Graph Algorithms - I

CS2800: Design and Analysis of Algorithms

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IIT Madras

Module plan

1. Graph representations
2. Graph traversals and applications
3. Directed graphs
4. Shortest paths in graphs

Graph representations

Graphs - basics

Graph: A set of vertices V, and a set of edges $E \subseteq V \times V$

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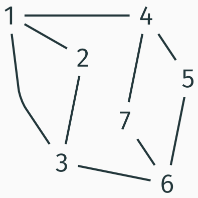

- Transportation networks
- Social networks and the web graph
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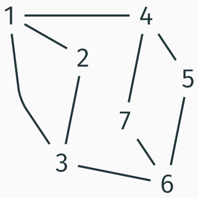
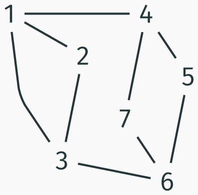


- Transportation networks
- Social networks and the web graph
- Circuits
- Ubiquitous abstraction for a variety of problems!

Graph representations - adjacency matrices



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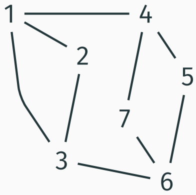

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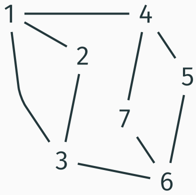

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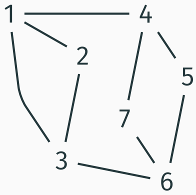

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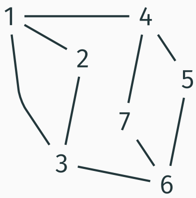

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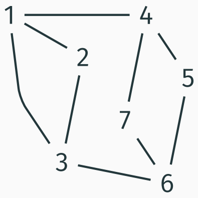
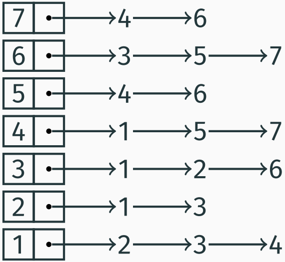
- More useful for representing dense graphs $(|E| = \Omega(n^2))$

Graph representations - adjacency lists





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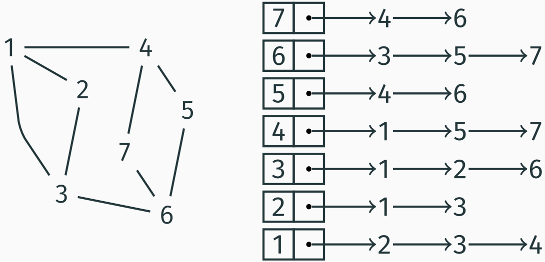

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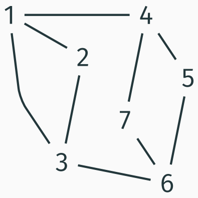

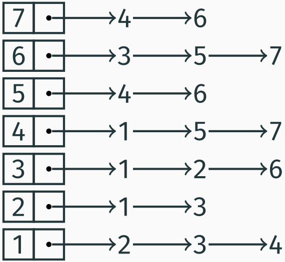

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Graph traversals and applications

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- Perhaps the most important primitive of graph operations
- Two important variants:
  - Depth-First Search (DFS)
  - Breadth-First Search (BFS)
- Can be used to collect a lot of auxilliary information about the structure of the graph, and
- both BFS and DFS are highly efficient in terms of time and space complexity

Depth-First Search

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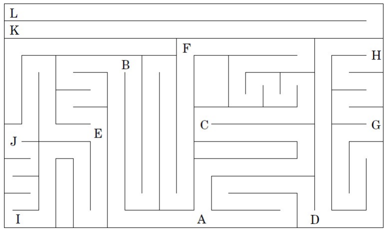

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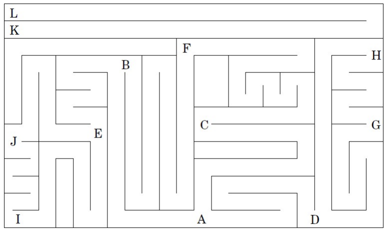

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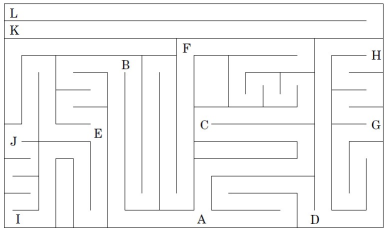

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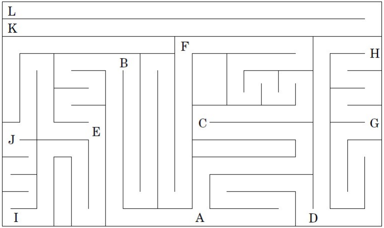

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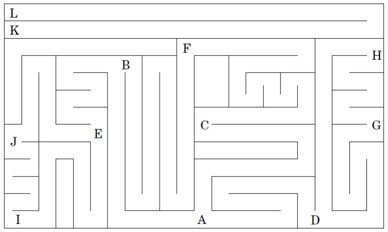

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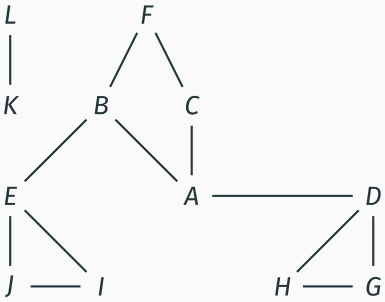

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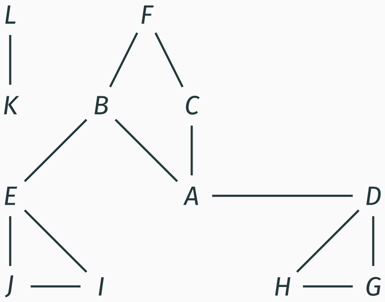

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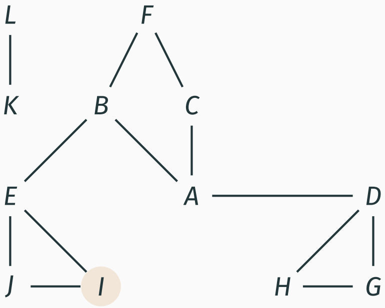

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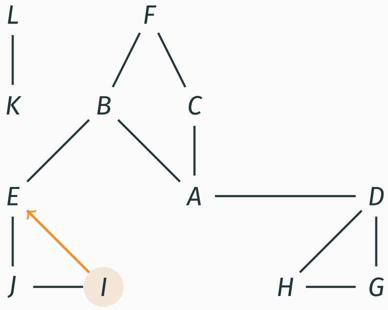

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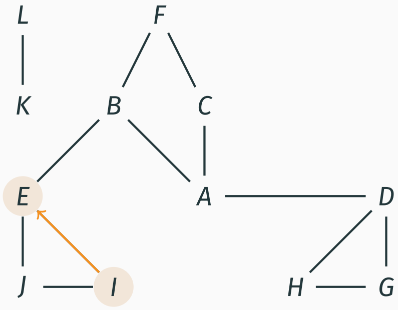

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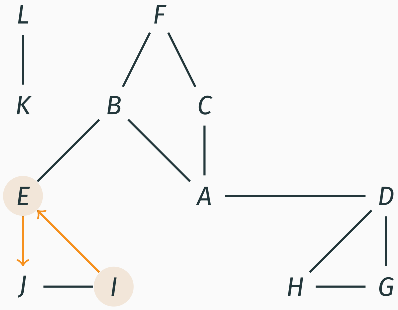

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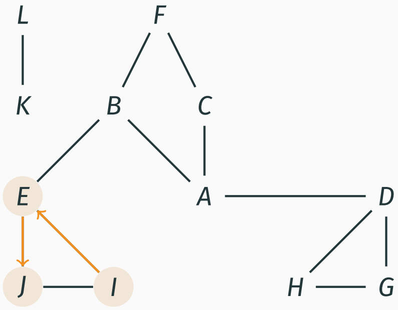

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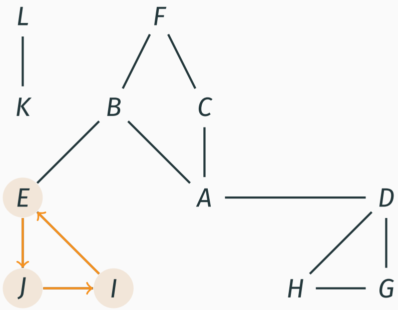

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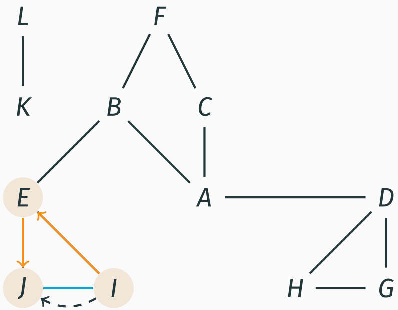

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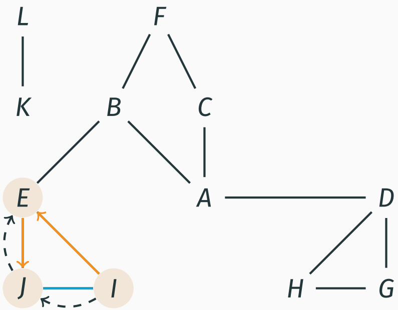

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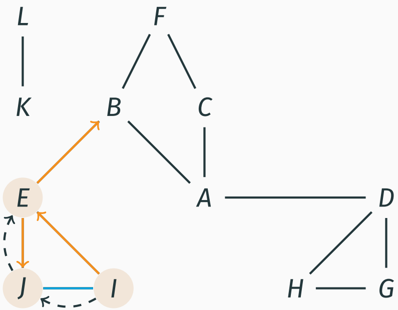

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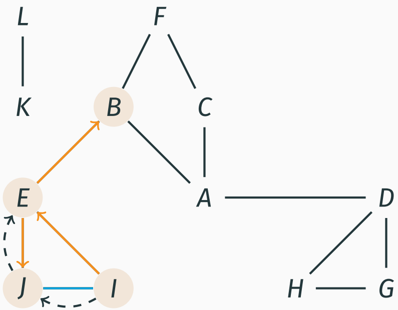

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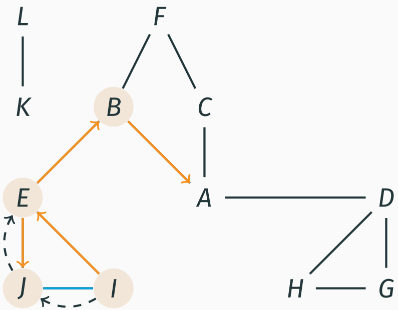

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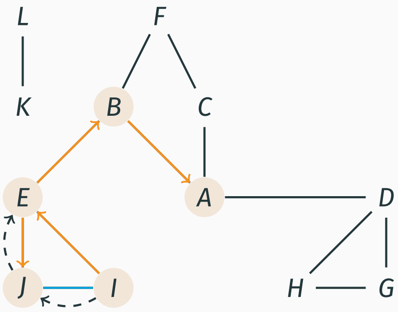

