Applied differential equations

TW244 - Lecture 12

3.3: Systems of first-order DEs

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Until now we've had only one dependent variable (one DE), e.g.,

$$\frac{dx}{dt} = f(t,x), \qquad \text{solution: } x = x(t).$$

Now consider more than one dependent variable* (system of DEs), e.g.,

$$\begin{cases} \frac{dx}{dt} = f(t,x,y) \\ \frac{dy}{dt} = g(t,x,y) \end{cases}, \qquad \text{solution: } x = x(t) \;\&\; y = y(t).$$

Example:

$$\begin{cases} \frac{dx}{dt} = 4x + 2y, & x(0) = 10 \\ \frac{dy}{dt} = 3x + 3y, & y(0) = -5 \end{cases}.$$

Systems of DEs will allow us to model interactions between different populations / species / substances / etc. (e.g., predator-prey, SIR.)

*Not to be confused with more than one independent variable, which gives a PDE!

3.3: Systems of first-order DEs
An example

Example:

$$\begin{cases} \frac{dx}{dt} = 4x + 2y, & x(0) = 10 \\ \frac{dy}{dt} = 3x + 3y, & y(0) = -5 \end{cases}, \qquad \text{solution: } \begin{cases} x(t) = 4e^{6t} + 6e^{t} \\ y(t) = 4e^{6t} - 9e^{t} \end{cases}.$$

Let's verify this:

$\frac{dx}{dt} = 24e^{6t} + 6e^{t}$ and $4x + 2y = 16e^{6t} + 24e^{t} + 8e^{6t} - 18e^{t} = 24e^{6t} + 6e^{t}$. ✓
$\frac{dy}{dt} = 24e^{6t} - 9e^{t}$ and $3x + 3y = 12e^{6t} + 18e^{t} + 12e^{6t} - 27e^{t} = 24e^{6t} - 9e^{t}$. ✓

and

$$x(0) = 4e^{0} + 6e^{0} = 4 + 6 = 10. \checkmark$$
$$y(0) = 4e^{0} - 9e^{0} = 4 - 9 = -5. \checkmark$$

In general...

For convenience we often write a system with n independent variables:

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n),$$
$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n),$$

$$\underline{f}(t, \underline{x}) = \begin{bmatrix} f_1(t, x_1, x_2, \ldots) \\ f_2(t, x_1, x_2, \ldots) \\ \vdots \\ f_n(\ldots) \end{bmatrix}$$

in vector form:

$$\frac{d}{dt}(\underline{x}) = \underline{f}(t, \underline{x}) \qquad \text{where} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Be aware that some other authors may use bold fonts to denote vectors, rather than underlines, i.e., $\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$.

Aside

Systems of first-order DEs are very important for numerical work because any n^{th}-order DE can be written as a system of n first-order equations, and so most algorithms† for IVP DEs are designed to solve systems of the form $\frac{dx}{dt} = \underline{f}(t, \underline{x}),\ \ \underline{x}(t_0) = \underline{x}_0$. For more details, see TW324 next year.

Exercise: By introducing a new variable, $y = x'$, show that the second-order DE‡

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = \cos(t),$$

can be written as the system of DEs:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \cos(t) - 2\gamma y - \omega^2 x. \end{aligned}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2}$$

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†Including MATLAB's `ode45` - but there are exceptions.

‡We will see this DE again in Lectures [illegible]3.

A substance decays by radioactivity to form another substance, which also decays to form a third substance, until a stable element is reached. [See discussion on p. 106]

Suppose initially that there is an amount x_0 of x, which decays to form y, which decays to form z, which is stable, and let $x(t)$, $y(t)$, and $z(t)$ be the amount of the substances at time t. Therefore

$$\begin{aligned} \frac{dx}{dt} &= \underline{-\lambda_1 x} \\ \frac{dy}{dt} &= \underline{\lambda_1 x} - \underline{\lambda_2 y} \\ \frac{dz}{dt} &= \underline{\lambda_2 y}. \end{aligned}$$

The DEs are linear, so we may write (see Section 8.1 in Z&W):

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}} = \overbrace{\underbrace{\begin{pmatrix} -\lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}}}^{f(t,\underline{x})}$$

Solution: (by forward substitution)

$$\frac{dx}{dt} = -\lambda_1 x, \quad x(0) = x_0 \implies \boxed{x(t) = x_0 e^{-\lambda_1 t}.}$$

Now solve for y:

$$\begin{aligned} \frac{dy}{dt} &= \lambda_1 x - \lambda_2 y, \qquad y(0) = 0 \\ &= \lambda_1 x_0 e^{-\lambda_1 t} - \lambda_2 y \\ \implies \frac{dy}{dt} + \lambda_2 y &= \lambda_1 x_0 e^{-\lambda_1 t} \end{aligned}$$

Application 1: Radioactive series (cont.)

Integrating factor: $e^{\int \lambda_2 \, dt} = e^{\lambda_2 t}$

$$
\begin{aligned}
\Longrightarrow \quad \frac{d}{dt}\left(e^{\lambda_2 t} y\right) &= \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t} \\
\Longrightarrow \quad e^{\lambda_2 t} y &= \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C
\end{aligned}
$$

$\lambda_1 \neq \lambda_2.$

Initial condition: $y(0) = 0 \implies 0 = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} + C \implies C = -\frac{x_0 \lambda_1}{\lambda_2 - \lambda_1}$.

$$
\Longrightarrow \quad e^{\lambda_2 t} y = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{(\lambda_2 - \lambda_1)t} - 1\right)
$$

$$
\Longrightarrow \quad \boxed{y(t) = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).}
$$

Did you notice that in step 2 we assumed $\lambda_1 \neq \lambda_2$?

