Diagonalization & Quadratic Forms

DEFINITION 1 A square matrix A is said to be ***orthogonal*** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^TA = I \tag{1}$$

Example 1: 3 x 3 orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is orthogonal for all choices of θ since

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THEOREM 7.1.1 *The following are equivalent for an $n \times n$ matrix A.*

(a) *A is orthogonal.*

(b) *The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

(c) *The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

THEOREM 7.1.2

(a) *The transpose of an orthogonal matrix is orthogonal.*

(b) *The inverse of an orthogonal matrix is orthogonal.*

(c) *A product of orthogonal matrices is orthogonal.*

(d) *If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.*

Example 3:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Check $A^T A$ = Identity &
- Det (A) = 1

- The given matrix is orthogonal since its row (and column) vectors form orthonormal sets in R^2 with the Euclidean inner product.

THEOREM 7.1.3 *If A is an $n \times n$ matrix, then the following are equivalent.*

(*a*) *A is orthogonal.*

(*b*) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(*c*) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.

1. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4. (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

THEOREM 7.1.1 *The following are equivalent for an $n \times n$ matrix A.*

(*a*) *A is orthogonal.*

(*b*) *The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

(*c*) *The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating A^TA, then by using part (*b*) of Theorem 7.1.1, and then by using part (*c*) of Theorem 7.1.1.

5. $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$ **6.** $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

T / F

(a) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is orthogonal. **FALSE** (only sq. matrices can be orthogonal)

(b) The matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ is orthogonal. **FALSE** (two row and column vectors are not unit vectors)

(c) An $m \times n$ matrix A is orthogonal if $A^TA = I$. **FALSE** (only sq. matrices can be orthogonal)

(d) A square matrix whose columns form an orthogonal set is orthogonal. **FALSE** (the column vectors must form an orthonormal set.)

(e) Every orthogonal matrix is invertible. **TRUE** (since AtT = I & inverse(A) = transpose(A))

(f) If A is an orthogonal matrix, then A^2 is orthogonal and $(\det A)^2 = 1$. **TRUE** (a product of orthogonal matrices is orthogonal, so A square is orthogonal)

(g) Every eigenvalue of an orthogonal matrix has absolute value 1. **TRUE** (since **norm of (Ax)** = **norm of (x)**)

(h) If A is a square matrix and $\|A\mathbf{u}\| = 1$ for all unit vectors $\mathbf{u}$, then A is orthogonal. **TRUE** (from theorem 7.1.3)

Orthogonal Diagonalization

- We have defined two square matrices, **A** and **B**, to be similar if there is an invertible matrix **P** such that $P^{-1}AP = \mathbf{B}$.

- Here, we will be concerned with the special case in which it is possible to find an orthogonal matrix **P** for which this relationship holds.

DEFINITION 1 If A and B are square matrices, then we say that B is ***orthogonally similar*** to A if there is an orthogonal matrix P such that $B = P^T AP$.

- Note that if **B** is orthogonally similar to **A**, then it is also true that **A** is orthogonally similar to **B**.
- we will say that A and B are **orthogonally similar matrices** if either is orthogonally similar to the other.

- If A is orthogonally similar to some diagonal matrix, say $P^T AP = D$
  - then we say that A is ***orthogonally diagonalizable*** and that P ***orthogonally diagonalizes*** A.

- Symmetric matrix $A^T = A$

THEOREM 7.2.1 *If A is an $n \times n$ matrix with real entries, then the following are equivalent.*

(a) *A is orthogonally diagonalizable.*

(b) *A has an orthonormal set of n eigenvectors.*

(c) *A is symmetric.*

Properties of Symmetric Matrices

THEOREM 7.2.2 *If A is a symmetric matrix with real entries, then:*

(a) *The eigenvalues of A are all real numbers.*

(b) *Eigenvectors from different eigenspaces are orthogonal.*

Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step 3. Form the matrix P whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

Example 1: Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$\det(\lambda I - A) = (\lambda - 2)^2(\lambda - 8) = 0$ $\longrightarrow$ the distinct eigenvalues of A are $\lambda = 2$ and $\lambda = 8$.

a basis for the eigenspace corresponding to $\lambda = 2$.

