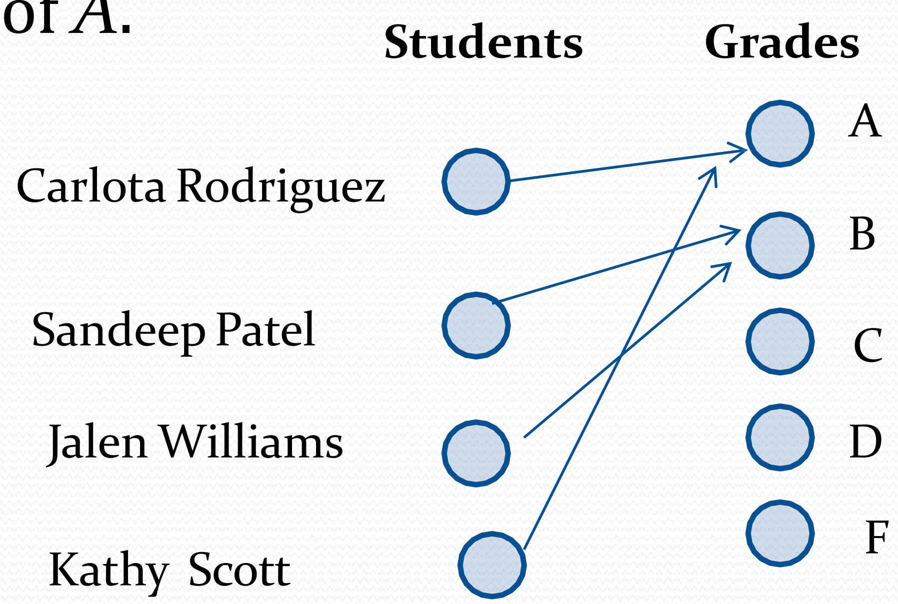
Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted f: $A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

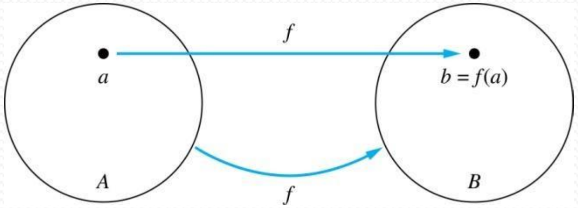
$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions

Given a function $f: A \to B$:



- We say f *maps A to B or*
  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Equal Functions

- Two functions are **equal** when they
  - have the same domain,
  - have the same codomain,
  - map each element of their common domain to the same element in their common codomain.
- If we change either the domain or the codomain of a function, then we obtain a different function.
- If we change the mapping of elements, then we also obtain a different function.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also inChapter 5).

Activity Time

What are the domain, codomain, and range of the following function

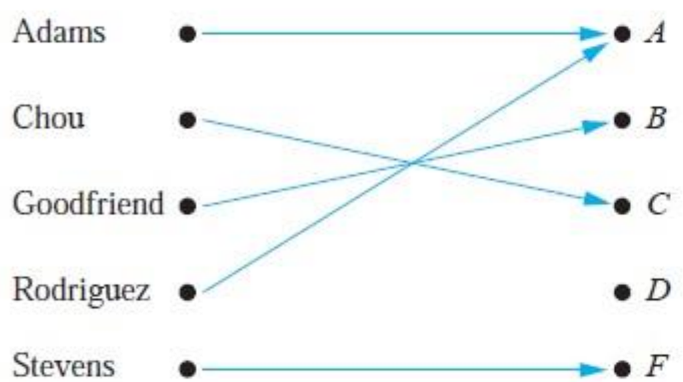


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Solution

- Let G be the function that assigns a grade to a student in our discrete mathematics class.
- Note that $G(\text{Adams}) = A$, for instance.
- The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens},
- The codomain is the set $\{A, B, C, D, F\}$.
- The range of G is the set $\{A, B, C, F\}$,

