Week 8
20.11 The M/G/s/GD/s/∞ System (Blocked Customers Cleared):

- When all servers are busy the customer will be lost te the system.
- e.g. if the firedepartment is not answering the call, then house wil burn down.
- interarrival times are exponential --> M/G/s/GD/s/∞
- Since a queue never occurs $L_q = W_q = 0$;

Since arrivals are turned away only when s customers(the same amount of customers as servers) are present, a fraction π_s of all arrivals will be turned away.

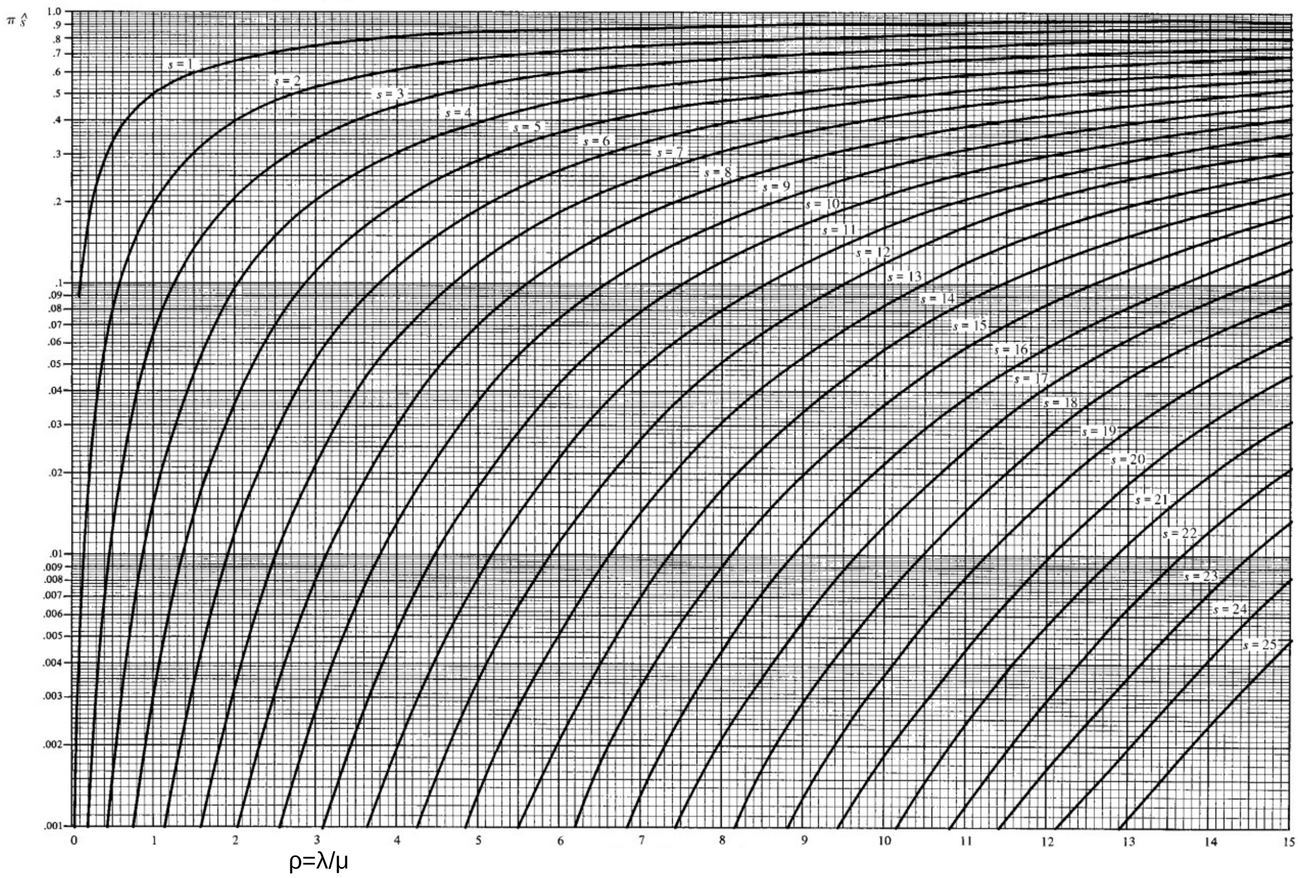
Average of Arrivals per unit time lost = $\lambda * \pi_s$
Average of Arrivals per unit time entering system = $\lambda * (1 - \pi_s)$

$$W = W_s = \frac{1}{\mu}.$$

$$L = L_s = \frac{\lambda(1 - \pi_s)}{\mu}$$

We will use the following graph to caculate π_s (see following example)

