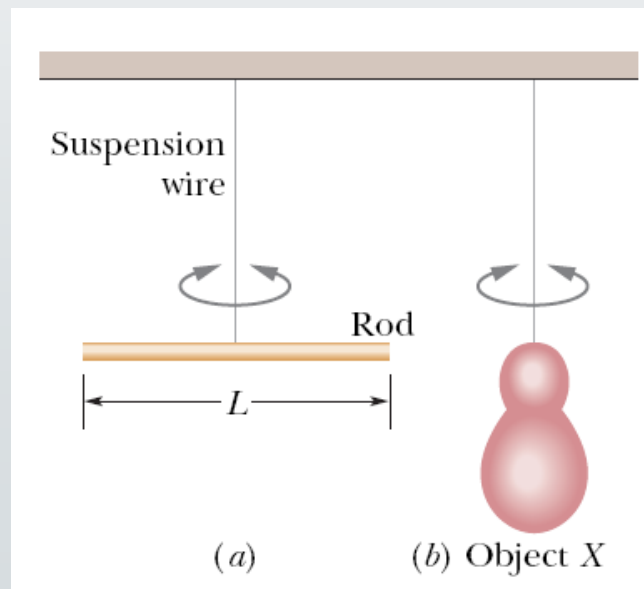


# ***Oscillations***



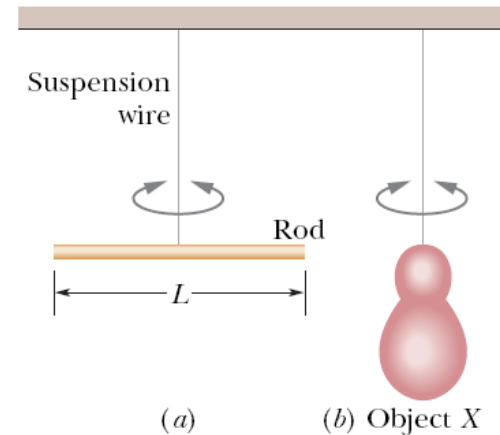
## Example, angular SHM:

Figure *a* shows a thin rod whose length  $L$  is 12.4 cm and whose mass  $m$  is 135 g, suspended at its midpoint from a long wire. Its period  $T_a$  of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object  $X$ , is then hung from the same wire, as in Fig. *b*, and its period  $T_b$  is found to be 4.76 s. What is the rotational inertia of object  $X$  about its suspension axis?



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**Answer:** The rotational inertia of either the rod or object  $X$  is related to the measured period. The rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as  $\frac{1}{12}mL^2$ . Thus, we have, for the rod in Fig. *a*,

$$I_a = \frac{1}{12}mL^2 = \left(\frac{1}{12}\right)(0.135 \text{ kg})(0.124 \text{ m})^2 \\ = 1.73 \times 10^{-4} \text{ kg} \cdot \text{m}^2.$$

Now let us write the periods, once for the rod and once for object  $X$ :

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

The constant  $\kappa$ , which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for  $I_b$ . The result is

$$I_b = I_a \frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4} \text{ kg} \cdot \text{m}^2) \frac{(4.76 \text{ s})^2}{(2.53 \text{ s})^2} \\ = 6.12 \times 10^{-4} \text{ kg} \cdot \text{m}^2. \quad (\text{Answer})$$

