
Problem 1:

(a) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, with $A = \begin{bmatrix} -2 & -2 \\ -2 & -5 \end{bmatrix}$. $\Delta = \det(A) = 6,\ \ \tau = \text{trace}(A) = -7$ and $\frac{1}{4}\tau^2 = 12\frac{1}{4} > \Delta$
$\implies$ The point $(0,0)$ is a stable node.

(b) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, with $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$. $\Delta = \det(A) = 2,\ \ \tau = \text{trace}(A) = 3$ and $\frac{1}{4}\tau^2 = 2\frac{1}{4} > \Delta$
$\implies$ The point $(0,0)$ is an unstable node.

(c) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, with $A = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$. $\Delta = \det(A) = -3,\ \ \tau = \text{trace}(A) = -2$
$\implies$ The point $(0,0)$ is a saddle point.

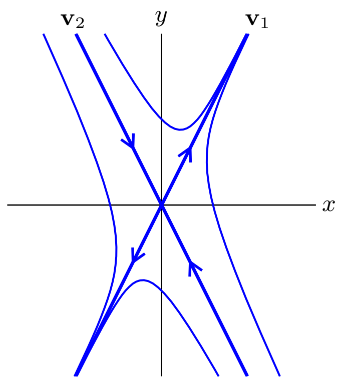
(d) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$, with $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $\Delta = \det(A) = 2,\ \ \tau = \text{trace}(A) = 2$ and $\frac{1}{4}\tau^2 = 1 < \Delta$
$\implies$ The point $(0,0)$ is an unstable spiral.

Problem 2:

For A in problem 2(c) the eigenvectors are given (in the hint) as:
$\mathbf{v}_1 = [1\ \ 2]^T$ and $\mathbf{v}_2 = [-1\ \ 2]^T$.
$A\mathbf{v}_1 = [1\ \ 2] = \mathbf{v}_1 \implies \lambda_1 = 1$,
and $A\mathbf{v}_2 = [3\ \ -6] = -3\mathbf{v}_2 \implies \lambda_2 = -3$.

Solution: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Note: $e^{-3t} \to 0$, so the direction of $\mathbf{v}_1$ dominates for large t.



Problem 3:

The eigenvectors of A in problem 2(d) are given (in the hint) as $\mathbf{v}_1 = [1,\ -i]^T$ and $\mathbf{v}_2 = [1,\ i]^T$, and
$A\mathbf{v}_1 = [1+i,\ 1-i]^T = (1+i)\mathbf{v}_1,\quad A\mathbf{v}_2 = [1-i,\ 1+i]^T = (1-i)\mathbf{v}_2 \implies \lambda_1 = 1+i,\quad \lambda_2 = 1-i.$

$$\begin{aligned}
\text{Hence } \begin{bmatrix} x \\ y \end{bmatrix} &= c_1 e^{(1+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= e^t \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ -ic_1 e^{it} + ic_2 e^{-it} \end{bmatrix} \\
&= e^t \begin{bmatrix} c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t) \\ -ic_1(\cos t + i\sin t) + ic_2(\cos t - i\sin t) \end{bmatrix} \\
&= e^t \begin{bmatrix} (c_1+c_2)\cos t + i(c_1-c_2)\sin t \\ (c_1+c_2)\sin t - i(c_1-c_2)\cos t \end{bmatrix} \\
&= e^t \left(d_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + d_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \right)
\end{aligned}$$

Now we can see that x and y both exhibit simple harmonic motion with growing amplitude (because of the e^t factor), and this indicates unstable spiral behaviour in the phase plane.

Problem 4:

(a) We seek solutions of the nonlinear system of equations: $-3x + 8xy = 0$ and $2y - 4xy = 0$, that is to say $x(8y-3) = 0$ and $y(2-4x) = 0$. Only real solutions: $(x, y) = (0, 0)$ and $(x, y) = \left(\frac{1}{2}, \frac{3}{8}\right)$.

(b) In order to classify these two critical points we first determine an expression for the Jacobian:

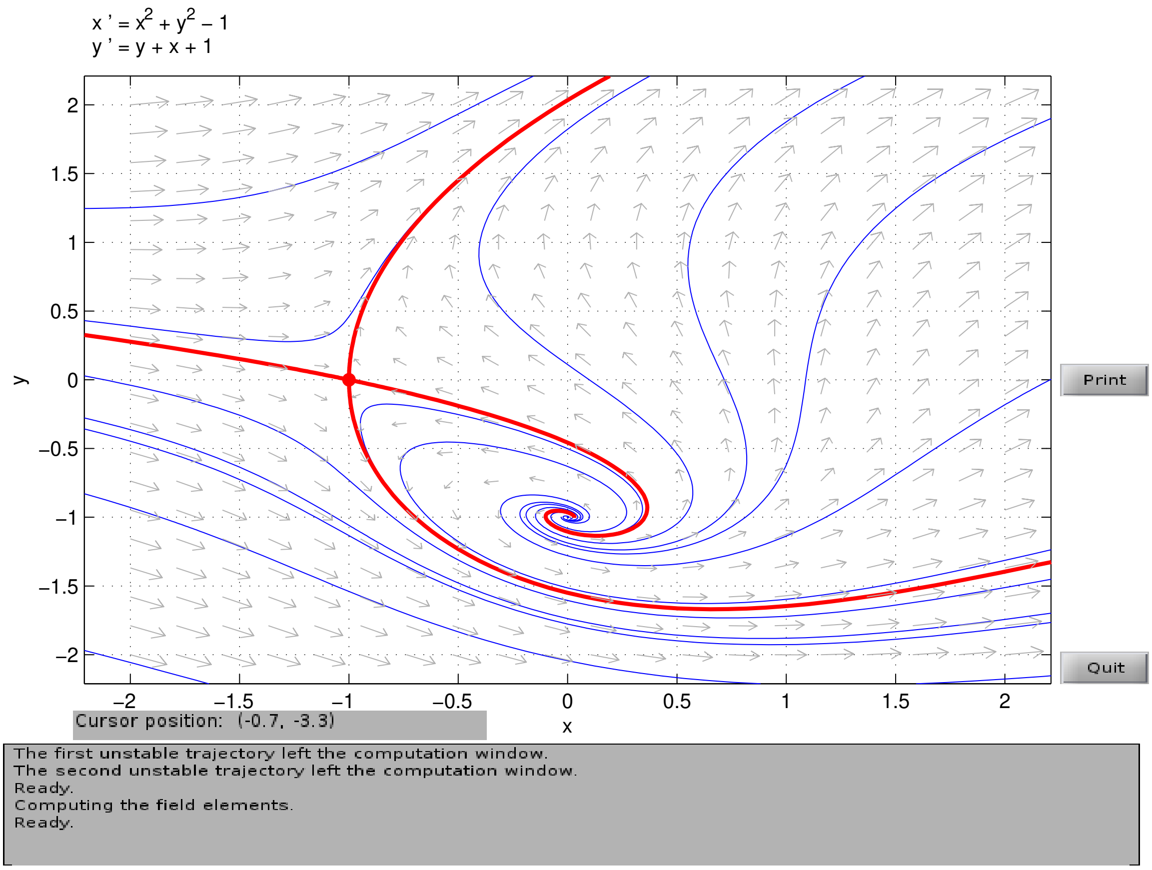
$$J = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} -3+8y & 8x \\ -4y & 2-4x \end{bmatrix}$$

