APSP for unweighted undirected graphs

(dense graphs)

- Repeated BFS: $O(nm + n^2) = O(n^3)$ for dense graphs

Seidel's algorithm

$A \to$ adjacency matrix of G

$G^2 = (V, E^2)$

$(u,v) \in E^2$ if $\exists$ path of length ≤ 2 in G

$\to$ Computing the adjacency A_2 matrix of G^2: Compute A^2 & set

$A_2(i,j) = 1$ if $A^2(i,j) > 0$, $i \neq j$ or $A(i,j) > 0$

$\to$ What is the relation between the shortest paths in G^2 & G

Claim: If d_{uv} & D_{uv} are the shortest path distances in G & G^2 resp.

Then $D_{uv} = \left\lceil \frac{d_{uv}}{2} \right\rceil$

Proof:

even case

$$u \to a_1 \to b_1 \to a_2 \to b_2 \to \cdots \to a_k \to b_k \to v$$

G

$$u \to b_1 \to b_2 \to \cdots \to b_k \to v \longrightarrow G^2$$

odd case

$$u \to a_1 \to b_1 \to a_2 \to b_2 \to \cdots \to a_k \to b_k \to a_{k+1} \to v$$

G

$$u \to b_1 \to b_2 \to \cdots \to b_k \to v$$

$$D_{uv} \leq \left\lceil \frac{d_{uv}}{2} \right\rceil$$

If $\exists$ path of length $l \leq \left\lceil \frac{d_{uv}}{2} \right\rceil$

Since each edge corr. to ≤ 2 edges in G

$\exists$ path of length $2l < d_{uv}$

Corollary: $d_{uv} \in \{2D_{uv}, 2D_{uv} - 1\}$

Lemma: ① If $d_{uv} = 2D_{uv}$, then $\forall\ w \in N_G(v)$

$$D_{uw} \geq D_{uv}$$

② If $d_{uv} = 2D_{uv} - 1$, then $\forall\ w \in N_G(v)$

$D_{uw} \leq D_{uv}$, and

$\exists\ z \in N_G(v)$ s.t. $D_{uz} < D_{uv}$

Proof: (1) Suppose $\exists\, w \in N_G(v)$ s.t

$$D_{uw} < D_{uv} \Rightarrow D_{uw} \leq D_{uv} - 1$$

$$2D_{uw} < 2D_{uv} - 1$$

Consider the shortest path from u to w

$+ (w, v)$

has length $\leq 2D_{uw} + 1 < 2D_{uv} = d_{uv}$

(2) For any $w \in N_G(v)$

Consider shortest path from u to v + (v, w)

$$d_{uw} \leq d_{uv} + 1$$

$$= 2D_{uv} \Longrightarrow D_{uv} \geq \frac{d_{uw}}{2}$$

$$\Rightarrow D_{uv} \geq D_{uw}$$

Consider z in the shortest path from u to v

s.t $z \in N_G(v)$

$$d_{uz} = d_{uv} - 1 = 2D_{uv} - 2$$

$$\frac{d_{uz}}{2} = D_{uv} - 1 \Rightarrow D_{uz} < D_{uv}$$

Corollary: $d_{uv} = 2D_{uv} \Longleftrightarrow \dfrac{\sum_{w \in N_G(v)} D_{uw}}{\deg(v)} \geq D_{uv}$

Consider the matrix $D \cdot A$

$$(D \cdot A)(u,v) = \sum_{w \in V} D(u,w) \cdot A(w,v)$$

$$= \sum_{w \in N_G(v)} D(u,w)$$