For examiner's use:

Naam/Name: Memo

Stud. Nr: ____________

Toegepaste Differensiaalvergelykings
TW244 Toets 1 2018

Dosent/Lecturer: Dr N Hale

Instruksies:

(a) 5 probleme (+1 bonus).

(b) 50 + 5 punte (50 maks).

(c) 2 uur, toeboek.

(d) Sakrekenaars **word toegelaat**. Selfone **nie**.

(e) Toon alle bewerkings. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(f) Daar is leë bladsye aan die agterkant van die vraestel as jou antwoorde nie inpas in die gegewe spasies nie. Dui duidelik aan as jou antwoord voortgaan op een van hierdie bladsye.

(g) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

Applied Differential Equations
TW244 Test 1 2018

Moderator: Dr W Brink

Instructions:

(a) 5 problems (+1 bonus).

(b) 50 + 5 marks (50 max).

(c) 2 hours, closed book.

(d) Calculators **are** allowed. Cell phones are **not**.

(e) Show all calculations. A correct answer does not earn full marks without the necessary explanation.

(f) There are blank pages at the back of the paper in case you cannot fit your answer in the space provided. Indicate clearly if your answer continues to one of these pages.

(g) The formulas below may be used without proof anywhere in the test.

Formules/Formulas:

- Fundamentele stelling van calculus/ Fundamental theorem of calculus :
$$\int \frac{df}{dx}\,dx = f(x) + C$$
- Produkreël vir differensiasie/ Product rule for differentiation :
$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$
- Deelsgewyse integrasie/ Integration by parts :
$$\int_a^b f\frac{dg}{dx}\,dx = [fg]_a^b - \int_a^b \frac{df}{dx}g\,dx$$
- Differensiasie van die logaritme/ Differentiation of the logarithm :
$$\frac{d}{dx}\ln(x-a) = \frac{1}{x-a}$$
- Wronskian:
$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Prob 1 (6 punte/marks)

(a) Vir die volgende differensiaalvergelykings, gee die orde, dui aan of dit lineêr is of nie, of dit 'n outonome DV is of nie, en of dit 'n homogene DV is of nie. (Verkeerde antwoorde sal die korrekte antwoorde kanselleer, tot 'n minimum van nulpunte vir deel (a).)

2 (a) For the following differential equation, give the order, state whether it is linear or not, whether it is autonomous or not, and whether it is homogenous or not. (Incorrect answers will cancel correct answers, to a minimum of zero marks for part (a).)

differensiaalvergelyking *differential equation*	orde *order*	lineêr? (ja/nee) *linear? (yes/no)*	outonome? (ja/nee) *autonomous? (yes/no)*	homogene (ja/nee) *homogeneous? (yes/no)*
$\frac{dy}{dx} + xy^2 = y$	1	×	×	✓

(b) Watter van die volgende DVs is geskik om deur die integrasie faktor metode opgelos te word.

1 **(b)** Which of the following DEs **would** be suitable for solving with the integrating factor method:

(A) $(\frac{dy}{dx})^2 + xy = y$ (B) $\frac{dy}{dx} + x^2y = y$ (C) $\frac{dy}{dx} + xy^2 = y$ (D) $\frac{dy}{dx} + xy = y^2$ (E) niks geskik nie/ none suitable

(c) Watter van die volgende DVs is nie geskik om deur skeiding van veranderlikes op gelos te word nie.

1 **(c)** Which of the following DEs would **not** be suitable for solving with separation of variables:

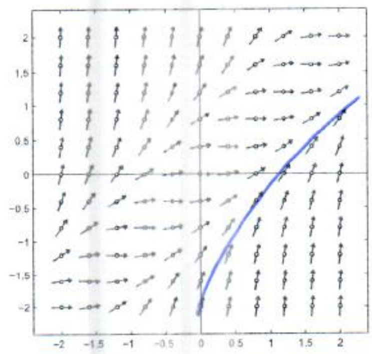
(A) $(\frac{dy}{dx})^2 + xy = y$ (B) $\frac{dy}{dx} + x^2y = y$ (C) $\frac{dy}{dx} + xy^2 = y$ (D) alles geskik/ all suitable (E) niks geskik nie/ none suitable

(d) Watter van die onderstaande **rigtingvelde** stem met die DV $\frac{dy}{dt} = (t-y)^2$.

