Lecture# 07

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \, \forall a \, \exists f \, (P(w,f) \land Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \, \forall a \, \exists f \, (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \, \forall a \, \exists f \, (P(w,f) \land Q(f,a))$
2. $\forall w \, \neg \forall a \, \exists f \, (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \, \exists a \, \neg \exists f \, (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \, \exists a \, \forall f \, \neg (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \, \exists a \, \forall f \, (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Some Questions about Quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

  Solution: Yes! The left and the right side will always have the same truth value. The order in which x and y are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

  Solution: No! The left and the right side may have different truth values for some propositional functions for P. Try "x + y = 0" for P(x,y) with U being the integers. The order in which the values of x and y are picked does matter.

- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

  Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by P(x) and Q(x).

  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

  Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

Summary

- Valid Arguments and Rules of Inference
- Proof Methods
- Proof Strategies

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Revisiting the Socrates Example

- We have the two premises:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the conclusion:
  - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\frac{\begin{array}{c}\forall x (Man(x) \rightarrow Mortal(x)) \\ Man(Socrates)\end{array}}{\therefore \quad Mortal(Socrates)}$$

- We will see shortly that this is a valid argument.

Valid Arguments

- We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

1. Propositional Logic

   Inference Rules

2. Predicate Logic

   Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Valid Arguments (Propositional Logic)

- “If you have a current password, then you can log onto the network.”
- “You have a current password.”
- Therefore, “You can log onto the network.”

- Sequence of propositions (argument) is valid; the conclusion must be true when both premise are true.

Argument Form

- p: “You have a current password”
- q: “You can log onto the network.”
- the argument has the form

$$p \rightarrow q$$
$$p$$
$$\therefore q$$

the statement $((p \rightarrow q) \land p) \rightarrow q$ is a tautology

Argument Form

- p: ““You have access to the network”
- q: “You can change your grade”
- the argument has the form

“If you have access to the network, then you can change your grade.”

“You have access to the network.”

∴ “You can change your grade.”

- The argument is valid, but if premise is false, we cannot conclude that the conclusion is true.

Arguments in Propositional Logic

- A argument in propositional logic is a sequence of propositions. All but the final proposition are called premises.
- The last statement is the conclusion.
  The argument is valid if the premises imply the conclusion.
- An argument form is an argument that is valid no matter
- what propositions are substituted into its propositional variables.

If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.
Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $\therefore\ p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore\ p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore\ p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore\ q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

Rules of Inference for Propositional Logic: Modus Ponendo Ponens

$$\frac{P \to Q, P}{\therefore Q}$$

- Latin for "the way that affirms by affirming"

- Implication elimination
- Abbreviated to MP or modus ponens

- whenever $p \to q$ is true
- and p is true,
- q must also be true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Example:

Let *p* be "It is snowing."
Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

Modus Ponens

- State which rule of inference is the basis of the following argument:
- “If it snows today, then we will go skiing”. “It is snowing today”, “therefore we will go skiing”.
- Let p be the proposition “It is snowing today” and q the proposition “We will go skiing” Then this argument is of the form

$$
\begin{array}{l}
p \rightarrow q \\
p \\
\therefore q
\end{array}
$$

- This is an argument that uses the Modus Ponens rule.

Modus Tollendo Tollens

$$\frac{P \to Q, \neg Q}{\therefore \neg P}$$

- Latin for "the way that denies by denying"

- Denying the consequent
- Abbreviated to modus tollens

- in every instance in which
- p → q is true and q is false,
- p must also be false.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg p \wedge (p \to q)) \to \neg q$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Corresponding Tautology:
((p →q) ∧ (q→r))→(p→ r)

Example:
Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Hypothetical Syllogism

- State which rule of inference is used in the argument:
- If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

$$p \to q$$
$$q \to r$$
$$\therefore p \to r$$

- Hence, this argument is a hypothetical syllogism.

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:
$p \rightarrow (p \vee q)$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Addition Rule

- State which rule of inference is the basis of the following argument:
- “It is below freezing now. Therefore, it is either below freezing or raining now.”
- Let p be the proposition “It is below freezing now” and q the proposition “It is raining now.” Then this argument is of the form

$$\begin{array}{l} p \\ \therefore p \vee q \end{array}$$

- This is an argument that uses the addition rule.

Simplification

$$\frac{p \wedge q}{\therefore\ q}$$

Corresponding Tautology:
$(p \wedge q) \rightarrow p$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Simplification Rule

- State which rule of inference is the basis of the following argument:
- “It is below freezing and raining now. Therefore, it is below freezing now.”
- Let p be the proposition “It is below freezing now,” and let q be the proposition “It is raining now.” This argument is of the form

$$\begin{array}{l} p \land q \\ \therefore p \end{array}$$

- This argument uses the simplification rule.