Exercise: What happens if $\lambda_1 = \lambda_2$?

Finally, we solve for z:

$$
\begin{aligned}
\frac{dz}{dt} &= \lambda_2 y, \qquad z(0) = 0 \\
&= \frac{x_0 \lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \\
\Longrightarrow z &= \frac{x_0 \lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(-\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t}\right) + C
\end{aligned}
$$

Initial condition: $z(0) = 0 \Longrightarrow C = -\frac{x_0 \lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(-\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)$

$$
\Longrightarrow \boxed{z(t) = \frac{x_0 \lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1}\left(1 - e^{-\lambda_1 t}\right) - \frac{1}{\lambda_2}\left(1 - e^{-\lambda_2 t}\right)\right).}
$$

Solutions:

$$
\begin{aligned}
x(t) &= x_0 e^{-\lambda_1 t} \\
y(t) &= \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \\
z(t) &= \frac{x_0 \lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1}\left(1 - e^{-\lambda_1 t}\right) - \frac{1}{\lambda_2}\left(1 - e^{-\lambda_2 t}\right)\right)
\end{aligned}
$$

Note that (exercise: show this):

$$
\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} z(t) = x_0
$$

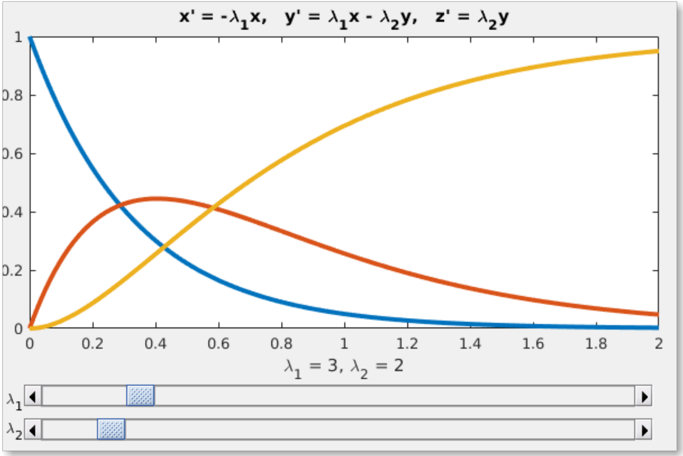
which is what we expect from the physics, and so is a good validation of our model / solution.

Application 1: Radioactive series (cont.)

Solution to the system of DEs:

$$\dot{x} = -\lambda_1 x, \quad \dot{y} = \lambda_1 x - \lambda_2 y, \quad \dot{z} = \lambda_2 y$$

with $\lambda_1 = 3, \quad \lambda_2 = 2, \quad x(0) = 1, \quad y(0) = 0, \quad z(0) = 0.$

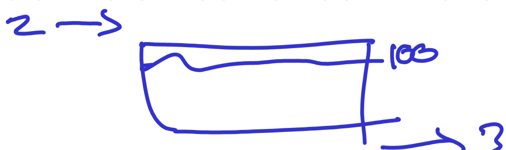


Application 2:

$v(t)$ = volume of water
$d(t)$ = mass of dirt in tank

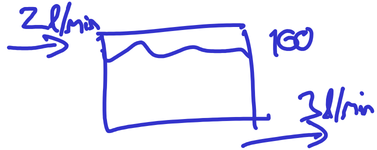
Exercise: (a) A 100 litre water tank is filled by a stream at a constant rate of 2 litres per minute. If the tank starts off full and water is pumped out at a constant rate of 3 litres per minute, write down an IVP describing the volume of water $v(t)$ (litres) in the tank at any given time t (mins), and solve the IVP.

(b) Assume now that the water in the tank is dirty, with an initial concentration of $0.2kg$ of dirt per litre. The water coming in from the stream is cleaner, and only contains $0.1kg$ of dirt per litre. Assuming that the water in the tank is well-mixed, write down and solve an IVP describing the mass of the dirt contained in the tank at any given time t.



$M(0) = 0.2\,kg/\ell * 100\,\ell$
$= 20\,kg$

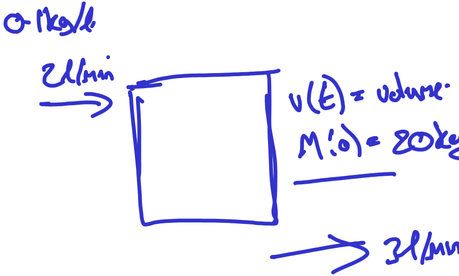
a)



$$\frac{dv}{dt} = \text{rate in} - \text{rate in} = 2 - 3 = -1 \ \ l/\text{min}$$

$$\left.\begin{aligned} v(0) &= 100 \\ v(t) &= -t + C_1 \end{aligned}\right\} \Rightarrow \boxed{v(t) = 100 - t.}$$

b)



$v(t)$ = volume

$M(0) = 20\text{kg}$

$$\frac{dM}{dt} = \text{rate in} - \text{rate out}$$

$$= (2l/\text{min}) \times (0.1\text{kg}/l) - (3l/\text{min}) \times \frac{M(t)}{v(t)}$$

$$= 0.2\text{kg}/\text{min} - \frac{M(t)}{v(t)} \ \text{kg}/\text{m}$$

$$* \quad \boxed{\begin{aligned} \frac{dv}{dt} &= -1, & v(0) &= 100 \\ dM/dt &= 0.2 - \frac{M(t)}{100-t} & M(0) &= 20 \end{aligned}}$$

Exercise: solve * using IF.