3

3.1

3.2

3.3

3.4

3.5

3.1

-

  -
  -
  -

- n

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

n $\boldsymbol{Ax} = \boldsymbol{b}$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

3.1

- (Gramer) A

  $\det(A)\neq 0$ $Ax = b$

$$x_1 = \frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})} \quad x_2 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} \quad \cdots \quad x_n = \frac{\det(\boldsymbol{A}_n)}{\det(\boldsymbol{A})}$$

$$\boldsymbol{A}_j = \begin{pmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix} (j = 1,2,\ldots,n)$$

b $\boldsymbol{A}$ j n $a_{1(j-1)}$

1 $j-1$

- $\det(\boldsymbol{A}), \det(\boldsymbol{A}_1), \cdots, \det(\boldsymbol{A}_n)$, $n+1$

  $n!$ $n-1$

3.1

-
  -
    -
    -
    -
  -
    -
    -
    -

3.2

例3.1

$$\begin{cases} x_1 + x_2 + x_3 = 11 & (1) \\ x_1 + 4x_2 - 2x_3 = 5 & (2) \\ 3x_1 - 3x_2 + 4x_3 = 15 & (3) \end{cases}$$

:

- 1 2 3 x_1 $(1) \times (-1) + (2), (1) \times (-3) + (3),$ 1

$$\begin{cases} x_1 + x_2 + x_3 = 11 & (1) \\ \quad 3x_2 - 3x_3 = -6 & (4) \\ \quad -6x_2 + x_3 = -18 & (5) \end{cases}$$

- 4 5 x_2 $(4) \times 2 + (5)$

1 4

3.2

$$\begin{cases} x_1 + x_2 + x_3 = 11 & (1) \\ 3x_2 - 3x_3 = -6 & (4) \\ -5x_3 = -30 & (6) \end{cases}$$

6 $x_3 = 6$ 4 $x_2 = x_3 - 2 =$

4 x_2 x_3 1 $x_1 = 1.$

3.2

- ❑
  - ➢
  - ➢
  - ➢ 3.1

$$(\boldsymbol{A}|\boldsymbol{b}) = \begin{pmatrix} 1 & 1 & 1 & 11 \\ 1 & 4 & -2 & 5 \\ 3 & -3 & 4 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 11 \\ 0 & 3 & -3 & -6 \\ 0 & -6 & 1 & -18 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 1 & 1 & 11 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & -5 & -30 \end{pmatrix}$$

3.2

- n

$$\begin{cases} a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 + \cdots + a_{1n}^{(0)}x_n = b_1^{(0)} \\ a_{21}^{(0)}x_1 + a_{22}^{(0)}x_2 + \cdots + a_{2n}^{(0)}x_n = b_1^{(0)} \\ \vdots \\ a_{n1}^{(0)}x_1 + a_{n2}^{(0)}x_2 + \cdots + a_{nn}^{(0)}x_n = b_n^{(0)} \end{cases} \tag{3.1}$$

$$\boldsymbol{A}^{(0)}\boldsymbol{x} = \boldsymbol{b}^{(0)}$$

3.1

$$\left(\boldsymbol{A}^{(0)}\middle|\boldsymbol{b}^{(0)}\right) = \begin{pmatrix} a_{11}^{(0)} & \cdots & a_{1n}^{(0)} & b_1^{(0)} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1}^{(0)} & \cdots & a_{n2}^{(0)} & b_n^{(0)} \end{pmatrix}$$

3.2

❑

➢ 1 $a_{11}^{(0)} \neq 0$ $l_{i1} = a_{i1}^{(0)} / a_{11}^{(0)} (i = 2,3 \cdots n)$

$\left(\boldsymbol{A}^{(0)} \middle| \boldsymbol{b}^{(0)}\right)$ $-l_{i1}$ i $(i = 2,3 \cdots n)$

$$\left(\boldsymbol{A}^{(1)} \middle| \boldsymbol{b}^{(1)}\right) = \begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \cdots & a_{1n}^{(0)} & b_1^{(0)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & a_n^{(0)} & & & \end{pmatrix}$$

3.2

- 2 1

  2 n 0

$$a_{ij}^{(1)} = a_{ij}^{(0)} - l_{i1}a_{1j}^{(0)}(i,j = 2,3 \cdots n)$$

$$b_i^{(1)} = b_i^{(0)} - l_{i1}b_1^0(i = 2,3 \cdots n)$$

$\boldsymbol{A}^{(0)}\boldsymbol{x} = \boldsymbol{b}^{(0)}$ $\boldsymbol{A}^{(1)}\boldsymbol{x} = \boldsymbol{b}^{(1)}$

- k-1

  $\boldsymbol{A}^{(0)}\boldsymbol{x} = \boldsymbol{b}^{(0)}$ $\boldsymbol{A}^{(k-1)}\boldsymbol{x} = \boldsymbol{b}^{(k-1)}$

3.2

❑

$$\left(\boldsymbol{A}^{(k-1)}\middle|\boldsymbol{b}^{(k-1)}\right)=\begin{pmatrix} a_{11}^{(0)} & \cdots & a_{1k}^{(0)} & \cdots & a_{1n}^{(0)} & b_1^{(0)} \\ & \ddots & \vdots & \ddots & \vdots & \vdots \\ & & a_{kk}^{(k-1)} & \cdots & a_{kn}^{(k-1)} & b_k^{(k-1)} \\ & & \vdots & \ddots & \vdots & \vdots \\ & & a_{nk}^{(k-1)} & \cdots & a_{nn}^{(k-1)} & b_n^{(k-1)} \end{pmatrix}$$

➢ k $a_{kk}^{(k-1)}\neq 0$ $l_{ik}=a_{ik}^{(k-1)}/a_{kk}^{(k-1)}(i=k+1,k+2,\cdots,n)$
$\left(\boldsymbol{A}^{(k-1)}\middle|\boldsymbol{b}^{(k-1)}\right)$ k $-l_{ik}$ i $(i=k+1,k+2,\cdots,n)$

$$\left(\boldsymbol{A}^{(k)}\middle|\boldsymbol{b}^{(k)}\right)=\begin{pmatrix} a_{11}^{(0)} & \cdots & a_{1k}^{(0)} & a_{1(k+1)}^{(0)} & \cdots & a_{1n}^{(0)} & b_1^{(0)} \\ & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & a_{kk}^{(k-1)} & a_{k(k+1)}^{(k-1)} & \cdots & a_{kn}^{(k-1)} & b_k^{(k-1)} \\ & & & a_{(k+1)(k+1)}^{(k)} & \cdots & a_{(k+1)n}^{(k)} & b_{k+1}^{(k)} \\ & & & \vdots & \ddots & \vdots & \vdots \\ & & & a_{n(k+1)}^{(k)} & \cdots & a_{nn}^{(k)} & b_n^{(k)} \end{pmatrix}$$

