Recursion and Divide-and-Conquer

CS2800: Design and Analysis of Algorithms

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Module plan

1. Asymptotic analysis
2. Integer Multiplication Redux
3. Order statistics
4. Fast Fourier Transform
5. Closest pair of points

Asymptotic analysis

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Definitions of O, Ω, and Θ

- A function $f(n) = O(g(n))$ if $\exists c > 0$ and $n_0 \geq 1$ such that $f(n) \leq cg(n), \forall n \geq n_0$
- A function $f(n) = \Omega(g(n))$ if $\exists c > 0$ and $n_0 \geq 1$ such that $f(n) \geq cg(n), \forall n \geq n_0$
- A function $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

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Little-omega

- A function $f(n) = \omega(g(n))$ if $\forall c > 0$, there exists an n_0 s.t $f(n) > cg(n), \forall n \geq n_0$
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$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$
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- If n is even (check the LSB), answer ”no”
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$$f(n) = \begin{cases} 1 & \text{if n is even,} \\ \log n & \text{otherwise} \end{cases}$$

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$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \log n & \text{otherwise} \end{cases}$$

- $f(n) = O(\log n)$, but not $\Theta(\log n)$
- $f(n) = \Omega(1)$, but not $\Omega(g(n))$ for any function that grows with n
- $f(n) \neq o(\log n)$

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- iterated logarithm - $\log^* n$: number of times log has to be taken before n is at most 1
    - $\log^* n \leq 4$ for all practical values of n
- Ackermann function
    - $\alpha(n, 0)$ - increment by 1
    - $\alpha(n, 1)$ - perform incremement n times: addition
    - $\alpha(n, 2)$ - perform addition n times: multiplication
    - $\alpha(n, 3)$ - perform multiplication n times: exponentiation
    - $\alpha(n, 4)$ - perform exponentiation n times: ? $\ldots$

Integer Multiplication Redux

Recall: Peasant multiplication

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Input: Integers a and b
Output: $c = a \cdot b$
Set $c \leftarrow 0$
repeat
  if a is odd **then** $c \leftarrow c + b$;
  $b \leftarrow 2 \cdot b$
  $a \leftarrow \lfloor \frac{a}{2} \rfloor$
until $a = 0$;

Running time:

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Running time:

- The main while loop runs for $\log a$ many iterations
- Each operation in the while loop takes at most $\log a + \log b$ steps
- $O(n^2)$ time to multiply two n-digit numbers

Faster integer multiplication

Recursive solution:

$$(10^{n/2}a + b)(10^{n/2}c + d) = 10^{n}(ac) + 10^{n/2}(ad + bc) + (bd)$$

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Running time?

Analyzing recurrence relations

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$$\underbrace{T(n)}_{\text{running time on inputs of size n}} = \underbrace{4}_{\text{\# recursive calls}}\, T(\underbrace{\tfrac{n}{2}}_{\text{size of sub-problems}}) + \underbrace{O(n)}_{\text{adding the sub-products}}$$

Analyzing recurrence relations: Recursion trees

n

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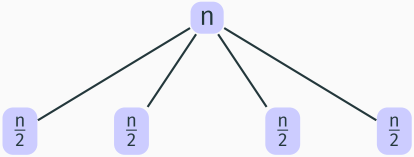

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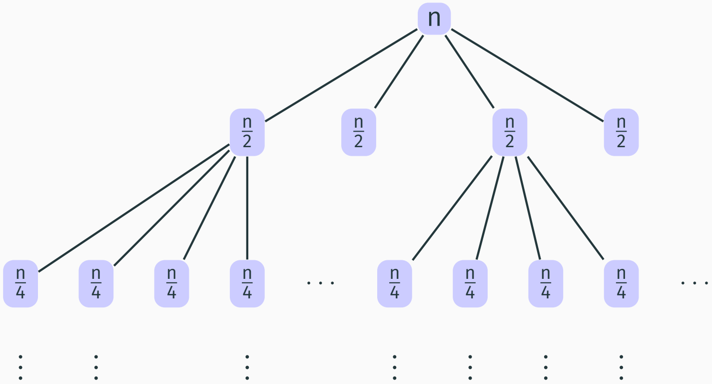

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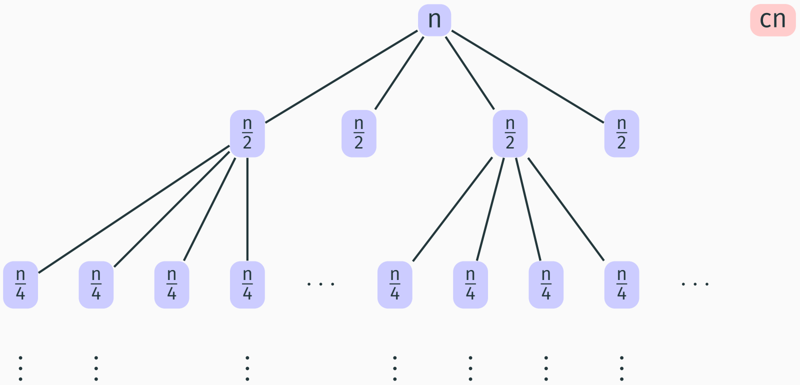

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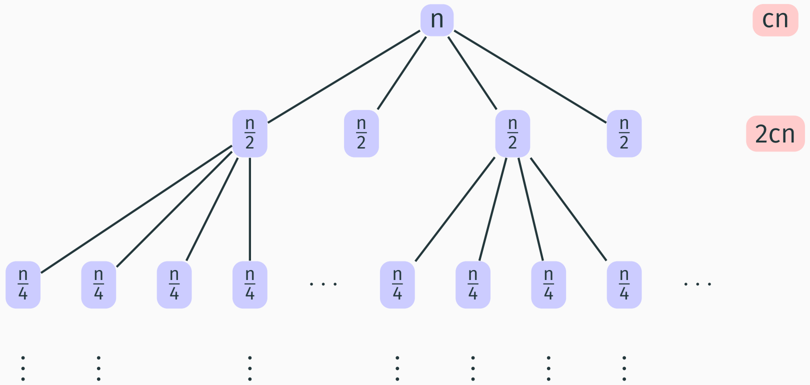