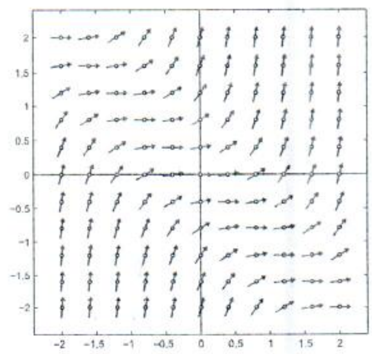
1 **(d)** Which of the **direction fields** below corresponds to the DE $\frac{dy}{dt} = (t-y)^2$ ooreen.

(A) (B) (C) (D) (E) none of the below

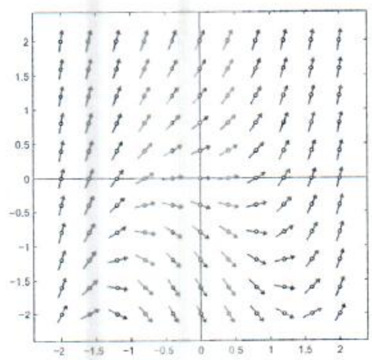
(A)



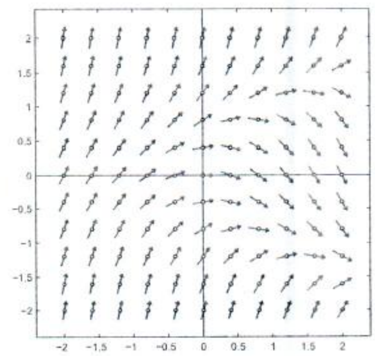
(B)



(C)



(D)



(e) Teken die oplossingskurwe op jou antwoord in (d) wat ooreenstem met die aanvangsvoorwaarde $y(0) = -2$ op jou antwoord in (d).

1 **(e)** On your answer to (d), draw the solution curve corresponding to the initial condition $y(0) = -2$.

Prob 2 (8 punte/marks)

Wanneer vergeetagtigheid in ag geneem word, kan die tempo van memorisering van 'n toegepaste wiskunde onderwerp gemodelleer word deur

When forgetfulness is taken in to account, the rate of memorization of an applied mathematics subject can be modelled by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A, \qquad k_1, k_2 > 0,$$

waar $A(t)$ die aantal voorstel wat op tyd t gememoriseer is, en M die aantal voorstel wat in totaal gememoriseer moet word.

where $A(t)$ is the amount memorized at time t and M is the total amount to be memorized.

(a) Los de DE onderhewig aan $A(0) = 0$ en **toon aan** dat die oplossing gegee word deur

3 **(a)** Solve the DE subject to $A(0) = 0$ and **show** the solution is given by

$$A(t) = (1 - e^{-(k_1+k_2)t})\frac{k_1 M}{k_1 + k_2}.$$

$dA/dt + (k_1+k_2)A = k_1 M$

$IF = e^{(k_1+k_2)t} \Rightarrow d/dt(e^{(k_1+k_2)t}A) = e^{(k_1+k_2)t}k_1 M$

$\Rightarrow e^{(k_1+k_2)t}A = \frac{k_1}{k_1+k_2}e^{(k_1+k_2)t}M + C$

$\Rightarrow A = \frac{k_1}{k_1+k_2}M + Ce^{-(k_1+k_2)t}$

$A(0)=0 \Rightarrow C = -\frac{k_1}{k_1+k_2}M \Rightarrow A(t) = (1-e^{-(k_1+k_2)t})\frac{k_1}{k_1+k_2}M$

(b) Is dit volgens die model moontlik om **al** die material te ken? **Motiveer**.

