Vector Operations

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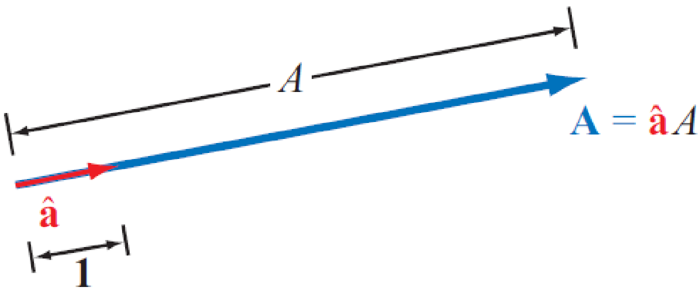
Laws of Vector Algebra

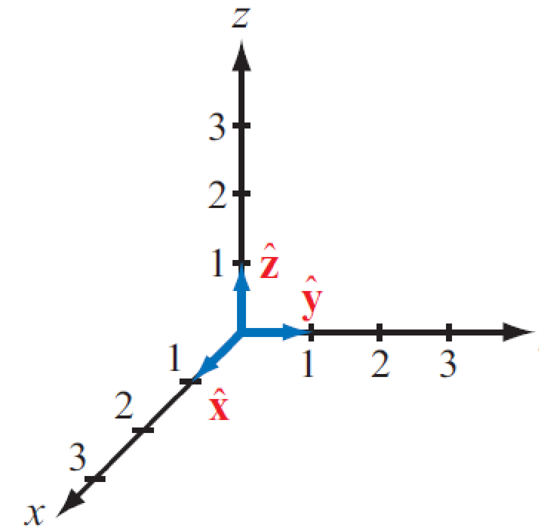


$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A$$

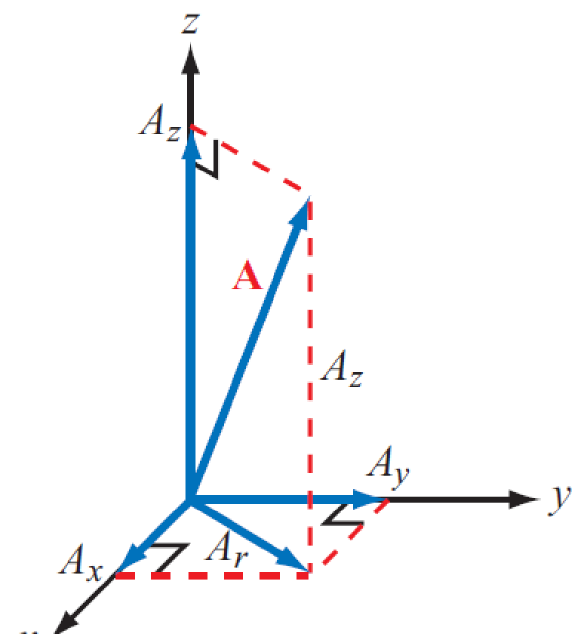
$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$



(a) Base vectors



(b) Components of **A**

Properties of Vector Operations

Equality of Two Vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.6a}$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \tag{3.6b}$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

Commutative property

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

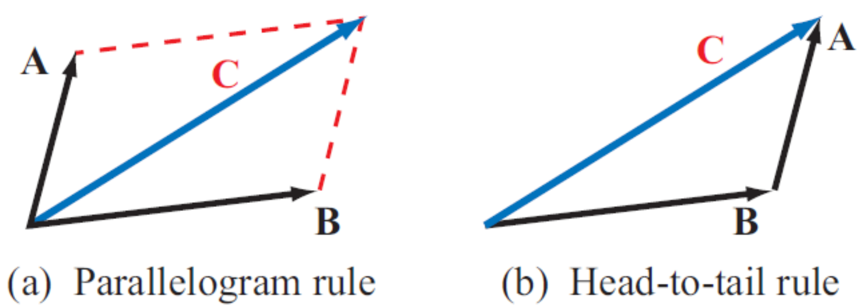


Figure 3-3: Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

Position & Distance Vectors

Position Vector: From origin to point P

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2$$

Distance Vector: Between two points

$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1P_2} \\ &= \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)\end{aligned}$$

the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$\begin{aligned}d &= |\mathbf{R}_{12}| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \qquad (3.12)\end{aligned}$$

