Design and Analysis of Algorithms

CS2800

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IIT Madras

Module plan

1. Introduction
2. Administrivia
3. SEAT algorithm

Introduction

Welcome to the course

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  - Prove correctness - All the implementations that I write work for all the test cases!
  - Analyze complexity - All the programs I write run very fast on my computer!

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  - Given $a = a_m a_{m-1} \ldots a_0$ and $b = b_n b_{n-1} \ldots b_0$ do the following:
    - Do $c_0 = a \times b_0$, $c_1 = a \times b_1$, . . ., $c_n = a \times b_n$

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    - Do $c_0 = a \times b_0$, $c_1 = a \times b_1$, ..., $c_n = a \times b_n$
    - Write down the number $\sum_{i=0}^{n} c_i \times 10^i$.

Integer multiplication

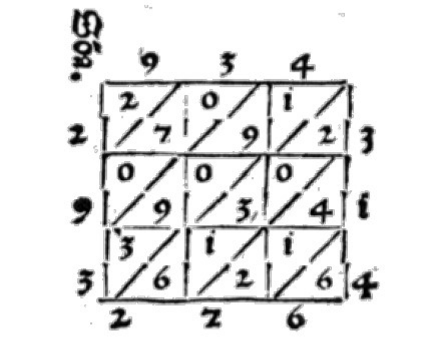

Figure 1: Anonymous 1458 textbook - Treviso Arithmetic

Integer multiplication

Peasant's multiplication (1650 B.C)
Given two numbers a, b
Start with $c = 0$, until $a = 0$

- Check parity of a
- Addition - If a is odd, do $c = c + b$
- Duplation - $b = 2 \times b$
- Mediation - $a = \lfloor \frac{a}{2} \rfloor$

a	b	c
		0
35	+46	46
17	+92	138
8	184	138
4	368	138
2	736	138
1	+1472	1610

Question Can you perform multiplication using a compass and straight-edge?

> Given two line segments of length a and b, and another line segment of unit length, construct a line segment of length a $\times$ b

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What if a and b are not integers

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Integer multiplication

- Which is a better algorithm or are they all equally good?
- How do we measure the goodness of an algorithm?
- Are there algorithms for integer multiplication that are better?
    - When do we decide to stop searching for better algorithms?

Administrivia

Administrative details

- When: ’B’ slot
  - 3 lectures (Mon, Tue, Wed) + 1 tutorial (Fri)
- Contact: Sign-up on Zulip
- Course page: `https://yaduvasudev.github.io/2800-23/`
- TAs: Sampriti, Keshav, Abhijit, Bibhuti, Ravi, Rayan, Barenya, Souradipto, Ankit, Anmol, Susmit

Administrative details

Relative grading

Grading scheme:

- Tutorial quizzes: best 2 out of 3 = 20%
- Quizzes 1 & 2: $2 \times 20\% = 40\%$
- End-sem: 40%

Attendance requirements: 85% (as mandated by the institute)

SEAT algorithm

SEAT algorithm at IITM

Student **E**lective **A**llocation **T**ool :

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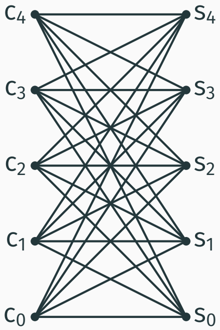
  - How should we define ”happy”?

The Stable Marriage problem



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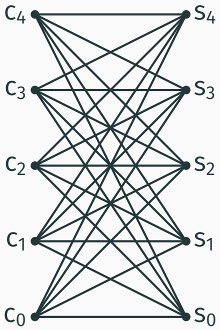

Preference orders for students and courses

- $S_1 : (C_0, C_1, C_2, C_3, C_4)$,
  $S_2 : (C_0, C_3, C_4, C_1, C_2), \dots$
- $C_0 : (S_1, S_3, S_4, S_0, S_2)$,
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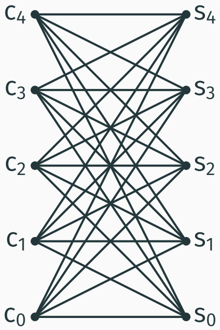

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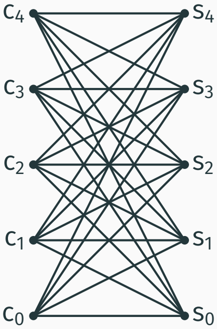

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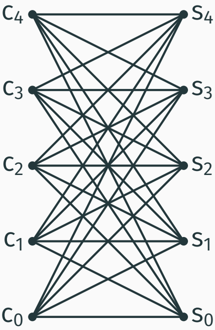