1 **(b)** Is it possible (under this model) to learn **all** the material? **Justify** your answer.

No. As $t \to \infty$, $A \to \frac{k_1}{k_1+k_2}M < M$

(c) Kom ons verbeel onsself dat jou geheue perfek is en dat jy nooit iets vergeet nie. Hoe sal die model nou lyk? (Geen motivering nodig nie.)

1 **(c)** Imagine now that you have a **perfect** memory and never forget anything. Adjust the model accordingly. (No justification required.)

$$\frac{dA}{dt} = k_1(M-A) \qquad (k_2=0)$$

(d) Ons verbeel onsself steeds dat jou geheue perfek is. Gestel dit neem jou 8 ure om 1/3 van die material te leer, gebruik jou model in (c) en bepaal hoe lank dit jou sal neem om 70% van die materiaal te leer.

3 **(d)** Still imagining that you have a perfect memory, suppose that you are studying for a test and it has taken you 8 hours to memorise 1/3 of the material. Use your model in part (c) to determine how long it will take you to memorise 70% of the material.

From above, $A(t) = (1-e^{k_1 t})M$

10.0507

$A(8) = (1-e^{-8k})M = M/3 \Rightarrow e^{-8k} = 1/3 - 1 = -2/3 \Rightarrow k = \frac{1}{8}\ln 3/2$

$A(t^*) = (1-e^{-kt^*})M = 7M/10 \Rightarrow t^* = \frac{1}{k}\ln(10/3) = \frac{8\ln(10/3)}{\ln(3/2)}$

≈ 23.75

24 ure/hours

Prob 3 (13 punte/marks)

Twee chemikalieë A en B word gekombineer om 'n chemikalie C te vorm. Die reaksietempo is **direkeweredig aan die produk** van die hoeveelhede van A en B wat nog nie na C omgeskakel is nie. Aanvanklik is daar 20 gram van A en 50 all suitablegram van B. Om 3 gram van C te maak word 2 gram van B en 1 gram van A benodig.

Two chemicals A and B are combined to form a chemical C. The rate of the reaction is **proportional to the product** of the instantaneous amounts of A and B not yet converted to C. Initially, there are 20 grams of A and 50 grams of B. To make 3 grams of C requires 2 grams of B and 1 gram of A.

(a) Gee vergelykings vir die hoeveelhede van A en B op enige tydstip in terme van nie hoeveelheid nuwe chemikalieë C.

1 (a) Give equations for the amount of A and B at any given time in terms of the newly created chemical C.

$A(t) = 20 - \frac{1}{3}C(t)$ $\qquad$ $B(t) = 50 - \frac{2}{3}C(t)$

(b) Skryf die DV neer wat die hoeveelheid van C in die stelsel wat hieruit volg, beskryf.

1 **(b)** Hence write down a DE describing the amount of C in the system..

$$\frac{dC}{dt} = k(20 - \tfrac{1}{3}C)(50 - \tfrac{2}{3}C) \quad \text{or} \quad \hat{k}(C-75)(C-60)$$

'n **Ander** stelsel van chemiese reaksies kan deur die volgende AWP gemodelleer word

A **different** system of chemical reactions can be modelled by the IVP

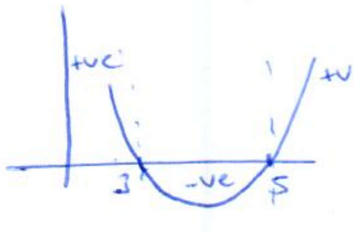
$$\frac{dC}{dt} = k(C-5)(C-3), \qquad C(0) = C_0, \quad k > 0,$$

waar C in gram en t in minute gemeet word.

where C is measured in grams and t in minutes.

(c) Vir watter waardes van C is $\frac{dC}{dt}$ positief / negatief?

1 **(c)** For what values of C is $\frac{dC}{dt}$ positive / negative?