FIGURE 32
Loss Probabilities for *M/G/s/GD/s/∞* Queuing System

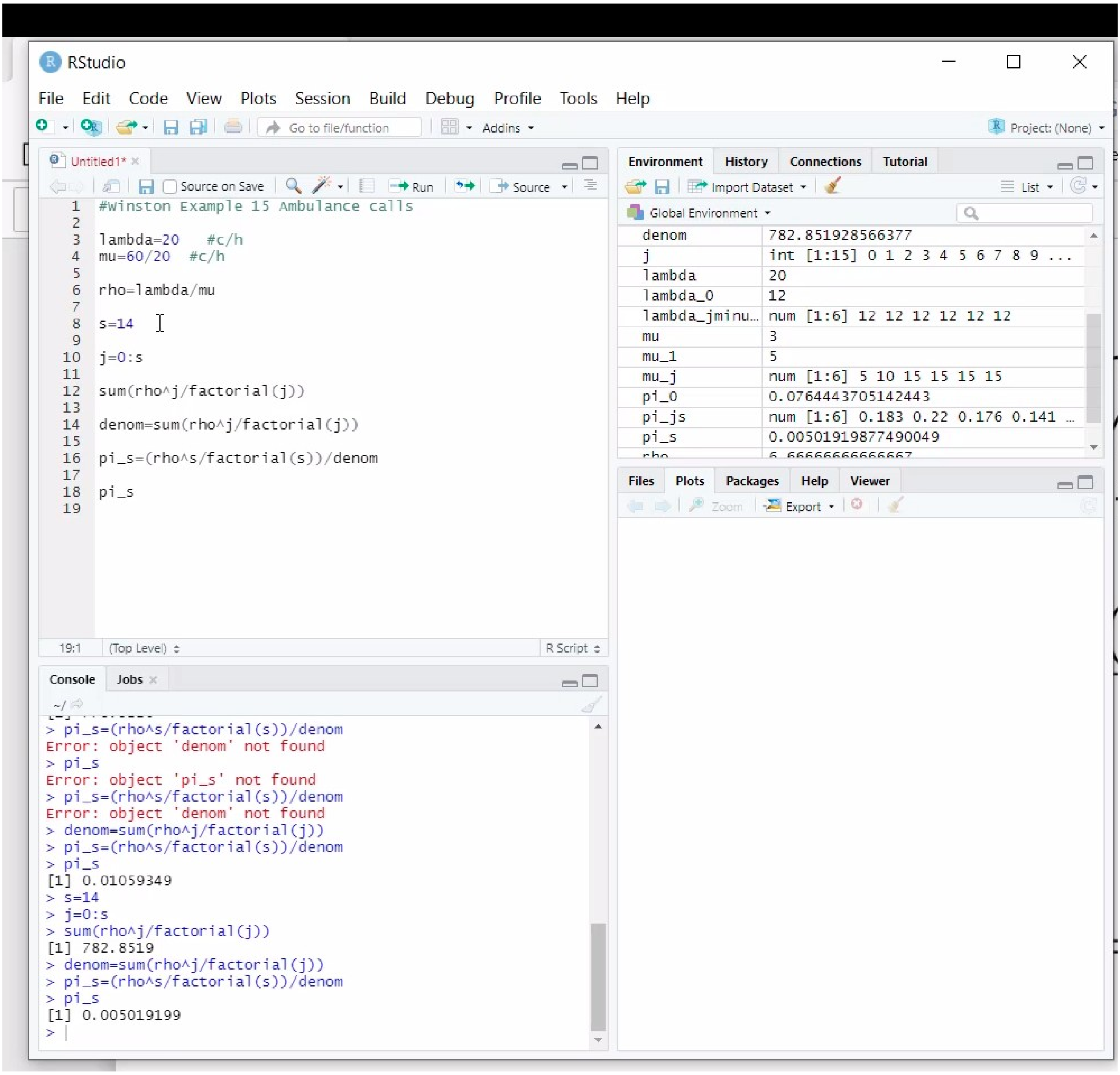


EXAMPLE 15 Ambulance Calls

An average of 20 ambulance calls per hour are received by Gotham City Hospital. An ambulance requires an average of 20 minutes to pick up a patient and take the patient to the hospital. The ambulance is then available to pick up another patient. How many ambulances should the hospital have to ensure that there is at most a 1% probability of not being able to respond immediately to an ambulance call? Assume that interarrival times are exponentially distributed.