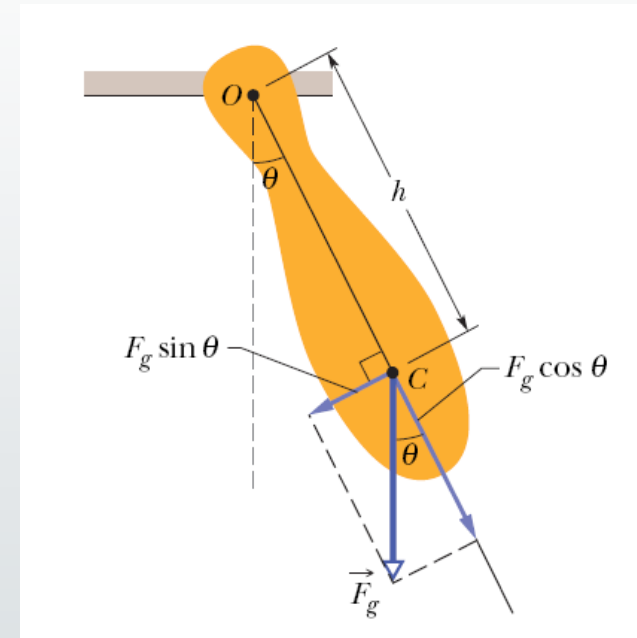
## Pendulums

A physical pendulum can have a complicated distribution of mass. If the center of mass, C, is at a distance of  $h$  from the pivot point (figure), then for *small angular amplitudes*, the motion is simple harmonic.

The period,  $T$ , is:

$$T = 2\pi \sqrt{\frac{I}{mgh}}$$

Here,  $I$  is the rotational inertia of the pendulum about O.



# Pendulums

In a *simple pendulum*, a particle of mass  $m$  is suspended from one end of an unstretchable massless string of length  $L$  that is fixed at the other end.

The restoring torque acting on the mass when its angular displacement is  $\theta$ , is:

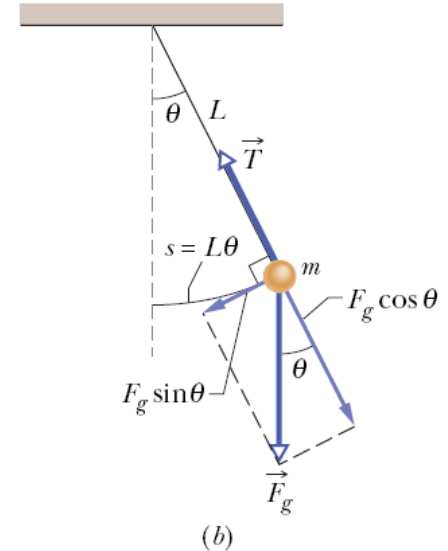
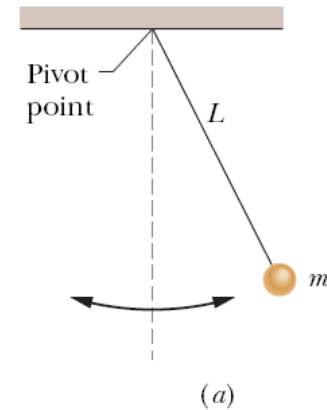
$$\tau = -L(F_g \sin \theta) = I\alpha$$

$\alpha$  is the angular acceleration of the mass. Finally,

$$\alpha = -\frac{mgL}{I}\theta, \text{ and}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is true for *small angular displacements*,  $\theta$ .



## Pendulums

In the **small-angle approximation** we can assume that  $\theta \ll 1$  and use the approximation  $\sin \theta \cong \theta$ . Let us investigate up to what angle  $\theta$  is the approximation reasonably accurate?

| $\theta$ (degrees) | $\theta$ (radians) | $\sin \theta$  |
|--------------------|--------------------|----------------|
| 5                  | 0.087              | 0.087          |
| 10                 | 0.174              | 0.174          |
| 15                 | 0.262              | 0.259 (1% off) |
| 20                 | 0.349              | 0.342 (2% off) |