3.2

❑

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik}a_{kj}^{(k-1)}(i, j = k+1, k+2, \cdots, n)$$

$$b_i^{(k)} = b_i^{(k-1)} - l_{ik}b_k^{k-1}(i = k+1, k+2, \cdots, n)$$

n-1

$\boldsymbol{A}^{(n-1)}\boldsymbol{x} = \boldsymbol{b}^{(n-1)}$

$$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \cdots & a_{1n}^{(0)} \\ & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ & & \ddots & \vdots \\ & & & a_{nn}^{(n-1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(1)} \\ \vdots \\ b_n^{(n-1)} \end{pmatrix}$$

3.2

❑

$$k = 1,2,\cdots,n-1$$

$$\begin{cases} l_{ik} = a_{ik}^{(k-1)}/a_{kk}^{(k-1)}(i = k+1,\cdots,n) \\ a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik}a_{kj}^{(k-1)}(i,j = k+1,\cdots,n) \\ b_i^{(k)} = b_i^{(k-1)} - l_{ik}b_k^{k-1}(i = k+1,\cdots,n) \end{cases} \qquad (3.2)$$

3.2

-

  -

$$\begin{cases} x_n = b_n^{(n-1)} / a_{nn}^{(n-1)} \\ x_k = \left(b_k^{(k-1)} - \sum_{j=k+1}^{n} a_{kj}^{(k-1)} x_j\right) / a_{kk}^{(k-1)} (k = n-1, n-2, \cdots, 1) \end{cases} \quad (3.3)$$

  -

$a_{kk}^{(k-1)} \neq 0 (k = 1,2,\cdots,n-1)$ $\boldsymbol{A}$

3.2

定理3.1 n $\boldsymbol{A}$

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \cdots, \det(\boldsymbol{A}) \neq 0$$

$$\boldsymbol{A}x = b$$

$$a_{kk}^{(k-1)} \neq 0 \ (k = 1,2,\cdots,n)$$

❑

➢

- ❖ k l_{ik} $n-k$
- ❖ $a_{ij}^{(k)}$ $(n-k)^2$
- ❖ $b_i^{(k)}$ $n-k$

3.2

$\sum_{k=1}^{n-1}\left[(n-k)+(n-k)^2+(n-k)\right] = 2\sum_{k=1}^{n-1}(n-k)+\sum_{k=1}^{n-1}(n-k)^2$

$= 2[(n-1)+(n-2)+\cdots+2+1]+\left[(n-1)^2+(n-2)^2+\cdots+2^2+1^2\right]$

$= 2\times\frac{1}{2}n(n-1)+\frac{1}{6}n(n-1)(2n-1)$

$= \frac{1}{3}n^3+\frac{1}{2}n^2-\frac{5}{6}n$

- ➢
  - ❖ x_k 1 $n-k$

$$\sum_{k=1}^{n}(n-k)+n = \frac{1}{2}n(n-1)+n = \frac{1}{2}n^2+\frac{1}{2}n$$

- ➢

$$\frac{1}{3}n^3+\frac{1}{2}n^2-\frac{5}{6}n+\frac{1}{2}n^2+\frac{1}{2}n = \frac{1}{3}n^3+n^2-\frac{1}{3}n$$

3.2

❑

$$(n+1)n!\,(n-1)+n$$

	3	10	20	50
	17	430	3060	44150
	51	359251210	9.7×10^{20}	7.9×10^{67}

➢ 12.5

❖ 20 $\frac{3060}{1.25\times10^{5}}\approx$

$0.02448s$ 0.02s

❖ 2 4

3.2

例3.2

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 4 \\ 4x_1 + 9x_2 + 5x_3 - 7x_4 = 7 \\ -x_1 + 2x_2 + 4x_3 - 6x_4 = 3 \\ x_1 + 3x_2 - 3x_3 + 4x_4 = -11 \end{cases}$$

:

$$(\boldsymbol{A}|\boldsymbol{b}) = \begin{pmatrix} 1 & 2 & 1 & 1 & 4 \\ 4 & 9 & 5 & -7 & 7 \\ -1 & 2 & 4 & -6 & 3 \\ 1 & 3 & -3 & 4 & -11 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & -11 & -9 \\ 0 & 0 & 1 & 39 & 43 \\ 0 & 0 & 0 & 209 & 209 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 4 \\ x_2 + x_3 - 11x_4 = -9 \\ x_3 + 39x_4 = 43 \\ 209x_4 = 209 \end{cases}$$

$x_1 = 3 \quad x_2 = -2 \quad x_3 = 4$

$x_4 = 1$

3.2

❑

➢

➢ $a_{kk}^{(k-1)} = 0$

$a_{kk}^{(k-1)} \neq 0$

例3.3

$$\begin{cases} 0.0001x_1 + x_2 = 1 \\ \qquad\qquad x_1 + x_2 = 2 \end{cases}$$

3.2

-

x_1 =10000/9999 x_2 =9998/9999

- 3 x_1

$$\begin{cases} 0.0001x_1 + x_2 = 1 \\ -10000x_2 = -10000 \end{cases}$$

$x_2 = 1$ x_1=0

- 0.0001

-

$$\begin{cases} x_1 + x_2 = 2 \\ 0.0001x_1 + x_2 = 1 \end{cases}$$

3.2

- x_1

$$\begin{cases} x_1 + x_2 = 2 \\ \quad\;\; x_2 = 1 \end{cases}$$

- $x_1 = x_2 = 1$

❑ $\left|\frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}\right| > 1$时

- $a_{kj}^{(k-1)}$ $e_{\mathrm{k}-1}$, $a_{ik}^{(k)} = a_{ik}^{(k-1)}$ - $l_{ik}\, a_{kj}^{(k-1)}$, $a_{ik}^{(k)}$ $e_{\mathrm{k}-1}$ $e_{\mathrm{k}} = |l_{ik}|\, e_{\mathrm{k}-1}$

