Applied differential equations

TW244 - Lecture 27

7.6: Systems of linear DEs via LTs.

Prof Nick Hale - 2020

SCIENCE
NATUURWETENSKAPPE
EYOBUNZULULWAZI

$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$

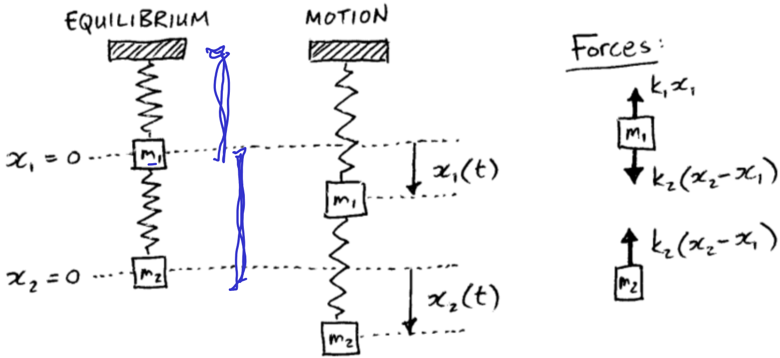
Recall: Laplace transform method for systems

The outline of the procedure is...

1. Take Laplace Transform (LT) of both equations.
2. Solve for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ simultaneously.
3. Get the inverse transform in one unknown (e.g., x)
4. Substitute back and solve for the other unknown (e.g., y)

We can use the Laplace transform technique to solve systems of second-order DEs!

Recall Hooke's law ($mx'' = -kx$) and consider two coupled springs:



Therefore

$$\left.\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' &= -k_2(x_2 - x_1) \end{aligned}\right\} \implies \boxed{\begin{aligned} x_1'' + \frac{k_1+k_2}{m_1}x_1 - \frac{k_2}{m_1}x_2 &= 0 \\ x_2'' - \frac{k_2}{m_2}x_1 + \frac{k_2}{m_2}x_2 &= 0 \end{aligned}}$$

7.6: Systems of linear DEs (cont.)

Application 1: Coupled springs (cont.)

$X_1 = \mathcal{L}\{x_1\}$
$X_2 = \mathcal{L}\{x_2\}$

Example: Consider

$$x_1'' + 10x_1 - 4x_2 = 0 \qquad (1)$$
$$x_2'' - 4x_1 + 4x_2 = 0 \qquad (2)$$

with $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0$ and $x_2'(0) = -1$.

1

$x'(0)$

LT of (1): $s^2X_1 - sx_1(0) - x_1'(0) + 10X_1 - 4X_2 = 0 \implies (s^2+10)X_1 - 4X_2 = 1$ (3)

LT of (2): $s^2X_2 - sx_2(0) - x_2'(0) - 4X_1 + 4X_2 = 0 \implies -4X_1 + (s^2+4)X_2 = -1$ (4)

2

$4 \times (3)$: $4(s^2+10)X_1 - 16X_2 = 4$ (5)

$(s^2+10)\times(4)$: $-4(s^2+10)X_1 + (s^2+4)(s^2+10)X_2 = -(s^2+10)$ (6)

$(5)+(6)$: $[(s^2+4)(s^2+10) - 16]X_2 = 4 - (s^2+10) \implies$

3

$$\mathcal{L}\{x_2\} = X_2 = -\frac{s^2+6}{(s^2+12)(s^2+2)} \overset{\text{partial fractions}}{=} -\frac{3}{5}\frac{1}{s^2+12} - \frac{2}{5}\frac{1}{s^2+2}$$

$$+\frac{s^2+6}{(s^2+12)(s^2+2)} = \frac{A}{s^2+12} + \frac{B}{s^2+2} \Rightarrow s^2+6 = A(s^2+2) + B(s^2+12)$$

$$= s^2\underbrace{(A+B)}_{=1} + \underbrace{(2A+12B)}_{=6}$$

$$\Rightarrow 6 = 2A+12B = 2A+12(1-A)$$

$$= 2A+12-12A$$

$$= 12-10A$$

$$\Rightarrow -6 = -10A \Rightarrow A = 6/10 = 3/5$$

$$A+B=1 \Rightarrow B = 1-A = 2/5.$$

$$= \frac{3}{5}\frac{1}{s^2+12} + \frac{2}{5}\frac{1}{s^2+2}$$

$$\Rightarrow -\frac{s^2+6}{(s^2+12)(s^2+2)} = -\frac{3}{5}\frac{1}{s^2+12} - \frac{2}{5}\frac{1}{s^2+2} = \mathcal{L}\{x_2\}$$

$$= -\frac{3}{5}\left[\frac{\sqrt{12}}{s^2+(\sqrt{12})^2}\right]\frac{1}{\sqrt{12}} - \frac{2}{5}\frac{1}{\sqrt{2}}\left[\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right]$$

7.6: Systems of linear DEs (cont.)