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2\}$ yields the following orthonormal eigenvectors (verify):

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

The eigenspace corresponding to $\lambda = 8$ has

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a basis. Applying the Gram–Schmidt process to $\{\mathbf{u}_3\}$ (i.e., normalizing $\mathbf{u}_3$) yields

$$\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Finally, using $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as column vectors, we obtain

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

► In Exercises 1–6, find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces. ◄

1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$ **10.** $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Linear Form on R^n

- All variables in a linear form occur to the first power &
- There are no products of variables.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Quadratic Form on R^n

- Expression of the form given below is called quadratic form on $\mathbf{R}^n$.

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 + (\text{all possible terms } a_kx_ix_j \text{ in which } i \neq j)$$

The terms of the form $a_kx_ix_j$ are called ***cross product terms***.

$$x_ix_j = x_jx_i$$

General Quadratic Form in $\mathbf{R}^2$ and $\mathbf{R}^3$

- $R^2 \rightarrow a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

- $R^3 \rightarrow a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 \rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

- Note that the matrix **A** in these formulas is **symmetric**, that its
- **diagonal entries** are the coefficients of the squared terms, and
- its **off-diagonal entries** are half the coefficients of the cross product terms.

Quadratic form in general

- In general, if **A** is a symmetric **n × n** matrix and **x** is an **n × 1** column vector of variables, then we call the function:

  $$Q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

  the quadratic form associated with A.
- The above function can be represented in a dot product notation as:

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x} \cdot A\mathbf{x} = A\mathbf{x} \cdot \mathbf{x}$$

what If A is diagonal matrix?

- In case where **A** is a diagonal matrix, the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ has no cross product terms; for example,
- If A has diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

Example 1: Expressing Quadratic Forms in Matrix Notation

In each part, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where A is symmetric.

(a) $2x^2 + 6xy - 5y^2$ (b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_2$

$$2x^2 + 6xy - 5y^2 = [x \quad y] \begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3 = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3 problems that occurs in a quadratic form

Problem 1 If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on R^2 or R^3, what kind of curve or surface is represented by the equation $\mathbf{x}^T A\mathbf{x} = k$?

Problem 2 If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on R^n, what conditions must A satisfy for $\mathbf{x}^T A\mathbf{x}$ to have positive values for $\mathbf{x} \neq \mathbf{0}$?

Problem 3 If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on R^n, what are its maximum and minimum values if $\mathbf{x}$ is constrained to satisfy $\|\mathbf{x}\| = 1$?

Change of Variable in a Quadratic Form

- We can simplify the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ by making a substitution

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{5}$$

- that expresses the variables $x_1, x_2, \ldots, x_n$ in terms of new variables $y_1, y_2, \ldots, y_n$.

- If P is invertible, then we call (5) a ***change of variable***, and
- if P is orthogonal, then we call (5) an ***orthogonal change of variable***.

If we make the change of variable $\mathbf{x} = P\mathbf{y}$ in the quadratic form $\mathbf{x}^T A\mathbf{x}$, then we obtain

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T A P\mathbf{y} = \mathbf{y}^T(P^T A P)\mathbf{y} \tag{6}$$

the matrix $B = P^T A P$ is symmetric

- the effect of the change of variable is to produce a new quadratic form $\mathbf{y}^T B\mathbf{y}$ in the variables $y_1, y_2, \ldots, y_n$.

- In particular, if we choose **P** to orthogonally diagonalize A, then

- the new quadratic form will be $\mathbf{y}^T D\mathbf{y}$, where **D** is a diagonal matrix with the eigenvalues of **A** on the main diagonal; that is,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

THEOREM 7.3.1 The Principal Axes Theorem

If A is a symmetric $n \times n$ matrix, then there is an orthogonal change of variable that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross product terms. Specifically, if P orthogonally diagonalizes A, then making the change of variable $\mathbf{x} = P\mathbf{y}$ in the quadratic form $\mathbf{x}^T A\mathbf{x}$ yields the quadratic form

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the eigenvectors that form the successive columns of P.

EXAMPLE 2: An Illustration of the Principal Axes Theorem

Find an orthogonal change of variable that eliminates the cross product terms in the quadratic form $Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$, and express Q in terms of the new variables.

