Applied differential equations

TW244 - Lecture 04

1st-order DEs: Separable equations and the Integrating Factor method

Prof Nick Hale - 2020

1918 · 2018

SCIENCE
NATUURWETENSKAPPE
EYOBUNZULULWAZI

Separable equations

2.2 Separable equations
Solution by integration

Consider the (trivial) DE $\frac{dy}{dx} = g(x)$.

We then have that

$$\int dy = y = \int g(x)\,dx = G(x) + C$$

where $G(x)$ is the anti-derivative of $g(x)$, i.e., $\frac{d}{dx}G(x) = g(x)$.

Example:

$$\begin{aligned} \frac{dy}{dx} &= \sin(5x) \\ y &= \int \sin(5x)\,dx = -\tfrac{1}{5}\cos(5x) + C \end{aligned}$$

We can use this approach to help us solve separable DEs.

2.2 Separable equations
Definition & solution by integration

A first-order DE of the form

$$\boxed{\frac{dy}{dx} = g(x)h(y)}$$

$\frac{dy}{dx} = x \cdot y$ ✓

$\frac{dy}{dx} = x + y$ ✗

is called "separable" (or has "separable variables").

To solve, we can proceed in a similar way to on the previous slide:

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ \int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \implies P(y) = G(x) + c, \end{aligned}$$

where $\frac{d}{dy}P(y) = \frac{1}{h(y)}$ and $\frac{d}{dx}G(x) = g(x)$.

$$P(y) = G(x) + C$$

$$\Rightarrow \frac{d}{dx}P(y) = \frac{d}{dx}G(x) + \cancel{\frac{d}{dx}C}$$

$$\Rightarrow \frac{dy}{dx}\frac{d}{dy}P(y) = \frac{d}{dx}G(x)$$

$$\Rightarrow \frac{dy}{dx}\frac{1}{h(y)} = g(x)$$

Solve $\frac{dy}{dx} = x\sqrt{1-y^2}$ by separation of variables:

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int x\,dx$$

Exercise
$y(1) = 0$

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int x\,dx^{*} + C$$

Don't forget the constant.

$$\arcsin(y) = \tfrac{1}{2}x^2 + C$$

$$y = \sin(\tfrac{1}{2}x^2 + C)^{\dagger}$$

$\neq \sin(\tfrac{1}{2}x^2) + C.$ ✗

Verify:

$$\frac{dy}{dx} = x\cos(\tfrac{1}{2}x^2 + C)$$

$$x\sqrt{1-y^2} = x\sqrt{1-\sin^2(\tfrac{1}{2}x^2 + C)} = x\cos(\tfrac{1}{2}x^2 + C)$$ ✓

---

*Exercise: Compute this step. (Hint: Let $y = \sin(u)$.)

†Note where the constant goes. This is not the same as $\sin(\frac{1}{2}x^2) + C$.

Beware: Losing a solution

$dy/dx = g(x)h(y)$

$y(t) = r$

Singular solutions:

If r is a zero of $h(y)$, then $\frac{dr}{dx} = g(x)h(r) = 0$, so $y = r$ is a solution to the DE.

But, in separating variables we divide by $h(y)$, so we're excluding $y = r$.‡

The solution $y = r$ may therefore not be a member of the family of the solutions we obtain (i.e., it is a singular solution).

For example, $y = 1$ satisfies the DE on the previous slide, but it is not a member of the family $y = \sin(\frac{1}{2}x^2 + C)$.

or $y = -1$

$dy/dx = x\sqrt{1-y^2}$

$y(t) = 1$

$y = \pm 1$

‡Otherwise we'd be dividing by zero!

Linear equations

and the integrating factor method

A first-order linear DE has the form

$$a_1(x)\frac{dy}{dx} + a_2(x)y = a_3(x).$$

If $a_1(x) \neq 0$, we may write this in standard form

$$\frac{dy}{dx} + p(x)y = q(x).$$

(Note that our notation has deviated slightly from the textbook.)

$a_1(x)y' + a_2(x)y = a_3(x)$ ←

Suppose our coefficient functions, $a_i(x)$, are such that our DE became

$$f(x)\frac{dy}{dx} + f'(x)y = f(x)q(x).$$

Observing that the left-hand side of this equation may be written as

$$\frac{d}{dx}\left(f(x)y(x)\right) = f(x)\frac{dy}{dx} + f'(x)y \quad = f(x)q(x)$$

then we may use solution by integration to obtain

$$y = \frac{1}{f(x)}\left[\int f(x)q(x)\,dx. + C\right]$$

$f(x)y(x) = \int f(x)q(x)\,dx$

$\Rightarrow y(x) \; \frac{1}{f(x)}\int f(x)q(x)\,dx$

$f(x)y'(x) + f'(x)y(x) = f(x)q(x)$ -

So, if we can find a function f for which $\frac{df}{dx} = f(x)p(x)$, then for any DE

$$\frac{dy}{dx} + p(x)y = q(x) \implies f(x)\frac{dy}{dx} + \underbrace{f(x)p(x)}_{f'(x)}y = f(x)q(x).$$

