All Pairs Shortest Paths (APSP)

Johnson's algorithm

- Repeat the SSSP from each vertex in the graph.
  - If the edge weights are non-negative
    - * $O(VE\log V)$ time using Binary heaps $\leftarrow O(V^3\log V)$
    - * $O(VE + V^2\log V)$ time using Fibonacci heaps $\leftarrow O(V^3)$
  - If the edge weights are negative
    - * $O(V^2E)$ - repeated Bellman-Ford
      - ↳ $O(V^4)$ - time algorithm

Convert the edge weights so that they are non-negative & perform Dijkstra

Observation: Let $\pi : V \to \mathbb{R}$ be any function

Define $w'(u,v) = \pi(u) + w(u,v) - \pi(v)$

Then $G(V,E,w')$ has the same shortest paths as in $G(V,E,w)$

Proof: For any path $u = u_0 \to u_1 \to u_2 \to \cdots \to u_k = v$

$$w'(u \rightsquigarrow v) = \sum_{i=0}^{k-1} \pi(u_i) + w(u_i, u_{i+1}) - \pi(u_{i+1})$$

$$= \pi(u_0) + w(u \rightsquigarrow v) - \pi(u_k)$$

Thus all the path lengths with w' are just $\pi(u) - \pi(v)$ + original lengths

Goal: Construct $\pi : V \to \mathbb{R}$ s.t

$\forall (u,v) \in E \quad \pi(u) + w(u,v) - \pi(v) \geq 0$

Construct a new vertex s & add edges (s,u) $\forall$ $u \in V$

Let $d(s,u)$ be the shortest path from s to u (Say obtained by BF)

Then $\pi(u) = d(s,u)$

$\pi(u) + w(u,v) - \pi(v) \geq 0$ because BF halted

Outline of Johson's algo:

Running time: $O(VE)$

1. Run BF and compute π
2. Change the weight to $\pi(u) + w(u,v) - \pi(v)$
3. Run Dijkstra (using Fib heaps) $\forall u \in V$

Dynamic Programming - based solutions

1. Bellman-Ford - based recurrence

$d_{\le i}(u,v)$ - shortest distance from u to v with $\le i$ edges

$$d_{\le i}(u,v) = \min \begin{cases} d_{\le i-1}(u,v) \\ \min\limits_{(w,v)\in E} \left\{ d_{\le i-1}(u,w) + wt(w,v) \right\} \end{cases}$$

Running-time based on this recurrence: $O(V^2 E)$

$\forall\ i \in \{0, n-1\}$

$\quad \forall\ u \in V$

$\quad\quad \forall\ e = (x,y) \in E$

$\quad\quad\quad$ if $d(u,x) + w(x,y) < d(u,y)$

$\quad\quad\quad\quad d(u,y) = d(u,x) + w(x,y)$

② Fischer-Meyer recurrence

(Divide-&-conquer)

$d(u,v;i)$ - shortest path from u to v with $\leq 2^i$ edges

$$d(u,v;i) = \min_{w \in V} \{ d(u,w,i-1) + d(w,v,i-1) \}$$

Base case: $d(u,v,0) = \begin{cases} \infty & \text{if } (u,v) \notin E \\ w(u,v) & \text{o/w} \end{cases}$

$d(u,u,0) = 0$

$V^3 \log V$

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∀ i ∈ {0, 1, .., log V}
    for ∀ u ∈ V
        for ∀ v ∈ V
            for ∀ w ∈ V
                if d(u,v) > d(u,w) + d(w,v)
                    d(u,v) = d(u,w) + d(w,v)
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