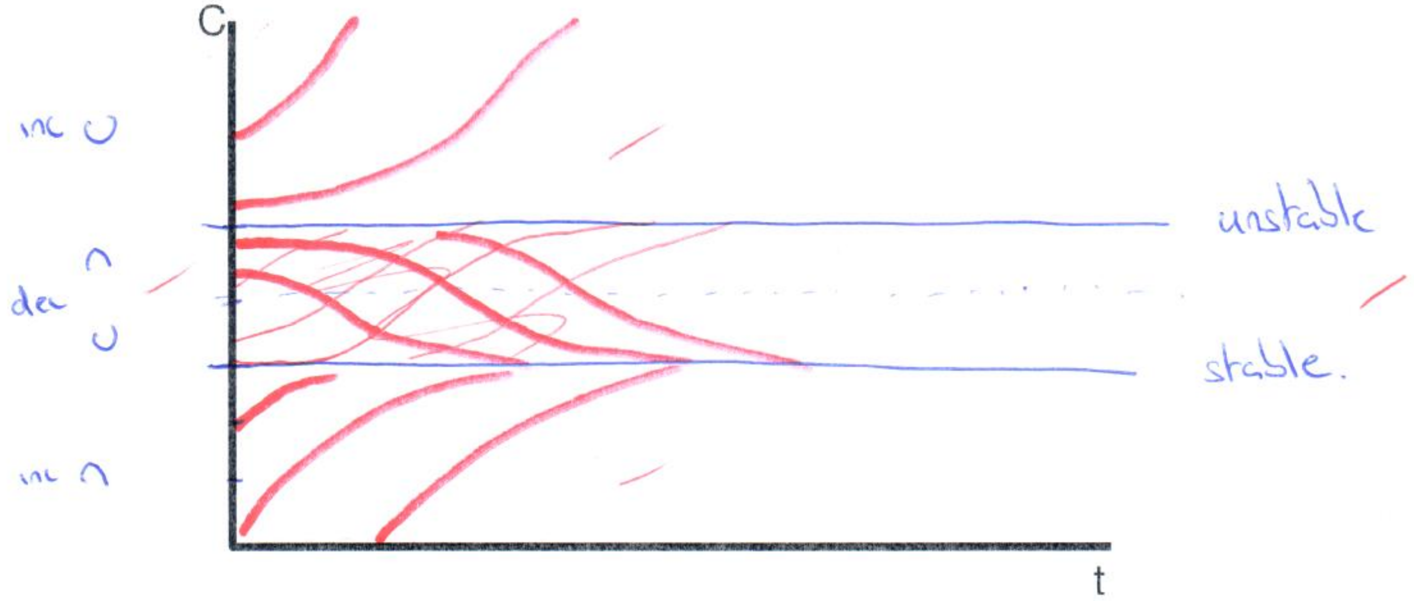
(d) Vir watter waardes van C is $C(t)$ konkaaf op / af?

1 **(d)** For what values of C is $C(t)$ concave up / down?

$$\frac{d^2C}{dt^2} = k\frac{dC}{dt}(C-3+C-5)$$
$$= \frac{k^2}{2}(C-5)(C-4)(C-3)$$

(e) Gebruik die informasie vanuit dele (c) en (d) om die 'familieportret' van oplossings vir die DV te teken op die oorkantste bladsy. Wys die gebiede waarin oplossings konkaaf na bo of konkaaf na onder is, en merk die infleksiepunt indien daar een is. Dui verteenwoordigende oplossingskurwes vir 'n paar aanvangsvoorwaardes aan, en klassifiseer die kritieke oplossings as stabiel, semi-stabiel, of onstabiel.

3 **(e)** Use the information from parts (c) and (d) to draw the 'family portrait' of solutions for the DE on the page opposite. Show the regions in which solutions are concave up or concave down, and mark the point of inflection if there is one. Include representative solution curves for some initial conditions, and classify the critical solutions as stable, semi-stable, or unstable.