Questions

f(a) = ? z

The image of d is ? z

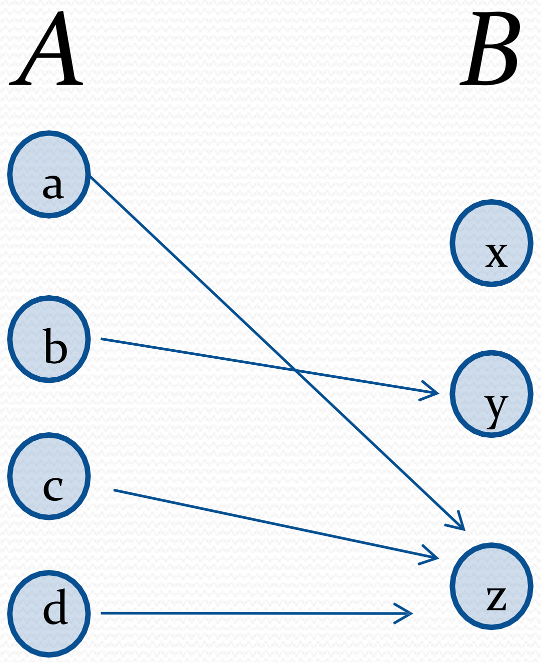
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A)$ = ?

The preimage(s) of z is (are) ? {a,c,d}



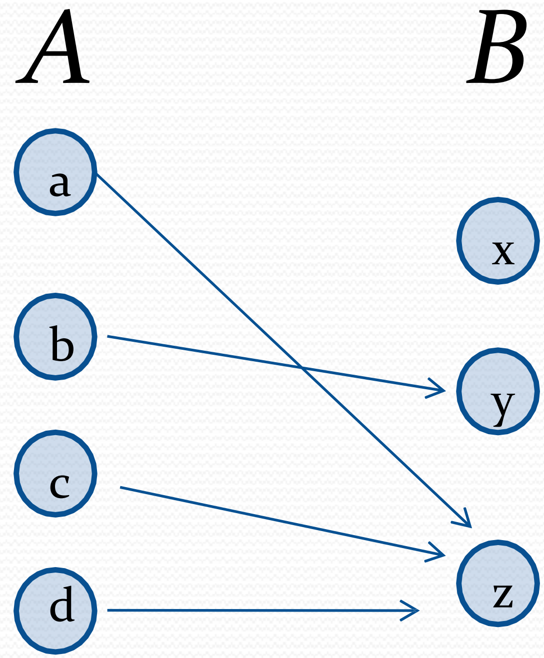
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

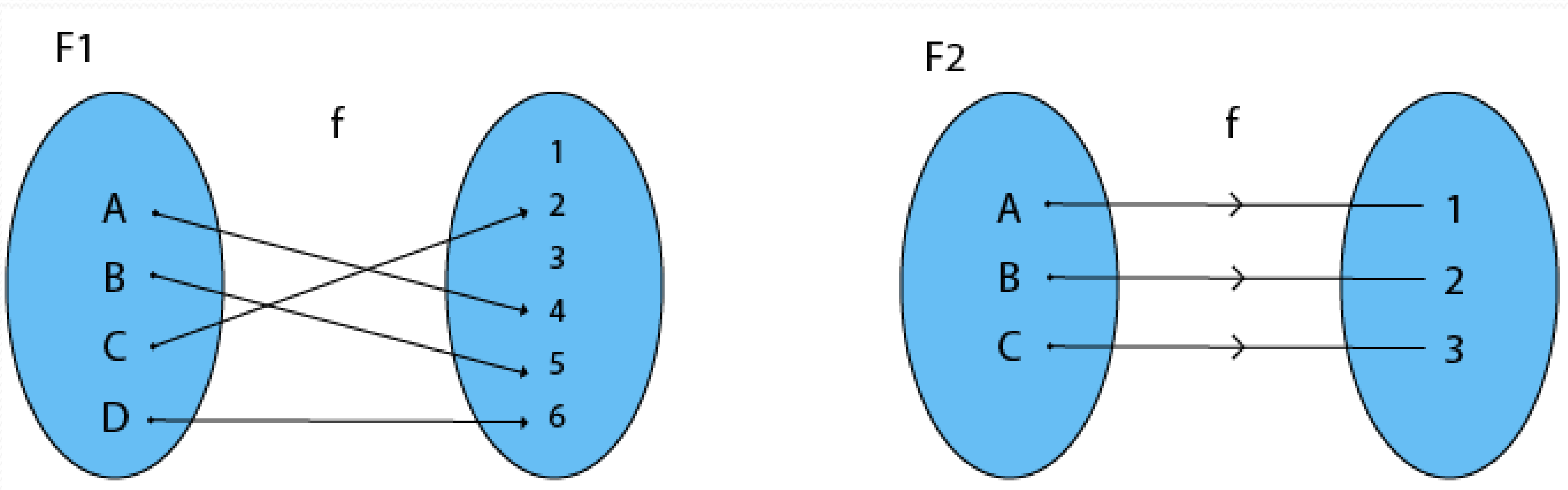
f {a,b,c,} is ?   {y,z}

f {c,d} is ?   {z}



Injections

Injective (One-to-One) Functions:
A function in which one element of Domain Set is connected to one element of Co-Domain Set.