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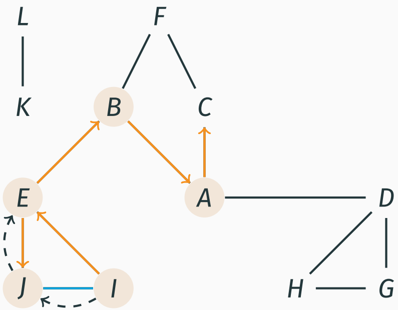

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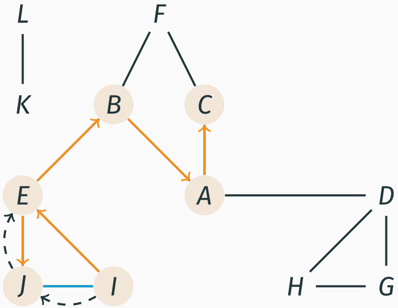

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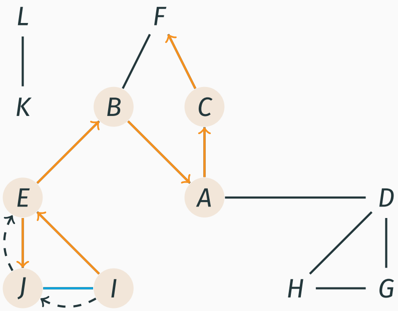

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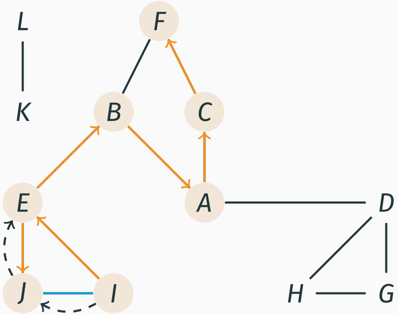

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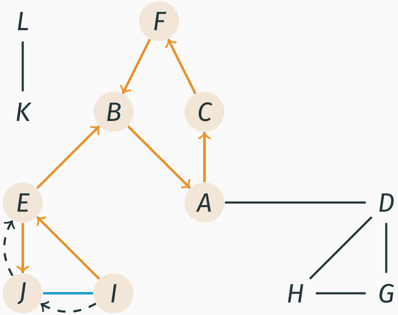

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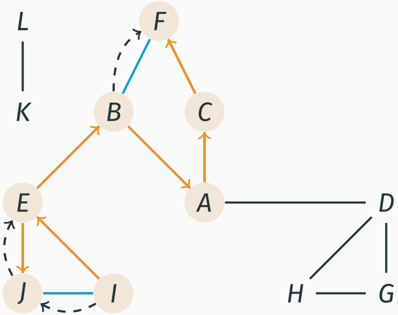

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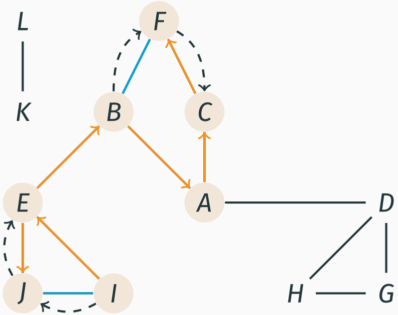

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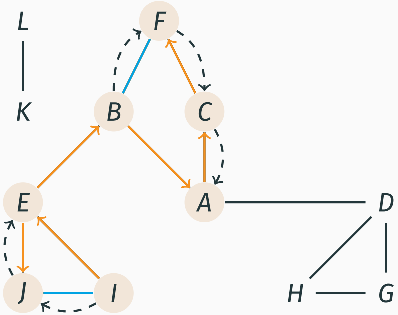

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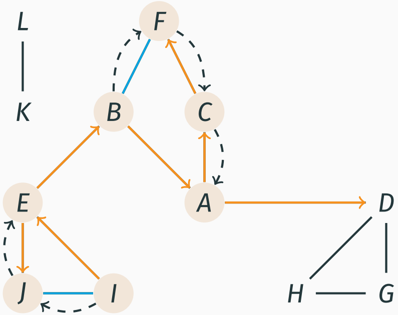

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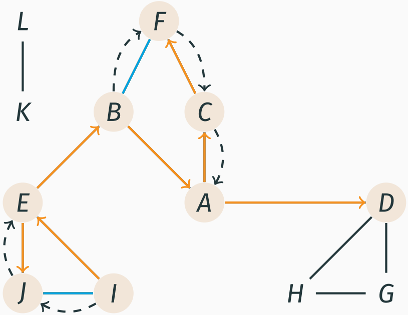

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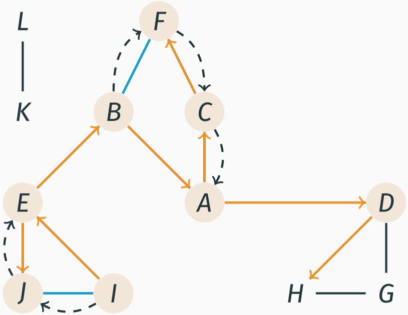

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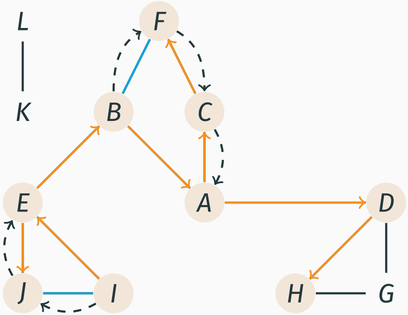

RecursiveDFS

if v is unmarked **then**
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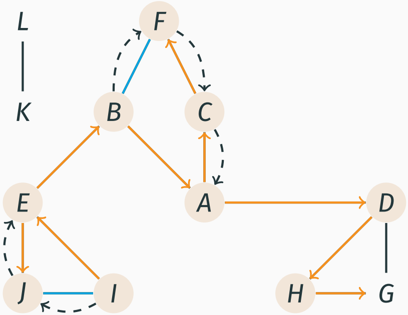

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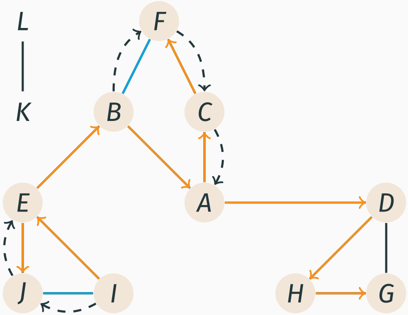

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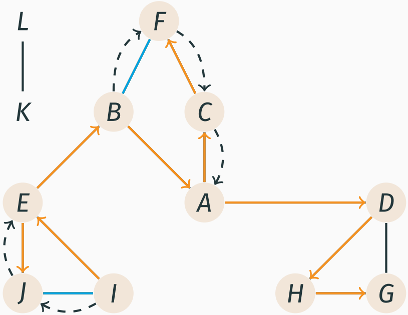

ITERATIVEDFS

push(s)
while stack is not empty **do**
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    foreach edge vw **do**
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Depth-First Search

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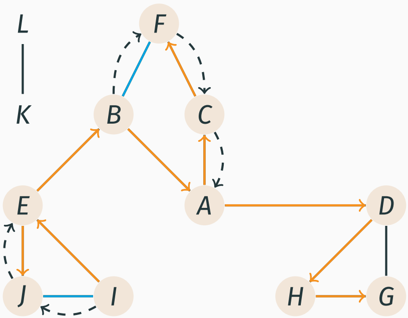

IterativeDFS

push($\varnothing, s$)
while stack is not empty **do**
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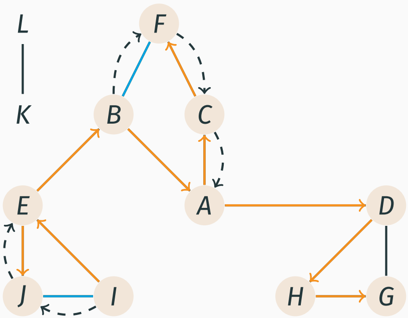

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$O(d(v))$/vertex

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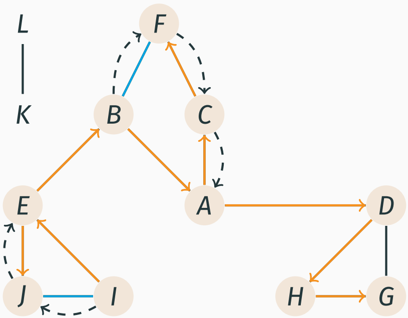

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Running time $O(|V| + |E|)$ in adjacency list model

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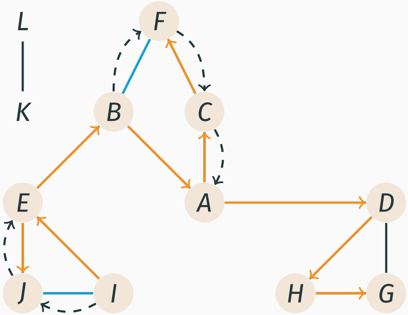

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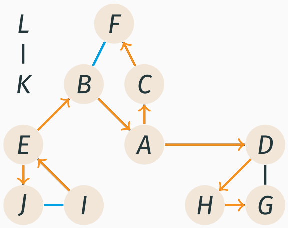
Question What is the running time in the adjacency matrix model?

Depth-First Search



Lemma: The following statements are true for DFS:

1. DFS marks those vertices reachable from s and only those vertices.
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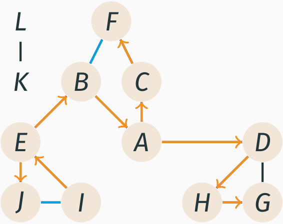

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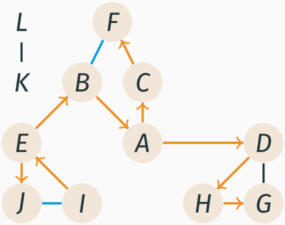

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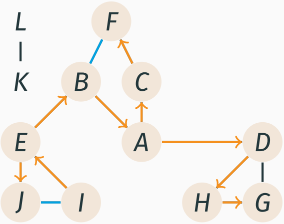

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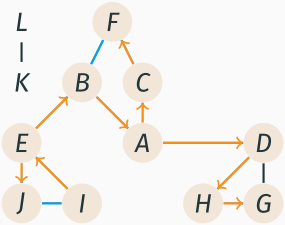

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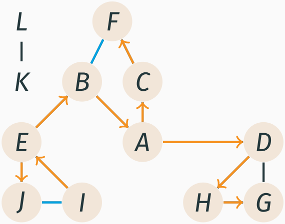

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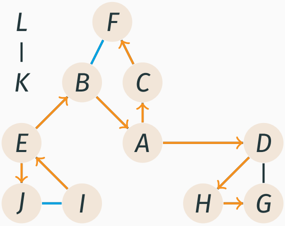
By the induction hypothesis, u is marked by the DFS

If u is marked, then all the neighbors of u are placed in the stack. Therefore when they are taken out of the stack, either

- They are already marked, or
- They are unmarked and will get marked at that stage

Depth-First Search



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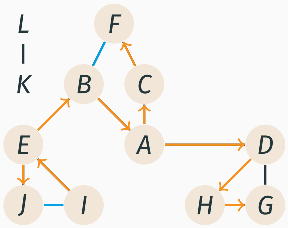

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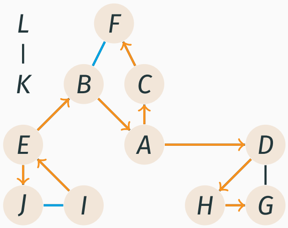

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If $w = \text{parent}(v)$, then w is marked before v - There must be a path from w to s and hence from v to s
Every vertex v has a unique parent (except s that has no parent), and hence the subgraph has $n - 1$ edges $\Rightarrow$ spanning tree

Breadth-First Search

How do you find the shortest path in a maze?

