Applied differential equations

TW244 - Lecture 24

5.1: Spring-mass systems (cont.)

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Driven motion without damping: Recall

The undamped, forced system

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \quad x(0) = x'(0) = 0, \quad 0 < \gamma \neq \omega$$

has solution

$$\boxed{x(t) = \frac{F_0}{\omega^2 - \gamma^2}\big(\cos(\gamma t) - \cos(\omega t)\big).}$$

We require $\gamma \neq \omega$, but what if

- $\gamma \approx \omega$?
- $\gamma \to \omega$?

$x(t) = \frac{F_0}{\omega^2-\gamma^2}(\cos\gamma t - \cos\omega t)$

$= \frac{2F_0}{\omega^2-\gamma^2}\sin\theta\sin\phi$

Let's first write x in a different form. Remember that

$$\begin{aligned}
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\cos(\theta-\phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi \\
\therefore \underline{\cos(\theta-\phi)-\cos(\theta+\phi)} &= \underline{2\sin\theta\sin\phi}
\end{aligned}$$

In our case,

$$\left.\begin{aligned} \theta-\phi &= \gamma t \\ \underline{\theta+\phi} &\underline{= \omega t} \end{aligned}\right\} \implies \begin{aligned} \theta &= \tfrac{1}{2}(\omega+\gamma)t \\ \phi &= \tfrac{1}{2}(\omega-\gamma)t \end{aligned}$$

Hence

$$\boxed{x(t) = \frac{2F_0}{\omega^2-\gamma^2}\sin\left[\tfrac{1}{2}(\omega+\gamma)t\right]\sin\left[\tfrac{1}{2}(\omega-\gamma)t\right].}$$

$x(t) = \frac{2F_0}{\omega^2-\gamma^2} \sin\left(\frac{\omega+\gamma}{2}t\right) \sin\left(\frac{\omega-\gamma}{2}t\right)$

Case 1: Let's consider $\boxed{\gamma = \omega + 2\varepsilon}$ where $\varepsilon \ll 1$, then we have that

$$\begin{aligned}
\tfrac{1}{2}(\omega + \gamma) &= \tfrac{1}{2}(\gamma - 2\varepsilon + \gamma) = \gamma - \varepsilon \approx \gamma \\
\tfrac{1}{2}(\omega - \gamma) &= \tfrac{1}{2}(\gamma - 2\varepsilon - \gamma) = -\varepsilon \\
(\omega^2 - \gamma^2) &= (\omega + \gamma)(\omega - \gamma) \approx -4\gamma\varepsilon
\end{aligned}$$

Substituting these approximations to

$$x(t) = \frac{2F_0}{\omega^2 - \gamma^2} \sin\left[\tfrac{1}{2}(\omega + \gamma)t\right] \sin\left[\tfrac{1}{2}(\omega - \gamma)t\right]$$

$$= \frac{2F_0}{-4\gamma\varepsilon} \sin(\gamma t) \sin(-\varepsilon t)$$

gives

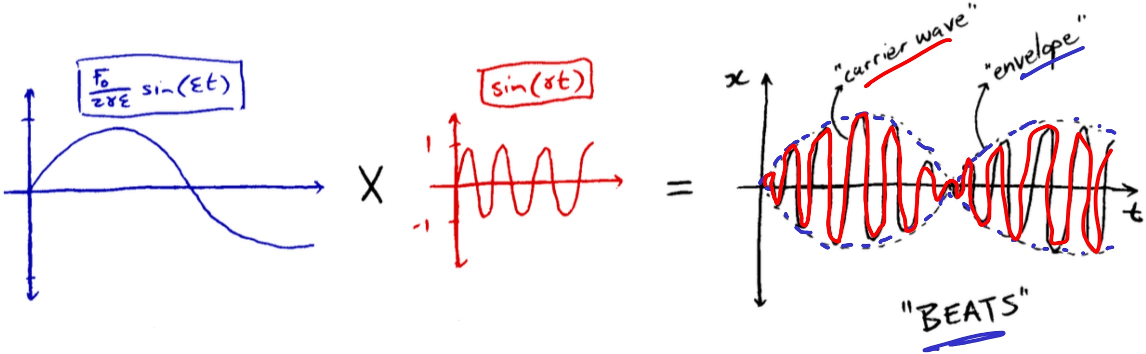
$$x(t) \approx \frac{F_0}{2\gamma\varepsilon} \sin[\varepsilon t] \sin[\gamma t].$$

What would the graph of such a function look like?