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:
Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Using the Rules of Inference to Build Valid Arguments

- A valid argument is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
\vdots \\
\therefore C \quad S_n
\end{array}
$$

Fallacies

- Arguments are based on tautologies.
- Fallacies are based on contingencies.
- The proposition $((p \rightarrow q) \land q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true.
- This type of incorrect reasoning is called the fallacy of affirming the conclusion.
- Example:
- If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.

Fallacy of denying the hypothesis.

- The proposition $((p \to q) \wedge \neg p) \to \neg q$ is not a tautology, because it is false when p is false and q is true.
- Is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?
- It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \to q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis.

Using Rules of Inference

Fallacies

- Are the following arguments correct?
  - Example 1 (Fallacy of affirming the conclusion)

    Hypothesis
    - If you success, you work hard $\quad p \rightarrow q$
    - You work hard $\quad q$

    Conclusion
    - You success $\quad \therefore\ p$

  - Example 2 (Fallacy of denying the hypothesis)

    Hypothesis
    - If you success, you work hard $\quad p \rightarrow q$
    - You do not success $\quad \neg p$

    Conclusion
    - You do not work hard $\quad \therefore\ \neg q$

V. Tell the validity of each argument by choosing one of the following:

MP- Modus Ponens **LS- Law of Syllogism (Hypothetical)**
MT- Modus Tollens **LC- Law of Contrapositive**
NVC- No Valid Conclusion

a. ____

$p \rightarrow \sim r$
$\sim p$

$\therefore r$

b. ____

$s \rightarrow t$
$\sim t$

$\therefore \sim s$

c. ____

$r \rightarrow \sim t$
$\sim t \rightarrow v$

$\therefore r \rightarrow v$

d. ____

$r \rightarrow p$
p

$\therefore r$

e. ____

$\sim w \rightarrow \sim s$
$\sim w$

$\therefore \sim s$

f. ____

$t \rightarrow (r \rightarrow s)$
$\sim(r \rightarrow s)$

$\therefore \sim t$

g. ____

$p \rightarrow \sim r$

$\therefore r \rightarrow \sim p$

h. ____

$r \rightarrow q$
$s \rightarrow q$

$\therefore r \rightarrow s$

V. Tell the validity of each argument by choosing one of the following:

MP- Modus Ponens
MT- Modus Tollens
LS- Law of Syllogism (Hypothetical)
LC- Law of Contrapositive
NVC- No Valid Conclusion

a. NVC

$p \rightarrow \sim r$
$\sim p$

$\therefore r$

b. M.T

$(s \rightarrow t)$ ✓
$\wedge (\sim t)$

$\therefore \boxed{\sim s}$

c. LS (H.S)

$r \rightarrow \cancel{\sim t}$
$\cancel{\sim t} \rightarrow v$

$\therefore r \rightarrow v$

d. NVC

$r \rightarrow p$
p

$\therefore r$

e. M.P

$\sim w \rightarrow \sim s$ ✓
$\sim w$

$\therefore \sim s$

f. M.T

$t \rightarrow (r \rightarrow s)$ ✓
$\sim(r \rightarrow s)$

$\therefore \sim t$

g. L.C

$p \rightarrow \sim r$

$\therefore r \rightarrow \sim p$

h. N.V.C

$r \rightarrow q$
$s \rightarrow q$

$\therefore r \rightarrow s$

Comparison between Inference and Equivalence

- Inference ($p \rightarrow q$)
  - Meaning:
    If p, then q
  - $p \rightarrow q$ does not mean $q \rightarrow p$
  - Either inference or equivalence rules can be used
  - $p \leftrightarrow q$ implies $p \rightarrow q$
  - $\Rightarrow$ is used in proof
- Equivalence ($p \leftrightarrow q$)
  - Meaning:
    p is equal to q
  - $p \leftrightarrow q$ mean $q \leftrightarrow p$
  - Only equivalence rules can be used
  - $p \leftrightarrow q$ can be proved by showing $p \rightarrow q$ and $q \rightarrow p$
  - $\Leftrightarrow$ is used in proof
- Equivalence ($\leftrightarrow$) is a **more restrictive** relation than Inference ($\rightarrow$)

Using Rules of Inference

- Example 1:
  - **Given:**
    - It is not sunny this afternoon and it is colder than yesterday.
    - We will go swimming only if it is sunny
    - If we do not go swimming, then we will take a canoe trip
    - If we take a canoe trip, then we will be home by sunset
  - Can these propositions lead to the **conclusion** "We will be home by sunset" ?