(f) **Los** hierdie AWP op vir die geval $C_0 = 0$ en wys dat die oplossing gegee word deur

4 (f) **Solve** this IVP in the case $C_0 = 0$ and show that the solution is given by

$$C(t) = \frac{15(1 - e^{-2kt})}{5 - 3e^{-2kt}}.$$

$$\int \frac{dC}{(C-5)(C-3)} = \int k\,dt = kt + C$$

$$\frac{1}{(C-5)(C-3)} = \frac{\alpha}{C-5} + \frac{\beta}{C-3}$$

$$\Rightarrow 1 = (C-3)\alpha + (C-5)\beta = C(\alpha+\beta) - 3\alpha - 5\beta$$

$$\Rightarrow \begin{cases} \alpha+\beta = 0 \\ 1 = -3\alpha - 5\beta \end{cases} \Rightarrow -2\beta = 1 \Rightarrow \beta = -1/2 \Rightarrow \alpha = 1/2$$

$$\frac{1}{2}\int \left(\frac{1}{C-5} - \frac{1}{C-3}\right) dC = \frac{1}{2}\ln\left|\frac{C-5}{C-3}\right|$$

$$\Rightarrow \frac{C-5}{C-3} = \pm C_2 e^{2kt} \qquad C(0) = 0 \Rightarrow C_2 = 5/3$$

$$\Rightarrow C - 5 = 5/3\, e^{2kt}(C-3)$$

$$\Rightarrow C(1 - 5/3\, e^{2kt}) = 5(1 - e^{2kt})$$

$$\Rightarrow C = \frac{5(1-e^{2kt})}{1 - 5/3\, e^{2kt}} = \frac{15(1-e^{-2kt})}{5 - 3e^{-2kt}}$$

(g) Dit word waargeneem dat 0.15 gram van C in 5 minute gevorm word. Hoeveel word gevorm in 1 uur?

2 (g) It is observed that 0.15 grams of C is formed in 5 minutes. How much is formed in 1 hour?

$$C(5) = \frac{15(1-e^{-10k})}{5-3e^{-10k}} = 0.15 = \frac{15}{100}$$

$$\Rightarrow 1 - e^{-10k} = \frac{1}{100}(5 - 3e^{-10k}) \Rightarrow e^{-10k}\left(\frac{3}{100} - 1\right) = \frac{5}{100} - 1$$

$$\Rightarrow k = \frac{1}{10}\log(97/95) \doteq 0.0021$$

$$\Rightarrow C(60) = \frac{15(1-e^{-120k})}{5 - 3e^{-120k}} \doteq 1.2457\text{g}.$$

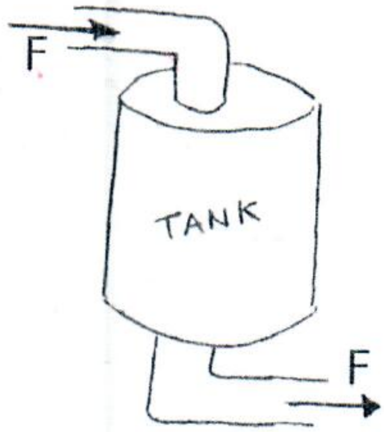
Prob 4 (11 punte/marks)

(a) 'n 100 liter watertenk is vol besoedelde water waarvan die aanvangs konsentrasie 0.3kg kontaminant per liter is. Water word in en uit die tenk gepomp teen 'n konstante tempo van F liters per minuut. Die water wat wanuit die stroom invloei is skoner en bevat $0.1kg$ vuilgoed per liter. Neem aan dat die water goed gemeng is en skryf die AW neer (moenie oplos nie) wat die massa vuilgoed in die tenk op enige tydstip beskryf.