**Conclusion:** If we keep  $\theta < 10^\circ$  we make less than 1 % error.

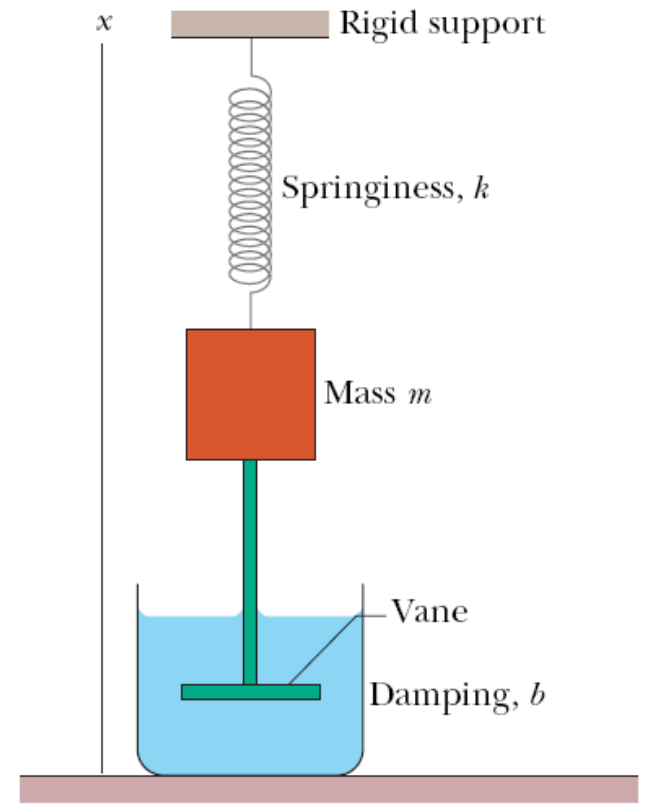
## Damped Oscillations

In a damped oscillation, the motion of the oscillator is reduced by an external force.

Example: A block of mass  $m$  oscillates vertically on a spring with spring constant  $k$ .

From the block a rod extends to a vane which is submerged in a liquid.

The liquid provides the external damping force,  $F_d$ .



## Damped Oscillations

Often the damping force,  $F_d$ , is proportional to the 1<sup>st</sup> power of the velocity  $v$ . That is,

$$F_d = -gv$$

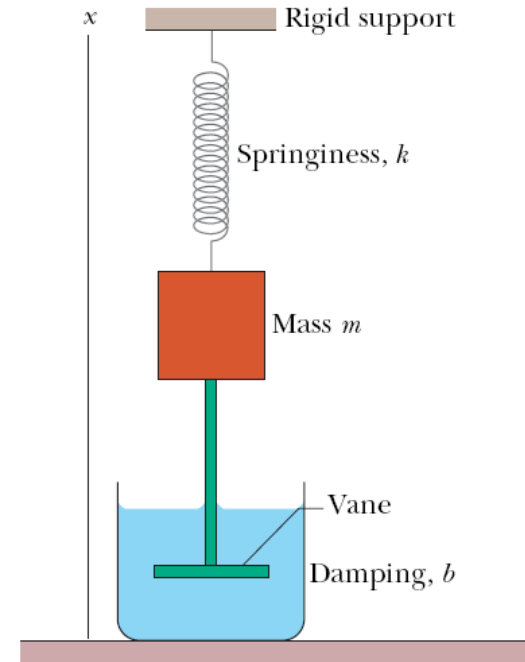
$$m \frac{d^2 x}{dt^2} + g \frac{dx}{dt} + kx = 0$$

The solution is:

$$x(t) = x_0 e^{\frac{-gt}{2m}} \cos(W't + j)$$

$$W' = \sqrt{W_0^2 - \frac{g^2}{4m^2}}$$

$$W_0 = \sqrt{\frac{k}{m}}$$





## Damped SHM

Often the damping force,  $F_d$ , is proportional to the 1<sup>st</sup> power of the velocity  $v$ . That is,

$$F_d = -bv$$

From Newton's 2<sup>nd</sup> law, the following DE results:

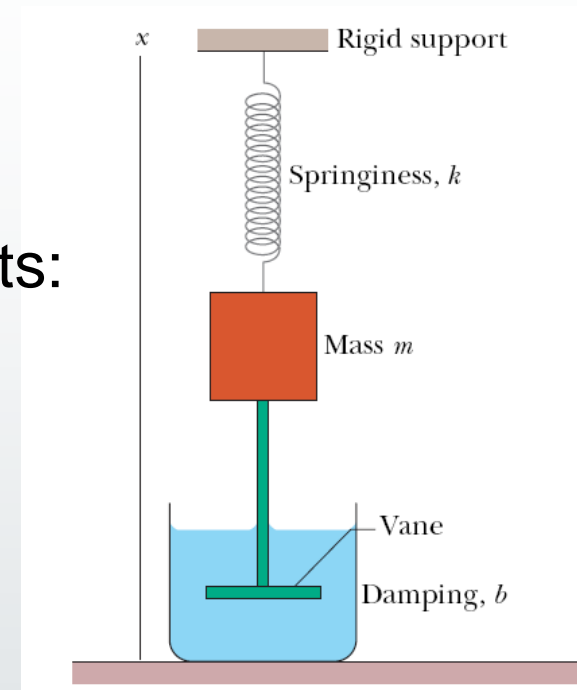
$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

