Semiconductor Laser Rate Equations: Fourth-Order Runge-Kutta Solution

Detailed Analysis of `Chapt9Exercise4.m`

Generated Documentation

Abstract

This document provides comprehensive theoretical foundations and line-by-line code analysis for `Chapt9Exercise4.m`, which solves the single-mode semiconductor laser rate equations using fourth-order Runge-Kutta integration. The code simulates carrier and photon density dynamics in response to step current injection, revealing fundamental laser phenomena including turn-on delay, relaxation oscillations, and steady-state clamping.

Contents

1 Theoretical Foundation

1.1 Axioms of Laser Rate Equation Theory

Axiom 1 (Conservation of Carriers). *In a semiconductor active region, the rate of change of carrier density n equals the injection rate minus recombination rates:*

$$\frac{dn}{dt} = R_{inj} - R_{rec} \tag{1}$$

where carriers are neither created nor destroyed except through well-defined physical processes.

Axiom 2 (Conservation of Photons). *The rate of change of photon density S equals the generation rate (stimulated emission) minus loss rate:*

$$\frac{dS}{dt} = R_{gen} - R_{loss} \tag{2}$$

Axiom 3 (Einstein Relations). *Stimulated emission and absorption rates are proportional to photon density, with the coefficient determined by material gain.*

1.2 Fundamental Definitions

Definition 1 (Carrier Injection Rate). *For current I injected into an active volume V:*

$$R_{inj} = \frac{I}{eV} = \frac{\eta_i I}{eV} \tag{3}$$

where η_i is the internal quantum efficiency (assumed unity here).

Definition 2 (Carrier Recombination Rate). *Total recombination includes non-radiative, radiative, and Auger processes:*

$$R_{rec} = \frac{n}{\tau_n} = An + Bn^2 + Cn^3 \tag{4}$$

where A is the non-radiative rate, B is the radiative coefficient, and C is the Auger coefficient.

Definition 3 (Optical Gain with Compression). *The material gain including gain compression is:*

$$g = \Gamma g_0(n - n_0)(1 - \varepsilon S) \tag{5}$$

where Γ is the confinement factor, g_0 is the gain slope, n_0 is the transparency density, and ε is the gain compression factor.

Definition 4 (Photon Decay Rate). *The total photon loss rate in the cavity:*

$$\kappa = \frac{v_g}{n_g}(\alpha_i + \alpha_m) = \frac{c}{n_g^2}(\alpha_i + \alpha_m) \tag{6}$$

where $\alpha_m = \frac{1}{2L} \ln \frac{1}{R_1 R_2}$ is the mirror loss.

1.3 Core Theorems

Theorem 1 (Single-Mode Rate Equations). *The coupled rate equations for carrier density n and photon density S are:*

$$\frac{dn}{dt} = \frac{I}{eV} - \frac{n}{\tau_n} - g \cdot S \tag{7}$$

$$\frac{dS}{dt} = (g - \kappa)S + \beta R_{sp} \tag{8}$$

where β is the spontaneous emission coupling factor.

Proof. From Axiom 1, the carrier equation follows from:

- Injection: $I/(eV)$
- Spontaneous recombination: $n/\tau_n = An + Bn^2 + Cn^3$
- Stimulated emission: gS (removes carriers to create photons)

From Axiom 2, the photon equation includes:

- Net stimulated emission: $(g - \kappa)S$ where gain g provides amplification
- Spontaneous emission into the mode: $\beta R_{sp} = \beta Bn^2$

□

Theorem 2 (Threshold Condition)**.** *Lasing threshold occurs when gain equals loss:*

$$g_{th} = \kappa = \frac{v_g}{n_g}(\alpha_i + \alpha_m) \tag{9}$$

The threshold carrier density is:

$$n_{th} = n_0 + \frac{\kappa}{\Gamma g_0 v_g / n_g} \tag{10}$$

Theorem 3 (Threshold Current)**.** *The threshold current satisfies the steady-state condition:*

$$I_{th} = eV \cdot \frac{n_{th}}{\tau_n} = eV(An_{th} + Bn_{th}^2 + Cn_{th}^3) \tag{11}$$

Theorem 4 (Relaxation Oscillation Frequency)**.** *Above threshold, small perturbations oscillate at frequency:*

$$\omega_R = \sqrt{\frac{gS_0}{\tau_p} - \frac{1}{4\tau_n^2}} \approx \sqrt{\frac{\Gamma g_0 v_g S_0}{n_g}} \tag{12}$$

where S_0 is the steady-state photon density and $\tau_p = 1/\kappa$ is the photon lifetime.