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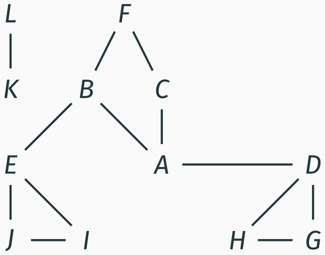

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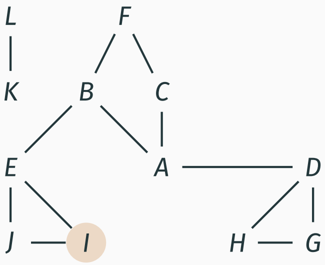

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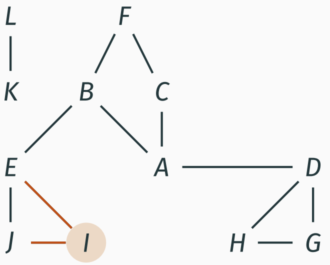

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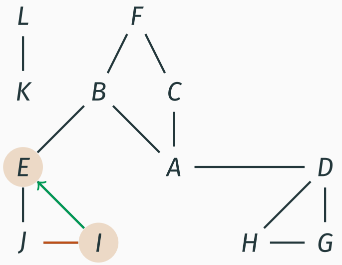

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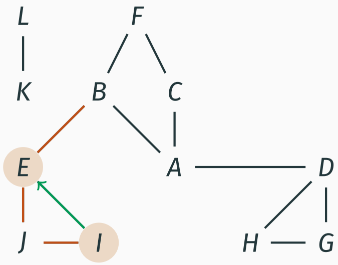

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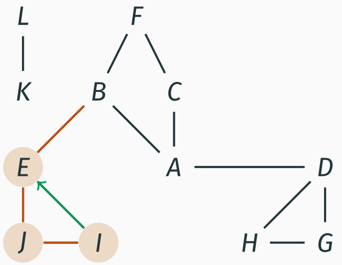

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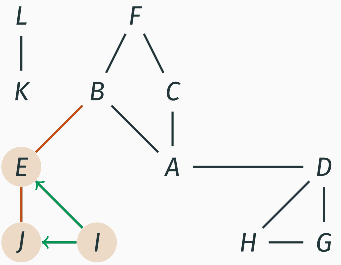

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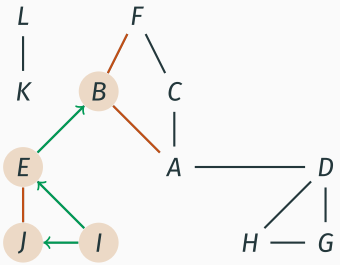

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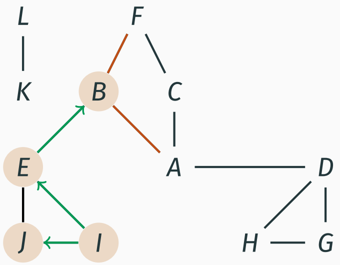

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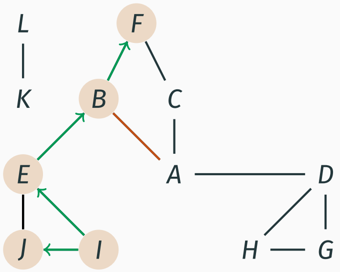

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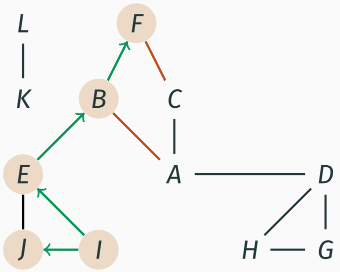

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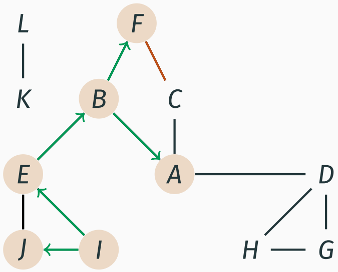

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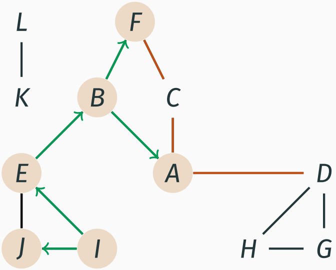

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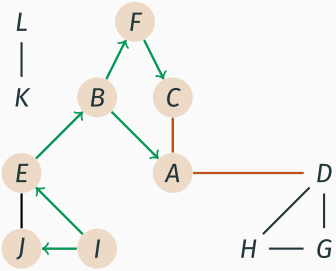

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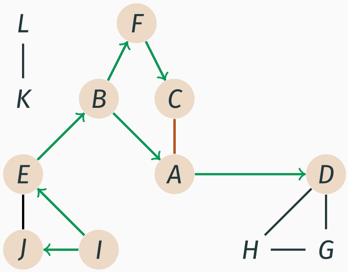

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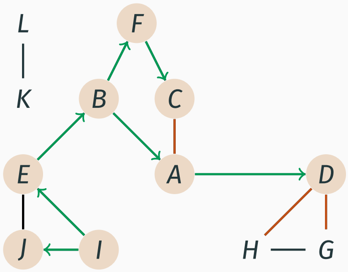

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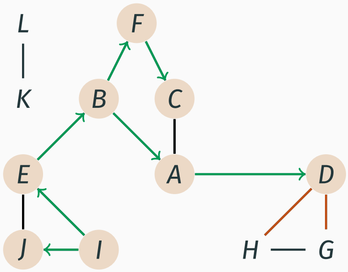

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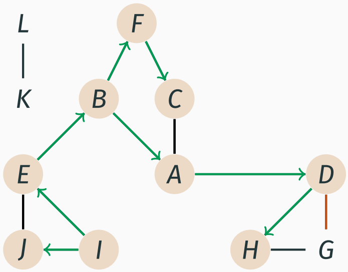

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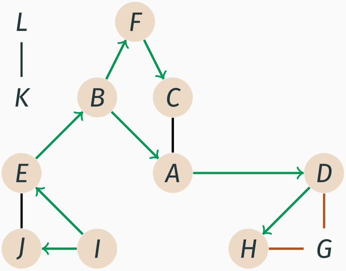

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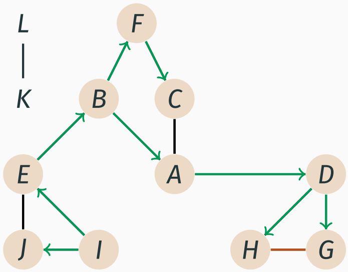

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DFS vs BFS

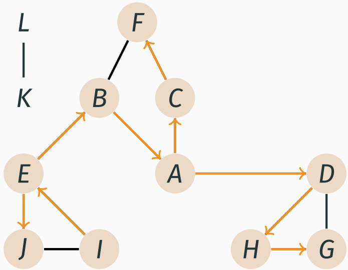

Figure 1: DFS tree

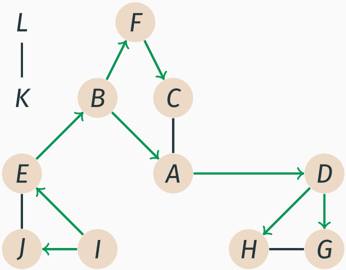

Figure 2: BFS tree

Applications: Connected compenents

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DFSALL(G)

foreach vertex $v \in G$ **do**
    unmark v

foreach vertex $v \in G$ **do**
    if v is unmarked **then**

        DFS(v)

DFS(s)

push(s)
while stack is not empty **do**
    $v \leftarrow$ pop()
    if v is unmarked **then**
        mark v

        foreach edge vw **do**
            push(w)

Applications: Connected compenents

Question: How many connected components are there in the graph?

- DFS/BFS starting from a vertex s visits those and only those vertices that lie in the same connected component as s

DFSALL(G)

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foreach vertex v ∈ G do
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cc ← 0
foreach vertex v ∈ G do
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unmarked vertex means new component

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```
DFS(s)
push(s)
while stack is not empty do
    v ← pop()
    if v is unmarked then
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        comp[v] = cc
        foreach edge vw do
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    $comp[v] = cc$ $\longrightarrow$ component label
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Directed graphs

Directed graphs: Basics

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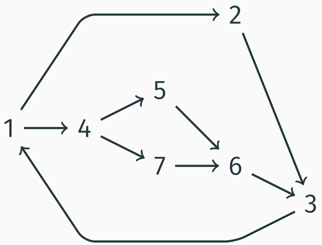
- $G(V, E)$ – $E \subseteq V \times V$: Implicitly assumed a symmetric relation earlier



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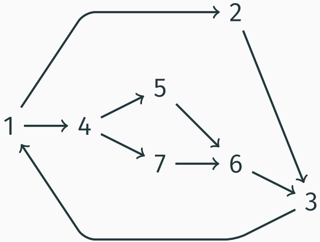

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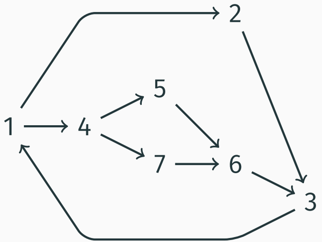

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    - Pairwise preferences - directed edges
    - Rank-ordering - Hamiltonian path

DFS in directed graphs: Classification of edges

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DFS(s)

mark s
foreach edge $s \to w$ **do**
  if w is unmarked **then**
    parent[w] $\leftarrow s$
    DFS(w)

DFS in directed graphs: Classification of edges

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clock ← 0
foreach vertex $v \in G$ **do**
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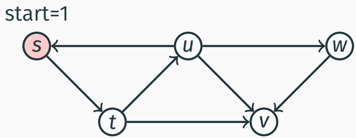

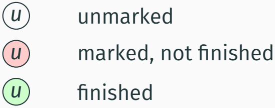

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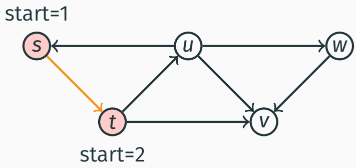

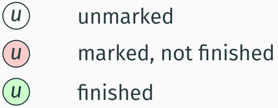

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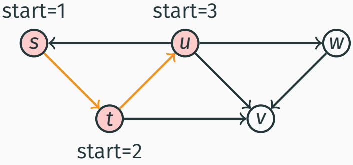