2 **(a)** A 100 litre water tank is full of contaminated water with an initial concentration of $0.3kg$ of contaminant per litre. Water is pumped in and out of the tank a constant rate of F litres per minute. The water coming in from the stream is cleaner, and only contains $0.1kg$ of dirt per litre. Assuming that the water in the tank is well-mixed, write down (but do not solve) an IVP describing the mass of the dirt contained in the tank at any given time t.

$$\frac{dM}{dt} = \text{rate in} - \text{rate out}$$
$$= F\times 0.1 - F\times C$$
$$= \frac{F}{10} - \frac{F\times M}{100}$$

$$M(0) = 100\times C(0) = 30kg$$



$$\frac{dm}{dt} = \frac{-MF}{100} + \frac{F}{10}, \qquad m(0) = 30$$

(b) Deur gebuik te maak van die integrasie faktor metode, wys dat die oplossing vir die model in (a) gegee word deur

2 **(b)** Use the method of integrating factor to show that the solution to the model in part (a) is given by

$$m(t) = 10 + 20e^{-\frac{Ft}{100}}.$$

$$\frac{dM}{dt} + \frac{MF}{100} = \frac{F}{10} \Rightarrow IF = e^{Ft/100}$$
$$\Rightarrow \frac{d}{dt}\left(e^{MFt/100} M\right) = e^{Ft/100}\,\frac{F}{10}$$
$$\Rightarrow e^{MFt/100} M = \frac{100}{F}\cdot\frac{F}{10}\cdot e^{Ft/100} + C = 10e^{Ft/100} + C$$
$$\Rightarrow M = 10 + Ce^{-Ft/100}$$
$$M(0) = 30 = 10 + C \Rightarrow C = 20 \Rightarrow M(t) = 10 + 20e^{-Ft/100}$$

Gestel die water word drinkbaar wanneer die konsentrasie vuilgoed in die water onder $0.15kg/\ell$ is. Wat is die nodige vloeitempo F sodat die water oor 30 minute drinkbaar is?

1 (b) Suppose the water becomes drinkable when the concentration of the dirt in the water drops below $0.15kg/\ell$. What flow rate F is required so that the water is drinkable in 30 minutes time?

$$M(30) = 15 = 10 + 20e^{-F.30/100}$$
$$\Rightarrow \frac{1}{4} = e^{-3F/10} \Rightarrow F = \frac{10}{3}\log 4 \approx \cancel{0.4519\ \ell/min}$$
$$\approx 4.6210\ \ell/min$$

Beskou nou 'n meer breedvoerige tenk opstelling wat deur die volgende stelsel DVs gemodelleer word

Consider now a more elaborate tank setup modelled by the system of DEs

$$① \quad \frac{da}{dt} = -2a + 2b, \quad ② \quad \frac{db}{dt} = 2a - 5b, \qquad a(0) = 1, \quad b(0) = 0,$$

waar a en b die massa van die vuilgoed in tenk A en B onderskeidelik voorstel.

where a and b are the masses of contaminant in tank A and tenk B, respectively.

(c) Gebruik die metode van eliminasie om die b veranderlike te elimineer, en die tweede order DV

2 **(c)** Use the method of elimination to eliminate the b variable and find a second-order DE

$$\frac{d^2a}{dt^2} + 7\frac{da}{dt} + 6a = 0.$$

wat deur $a(t)$ bevredig word, te vind.

satisfied by $a(t)$.

$$d/dt\,① \Rightarrow a'' = -2a' + 2b' \overset{②}{=} -2a' + 2(2a-5b) = -2a' + 4a - 10b$$

$$\overset{①}{=} -2a' + 4a - \tfrac{10}{2}(a' + 2a) = -7a' - 6a.$$

$$\Rightarrow a'' + 7a' + 6a = 0.$$

(d) Vind massas $a(t)$ en $b(t)$ deur die DV in (c) op te los (of op 'n ander manier).

4 **(d)** By solving the DE in part (c) (or otherwise) hence obtain solutions for the masses $a(t)$ and $b(t)$.

