# Problem 5

In [ ]:
```
import matplotlib.pyplot as plt
import numpy as np
```

Define hyperparameters and datasets. To avoid confusion, we express matrices by familiar ways.

To be more precise, we are accustomed to the case each $X_i$ and $Y$ is a column vector. To do this, we use transpose of $X$ mainly later and define $Y$ to be column vector by np.expand_dims.

In [ ]:
```
d = 35
n_train, n_val, n_test = 300, 60, 30
np.random.seed(0)
beta = np.random.randn(d)
beta_true = beta / np.linalg.norm(beta)
# Generate and fix training data
X_train = np.array([np.random.multivariate_normal(np.zeros(d), np.identity(d)) for
Y_train = X_train @ beta_true + np.random.normal(loc = 0.0, scale = 0.5, size = n_t
# Generate and fix validation data (for tuning lambda).
X_val = np.array([np.random.multivariate_normal(np.zeros(d), np.identity(d)) for _
Y_val = X_val @ beta_true; Y_val = np.expand_dims(Y_val, axis=1)
# Generate and fix test data
X_test = np.array([np.random.multivariate_normal(np.zeros(d), np.identity(d)) for _
Y_test = X_test @ beta_true; Y_test = np.expand_dims(Y_test, axis=1)
```

Our "familiar" construction applies again. To fit the dimension of columns and rows, linear weight $W$ has shape $(p, d)$, and the product $\tilde{X}$ becomes of the shape $(p, n)$. Also we denote $\theta$ as column vector.

Unlike usual least-squre case, our loss contains weight decay term. By differentiating to $\theta$ as usual we get $\tilde{X}(\tilde{X}^{\top}\theta - Y) + \lambda I_p \theta = 0.$

Therefore we get $\theta = (\tilde{X}\tilde{W}^{\top} + \lambda I_p)^{-1}\tilde{X}Y$, provided $\tilde{X}\tilde{X}^{\top} + \lambda I_p$ invertible.

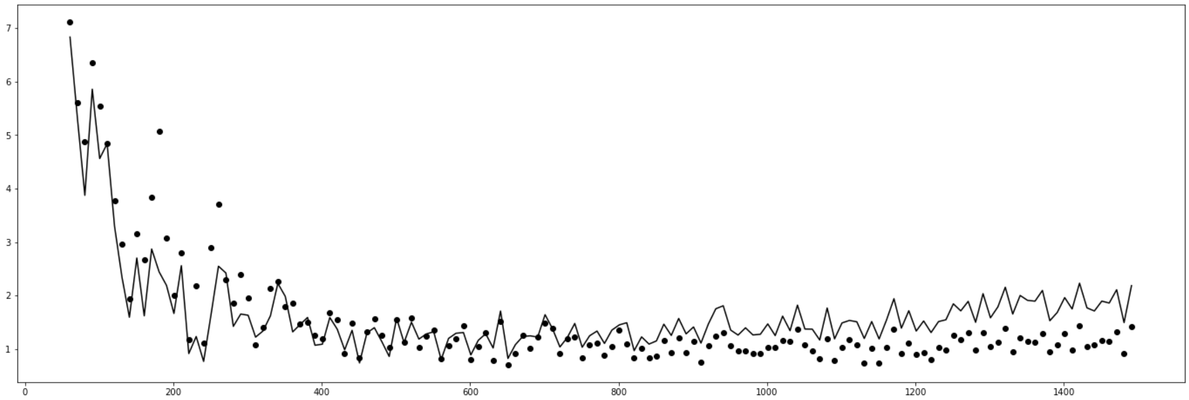
Of course that weight decay term should appears only at training step.

In [ ]:
```
fixed_lambda = 0.01
lambda_list = [2 ** i for i in range(-6, 6)]
num_params = np.arange(1,1501,10)

errors_opt_lambda = []
errors_fixed_lambda = []
for p in num_params :
    # fixed lambda
    W = np.random.normal(0, 1/p*np.ones((p,d)), size=(p,d))    # (p,d)
    X_tilde = np.maximum(np.zeros((p,n_train)), W@X_train.T)    # (p,n)
    theta = np.linalg.inv(X_tilde@X_tilde.T + fixed_lambda*np.identity(p))@X_tilde@

    X_tilde = np.maximum(np.zeros((p,n_test)), W@X_test.T)    # (p,n)
    errors_fixed_lambda.append( np.linalg.norm(X_tilde.T@theta - Y_test)**2/2 )

    # tuned lambda
    thetas = []
    errors = []
    for lamb in lambda_list:
        X_tilde = np.maximum(np.zeros((p,n_train)), W@X_train.T)
```

```
        theta = np.linalg.inv(X_tilde@X_tilde.T + lamb*np.identity(p))@X_tilde@Y_tr
        thetas.append(theta)

        X_tilde = np.maximum(np.zeros((p,n_val)), W@X_val.T)
        errors.append( np.linalg.norm(X_tilde.T@theta - Y_val)**2/2 )

    min_index = np.argmin(np.array(errors))
    X_tilde = np.maximum(np.zeros((p,n_test)), W@X_test.T)
    theta = thetas[min_index]
    errors_opt_lambda.append( np.linalg.norm(X_tilde.T@theta - Y_test)**2/2 )
```

Because of technical issues, I unwillingly exclude tex grammer in source code. But everything is same.
scatter plot is fixed model and line plot is tuned model.
To visualize better, I omit first six data. These errors are significantly high due to the number of parameters.

In [ ]:
```
plt.figure(figsize = (24, 8))
plt.rc('text', usetex = False)

cut_from = 6
plt.scatter(num_params[cut_from:], errors_fixed_lambda[cut_from:], color = 'black',
            label = "Test error with fixed lambda = 0.01",
            )

plt.plot(num_params[cut_from:], errors_opt_lambda[cut_from:], 'k', label = "Test err
plt.show()
```




On x-axis (number of parameters) near 200, fixed model(scatter plot) shows relatively high loss compared to previous adjacent points.
Tuned model, however, shows gradually decreasing losses.