Applied differential equations

TW244 - Lecture 26

7.2: Laplace transforms of derivatives

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7.2: Laplace transforms of derivatives
First derivatives

How does the Laplace transform behave with derivatives?
Claim:*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

$\int uv' = [uv] - \int u'v$

Proof:

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\, dt \\ &= \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty s e^{-st} f(t)\, dt \\ &= \left[0 - f(0)\right] + s\mathcal{L}\{f(t)\} \\ &= s\mathcal{L}\{f(t)\} - f(0).\end{aligned}$$

This remarkable property (and the others which follow below) make Laplace transforms incredibly useful for solving DEs as they turn **differential** equations in to more familiar **algebraic** equations.

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*Under some fairly general assumptions on f that we won't concern ourselves with..

7.2: Laplace transforms of derivatives
Second (and higher) derivatives

$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0).$

Claim:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Proof:

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \quad \text{(from above)} \\
&= s\big[s\mathcal{L}\{f(t)\} - f(0)\big] - f'(0) \\
&= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).
\end{aligned}$$

In general[†]

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-1}f'(0) - \ldots - f^{(n-1)}(0).$$

Now how can we use these results to solve initial value problems...?

[†] Exercise: Prove this by induction.

Solve $y' = 2y + 4$ with $y(0) = 3$.

Taking the Laplace transform (LT) of both sides, we have

$3e^{2t}$

$$
\begin{aligned}
\mathcal{L}\{y'\} &= \mathcal{L}\{2y+4\} \\
\implies s\mathcal{L}\{y\} - y(0) &= 2\mathcal{L}\{y\} + 4\mathcal{L}\{1\} \\
\implies (s-2)\mathcal{L}\{y\} &= 3 + \tfrac{4}{s} \implies \mathcal{L}\{y\} = \tfrac{3}{s-2} + \tfrac{4}{s(s-2)} \\
\therefore y &= \mathcal{L}^{-1}\{\tfrac{3}{s-2} + \tfrac{4}{s(s-2)}\} = 3\mathcal{L}^{-1}\{\tfrac{1}{s-2}\} + \mathcal{L}^{-1}\{\tfrac{4}{s(s-2)}\} \\
&= \mathcal{L}^{-1}\{\tfrac{3}{s-2}\} - \mathcal{L}^{-1}\{\tfrac{2}{s}\} + \mathcal{L}^{-1}\{\tfrac{2}{s-2}\} \quad \text{(partial fractions)} \\
&= 5\mathcal{L}^{-1}\{\tfrac{1}{s-2}\} - 2\mathcal{L}^{-1}\{\tfrac{1}{s}\} \\
y(t) &= 5e^{2t} - 2
\end{aligned}
$$

$$\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$4 = A(s-2) + Bs = s(A+B) - 2A$$

$$\Rightarrow \begin{cases} B = -A \\ A = -2 \end{cases} \Rightarrow B = 2$$

$$\frac{4}{s(s-2)} = \frac{-2}{s} + \frac{2}{s-2}$$

Therefore $\boxed{y(t) = 5e^{2t} - 2.}$

Exercise: Use partial fraction expansion to show $\frac{4}{s(s-2)} = -\frac{2}{s} + \frac{2}{s-2}$.

7.2: Laplace transforms of derivatives
Example 2

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

Let $\mathcal{L}\{y\} := Y$

Solve $y'' + 9y = 80\cos(5t)$ with $y(0) = y'(0) = 0$.

Taking the Laplace transform (LT) of both sides, we have

$$\begin{aligned}
\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} &= 80\mathcal{L}\{\cos(5t)\} \\
\implies \underline{s^2\mathcal{L}\{y\} - s\cancel{y(0)} - \cancel{y'(0)}} + 9\underline{\mathcal{L}\{y\}} &= \underline{(s^2+9)\mathcal{L}\{y\}} = 80\left(\frac{s}{s^2+25}\right) \\
\therefore y &= \mathcal{L}^{-1}\left\{\frac{80s}{\underline{(s^2+25)(s^2+9)}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-5s}{s^2+25} + \frac{5s}{s^2+9}\right\} \quad \text{(partial fraction expansion)} \\
&= -5\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}
\end{aligned}$$