The solution is:

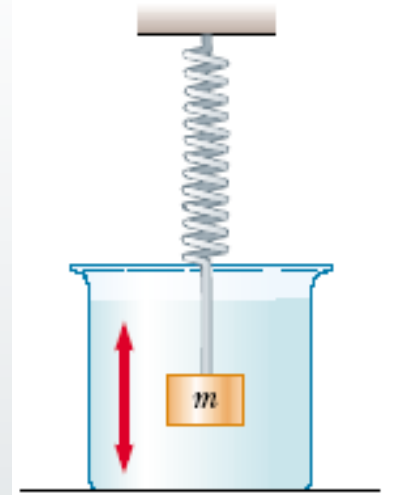
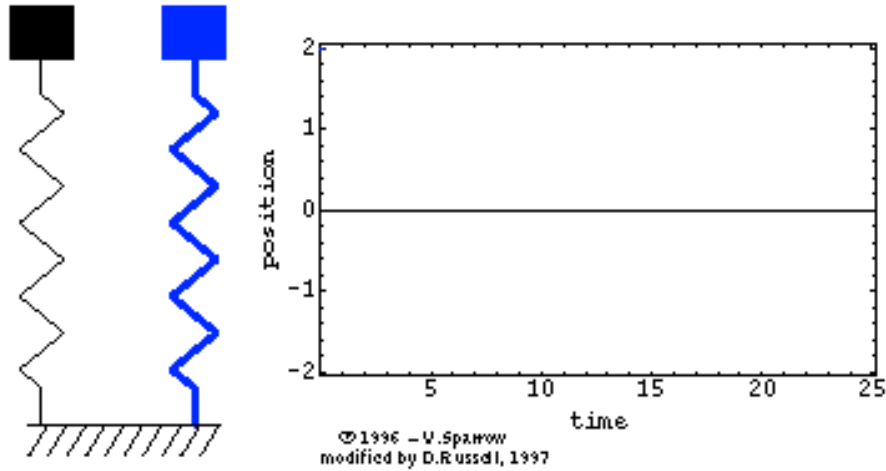
$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$

Here  $\omega'$  is the angular frequency, and is given by:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