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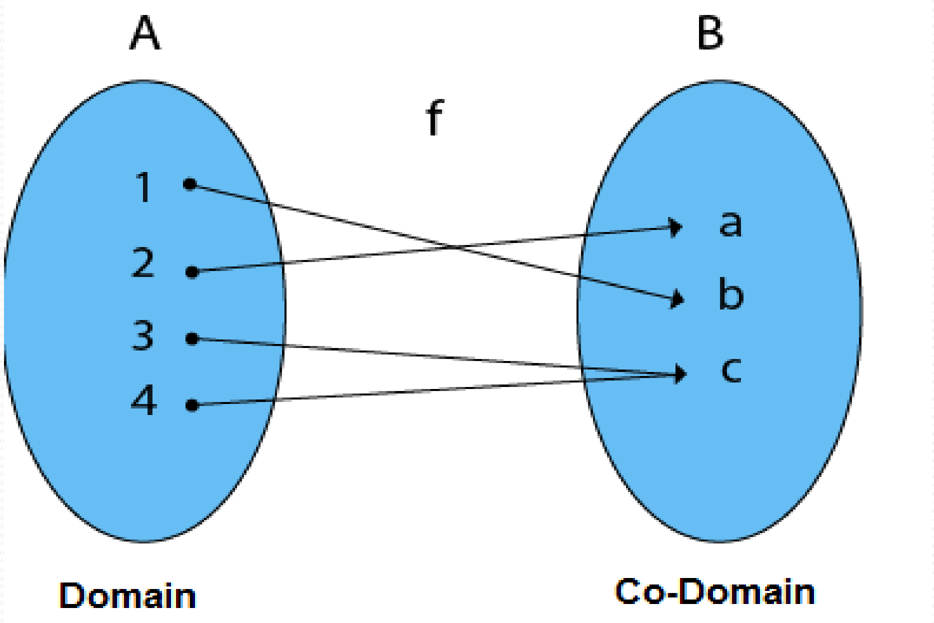
Surjections

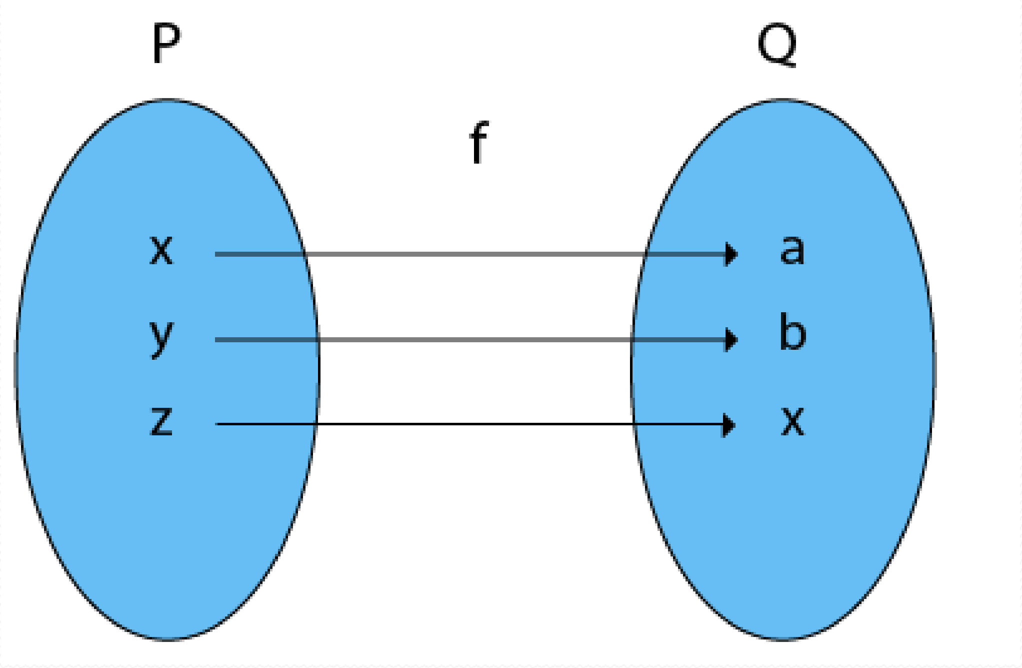
Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

Example:
Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.