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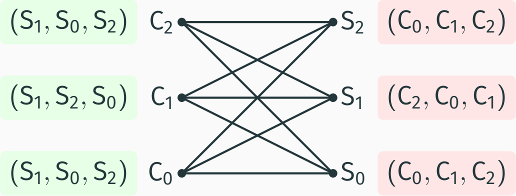
- If S_1 is assigned the course C_0, then S_1 and C_0 are happy!
- If S_2 is assigned the course C_3, he/she may not be happy but cannot do anything about it! (why?)
- The notion of happiness has to be modified to one of stability

Stable marriages



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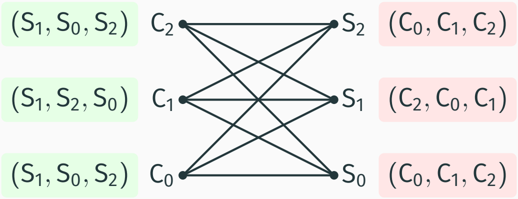

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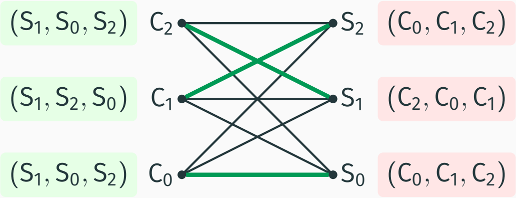

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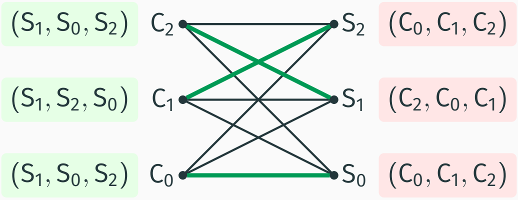

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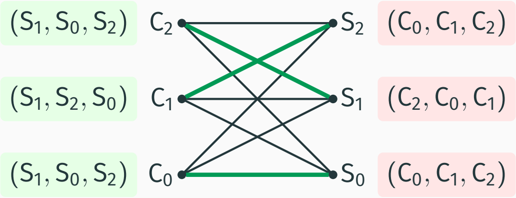

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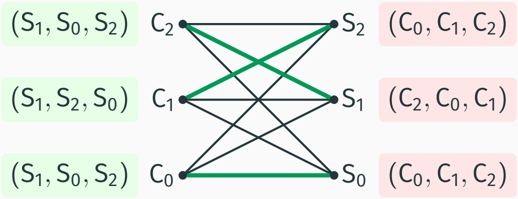

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- If a mapping exists, how will we find it?

Stable marriages

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept.

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We contend that the difficulties here described can be avoided. We shall describe a procedure for assigning applicants to colleges which should be satisfactory to both groups, which removes all uncertainties and which, assuming there are enough applicants, assigns to each college precisely its quota.

Stable marriages

How Game Theory Helped Improve New York City's High School Application Process

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- Nobel prize in Economics (2012) for Shapley and Roth - ”for the theory of stable allocations and the practice of market design”

Matchings and unstable pairs

Input: Set of students, courses and the list of preferences

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An unstable pair can improve their mutual happiness by breaking the current matching!

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  C $\leftarrow$ first course in list of S to which (s)he has not applied

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  if C is unmatched **then**
    $\mathsf{M} \leftarrow \mathsf{M} + \{(\mathsf{C}, \mathsf{S})\}$
  else

Gale-Shapley algorithm: A pseudocode

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    if C prefers S to its current student S' **then**
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    else
      C rejects S

Gale-Shapley algorithm: an example

	1st	2nd	3rd	4th
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1st	2nd	3rd	4th
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
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C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_0 applies to C_1:

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1^{st}	2^{nd}	3^{rd}	4^{th}
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

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S_3	C_0	C_3	C_2	C_1

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C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_1 applies to C_3:

Gale-Shapley algorithm: an example

	1st	2nd	3rd	4th
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1st	2nd	3rd	4th
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_1 applies to C_3: C_3 accepts

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1^{st}	2^{nd}	3^{rd}	4^{th}
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_2 applies to C_1:

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1^{st}	2^{nd}	3^{rd}	4^{th}
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

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Gale-Shapley algorithm: an example

	1st	2nd	3rd	4th
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
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C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_0 applies to C_0:

Gale-Shapley algorithm: an example

	1st	2nd	3rd	4th
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

	1st	2nd	3rd	4th
C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
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S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
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C_0	S_0	S_1	S_3	S_2
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C_3	S_0	S_3	S_2	S_1

S_3 applies to C_0:

Gale-Shapley algorithm: an example

	1st	2nd	3rd	4th
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
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C_0	S_0	S_1	S_3	S_2
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S_3 applies to C_0: C_0 rejects

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

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C_0	S_0	S_1	S_3	S_2
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C_3	S_0	S_3	S_2	S_1

S_3 applies to C_3:

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	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
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C_0	S_0	S_1	S_3	S_2
C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_3 applies to C_3: C_3 accepts

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	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
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S_1 applies to C_1:

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
S_3	C_0	C_3	C_2	C_1

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C_0	S_0	S_1	S_3	S_2
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C_3	S_0	S_3	S_2	S_1

S_1 applies to C_1: C_1 rejects

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	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
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S_1	C_3	C_1	C_0	C_2
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C_1	S_2	S_1	S_3	S_0
C_2	S_1	S_2	S_3	S_0
C_3	S_0	S_3	S_2	S_1

S_1 applies to C_2:

Gale-Shapley algorithm: an example

	1^{st}	2^{nd}	3^{rd}	4^{th}
S_0	C_1	C_0	C_3	C_2
S_1	C_3	C_1	C_0	C_2
S_2	C_1	C_2	C_3	C_0
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Gale-Shapley algorithm: pseudocode

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while $\exists$ unmatched student S and course to which (s)he has not applied **do**
    $C \leftarrow$ first course in list of S to which (s)he has not applied
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            C rejects S

Gale-Shapley algorithm: termination

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Lemma: Once a course is matched to a student, the following are true

- The course is never unmatched
- If the course is assigned a new student, (s)he will be higher in preference order for the course

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Lemma: Once a course is matched to a student, the following are true

- The course is never unmatched
- If the course is assigned a new student, (s)he will be higher in preference order for the course

- Only reason for a course to change the assigned student is if a student who is higher in its preference order applies

Gale-Shapley algorithm: termination

Lemma: The Gale-Shapley algorithm terminates after n^2 iterations of the while-loop with a matching

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- Each student applies to a course at most once
- After n^2 steps, every course has been applied to at least once
- Once a course is applied to, it remains matched
- If all courses are matched, then all students are also matched

Gale-Shapley algorithm: stability

Lemma: If M is the matching returned by the Gale-Shapley algorithm, then M has no unstable pairs

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  - S applies to courses in the decreasing order of preference
  - S is matched to a course C' higher in its preference order

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  - S applies to courses in the decreasing order of preference
  - S is matched to a course C' higher in its preference order
- S applied to C at some iteration

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- S applied to C at some iteration
  - S was rejected by C because C was assigned a more preferred student

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Question Do all executions of the Gale-Shapley algorithm lead to the same stable matching?

Gale-Shapley algorithm: running time

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The while-loop executes at most n^2 times

- How long does it take to find an unmatched student and a course that (s)he has not applied to?

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    One bit per course/student to indicate whether the course is matched, and id of the matched course/student

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    One bit per course/student to indicate whether the course is matched, and id of the matched course/student
- The data structure storing the matching should allow an update of the matching in O(1) time

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  - Queue of students who are not matched
  - Queue of courses not applied to (for every student)
- How should the matching be stored so that you can check if C is matched in O(1) time?

  One bit per course/student to indicate whether the course is matched, and id of the matched course/student
- The data structure storing the matching should allow an update of the matching in O(1) time

  Update the corresponding bits, and change the course/student information

Gale-Shapley algorithm: running time

The while-loop executes at most n^2 times

- How long does it take to find an unmatched student and a course that (s)he has not applied to? O(1)-time
- Checking if C is a matched course O(1)-time
- Update the matching O(1)-time

Gale-Shapley algorithm: running time

The while-loop executes at most n^2 times

- How long does it take to find an unmatched student and a course that (s)he has not applied to? $O(1)$-time
- Checking if C is a matched course $O(1)$-time
- Update the matching $O(1)$-time

Gale-Shapley algorithm runs in time $O(n^2)$

Gale-Shapley algorithm: properties of the solution

	1st	2nd	3rd
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

	1st	2nd	3rd
S_0	C_1	C_0	C_2
S_1	C_0	C_1	C_2
S_2	C_0	C_1	C_2

$$\left.\begin{array}{l} S_0 \leftrightarrow C_0,\ S_1 \leftrightarrow C_1,\ S_2 \leftrightarrow C_2 \\ S_0 \leftrightarrow C_1,\ S_1 \leftrightarrow C_0,\ S_2 \leftrightarrow C_2 \end{array}\right\} \text{– two stable matchings}$$