1.4 Fourth-Order Runge-Kutta Method

Theorem 5 (Classical RK4 Algorithm)**.** *For the system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, the fourth-order Runge-Kutta update is:*

$$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n) \tag{13}$$
$$\mathbf{k}_2 = \mathbf{f}(t_n + h/2, \mathbf{y}_n + h\mathbf{k}_1/2) \tag{14}$$
$$\mathbf{k}_3 = \mathbf{f}(t_n + h/2, \mathbf{y}_n + h\mathbf{k}_2/2) \tag{15}$$
$$\mathbf{k}_4 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_3) \tag{16}$$
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \tag{17}$$

This achieves $O(h^5)$ local truncation error and $O(h^4)$ global error.

2 Line-by-Line Code Analysis

2.1 Header and Physical Constants

```
%  Chapt9Exercise4.m
%  solves single-mode laser diode rate equations
%  calls runge4.m which is a 4th order Runge-Kutta integrator
%  plots carrier density and photon density as function of time.
```

Documentation header describing the rate equation solver.

```
%  gain=g=gamma*gslope*(n-n0)*(1-epsi*s)
%  carrier density, n, current, I, and photon density, s:
%  fcn=n'=(I/ev)-(n/tau_n)-(g*s)
%  fcs=s'=(g-K)*s+beta*Rsp
```

Equations implemented:

$$g = \Gamma g_0(n - n_0)(1 - \varepsilon S) \tag{18}$$

$$\frac{dn}{dt} = \frac{I}{eV} - \frac{n}{\tau_n} - gS \tag{19}$$

$$\frac{dS}{dt} = (g - \kappa)S + \beta R_{sp} \tag{20}$$

```
clear;
clf;

%declare constants
hconstjs=6.6262e-34;       %Planck's constant (J s)
echargec=1.6021e-19;       %electron charge (C)
vlightcm=2.997925e10;      %velocity of light (cm s-1)
```

Physical constants:

$$h = 6.6262 \times 10^{-34} \text{ J·s} \tag{21}$$

$$e = 1.6021 \times 10^{-19} \text{ C} \tag{22}$$

$$c = 2.997925 \times 10^{10} \text{ cm/s} \tag{23}$$

```
nsteps=10000;              %number of time-steps
```

Total integration steps: 10,000 for 10 ns simulation (1 ps per step).

2.2 Device Parameters

```
ngroup     = 4;            %refractive index

clength    = 3.00E-02;     %cavity length (cm)
thick      = 1.40E-05;     %thickness of active region (cm)
width      = 8.00E-05;     %width of active region (cm)
```

Cavity geometry:

$$n_g = 4 \quad \text{(group index)} \tag{24}$$

$$L = 300 \ \mu\text{m} \tag{25}$$

$$d = 140 \text{ nm (active layer)} \tag{26}$$

$$w = 800 \text{ nm (ridge width)} \tag{27}$$

Active volume: $V = L \cdot w \cdot d = 3.36 \times 10^{-11}$ cm^3.

```
tincrement = 1.00E-12;     %time increment (s)
initialn   = 0.00E+18;     %initial carrier density (cm-3)
```

Time step $\Delta t = 1$ ps, starting from empty cavity.

```
Anr        = 2.00E+08;     %non-radiative rate (s-1)
Bcons      = 1.00E-10;     %radiative coefficient (cm3 s-1)
Ccons      = 1.00E-29;     %Auger coefficient (cm6 s-1)
```

Recombination parameters:

$$A = 2 \times 10^8 \text{ s}^{-1} \quad \text{(SRH recombination)} \tag{28}$$

$$B = 10^{-10} \text{ cm}^3/\text{s} \quad \text{(radiative)} \tag{29}$$

$$C = 10^{-29} \text{ cm}^6/\text{s} \quad \text{(Auger)} \tag{30}$$

Carrier lifetime at $n = 10^{18}$ cm^{-3}:

$$\tau_n = \frac{1}{A + Bn + Cn^2} \approx 3.3 \text{ ns} \tag{31}$$

```
n0density  = 1.00E+18;     %transparency density (cm-3)
gslope     = 2.50E-16;     %gain-slope coefficient (cm2 s-1)
epsi       = 5.00E-18;     %gain compression (cm3)
beta       = 1.00E-04;     %spontaneous emission factor
gamma_cons = 0.25;         %optical confinement factor
wavelength = 1.31;         %emission wavelength (um)
```