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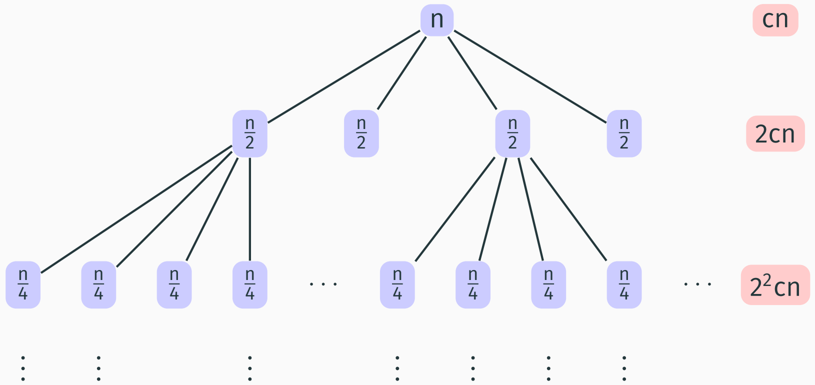

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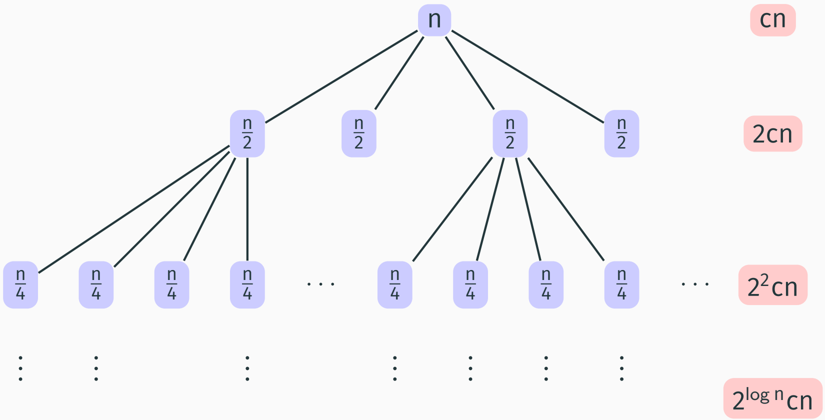

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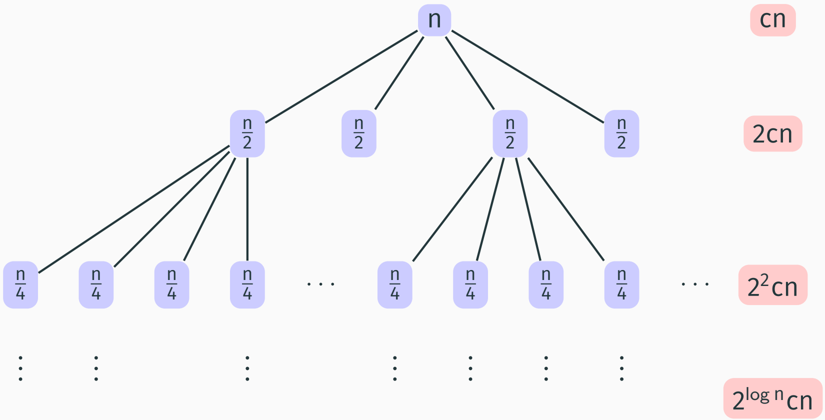

Running time: $cn(1 + 2 + 2^2 + \cdots + 2^{\log n}) = O(n^2)$

Recursion trees: The master theorem

Solving recurrence relations of the form $T(n) = aT(\frac{n}{b}) + f(n)$?

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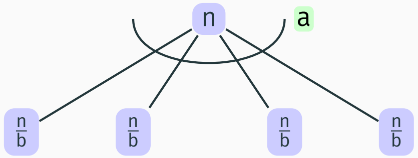

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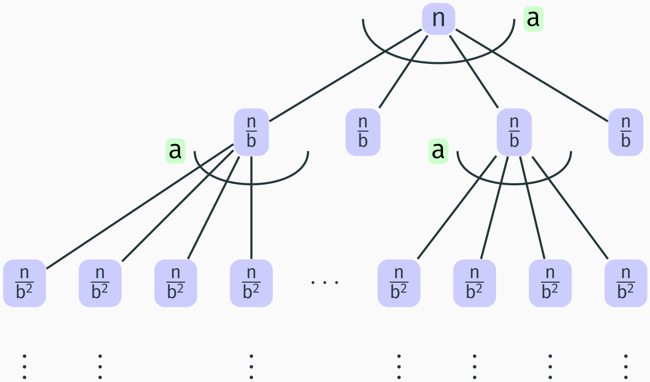

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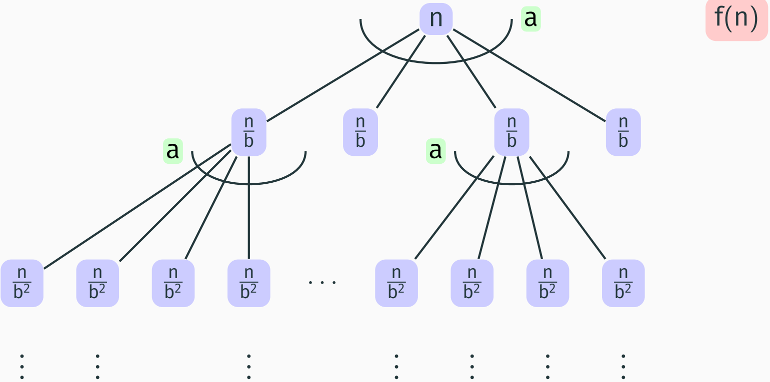

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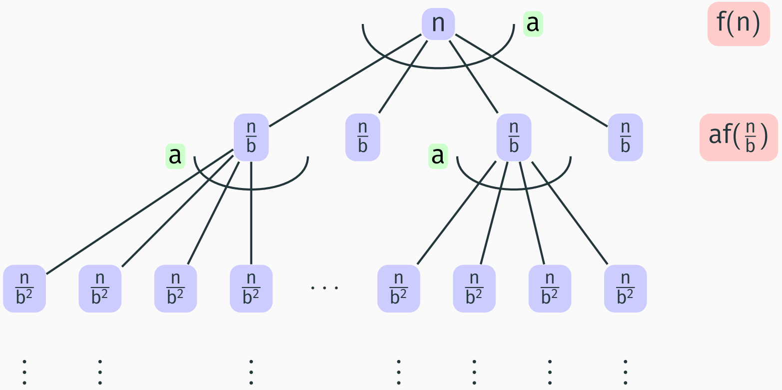