Solution: Separate variables

solve $df/dx = f\, p(x)$

$$\int \frac{1}{f}\,df = \int p(x)\,dx \implies \ln|f| = \int p(x)\,dx + C \implies f = [\cancel{\pm e^C}]e^{\int p(x)\,dx}$$

Since any such function will do, we choose

$$\boxed{f(x) = e^{\int p(x)\,dx}}$$

and call this the "integrating factor".

2.3 Linear equations
Integrating factor method (cont.)

$a_1(x)\, y'(x) + a_2(x)\, y(x) = a_3(x)$

To solve a first-order linear DE with an integrating factor:

1. Write the DE in standard form, $\frac{dy}{dx} + p(x)y = q(x)$
2. Multiply both sides of the equation by the function $f(x) = e^{\int p(x)\,dx}$
3. The LHS reduces to $\frac{d}{dx}(f(x)y)$
4. Solve for y by integrating RHS and then dividing by f, i.e.,

$$y = e^{-\int p(x)\,dx}\left[\int q(x) e^{\int p(x)\,dx}\, dx + C\right]$$

Remarks

1. Do not simply remember this final result! Learn the procedure.
2. We (and the textbook) are being very sloppy with our integration variables. More properly we should write, for example, $f(x) = e^{\int^x p(\xi)\,d\xi}$.

Integrating factor method: example

Solve the first-order linear DE $\frac{dy}{dx} + 3x^2 y = x^2$ with an integrating factor.

Integrating factor: $f(x) = e^{\int p(x)\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$.

Hence

$$\begin{aligned} & e^{x^3}\frac{dy}{dx} + 3e^{x^3}x^2 y = \frac{d}{dx}\left(e^{x^3}y\right) = e^{x^3}x^2 \\ \Longrightarrow \quad & e^{x^3}y = \int e^{x^3}x^2\,dx + C = \tfrac{1}{3}e^{x^3} + C \\ \Longrightarrow \quad & y = \tfrac{1}{3} + Ce^{-x^3} \end{aligned}$$

$y(0) = 1/2 \Rightarrow 1/2 = 1/3 + Ce^0$

$C = 1/2 - 1/3 =$

Exercise: Solve the initial value problem $\frac{dy}{dx} + 3x^2 y = x^2, \quad y(0) = 0$.
Exercise: Solve the DE above using separation of variables.
Exercise: Will separation of variables work on the DE $\frac{dy}{dt} + 3x^2 y = x^3$?
Exercise: Use the integrating factor method to solve the DE $\frac{dy}{dx} - 3y = 6$.

Exercise 1 Solve $y' + 3x^2 y = x^2$ using separation of variables.

$y = 1/3$ is singular soln?

$$\Rightarrow y' = x^2(1-3y)$$

$$\Rightarrow \int \frac{dy}{1-3y} = \int x^2\,dx$$

$$\Rightarrow -\frac{1}{3}\int \frac{dy}{y-1/3} = -\frac{1}{3}\ln|y-1/3| = \frac{1}{3}x^3 + C$$

$$\Rightarrow \ln|y-1/3| = -x^3 - 3C = -x^3 + \hat{C}$$

$$\Rightarrow y - 1/3 = e^{-x^3+\hat{C}} = \bar{C}e^{-x^3}$$

$$\Rightarrow y = \bar{C}e^{-x^3} + 1/3.$$

__Exercise II__ Will SoV work for $y' + 3x^2y = x^3$?

$$\Rightarrow y' = x^3 - 3x^2y = \underbrace{x^2}_{\neq\, g(x)}\underbrace{(x-3y)}_{h(y)}$$

__Ans__ No, the DE is not separable.

You can still use IF method.

Exercise III Solve $\frac{dy}{dx} - 3y = 6$ using integrating factor.

Integrating factor $f = e^{\int p(x)dx} = e^{\int -3dx} = e^{-3x}$

$$\Rightarrow \quad e^{-3x}y' - 3e^{-3x}y = \underbrace{\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}}$$

$$\Rightarrow \quad e^{-3x}y = \int 6e^{-3x}dx = -\frac{6}{3}e^{-3x} + C$$

$$= -2e^{-3x} + C$$

$$\Rightarrow \quad y = e^{3x}\left(-2e^{-3x} + C\right)$$

$$\underline{y = -2 + Ce^{3x}}$$