Therefore $\boxed{y(t) = -5\cos(5t) + 5\cos(3t).}$

Exercise: Use partial fraction expansion to show $\frac{80}{(s^2+25)(s^2+9)} = -\frac{5}{s^2+25} + \frac{5}{s^2+9}$.

$$\frac{80s}{(s^2+25)(s^2+9)} = \frac{A+Bs}{s^2+25} + \frac{C+Ds}{s^2+9}$$

$$\Rightarrow \quad 80s = (A+Bs)(s^2+9) + (C+Ds)(s^2+25)$$

$$= (\underbrace{9A+25C}_{=0}) + s(\underbrace{9B+25D}_{=80}) + s^2(\underbrace{A+C}_{=0}) + s^3(\underbrace{B+D}_{=0})$$

$$\Rightarrow A = C = 0.$$

$$\Rightarrow D = -B \Rightarrow \quad 9B + 25(-B) = 80 = (9-25)B = 80$$

$$\Rightarrow B = -80/16 = -5$$

$$\Rightarrow D = 5$$

$$\frac{80s}{(s^2+25)(s^2+9)} = \frac{-5s}{s^2+25} + \frac{5s}{s^2+9}$$

7.2: Laplace transforms of derivatives
The punchline

1. Take Laplace Transform (LT) of both sides of the DE.
2. Use results from p. 1–2 to write LT of derivatives in terms of LT of y.
3. Make $\mathcal{L}\{y\}$ the subject of the equation via algebraic manipulation.
4. Take the inverse LT of both sides to obtain $y(t)$.‡
5. ~~Make money.~~

Exercise: Use this technique solve the IVP $y'' - 3y' + 2y = e^{-4t}$ with $y(0) = 1$ and $y'(0) = 5$.
(Hint: Solution $y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$.)

‡ As we've seen, the ability to compute partial fraction expansions is important here, so practice!

7.6: Systems of linear DEs (revisited)

Recall: Method of elimination

Recall the "method of elimination" for solving systems of DEs:

Example: $\frac{dx}{dt} = -\frac{4}{25}x + \frac{1}{25}y, \frac{dy}{dt} = \frac{4}{25}x - \frac{4}{25}y, \quad x(0) = 4, y(0) = 0.$

$$\begin{aligned} (D+\tfrac{4}{25})x - \tfrac{1}{25}y = 0 \quad &\Longrightarrow \quad (D+\tfrac{4}{25})^2 x - \tfrac{1}{25}(D+\tfrac{4}{25})y = 0 && (1)\\ -\tfrac{4}{25}x + (D+\tfrac{4}{25})y = 0 \quad &\Longrightarrow \quad -\tfrac{4}{25}\tfrac{1}{25}x + \tfrac{1}{25}(D+\tfrac{4}{25})y = 0 && (2)\\ (1)+(2) \quad &\Longrightarrow \quad x'' + \tfrac{8}{25}x' + \tfrac{12}{625}x = 0 \end{aligned}$$

Try $x = e^{mt} \implies m^2 + \frac{8}{25}m + \frac{12}{625} = 0 \implies m = -\frac{2}{25}, -\frac{6}{25}$ then

$$x = c_1 e^{-\frac{2}{25}t} + c_2 e^{-\frac{6}{25}t} \quad \text{and} \quad y = 25\frac{dx}{dt} + 4x = 2c_1 e^{\frac{-2}{25}t} - 2c_2 e^{-\frac{6}{25}t}.$$

Initial conditions imply $c_1 = c_2 = 2$ (exercise) so

$$\boxed{\begin{aligned} x(t) &= 2e^{-\frac{2}{25}t} + 2e^{-\frac{6}{25}t}\\ y(t) &= 4e^{-\frac{2}{25}t} - 4e^{-\frac{6}{25}t}. \end{aligned}}$$

Laplace transform method for systems

The outline of the procedure is...