Solution We are given that $\lambda = 20$ calls per hour, and $\frac{1}{\mu} = \frac{1}{3}$ hour. Thus, $\rho = \frac{\lambda}{\mu} = \frac{20}{3} = 6.67$. For $\rho = 6.67$, we seek the smallest value of s for which π_s is .01 or smaller. From Figure 32, we see that for $s = 13$, $\pi_s = .011$; and for $s = 14$, $\pi_s = .005$. Thus, the hospital needs 14 ambulances to meet its desired service standards.

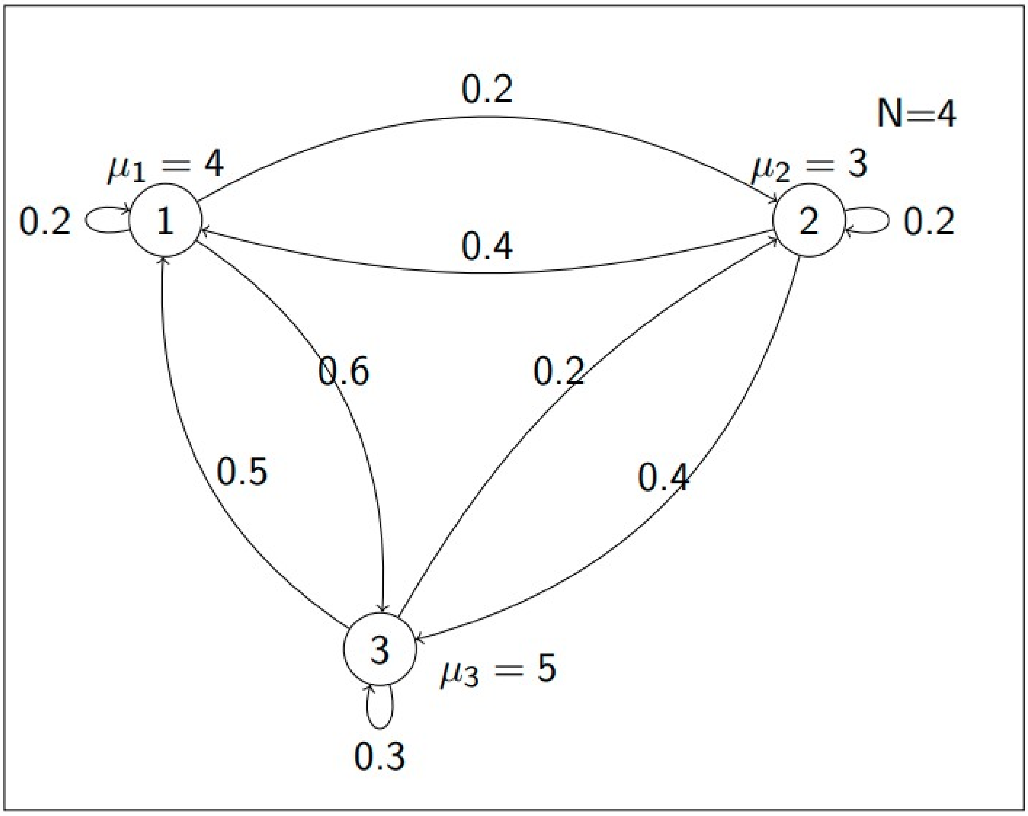
LINGO :



20.13 Closed Queuing Networks:

- Systems, where there are a constant number of jobs present.
- A network where workstations are linked, where a fixed number of customers are distributed over the network and the customers go from one workstation to the next with certain probabilities.

DezinerZ example :



-Nothing leavers or enters the system

- The probability of job leaving station i to join j is p_{ij} and since the system has nothing entering or leaving , then :

$$\sum_{j=1}^{S} p_{ij} = 1$$

Where S = number of servers and i is element of {1,...,S}

- the arrival rate at workstation j is :

$$\lambda_j = \sum_{i=1}^{S} p_{ij}\lambda_i.$$

- Service times at workstations are exponential distributed with parameter μ_s, where s is element of {1,...,S}.

- For each workstation define $$\rho_s = \frac{\lambda_s}{\mu_s}.$$

- The system has s servers,and at all times, exactly N jobs are present.
- We let n_i be the number of jobs present at server i.
- State of the system can be defined by : vector **n** = $(n_1, n_2, ..., n_s)$
- The set of possible states is given by $S_N = n_1+n_2+...+n_s= N$

steady-state probability :

$$\Pi_N(\mathbf{n}) = \frac{\rho_1^{n_1}\, \rho_2^{n_2} \cdots \rho_n^{n_s}}{G(N)}$$

Here, $G(N) = \sum_{\mathbf{n} \in S_N} \rho_1^{n_1}\, \rho_2^{n_2} \cdots \rho_s^{n_s}$.