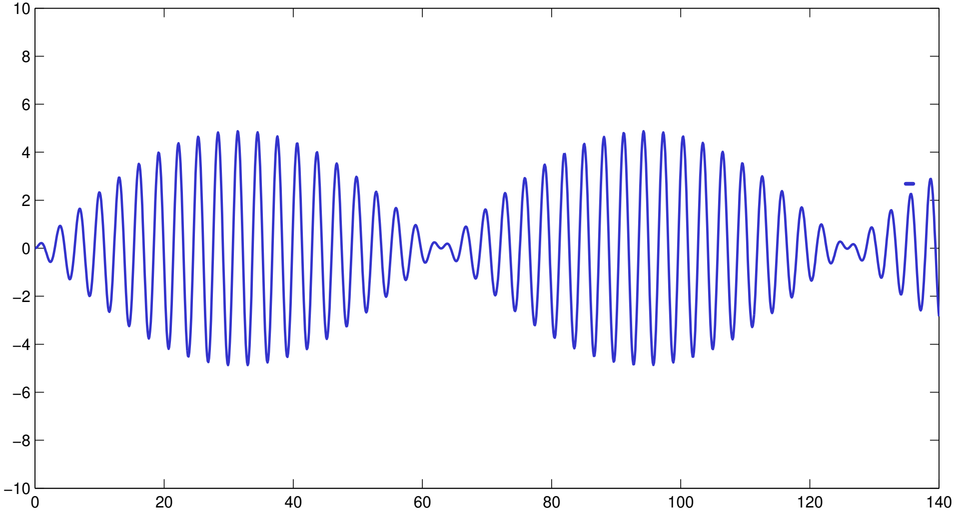
5.1: Spring-mass systems (cont.)

Driven motion without damping

$$x(t) \approx \underbrace{\frac{F_0}{2\gamma\varepsilon} \sin[\varepsilon t]}\, \underbrace{\sin[\gamma t]}\,.$$



5.1: Spring-mass systems (cont.)
Driven motion without damping



5.1: Spring-mass systems (cont.)
Driven motion without damping

$\gamma = \omega + 2\varepsilon$

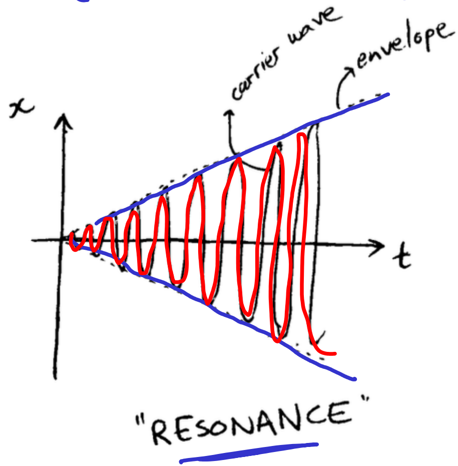
$$\frac{\sin \varepsilon t}{\varepsilon} = \frac{\varepsilon t - \frac{\varepsilon^3}{3!}t^3 + O(\varepsilon^5)}{\varepsilon} = t - \frac{\varepsilon^2}{6}t^3 + \dots \approx t$$

Case 2: Let's now consider the case $\boxed{\gamma \to \omega}$. Equivalently, let $\varepsilon \to 0$ then we have that

$$\begin{aligned} x(t) &= \lim_{\varepsilon \to 0} \frac{F_0}{2\gamma} \frac{\sin(\varepsilon t)}{\varepsilon} \sin(\gamma t) \\ &= \frac{F_0 t}{2\omega} \sin(\omega t) \end{aligned}$$