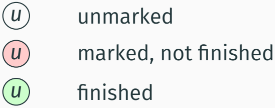

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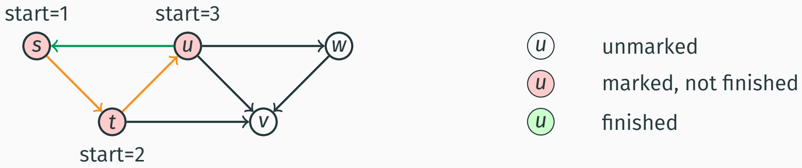

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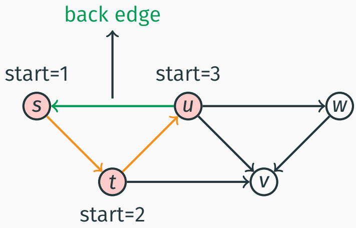

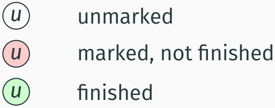

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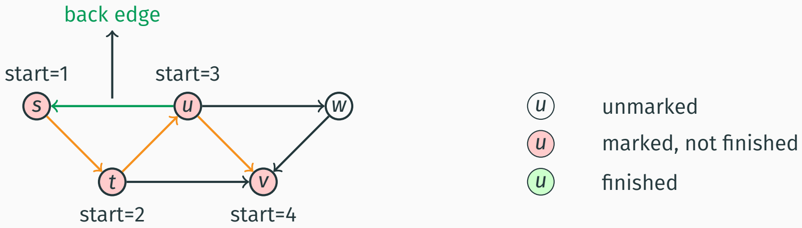

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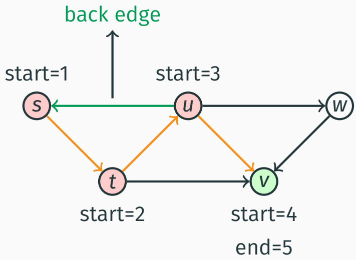

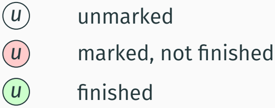

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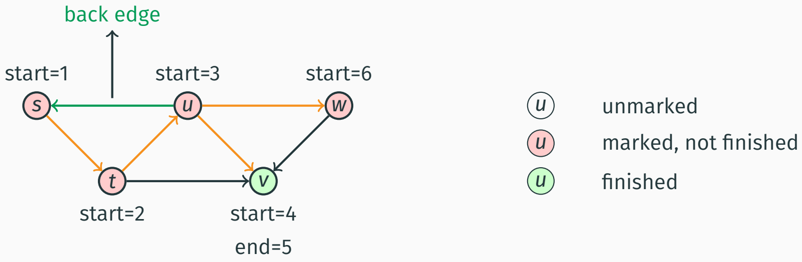

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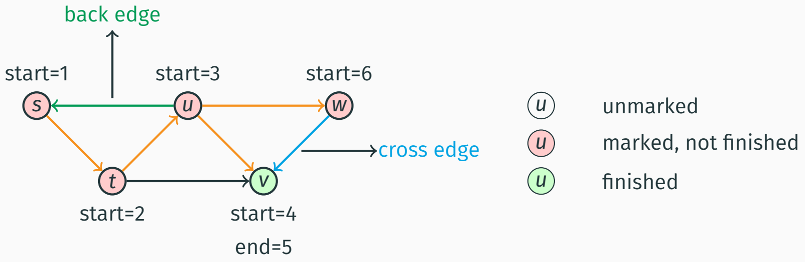

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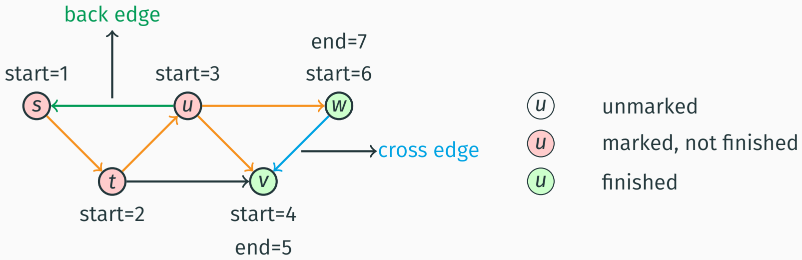

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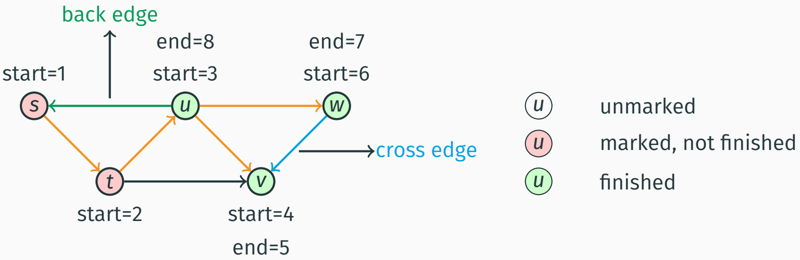

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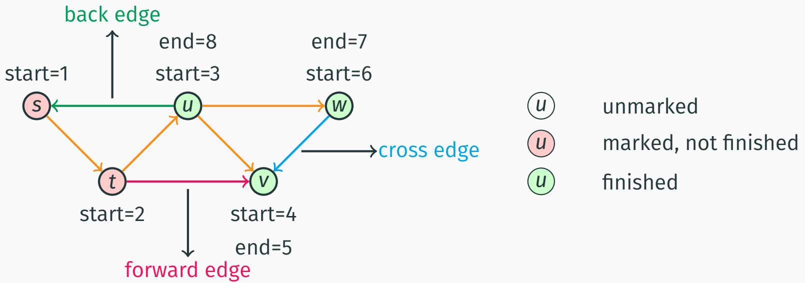

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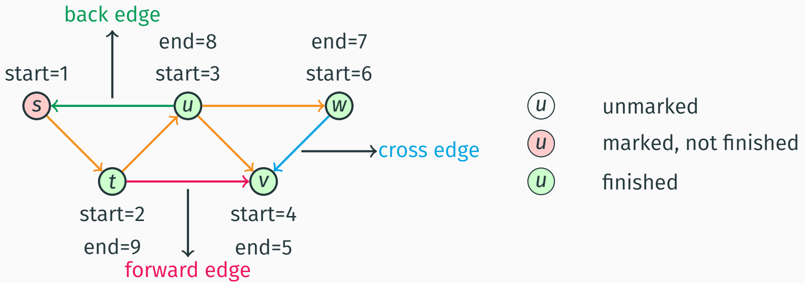

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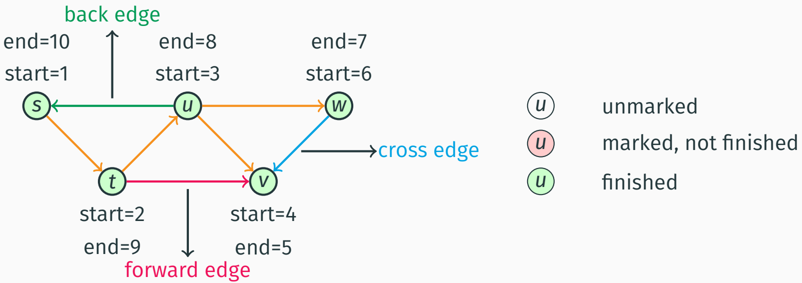

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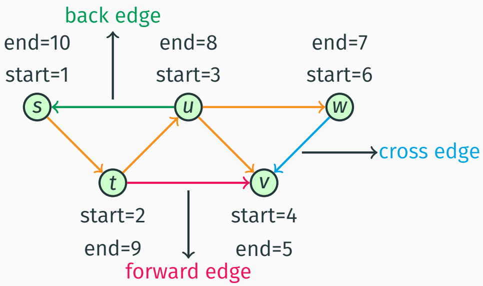

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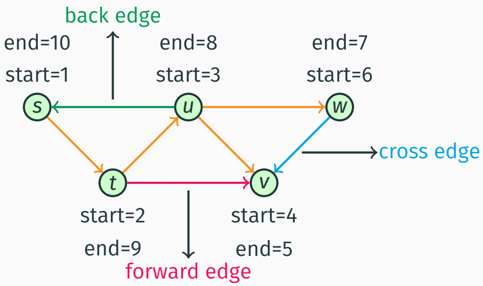

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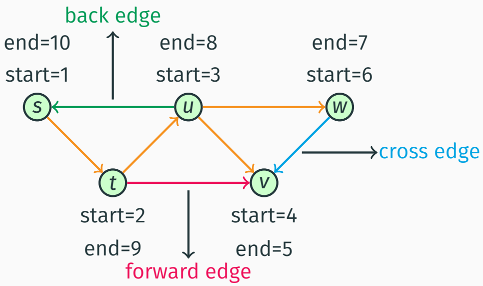

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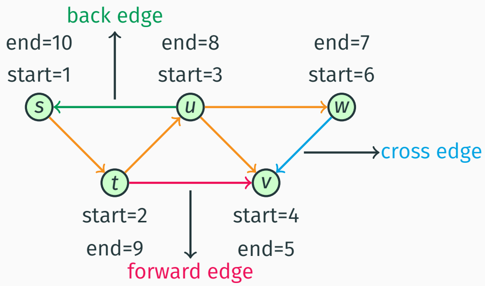

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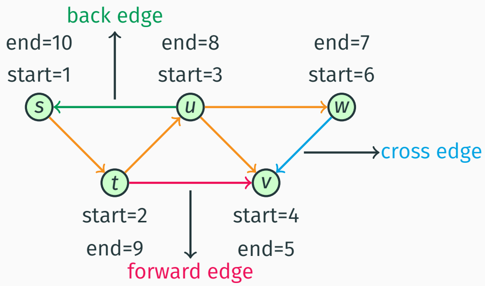

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- **Cross edge:** v is finished when DFS(w) starts - end[v] < start[w]

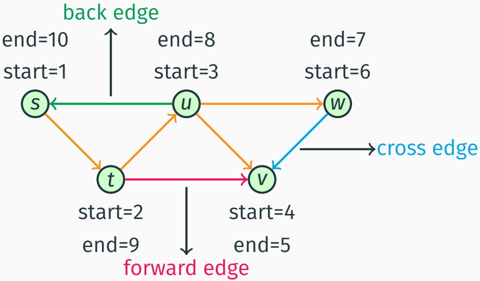
Cycles in directed graphs

Cycles in directed graphs

When does a digraph have a cycle?

Cycles in directed graphs

When does a digraph have a cycle?



