Lecture# 06

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$
$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.
- Negating the original statement gives "It is not the case that every student in your class has taken a course in Java." This implies that "There is a student in your class who has not taken a course in Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken a course in Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let M(x) denote "x has visited Mexico" and S(x) denote "x is a student in this class," and U be all people.

   $$\exists x\,(S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add C(x) denoting "x has visited Canada."

   and U be all people.

   $$\forall x\,(S(x) \rightarrow (M(x) \vee C(x)))$$

Some Fun with Translating from English into Logical Expressions

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

  Translate "Everything is a Nitwit"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

  “Nothing is a Blubber.”

  Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

  Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

"All Nitwits are Blubbers."

Solution: $\forall x\, (F(x) \rightarrow S(x))$

Translation (cont)

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

"Some Nitwits are Oddments."

Solution: $\exists x\,(F(x) \wedge T(x))$

Translation (cont)

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

“No Blubber is a Oddment.”

Solution: $\neg\exists x\,(S(x) \wedge T(x))$ What is this equivalent to?

Solution: $\forall x\,\neg\,(S(x) \wedge T(x))$

$\forall x\,(\neg S(x) \vee \neg T(x))$

Translation (cont)

- U = {Nitwits, Blubbers, Oddments}

  F(x): x is a Nitwit

  S(x): x is a Blubber

  T(x): x is a Oddment

"If any Nitwit is a Blubber then it is also a Oddment."

Solution: $\forall x\, ((F(x) \land S(x)) \rightarrow T(x))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let L(m, y) be "Mail message m is larger than y megabytes."
  - Let C(m) denote "Mail message m will be compressed."
  - Let A(u) represent "User u is active."
  - Let S(n, x) represent "Network link n is state x.
- Now we have: $\forall m(L(m,1) \rightarrow C(m))$

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

- The first two are called *premises* and the third is called the *conclusion*.
  1. “All lions are fierce.”
  2. “Some lions do not drink coffee.”
  3. “Some fierce creatures do not drink coffee.”
- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions “x is a lion,” “x is fierce,” and “x drinks coffee,” respectively.
  1. $\forall x\,(P(x) \rightarrow Q(x))$
  2. $\exists x\,(P(x) \land \neg R(x))$
  3. $\exists x\,(Q(x) \land \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.

Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \exists y (x + y = 0)$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ *is* $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x\, \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\, \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x\, \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall_x \exists_y P(x,y)$ is true, but $\exists y\, \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\ P(x,y)$
   Answer: True
4. $\exists x \exists\ y\ P(x,y)$
   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the real numbers,
Define $P(x,y)$: $x / y = 1$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\, P(x,y)$
   Answer: False
4. $\exists x\, \exists\, y\, P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	**When True?**	**When False**
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x, y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x, y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\,(C(x) \vee \exists y\,(C(y) \wedge F(x, y)))$$

where C(x) is “x has a computer,” and F(x,y) is “x and y are friends,” and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, $x + y$ is positive."

3. The result is:

   $$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \exists f \; \forall a \; (P(w,f) \land Q(f,a))$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x \, \forall y \, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \, \exists y \, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \, \forall x \, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \, \exists y \, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x \, L(x,x)$