EXAMPLE 17 Flexible Manufacturing System

Consider a flexible manufacturing system in which 10 parts are always in process. Each part requires two operations. Each part begins by having operation 1 done at machine 1. Then, with probability .75 the part has operation 2 processed on machine 2, and with probability .25 the part has operation 2 processed on machine 3. Once a part completes operation 2, the part leaves the system and is immediately replaced by another part. We are given the following machine rates (the time for each operation is exponentially distributed): $\mu_1 = .25$ minute, $\mu_2 = .48$ minute, and $\mu_3 = .08$ minute.

a Find the probability distribution of the number of parts at each machine.

b Find the expected number of parts present at each machine.

c What fraction of the time is each machine busy?

d How many parts per minute are completed by each machine?

Buzen.xls **Solution** Our work is in file Buzen.xls. To begin, we need to compute one solution to the equations (59) defining λ_1, λ_2, and λ_3. We must solve

$$\lambda_1 = \lambda_2 + \lambda_3$$
$$\lambda_2 = .75\lambda_1$$
$$\lambda_3 = .25\lambda_1$$

There are an infinite number of solutions to this system. Arbitrarily choosing $\lambda_1 = 1$ yields the solution $\lambda_2 = .75$ and $\lambda_3 = .25$. In cells G8:I8, we compute $\rho_i = \frac{\lambda_i}{\mu_i}$. In G10:G20, we compute $C_1(k) = \rho_1^k, k = 0, 1, \ldots, 10$, and in G10:I10, we enter $C_i(0) = 1, i = 1, 2, 3$. Copying from H11 to H11:I20 the formula

=G11+H$8*H10

implements the recursion $C_i(k) = C_{i-1}(k) + \rho_i C_i(k - 1)$. Then we can find G(10) = 7,231,883 from the value of $C_3(10)$ in cell H20. See Figure 34.

We can now generate all possible system states efficiently by starting with $n_1 = 0$ and listing those states in order of increasing values of n_2. Then we increase n_1 to 1 and list all states in increasing values of n_2, etc. Once we have $n_1 = 10$, we will have listed all states. (See Figure 35.) To efficiently generate all possible states, we copy down from C25 the formula

=IF(D25=0,B25+1,B25)

This formula increments n_1 by 1 if $n_3 = 0$ (which is the same as having $n_2 = 10 - n_1$). Otherwise, the formula keeps n_1 constant.

Then we copy down from D25 the formula

=IF(B25-B24=1,0,C24+1)

This formula makes $n_2 = 0$ if we have just increased the value of n_1; otherwise, the formula increments the value of n_2 by 1.

Finally, from E25, we copy down the formula

=10-B24-C24

This ensures that $n_3 = 10 - n_1 - n_2$.

In E24:E89 we use (60) to compute the steady-state probability for each state by copying from E24 to E25:E89 the formula

=($G$8^B24)*($H$8^C24)*($I$8^D24)/$I$20

Part (a) Next, we answer part (a) by determining the probability distribution of the number of parts at each machine. We use the SUMIF function and a one-way data table to accomplish this goal. To begin, compute in H24 the probability of 0 parts at machine 1 with the formula

=SUMIF($B$24:$B$89,I23,E24:E89)

This formula adds up every number in column D (which contains state probabilities) for the rows in which column B (which is parts at machine 1) has a 0 entry. See Figure 36.

	F	G	H	I
7	Mui	0.25	0.48	0.08
8	phoi	4	1.5625	3.125
9		1	2	3
10	0	1	1	1
11	1	4	5.5625	8.6875
12	2	16	24.6914063	51.83984
13	3	64	102.580322	264.5798
14	4	256	416.281754	1243.094
15	5	1024	1674.44024	5559.108
16	6	4096	6712.31287	24084.53
17	7	16384	26871.9889	102136.1
18	8	65536	107523.483	426698.9
19	9	262144	430149.442	1763583
20	10	1048576	1720684.5	7231883

FIGURE 34

	B	C	D	E
23	Parts at 1	Parts at 2	Parts at 3	Probability
24	0	0	10	0.01228143
25	0	1	9	0.00614071
26	0	2	8	0.00307036
27	0	3	7	0.00153518
28	0	4	6	0.00076759
29	0	5	5	0.00038379
30	0	6	4	0.0001919
31	0	7	3	9.5949E-05
32	0	8	2	4.7974E-05
33	0	9	1	2.3987E-05
34	0	10	0	1.1994E-05
35	1	0	9	0.01572023
36	1	1	8	0.00786011