- $b_i^{(k)}$ $b_i^{(k-1)}$

- $|l_{ik}| < 1$ $\left|a_{kk}^{(k-1)}\right| > \left|a_{ik}^{(k-1)}\right| (i = k+1, k+2, \dots, n)$

3.2

例3.4

$$\begin{cases} 10x_1 - 3x_2 + x_3 = -1 \\ -20x_1 - 3x_2 - 2x_3 = 8 \\ x_1 + x_2 + x_3 = 4 \end{cases}$$

： $(\boldsymbol{A}|\boldsymbol{b}) = \begin{pmatrix} 10 & 3 & 1 & -1 \\ -20 & -3 & -2 & 8 \\ 1 & 1 & 1 & 4 \end{pmatrix} \xrightarrow{1,2} \begin{pmatrix} -20 & -3 & -2 & 8 \\ 10 & 3 & 1 & -1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -20 & -3 & -2 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 7 & 9 & 41 \end{pmatrix} \xrightarrow{2,3} \begin{pmatrix} -20 & -3 & -2 & 8 \\ 0 & 7 & 9 & 41 \\ 0 & 1 & 0 & 2 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -20 & -3 & -2 & 8 \\ 0 & 7 & 9 & 41 \\ 0 & 0 & -9 & 27 \end{pmatrix}$

x_1=-1 x_2=2 x_3=3

3.2 -

-

- Gauss-Jordan

3.3

-
- 3.1

  $\boldsymbol{A}$

-

$a_{11} \neq 0, l_{i1} \neq a_{i1}^{(0)}/a_{11}^{(0)}(i = 2,3,\cdots,n)$

$$\boldsymbol{L}_1 = \begin{pmatrix} 1 & & & \\ -l_{21} & 1 & & \\ \vdots & & \ddots & \\ -l_{n1} & & & 1 \end{pmatrix}$$

- L_1

  $(A^{(0)}, b^{(0)})$

$$\boldsymbol{L}_1(\boldsymbol{A}^{(0)}, \boldsymbol{b}^{(0)}) = (\boldsymbol{A}^{(1)}, \boldsymbol{b}^{(1)})$$

3.3

- $a_{kk}^{(k-1)} \neq 0,\ l_{ik} \neq a_{ik}^{(k-1)}/a_{kk}^{(k-1)}(i = k+1, k+2, \cdots, n),$

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{(k+1)k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{nk} & & & 1 \end{pmatrix}$$

- k $\boldsymbol{L}_k$ $(\boldsymbol{A}^{(k-1)}, \boldsymbol{b}^{(k-1)}),$

$$\boldsymbol{L}_k(\boldsymbol{A}^{(k-1)}, \boldsymbol{b}^{(k-1)}) = (\boldsymbol{A}^{(k)}, \boldsymbol{b}^{(k)})$$

- n-1

$$\boldsymbol{L}_{n-1}\, \boldsymbol{L}_{n-2} \cdots (\boldsymbol{A}^{(0)}, \boldsymbol{b}^{(0)}) = (\boldsymbol{A}^{(n-1)}, \boldsymbol{b}^{(n-1)})$$

3.3

- $\boldsymbol{L}_k(k=1,2,\cdots,n-1)$

$$\boldsymbol{L}_k^{-1}=\begin{pmatrix}1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{(k+1)k} & 1 & & \\ & & \vdots & & \ddots & \\ & & l_{nk} & & & 1\end{pmatrix}$$

-

$$(\boldsymbol{A}^{(0)},\boldsymbol{b}^{(0)})=\boldsymbol{L}_1^{-1}\,\boldsymbol{L}_2^{-1}\cdots\boldsymbol{L}_{n-1}^{-1}(\boldsymbol{A}^{(n-1)},\boldsymbol{b}^{(n-1)})$$

$$\boldsymbol{L}=\boldsymbol{L}_1^{-1}\,\boldsymbol{L}_2^{-1}\cdots\boldsymbol{L}_{n-1}^{-1}=\begin{pmatrix}1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{n(n-1)} & 1\end{pmatrix}$$

3.3

➢

$$U=A^{(n-1)},Y=b^{(n-1)}$$

➢

$$(A^{(0)},b^{(0)})=L(A^{(n-1)},b^{(n-1)})=(LU,LY)$$

➢ $a_{kk}^{(k-1)}\neq 0(k=1,2,\cdots,n-1)$

A

L U

3.3

定理3.2 $\boldsymbol{A}$ n $\boldsymbol{A}$ $\boldsymbol{D_i} \neq 0(i = 1,2,\cdots,n)$, $\boldsymbol{A}$ $\boldsymbol{L}$ $\boldsymbol{U}$

- $\boldsymbol{Ax}=\boldsymbol{b}$,

$$\boldsymbol{Ax}=\boldsymbol{b} \underset{A=LU}{\longleftrightarrow} \boldsymbol{LUx}=\boldsymbol{b} \underset{y=Ux}{\longleftrightarrow} \begin{cases} \boldsymbol{Ly}=\boldsymbol{b} \\ \boldsymbol{Ux}=\boldsymbol{y} \end{cases}$$

- $\boldsymbol{A}=\boldsymbol{LU}$

  $\boldsymbol{Ly}=\boldsymbol{b}$

- $\boldsymbol{Ux}=\boldsymbol{y}$,

3.3

-

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & a_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ & & & \ddots & \vdots \\ & & & & u_{nn} \end{pmatrix} = \boldsymbol{LU} \quad (3.3)$$