Gain parameters:

$$n_0 = 10^{18} \text{ cm}^{-3} \tag{32}$$

$$g_0 = 2.5 \times 10^{-16} \text{ cm}^2 \tag{33}$$

$$\varepsilon = 5 \times 10^{-18} \text{ cm}^3 \tag{34}$$

$$\beta = 10^{-4} \tag{35}$$

$$\Gamma = 0.25 \tag{36}$$

$$\lambda = 1.31\ \mu\text{m} \tag{37}$$

```
mirrone    = 0.32;         %reflectivity mirror 1
mirrtwo    = 0.32;         %reflectivity mirror 2
alfa_i     = 40;           %internal loss (cm-1)
```

Optical losses: For cleaved facets $R_1 = R_2 = 0.32$, internal loss $\alpha_i = 40$ cm^{-1}.

```
pstart1    = 2.5;          %start of first current step (ns)
pstart2    = 3.5;          %start of second current step (ns)

rioff1     = 0;            %off value of current (mA)
rion1      = 20;           %first step current (mA)
rion2      = 20;           %second step current (mA)
```

Current pulse profile: Step from 0 to 20 mA at $t = 2.5$ ns, maintained at 20 mA.

2.3 Initialization and Loss Calculations

```
sdensity = 0.0;
ndensity = initialn * 1.0e-21;

alfa_m   = log( 1.0/(mirrone * mirrtwo) ) / ( 2.0 * clength );
alfasum  = alfa_m + alfa_i;
```

Mirror loss calculation:

$$\alpha_m = \frac{1}{2L} \ln \frac{1}{R_1 R_2} = \frac{\ln(1/0.1024)}{0.06} = 38.0 \text{ cm}^{-1} \tag{38}$$

Total loss: $\alpha_{tot} = 40 + 38 = 78 \text{ cm}^{-1}$.

2.4 Unit Rescaling

```
%rescale: time(ns), current(mA), length(nm)
echarge     = echargec*1.0e9*1.0e3;      % rescaled charge
vlight      = vlightcm/1.0e9/1.0e-7;     % nm/ns

tincrement = tincrement * 1.0e9;
gslope      = gslope * 1.0e14;
n0density  = n0density * 1.0e-21;
```

Unit conversions for numerical stability:

- Time: s $\to$ ns
- Length: cm $\to$ nm
- Density: $\text{cm}^{-3} \to \text{nm}^{-3}$ (factor 10^{-21})
- Current: A $\to$ mA

```
clength     = clength / 1.0e-7;
width       = width / 1.0e-7;
thick       = thick / 1.0e-7;

Anr         = Anr / 1.0e9;
Bcons       = Bcons / (1.0e9*1.0e-21);
Ccons       = Ccons / (1.0e9*1.0e-42);
```

Length in nm, recombination coefficients rescaled for ns timescale.

```
epsi        = epsi / 1.0e-21;
gslope      = gslope * vlight / ngroup;
alfasum     = alfasum * 1.0e-7;
kappa       = alfasum * vlight / ngroup;
volume      = clength * width * thick;
```

Key derived quantities:

$$\kappa = \frac{\alpha_{tot} \cdot v_g}{n_g} = \frac{78 \times 10^{-7} \times 3 \times 10^{17}}{4} \text{ ns}^{-1} \tag{39}$$

Photon lifetime: $\tau_p = 1/\kappa \approx 1.7$ ps.

2.5 Light Output Scaling

```
tmp    = hconstjs * (vlightcm * 1.0e-2)/(wavelength * 1.0e-6);   % photon energy
tmp    = tmp * alfa_m * vlightcm;                                 % rate factor
tmp    = tmp * 1000.0*volume/ngroup;                              % mW scaling
lightout = tmp*(1.0-mirrone)/(2.0-mirrone-mirrtwo);               % mirror 1
```

Power output formula: The output power from mirror 1:

$$P_1 = \frac{hc}{\lambda} \cdot \alpha_m v_g S V \cdot \frac{1 - R_1}{(1 - R_1) + (1 - R_2)} \tag{40}$$

2.6 Data Array for RK4

```
data(1)=gamma_cons;
data(2)=gslope;
data(3)=n0density;
data(4)=epsi;
data(5)=kappa;
data(6)=Bcons;
data(7)=Anr;
data(8)=Ccons;
data(9)=volume;
data(10)=beta;
```

Parameters passed to the Runge-Kutta integrator function.