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	1^{st}	2^{nd}	3^{rd}
C_0	S_0	S_1	S_2
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- There is no stable matching that matches S_2 to C_0 or C_1

Gale-Shapley algorithm: properties of the solution

	1^{st}	2^{nd}	3^{rd}
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

	1^{st}	2^{nd}	3^{rd}
S_0	C_1	C_0	C_2
S_1	C_0	C_1	C_2
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$$\left.\begin{array}{l} S_0 \leftrightarrow C_0,\ S_1 \leftrightarrow C_1,\ S_2 \leftrightarrow C_2 \\ S_0 \leftrightarrow C_1,\ S_1 \leftrightarrow C_0,\ S_2 \leftrightarrow C_2 \end{array}\right\} \text{– two stable matchings}$$

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Gale-Shapley algorithm: properties of the solution

	1^{st}	2^{nd}	3^{rd}
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

	1^{st}	2^{nd}	3^{rd}
S_0	C_1	C_0	C_2
S_1	C_0	C_1	C_2
S_2	C_0	C_1	C_2

$$\left.\begin{array}{l} S_0 \leftrightarrow C_0,\ S_1 \leftrightarrow C_1,\ S_2 \leftrightarrow C_2 \\ S_0 \leftrightarrow C_1,\ S_1 \leftrightarrow C_0,\ S_2 \leftrightarrow C_2 \end{array}\right\} \text{– two stable matchings}$$

- There is no stable matching that matches S_2 to C_0 or C_1
- There is no stable matching that matches C_2 to S_0 or S_1
- M is the best matching for the students, and the worst for the courses

Gale-Shapley algorithm: properties of the solution

	1st	2nd	3rd
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

	1st	2nd	3rd
S_0	C_1	C_0	C_2
S_1	C_0	C_1	C_2
S_2	C_0	C_1	C_2

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- M' is the best matching for the courses, and the worst for the students

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	1st	2nd	3rd
C_0	S_0	S_1	S_2
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S_1	C_0	C_1	C_2
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Theorem:

- If the students apply to the courses, then the output obtained will be the best matching for the students, and the worst for the courses.
- If the courses propose to the students, then the output obtained will be the best matching for the courses, and the worst for the students.

Gale-Shapley algorithm: Optimality

	1^{st}	2^{nd}	3^{rd}
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

	1^{st}	2^{nd}	3^{rd}
S_0	C_1	C_0	C_2
S_1	C_0	C_1	C_2
S_2	C_0	C_1	C_2

- A course C_j is a valid match for S_i if there exists a stable matching that matches C_j to S_i

Gale-Shapley algorithm: Optimality

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C_1	S_1	S_0	S_2
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  C_0 is a valid match for S_0 and S_1, but not for S_2

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C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

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- C_j is the best valid match for S_i if in every stable matching M', either C_j is matched to S_i, or if C' is matched to S_i, then C' is lower in the preference order than C_j

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	1^{st}	2^{nd}	3^{rd}
C_0	S_0	S_1	S_2
C_1	S_1	S_0	S_2
C_2	S_0	S_1	S_2

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Theorem: Let $M^* = \{(S_i, \text{best}(S_i)) | 1 \leq i \leq n\}$. If Gale-Shapley algorithm is run with students making requests, then it always returns M^* irrespective of the order in which the students make the requests.

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Proof: Suppose the GS algorithm returns matching $M \neq M^*$

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- S_i is rejected by a valid match C_j
- C_j matched to S', higher in preference order

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Question Which course does S' prefer more: C_j or C'?

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Question Which course does S' prefer more: C_j or C'?

- S' was not rejected by a valid match when it is matched to C_j
- S' prefers C_j to $C' \Rightarrow (S', C_j)$ is unstable in $M' \rightarrow$ contradicts the fact that M' is a stable matching

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Theorem: Let $\widehat{M} = \{(\mathrm{worst}(C_j), C_j) | 1 \leq j \leq n\}$. Then $\widehat{M} = M^*$

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$C_j = \text{best}(S_i)$

- S_i prefers C_j to C'

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- $\exists$ a stable matching M' such that $(S, C_j) \in M'$, and
- C_j prefers S_i to S

Suppose $(S_i, C') \in M'$

$C_j = \text{best}(S_i)$

- S_i prefers C_j to C'
- (S_i, C_j) is an unstable pair in M'