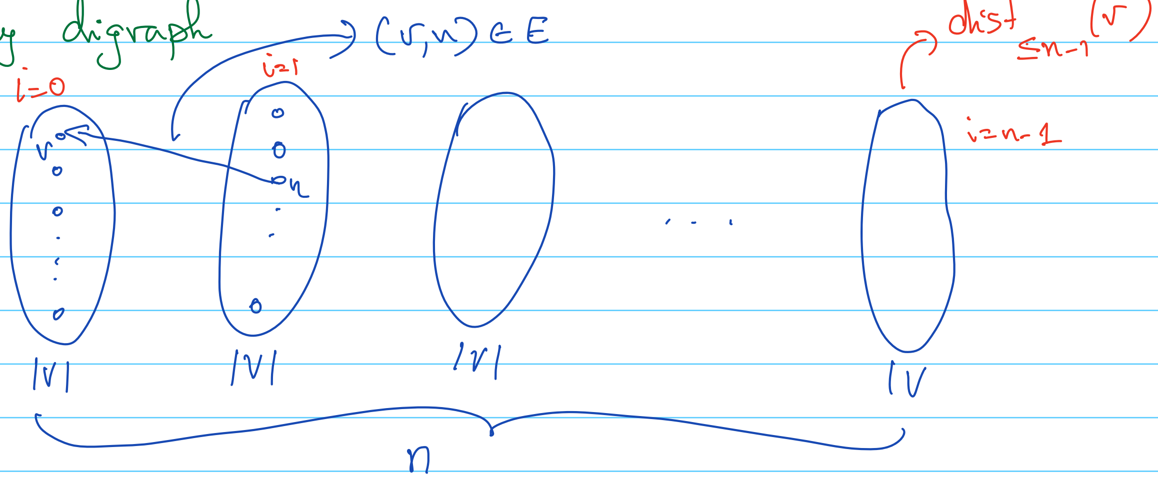
Bellman-Ford recurrence

$$\mathrm{dist}_{\le i}(v) = \begin{cases} 0 & \text{if } i=0 \ \& \ v=s \\ \infty & \text{if } i=0 \ \& \ v\neq s \\ \min\begin{cases} \mathrm{dist}_{\le i-1}(v) \\ \min\limits_{(u,v)\in E} \left\{ \mathrm{dist}_{\le i-1}(u) + w(u,v) \right\} \end{cases} \end{cases}$$

$\mathrm{dist}_{\le i}(v)$ ↓ shortest path from s to v that contain $\le i$ edges

Dependency digraph



Bellman-Ford algorithm

$\forall v \in G \quad \mathrm{dist}(v) = \infty$

$\mathrm{dist}(s) = 0$

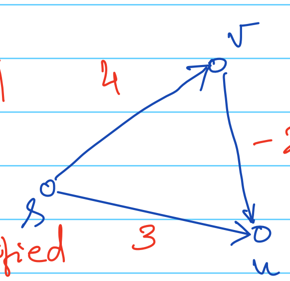
add an additional check

$O(|V|.|E|) \leftarrow$ Repeat $|V|-1$ times

for each $(u,v) \in G$ → order not specified

if $\mathrm{dist}(v) > \mathrm{dist}(u) + w(u,v)$

Relaxing an edge ← $\mathrm{dist}(v) = \mathrm{dist}(u) + w(u,v)$



Repeat loop: (i) $(s,u), (v,u), (s,v)$ $\quad$ $\mathrm{dist}(s)=0$, $\mathrm{dist}(u)=3$

1st iteration $\quad$ $\mathrm{dist}(v)=4$

(ii) $(s,v), (s,u), (v,u)$

↳ $\underset{3}{\mathrm{dist}(u)} > \underset{4}{\mathrm{dist}(v)} + \underset{-2}{w(v,u)}$

Ex: Check if G has a negative wt. cycle

Theorem: $\forall\, v \in G$ & $i \geq 0$. After the i^{th} iteration of the Repeat loop

$$dist(v) \leq dist_{\leq i}(v)$$

Proof: Base case: $i=0$ $\quad dist(v) = \infty$ if $v \neq s$

$$dist(s) = 0$$

Induction step: $s \to u_1 \to u_2 \to \cdots \to u_k \to v$

↳ shortest path from s to v with $\leq i$ edges

After the $(i-1)^{st}$ iteration

$$dist(u_k) \leq dist_{\leq i-1}(u_k)$$

Instant when (u_k, v) is considered in the inner loop

$$dist(v) \leq dist(u_k) + w(u_k, v) \leq dist_{\leq i-1}(u_k) + w(u_k, v)$$

$$= dist_{\leq i}(v)$$

After $n-1$ iterations, $\forall\, v \in G$ $dist(v) \leq dist_{\leq n-1}(v)$

shortest path