Try $a = e^{mt} \Rightarrow m^2 + 7m + 6 = 0 \Rightarrow (m+6)(m+1) = 0$

$\Rightarrow m = -1, 6$

$\Rightarrow a = c_1 e^{-t} + c_2 e^{-6t}$

$b = \tfrac{1}{2}(a' + 2a) = \tfrac{1}{2}(-c_1 e^{-t} - 6c_2 e^{-6t} + 2c_1 e^{-t} + 2c_2 e^{-6t})$

$= \tfrac{1}{2} c_1 e^{-t} - 2c_2 e^{-6t}$

$a(0) = 1 \Rightarrow c_1 + c_2 = 1$
$b(0) = 0 \Rightarrow c_1 - 4c_2 = 0$
$\Rightarrow 5c_2 = 1 \Rightarrow c_2 = 1/5$
$\Rightarrow c_1 = 4/5$

$a(t) = \frac{4}{5}e^{-t} + \frac{1}{5}e^{-6t}$, $\quad b(t) = \frac{2}{5}e^{-t} - \frac{2}{5}e^{-6t}$

Prob 5 (12 punte/marks)

Beskou die DV: $y'' + 4y' + 4y = 0$.

Consider the DE: $y'' + 4y' + 4y = 0$.

(a) Bevestig dat $y_1 = e^{-2x}$ en $y_2 = xe^{-2x}$ oplossings is.

(2) **(a) Verify** that $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are solutions.

$y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x} \Rightarrow y_1'' + 4y_1' + 4y = (4-8+4)e^{-2x} = 0$ ✓

$y_2' = e^{-2x}(1-2x)$, $y_2'' = e^{-2x}(-2+4x-2) = 4e^{-2x}(x-1)$

$\Rightarrow y_2'' + 4y_2' + 4y_2 = e^{-2x}(4x-4+4-8x+4x) = 0$ ✓

(b) Wys dat die oplossings in (a) fundamentele oplossings is deur die gepaste Wronskian te bereken, en skryf die **algemene** oplossing vir die DV neer.

(2) **(b)** By calculating the appropriate Wronskian, show that the solutions in (a) are **fundamental** solutions and write down the general solution to the DE.

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = e^{-2x}e^{-2x}(1-2x) + 2\cancel{x}e^{-2x}e^{-2x}$

$= e^{-4x} \neq 0 \Rightarrow$ fundamental

$\Rightarrow y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

(c) Los die aanvangswaardeprobleem op:

(2) **(c)** Solve the initial value problem:

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

$y(0) = c_1 + 0.c_2 = 1 \Rightarrow c_1 = 1$

$y'(0) = -2c_1 + c_2 = 0 \Rightarrow c_2 = 2c_1 = 2$

$\Rightarrow y(x) = e^{-2x} + 2xe^{-2x}$

$= e^{-2x}(1+2x)$.

(3) **(d)** Vind die spesifieke oplossing van die DV

(d) Find a particular solution to the DE

$$y'' + 4y' + 4y = 4x + 25\sin(x).$$

Try $y_p = A + Bx + C\cos x + D\sin x$

$y_p' = B - C\sin x + D\cos x$

$y_p'' = -C\cos x - D\sin x$

$\Rightarrow y_p'' + 4y_p' + 4y_p = 4A + 4Bx + 4C\cos x + 4D\sin x + 4B - 4C\sin x + 4D\cos x - C\cos x - D\sin x$

$= (4A + 4B) + 4Bx + (3C + 4D)\cos x + (3D - 4C)\sin x$

$= 4x + 25\sin x$

$\Rightarrow \left.\begin{array}{l} 4A + 4B = 0 \\ 4B = 4 \end{array}\right\} \Rightarrow B = 1 \Rightarrow A = -1$

$\left.\begin{array}{l} 3D - 4C = 25 \\ 3C + 4D = 0 \end{array}\right\} \Rightarrow \cancel{16D} \left.\begin{array}{l} 9D - 12C = 75 \\ 12C + 16D = 0 \end{array}\right\} \Rightarrow 25D = 75 \Rightarrow D = 3$