32	0	8	2	4.7974E-05
33	0	9	1	2.3987E-05
34	0	10	0	1.1994E-05
35	1	0	9	0.01572023
36	1	1	8	0.00786011
37	1	2	7	0.00393006
38	1	3	6	0.00196503
39	1	4	5	0.00098251
40	1	5	4	0.00049126
41	1	6	3	0.00024563
42	1	7	2	0.00012281
43	1	8	1	6.1407E-05
44	1	9	0	3.0704E-05
45	2	0	8	0.02012189
46	2	1	7	0.01006094
47	2	2	6	0.00503047
48	2	3	5	0.00251524
49	2	4	4	0.00125762
50	2	5	3	0.00062881
51	2	6	2	0.0003144
52	2	7	1	0.0001572
53	2	8	0	7.8601E-05
54	3	0	7	0.02575602
55	3	1	6	0.01287801
56	3	2	5	0.006439
57	3	3	4	0.0032195
58	3	4	3	0.00160975
59	3	5	2	0.00080488
60	3	6	1	0.00040244
61	3	7	0	0.00020122
62	4	0	6	0.0329677
63	4	1	5	0.01648385
64	4	2	4	0.00824193
65	4	3	3	0.00412096
66	4	4	2	0.00206048
67	4	5	1	0.00103024
68	4	6	0	0.00051512
69	5	0	5	0.04219866
70	5	1	4	0.02109933
71	5	2	3	0.01054967
72	5	3	2	0.00527483
73	5	4	1	0.00263742
74	5	5	0	0.00131871
75	6	0	4	0.05401429
76	6	1	3	0.02700714
77	6	2	2	0.01350357
78	6	3	1	0.00675179
79	6	4	0	0.00337589
80	7	0	3	0.06913829
81	7	1	2	0.03456914
82	7	2	1	0.01728457
83	7	3	0	0.00864229
84	8	0	2	0.08849701
85	8	1	1	0.0442485
86	8	2	0	0.02212425
87	9	0	1	0.11327617
88	9	1	0	0.05663809
89	10	0	0	0.1449935

FIGURE 35

Selecting the table range G24:H35 and column input cell I23 enables us to loop through and compute the steady-state probabilities for each number of parts at machine 1. In a similar fashion, we obtain the following steady-state probability distributions for machines 2 and 3. See Figure 37.

Part (b) The mean number of parts present at machine 1 may be computed as $\sum_{i=0}^{i=10} i*$ (Probability of i parts at machine 1). In cell K31, we compute the mean number of parts at machine 1 with the formula

=SUMPRODUCT(G25:G35,H25:H35)

In a similar fashion, we compute the mean number of parts at machines 2 and 3 in cells K32 and K33. See Figure 38. Note that machine 1 is clearly the bottleneck.

Part (c) To compute the probability that each machine is busy, we just subtract from 1 the probability that each machine has 0 parts. These computations are done in L31:L33. We find that machine 1 is busy 97% of the time, machine 2 38% of the time, and machine 3 76% of the time.

	G	H	I
21			
22			Parts
23		Prob	0
24	Machine 1 parts	0.02455086	
25	0	0.02455086	
26	1	0.03140975	
27	2	0.04016518	
28	3	0.05131082	
29	4	0.06542029	
30	5	0.08307862	
31	6	0.10465268	
32	7	0.12963429	
33	8	0.15486976	
34	9	0.16991426	
35	10	0.1449935	

FIGURE 36

	G	H
37	Machine 2 Parts	0.61896518
38	0	0.61896518
39	1	0.23698584
40	2	0.09017388
41	3	0.03402481
42	4	0.01269126
43	5	0.00465769
44	6	0.00166949
45	7	0.00057718
46	8	0.00018798
47	9	5.4691E-05
48	10	1.1994E-05

	G	H
50	Machine 3 Parts	0.23793036
51	0	0.23793036
52	1	0.18587372
53	2	0.14519511
54	3	0.1133962
55	4	0.08851582
56	5	0.06900306
57	6	0.0536088
58	7	0.0412822
59	8	0.03105236
60	9	0.02186094
61	10	0.01228143

FIGURE 37

	J	K	L	M
29				
30	Mean Number	Mean Number	Prob busy	Completions per second
31	Machine 1	6.696224299	0.97544914	0.243862285
32	Machine 2	0.609634749	0.38103482	0.182896714
33	Machine 3	2.694140952	0.76206964	0.060965571

FIGURE 38

Part (d) To compute the mean number of service completions per minute by each machine, we simply multiply the probability that a machine is busy by the machine's service rate. These computations are done in M31:M33. We find that machine 1 on average completes .24 part/minute, machine 2 .18 part/minute, and machine 3 .06 part/minute.