Application 1: Coupled springs (cont.)

$x_2'' + 4x_1 - 4x_2 = 0 \qquad (2)$

$\Rightarrow x_1 = x_2 - \tfrac{1}{4}x_2''$

$$\therefore x_2 = -\underset{\sqrt{12}}{\underline{\tfrac{3}{5}}}\mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\} - \underset{\sqrt{2}}{\underline{\tfrac{2}{5}}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$$

$$\Longrightarrow \boxed{x_2(t) = -\tfrac{3}{5}\left(\tfrac{1}{\sqrt{12}}\right)\sin(\sqrt{12}t) - \tfrac{2}{5}\left(\tfrac{1}{\sqrt{2}}\right)\sin(\sqrt{2}t)}$$

4

To obtain x_1, let's substitute X_2 into (4) so that

$$\mathcal{L}\{x_1\} = X_1 = \frac{1}{4}\Big[(s^2+4)X_2 + 1\Big] = \frac{1}{4}\left[\frac{-(s^2+4)(s^2+6)}{(s^2+12)(s^2+2)} + 1\right]$$

$$= \frac{s^2}{(s^2+12)(s^2+2)} \overbrace{=}^{\text{partial fractions}} \frac{6}{5}\frac{1}{s^2+12} - \frac{1}{5}\frac{1}{s^2+2}$$

$$\Longrightarrow \boxed{x_1(t) = \tfrac{6}{5}\left(\tfrac{1}{\sqrt{12}}\right)\sin(\sqrt{12}t) - \tfrac{1}{5}\left(\tfrac{1}{\sqrt{2}}\right)\sin(\sqrt{2}t)}$$

$$x_2'' - 4x_1 + 4x_2 = 0 \Rightarrow x_1 = x_2 + \tfrac{1}{4}x_2''$$

$$\boxed{x_2(t) = -\frac{3}{5}\left(\frac{1}{\sqrt{12}}\right)\sin(\sqrt{12}t) - \frac{2}{5}\left(\frac{1}{\sqrt{2}}\right)\sin(\sqrt{2}t)}$$

$$-\frac{1}{12} + \frac{3}{12} = \frac{2}{12}$$

$$x_2' = -\frac{3\cancel{\sqrt{12}}}{5\cancel{\sqrt{12}}}\cos(\sqrt{12}t) - \frac{2}{5}\frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\cos(\sqrt{2}t)$$

$$x_2'' = +\frac{3}{5}\sqrt{12}\sin(\sqrt{12}t) + \frac{2}{5}\sqrt{2}\sin(\sqrt{2}t)$$

$$x_1 = x_2 + \tfrac{1}{4}x_2''$$

$$= -\frac{3}{5}\frac{1}{\sqrt{12}}\sin\sqrt{12}t - \frac{2}{5}\frac{1}{\sqrt{2}}\sin\sqrt{2}t + \tfrac{1}{4}\left[\frac{3}{5}\sqrt{12}\sin\sqrt{12}t + \frac{2}{5}\sqrt{2}\sin\sqrt{2}t\right]$$

$$= \left(-\frac{3}{5}\frac{1}{\sqrt{12}} + \frac{3}{4\cdot 5}\sqrt{12}\right)\sin\sqrt{12}t + \left(-\frac{2}{5}\frac{1}{\sqrt{2}} + \frac{2}{4\cdot 5}\sqrt{2}\right)\sin\sqrt{2}t$$