3.3

- (3.3)
  - 1
    - ***L*** 1 ***U*** j ***A***

      $a_{1j} = u_{1j}$, ***U***

      $$a_{1j} = u_{1j}\ (j = 1,2,\cdots,n)$$
    - L i **U** 1 ***A***

      $a_{i1} = l_{i1}u_{11}$, ***L*** 1

      $$l_{i1} = a_{i1}/u_{11}\ (i = 2,3,\cdots,n)$$
    - 1 ***U*** 1 ***L*** 1
  - 2
    - $a_{2j} = l_{21}u_{1j} + u_{2j}$ ***U*** 2

      $$u_{2j} = a_{2j} - l_{21}u_{1j}\ (j = 2,3,\cdots,n)$$

3.3

❖ $a_{i2} = l_{i1}u_{12} + l_{i2}u_{22}$ **L**

$$l_{i2} = (a_{i2} - l_{i2}u_{22})/u_{22}\ (i = 3,4,\cdots,n)$$

U 2 **L** 2

➢ *k* **U** *k*-1 **L** *k*-1 $1 \le k \le n$

$$a_{kj} = \sum_{r=1}^{n} l_{kr}u_{rj} (j \le k),$$ **L**

$r > k$时 $l_{kr} = 0$ $l_{kk} = 1$

$$a_{kj} = \sum_{r=1}^{k-1} l_{kr}u_{rj} + u_{kj}$$

$$u_{kj} = a_{kj} - \sum_{r=1}^{k-1} l_{kr}u_{rj}\ (j = k, k+1, \cdots n),$$

U *k*

3.3

- ➢ $a_{ik} = \sum_{r=1}^{n} l_{ir} u_{rk} (i > k)$, $\boldsymbol{U}$ $r > k$时 $u_{rk} = 0$

$$a_{ik} = \sum_{r=1}^{k} l_{ir} u_{rk} = \sum_{r=1}^{k-1} l_{ir} u_{rk} + l_{ik} u_{kk},$$

$$l_{ik} = (a_{ik} - \sum_{r=1}^{k-1} l_{ir} u_{rk}) / u_{kk} \ (i = k + 1, k + 2, \cdots n),$$

L k

- ❑ n $\boldsymbol{L}$ $\boldsymbol{U}$

$\boldsymbol{A}$ $\boldsymbol{LU}$

- ➢ $k = 1,2, \cdots n,$

$$\begin{cases} u_{kj} = a_{kj} - \sum_{r=1}^{k-1} l_{kr} u_{rj} (j = k, k + 1, \cdots n) \\ l_{ik} = (a_{ik} - \sum_{r=1}^{k-1} l_{ir} u_{rk}) / u_{kk} \ (i = k + 1, k + 2, \cdots n) \end{cases}$$

3.3

- $\boldsymbol{A}$ $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}$ $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$

  $\boldsymbol{L}(\boldsymbol{U}\boldsymbol{x})=\boldsymbol{b}$ $\boldsymbol{L}\boldsymbol{y}=\boldsymbol{b}$ $\boldsymbol{U}\boldsymbol{x}=\boldsymbol{y}$

$$\begin{cases} y_1=b_1 \\ y_k=b_k-\sum_{j=1}^{k-1} l_{kj}y_j \quad (k=2,3,\cdots n) \end{cases}$$

$$\begin{cases} x_n=y_n/u_{nn} \\ x_k=\left(y_k-\sum_{j=k+1}^{n} u_{kj}x_j\right)/u_{kk} \quad (k=n-1,n-2,\cdots 1) \end{cases}$$

例3.5

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 9 & 5 & -7 \\ -1 & 2 & 4 & -6 \\ 1 & 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 3 \\ -11 \end{bmatrix}$$

$\boldsymbol{A}$ $\boldsymbol{LU}$

$$\boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 1 & 1 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -11 \\ 0 & 0 & 1 & 39 \\ 0 & 0 & 0 & 209 \end{bmatrix}$$

$\boldsymbol{Ly} = \boldsymbol{b}$

$$y_1 = 4, y_2 = -9, y_3 = 43, y_4 = 209$$

$\boldsymbol{Ux} = \boldsymbol{y}$

$$x_1 = 3, x_2 = -2, x_3 = 4, x_4 = 1$$

3.3

-

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\mathrm{O}(n^3)$

u_{ij} , l_{ij}, y_i a_{ij} b_j

$\boldsymbol{L}, \boldsymbol{U}$ y $\boldsymbol{A}, \boldsymbol{b}$ x y

3.3

❑

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

$$det(\boldsymbol{A}) = det(\boldsymbol{L}) \times det(\boldsymbol{U}) = u_{11}u_{22}\cdots u_{nn}$$

A

Crout

3.3

❑ Cholesky

定理3.2 $\boldsymbol{A}$

$\boldsymbol{L} \in \mathrm{R}^{n\times n}$

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^{T},$$

$\boldsymbol{L}$ $\boldsymbol{A}$ Cholesky

3.3

❑

➢

$$\boldsymbol{Ax} = \boldsymbol{d}$$

$$\boldsymbol{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$\boldsymbol{A}$

$$\begin{cases} |b_1| > |c_1| > 0 \\ |b_i| \geq |a_i| + |c_i| \qquad (a_i c_i \neq 0) \\ |b_n| > |a_n| > 0 \end{cases}$$

$\boldsymbol{A}$

3.3

- **A** **L** **U** $\boldsymbol{A = LU}$
  - **L** 1 Doolittle
  - **U** 1 Crout
- **A** Crout

$$\boldsymbol{L} = \begin{bmatrix} l_1 & & & & \\ a_2 & l_2 & & & \\ & a_3 & l_3 & & \\ & & \ddots & \ddots & \\ & & & a_n & l_n \end{bmatrix}, \boldsymbol{U} = \begin{bmatrix} 1 & u_1 & & & \\ & 1 & u_2 & & \\ & & 1 & \ddots & \\ & & & \ddots & u_{n-1} \\ & & & & 1 \end{bmatrix}$$

$$\begin{cases} b_1 = l_1 \\ c_i = l_i u_i \\ b_{i+1} = a_{i+1} u_i + l_{i+1} \end{cases} \qquad (i = 1,2,3,\cdots,n-1)$$

3.3

➢

$$\begin{cases} l_1 = b_1 \\ u_i = c_i / l_i \\ l_{i+1} = b_{i+1} - a_{i+1} u_i \end{cases} \qquad (i = 1,2,3,\cdots,n-1)$$

➢ $\boldsymbol{A}$ l_i, u_i $l_1 \to u_1 \to l_2 \to u_2 \to \cdots \to l_{n-1} \to u_{n-1} \to l_n$ “ ”

➢ $\boldsymbol{LUx} = \boldsymbol{d}$ $\boldsymbol{Ux} = \boldsymbol{y}$ $\boldsymbol{Ly} = \boldsymbol{d}$