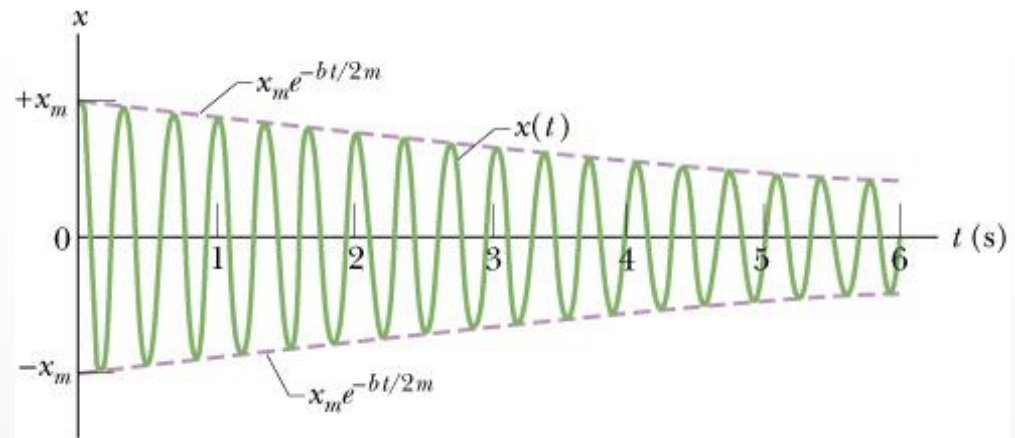
# DAMPED OSCILLATIONS



<http://www.lon-capa.org/~mmp/applist/damped/d.htm>

## 15.5 Damped Oscillations

$$x(t) = x_0 e^{\frac{-\gamma t}{2m}} \cos(W' t + j)$$

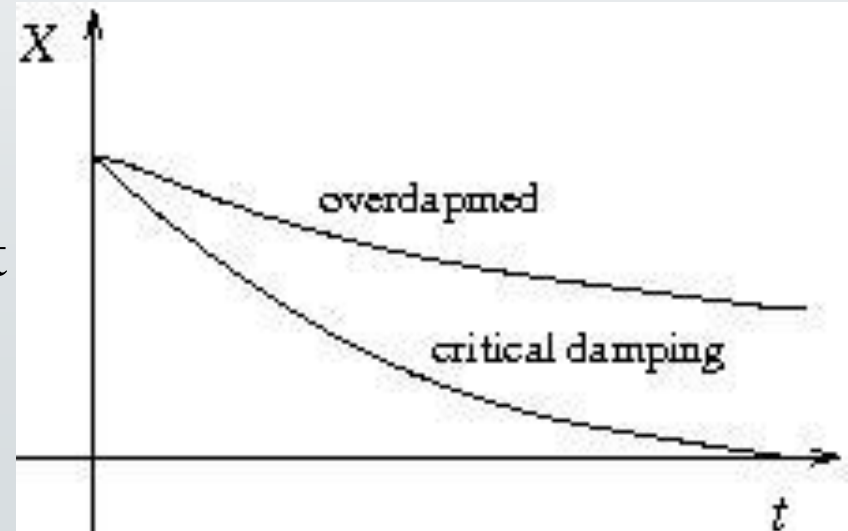


The above figure shows the displacement function  $x(t)$  for the damped oscillator described before.

The amplitude decreases as  $x_0 \exp(-\gamma t / 2m)$  with time.

# The above is for  $\gamma < 2m\omega_0$  (underdamped).

# For  $\gamma > 2m\omega_0$  (overdamped)  
and  $\gamma = 2m\omega_0$  (critical damping),  
the oscillation goes like the right  
figure.



# DAMPED OSCILLATIONS

***In many real systems, dissipative forces, such as friction, retard the motion.***

***Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be Damped.***

***Retarding force***  $\mathbf{R} = -b\mathbf{v}$

, we can write Newton's second law as

$$\sum F_x = -kx - bv = ma_x$$

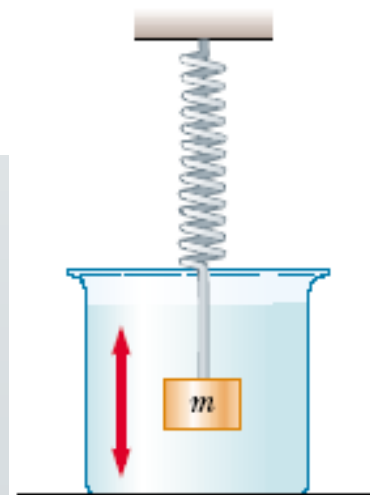
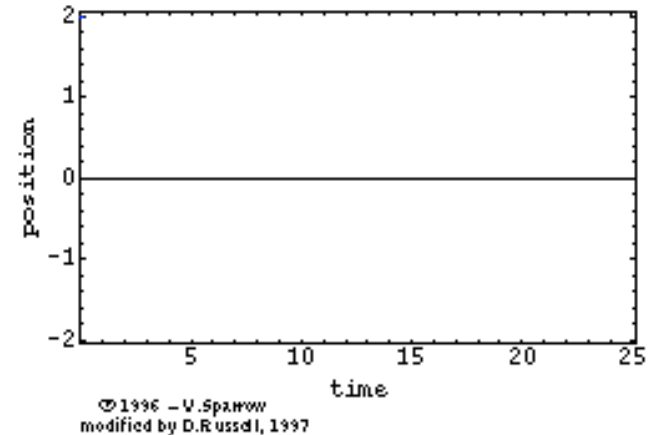
$$-kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$

The solution of this equation  $x = Ae^{-\frac{b}{2m}t} \cos(\omega t + \phi)$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