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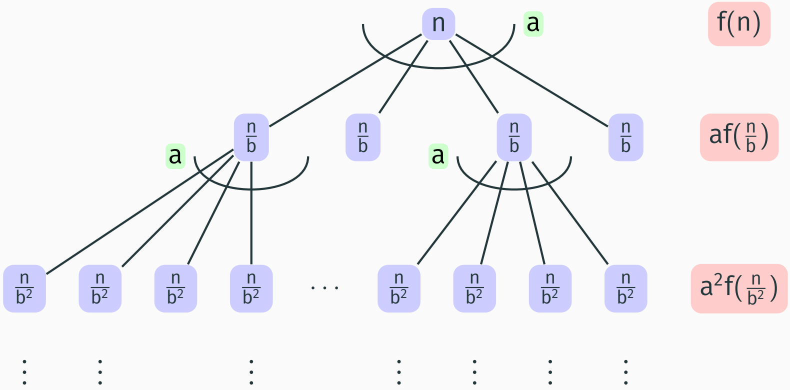

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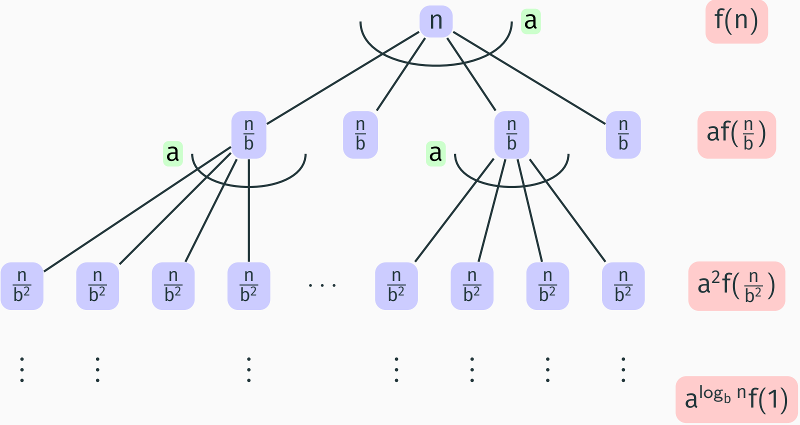

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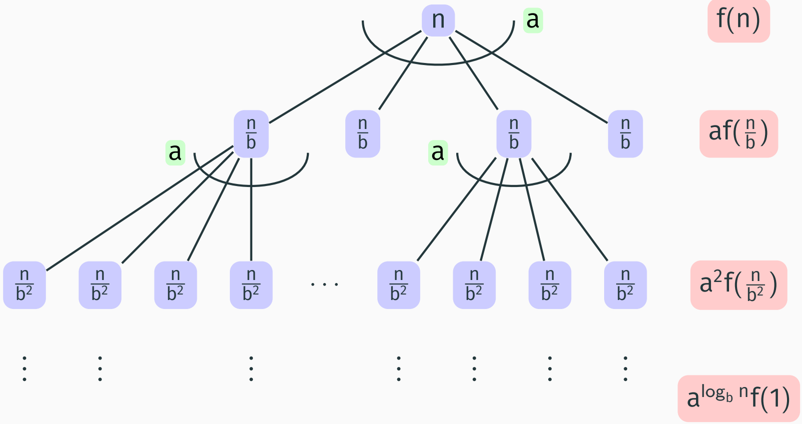

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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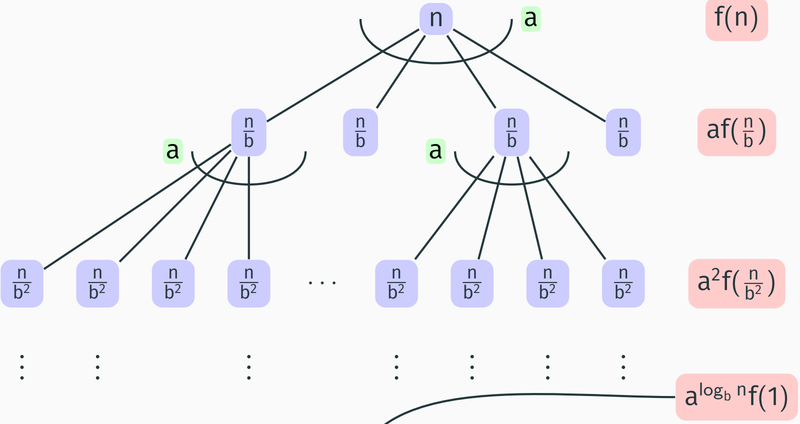

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- If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- If $f(n) = \Omega(n^{\log_b a + \epsilon})$ and $af(n/b) \leq cf(n)$ for some constant $c < 1$, then $T(n) = \Theta(f(n))$

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Points to remember:

- Floors and ceilings are not important in proving the upper bounds (for most cases)
- Use recursion trees directly when the master theorem cannot be applied

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Running time: $T(n) = 3T(n/2) + O(n)$

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Is the master theorem applicable here?

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Is the master theorem applicable here? $T(n) = \Theta(n^{\log 3})$

Integer multiplication: A short history

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Peasant multiplication - $O(n^2)$	1600 BC
Lattice multiplication - $O(n^2)$	1200
Karatsuba's method - $O(n^{\log 3})$	1960
Schönhage-Strassen - $O(n \log n \log\log n)$	1971
Conjecture: $\Theta(n \log n)$	
Fürer/De-Saha-Kurur-Saptharishi - $O(2^{\Theta(\log^* n)} n \log n)$	2008
Harvey-van der Hoeven - $O(2^{2\log^* n} n \log n)$	2018
Harvey-van der Hoeven - $O(n \log n)$	2019

"...our work is expected to be the end of the road for this problem, although we don't know yet how to prove this rigorously."