Cycles in directed graphs

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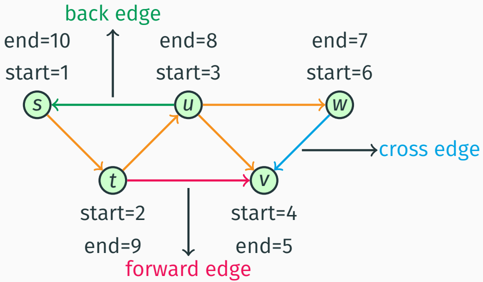

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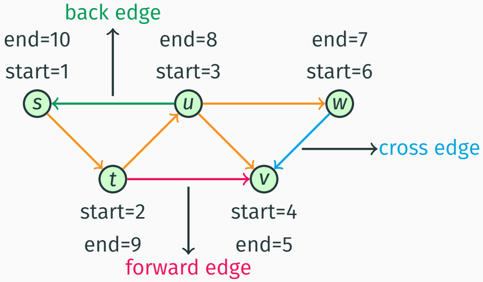

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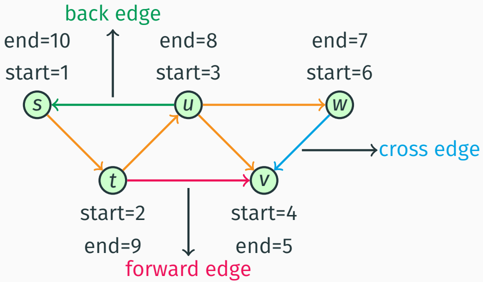

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Exercise: Write the pseudo-code for the $O(|V| + |E|)$ algorithm to check if G is a DAG (Directed Acyclic Graph)?

Recursion and DAGs

$\textsc{Fib}(n)$

if $n = 1$ or 2 **then return** 1
return $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

Recursion and DAGs

Dependency digraph

FIB(n)

if n = 1 or 2 **then return** 1
return FIB(n – 1) + FIB(n – 2)

$$
\begin{tikzcd}
 & \textsc{Fib}(6) \arrow[dl] \arrow[dr] & \\
\textsc{Fib}(5) \arrow[rr] \arrow[d] & & \textsc{Fib}(4) \arrow[dll] \arrow[dl] \\
\textsc{Fib}(3) \arrow[r] \arrow[rr, bend right] & \textsc{Fib}(2) & \textsc{Fib}(1)
\end{tikzcd}
$$

Recursion and DAGs

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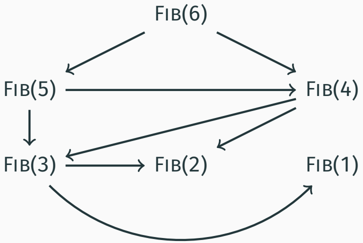
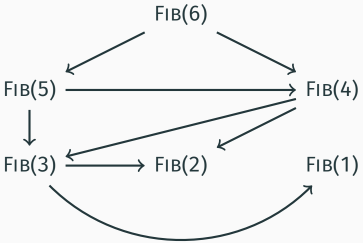


- Recursive function calls can be simulated by a DFS of the dependency graph – memoization

Recursion and DAGs

Dependency digraph



FIB(n)

if n = 1 or 2 **then return** 1
return FIB(n − 1) + FIB(n − 2)

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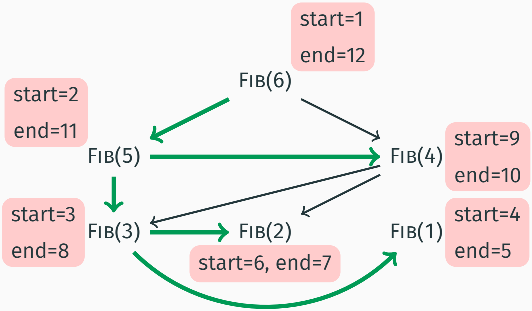

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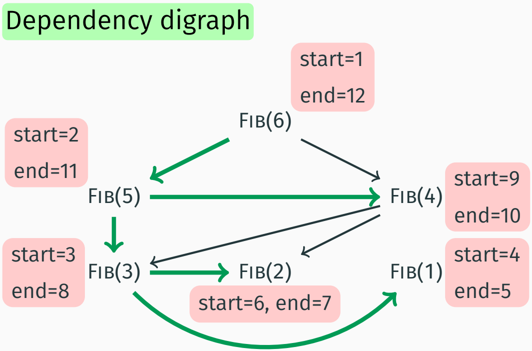

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      Fɪʙ(6) → Fɪʙ(5) → Fɪʙ(4) → Fɪʙ(3) → Fɪʙ(2) → Fɪʙ(1)

Topological ordering of DAGs

$$\begin{tikzcd}
\textsc{Fib}_6 \arrow[r] \arrow[rr, bend left] & \textsc{Fib}_5 \arrow[r] \arrow[rr, bend left] & \textsc{Fib}_4 \arrow[r] \arrow[rr, bend left] & \textsc{Fib}_3 \arrow[r] \arrow[rr, bend left] & \textsc{Fib}_2 & \textsc{Fib}_1
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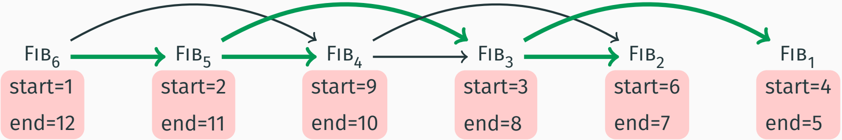

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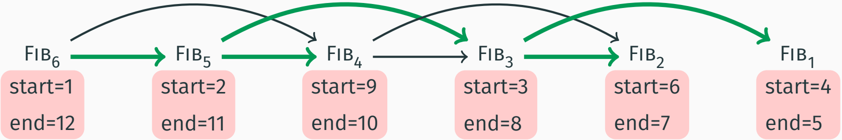

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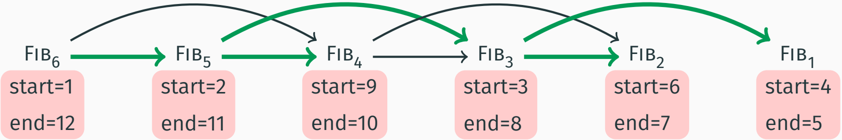

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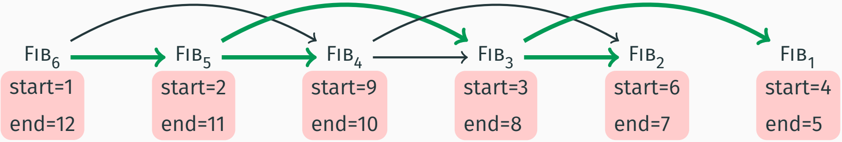

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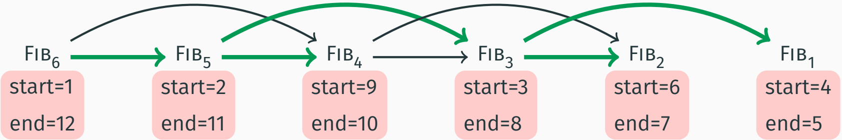

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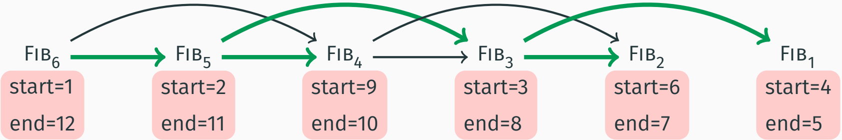

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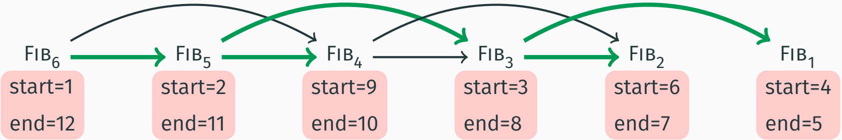

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Three cases possible about their start times

- start$[u]$ < end$[u]$ < start$[v]$ < end$[v]$: not possible because if DFS from u starts before v, then it cannot end before the DFS of v since $(u, v) \in E$
- start$[u]$ < start$[v]$ < end$[u]$ < end$[v]$: not possible for any DFS
- start$[v]$ < start$[u]$ < end$[u]$ < end$[v]$: path from v to u $\Rightarrow$ there must be a cycle that includes the edge (u, v)

Topological ordering of DAGs

Observation: In a DAG G, if $(u, v) \in E$ then $\mathrm{end}[u] > \mathrm{end}[v]$

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Exercise: Implement this algorithm to run in time $O(|V| + |E|)$

DAGs and the structure of digraphs

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DFS from v on a digraph covers all vertices u that are reachable from v - reach(v)

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Observation: For every u and v, either

$$\left.\begin{array}{l} \cdot \quad \text{scc}(u) = \text{scc}(v), \text{ or} \\ \cdot \quad \text{scc}(u) \cap \text{scc}(v) = \emptyset \end{array}\right\}$$

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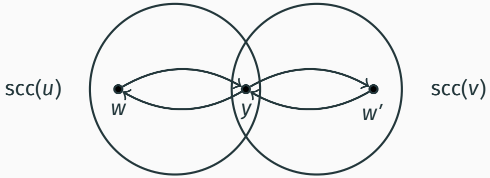

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$$
\begin{tikzcd}
v_{00} \arrow[r] & v_{10} \arrow[d] & v_{20} \arrow[dr] & v_{30} \arrow[l] \\
v_{01} \arrow[r] \arrow[d] & v_{11} \arrow[r] \arrow[drr] & v_{21} \arrow[u] \arrow[d] \arrow[ull] & v_{31} \arrow[u] \arrow[d] \\
v_{02} \arrow[dr] & v_{12} \arrow[r] \arrow[dl] & v_{22} \arrow[r] \arrow[ur] & v_{32} \arrow[d] \arrow[dl] \\
v_{03} \arrow[u] & v_{13} \arrow[u] \arrow[r] & v_{23} \arrow[u] & v_{33}
\end{tikzcd}
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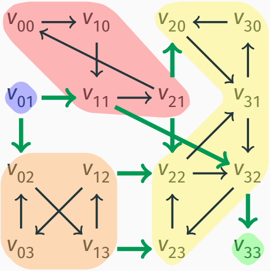

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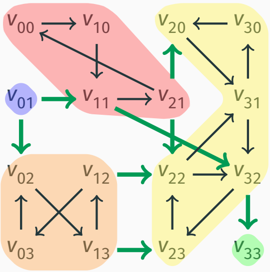
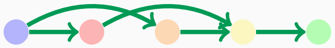


Digraph of strongly connected components: SCC(G)



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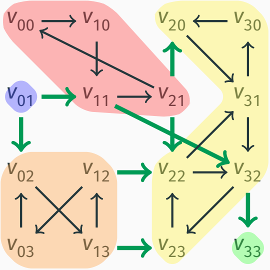
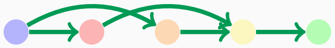


Digraph of strongly connected components: $\text{SCC}(G)$



Question: Is $\text{SCC}(G)$ always a DAG?