➢ $\boldsymbol{Ly} = \boldsymbol{d}$

$$\begin{cases} l_1 y_1 = d_1 \\ a_i y_{i-1} + l_i y_i = d_i \end{cases} \qquad (i = 2,\cdots,n)$$

$$\begin{cases} y_1 = d_1 / l_1 \\ y_i = (d_i - a_i y_{i-1}) / l_i \end{cases} \qquad (i = 2,\cdots,n)$$

3.3

➢ $\boldsymbol{Ux} = \boldsymbol{y}$

$$\begin{cases} x_i + u_i x_{i+1} = y_i \\ x_n = y_n \end{cases} \qquad (i = 2, \cdots, n)$$

$$\begin{cases} x_i = y_i - u_i x_{i+1} \\ x_n = y_n \end{cases} \qquad (i = n-1, \cdots, 2, 1)$$

➢ “ ”

3.3

例3.6

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

A LU

$$\boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 7/3 & 0 & 0 \\ 0 & 1 & 15/7 & 0 \\ 0 & 0 & 1 & 31/15 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 6/7 & 0 \\ 0 & 0 & 1 & 14/15 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{Ly} = \boldsymbol{d}$

$$\boldsymbol{y} = [1/3 \quad 5/7 \quad 16/15 \quad 44/31]^{\mathrm{T}}$$

$\boldsymbol{Ux} = \boldsymbol{y}$

$$\boldsymbol{x} = [-9/31 \quad 29/31 \quad -8/31 \quad 44/31]^{\mathrm{T}}$$

3.4

- well-posed
  -
  -
  -
- ill-posed
- ill-conditioned
  -
  -
  - Ax=b A
    ill-conditioned well-conditioned
-

3.4

❑

➢ A b

例3.7

$$Ax = b$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3.0001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6.0001 \end{pmatrix}$$

$x^* = (1,1)^T$ A b

$$\begin{pmatrix} 3 & 3 \\ 3 & 2.9997 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6.0003 \end{pmatrix}$$

$\tilde{x} = (3, -1)^T$

3.4

定义3.1 $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{A}$ $\boldsymbol{b}$ $\|\delta \boldsymbol{A}\|$ $\|\delta \boldsymbol{b}\|$ x $\|\delta \boldsymbol{x}\|$

A

$\|\delta \boldsymbol{x}\|$

A

定义3.2 $\frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|}$ $\frac{\|\delta \boldsymbol{A}\|}{\|\boldsymbol{A}\|}$ $\frac{\|\delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|}$ $\boldsymbol{x}$ $\boldsymbol{A}$

$\boldsymbol{b}$

3.4

- “ ”

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad (3.4)$$

$$(\boldsymbol{A} + \delta\boldsymbol{A})(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b} + \delta\boldsymbol{b} \qquad (3.5)$$

δA A δx δb x b

$A + \delta A$

3.4

❑

➢ $\delta \boldsymbol{A} = 0, \delta \boldsymbol{b} \neq 0, \boldsymbol{b} \neq 0$

❖

$$\boldsymbol{A}(\boldsymbol{x} + \delta \boldsymbol{x}) = \boldsymbol{b} + \delta \boldsymbol{b} \qquad (3.6)$$

❖ (3.6) (3.4)

$$\boldsymbol{A}\delta \boldsymbol{x} = \delta \boldsymbol{b}$$

❖ $\delta \boldsymbol{x} = \boldsymbol{A}^{-1}\delta \boldsymbol{b}$

$$\|\delta \boldsymbol{x}\| \leq \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\delta \boldsymbol{b}\|$$

❖ $Ax = b$

$$\|\boldsymbol{b}\| \leq \|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|$$

❖

$$\|\delta \boldsymbol{x}\| \cdot \|\boldsymbol{b}\| \leq \|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\boldsymbol{x}\| \cdot \|\delta \boldsymbol{b}\|$$

3.4

❖ $\boldsymbol{b} \neq 0$，$\boldsymbol{x} \neq 0$， $\|\boldsymbol{b}\| > 0$，$\|\boldsymbol{x}\| > 0$，

$$\frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\| \cdot \frac{\|\delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|} \quad (3.7)$$

❖ (3.7) b $\|\delta \boldsymbol{b}\|$ $\boldsymbol{x}$ $\boldsymbol{b}$

$\|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\|$

➢ $\delta \boldsymbol{A} \neq 0, \delta \boldsymbol{b} = 0, \boldsymbol{A} \neq 0$

❖

$$(\boldsymbol{A} + \delta \boldsymbol{A})(\boldsymbol{x} + \delta \boldsymbol{x}) = \boldsymbol{b} \quad (3.8)$$

❖ (3.8) (3.4)

$$\delta \boldsymbol{A}(\boldsymbol{x} + \delta \boldsymbol{x}) + \boldsymbol{A} \delta \boldsymbol{x} = 0$$

$$\delta \boldsymbol{x} = -\boldsymbol{A}^{-1} \delta \boldsymbol{A}(\boldsymbol{x} + \delta \boldsymbol{x})$$

❖

$$\|\delta \boldsymbol{x}\| \leq \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\delta \boldsymbol{A}\| \cdot \|\boldsymbol{x} + \delta \boldsymbol{x}\|$$

3.4

❖

$$\frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}+\delta \boldsymbol{x}\|} \le \|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\| \cdot \frac{\|\delta \boldsymbol{A}\|}{\|\boldsymbol{A}\|} \qquad (3.9)$$

(3.9) $\boldsymbol{A}$ $\|\delta \boldsymbol{A}\|$ $\boldsymbol{x}+\delta \boldsymbol{x}$

$\boldsymbol{A}$ $\|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\|$

➢ (3.4) $\boldsymbol{A}$ $\boldsymbol{b}$ $\|\delta \boldsymbol{A}\|$

$\|\delta \boldsymbol{b}\|$ $\left\|\boldsymbol{A}^{-1} \cdot \delta \boldsymbol{A}\right\| \le \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\delta \boldsymbol{A}\| < 1$

$$\frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \le \frac{\|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\|}{1-\|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\| \cdot \frac{\|\delta \boldsymbol{A}\|}{\|\boldsymbol{A}\|}}\left(\frac{\|\delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|}+\frac{\|\delta \boldsymbol{A}\|}{\|\boldsymbol{A}\|}\right) \qquad (3.10)$$