Order statistics

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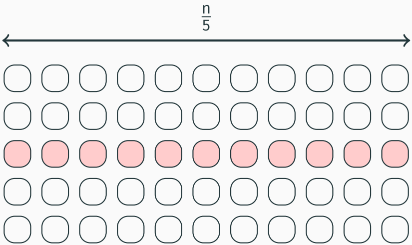
- Sort the array, and return $A[k] \longrightarrow O(n \log n)$
- If we find a pivot partitioning A into two parts of size αn and $(1 - \alpha)n$ in $O(n)$ time, then we have a recursive algorithm with running time

$$T(n) = T(\alpha n) + O(n)$$

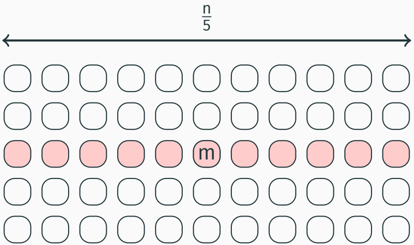
Median of medians

Median of medians



- Divide the array A into n/5 subarrays of length 5 and find their medians

Median of medians



- Divide the array A into n/5 subarrays of length 5 and find their medians
- How good is the median of these medians as a pivot?

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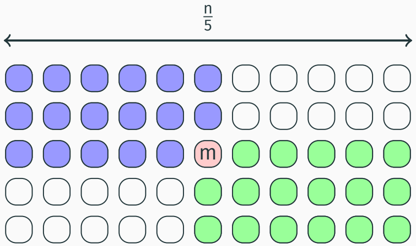

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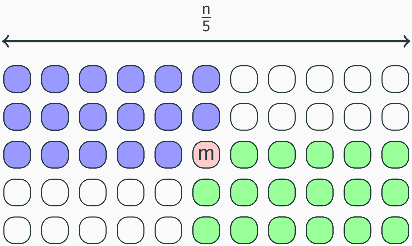
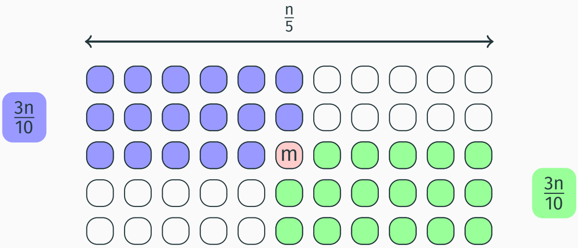


- Divide the array A into n/5 subarrays of length 5 and find their medians
- How good is the median of these medians as a pivot?
- All the blue elements are $\leq$ m and all the green elements are $\geq$ m - How many are there?

Median of medians



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A recursive algorithm

Input: Array $A[1, 2, \ldots, n]$, k

if $n \leq 10$ **then**
    brute-force search
else
    set $d \leftarrow n/5$
    for $1 \leq i \leq d$ **do**
        set $M[i] \leftarrow$ median of $A[5i - 4, \ldots, 5i]$
    set $m \leftarrow$ median of M recursively
    $r \leftarrow \textsc{Partition}(A[1, 2, \ldots, n], m)$
    if $k < r$ **then**
        recursively search in $A[1, 2, \ldots, m - 1]$ for k
    else
        if $k > r$ **then**
            recursively search in $A[m + 1, \ldots, n]$ for $k - r$
        else
            return m

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  for $1 \leq i \leq d$ **do**

    set $M[i] \leftarrow$ median of $A[5i - 4, \ldots, 5i]$

  set $m \leftarrow$ median of M recursively $T(\frac{n}{5})$

  $r \leftarrow$ PARTITION$(A[1, 2, \ldots, n], m)$ $O(n)$

  if $k < r$ **then**

    recursively search in $A[1, 2, \ldots, m - 1]$ for k

  else $T(\frac{7n}{10})$

    if $k > r$ **then**

      recursively search in $A[m + 1, \ldots, n]$ for $k - r$

    else

      return m

Running time - median of medians algorithm

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- Is our algorithm bad or analysis not tight?

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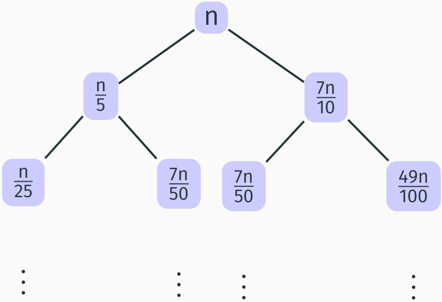

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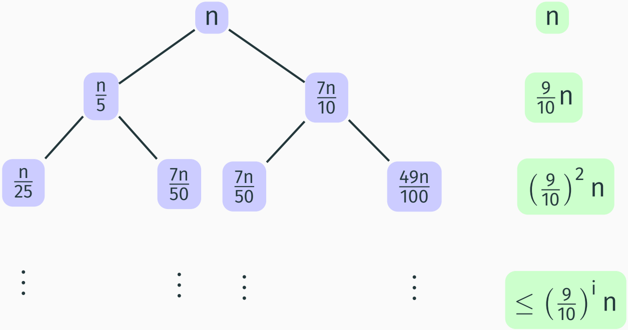

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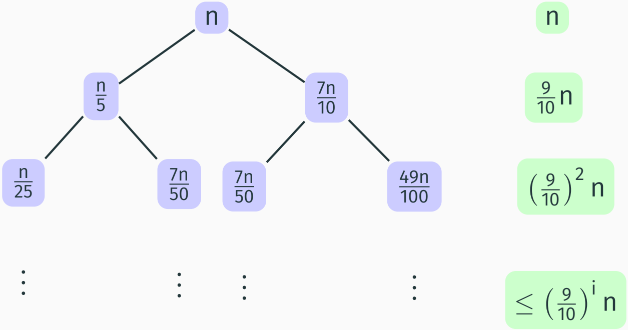

$$T(n) \leq \left(1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots\right) n = O(n)$$

Fast Fourier Transform

Convolution of vectors

Given $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$, $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})$, compute the value

$$\mathbf{a} * \mathbf{b} = (c_0, c_1, \ldots, c_{2n-1}), \text{ where}$$
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- **Signal processing:** - smoothing discrete measurements, analyzing response of a linear time-invariant system