1. Take Laplace Transform (LT) of both equations.
2. Solve for $\overset{X}{\mathcal{L}\{x\}}$ and $\overset{Y}{\mathcal{L}\{y\}}$ simultaneously.
3. Get the inverse transform in one unknown (e.g., x)
4. Substitute back and solve for the other unknown (e.g., y)

Let's try this on the example from above!

7.6: Systems of linear DEs (revisited)

Laplace transform method for systems: example

Consider again the IVP (with $x(0) = 4$, $y(0) = 0$)

$$\frac{dx}{dt} = -\frac{4}{25}x + \frac{1}{25}y \qquad (1)$$

$$\frac{dy}{dt} = \frac{4}{25}x - \frac{4}{25}y \qquad (2)$$

(For convenience, we denote $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.)

Step 1:

LT of (1) $\implies \overbrace{sX - 4}^{\mathcal{L}\{x'\}} = -\frac{4}{25}X + \frac{1}{25}Y \implies (s + \frac{4}{25})X - \frac{1}{25}Y = 4 \qquad (3)$

LT of (2) $\implies \underbrace{sY - 0}_{\mathcal{L}\{y'\}} = \frac{4}{25}X - \frac{4}{25}Y \implies -\frac{4}{25}X + (s + \frac{4}{25})Y = 0 \qquad (4)$

Step 2:

$\frac{4}{25} \times (3) \implies +\frac{4}{25}(s + \frac{4}{25})X - \frac{4}{625}Y = \frac{16}{25} \qquad (5)$

$(s + \frac{4}{25}) \times (4) \implies -\frac{4}{25}(s + \frac{4}{25})X + (s + \frac{4}{25})^2 Y = 0 \qquad (6)$

$$\implies (s + \frac{4}{25})^2 Y - \frac{4}{625}Y = \frac{16}{25} \implies y = \mathcal{L}^{-1}\left\{\frac{\frac{16}{25}}{(s+\frac{2}{25})(s+\frac{6}{25})}\right\}$$

$\left((s+\frac{4}{25})^2 - \frac{4}{625}\right) Y = 16/25$

$$\underbrace{\left(\left(s+\tfrac{4}{25}\right)^2-\tfrac{4}{625}\right)}Y = 16/25$$

$$\hookrightarrow s^2 + \tfrac{8}{25}s + \tfrac{16}{625} - \tfrac{4}{625}$$

$$= s^2 + \tfrac{8}{25}s + \tfrac{12}{625} = (s+2/25)(s+6/25)$$

Partial fractions:

$$\mathcal{L}\{y\} = Y = \frac{16/25}{(s+2/25)(s+6/25)} = \frac{A}{s+2/25} + \frac{B}{s+6/25}$$

$$\Rightarrow 16/25 = A(s+6/25) + B(s+2/25) = s\underbrace{(A+B)}_{=0} + \underbrace{(6A+2B)}_{=16}/25$$

$$\Rightarrow \begin{cases} B=-A \\ 6A-2A=4A=16 \Rightarrow A=4,\ B=-4 \end{cases}$$

$$\Rightarrow Y = \frac{4}{s+2/25} - \frac{4}{s+4/25} = \mathcal{L}\{y\}$$

Step 3:

$$y = \mathcal{L}^{-1}\left\{\frac{\frac{16}{25}}{(s+\frac{2}{25})(s+\frac{6}{25})}\right\} \overbrace{=}^{\text{partial fractions}} 4\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{2}{25}}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{6}{25}}\right\}$$

$$\therefore y(t) = 4e^{-\frac{2}{25}t} - 4e^{-\frac{6}{25}t}.$$

Step 4: From the DE in (2) we have that

$$\begin{aligned} x &= \frac{25}{4}\frac{dy}{dt} + y \\ &= \frac{25}{4}\left[-\frac{8}{25}e^{-\frac{2}{25}t} + \frac{24}{25}e^{\frac{-6}{25}t}\right] + 4e^{-\frac{2}{25}t} - 4e^{-\frac{6}{25}t} \\ \therefore x(t) &= 2e^{\frac{-2}{25}t} + 2e^{-\frac{6}{25}t}. \end{aligned}$$

Which agrees with our answer from the method of elimination!