Bijections

Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



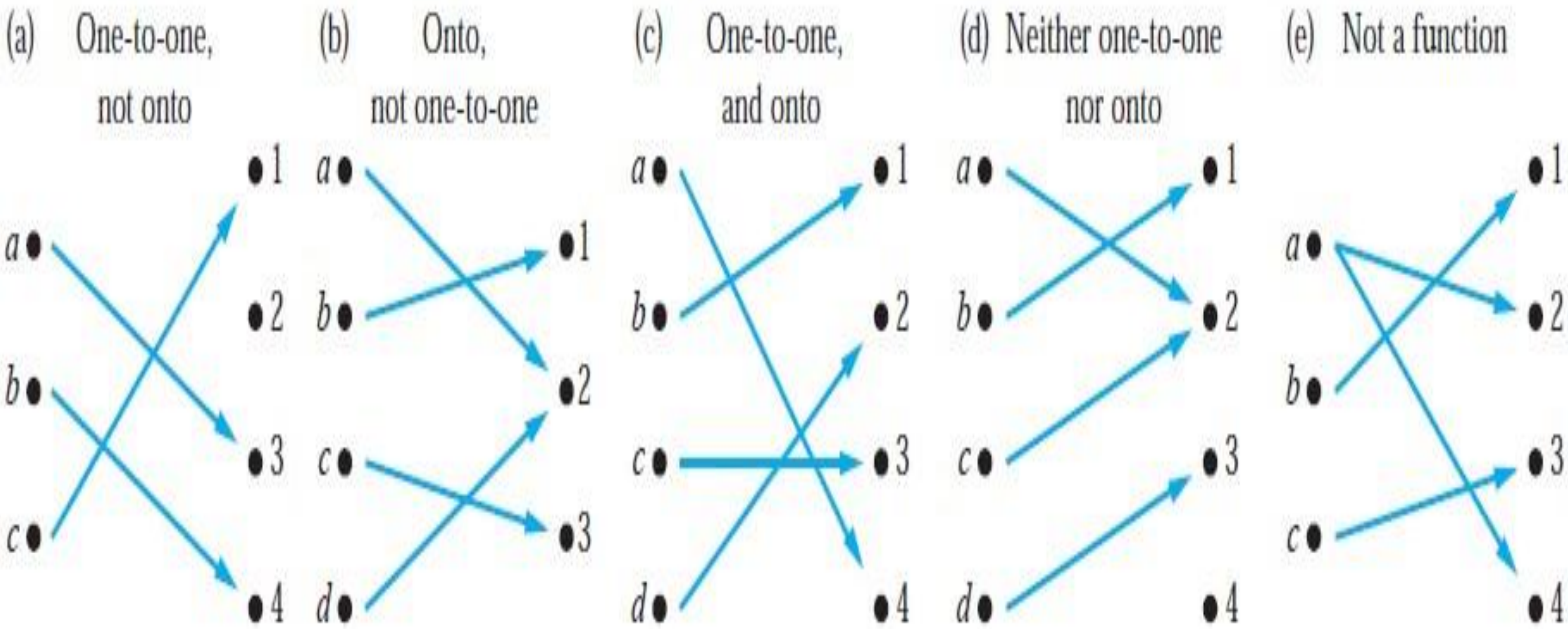


FIGURE 5 Examples of Different Types of Correspondences.

real-valued

- A function is called **real-valued** if its codomain is the set of real numbers.

- Let f_1 and f_2 be functions from A to $\mathbf{R}$.
- Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
  - $(f_1 f_2)(x) = f_1(x) f_2(x)$.

real-valued Functions

Examples:

- **Assume**
  - $f1(x) = x - 1$
  - $f2(x) = x^3 + 1$

  then
  - $(f1 + f2)(x) = x^3 + x$
  - $(f1 * f2)(x) = x^4 - x^3 + x - 1.$

Showing that *f* is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution:

Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2:

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.
Solution:

The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

The *identity function*

- Let A be a set. The *identity function* on A is the function $i\,A : A \rightarrow A$, where

$$i\,A(x) = x$$

- for all $x \in A$.

