Applied differential equations

TW244 - Lecture 15

4.3: Homogeneous linear DEs

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Consider the DE

$$ay'' + by' + cy = 0$$

with a, b, c constant.

TRICK: To solve DEs of this form, try the ansatz $\boxed{y = e^{mx}.}$

Substituting this to the DE gives

$$\begin{aligned} am^2\cancel{e}^{mx} + bm\cancel{e}^{mx} + c\cancel{e}^{mx} &= 0 \\ \Longrightarrow \quad am^2 + bm + c &= 0. \end{aligned}$$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

We call this the "auxiliary equation".

We solve for m and consider three different cases...

(1) **Two distinct real roots.** (m_1 and m_2)

$$\implies y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(2) **One real root** (m)

$$\implies y = c_1 e^{mx} + c_2 x e^{mx}$$

(3) **Two complex roots** ($\alpha + i\beta$ and $\alpha - i\beta$)

$$\implies y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

The first case is immediate, but let's see why the other two hold...

Case (2) – repeated root.

Let $\boxed{y = xe^{mx}}$ then $y' = (mx+1)e^{mx}$ and $y'' = m(mx+2)e^{mx}$.

Substituting this to the LHS of the DE we find

$$\begin{aligned} & am(mx+2)e^{mx} + b(mx+1)e^{mx} + cxe^{mx} \\ = \; & e^{mx}\left[(am^2+bm+c)x + (2am+b)\right] \\ = \; & e^{mx}\left[0x+0\right] = 0 \end{aligned}$$

where the first term is zero by the definition of m.

To see that the second term is zero, note

$$f(m) = am^2 + bm + c \implies f'(m) = 0 \implies 2am + b = 0$$

(i.e., a double root at m implies a turning point at m).

Hence when $am^2 + bm + c = 0$ has a single real solution, $y = xe^{mx}$ is a solution of $ay'' + by' + c = 0$.

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Case (2) (cont.)

$ay'' + by' + cy = 0 \quad y = e^{mx}$

It remains to show that $y_1 = e^{mx}$ and $y_2 = xe^{mx}$ are fundamental solutions.

$y_c = c_1 y_1 + c_2 y_2$

Consider the Wronskian:

$$
\begin{aligned}
W(x) &= \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & (mx+1)e^{mx} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= (\cancel{mx} + 1)e^{2mx} - \cancel{mxe^{2mx}} \\
&= e^{2mx} \\
&\neq 0 \ \forall \ -\infty < x < \infty.
\end{aligned}
$$

Therefore $y_1 = e^{mx}$ and $y_2 = xe^{mx}$ are indeed fundamental solutions and

$$\boxed{y = c_1 e^{mx} + c_2 xe^{mx}}$$

is the general solution.

Case (3)

$ay'' + by' + cy = 0$

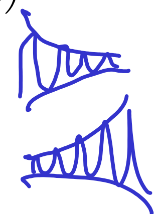
$y = e^{mx}$

Recalling that $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ (Euler) we have

$m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$

$$
\begin{aligned}
y &= d_1 e^{(\alpha+i\beta)x} + d_2 e^{(\alpha-i\beta)x}\\
&= e^{\alpha x}\left(d_1 e^{i\beta x} + d_2 e^{-i\beta x}\right)\\
&= e^{\alpha x}\left(d_1[\cos(\beta x) + i\sin(\beta x)] + d_2[\cos(\beta x) - i\sin(\beta x)]\right)\\
&= e^{\alpha x}\left([d_1 + d_2]\cos(bx) + i[d_1 - d_2]\sin(\beta x)\right)\\
&= e^{\alpha x}\left(c_1\cos(\beta x) + c_2\sin(\beta x)\right)
\end{aligned}
$$



as required.

The advantage of the final form is that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ will be real (if β and x are real). This is not true of the terms in the first expression.

Exercise: Verify that these two solutions are fundamental.

Show that $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$ are fundamental soln.