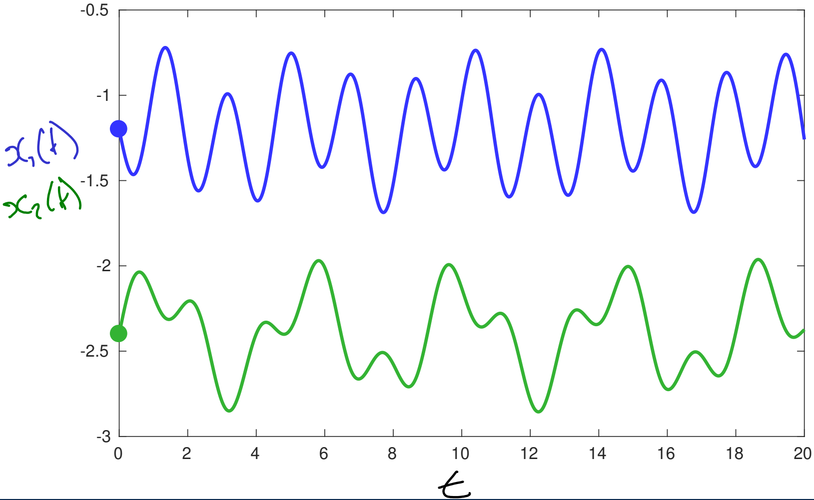
$$= \frac{3}{5}\frac{1}{\sqrt{12}}\left(-1 + \frac{12}{4}\right)\sin\sqrt{12}t + \frac{1}{\sqrt{2}}\frac{2}{5}\left(-1 + \frac{2}{4}\right)\sin\sqrt{2}t$$

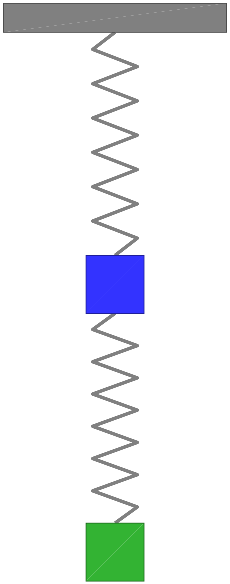
$$= \frac{3}{5}\frac{1}{\sqrt{12}}2\sin\sqrt{12}t + \frac{1}{\sqrt{2}}\frac{2}{5}\left(-\tfrac{1}{2}\right)\sin\sqrt{2}t$$

$$= \frac{6}{5}\frac{1}{\sqrt{12}}\sin\sqrt{12}t - \frac{1}{5}\frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

Application 1: Coupled springs (cont.)

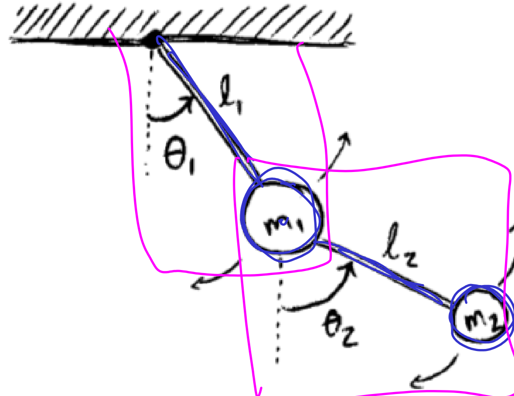
$$x_1(t) = \frac{6}{5}\left(\frac{1}{\sqrt{12}}\right)\sin(\sqrt{12}t) - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)\sin(\sqrt{2}t)$$
$$x_2(t) = -\frac{3}{5}\left(\frac{1}{\sqrt{12}}\right)\sin(\sqrt{12}t) - \frac{2}{5}\left(\frac{1}{\sqrt{2}}\right)\sin(\sqrt{2}t)$$





7.6: Systems of linear DEs (cont.)
Application 2: Double pendulum

Consider the double pendulum with $\ell_1 = \ell_2 = 16, m_1 = 3$, and $m_2 = 1$.



It can be shown (see p. 318, self-study) that the motion of the pendulum can be described (approximately) by:

$$\begin{aligned} 4x'' + y'' + 8x &= 0 \qquad (1) \\ x'' + y'' + 2y &= 0 \qquad (2) \end{aligned}$$

with $x(t) = \theta_1(t)$ & $y(t) = \theta_2(t)$ in radians.

