Projects: Parallel Computing
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The project consists in solving one of the following proposal using parallel computing techniques. The final project has to contain the source code and the necessary information for the reproducibility of the results; a short scientific document describing the problem, the solution and the adopted methodology. The project has to be sent via email to `mtavelli@unibz.it`.

You finally have to prepare a short oral presentation of the project. After the presentation will eventually follow a Q&A session.

In some projects you can choose the algorithm and the framework for the parallelization. Every option has an associated score. The minimum score in order to complete the project is 1.0. For example if OpenMP parallelization has score (0.5), MPI (score 1.0) and CUDA (score 0.5). I.e. you can include in the project an MPI implementation (1.0) or an OpenMP and CUDA (0.5+0.5) parallelization.

PROJECT I. Heat equation on structured meshes. Consider the following two-dimensioal PDE describing heat conduction

$$\frac{\partial T}{\partial t} - k\nabla^2 T = 0, \qquad k = \frac{\lambda}{\rho c}, \tag{1}$$

and consider the following finite difference approximation

$$\begin{aligned} T^{n+1}_{i,j} = T^n_{i,j} \quad &+ \quad k\frac{\Delta t}{\Delta x^2}\left(T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}\right) \\ &+ \quad k\frac{\Delta t}{\Delta y^2}\left(T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1}\right). \end{aligned} \tag{2}$$

Let $\Omega = [-1, 1] \times [0, 4]$ and the conductivity coefficient $k = 0.2$. The initial condition is given by

$$T(0) = \begin{cases} T_L & \text{if} \quad x \leq 1 \\ T_R & \text{if} \quad x > 1 \end{cases}, \qquad T_L = 60, \quad T_R = 30. \tag{3}$$

1. Verify the following points in a **serial code**:
   - Evolve the PDE up to the final time $t_f = 0.05$;
   - Perform a comparison between the obtained solution and the exact one
   $$T_e = \frac{1}{2}(TR + TL) + \frac{1}{2}erf\left(\frac{x-1}{2\sqrt{k\,t}}\right)(TR - TL) \tag{4}$$
   - (score 0.4) Use an implicit discretization in (2) and solve, using the conjugate gradient method, the resulting linear system.
2. Parallelize the code
   - (score 0.6) Using shared memory OpenMP;
   - (score 1.0) Using MPI directives;
3. Produce a graph or a table with the following measures:
   - Relative Speedup
   $$S(p) = \frac{t_{(p=1)}}{t(p)}$$
   - Efficiency
   $$E(p) = \frac{S(p)}{p}$$
   - Kuck function
   $$K(p) = S(p)\,E(p)$$
4. (score 0.5) Write a CUDA-FORTRAN program that solves the one-dimensional problem on a GPU.

PROJECT II. Free sufrace flow on structured meshes. Consider the following two-dimensional PDE system:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{5}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \tag{6}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(H\,u) + \frac{\partial}{\partial y}(H\,v) = 0. \tag{7}$$

Assume $\nu = 0$ and the following finite difference discretization:

$$u^{n+1}_{i+\frac{1}{2},j} = Fu^{n}_{i+\frac{1}{2},j} - g\frac{\Delta t}{\Delta x}\left(\eta^{n+1}_{i+1,j} - \eta^{n+1}_{i,j}\right) \tag{8}$$

$$v^{n+1}_{i,j+\frac{1}{2}} = Fv^{n}_{i,j+\frac{1}{2}} - g\frac{\Delta t}{\Delta y}\left(\eta^{n+1}_{i,j+1} - \eta^{n+1}_{i,j}\right) \tag{9}$$

$$\begin{aligned} \eta^{n+1}_{i,j} &= \eta^{n}_{i,j} - \frac{\Delta t}{\Delta x}\left(H^{n}_{i+\frac{1}{2},j}\,u^{n+1}_{i+\frac{1}{2},j} - H^{n}_{i-\frac{1}{2},j}\,u^{n+1}_{i-\frac{1}{2},j}\right) \\ &- \frac{\Delta t}{\Delta y}\left(H^{n}_{i,j+\frac{1}{2}}\,v^{n+1}_{i,j+\frac{1}{2}} - H^{n}_{i,j-\frac{1}{2}}\,v^{n+1}_{i,j-\frac{1}{2}}\right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} H^{n}_{i+\frac{1}{2},j} &= \max\left(0,\, b^{n}_{i+\frac{1}{2},j} + \eta^{n}_{i,j},\, b^{n}_{i+\frac{1}{2},j} + \eta^{n}_{i+1,j}\right), \\ H^{n}_{i,j+\frac{1}{2}} &= \max\left(0,\, b^{n}_{i,j+\frac{1}{2}} + \eta^{n}_{i,j},\, b^{n}_{i,j+\frac{1}{2}} + \eta^{n}_{i,j+1}\right). \end{aligned} \tag{11}$$

The domain is $\Omega = [-1.0, 1.0] \times [-3.0, 3.0]$, consider $b = -0.9e^{-10(x^2+(y-1)^2)}$ and the initial condition

$$\eta(0,x,y) = 1 + e^{-\frac{1}{2\,s^2}\left(x^2+y^2\right)}, \qquad s = 0.1. \tag{12}$$

Furthermore, assume that the convective terms can be neglected $Fu^{n}_{i+\frac{1}{2},j} = u^{n}_{i+\frac{1}{2},j}$ and $Fv^{n}_{i,j+\frac{1}{2}} = v^{n}_{i,j+\frac{1}{2}}$.

1. Write a **serial code** that:
   - Evolve the PDE up to the final time $t_f = 0.1$ using the conjugate gradient method;
2. Parallelize the code
   - (score 1.0) Using shared memory OpenMP
   - (score 1.0) Using MPI directives

3. Produce a graph or a table with the following measures:

   - Relative Speedup

$$S(p) = \frac{t_{(p=1)}}{t(p)}$$

   - Efficiency

$$E(p) = \frac{S(p)}{p}$$

   - Kuck function

$$K(p) = S(p)\,E(p)$$

PROJECT III. Page ranking. Consider a graph with N nodes and G the sparse adjacency matrix of dimension $N \times N$ describing the links between the pages (i.e. $g_{ij} = 1$ if the node i links to node j).

1. Consider $N = 5$ and the following links: $g_{1,4} = g_{2,1} = g_{3,1} = g_{4,2} = g_{4,3} = g_{4,5} = g_{5,3} = 1$, zero otherwise.
   - Build the matrix G and the transition matrix $M = p(G_{ij}/r_i) + (1-p)/N$ where $p = 0.85$;
   - Use the power method to determine the solution u such that $u^\top M = u^\top$;
   - Order the pages according to the rank value and verify the value of the obtained maximum eigenvalue.
2. Consider the following problems:
   - Let N be a large number of pages (i.e. $N = 65536$ pages), $p = 0.999$ and G defined by $G(j-1, j) = 1$ for all $j \geq 1$ furthermore $G_{2:N,2} = 1$, $G_{2,2} = 0$.
   - Consider the file "NetworkN.dat" that contains the following information about the sparse adjacency matrix: Line 1: N_x, Line 2: N_y; Line 3 N_{links}. From line 4: $i\ j\ val$. Read the file and perform a page ranking for the given graph and $p = 0.999$.
   - (score 1.0) Parallelize the code using MPI directives and solve one of the setup described above.
3. Report a plot or a table with the following measures:
   - Relative Speedup
   $$S(p) = \frac{t(p_0)}{t(p)}$$
   - Efficiency
   $$E(p) = \frac{S(p)}{p}$$
   - Kuck function
   $$K(p) = S(p)\, E(p)$$