$y_1' = e^{\alpha x}(\alpha\cos(\beta x) - \beta\sin(\beta x))$, $y_2' = e^{\alpha x}(\alpha\sin(\beta x) + \beta\cos(\beta x))$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= e^{\alpha x}\cos(\beta x)\, e^{\alpha x}(\alpha\sin\beta x + \beta\cos\beta x) - e^{\alpha x}\sin\beta x\, e^{\alpha x}(\alpha\cos\beta x - \beta\sin\beta x)$$

$$= e^{2\alpha x}\left[\cos(\beta x)(\alpha \cancel{\sin\beta x} + \beta\cos\beta x) - \sin\beta x(\alpha\cancel{\cos\beta x} - \beta\sin\beta x)\right]$$

$$= e^{2\alpha x}\beta\left[\cos^2\beta x + \sin^2\beta x\right] = e^{2\alpha x}\beta \neq 0$$

$\therefore$ Wronskian case 2 $\Rightarrow$ fundamental solutions

Example 1: $y'' + 6y' + 5y = 0.$

$y_1 = e^{-x}$
$y_2 = e^{-5x}$

Try $y = e^{mx} \implies$

$$m^2\cancel{e^{mx}} + 6m\cancel{e^{mx}} + 5\cancel{e^{mx}} = 0$$
$$m^2 + 6m + 5 = 0$$
$$(m+1)(m+5) = 0$$

Two real roots, $m = -1, -5 \implies$ general solution $\boxed{y = c_1e^{-x} + c_2e^{-5x}.}$

Verify:

$$\begin{aligned} & y'' + 6y' + 5y \\ = \; & (c_1e^{-x} + 25c_2e^{-5x}) + 6(-c_1e^{-x} - 5c_2e^{-5x}) + 5(c_1e^{-x} + c_2e^{-5x}) \\ = \; & (1 - 6 + 5)c_1e^{-x} + (25 - 30 + 5)c_2e^{-5x} = 0 \qquad \checkmark \end{aligned}$$

Example 2: $y'' + 6y' + 9y = 0$.

Try $y = e^{mx} \implies$

$$m^2 \cancel{e^{mx}} + 6m\cancel{e^{mx}} + 9\cancel{e^{mx}} = 0$$
$$m^2 + 6m + 9 = 0$$
$$(m+3)^2 = 0$$

One real root, $m = -3 \implies$ general solution $\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}.}$

Exercise: Verify this solution.

$y_1 = e^{-3x}$, $y_2 = xe^{-3x}$

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Examples

Example 3: $y'' + 4y' + 13y = 0.$

Try $y = e^{mx} \implies$

$$m^2\cancel{e^{mx}} + 4m\cancel{e^{mx}} + 13\cancel{e^{mx}} = 0$$
$$m^2 + 4m + 13 = 0$$
$$m = \tfrac{1}{2}(-4 \pm \sqrt{16 - 4 \times 13})$$
$$m = -2 \pm \tfrac{1}{2}\sqrt{-36}$$
$$m = -2 \pm 3i$$

$y_1 = e^{-2x}\cos 3x$
$y_2 = e^{-2x}\sin 3x$

Two complex roots, $m = -2 \pm 3i \implies$ general solution

$$\boxed{y = e^{-2x}\Big(c_1\cos(3x) + c_2\sin(3x)\Big).}$$

Consider the following two important DEs: (we will see these again later)

#1: $y'' = k^2 y, \quad k \in \mathbb{R}.$

Try $y = e^{mx} \implies m^2 e^{mx} = k^2 e^{mx} \implies m = \pm k$, $m_1 = k$, $m_2 = -k$

Two real roots $\implies$ general solution:

$$y = d_1 e^{kx} + d_2 e^{-kx} = \boxed{c_1 \cosh(kx) + c_2 \sinh(kx)}$$

#2: $y'' = -k^2 y, \quad k \in \mathbb{R}.$

Try $y = e^{mx} \implies m^2 e^{mx} = -k^2 e^m x \implies m = \pm ik$

Two complex roots $\implies$ general solution:

$$y = d_1 e^{ikx} + d_2 e^{-ikx} = \boxed{c_1 \cos(kx) + c_2 \sin(kx)}$$