Convolution of vectors

The naive algorithm:

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- First described by Gauss in 1805
- Modern version by Cooley and Tukey in 1965

Vectors and polynomials

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$$P(x) = \sum_{i=0}^{n-1} a_i x^i,\ Q(x) = \sum_{i=0}^{n-1} b_i x^i \Rightarrow \mathbf{a} * \mathbf{b} \equiv P(x)Q(x)$$

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Naive algorithm: Multiply $a_i x^i$ with $Q(x)$ for each i and then add

Polynomials - an alternate view

A polynomial $P(x)$ of degree $n - 1$ is uniquely defined by its evaluation on n points

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$$\text{vector of coefficients} \longleftarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \equiv \begin{pmatrix} P(\alpha_0) \\ P(\alpha_1) \\ P(\alpha_2) \\ \vdots \\ P(\alpha_{n-1}) \end{pmatrix} \longrightarrow \text{vector of evaluations}$$

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$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} * \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix} \equiv \begin{pmatrix} P(\alpha_0)Q(\alpha_0) \\ P(\alpha_1)Q(\alpha_1) \\ P(\alpha_2)Q(\alpha_2) \\ \vdots \\ P(\alpha_{2n-2})Q(\alpha_{2n-2}) \end{pmatrix}$$

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$$\Big\downarrow \text{pointwise product}$$

$$\begin{array}{c} P(\alpha_0)Q(\alpha_0), \\ P(\alpha_1)Q(\alpha_1), \\ \ldots \\ P(\alpha_{2n-2})Q(\alpha_{2n-2}) \end{array}$$

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$$\begin{tikzcd}
\begin{matrix}(a_0, a_1, \ldots, a_{n-1}) \\ (b_0, b_1, \ldots, b_{n-1})\end{matrix} \arrow[r, "\text{evaluation}"] & \begin{matrix}P(\alpha_0), P(\alpha_1), \ldots, P(\alpha_{2n-2}) \\ Q(\alpha_0), Q(\alpha_1), \ldots, Q(\alpha_{2n-2})\end{matrix} \arrow[d, "\text{pointwise product}"'] \\
P(x)Q(x) & \begin{matrix}P(\alpha_0)Q(\alpha_0), \\ P(\alpha_1)Q(\alpha_1), \\ \ldots \\ P(\alpha_{2n-2})Q(\alpha_{2n-2})\end{matrix} \arrow[l, "\text{interpolation}"']
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evaluation of two $n/2$-degree polynomials at $n/2$ points

Polynomial evaluation - a “complex” approach

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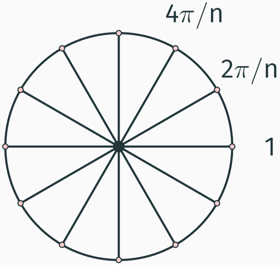

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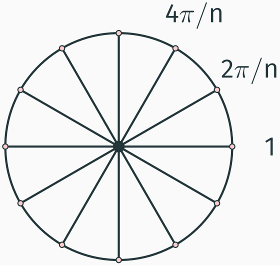

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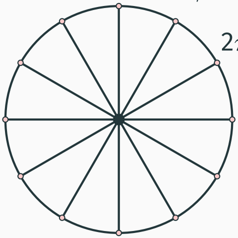

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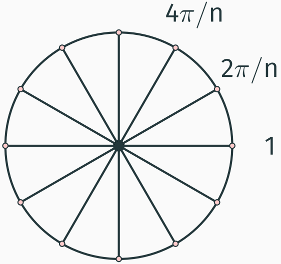

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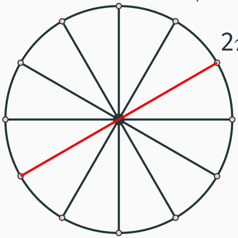

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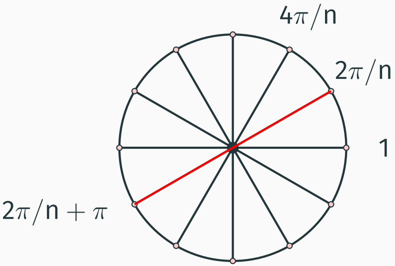

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- $(\omega^j)^2$ are $\frac{n}{2}$-th roots of unity

Polynomial evaluation - the Fast Fourier Transform

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$(n = 2^k)$, ω - n^{th}root of unity
Output: $P(\omega^0), P(\omega), P(\omega^2), \ldots, P(\omega^{n-1})$

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Evaluate $P_e(\omega^2)$
Evaluate $P_o(\omega^2) \longrightarrow$ recursive calls - divide step

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combining solutions - conquer step

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express $P(x) = P_e(x^2) + xP_o(x^2)$

Evaluate $P_e(\omega^2)$

Evaluate $P_o(\omega^2)$ $\longrightarrow$ recursive calls - divide step $2T(\frac{n}{2})$

foreach $j \in \{0, 1, \ldots, n-1\}$ **do**

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Output: $P(\omega^0), P(\omega), P(\omega^2), \ldots, P(\omega^{n-1})$

if $\omega = 1$ **then return** $P(1) \longrightarrow$ base case of recursion $O(n)$

express $P(x) = P_e(x^2) + xP_o(x^2)$

Evaluate $P_e(\omega^2)$

Evaluate $P_o(\omega^2) \longrightarrow$ recursive calls - divide step $2T(\frac{n}{2})$

foreach $j \in \{0, 1, \ldots, n-1\}$ **do**

$P(\omega^j) = P_e(\omega^{2j}) + \omega^j P_o(\omega^{2j})$

$O(n)$

combining solutions - conquer step

Where are we now?

$$P(x) = \sum_{i=0}^{n-1} a_i x^i \quad Q(x) = \sum_{i=0}^{n-1} b_i x^i$$

$$\begin{tikzcd}
\begin{matrix}(a_0, a_1, \ldots, a_{n-1})\\(b_0, b_1, \ldots, b_{n-1})\end{matrix} \arrow[r, "\text{evaluation}", "O(n \log n)"'] & \begin{matrix}P(\alpha_0), P(\alpha_1), \ldots, P(\alpha_{2n-2})\\Q(\alpha_0), Q(\alpha_1), \ldots, Q(\alpha_{2n-2})\end{matrix} \arrow[d, "\text{pointwise product}"', "O(n)"] \\
P(x)Q(x) & \begin{matrix}P(\alpha_0)Q(\alpha_0),\\P(\alpha_1)Q(\alpha_1),\\\ldots\\P(\alpha_{2n-2})Q(\alpha_{2n-2})\end{matrix} \arrow[l, "\text{interpolation}"']
\end{tikzcd}$$