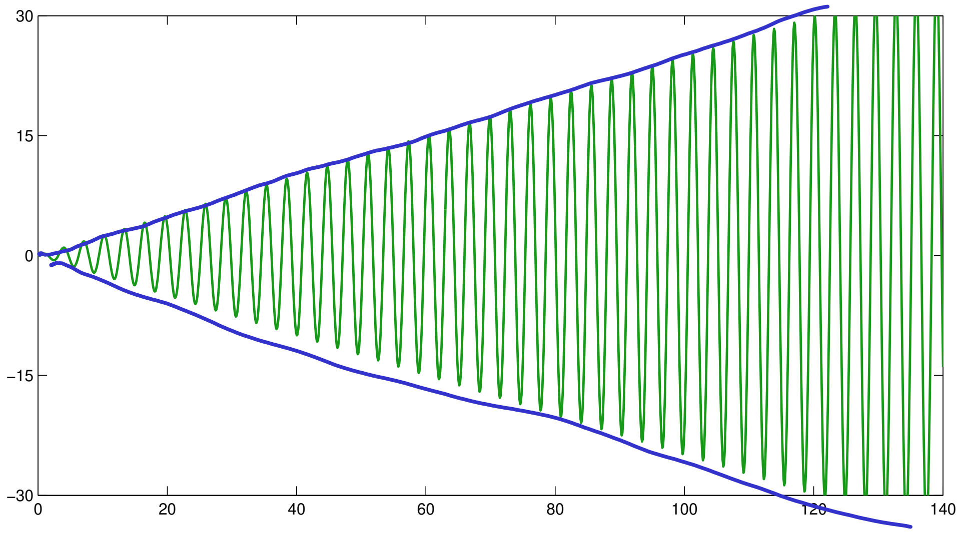
i.e., a sine wave with pseudo-amplitude $\frac{F_0}{2\omega}t$, which **grows** linearly with time!

We call this "*pure resonance*" (as there is no damping present, which is unphysical).



See CA04 for damped resonance.

5.1: Spring-mass systems (cont.)
Driven motion without damping



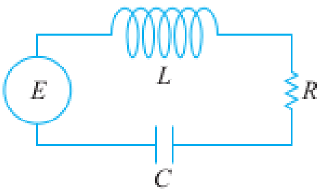
5.14: Series circuit analogue

Self-study

Many different physical systems can be described by a linear second order differential equation similar to the differential equation of forced motion with damping.

For example, if $i(t)$ denotes current in the LRC-series electrical circuit shown in the Figure, then, by Kirchhoff's second law, the sum of the voltages drops equals the voltage $E(t)$ impressed on the circuit; that is,

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t).$$



But $i = \frac{dq}{dt}$, so

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Although the letters are different, this equation has exactly the same form as the forced-damped spring-mass system we have been considering, and we can conclude that such circuits exhibit the same behaviour.

5.3: NONLINEAR MODELS

Example 1a: Nonlinear spring

Assume a spring force of the form $F = k_1 x + k_2 x^3$

$$\begin{aligned} \implies \quad m\frac{d^2x}{dt^2} &= -(k_1 x + k_2 x^3) \\ \implies \quad x'' + \omega_1 x + \omega_2 x^3 &= 0. \end{aligned}$$

This is known as 'Duffing's equation'. We cannot solve it analytically.)=

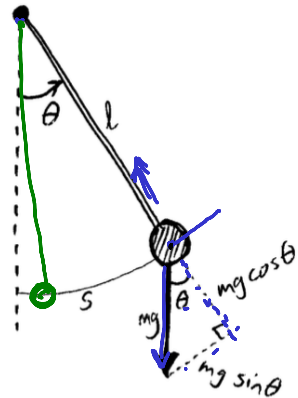
Example 1b: Quadratic damping

Assume the magnitude of the damping force is proportional to the square of the velocity (and direction opposite to motion)

$$\implies \quad m\frac{d^2x}{dt^2} = -kx - \beta\left|\frac{dx}{dt}\right|\frac{dx}{dt}.$$

(Note that $\left(\frac{dx}{dt}\right)^2$ won't work. $\left|\frac{dx}{dt}\right|\frac{dx}{dt}$ incorporates the direction of motion.)

This DE is also not analytically solveable.)=



Example 2: Nonlinear pendulum

Let $\theta(t)$ be the (signed) angle that the pendulum forms with the vertical and $s(t)$ be the arc length.

Tangential force: $F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$

But $s = \ell\theta \implies \frac{d^2s}{dt^2} = \ell\frac{d^2\theta}{dt^2} \implies m\ell\frac{d^2\theta}{dt^2} = -mg\sin\theta$

i.e.,

$$\boxed{\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0} \quad : \quad \omega^2 = \frac{g}{l}.$$

This DE can be solved 'analytically': $\theta(t) = 2\arcsin(\sqrt{m}\,\mathrm{sn}(K(m) - \omega t; m))$ where $m = \sin^2(\theta_0/2)$, $\mathrm{sn}(\cdot; m)$ is a Jacobi elliptic function (`ellipj(·,m)` in MATLAB), and $K(m)$ is the complete elliptic integral (`ellipke(m)`).

See here or here for a partial derivation (not examinable). Note $m = k^2$.

For $\theta \ll 1$ the approximation $\sin\theta \approx \theta$ gives the linear pendulum $\boxed{\frac{d^2\theta}{d\theta^2} + \omega^2\theta = 0.}$