Finding the strongly connected components

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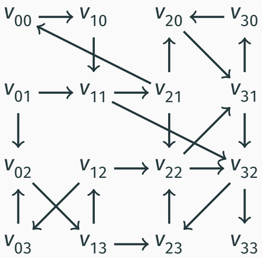

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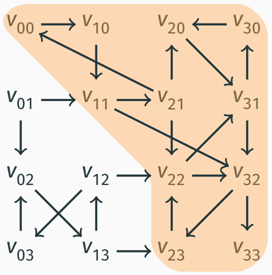

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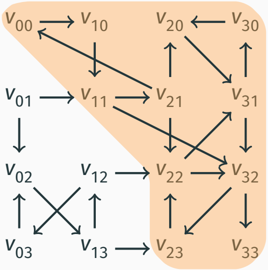

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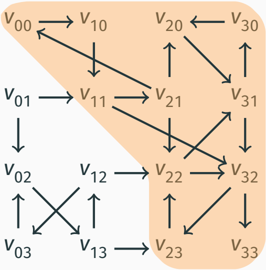

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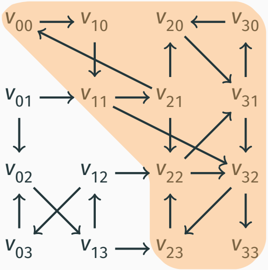

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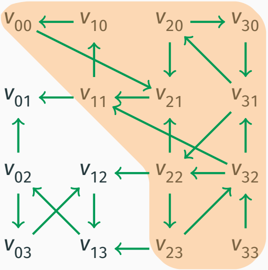

$\mathrm{rev}(G)$

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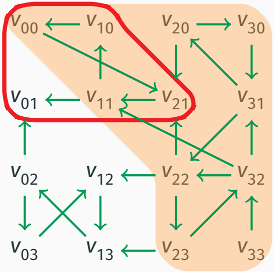

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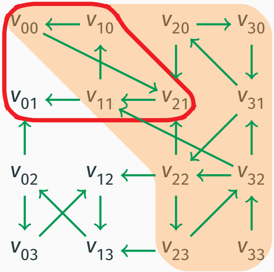

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$$\mathrm{scc}(v) = \mathrm{reach}(v) \cap \mathrm{reach}^{-1}(v)$$

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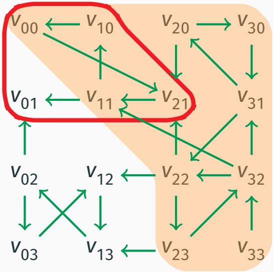

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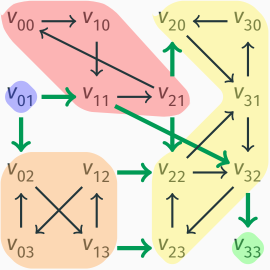
Exercise: How do we find $\mathrm{rev}(G)$ in $O(|V| + |E|)$ time?

Finding the strongly connected components



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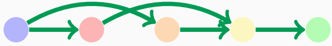
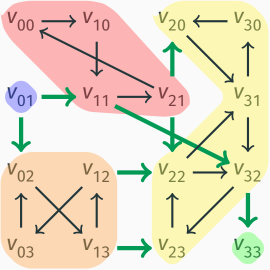


Finding the strongly connected components



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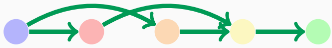
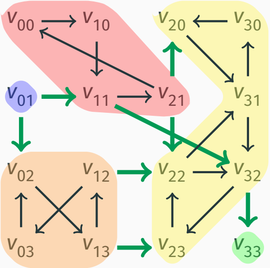


Question: For which vertices v, do we have scc(v) = reach(v)?

Finding the strongly connected components



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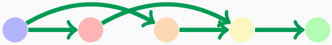

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iff v is a vertex in a sink component of SCC(G)

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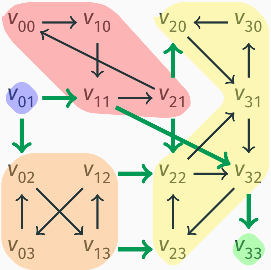

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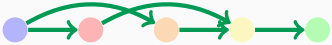

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Algorithm idea:

- Find a vertex v in a sink component of SCC(G)
- Find reach(v)
- Remove it from the graph and continue...

Finding the strongly connected components

Question: How do we find a vertex v in a sink component of $\mathrm{SCC}(G)$?

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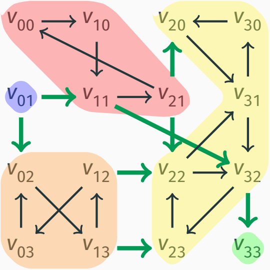
A different question: How do you find a vertex v in a source component of $\mathrm{SCC}(G)$?



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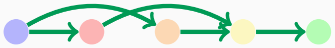
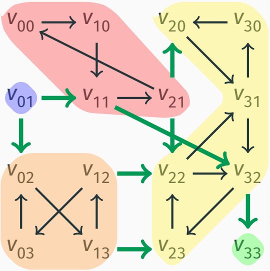


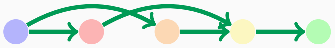
Finding the strongly connected components

Question: How do we find a vertex v in a sink component of SCC(G)?

A different question: How do you find a vertex v in a source component of SCC(G)?



Digraph of strongly connected components: SCC(G)



Which vertex v has the largest end[v] among all the vertices?

Finding the strongly connected components

Theorem: The vertex v with the largest value of end[v] must lie in a source component of SCC(G)

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- If DFS visits vertex in C_1 before C_2, then the first vertex w visited in C_1 has end[v] greater than all the vertices in C_2
- If DFS visits a vertex in C_2 first, then first vertex w visited in C_1 has start[w] > end time of all vertices in C_2

Finding the strongly connected components

But, how do we get a vertex in the sink component?

Finding the strongly connected components

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Proof sketch:

- A subset $C \subseteq V$ is a strongly connected component of G iff it is a strongly connected component of rev(G)
- SCC(G) and SCC(rev(G)) have the same set of vertices
- An edge goes from component C_i to C_j in rev(SCC(G)) iff an edge goes from C_i to C_j in rev(G)

Finding the strongly connected components

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Lemma: rev(SCC(G)) = SCC(rev(G))

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- An edge goes from component C_i to C_j in rev(SCC(G)) iff an edge goes from C_i to C_j in rev(G)

To obtain the source components of rev(SCC(G)), it is sufficient to perform DFS on rev(G)

The Kosaraju-Sharir algorithm

The Kosaraju-Sharir algorithm

Phase 1: Perform DFS on rev(G)

DFS(v)
mark v
foreach edge $u \to w$ **do**
    if w is unmarked **then**
        DFS(w)
push(v)

The Kosaraju-Sharir algorithm

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Phase 2: Perform DFS in the order of the stack

DFS(G)
count $\leftarrow$ 1
foreach $v \in G$ **do**
    set cc[v] $\leftarrow$ 0
while stack is non-empty **do**
    $v \leftarrow$ pop()
    if cc[v] = 0 **then**
        LabelCC(v, count)
        count $\leftarrow$ count +1

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The Kosaraju-Sharir algorithm

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DFS(v)
mark v
foreach edge u → w do
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vertices pushed in the order of ending times

- Perform DFS on rev(G), and order according to end times
- Perform DFS on G in this order and label components
- Total running time: $O(|V| + |E|)$

Phase 2: Perform DFS in the order of the stack

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Shortest paths in graphs

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- The graphs could be weighted or unweighted - shortest path in terms of edge-weights
- All the algorithms actually solve a more general problem - compute the shortest distances from s to all the other nodes (Single-Source Shortest Path)
- Unweighted case - already seen. But, let's recall...

Unweighted graphs: Breadth-First Search

$\textsc{ShortestPath}(G)$

foreach $v \in G$ **do**
    $\text{dist}(v) \leftarrow \infty$
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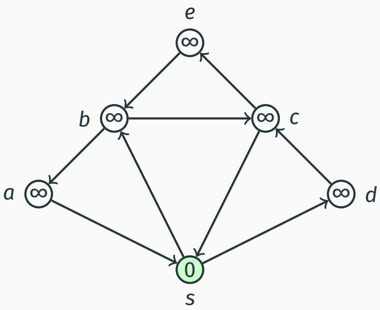

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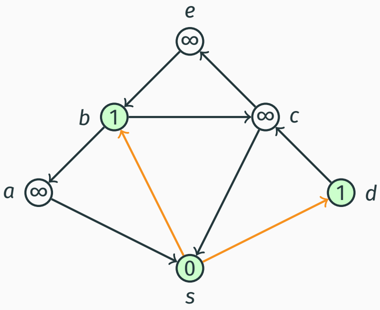

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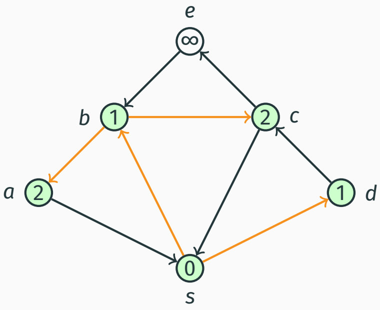

$d\ a\ c$

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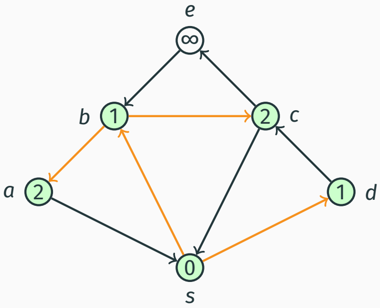

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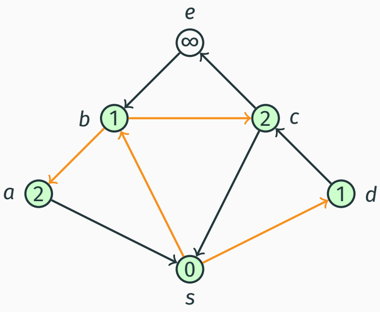

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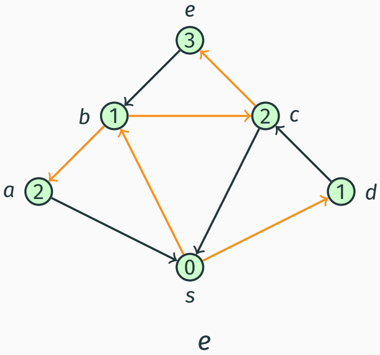

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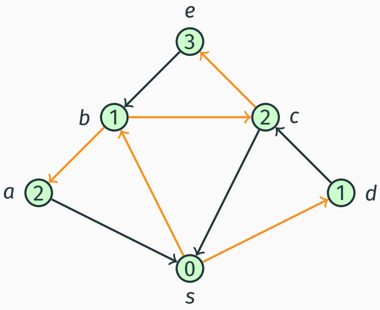

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Unweighted graphs: Breadth-First Search

Theorem: When ShortestPath(G) ends, dist(v) is the length of the shortest path from s to v for every $v \in G$

Proof:

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dist(v) must be at most the length of the shortest path from s to v, and
dist(v) is the length of an actual path from s to v

Weighted DAGs

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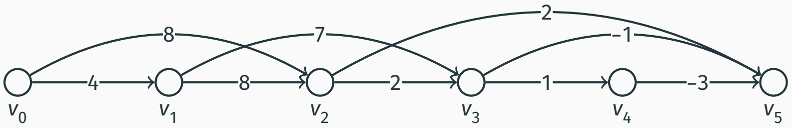

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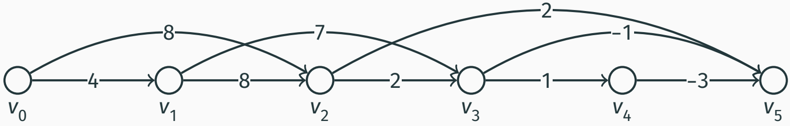

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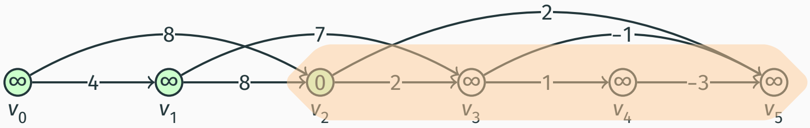