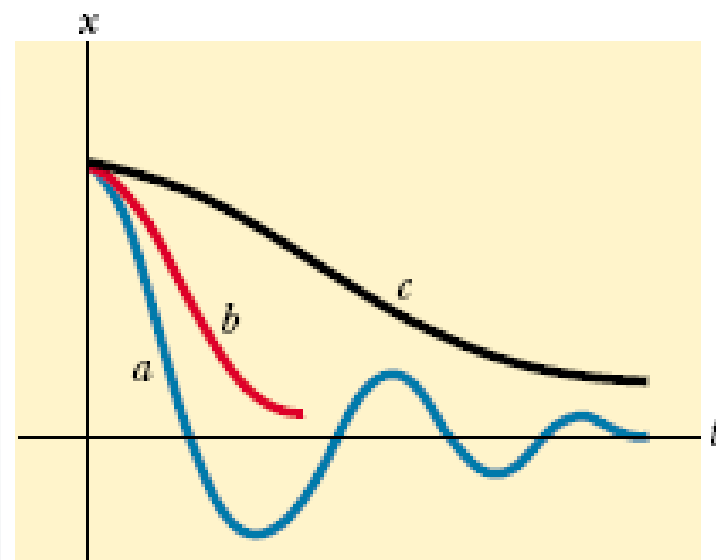
<http://www.lon-capa.org/~mmp/applist/damped/d.htm>



# DAMPED OSCILLATIONS

$$\omega' = \sqrt{\omega_0^2 - \frac{c^2}{4m^2}}$$

where  $\omega_0 = \sqrt{k/m}$  represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency**



Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

# *Forced Oscillations and Resonance*

When the oscillator is subjected to an external force that is periodic, the oscillator will exhibit forced/driven oscillations.

There are two frequencies involved in a forced oscillator:

- I.  $\omega_0$ , the natural angular frequency of the oscillator, without the presence of any external force, and
- II.  $\omega_e$ , the angular frequency of the applied external force.

The equation of motion is like the following:

$$m \frac{d^2 x}{dt^2} + g \frac{dx}{dt} + kx = F_0 \cos(\omega_e t)$$

# Forced Oscillations and Resonance

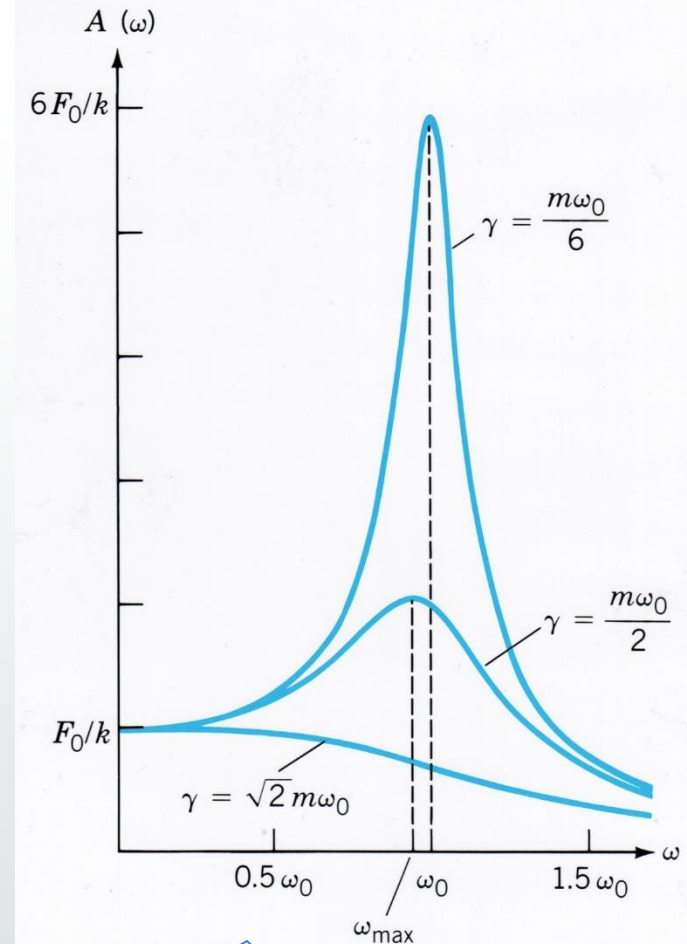
$$m \frac{d^2 x}{dt^2} + g \frac{dx}{dt} + kx = F_0 \cos(\omega_e t)$$