Let		
	p:	It is sunny this afternoon
	q:	It is colder than yesterday
	r:	We go swimming
	s:	We take a canoe trip
	t:	We will be home by sunset

$\neg p \wedge q$	It is not sunny this afternoon and it is colder than yesterday
$r \rightarrow p$	We will go swimming only if it is sunny
$\neg r \rightarrow s$	If we do not go swimming, then we will take a canoe trip
$s \rightarrow t$	If we take a canoe trip, then we will be home by sunset
t	We will be home by sunset

Using Rules of Inference

Hypothesis:

$\neg p \wedge q$

$r \to p$

$\neg r \to s$

$s \to t$

Conclusion:

t

Step		Reason
1.	$\neg p \wedge q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Premise
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \to s$	Premise
6.	s	Modus ponens using (4) and (5)
7.	$s \to t$	Premise
8.	t	Modus ponens using (6) and (7)

Therefore, the propositions can lead to the conclusion
We will be home by sunset

Using Rules of Inference

- Or, another presentation method:

Hypothesis:

$\neg p \wedge q$

$r \rightarrow p$

$\neg r \rightarrow s$

$s \rightarrow t$

Conclusion:

t

$\boxed{(\neg p \wedge q)} \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$

$\Rightarrow \boxed{\neg p \wedge (r \rightarrow p)} \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ By Simplification

$\Rightarrow \boxed{\neg r \wedge (\neg r \rightarrow s)} \wedge (s \rightarrow t)$ By Modus Tollens

$\Rightarrow \boxed{s \wedge (s \rightarrow t)}$ By Modus Ponens

$\Rightarrow t$ By Modus Ponens

EXAMPLE #2

- **Given:**
  - If you send me an e-mail message, then I will finish writing the program
  - If you do not send me an e-mail message, then I will go to sleep early
  - If I go to sleep early, then I will wake up feeling refreshed
- Can these propositions lead to the **conclusion** "If I do not finish writing the program, then I will wake up feeling refreshed."

Let		
	p:	you send me an e-mail message
	q:	I will finish writing the program
	r:	I will go to sleep early
	s:	I will wake up feeling refreshed

$p \to q$

- If you send me an e-mail message, then I will finish writing the program

$\neg p \to r$

- If you do not send me an e-mail message, then I will go to sleep early

$r \to s$

- If I go to sleep early, then I will wake up feeling refreshed

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$\neg q \to s$

- If I do not finish writing the program, then I will wake up feeling refreshed

Hypothesis:

$p \to q$

$\neg p \to r$

$r \to s$

Conclusion:

$\neg q \to s$

Step		Reason
1.	$p \to q$	Premise
2.	$\neg q \to \neg p$	Contrapositive of (1)
3.	$\neg p \to r$	Premise
4.	$\neg q \to r$	Hypothetical Syllogism using (2) and (3)
5.	$r \to s$	Premise
6.	$\neg q \to s$	Hypothetical Syllogism using (4) and (5)

Therefore, the propositions can lead to the conclusion

If I do not finish writing the program,

then I will wake up feeling refreshed

- Or, another presentation method:

Hypothesis:

$p \to q$

$\neg p \to r$

$r \to s$

Conclusion:

$\neg q \to s$

$$\boxed{(p \to q)} \wedge (\neg p \to r) \wedge (r \to s)$$

$$\Leftrightarrow \boxed{(\neg q \to \neg p) \wedge (\neg p \to r)} \wedge (r \to s)$$ Contrapositive

$$\Rightarrow \boxed{(\neg q \to r) \wedge (r \to s)}$$ By Hypothetical Syllogism

$$\Rightarrow (\neg q \to s)$$ By Hypothetical Syllogism

Valid Arguments

Example 3: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Conjunction using (1)
3. $p \rightarrow q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments (Predicate Logic)

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Universal Instantiation

- Universal instantiation is the rule of inference used to conclude that P(c) is true, where c is a particular member of the domain, given the premise $\forall x\, P(x)$.
- •Universal instantiation is used when we conclude from the statement “All women are wise” that “Lisa is wise,” where Lisa is a member of the domain of all women.

Universal Generalization

- Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$is true for all elements c in the domain.
- Universal generalization is used when we show that $\forall x\, P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true.
- Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential Instantiation

- Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that $\exists x$ P(x) is true.
- We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true.
- Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential Generalization

- Existential generalization is the rule of inference that is used to conclude that $\exists x\, P(x)$ is true when a particular element c with P(c) true is known.
- That is, if we know one element c in the domain for which P(c) is true, then we know that $\exists x\, P(x)$ is true.