

$$\Rightarrow \Delta < \tau^2/4$$