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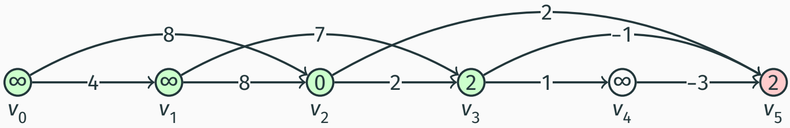

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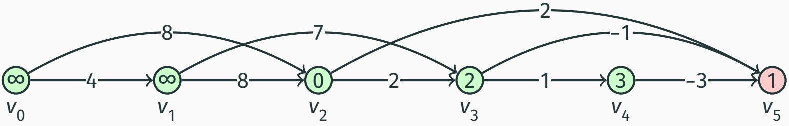

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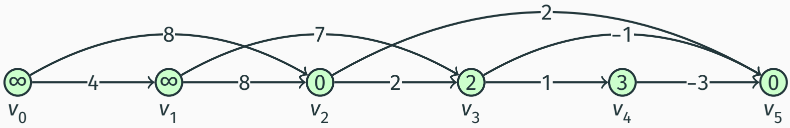

$s = v_2$

Weighted DAGs

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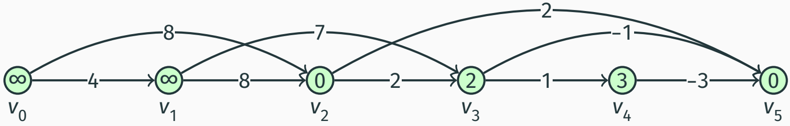

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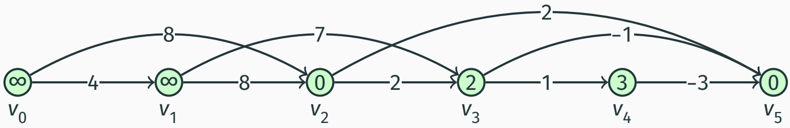

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What is the dependency digraph?

Weighted DAGs

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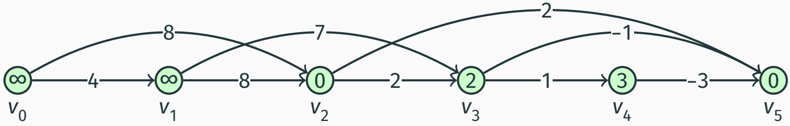

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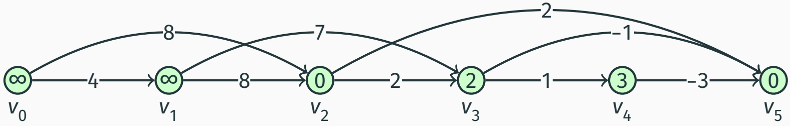
$$\text{dist}(v) = \min_{(u,v)\in E} \{\text{dist}(u) + w(u, v)\}$$

- The dependency digraph of the recurrence is $\text{rev}(G)$
- The order in which the recursive solutions are computed corresponds to the topological order of G

Weighted DAGs

- Only need to compute for vertices v after s in the topological order
- Multiple iterations before dist(v) is correctly computed



$s = v_2$

SSSP-DAG(G, s)

foreach $v \in G$ **do**
  dist(v) $\leftarrow \infty$
dist(s) $\leftarrow 0$
foreach $v \in G$ in topological order **do**
  foreach edge $v \to w$ **do**
    if dist(w) > dist(v) + wt(v, w) **then**
      dist(w) $\leftarrow$ dist(v) + wt(v, w)

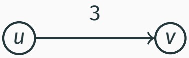
General weighted (di)graphs

Points to remember:

- Graphs cannot contain negative weight cycles
- For now, assume that there are no negative weight edges also

A BFS-based idea

A BFS-based idea



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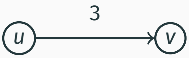


A BFS-based idea

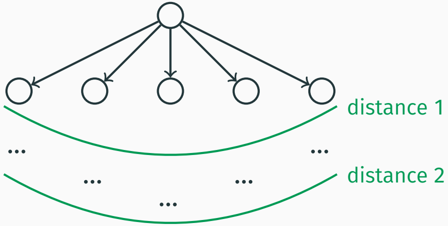
- Replace an edge of weight w with a path of length w with edge weight 1
- Perform BFS on the new graph

A BFS-based idea

- Replace an edge of weight w with a path of length w with edge weight 1
- Perform BFS on the new graph

Question: Does this algorithm compute the shortest paths correctly?

Generalizing the BFS idea



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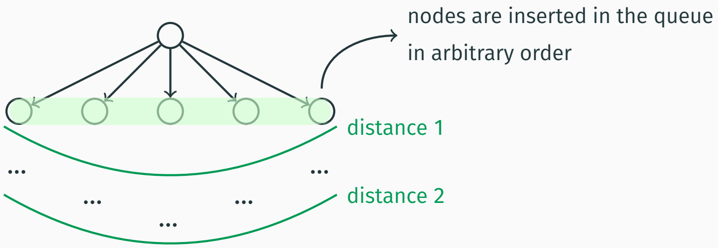

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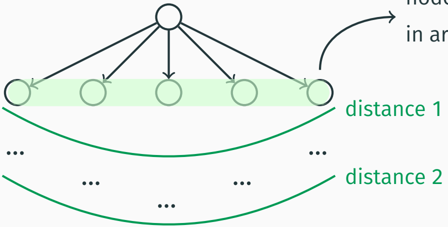
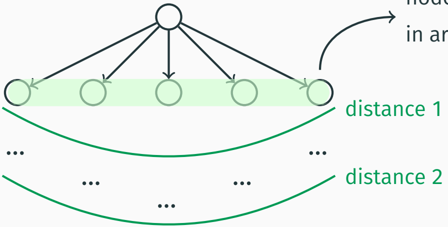


nodes are inserted in the queue
in arbitrary order

What happens when edges have weights?

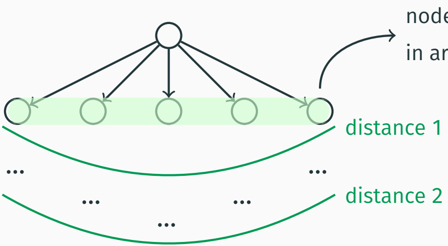
Generalizing the BFS idea



What happens when edges have weights?

- Maintain a set S of vertices whose shortest distances are computed, and expand this set one vertex at a time.

Generalizing the BFS idea



What happens when edges have weights?

- Maintain a set S of vertices whose shortest distances are computed, and expand this set one vertex at a time.
- For BFS - add a neighbor of a vertex in S
- What should we do for weighted graphs?

Dijkstra's algorithm for SSSP

How do we choose the next vertex to add to the set S?

Dijkstra's algorithm for SSSP

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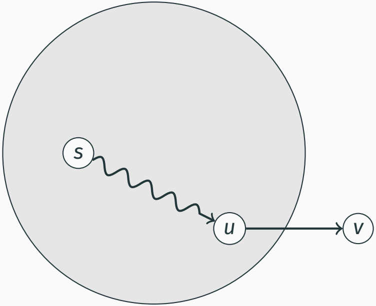

Choose v that minimizes

$\text{dist}(u) + \text{wt}(u, v)$ over all $u \in S$

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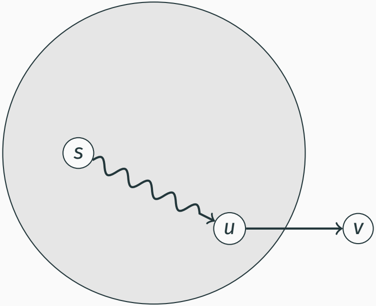

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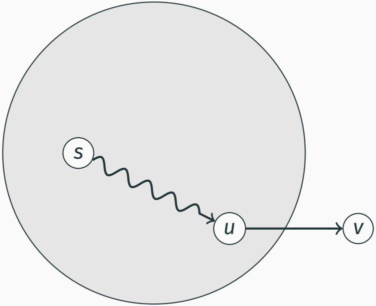

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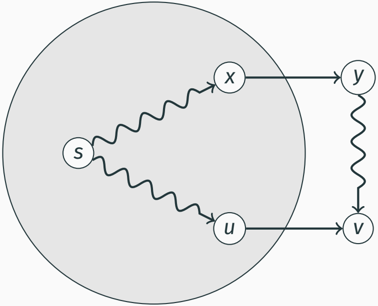

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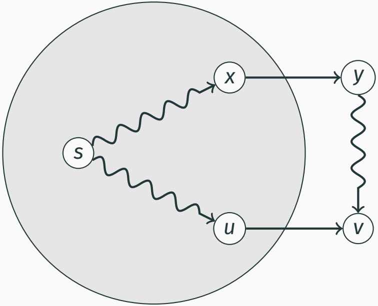

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$\mathrm{dist}(x) + \mathrm{wt}(x, y) \geq dist(u) + \mathrm{wt}(u, v)$ and $\mathrm{dist}(y, v) \geq 0$

Dijkstra's algorithm for SSSP

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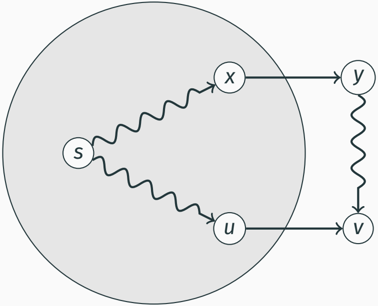

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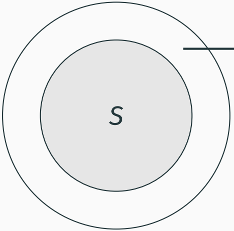
Proof: Suppose not!

$dist(x) + wt(x, y) \geq dist(u) + wt(u, v)$ and $dist(y, v) \geq 0$

Question: How do we find the vertex v efficiently?