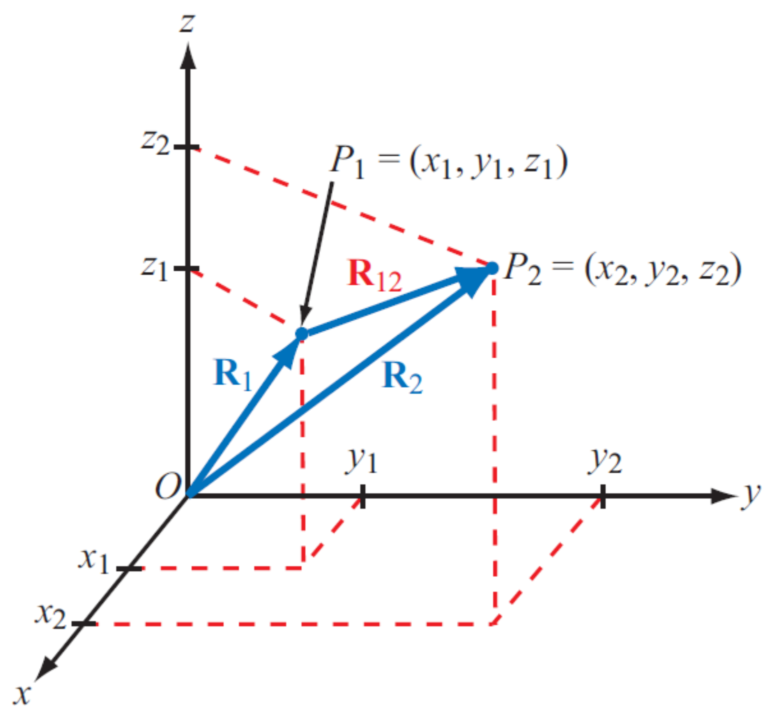


Figure 3-4: Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

- So far, we have added two vectors and multiplied a vector by a scalar
- The question arises:
  - Is it possible to multiply two vectors so that their product is a useful quantity?

- **One such product is the DOT PRODUCT, which we will discuss in this section.**
- **Another is the cross product, which we will discuss later.**

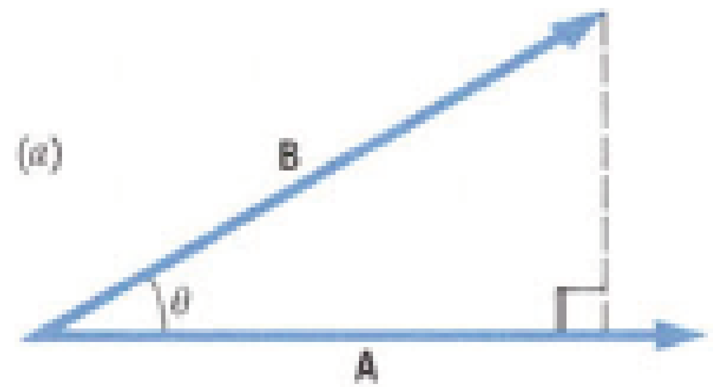
Fundamental concept of DOT PRODUCT

Vector Multiplication: Scalar Product or "Dot Product"

Very often in physics we have two vectors with an angle θ between them, and we wish to find the product of their components that lie in the direction of one or the other vector.

Consider Fig. 2-12. If we, for instance, select the **A** direction, then the component of vector **B** in that direction is given by dropping a perpendicular (Fig. 2-12a) and noting from the resulting right triangle that the component of **B** in the **A** direction is B cos θ and the product of this component and vector **A** is

$$AB\cos\theta$$



Vector Multiplication: Scalar Product or "Dot Product"

If, instead, we had selected the **B** direction we could equally have dropped a perpendicular from vector A to the line of vector **B** (Fig. 2-12b) and obtained the identical result. Because there is no specified direction for the resulting product, we define such a product as a scalar. We use the shorthand notation of a dot $(\cdot)$ to represent this type of product, which is referred to as the *dot product*

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \tag{2.1}$$

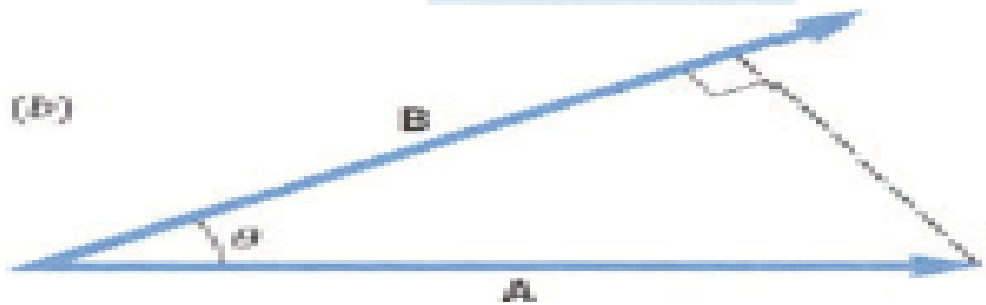


FIGURE 2-12 Geometric representation of two ways of forming a dot product of vectors **A** and **B**.