Let's us assume the initial conditions

$$x(0) = 1, x'(0) = 0, y(0) = -1, y'(0) = 0.$$

$$\left.\begin{aligned} 4x'' + y'' + 8x &= 0 \\ x'' + y'' + 2y &= 0 \end{aligned}\right\}$$

Using these initial conditions, we then have that

$$\mathcal{L}\{x''\} = s^2X - sx(0) - x'(0) = s^2X - s\cdot 1 - 0 = s^2X - s$$
$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s\cdot(-1) - 0 = s^2Y + s$$

1

$\therefore$ LT of (1):

$$\begin{aligned} 4(s^2X - s) + (s^2Y + s) + 8X &= 0 \\ \implies (4s^2+8)X + s^2Y &= 3s \qquad (3) \end{aligned}$$

and LT of (2):

$$\begin{aligned} (s^2X - s) + (s^2Y + s) + 2Y &= 0 \\ \implies s^2X + (s^2+2)Y &= 0 \qquad (4) \end{aligned}$$

Therefore

2

$$\begin{aligned} -s^2\times(3) &\implies \cancel{-s^2(4s^2+8)}X - s^4Y = -3s^3 && (5) \\ (4s^2+8)\times(4) &\implies \cancel{s^2(4s^2+8)}X + (4s^2+8)(s^2+2)Y = 0 && (6) \\ (5)+(6) &\implies 3(s^2+\tfrac{4}{3})(s^2+4)Y = -3s^3 && (7) \end{aligned}$$

check.

$$s\left[\frac{-s^2}{(s^2+4/3)(s^2+4)}\right] = s\left[\frac{A}{s^2+4/3}+\frac{B}{s^2+4}\right]$$

$$-s^2 = A(s^2+4)+B(s^2+4/3)$$

From (7) we have that

$$3(s^2+\tfrac{4}{3})(s^2+4)Y = -3s^3$$

$$\implies \boxed{Y = \frac{-s^3}{(s^2+\frac{4}{3})(s^2+4)}} \overbrace{=}^{\text{partial fractions}} \frac{\frac{1}{2}s}{s^2+\frac{4}{3}} - \frac{\frac{3}{2}s}{s^2+4}$$

Check

$$\implies y = \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{4}{3}}\right\} - \tfrac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \implies \boxed{y(t) = \tfrac{1}{2}\cos(\tfrac{2}{\sqrt{3}}t) - \tfrac{3}{2}\cos(2t)}$$

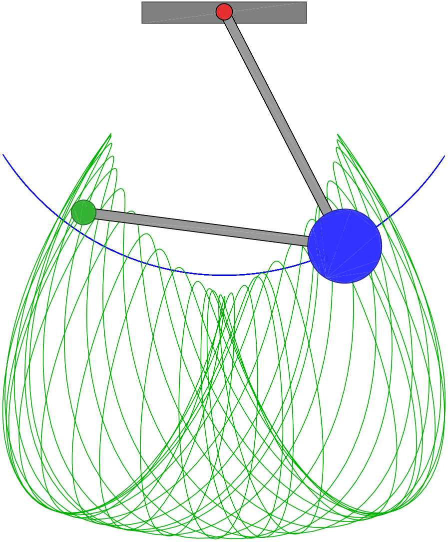
From (4) we then have

Eliminate y from 3/4 and solve for x.

$$X = -\frac{s^2+2}{s^2}Y = -Y - \tfrac{2}{s^2}Y = \frac{-\frac{1}{2}s}{s^2+\frac{4}{3}} + \frac{\frac{3}{2}s}{s^2+4} + \frac{2s}{(s^2+\frac{4}{3})(s^2+4)} \overbrace{=}^{p.f.} \frac{\frac{1}{4}s}{s^2+\frac{4}{3}} + \frac{\frac{3}{4}s}{s^2+4}$$

$$\implies x = \tfrac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{4}{3}}\right\} + \tfrac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \implies \boxed{x(t) = \tfrac{1}{4}\cos(\tfrac{2}{\sqrt{3}}t) + \tfrac{3}{4}\cos(2t)}$$

$$
\begin{aligned}
x(t) &= \tfrac{1}{4}\cos\left(\tfrac{2}{\sqrt{3}}t\right) + \tfrac{3}{4}\cos(2t) \\
y(t) &= \tfrac{1}{2}\cos\left(\tfrac{2}{\sqrt{3}}t\right) - \tfrac{3}{2}\cos(2t)
\end{aligned}
$$



☑ Trace the paths?