Polynomial evaluation - a matrix view

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Multiplying the Vandermonde matrix with the i^{th} vector of the standard basis gives the i^{th} column of the Vandermonde matrix

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Lemma: The columns of the Vandermonde matrix $V(\omega)$ are orthogonal vectors

Polynomial evaluation - a geometric view

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complex conjugate

$$\begin{pmatrix} 1 & \omega^j & (\omega^j)^2 & (\omega^j)^3 & \cdots & (\omega^j)^{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ \omega^{-k} \\ (\omega^{-k})^2 \\ (\omega^{-k})^3 \\ \vdots \\ (\omega^{-k})^{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} \omega^{i(j-k)}$$

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- If $j = k$, then $\sum_{i=0}^{n-1} \omega^{i(j-k)} = n$
- If $j \neq k$,

$$\sum_{i=0}^{n-1} \omega^{i(j-k)} = \frac{1 - \omega^{n(j-k)}}{1 - \omega^{j-k}}$$

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- If $j = k$, then $\sum_{i=0}^{n-1} \omega^{i(j-k)} = n$
- If $j \neq k$,

$\omega^n = 1$ - n^{th} root of unity

$$\sum_{i=0}^{n-1} \omega^{i(j-k)} = \frac{1 - \omega^{n(j-k)}}{1 - \omega^{j-k}} = 0$$

Polynomial evaluation - a geometric view

Lemma: The columns of the Vandermonde matrix V(ω) are orthogonal vectors

- The Vandermonde matrix V(ω) rotates the standard basis to the Fourier basis
- The Fourier basis consists of the columns of the Vandermonde matrix V(ω)

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$$V(\omega)V(\omega)^* = n\mathbf{I}$$

complex conjugate

Interpolation - Inverting the Vandermonde matrix

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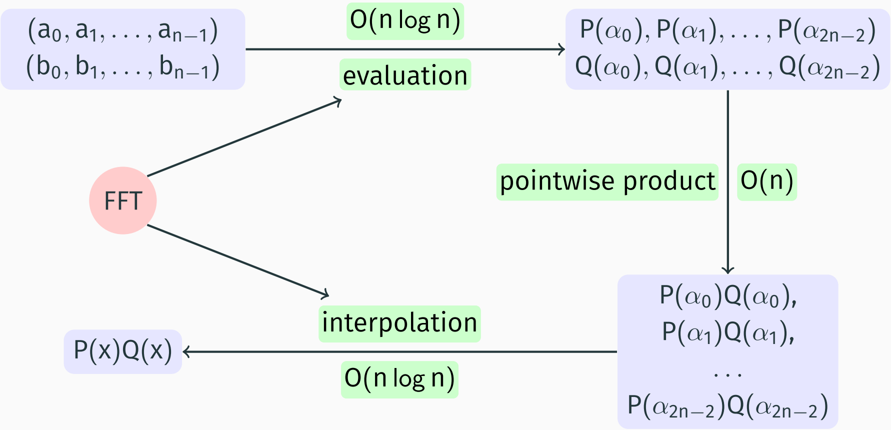
- Conjugates of the roots of unity are themselves roots of unity!
- Compute the FFT, and reverse the output!

$\omega^{-j} = \omega^{n-j}$

Convolution - the final algorithm

$$P(x) = \sum_{i=0}^{n-1} a_i x^i \quad Q(x) = \sum_{i=0}^{n-1} b_i x^i$$



Closest pair of points

Pairwise distances of points

Given n points $\{p_i = (x_i, y_i)\}_{1 \le i \le n}$ on the plane, find a pair of points that are closest to each other

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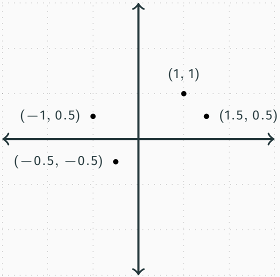

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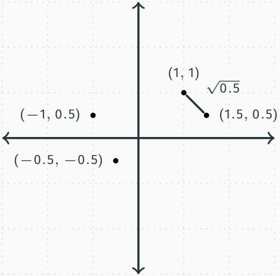

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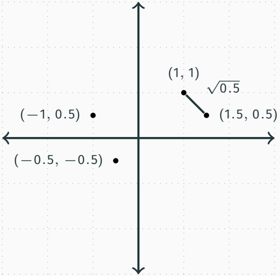

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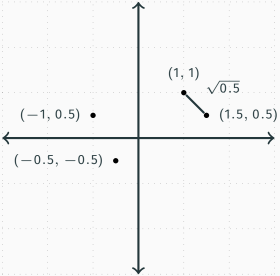

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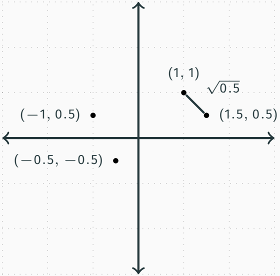

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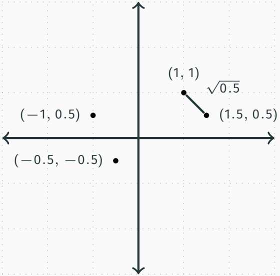

- (Most?) Important primitive for geometric algorithms
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Pairwise distances - the 1D case

Given n points on the real line - find the pair of points that are closest to each other

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Question: Does a similar idea work in the 2D case - sorting based on the x and y coordinates

Pairwise distances - the 2D case

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Sorting according to x-coordinate and y-coordinate and checking in order

Pairwise distances - the 2D case

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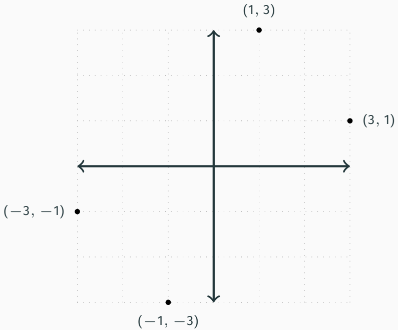