- The function $i\,A$ is one-to-one and onto, so it is a bijection.

The *identity function*

Example:

- *Let A ={1, 2, 3}*

Solution:

$$i_A(1) = 1$$

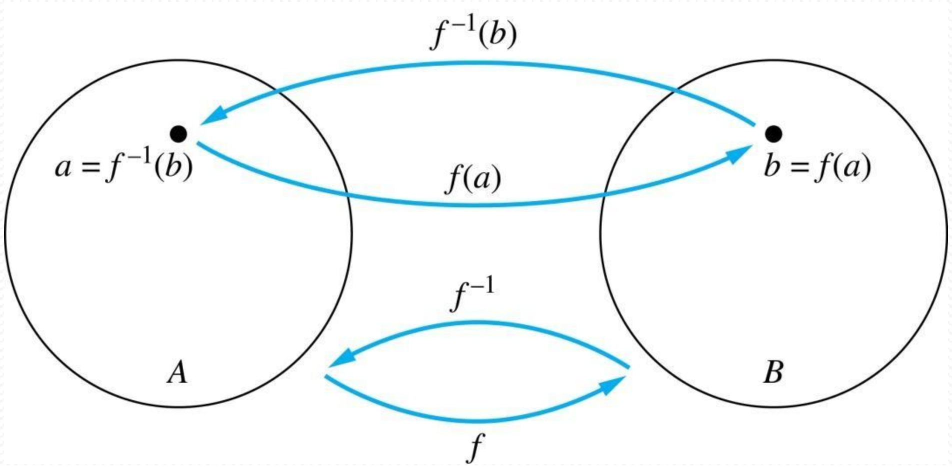
$$i_A(2) = 2$$

$$i_A(3) = 3$$

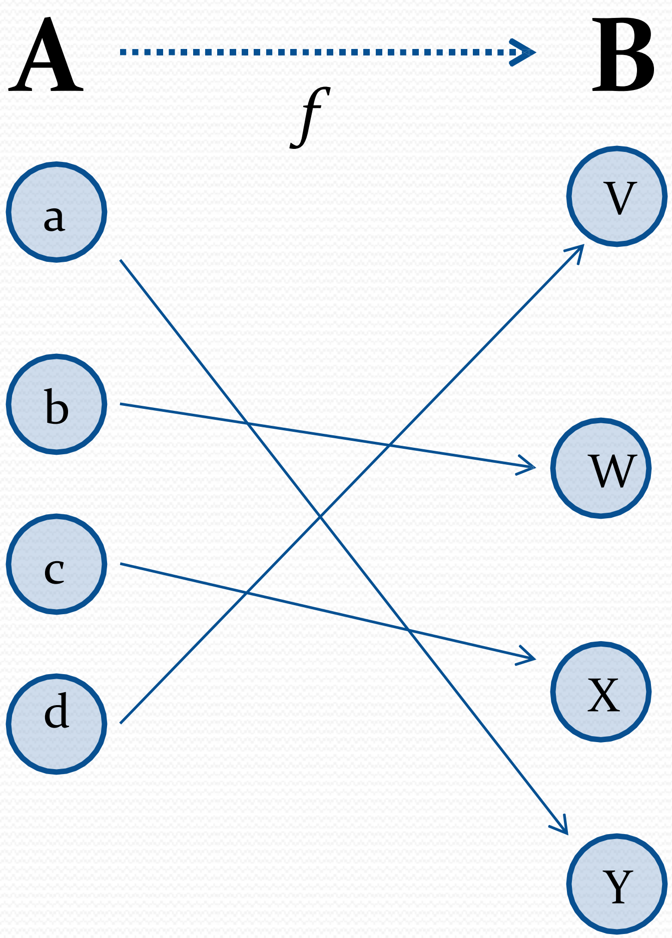
Inverse Functions

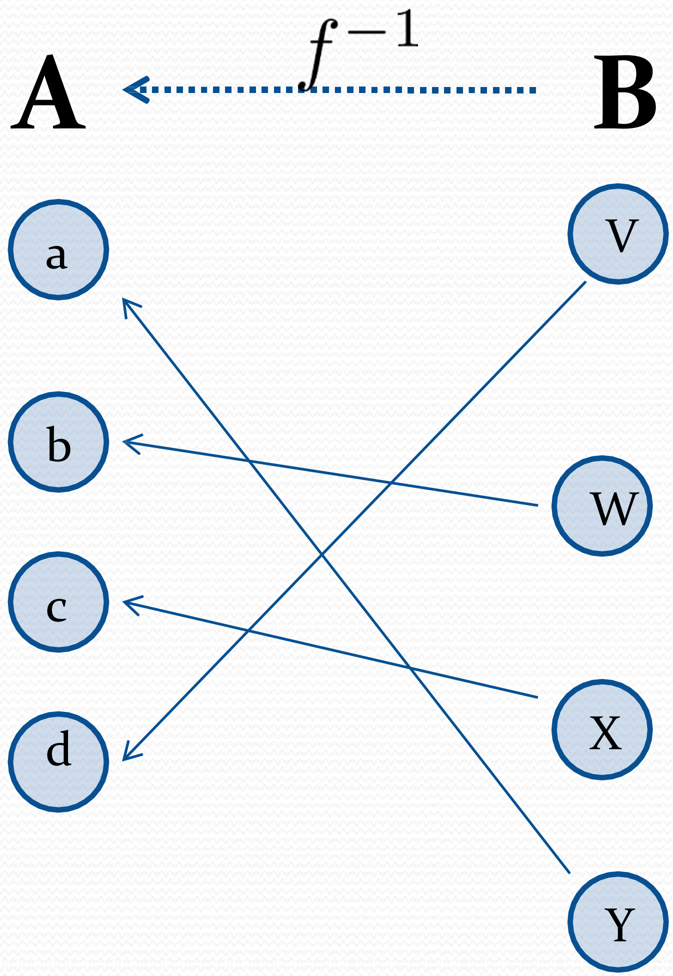
Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted ,f^{-1} is the function from B to A defined as $$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless f is a bijection. Why?