$\Rightarrow C = -4$

$\Rightarrow \boxed{y_p = x - 1 + 3\sin x - 4\cos x}$

(2) **(e)** Los die AWP op

(e) Solve the IVP

$$y'' + 4y' + 4y = 4x + 25\sin(x), \quad y(0) = 0, \quad y'(0) = 0.$$

$y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x - 1 + 3\sin x - 4\cos x$

$\Rightarrow y(0) = c_1 - 1 - 4 = 0 \Rightarrow c_1 = 5$

$y'(0) = -2c_1 + c_2 + 1 + 3 \Rightarrow c_2 = 2c_1 - 4 = 6$

$\Rightarrow y = 5e^{-2x} + 6xe^{-2x} + x - 1 + 3\sin x - 4\cos x$

(1) **(f)** Deur 'n nuwe veranderlike (of andersins) in te voer, herschryf die DE vanaf deel (e) as 'n stelsel van eerste orde vergelykings.

(f) By introducing a new variable (or otherwise) rewrite the DE from part (e) as a system of first order equations.

$\left.\begin{array}{l} z_1 = y \\ z_2 = y' \end{array}\right. \Rightarrow \begin{array}{l} z_1' = y' = z_2 \\ z_2' = y'' = 4x + 25\sin x - 4y' - 4y \\ \quad = 4x + 25\sin x - 4z_2 - 4z_1 \end{array}$

Bonus Prob (4 punte/marks)

Los die AWP op

Solve the IVP

$$y'' + 4y' + 4y = \frac{\log(x)}{xe^{2x}}, \qquad y(1) = 0, \quad y'(1) = 0.$$

~~Life is too short...~~

From Problem 5, $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$, $W(x) = e^{-4x}$

Method of variation of parameters:

$$y_p = u_1y_1 + u_2y_2 \text{ where } u_1 = \int -\frac{y_2 f}{W}\,dx = \int -\frac{xe^{-2x}\log x}{xe^{2x}e^{-4x}}\,dx = -\int \log x\,dx$$

$$\overset{i.p}{=} -[x\log x] + \int 1\,dx = x(1-\log x)$$

$$u_2 = \int \frac{y_1 f}{W}\,dx = \int \frac{e^{-2x}\log x}{xe^{2x}e^{-4x}}\,dx = \int \frac{\log x}{x}\,dx$$

$$= \frac{1}{2}\int \frac{d}{dx}(\log^2 x)\,dx = \tfrac{1}{2}\log^2 x.$$

$$\Rightarrow y_p = xe^{-2x}(1-\log x) + \tfrac{1}{2}xe^{-2x}\log^2 x = xe^{-2x}(1-\log x + \tfrac{1}{2}\log^2 x)$$

$$\Rightarrow y = y_c + y_p = c_1e^{-2x} + c_2xe^{-2x} + xe^{-2x}(1-\log x + \tfrac{1}{2}\log^2 x)$$

$$= c_1e^{-2x} + c_3xe^{-2x} + xe^{-2x}(-\log x + \tfrac{1}{2}\log^2 x)$$

Initial conditions:

$$y(1) = 0 = c_1e^{-2} + c_3e^{-2} + 0 \Rightarrow c_1 = -c_3$$

$$y'(x) = -2c_1e^{-2x} + c_3(1-2x)e^{-2x} + (1-2x)e^{-2x}(-\log x + \tfrac{1}{2}\log^2 x) + xe^{-2x}\left(-\tfrac{1}{x} + \frac{\log x}{x}\right)$$

$$y'(1) = -2c_1e^{-2} - c_3e^{-2} - e^{-2} = 0$$

$$\Rightarrow 2c_1 + c_3 = -1 \Rightarrow c_1 = -1,\ c_3 = 1$$

$$\Rightarrow \boxed{y(x) = e^{-2x}(-1 + x - x\log x + \tfrac{1}{2}\log^2 x).}$$