Dijkstra's algorithm for SSSP



precomputed $\text{dist}(v) + w(u, v)$ values

- Choose the v with the smallest $\text{dist}(v) + w(u, v)$ where $u \in S$
- Once a v is chosen, recompute the $\text{dist}(w) + w(u, w)$ values for vertices lying outside $S \cup \{v\}$

need to recompute this
for only neighbors of v

Dijkstra's algorithm for SSSP

```
ShortestPath(G, s)
foreach v ∈ G do
    dist(v) ← ∞
dist(s) ← 0
enqueue(s)
while  queue is not empty do
    v ← dequeue()
    foreach edge v → w do
        if dist(w) = ∞  then
            dist(w) ← dist(v) + 1
            enqueue(w)
```

Dijkstra's algorithm for SSSP

```
DIJKSTRA(G, s)

foreach v ∈ G do
    dist(v) ← ∞
dist(s) ← 0
INSERT(s, 0)
while priority queue is not empty do
    v ← EXTRACTMIN()
    foreach edge v → w do
        if dist(w) > dist(v) + wt(v, w) then
            dist(w) ← dist(v) + wt(v, w)
            if w is in the priority queue then
                DECREASEKEY(w, dist(w))
            else
                INSERT(w, dist(w))
```

Dijkstra's algorithm for SSSP

```
Dijkstra(G, s)
foreach v ∈ G do
    dist(v) ← ∞
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foreach v ∈ G do
    Insert(v, dist(v))
while priority queue is not empty do
    v ← ExtractMin()
    foreach edge v → w do
        if dist(w) > dist(v) + wt(v, w) then
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            DecreaseKey(w, dist(w))
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Dijkstra's algorithm for SSSP

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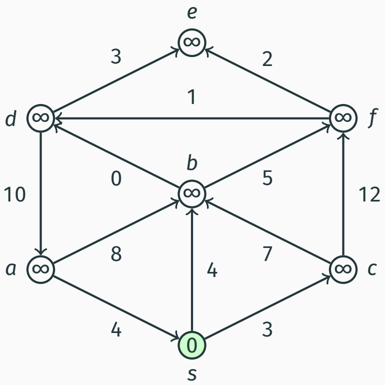

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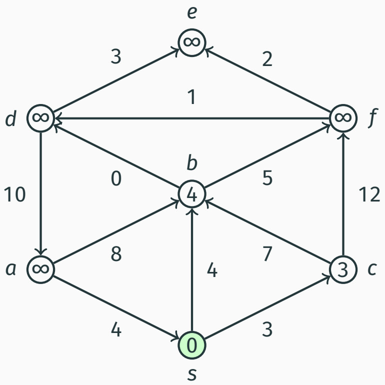

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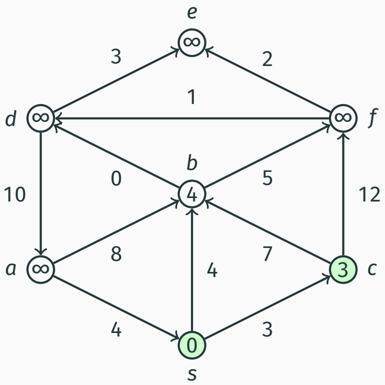

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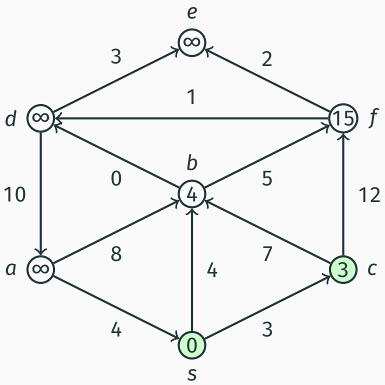

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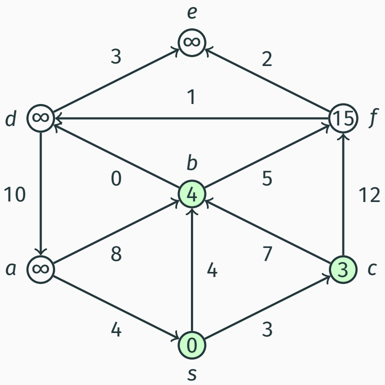

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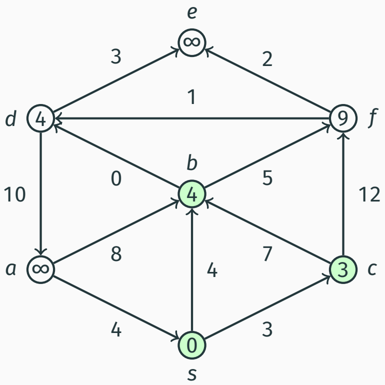

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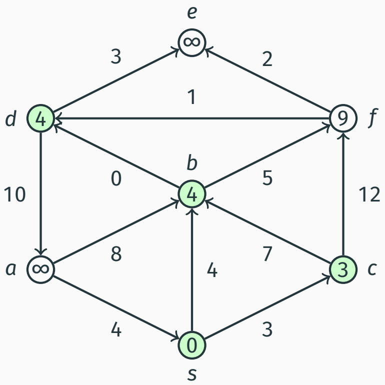

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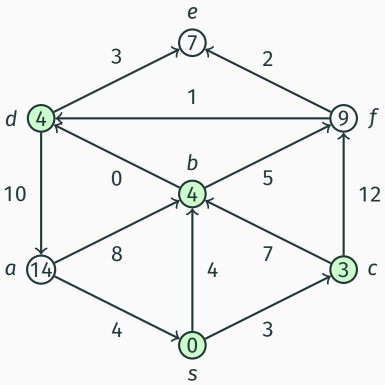

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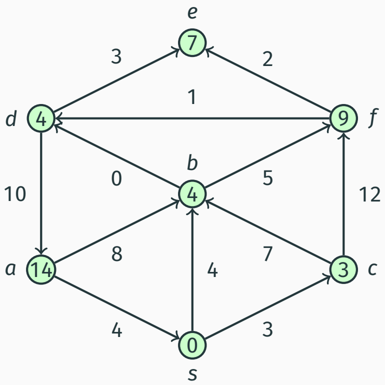

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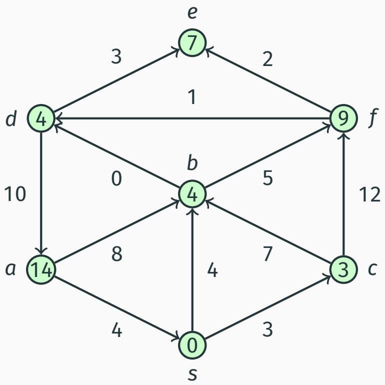

- Each vertex is inserted in the priority queue, and extracted exactly once

Dijkstra's algorithm for SSSP

$\textsc{Dijkstra}(G, s)$

foreach $v \in G$ **do**
    $\text{dist}(v) \leftarrow \infty$
$\text{dist}(s) \leftarrow 0$
foreach $v \in G$ **do**
    $\textsc{Insert}(v, \text{dist}(v))$
while priority queue is not empty **do**
    $v \leftarrow \textsc{ExtractMin}()$
    foreach edge $v \rightarrow w$ **do**
        if $\text{dist}(w) > \text{dist}(v) + \text{wt}(v, w)$ **then**
            $\text{dist}(w) \leftarrow \text{dist}(v) + \text{wt}(v, w)$
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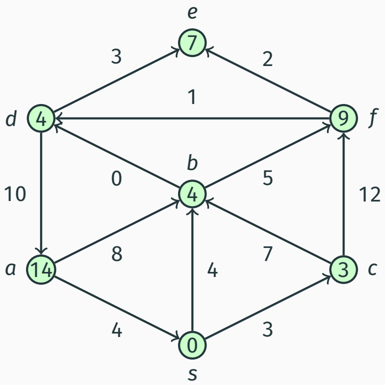

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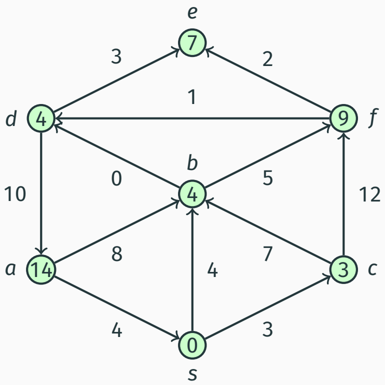

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Running time

- $O((|V| + |E|) \log |V|)$ (using binary heaps)
- $O(|V| \log |V| + |E|)$ (using Fibonacci heaps)

SSSP with negative-weight edges

SSSP with negative-weight edges

- Already seen in the case of DAGs

SSSP with negative-weight edges

- Already seen in the case of DAGs
- Dijkstra's algorithm fails when there are negative-weight edges - can you explain why?

SSSP with negative-weight edges

- Already seen in the case of DAGs
- Dijkstra's algorithm fails when there are negative-weight edges - can you explain why?
- Problem is ill-posed when the graph contains negative-weight cycles
- But, can you identify this situation and fail gracefully?

SSSP with negative-weight edges

Observations

- dist(v) is always at least as large as the shortest distance to v
- Every edge is relaxed at most once in a run of Dijkstra's algorithm

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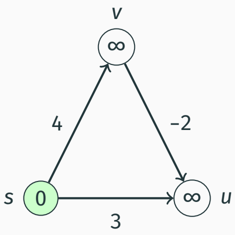

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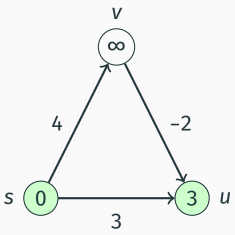

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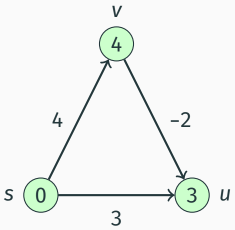

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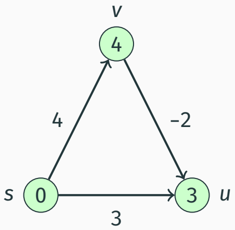
- dist(v) is always at least as large as the shortest distance to v
- Every edge is relaxed at most once in a run of Dijkstra's algorithm



one more update will set the distances correctly

The Bellman-Ford algorithm

Recall the recurrence

$$\text{dist}(v) = \min_{(u,v) \in E} \{\text{dist}(u) + w(u, v)\}$$

The Bellman-Ford algorithm

Modified recurrence

$$\mathrm{dist}_{\leq i}(v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v = s \\ \infty & \text{if } i = 0 \text{ and } v \neq s \\ & \\ & \end{cases}$$

shortest distance from s to v
by paths with $\leq i$ edges

The Bellman-Ford algorithm

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shortest distance from s to v

by paths with $\leq i$ edges

- Have to compute $\mathrm{dist}_{\leq i}(v)$ for $i \leq n - 1$
- In every iteration, relax all the edges

if $\mathrm{dist}(v) > \mathrm{dist}(u) + w(u,v)$ **then**
$\mathrm{dist}(v) = \mathrm{dist}(u) + w(u,v)$

The Bellman-Ford algorithm

```
BellmanFord(G, s)
foreach v ∈ G do
    dist(v) ← ∞
dist(s) ← 0
repeat |V| − 1 times
    foreach edge u → v do
        if dist(v) > dist(u) + wt(u, v) then
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Lemma: Let $\text{dist}_{\leq i}(v)$ be the distance of the shortest path from s to v using at most i edges. Then, after the i^{th} iteration of the **repeat** loop for every vertex v,

$$\text{dist}(v) \leq \text{dist}_{\leq i}(v)$$

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Proof: Induction on i
Base case $i = 0$: $\text{dist}(s)$ is set to 0, and the rest to ∞

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must be a simple path!

Proof: Induction on i
Induction step $s \to u_1 \to \dots u_k \to v$: shortest path to v of length $\le i$

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Induction step $s \to u_1 \to \dots u_k \to v$: shortest path to v of length $\le i$
After the $(i-1)^{st}$ iteration, $\text{dist}(u_k) \le \text{dist}_{i-1}(u_k)$
After the i^{th} iteration, $\text{dist}(v) \le \text{dist}(u_k) + \text{wt}(u, v) \le \text{dist}_{\le i-1}(u_k) + \text{wt}(u, v) \le \text{dist}_{\le i}(v)$

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Running time: $O(|V||E|)$

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Question: How do we detect if G has a negative weight cycle?