The *steady state* solution is

$$x(t) = A \cos(\omega_e t + \phi)$$

$$A = \frac{F_0 / m}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + \left(\frac{g}{m} \omega_e\right)^2}}$$

$$\tan \phi = \frac{g}{m} \frac{\omega_e}{\omega_0^2 - \omega_e^2} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

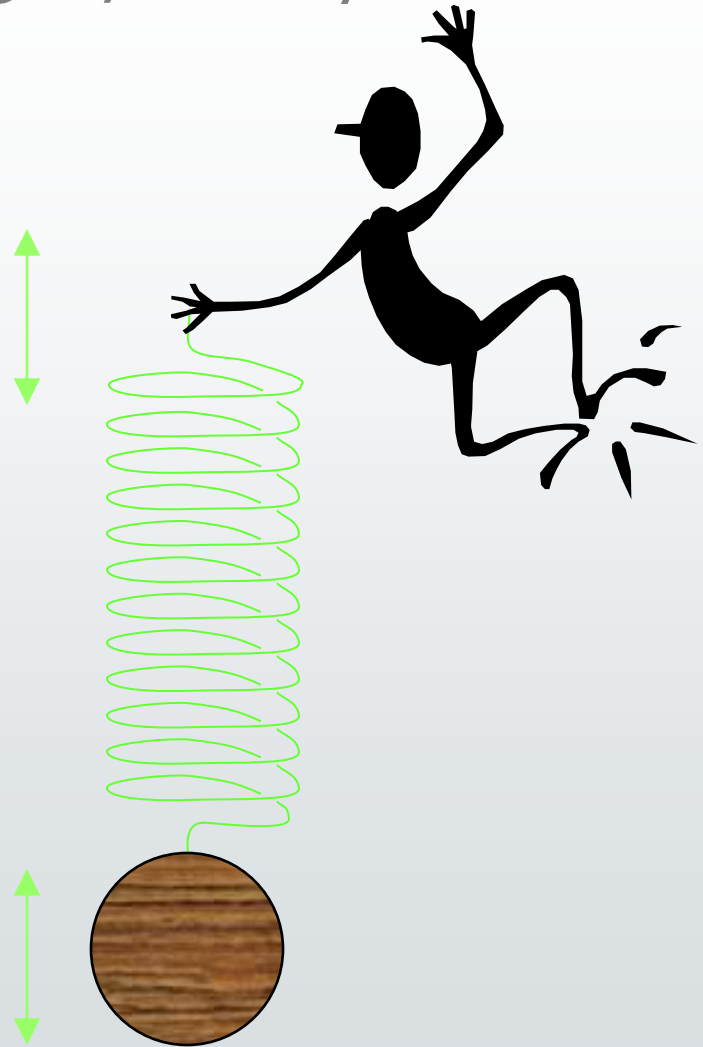


**Resonance** occurs at  $\omega_e \sim \omega_{\max} < \omega_0$ , for  $g < \sqrt{2}m\omega_0$

## Example (Mass-Spring System)

Periodic driving  
force of freq.  $f$

Oscillating with  
natural freq.  $f_0$





# Resonance

When a system is disturbed by a periodic driving force which frequency is ***equal to the natural frequency (  $f_0$  )*** of the system the system will oscillate with ***LARGE amplitude***.

***Resonance*** is said to occur.

<http://www.acoustics.salford.ac.uk/feschools/waves/shm3.htm>

# Example 1

Breaking Glass

System : **glass**

Driving Force :  
**sound wave**



**Breaking glass**  
*A wineglass can be shattered by a sound wave if the wave matches the natural vibrations of the glass.*



## Example 2

Collapse of the Tacoma Narrows suspension bridge in America in 1940

System : **bridge**

Driving Force :  
**strong wind**



# FORCED OSCILLATIONS

When a system is disturbed by a ***periodic driving force*** and then oscillate, this is called ***forced oscillation***.

The system will oscillate with ***its natural frequency (  $f_0$  )*** which is ***independent of*** the frequency of the driving force

$$x = A \cos(\omega t + \phi)$$

Where,

$$A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

$$F_{\text{ext}} \cos \omega t - kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$