Applied differential equations

TW244 - Lecture 17

4.6: Variation of parameters
4.8: Green´s functions

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THE METHOD OF VARIATION OF PARAMETERS

$ay'' + by' + cy = g(x)$

The undetermined coefficients method (Lecture 16) has some drawbacks:

- We must have a good guess of what y_p should be.
- Sometimes there are no good guesses.
- It applies only to linear constant coefficient problems.

In this lecture we briefly explore the method of variation of parameters.

Soon we will consider problems of the form

$$y'' + p(x)y' + q(x)y = f(x),$$

but for now recall the integrating factor method for DEs of the form

$$y' + p(x)y = f(x).$$

4.6: Variation of parameters $F(x) = e^{\int p(x)\,dx}$

Linear 1st-order DEs revisited

Recall the integrating factor method for 1st-order DEs in standard form:*

$$y' + p(x)y = f(x) \implies y(x) = \underbrace{Ce^{-\int p(x)\,dx}}_{y_C} + \underbrace{e^{-\int p(x)\,dx}\int e^{\int p(x)\,dx} f(x)\,dx}_{y_p}.$$

$$y = \frac{1}{F}\left[\int F f\,dx + C\right]$$

Although we didn't look at it in this way at the time, the integrating factor (more precisely its reciprocal) provides a solution to the homogeneous problem, since

$$y_C' + p(x)y_C = 0.^\dagger$$

The idea of the method of variation of parameters is to replace the constant C in y_C by some unknown function $u(x)$ and let

$$y_p = u(x)\underbrace{e^{-\int p(x)\,dx}}_{y_c}.$$

$y_c = e^{\int -p}$
$y_c' = \frac{d}{dx}\int -p \; e^{\int -p}$
$= -p(x)\, e^{-\int p(x)\,dx}$

*Note we are being a little lazy with our integration variables..

†Exercise: Verify this. Notice that $y_C = 1/(\text{integrating factor})$.

4.6: Variation of parameters

Linear 1st-order DEs revisited (cont.)

Recall: is y_1 is soln to the homogeneous problem $y'+py=f$

For convenience denote $y_1 = e^{-\int p(x)\,dx}$ so that $y_p = u(x)y_1$ and notice

$$\begin{aligned} y_p' + p(x)y_p &= u'(x)y_1 + u(x)y_1' + p(x)u(x)y_1 \\ &= u(x)\underbrace{\left[y_1' + p(x)y_1\right]}_{=0} + u'(x)y_1 \\ &= u'(x)y_1 = f(x). \end{aligned}$$

Therefore we have that

$$\boxed{u(x) = \int \frac{f(x)}{y_1(x)}\,dx,}$$

and the particular solution y_p is given by

$$\boxed{y_p = u(x)y_1 = y_1 \int \frac{f(x)}{y_1(x)}\,dx,}$$

which is the same as we obtained from the integrating factor method.

We can use the same idea to solve inhomogeneous linear 2nd-order DEs!

Now consider the linear 2nd-order equation in standard form, i.e.,

$$y'' + p(x)y' + q(x)y = f(x).$$

Suppose that we have already obtained the complementary solution $y_c = c_1y_1(x) + c_2y_2(x)$, for example using the method of Lecture 15 if p and q are constant.

Our approach will again be to replace the constants c_1 and c_2 by functions $u_1(x)$ and $u_2(x)$ and seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

What follows on the next slide is mostly just algebra. Don't panic! 🙈

$y'' + p(x)y' + q(x)y = f(x)$,

Dropping the dependence on x for brevity, we have that

$$y_p = u_1 y_1 + u_2 y_2 \Longrightarrow \begin{cases} y_p' &= u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2 \\ y_p'' &= u_1 y_1'' + y_1' u_1' + y_1 u_1'' + u_1' y_1' + \cdots \\ & \quad u_2 y_2'' + y_2' u_2' + y_2 u_2'' + u_2' y_2' \end{cases}$$

Substituting to the original DE (exercise) we have that

$$\begin{aligned} y_p'' + p y_p' + q y &= u_1 \overbrace{[y_1'' + p y_1' + q y_1]}^{=0} + u_2 \overbrace{[y_2'' + p y_2' + q y_2]}^{=0} + y_1 u_1'' + \cdots \\ & \quad u_1' y_1' + y_2 u_2'' + u_2' y_2' + p[y_1 u_1' + y_2 u_2'] + y_1' u_1' + y_2' + u_2' \\ &= \frac{d}{dx}[y_1 u_1' + y_2 u_2'] + p[y_1 u_1' + y_2 u_2'] + [y_1' u_1' + y_2' u_2'] = f(x) \end{aligned}$$

Notice that if we could find u_1 and u_2 satisfying

$$y_1 u_1' + y_2 u_2' = 0 \quad \underline{\text{and}} \quad y_1' u_1' + y_2' u_2' = f(x)$$

then the above would be satisfied.

Claim: The functions $u_1' = -\dfrac{y_2 f(x)}{y_1 y_2' - y_2 y_1'}$ and $u_2' = \dfrac{y_1 f(x)}{y_1 y_2' - y_2 y_1'}$ satisfy[‡]

$$y_1 u_1' + y_2 u_2' = 0 \quad \underline{\text{and}} \quad y_1' u_1' + y_2' u_2' = f(x)$$

Proof: Exercise. (It is easy to verify.)