Pairwise distances - the 2D case

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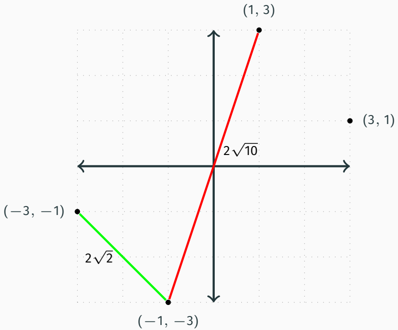

Pairwise distances - a divide and conquer approach

Assumption: No two points have the same x-coordinate

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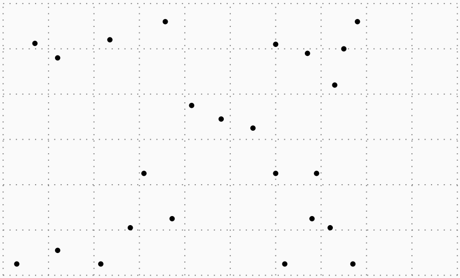

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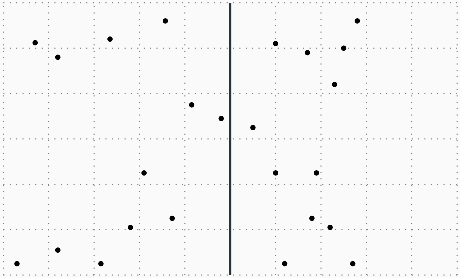

- Divide the set of points into two (almost)equal halves

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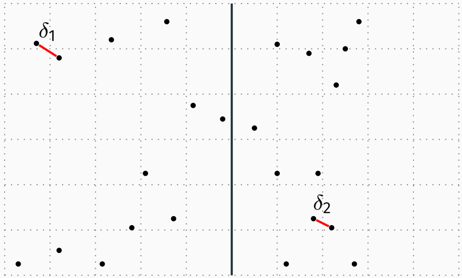

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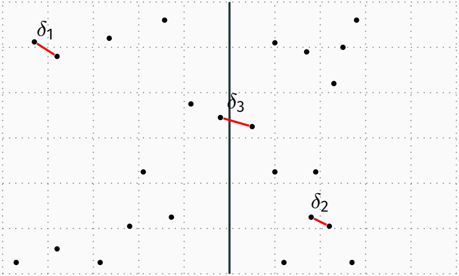

- Divide the set of points into two (almost)equal halves
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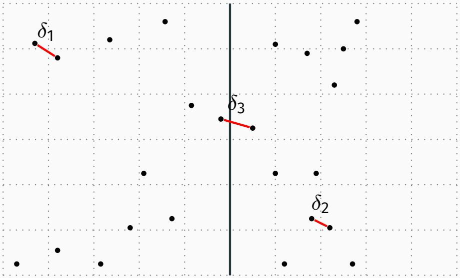

- Divide the set of points into two (almost)equal halves
- Find the closest pair of points in each part recursively
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- Return $\min(\delta_1, \delta_2, \delta_3)$

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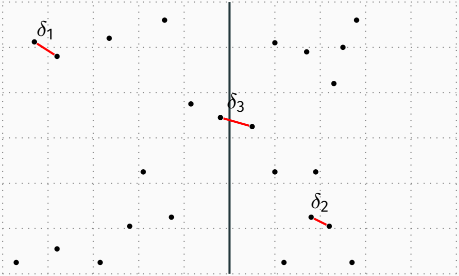

- Divide the set of points into two (almost)equal halves
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Pairwise distances - a divide and conquer approach

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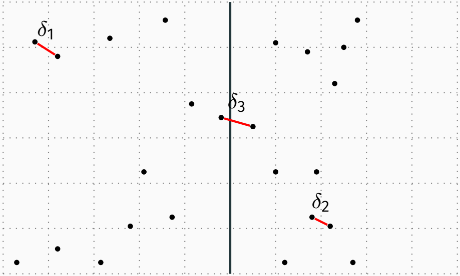

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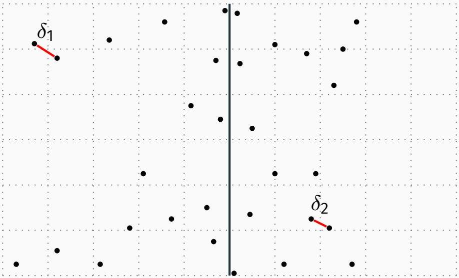
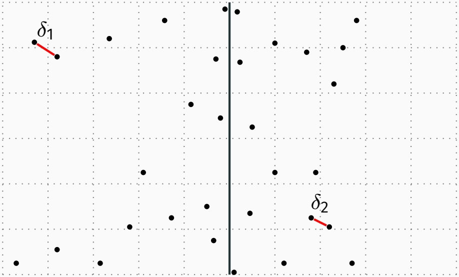
Pairwise distances - a divide and conquer approach

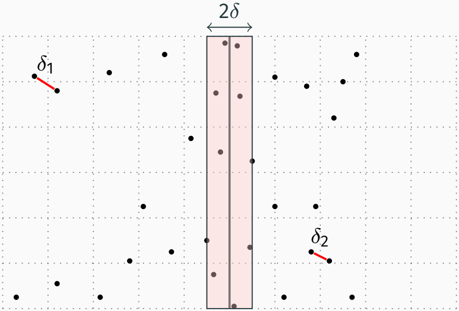


Pairwise distances - a divide and conquer approach



$\delta = \min(\delta_1, \delta_2)$

Pairwise distances - a divide and conquer approach



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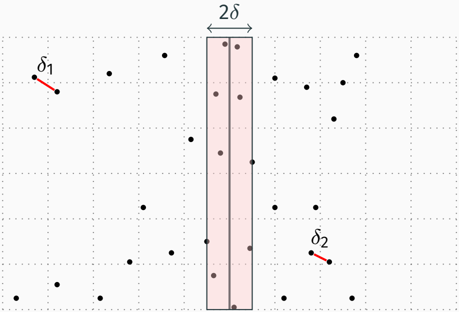
- Only points within δ of the midpoint line need to be considered

Pairwise distances - a divide and conquer approach



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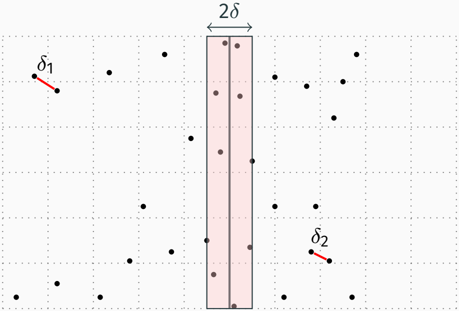

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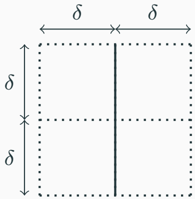
Question: For a fixed point p on one side, how many candidate points are there on the other side?