Exercise: Use this technique to solve $x' = -x + y,\ y' = 2x,\ x(0) = 0, y(0) = 1.$
(Hint: Solution $x(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t},\ y(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t}$.)

$y'' - 3y' + 2y = e^{-4t}$ with $y(0) = 1$ and $y'(0) = 5$. ②

① $\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = [s^2Y - sy(0) - y'(0)] - 3[sY - y(0)] + 2Y = \mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$

$(s^2 - 3s + 2)Y - s - 2 = (s-2)(s-1)Y - s - 2 = \frac{1}{s+4}$

③ $\mathcal{L}\{y\} = Y = \underbrace{\frac{1}{(s+4)(s-2)(s-1)} + \frac{s+2}{(s-2)(s-1)}} = *$

Partial fractions

$\frac{1}{(s+4)(s-2)(s-1)} = \frac{A}{s+4} + \frac{B}{s-2} + \frac{C}{s-1} \Rightarrow 1 = A(s-2)(s-1) + B(s+4)(s-1) + C(s+4)(s-2)$

$s = -4 \Rightarrow 1 = A(-6)(-5) \Rightarrow A = 1/30$

$s = 2 \Rightarrow 1 = B(6)(1) \Rightarrow B = 1/6$

$s = +1 \Rightarrow 1 = C(5)(-1) \Rightarrow C = -1/5$

$= \frac{1/30}{s+4} + \frac{1/6}{s-2} - \frac{1/5}{s-1}$

$\frac{s+2}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1} \Rightarrow s+2 = A(s-1) + B(s-2) = s\underbrace{(A+B)}_{=1} + \underbrace{(-A-2B)}_{=2}$

$\Rightarrow A+B = 1,\ -A-2B = 2 \Rightarrow -B = 3 \Rightarrow B = -3,\ A = 4$

$= \frac{4}{s-2} - \frac{3}{s-1}$

$y = \mathcal{L}^{-1}\{*\} = \mathcal{L}^{-1}\left\{\frac{1/30}{s+4}\right\} + \mathcal{L}^{-1}\left\{\frac{4+1/6}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{3+1/5}{s-1}\right\}$

$= \frac{1}{30}e^{-4t} + \frac{28}{6}e^{2t} - \frac{18}{5}e^{t}$

$X := \mathcal{L}\{x\}, \quad Y := \mathcal{L}\{y\}$

$x' = -x + y, \; y' = 2x, \; x(0) = 0, y(0) = 1$

① $\mathcal{L}\{x'\} = s\mathcal{L}\{x\} - x(0) = sX - 0 = -\mathcal{L}\{x\} + \mathcal{L}\{y\} = -X + Y$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY - 1 = 2\mathcal{L}\{x\} = 2X$

② $\Rightarrow \begin{cases} sX = -X + Y \\ sY - 1 = 2X \end{cases}$

$\Rightarrow Y = (s+1)X$

$\Rightarrow s[(s+1)X] - 1 = 2X \Rightarrow (s^2 + s - 2)X = (s+2)(s-1)X = 1$

$\Rightarrow X = \dfrac{1}{(s+2)(s-1)} = \mathcal{L}\{x\}$

Partial fractions

$\dfrac{1}{(s+2)(s-1)} = \dfrac{A}{(s+2)} + \dfrac{B}{(s-1)} \Rightarrow 1 = A(s-1) + B(s+2) = s\underbrace{(A+B)}_{=0} + \underbrace{(-A+2B)}_{=1}$

$\Rightarrow 3B = 1 \Rightarrow B = 1/3, \; A = -1/3.$

$= \dfrac{-1/3}{s+2} + \dfrac{1/3}{s-1}$

③ $x = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-1/3}{s+2} + \dfrac{1/3}{s-1}\right\} = -1/3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\} + 1/3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$

$= -1/3\,e^{-2t} + 1/3\,e^{t} = x$

④ $y = x' + x = \left[2/3\,e^{-2t} + 1/3\,e^{t}\right] + \left[-1/3\,e^{-2t} + 1/3\,e^{t}\right]$

$= 1/3\,e^{-2t} + 2/3\,e^{t}.$