Inverse Functions





Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

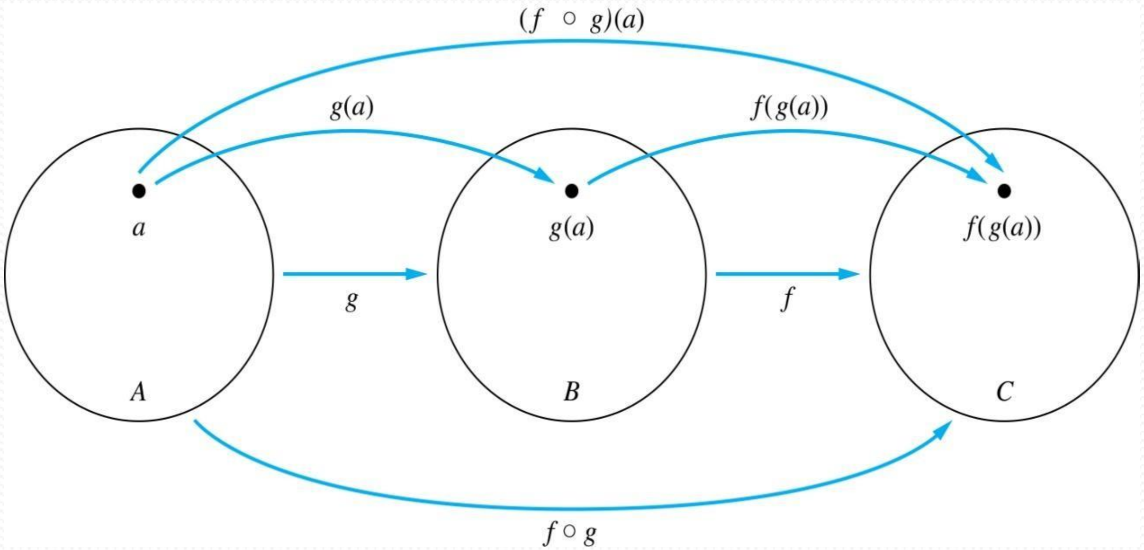
Questions

Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

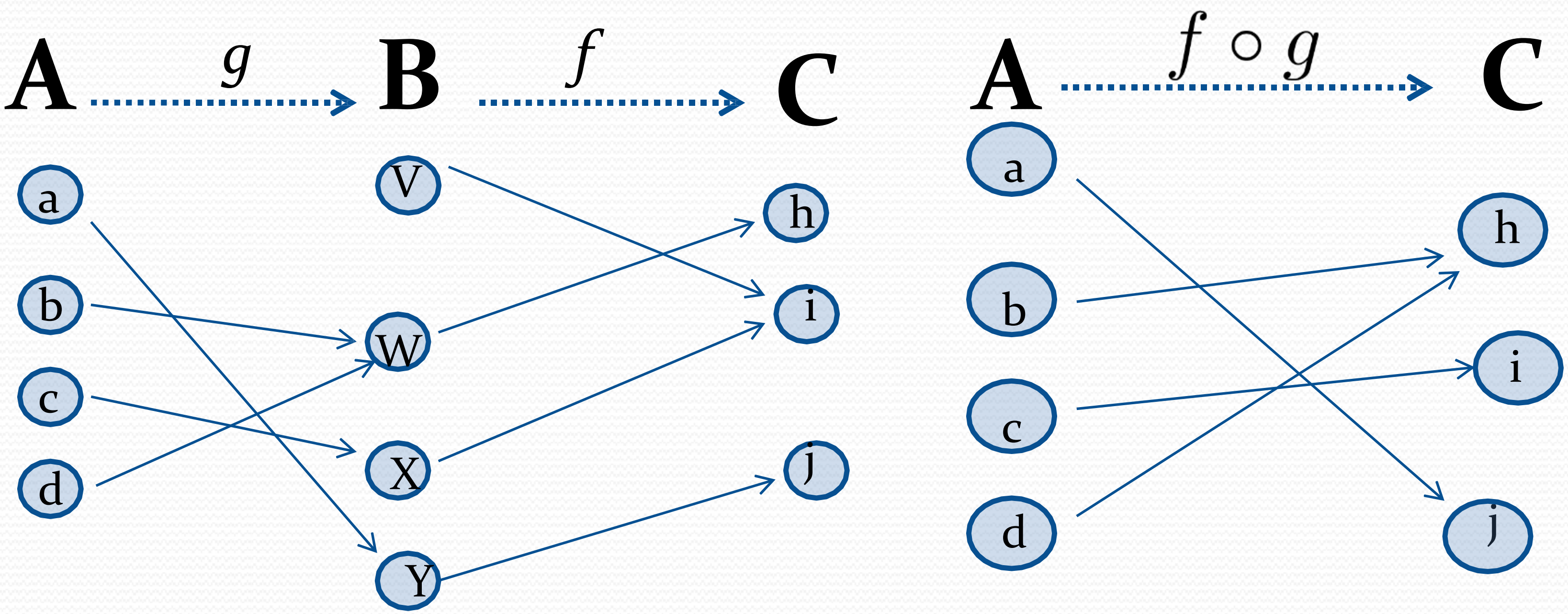
Solution: The function f is not invertible because it is not one-to-one .

Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $$f \circ g(x) = f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f?

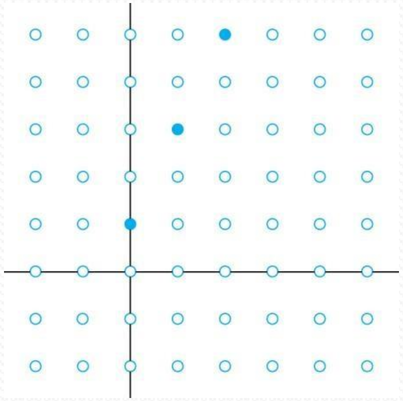
Solution:

$f \circ \mathrm{g}\ (x) = f(\mathrm{g}(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

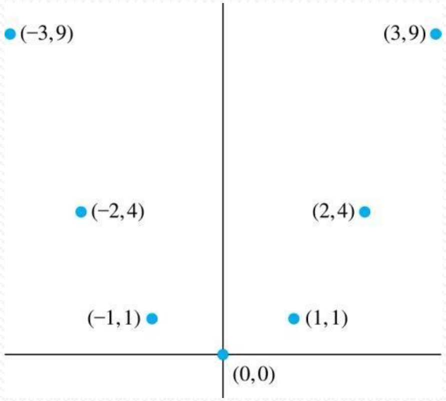
$g \circ \mathrm{f}\ (x) = g(\mathrm{f}(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