Pairwise distances - bounding the number of points



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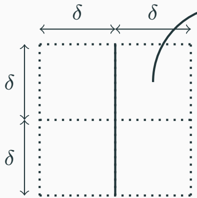

How many points can lie inside the $\delta \times \delta$ square?

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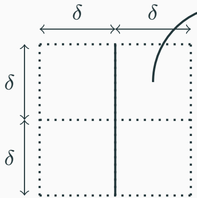

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- Every two points inside a $\delta \times \delta$ square must be at least δ apart!

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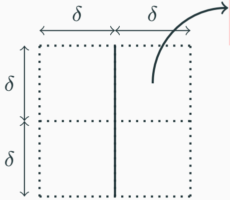

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there can be no more than 4 points inside a $\delta \times \delta$ square

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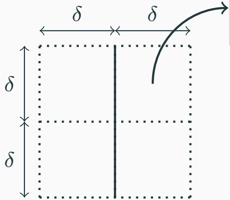

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- Every two points inside a $\delta \times \delta$ square must be at least δ apart!

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- For any point p in the 2δ strip, need to consider at most 15 points next to it, when the points are sorted according to the y coordinate

Closest pair of points - a divide and conquer algorithm

Input: Set of points $p_i = (x_i, y_i)$
Output: Pair of points closest to each other
Find the line $x = x_m$ dividing into two equal halves P_1 and P_2
$\delta_1 \leftarrow$ distance between closest pair of points in P_1
$\delta_2 \leftarrow$ distance between closes pair of points in P_2
$\delta \leftarrow \min(\delta_1, \delta_2)$
Sort the points P_δ between $x = x_m - \delta$ and $x_m + \delta$ according to the y coordinates
foreach point p in the sorted list of P_δ **do**
    foreach point p' that is at most 15 positions away from p **do**
        $\delta' \leftarrow \min(\delta', d(p, p'))$
return $\min(\delta', \delta)$

Closest pair of points - a divide and conquer algorithm

Input: Set of points $p_i = (x_i, y_i)$
Output: Pair of points closest to each other
Find the line $x = x_m$ dividing into two equal halves P_1 and P_2 $\longrightarrow$ $O(n)$
$\delta_1 \leftarrow$ distance between closest pair of points in P_1
$\delta_2 \leftarrow$ distance between closes pair of points in P_2 $\longrightarrow$ $2T(\frac{n}{2})$
$\delta \leftarrow \min(\delta_1, \delta_2)$
Sort the points P_δ between $x = x_m - \delta$ and $x_m + \delta$ according to the y coordinates $\longrightarrow$ $O(n \log n)$
foreach point p in the sorted list of P_δ **do** $\longrightarrow$ $O(n)$
    foreach point p' that is at most 15 positions away from p **do**
        $\delta' \leftarrow \min(\delta', d(p, p'))$
return $\min(\delta', \delta)$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n)$$

Closest pair of points - solving the recurrence

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$$a \quad b \quad f(n)$$

Recall the formula from the recursion tree:
$T(n) = n^{\log_b a} + \sum_{i=0}^{\log n - 1} a^i f(n/b^i)$

Closest pair of points - solving the recurrence

$$\boxed{T(n) = \underset{\substack{\downarrow \\ a}}{2}T\Big(\underset{\substack{\downarrow \\ b}}{\frac{n}{2}}\Big) + O(\underset{\substack{\downarrow \\ f(n)}}{n \log n})}$$

Recall the formula from the recursion tree:
$T(n) = n^{\log_b a} + \sum_{i=0}^{\log n - 1} a^i f(n/b^i)$

$$T(n) = n + \sum_{i=0}^{\log n - 1} 2^i \frac{n}{2^i} \log\left(\frac{n}{2^i}\right)$$

$$= n + n \log\left(\prod_{i=0}^{\log n - 1} \frac{n}{2^i}\right) = n + n \log\left(\frac{n^{\log n}}{2^{\sum_{i=0}^{\log n - 1} i}}\right)$$

$$= n + n \log\left(\frac{n^{\log n}}{2^{\log n (\log n - 1)/2}}\right) = O(n \log^2 n)$$

Closest pair of points - solving the recurrence

$$\underbrace{T(n) = \underset{\downarrow}{2}T\left(\underset{\downarrow}{\tfrac{n}{2}}\right) + \underset{\downarrow}{O(n \log n)}}_{}$$

a b $f(n)$

$\longrightarrow$ Can we avoid sorting at each step?

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Closest pair of points - preprocessing the input

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n) \longrightarrow$$

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- Sort the list of points P according to both the x and y coordinates and maintain two lists P_x and P_y

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- Sort the list of points P according to both the x and y coordinates and maintain two lists P_x and P_y
- At each recursive call, divide the the list into two lists maintaining the sorted order

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- Sort the list of points P according to both the x and y coordinates and maintain two lists P_x and P_y
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- Scan the sorted list to obtain the list of points in the 2δ-strip sorted according to y coordinates

Closest pair of points - preprocessing the input

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Closest pair of points - preprocessing the input

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n) \longrightarrow$$ Can we avoid sorting at each step?

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- At each recursive call, divide the the list into two lists maintaining the sorted order $\longrightarrow O(n)$
- Scan the sorted list to obtain the list of points in the 2δ-strip sorted according to y coordinates $\longrightarrow O(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n) \text{ + one-time cost of sorting}$$