➢ (3.7) (3.9) (3.10) $\|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\|$

$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ “ ” $\boldsymbol{A}, \boldsymbol{b}$

3.4

定义3.3 $\boldsymbol{A}$ n $cond(\boldsymbol{A}) = \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\boldsymbol{A}\|$

$\boldsymbol{A}$ $\|\boldsymbol{A}\|$ A

➢

$$cond_\infty(\boldsymbol{A}) = \|\boldsymbol{A}\|_\infty \cdot \left\|\boldsymbol{A}^{-1}\right\|_\infty$$

$$cond_1(\boldsymbol{A}) = \|\boldsymbol{A}\|_1 \cdot \left\|\boldsymbol{A}^{-1}\right\|_1$$

$$cond_2(\boldsymbol{A}) = \|\boldsymbol{A}\|_2 \cdot \left\|\boldsymbol{A}^{-1}\right\|_2$$

$\boldsymbol{A}$ ∞- 1- 2-

3.4

➢

$\boldsymbol{A}$ $cond(\boldsymbol{A}) \geq 1, cond(\boldsymbol{A}) = cond(\boldsymbol{A}^{-1})$

$\boldsymbol{A}$ $k \neq 0$ $cond(k\boldsymbol{A}) = cond(\boldsymbol{A})$

$\boldsymbol{A}$ $\boldsymbol{U}$

$$cond_2(\boldsymbol{A}) = cond_2(\boldsymbol{UA}) = cond_2(\boldsymbol{AU}), cond_2(\boldsymbol{U}) = 1$$

λ_1 λ_2 $\boldsymbol{A}$

$$cond_2(\boldsymbol{A}) = \frac{|\lambda_1|}{|\lambda_2|}$$

3.4

例 3.8 3.7 $Ax = b$, b $\delta b = (0,0.0001)^T$, $cond_\infty(A)$ δb x

$$A^{-1} = \begin{pmatrix} \frac{30001}{3} & -\frac{30000}{3} \\ -10000 & 10000 \end{pmatrix}$$

$$cond_\infty(A) = \|A\|_\infty \times \left\|A^{-1}\right\|_\infty = 6.0001 \times \frac{60001}{3} \approx 1.20 \times 10^5$$

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq cond_\infty(A) \frac{\|\delta b\|_\delta}{\|b\|_\infty} \approx 1.20 \times 10^5 \times \frac{0.0001}{6.0001} \approx 2 = 200\%$$

0.01% x 200%

“ ” A “ ”

3.4

❑ $\boldsymbol{Ax}=\boldsymbol{b}$

➢

$$cond(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \left\|\boldsymbol{A}^{-1}\right\|$$

➢

➢

$\boldsymbol{A}$

$\boldsymbol{A}$

$\boldsymbol{A}$ $\boldsymbol{A}$

3.4

-

A

P, Q $Ax = b$

$$\begin{cases} PAQy = Pb \\ y = Q^{-1}x \end{cases}$$

$cond(PAQ) < cond(A)$

A A

A $\infty -$ A

A

3.5

-
-

$$\boldsymbol{Ax} = \boldsymbol{b}$$

-

$$\boldsymbol{x} = \boldsymbol{Bx} + \boldsymbol{f}$$

($\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}, \boldsymbol{f} = \boldsymbol{b}$)

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{Bx}^{(k)} + \boldsymbol{f}$$

B f

- $\boldsymbol{x}^{(0)}$ $\left\{\boldsymbol{x}^{(k)}\right\}_0^{\infty}$

$\lim\limits_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{x}^*$ $\boldsymbol{x}^*$

B

3.5

例3.9

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 6 \end{pmatrix} \qquad (3.11)$$

3.11 : $x_1 = 1$ $x_2 = 2$ $x_3 = 3$

x_1 x_2 x_3

$$\begin{cases} x_1 = -\frac{1}{3}x_2 - \frac{1}{3}x_3 + \frac{8}{3} \\ x_2 = -\frac{1}{2}x_1 - \frac{1}{2}x_3 + 4 \\ x_3 = -x_1 - x_2 + 6 \end{cases}$$

$$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})^T = (0,0,0)^T$$

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{3}x_2^{(k)} - \frac{1}{3}x_3^{(k)} + \frac{8}{3} \\ x_2^{(k+1)} = -\frac{1}{2}x_1^{(k)} - \frac{1}{2}x_3^{(k)} + 4 \\ x_3^{(k+1)} = -x_1^{(k)} - x_2^{(k)} + 6 \end{cases}$$

3.5

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}^{(k+1)} = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}^{(k)} + \begin{pmatrix} \frac{8}{3} \\ 4 \\ 6 \end{pmatrix} \qquad (3.12)$$

3.1

3.1 3.9 1

k	**0**	**1**	**2**	⋯	**9**	**10**	⋯
$x_1^{(k)}$	0	2.6667	-0.6667	⋯	-3.8395	6.5062	⋯
$x_2^{(k)}$	0	4	-0.3333	⋯	-4.5185	9.4198	⋯
$x_3^{(k)}$	0	6	-0.6667	⋯	-7.0000	14.3580	⋯

3.1 3.12 $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^T$

3.11 $x^* = (1,2,3)^T$ 3.12

3.5

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i = 1,2,\cdots,n)$$

$\boldsymbol{Ax} = \boldsymbol{b}$ $\boldsymbol{A}$ $a_{ii} \neq 0(i = 1,2,\cdots,n)$, i x_i

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1,j\neq i}^{n} a_{ij}x_j\right) \quad (i = 1,2,\cdots,n)$$

$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})^T$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1,j\neq i}^{n} a_{ij}x_j^{(k)} + b_i\right) \quad (3.13)$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{B_J}\boldsymbol{x}^{(k)} + \boldsymbol{f_J} \qquad (3.14)$$

3.5

- $\boldsymbol{A}$ $\boldsymbol{A} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U}$

$$\boldsymbol{D} = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$-\boldsymbol{L} = \begin{pmatrix} 0 & & & & \\ a_{21} & 0 & & & \\ a_{31} & a_{32} & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ a_{n1} & a_{n2} & \cdots a_{nn-1} & & 0 \end{pmatrix}$$

$$-\boldsymbol{U} = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 0 & a_{23} & \cdots & a_{2n} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & a_{n-1n} \\ & & & & 0 \end{pmatrix}$$