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$\lambda^3 - 9\lambda = \lambda(\lambda + 3)(\lambda - 3) = 0$$

so the eigenvalues are $\lambda = 0, -3, 3$

orthonormal bases for the three eigenspaces are

$$\lambda = 0: \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \lambda = -3: \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \lambda = 3: \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

a substitution $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This produces the new quadratic form

$$Q = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -3y_2^2 + 3y_3^2$$

Remember!

- If $\mathbf{A}$ is a symmetric $\mathbf{n} \times \mathbf{n}$ matrix, then the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is a real-valued function whose range is the set of all possible values for $\mathbf{x}^T\mathbf{A}\mathbf{x}$ as $\mathbf{x}$ varies over $\mathbf{R}^n$.

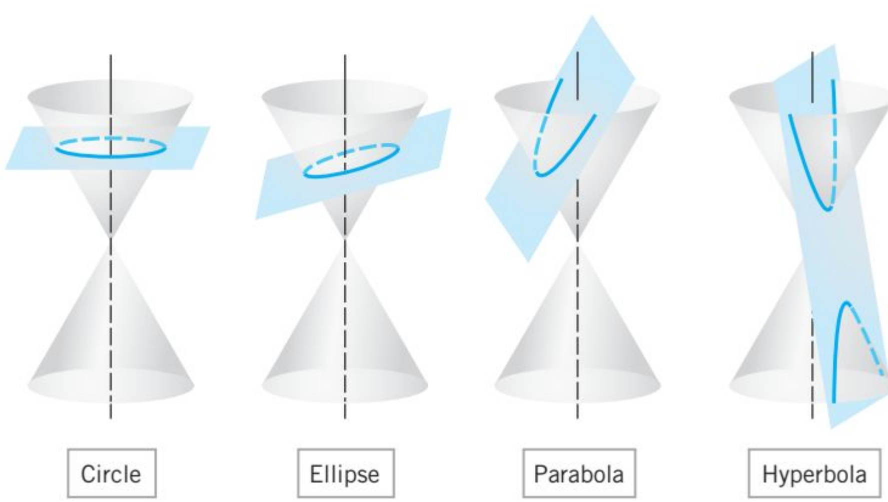
- It can be shown that an orthogonal change of variable $\mathbf{x} = \mathbf{P}\mathbf{y}$ does not alter the range of a quadratic form; that is,

- the set of all values for $\mathbf{x}^T\mathbf{A}\mathbf{x}$ as $\mathbf{x}$ varies over $\mathbf{R}^n$ is the same as the set of all values for $\mathbf{y}^T(\mathbf{P}^T\mathbf{A}\mathbf{P})\mathbf{y}$ as $\mathbf{y}$ varies over $\mathbf{R}^n$.

Conic Section or Conic

A **conic section** or **conic** is a curve that results by cutting a double-napped cone with a plane.

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$



Central Conic Equation

$$ax^2 + 2bxy + cy^2 + f = 0$$

- If there are no cross products term then the above eq is called **central conic in standard position**.

$$ax^2 + cy^2 + f = 0$$

Rewrite the conic equations in the matrix notation

(by letting k = -f to the right side of the eq.)

$$ax^2 + 2bxy + cy^2 + f = 0$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k$$

$$ax^2 + cy^2 + f = 0$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k$$

3-dimensional analogs of the previous slide

The thre

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \quad \text{and} \quad \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k$$

In Exercises 1–2, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where A is a symmetric matrix.

1. (a) $3x_1^2 + 7x_2^2$ (b) $4x_1^2 - 9x_2^2 - 6x_1x_2$

(c) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

2. (a) $5x_1^2 + 5x_1x_2$ (b) $-7x_1x_2$

(c) $x_1^2 + x_2^2 - 3x_3^2 - 5x_1x_2 + 9x_1x_3$

In Exercises 3–4, find a formula for the quadratic form that does not use matrices.

3. $[x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

4. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} -2 & \frac{7}{2} & 1 \\ \frac{7}{2} & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

In Exercises 5–8, find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables.

5. $Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$

6. $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$

7. $Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

8. $Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$