Vector Multiplication: Scalar Product or "Dot Product"

Let us apply our definition of the dot product to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$\mathbf{i} \cdot \mathbf{j} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{j} \cdot \mathbf{k} = (1)(1)\cos 90^\circ = 0$$

$$\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0^\circ \quad = 1$$

$$\mathbf{j} \cdot \mathbf{j} = (1)(1)\cos 0^\circ \quad = 1$$

$$\mathbf{k} \cdot \mathbf{k} = (1)(1)\cos 0^\circ \quad = 1$$

We see that when a unit vector is dotted with a different unit vector the result is zero, whereas when a unit vector is dotted with itself the result is unity.

Method to find DOT PRODUCT

- If **a** = $\langle a_1, a_2, a_3 \rangle$ and **b** = $\langle b_1, b_2, b_3 \rangle$, then the dot product of **a** and **b** is the number $\mathbf{a} \bullet \mathbf{b}$ given by:

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add
Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\,\sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]$$

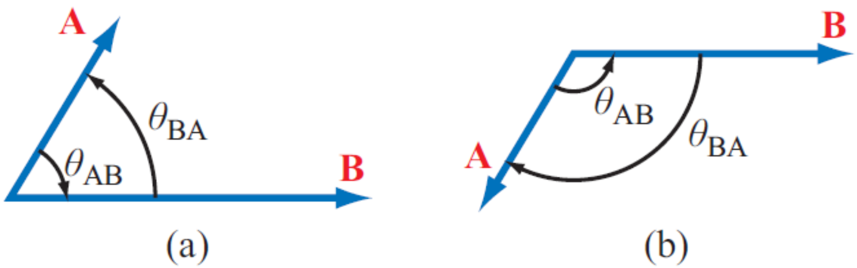


Figure 3-5: The angle θ_{AB} is the angle between **A** and **B**, measured from **A** to **B** between vector tails. The dot product is positive if $0 \leq \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \leq 180°$, as in (b).

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1,$$
$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z).$$

Hence:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Example to solve dot product

- $\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1) = 2$
- $\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2})$
  $= 6$
- $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1)$
  $= 7$

Example 2

Solve:

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$
- **Ans : 12**

Example 3

Solve:

- Find the angle between the vectors

$$\mathbf{a} = \langle 2, 2, -1 \rangle \text{ and } \mathbf{b} = \langle 5, -3, 2 \rangle$$

Example 3: Solution

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

and

$$|\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

Also, $\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$

Example 3 Solution:Cont'd

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So, the angle between **a** and **b** is:

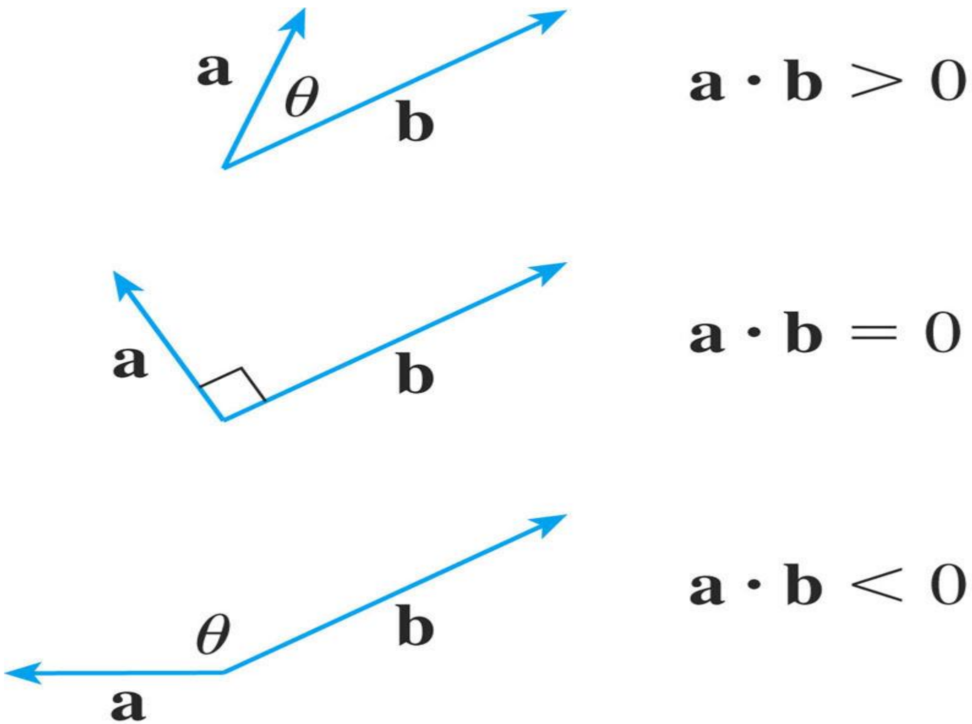
$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ)$$