3.5

- $\boldsymbol{Ax = b}$

$$(\boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U})\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{Dx} = (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{x} + \boldsymbol{b}$$

-

$$\boldsymbol{x} = \boldsymbol{D}^{-1}(\boldsymbol{L} + \boldsymbol{U})\boldsymbol{X} + \boldsymbol{D}^{-1}\boldsymbol{b} = \boldsymbol{D}^{-1}(\boldsymbol{D} - \boldsymbol{A})\boldsymbol{x} + \boldsymbol{D}^{-1}\boldsymbol{b}$$
$$= \left(\boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{A}\right)\boldsymbol{x} + \boldsymbol{D}^{-1}\boldsymbol{b} = \boldsymbol{B_J}\boldsymbol{x} + \boldsymbol{f_J}$$

- 3.14 $\boldsymbol{B_J} = \boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{A}, \boldsymbol{f_J} = \boldsymbol{D}^{-1}\boldsymbol{b}$
- 3.13 3.14 Jacobi

  -
  -

3.5

例3.10

$$\begin{pmatrix} -8 & 1 & 1 \\ 1 & -5 & 1 \\ 1 & 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 16 \\ 7 \end{pmatrix}$$

$$\begin{cases} -8x_1 + x_2 + x_3 = 8 \\ x_1 - 5x_2 + x_3 = 16 \\ x_1 + x_2 - 4x_3 = 7 \end{cases}$$

3 x_1 x_2 x_3

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{8}(1 - x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = -\frac{1}{5}(16 - x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{4}(7 - x_1^{(k)} - x_2^{(k)}) \end{cases}$$

$$\boldsymbol{x}^{(k+1)} = \begin{pmatrix} 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{16}{5} & 0 & \frac{16}{5} \\ \frac{7}{4} & \frac{7}{4} & 0 \end{pmatrix}\boldsymbol{x}^{(k)} + \begin{pmatrix} -\frac{1}{8} \\ -\frac{16}{5} \\ -\frac{7}{4} \end{pmatrix}$$

3.5

$\boldsymbol{x}^{(0)} = (0,0,0)^T$ [-1 -4 -3]

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{8}(1 - x_2^{(k)} - x_3^{(k)}) \\ x_2^{(k+1)} = -\frac{1}{5}(16 - x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{4}(7 - x_1^{(k)} - x_2^{(k)}) \end{cases}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0	5	-0.9855	-3.9803	-2.9766
1	-0.125	-3.2	-1.75	6	-0.9946	-3.9924	-2.9914
2	-0.74375	-3.575	-2.58125	7	-0.9979	-3.9972	-2.9967
3	-0.8945	-3.8649	-2.8296	8	-0.9992	-3.9989	-2.9988
4	-0.9618	-3.9448	-2.9398	9	-0.9997	-3.9996	-2.9995

3.5

- -　(Gauss-Seidel)
  -
  - $\boldsymbol{x}^{(k)}$ $\boldsymbol{x}^{(k+1)}$ $\boldsymbol{x}^{(k)}$
    $\boldsymbol{x}^{(k+1)}$
  -
    - i $x_i^{(k+1)}$ $x_1^{(k+1)} \cdots x_{i-1}^{(k+1)}$
      $x_1^{(k+1)} \cdots x_{i-1}^{(k+1)}$ $x_1^{(k)} \cdots x_{i-1}^{(k)}$
  -
  - - Gauss-Seidel

3.5

➢

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)} + b_i\right) (i = 1,2 \cdots, n)$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{B}_{G-S}\, \boldsymbol{x}^{(k)} + \boldsymbol{f}_{G-S} \qquad (3.15)$$

➢ $\boldsymbol{A}$ $\boldsymbol{A} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U}(\boldsymbol{D}, \boldsymbol{L}, \boldsymbol{U}$)

$$(\boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U})\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{D}\boldsymbol{x} = \boldsymbol{L}\boldsymbol{x} + \boldsymbol{U}\boldsymbol{x} + \boldsymbol{b}$$

➢

$$\boldsymbol{D}\boldsymbol{x}^{(k+1)} = \boldsymbol{L}\boldsymbol{x}^{(k+1)} + \boldsymbol{U}\boldsymbol{x}^{(k)} + \boldsymbol{b}$$

$$(\boldsymbol{D} - \boldsymbol{L})\boldsymbol{x}^{(k+1)} = \boldsymbol{U}\boldsymbol{x}^{(k)} + \boldsymbol{b},$$

$$\boldsymbol{x}^{(k+1)} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{x}^{(k)} + (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{b} = \boldsymbol{B}_{G-S}\, \boldsymbol{x}^{(k)} + \boldsymbol{f}_{G-S}$$

➢ 3.15 $\boldsymbol{B}_{G-S} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{U}$ $\boldsymbol{f}_{G-S} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{b}$

3.5

例3.11 -

$$\begin{pmatrix}3 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}8\\ 8\\ 6\end{pmatrix}$$

$$\begin{cases}3x_1+x_2+x_3=8\\ x_1+2x_2+x_3=8\\ x_1+x_2+x_3=6\end{cases}$$ $[1 \quad 2 \quad 3]$

3 x_1 x_2 x_3 -

$$\begin{cases}x_1^{(k+1)}=\dfrac{1}{3}(8-x_2^{(k)}-x_3^{(k)})\\ x_2^{(k+1)}=\dfrac{1}{2}(8-x_1^{(k+1)}-x_3^{(k)})\\ x_3^{(k+1)}=6-x_1^{(k+1)}-x_2^{(k+1)}\end{cases}$$

3.5

$\boldsymbol{x}^{(0)} = (0,0,0)^T$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0	5	1.0205	2.2088	2.7705
1	2.6666	2.6666	0.6666	6	1.0068	2.1112	2.8818
2	1.5555	2.8888	1.5555	7	1.0022	2.0579	2.9397
3	1.1851	2.6296	2.1851	8	1.0007	2.0297	2.9695
4	1.0617	2.3765	2.5617	9	1.0002	2.0151	2.9846

