Problem 1:

(a) $\dfrac{dy}{dx} = e^x e^{-2y} \implies \displaystyle\int e^{2y}dy = \int e^x dx \implies \tfrac{1}{2}e^{2y} = e^x + c' \implies e^{2y} = 2e^x + c \implies y = \tfrac{1}{2}\ln(2e^x + c).$

(b) $\dfrac{dy}{dx} - \dfrac{1}{x}y = x\sin(x)$. Integration factor: $e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \sin(x) \implies \frac{d}{dx}\left[\frac{y}{x}\right] = \sin(x) \implies \frac{y}{x} = -\cos(x) + c \implies y = x(c - \cos(x)).$$

(c) $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x^2}$. Integration factor: $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$x\frac{dy}{dx} + y = \frac{1}{x} \implies \frac{d}{dx}[xy] = \frac{1}{x} \implies xy = \ln|x| + c \implies y = \frac{1}{x}(\ln|x| + c).$$

Problem 2:

(a) $\dfrac{dN}{dt} = -\lambda N \implies N(t) = N_0 e^{-\lambda t}$. We also know that $N_0 = N(0) = 100$ and $N(6) = 0.97 \times N_0 = 97$, i.e. $97 = 100e^{-\lambda(6)}$, so that $\lambda = 0.0050765$. Therefore, $N(24) = 100e^{-\lambda(24)} = 88.53\,\text{mg}$.

(b) The half-life is the time t for which $100e^{-\lambda t} = 50$. Hence $t = \frac{1}{\lambda}\ln(100/50) = 136.54\,\text{hours}$.

Problem 3:

$$\frac{dT}{dt} + kT = kT_m. \text{ Integration factor: } e^{\int k\,dt} = e^{kt} \implies \frac{d}{dt}\left[e^{kt}T\right] = ke^{kt}T_m \implies e^{kt}T = e^{kt}T_m + c$$

$$T(0) = T_0 :\ T_0 = T_m + c \implies c = T_0 - T_m.\quad e^{kt}T = e^{kt}T_m + T_0 - T_m \implies T(t) = T_m + (T_0 - T_m)e^{-kt}.$$

Problem 4:

Let $T = T(t)$ represent the core temperature, where t is measured in hours from the time of death. According to Newton's law, with $T_0 = 37$ and $T_m = 21$, we have $T(t) = 21 + 16e^{-kt}$. Let $t = t^*$ be the time 16:00.

We are given that $T(t^*) = 29$ and $T(t^* + 1) = 26.5$, hence $29 - 21 = 16e^{-kt^*}$ and $26.5 - 21 = 16e^{-k(t^*+1)}$. Divide the second equation by the first, so that $\frac{26.5-21}{29-21} = e^{-k}$, that is $k = \ln(8/5.5) = 0.374693$.

Now we can determine t^* from $29 - 21 = 16e^{-kt^*} \implies t^* = \frac{1}{k}\ln(16/8) = 1.849905$.

The time of death ($t = 0$) is therefore 1.849905 hours before 16:00, which is approximately $\boxed{14{:}09}$.

Problem 5:

Let $T = T(t)$ represent the temperature of the thermometer.

We first consider t as being the time elapsed since the first minute (when the temperature was 55). Then $T_0 = 55$, $T_m = 5$, and we have $T(5-1=4) = 5 + (55-5)e^{-4k} = 30 \implies k = 0.25\ln(2) \approx 0.173$.

We next consider t as being the time elapsed since we left the room, and we seek T_0. Now we have $T(1) = 5 + (T_0 - 5)e^{-k} = 55$. Substituting the value of k we found above and solving, we have $T_0 = 50e^{\frac{1}{4}\ln 2} + 5 \approx 64.46$.

Problem 6:

While the bar is in A we have: $dT_1/dt = k_1(0 - T_1) = -k_1T_1$, with $T_1(0) = 100 \implies T_1(t) = 100e^{-k_1t}$.

After 1 min the bar's temperature is 90°C, hence $90 = 100e^{-k_1}$ so that $k_1 = \ln(100/90) = 0.105361$.

After 2 min the temperature is $T(2) = 100e^{-2k_1} = 81$. We now instantly transfer it to B.

The temperature of the bar is now described by the DE: $dT_2/dt = k_2(100 - T_2)$ with $T_2(0) = 81$.

Note: here we measure time from the moment of transfer. We therefore get $T_2(t) = 100 + (81 - 100)e^{-k_2 t}$. After 1 min the temperature increases by $10°$, hence $91 = 100 - 19e^{-k_2}$ so that $k_2 = \ln(19/9) = 0.747214$.

How long does it take to reach 99.9°C? We seek the time t for which $99.9 = 100 - 19e^{-k_2 t}$, therefore $t = 7.02$ min after transfer, i.e. $\boxed{9.02\,\text{min}}$ from the start of the entire process.

Problem 7:

We now have $\frac{dm}{dt} = 0.1F - \frac{m}{1500}F$, $m(0) = 450$. Solving again by integrating factor (omitted) we have $m(t) = 150 + 300e^{-tF/1500}$. We require that $m(60) = 0.2 \times 1500 = 300 = 150 + 300e^{-60F/1500}$ and solving for F we find $F = \frac{1500}{60}\ln(2) \approx 17.3l/min$.