Orthogonal Vectors

- Two nonzero vectors **a** and **b** are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$.
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$
- Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$; so, $\theta = \pi/2$.
- Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$
- The zero vector **0** is considered to be perpendicular to all vectors.

THE DOT PRODUCT

- The dot product **a** · **b** is:
  - Positive, if **a** and **b** point in the same general direction
  - Zero, if they are perpendicular
  - Negative, if they point in generally opposite directions



Example 3-1: Vectors and Angles

In Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** is directed from P_1 to point $P_2 = (1, -2, 2)$. Find

(a) vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$,
(b) the angle between **A** and the y-axis,
(c) vector **B**,
(d) the angle θ_{AB} between **A** and **B**, and
(e) the perpendicular distance from the origin to vector **B**.

Solution: (a) Vector **A** is given by the position vector of $P_1 = (2, 3, 3)$ as shown in Fig. 3-7. Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}\,,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}\,.$$

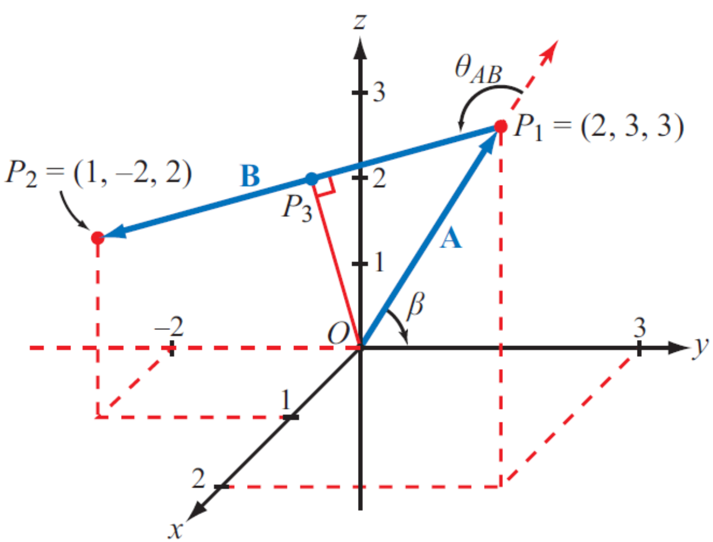


Figure 3-7: Geometry of Example 3-1.

(b) The angle β between **A** and the y-axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2°.$$

(c)

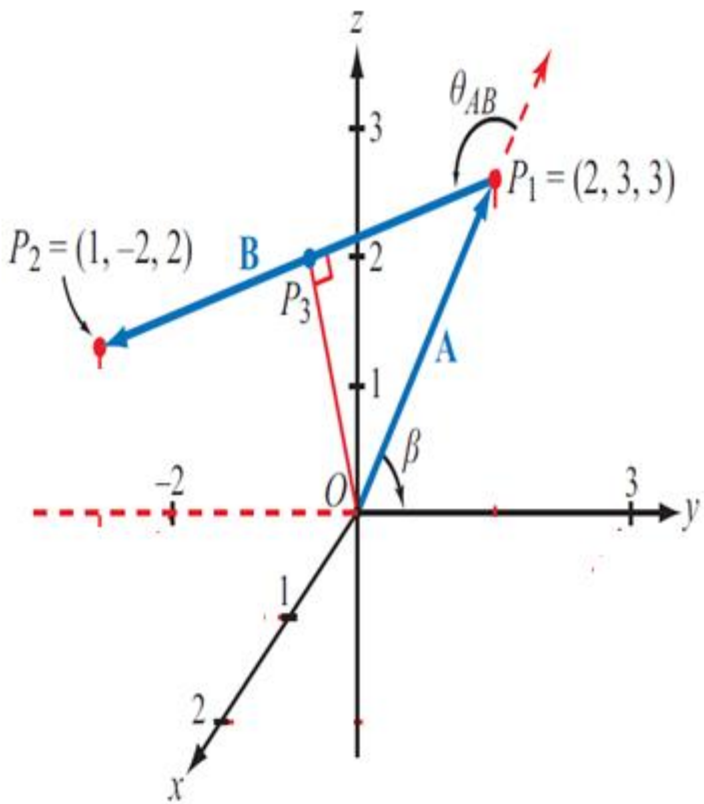
$$\mathbf{B} = \hat{\mathbf{x}}(1 - 2) + \hat{\mathbf{y}}(-2 - 3) + \hat{\mathbf{z}}(2 - 3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2 - 15 - 3)}{\sqrt{22}\,\sqrt{27}}\right] = 145.1°.$$