Summary: To solve $y'' + py' + qy = f$,

1. First find the complementary function $y_c = c_1 y_1 + c_2 y_2$ $y_1 = e^{mx}$, etc
2. Compute the Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
3. Find u_1 and u_2 by integrating $u_1' = -y_2 f/W$ and $u_2' = y_1 f/W$[§]
4. A particular solution is $y_p = u_1 y_1 + u_2 y_2$
5. The general solution is then $y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p$.

[‡]Notice that the numerator is the Wronskian W!

[§]The constant of integration is unnecessary. Exercise: Explain why?

Example 1

$y = e^{mx}$, $m^2 + 6m + 5 = 0$
$\Rightarrow (m+1)(m+5) = 0$

Exercise: Find the general solution of $y'' + 6y' + 5y = e^{2x}$.

1. From Lecture 16, we know that $y_c = c_1 e^{-x} + c_2 e^{-5x}$.
2. $W = \begin{vmatrix} e^{-x} & e^{-5x} \\ -e^{-x} & -5e^{-5x} \end{vmatrix} = -5e^{-6x} + e^{-6x} = -4e^{-6x}$
3. $u_1 = \int -y_2(x)f(x)/W(x)\,dx = \int \frac{-e^{-5x}e^{2x}}{-4e^{-6x}}\,dx = \frac{1}{4}\int e^{3x}\,dx = \frac{1}{12}e^{3x}$

   $u_2 = \int y_1(x)f(x)/W(x)\,dx = \int \frac{e^{-x}e^{2x}}{-4e^{-6x}}\,dx = \frac{1}{4}\int e^{7x}\,dx = \frac{1}{28}e^{7x}$
4. A particular solution is $y_p = u_1y_1 + u_2y_2 = \frac{1}{12}e^{3x}e^{-x} - \frac{1}{28}e^{7x}e^{-5x} = \frac{1}{21}e^{2x}$
5. The general solution is then $y = y_c + y_p = c_1e^{-x} + c_2e^{-5x} + \frac{1}{21}e^{2x}$

which is consistent with what we saw in Lecture 16.

Let's try a more complicated example...

4.6: Variation of parameters
Example 2

Exercise: Find the general solution of $y'' + y = \csc x$. $= f(x) = \frac{1}{\sin(x)}$

1 From Lecture 15, we know that $y_c = c_1 \cos x + c_2 \sin x$

2 $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

3 $u_1 = \int -y_2(x) f(x)\, dx = \int \frac{-\sin x}{\sin x}\, dx = -x$

$u_2 = \int y_1(x) f(x)\, dx = \int \frac{\cos x}{\sin x}\, dx = \log|\sin x|$

4 A particular solution is $y_p = u_1 y_1 + u_2 y_2 = -x \cos x + \sin x \log|\sin x|$

5 The general solution is then $y = (c_1 - x)\cos x + (c_2 + \log|\sin x|)\sin x$

We could not have solved this problem using the method of undetermined coefficients!

4.8: GREEN'S FUNCTIONS

4.8: Green's functions

Definition

Consider again the differential equation

$$y''(x) + p(x)y'(x) + q(x)y = f(x), \quad y(x_0) = y'(x_0) = 0,$$

with fundamental solutions $y_1(x)$ and $y_2(x)$. From above we have that

$$y_p(x) = y_1(x) \underbrace{\int_{x_0}^{x} \frac{-y_2(t)f(t)}{W(t)}\,dt}_{u_1} + y_2(x) \underbrace{\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt}_{u_2}$$

which we may rewrite as

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)\,dt$$

where

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

is known as the Green's function for the DE above.

Notice importantly that the Green's function depends only on the fundamental solutions $y_1(x)$ and $y_2(x)$ and not the forcing function $f(x)$.

4.8: Green's functions
Example 1

Example 1: (a) Compute the Green's function for the DE

$$y''(x) - y = 1/x, \quad y(0) = y'(0) = 0$$

check.

and (b) use it to determine the particular solution $y_p(x)$.

(a) We have $y_1(x) = e^x$ and $y_2 = e^{-x}$ with $W(x) = -2$, hence

$$G(x,t) = \frac{e^t e^{-x} - e^x e^{-t}}{-2} = \frac{e^{x-t} - e^{-(x-t)}}{2} = \sinh(x-t).$$

(b) From the result of the previous slide, we have that

$$y_p(x) = \int_0^x \frac{\sinh(x-t)}{t}\,dt,$$

In MATLAB

sinint

and therefore that

$$y = c_1 e^x + c_2 e^{-x} + \int_0^x \frac{\sinh(x-t)}{t}\,dt.$$

Example 2: (a) Compute the Green's function for the DE

$$y'' + y = \csc x$$

and (b) use it to determine the particular solution $y_p(x)$.

(a) We have $y_1(x) = \cos(x)$ and $y_2 = \sin(x)$ with $W(x) = 1$, hence

$$G(x,t) = \sin(x)\cos(t) - \cos(x)\sin(t) = \sin(x-t).$$

(b) From the result of the previous slide, we have that

$$y_p(x) = \int_0^x \csc(t)\sin(x-t)\,dt = \sin(x)\int_0^x \frac{\cos(t)}{\sin(t)}\,dt - \cos(x)\int_0^x \frac{\sin(t)}{\sin(t)}\,dt,$$

and therefore that

$$y = c_1\sin(x) + c_2\cos(x) + \sin(x)\log|\sin(x)| - x\cos(x),$$

which matches what we found on Slide 09.