At the point $(x, y) = (0, 0)$ we have $J = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$, and $\det(J) = -6 < 0 \implies (0, 0)$ is a saddle point.

At $(x, y) = \left(\frac{1}{2}, \frac{3}{8}\right)$ we have $J = \begin{bmatrix} 0 & 4 \\ -\frac{3}{2} & 0 \end{bmatrix}$, so $\det(J) = 6$ and $\mathrm{trace}(J) = 0$, which is a borderline case!

It may seem like a centre, but at this stage we cannot make a concrete claim about the classification.

Problem 5:



Problem 6:

Separate variables: $\displaystyle\int y\,dy = \int (x^2 + 3x^3)\,dx \implies \frac{1}{2}y^2 = x^2 + \frac{3}{4}x^4 + c_1 \implies y^2 = x^2(2 + \frac{3}{2}x^2) + c^2.$

Problem 7:

$$x'' = -4x^3 \implies \begin{array}{l} x' = y \\ y' = -4x^3 \end{array} \implies \frac{dy}{dx} = \frac{-4x^3}{y} \implies \frac{1}{2}y^2 = -\frac{1}{4}x^4 + c.$$

Choose an initial condition $(x_0, 0)$ so that $y(x_0) = 0$ then

$$\frac{1}{2}y^2(x_0) = 0 = -\frac{1}{4}x_0^4 + c \implies c = \frac{1}{4}x_0^4 \implies y^2(x) = \frac{1}{2}(x_0^4 - x^4) = \frac{1}{2}(x_0^2 - x^2)(x_0^2 + x^2).$$

Notice that

- $y(-x_0) = y(x_0) = 0$
- If $|x| < |x_0|$ then $y^2(x) > 0$, hence $y(x)$ has +ve and -ve solutions.

We conclude that a initial condition starting at $(x_0, 0)$ forms a closed orbit in the phase plane, therefore there is a periodic solution. In this case, this is true for any x_0.

Problem 8:

We have

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) = E\psi(x).$$

Taking $x = x_c\tilde{x}$ and $\tilde{\psi}(\tilde{x}) = \psi(x) \implies \dfrac{d^2}{dx^2} = \dfrac{1}{x_c^2}\dfrac{d^2}{\tilde{x}^2} = \dfrac{1}{x_c^2}\psi(x) = \dfrac{d^2}{d\tilde{x}^2}\tilde{\psi}(\tilde{x})$.

Hence

$$\left(-\frac{\hbar^2}{2m}\frac{1}{x_c^2}\frac{d^2}{d\tilde{x}^2} + \frac{1}{2}x_c^2 m\omega^2\tilde{x}^2\right)\tilde{\psi}(\tilde{x}) = E\tilde{\psi}(\tilde{x}).$$

Divide through by $\dfrac{\hbar^2}{2m}\dfrac{1}{x_c^2}$ to make the coefficient of the first term (negative) one, i.e.,

$$\left(-\frac{d^2}{d\tilde{x}^2} + \frac{m\omega^2 x_c^4}{\hbar^2}\tilde{x}^2\right)\tilde{\psi}(\tilde{x}) = \frac{2mx_c^2E}{\hbar^2}\tilde{\psi}(\tilde{x}).$$

Choose x_c to make the term in front of $\tilde{x}^2$ unitary, i.e,.

$$\frac{m\omega^2 x_c^4}{\hbar^2} = 1 \implies x_c = \sqrt{\frac{\hbar}{m\omega}}.$$

Hence

$$\left(-\frac{d^2}{d\tilde{x}^2} + \tilde{x}^2\right)\tilde{\psi}(\tilde{x}) = \frac{2m\left(\frac{\hbar}{m\omega}\right)x_c^2E}{\hbar^2}\tilde{\psi}(\tilde{x}) = \frac{2E}{\hbar\omega}\tilde{\psi}(\tilde{x}) = \tilde{E}\tilde{\psi}(\tilde{x}),$$

where $\tilde{E} = \frac{2}{\hbar\omega}E$.