(e) The perpendicular distance between the origin and vector **B** is the distance $|\overrightarrow{OP_3}|$ shown in Fig. 3-7. From right triangle OP_1P_3,

$$|\overrightarrow{OP_3}| = |\mathbf{A}| \sin(180° - \theta_{AB}) = \sqrt{22}\,\sin(180° - 145.1°) = 2.68.$$


z
3
θ_{AB}
$P_1 = (2, 3, 3)$
$P_2 = (1, -2, 2)$
B
2
P_3
A
1
β
−2
O
3
y
1
2
x

Properties of Dot Product

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated as follows

If a, b, and c are vectors in V_3 and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b}=c(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(c\mathbf{b})$
5. $0\cdot\mathbf{a}=0$

Proofs

- These properties are easily proved:

For example

$$*\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

$$= a_1^2 + a_2^2 + a_3^2$$

$$= |\mathbf{a}|^2$$

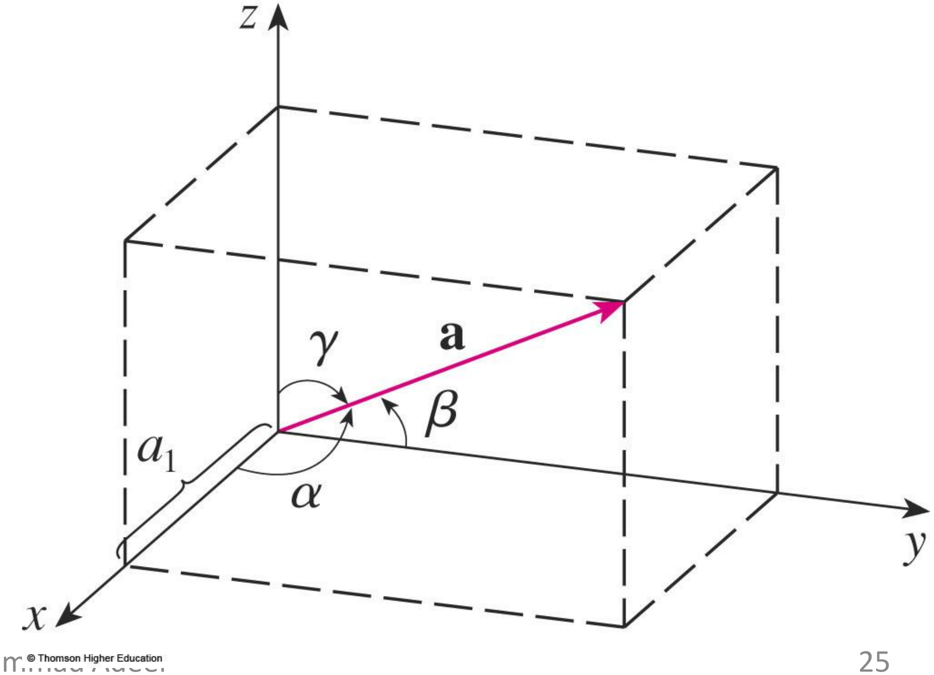
- $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$

$$= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$
$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$$
$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Task: The proofs of the remaining properties are left as exercises

Direction Angles

- The direction angles of a nonzero vector **a** are the angles *α*, *β*, and *γ* (in the interval [0, *π*]) that **a** makes with the positive *x*-, *y*-, and *z*-axes.



Direction Cosine

- The direction cosine of a vector are the cosines of the angles between the vector and the three co ordinate axis.
- The cosines of these direction angles—cos α, cos β, and cos γ

As $$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

b replaced by **i**, we obtain:

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

Similarly we also will have ,

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$