2.7 Main Integration Loop

```
for i = 1 : 1 : nsteps
   time(i) = ( double(i-1) +1.0 ) * tincrement;
   current(i) = rioff1;

   if time(i) > pstart1
      current(i) = rion1;
   end

   if time(i) > pstart2
      current(i) = rion2;
   end
```

Current pulse generation:

$$I(t) = \begin{cases} 0 & t < 2.5 \text{ ns} \\ 20 \text{ mA} & t \geq 2.5 \text{ ns} \end{cases} \tag{41}$$

```
   [sdensity, ndensity]=runge4(current(i), sdensity, ndensity, tincrement, data);
   carrier(i) = ndensity * 1.0e21/1.0e18;
   photon(i) = (lightout*sdensity);
end;
```

RK4 integration step: The function `runge4` implements:

$$\frac{dn}{dt} = \frac{I}{eV} - (A + Bn + Cn^2)n - \Gamma g_0(n - n_0)(1 - \varepsilon S)S \tag{42}$$

$$\frac{dS}{dt} = [\Gamma g_0(n - n_0)(1 - \varepsilon S) - \kappa]S + \beta Bn^2 \tag{43}$$

2.8 Output Analysis

```
photon_number = photon(nsteps-1)*volume/lightout  %photons at end
carrier_number = carrier(nsteps-1)*volume         %carriers at end
```

Final state analysis: total photon and carrier numbers in the cavity.

3 Numerical Methods

3.1 Runge-Kutta Implementation

The `runge4` function solves the coupled system with state vector $\mathbf{y} = (S, n)^T$:

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} (g - \kappa)S + \beta Bn^2 \\ I/(eV) - (A + Bn + Cn^2)n - gS \end{pmatrix} \tag{44}$$

The RK4 update for each time step:

$$\mathbf{k}_1 = \mathbf{f}(S_n, n_n) \tag{45}$$

$$\mathbf{k}_2 = \mathbf{f}(S_n + \frac{h}{2}k_1^{(S)}, n_n + \frac{h}{2}k_1^{(n)}) \tag{46}$$

$$\mathbf{k}_3 = \mathbf{f}(S_n + \frac{h}{2}k_2^{(S)}, n_n + \frac{h}{2}k_2^{(n)}) \tag{47}$$

$$\mathbf{k}_4 = \mathbf{f}(S_n + hk_3^{(S)}, n_n + hk_3^{(n)}) \tag{48}$$

$$\begin{pmatrix} S_{n+1} \\ n_{n+1} \end{pmatrix} = \begin{pmatrix} S_n \\ n_n \end{pmatrix} + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \tag{49}$$

3.2 Stability Considerations

The stiff nature of laser rate equations requires:

$$\Delta t < \min\left(\tau_p, \frac{1}{\omega_R}\right) \tag{50}$$

With $\tau_p \approx 1.7$ ps and $\omega_R \sim 10$ GHz, the 1 ps time step is adequate.

4 Physical Interpretation

4.1 Turn-On Dynamics

The simulation reveals several distinct phases:

1. **Pre-injection ($t < 2.5$ ns):** Both n and S remain at zero.
2. **Carrier buildup ($2.5 < t < t_{th}$ ns):** After current injection, carriers accumulate:

$$n(t) \approx \frac{I\tau_n}{eV}\left(1 - e^{-t/\tau_n}\right) \tag{51}$$

3. **Threshold crossing:** When $n = n_{th}$, gain equals loss and photon density begins exponential growth.

4. Relaxation oscillations: The carrier-photon coupling produces damped oscillations at frequency:

$$f_R = \frac{1}{2\pi}\sqrt{\frac{\Gamma g_0 v_g S_0}{n_g}} \approx 2 - 5 \text{ GHz} \tag{52}$$

5. Steady state: Carrier density clamps near n_{th}, photon density reaches equilibrium.

4.2 Gain Compression Effects

The factor $(1 - \varepsilon S)$ models spectral hole burning:

$$g_{eff} = \Gamma g_0 (n - n_0) \cdot \frac{1}{1 + \varepsilon S} \tag{53}$$

This limits maximum gain and affects modulation bandwidth.

5 Summary

This code implements a complete single-mode laser rate equation solver featuring:

- Fourth-order Runge-Kutta integration for high accuracy
- Realistic device parameters (InGaAsP 1310 nm laser)
- ABC recombination model
- Gain compression for nonlinear saturation
- Spontaneous emission coupling to lasing mode
- Power output calculation from cavity photon density

The simulation captures key laser dynamics: turn-on delay, relaxation oscillations at GHz frequencies, and steady-state operation above threshold.