Lecture # 3

6. Bi-implications:

DEFINITION 6

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
- "if and only if" can be expressed by "iff"
- Example:
  - Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement "You can take the flight if and only if you buy a ticket."

    Implication:

    If you buy a ticket you can take the flight.

    If you don't buy a ticket you cannot take the flight.

Bi-implications:

If p denotes "I am at home." and q denotes "It is raining." then
$p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

If both $P \Longrightarrow Q$ and $Q \Longrightarrow P$ are true, then we say "P if and only if Q" (abbreviated P iff Q). Formally, we write $P \iff Q$. P if and only if Q is true only when P and Q have the same truth values.

For example, if we let P be "3 is odd," Q be "4 is odd," and R be "6 is even," then $P \Longrightarrow R$, $Q \Longrightarrow P$ (vacuously), and $R \Longrightarrow P$. Because $P \Longrightarrow R$ and $R \Longrightarrow P$, P if and only if R.

Given an implication $P \Longrightarrow Q$, we can also define its

(a) **Contrapositive**: $\neg Q \Longrightarrow \neg P$
(b) **Converse**: $Q \Longrightarrow P$

The contrapositive of "If you got an A in this class, I gave you \$5," is "If I did not give you \$5, you didn't get an A in this class." The converse is "If I gave you \$5 you must have received an A in this class." Does the contrapositive say the same thing as the original statement? Does the converse?

The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Expressing the Biconditional

- Some alternative ways “p if and only if q” is expressed in English:
  - p is necessary and sufficient for q
  - if p then q , and conversely
  - p iff q

Without changing their meanings, convert each of the following sentences into a sentence having the form "p iff q"

For a matrix to be invertible, it is necessary and sufficient that its determinant is not zero.

Answer: A matrix is invertible if and only if its determinant is not zero.

If $xy = 0$ then $x = 0$ or $y = 0$, and conversely.

Answer: $xy = 0$ if and only if $x = 0$ or $y = 0$

For an occurrence to become an adventure, it is necessary and sufficient for one to recount it.

Answer: An occurrence becomes an adventure if and only if one recounts it.

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Example Truth Table

- Construct a truth table for $p \vee q \to \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \to \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Problem:

- How many rows are there in a truth table with n propositional variables?

  Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^{2^n} distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Translating English Sentences

(1) If P, then Q.
(2) Q if P.
(3) P only if Q.
(4) P is sufficient for Q.
(5) Q is necessary for P.

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

  "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

  Solution: Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old." The sentence can be translated into:

$$(r \wedge \neg s) \rightarrow \neg q.$$

You cannot ride the coaster if You are under 4 feet tall and you are not older than 16 Years old.

Translating English Sentences

(1) If P, then Q.
(2) Q if P.
(3) P only if Q.
(4) P is sufficient for Q.
(5) Q is necessary for P.

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.

If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Translating English Sentences

- Example: How can this English sentence be translated into a logical expression?

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman." The sentence can be translated into:

$$a \rightarrow (c \lor \neg f).$$

(1) If P, then Q.
(2) Q if P.
(3) P only if Q.
(4) P is sufficient for Q.
(5) Q is necessary for P.

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

  Example: Express in propositional logic:

  "The automated reply cannot be sent when the file system is full"

  Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

$$q \rightarrow \neg p$$

1.1 Propositional Logic

Logic and Bit Operations

- Computers represent information using bits.
- A bit is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators OR, AND, and XOR.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

1.1 Propositional Logic

> DEFINITION 7
>
> A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

- Example: Find the bitwise OR, bitwise AND, and bitwise XOR of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

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01 1011 0110
11 0001 1101
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11 1011 1111    bitwise OR
01 0001 0100    bitwise AND
10 1010 1011    bitwise XOR
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Propositional Equivalences

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a tautology.
A compound proposition that is always false is called a contradiction.
A compound proposition that is neither a tautology or a contradiction is called a contingency.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Tautologies and Contradictions

- Tautology is a statement that is always true regardless of the truth values of the individual logical variables

- Examples:
- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

Tautologies and Contradictions

- A Contradiction is a statement that is always false regardless of the truth values of the individual logical variables

Examples

- The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

Propositional Equivalences

DEFINITION 2

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositional Equivalences

- In general, 2^n rows are required if a compound proposition involves n propositional variables in order to get the combination of all truth values.

- Prove that $\neg(\neg p) \equiv p$

Solution

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

As you can see the corresponding truth values of p and $\neg(\neg p)$ are same, hence equivalence is justified.

Applications: Boolean Searches

- Logical connectives are used extensively in searches of large collections of information.
  - Example: indexes of Web pages.

- AND - used to match records that contain both of two search terms.
- OR - used to match one or both of two search terms.
- NOT - used to exclude a particular search term.

- Read about: Web Page Searching

Applications: Logic Puzzles

- Puzzles (important job interview question) that can be solved using logical reasoning
- [Sm78] Smullyan: An island that has two kinds of inhabitants.
  - knights, who always tell the truth.
  - knaves, who always lie.
- You encounter two people A and B.
- What are the types of A and B?:
  - A says "B is a knight" and
  - B says "The two of us are opposite types."

Example 1:

- p: A is a knight　　　　$\neg$p: A is a knave
- q: B is a knight　　　　$\neg$q: B is a knave

- Consider the possibility that A is a knight;
  - So, p is true. And he is telling truth.
  - & q is true. So, A and B are the same type.

- However, if B is a knight, then B 's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

Example 1: Solution (cont..)

- Consider the possibility that A is a knave,
  - everything a knave says is false; q is true, is a lie.
  - So, q is false. B is also a knave.
  - B ’s statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves.
- We can conclude that both A and B are knaves.

Example 2:

- A father tells his two children, a boy and a girl, to play in their backyard without getting dirty.
- However, while playing, both children get mud on their foreheads. When the children stop playing, the father says "At least one of you has a muddy forehead," and then asks the children to answer "Yes" or "No" to the question: "Do you know whether you have a muddy forehead?"
- The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead?
- Assume that both children are honest and that the children answer each question simultaneously.

Example 2:

Solution: S denotes "Son has a muddy forehead" and D denotes " Daughter has a muddy forehead". The father states that $S \vee D$ is True. Boy can know D is True but can't know S. Girl can know D is True but can't know S. So no for the first time. After that they can conclude that both D and S are True. Since one of them will say yes for the first time if one of D and S is not True.

Example 3:

6. Determine whether these system specifications are consistent:
   "the diagnostic message is stored in the buffer or it is retransmitted"
   "the diagnostic message is not stored in the buffer"
   "if the diagnostic message is stored in the buffer, then it is retransmitted"

Solution: p denotes "the diagnostic message is stored in the buffer", q denotes "the diagnostic message is retransmitted". Then the specifications can be written as $p \vee q$, $\neg p$ and $p \rightarrow q$. $\neg p$ is True, so p is False. $p \vee q$ is True, so q is True. So $p \rightarrow q$ is True. They are consistent.