3.5

❑

定理3.3 n $x = Bx + f$ $x^{(0)}$ f $\rho(B) < 1$。

- $\rho(B) < 1$ B $|\lambda_i| < 1(i = 1,2,\cdots n)$ $I - B$

$$u_i = 1 - \lambda_i \qquad (\lambda = 1,2,\cdots n)$$

-

$$\det(I - B) = \prod_{i=1}^{n}(1 - \lambda_i) \neq 0$$

$I - B$ $(I - B)x = f$

$$x = Bx + f$$

x^*

3.5

- $$e^{(k)} = x^{(k)} - x^*$$

$$e^{(k)} = x^{(k)} - x^* = \left(Bx^{(k-1)} + f\right) - (Bx^* - f)$$
$$= B\left(x^{(k-1)} - x^*\right) = Be^{(k-1)}$$

-

$$e^{(k)} = B^k e^{(0)}$$

- $$\rho(B) < 1$$

$$\lim_{k\to\infty} B^k = 0$$

$x^{(0)}$ f

$$\lim_{k\to\infty} e^{(k)} = 0$$

$$\lim_{k\to\infty} x^{(k)} = x^*$$

- $x^{(0)}$ f

$\lim_{k\to\infty} x^{(k)} = x^*$ $x^* = Bx^* + f$

3.5

-

$$x^{(k)} - x^* = B^k\left(x^{(0)} - x^*\right)$$

- $x^{(0)}$

$$\lim_{k\to\infty} B^k\left(x^{(0)} - x^*\right) = 0$$

$$\lim_{k\to\infty} B^k = 0$$

$$\rho(B) < 1$$

- ❑ B B

A

- ❑ $\rho(B)$ $\left\{x^{(k)}\right\}$

3.5

定义3.4

$$R(B) = -\ln \rho(B)$$

例3.12 3.9

3.12

$$\mathbf{B} = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & -1 & 0 \end{pmatrix}$$

$$\lambda_1 = -3.1701 \quad \lambda_2 = 0.4812 \quad \lambda_3 = -1.3111$$

$$\rho(\mathbf{B}) = |-3.1701| > 1$$

3.12

3.5

定理3.4 $B_J = I - D^{-1}A$

(1) $\rho(B_J) = \rho(B_{G-S}) = 0$

(2) $0 < \rho(B_{G-S}) < \rho(B_J) < 1$

(3) $\rho(B_J) = \rho(B_{G-S}) = 1$

(4) $1 < \rho(B_J) < \rho(B_{G-S})$

- 3.4 $B_J = I - D^{-1}A$

  -

  - -

3.5

定理3.5 （迭代收敛法的充分条件1） $\|B\| < 1$

$\boldsymbol{x}^{(k+1)} = \boldsymbol{B}\boldsymbol{x}^{(k)} + \boldsymbol{f}$ $\{x^{(k)}\}$ $x = Bx + f$

x^*

$$\left\|x^{(k)} - x^*\right\| \leq \frac{\|B\|}{1-\|B\|}\left\|x^{(k)} - x^{(k-1)}\right\| \quad (3.16)$$

$$\left\|x^{(k)} - x^*\right\| \leq \frac{\|B\|^k}{1-\|B\|}\left\|x^{(1)} - x^{(0)}\right\| \quad (3.17)$$

$\|B\| < 1$ $\rho(B) \leq \|B\|$ 3.3 $\boldsymbol{x}^{(k+1)} = \boldsymbol{B}\boldsymbol{x}^{(k)} + \boldsymbol{f}$

$$\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{x}^*,$$

$\boldsymbol{x}^* = \boldsymbol{B}\boldsymbol{x}^* + \boldsymbol{f}$ (3.18)

3.5

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{B}\boldsymbol{x}^{(k)} + \boldsymbol{f} \quad (3.18)$$

$$x^{(k+1)} - x^{(k)} = B\left(x^{(k)} - x^{(k-1)}\right)$$

$$x^{(k+1)} - x^{*} = B\left(x^{(k)} - x^{*}\right)$$

$$\left\|x^{(k+1)} - x^{*}\right\| \leq \|B\| \cdot \left\|x^{(k)} - x^{*}\right\| \quad 3.19$$

$$\left\|x^{(k+1)} - x^{(k)}\right\| \leq \|B\| \cdot \left\|x^{(k)} - x^{(k-1)}\right\| \quad 3.20$$

3.20

$$\left\|x^{(k+1)} - x^{(k)}\right\| \leq \|B\|^{k-1} \cdot \left\|x^{(1)} - x^{(0)}\right\| \quad (3.21)$$

3.19 ,

$$\begin{aligned}\left\|x^{(k)} - x^{*}\right\| &= \left\|x^{(k)} - x^{(k+1)} + x^{(k+1)} - x^{*}\right\| \\ &\leq \left\|x^{(k+1)} - x^{(k)}\right\| + \left\|x^{(k+1)} - x^{*}\right\| \\ &\leq \|B\| \cdot \left\|x^{(k)} - x^{(k-1)}\right\| + \|B\| \cdot \left\|x^{(k)} - x^{*}\right\|\end{aligned}$$

3.5

$\|B\| < 1, 1 - \|B\| > 0$

$$\left\|x^{(k)} - x^*\right\| \le \frac{\|B\|}{1-\|B\|}\left\|x^{(k)} - x^{(k-1)}\right\|$$

3.16 3.21 3.17

❑ 3.16 $\left\|x^{(k)} - x^{(k-1)}\right\| < \varepsilon$

例3.13 $x = Bx + f$

$$B = \begin{pmatrix} 0.9 & 0 \\ 0.3 & 0.8 \end{pmatrix}, f = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\|B\| > 1$ $x^{(k+1)} = Bx^{(k)} + f$

: $\|B\|_\infty = 1.1$, $\|B\|_1 = 1.2$, $\|B\|_F = \sqrt{1.54}$, $\|B\|_2 = 1.021$, $\|B\| > 1$ $\det(\lambda I - B) = \begin{vmatrix} \lambda - 0.9 & 0 \\ 0.3 & \lambda - 0.8 \end{vmatrix} = (\lambda - 0.9)(\lambda - 0.8) = 0$, $\lambda_1 = 0.9, \lambda_2 = 0.8$

$\rho(B) = 0.9 < 1,$

3.5

定理3.6（迭代法收敛的充分条件2） $Ax = b$

-

定理3.7 $Ax = b$ A -

A $2D - A$

D A $2D - A$ A

A $2D - A$