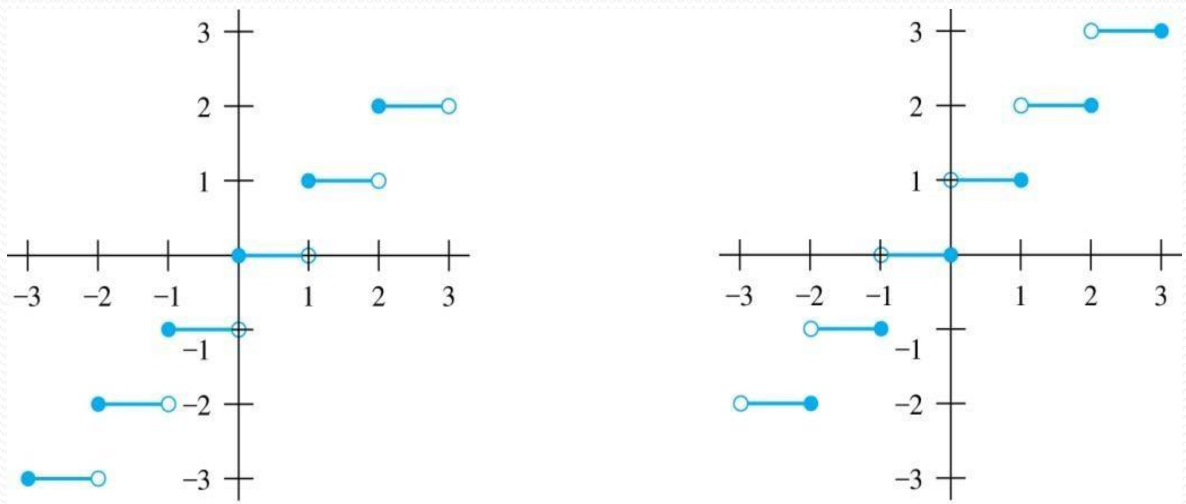
- The *floor* function, denoted
$$f(x) = \lfloor x \rfloor$$
is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted
$$f(x) = \lceil x \rceil$$
is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

Floor Functions

- Determine whether the function from R to Z is Injective OR Surjective.

$$f(\mathrm{a}) = \lfloor a/2 \rfloor$$

Solution:

- It is Surjective (onto function). This can be shown by an example; f (0) = 0, and f (1) = 0.

Ceiling Functions

- Determine whether the function from R to Z is Injective OR Surjective.

  $f(a) = \lceil a/2 \rceil$

Solution:

- It is Surjective (onto function). This can be shown by an example; $f(1) = 1$, and $f(2) = 1$.

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

Factorial Function

Definition: f: **N** → **Z**$^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n,$ $\quad f(0) = 0! = 1$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Remaining Topic from Sets:

- Inclusion-Exclusion Principle

Let A, B be any two finite sets.

Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Here "include" $n(A)$ and $n(B)$ and we "exclude" $n(A \cap B)$

Example 1:

Suppose A, B, C are finite sets.

Then $A \cup B \cup C$ is finite and $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

In a town of 10000 families it was found that 40% of families buy newspaper A, 20% family buy newspaper B, 10% family buy newspaper C, 5% family buy newspaper A and B, 3% family buy newspaper B and C and 4% family buy newspaper A and C. If 2% family buy all the newspaper. Find the number of families which buy

1. Number of families which buy all three newspapers:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$n(A \cup B \cup C) = 40 + 20 + 10 - 5 - 3 - 4 + 2 = 60\%$$

2. Number of families which buy newspaper A only

$$= 40 - 7 = 33\%$$

3. Number of families which buy newspaper B only

$$= 20 - 6 = 14\%$$

4. Number of families which buy newspaper C only

= 10 - 5 = 5%

5. Number of families which buy None of A, B, and C

$n\ (A\ \cup B\ \cup C)^c = 100 - n\ (A \cup B \cup C)$

$n\ (A\ \cup B\ \cup C)^c = 100 - [40 + 20 + 10 - 5 - 3 - 4 + 2]$

$n\ (A\ \cup B\ \cup C)^c = 100 - 60 = 40\ \%$