Lecture # 2

Chapter#1 : The Foundations: Logic and Proofs

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Proposition

Definition

proposition (or statement):
a declarative sentence that is either true or false

- law of the excluded middle:
  a proposition cannot be partially true or partially false
- law of contradiction:
  a proposition cannot be both true and false

propositions

- The Moon revolves around the Earth.
- Elephants can fly.
- $3 + 8 = 11$

not propositions

- What time is it?
- Exterminate!
- $x < 43$

Examples: Propositions

Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Islamabad is the capital of Pakistan.

This makes a declarative statement, and hence is a proposition. The proposition is TRUE (T).

Can Ali come with you?.

This is a question not the declarative sentence and hence not a proposition.

Take two aspirins.

This is an imperative sentence not the declarative sentence and therefore not a proposition.

$x + 4 > 9$.

Because this is true for certain values of x (such as $x = 6$) and false for other values of x (such as $x = 5$), it is not a proposition.

He is a college student.

Because truth or falsity of this proposition depend on the reference for the pronoun *he.* it is not a proposition.

Propositional Variable

- propositional variable:
  a name that represents the proposition

examples

- p_1: The Moon revolves around the Earth. (T)
- p_2: Elephants can fly. (F)
- p_3: $3 + 8 = 11$ (T)

Notations

- The small letters are commonly used to denote the propositional variables, that is, variables that represent propositions, such as, p, q, r, s, ...
- The truth value of a proposition is true, denoted by T or 1, if it is a true proposition and false, denoted by F or 0, if it is a false proposition.

Compound Propositions

Logical operators are used to form new propositions also called compound propositions from two or more existing propositions.

- compound propositions are obtained by applying logical operators

Propositional Logic – the area of logic that deals with propositions

- truth table:
  a table that lists the truth value of the compound proposition for all possible values of its variables

CONNECTIVE	MEANINGS	SYMBOLS	CALLED
Negation	not	$\sim$	Tilde
Conjunction	and	$\wedge$	Hat
Disjunction	or	$\vee$	Vel
Conditional	if…then…	$\rightarrow$	Arrow
Biconditional	if and only if	$\leftrightarrow$	Double arrow

1. Negation:

DEFINITION 1:

Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement "It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$ is the opposite of the truth value of p.

- Examples
  - Find the negation of the proposition "Today is Friday." and express this in simple English.

    Solution: The negation is "It is not the case that today is Friday."

    In simple English, "Today is not Friday." or "It is not Friday today."
  - Find the negation of the proposition "At least 10 mm of rain fell today in Karachi." and express this in simple English.

    Solution: The negation is "It is not the case that at least 10 mm of rain fell today in Karachi."

    In simple English, "Less than 10 mm of rain fell today in Karachi."

Negation:

- Note: Always assume fixed times, fixed places, and particular people unless otherwise noted.
- Truth table:

The Truth Table for the Negation of a Proposition.	
p	¬p
T	F
F	T

examples

- $\neg p_1$: The Moon does not revolve around the Earth.
  $\neg T : F$
- $\neg p_2$: Elephants cannot fly.
  $\neg F : T$

2. Conjunction:

DEFINITION 2

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition “p and q”. The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

- Examples
  - Find the conjunction of the propositions p and q where p is the proposition “Today is Friday.” and q is the proposition “It is raining today.”, and the truth value of the conjunction.

    Solution: The conjunction is the proposition “Today is Friday and it is raining today.” The proposition is true on rainy Fridays.

$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

examples

- $p_1 \wedge p_2$: The Moon revolves around the Earth and elephants can fly.
  $T \wedge F : F$
- $p_1 \wedge p_3$: The Moon revolves around the Earth and $3 + 8 = 11$.
  $T \wedge T : T$

3. Disjunction:

DEFINITION 3

Let p and q be propositions. The disjunction of p and q, denoted by p v q, is the proposition "p or q". The disjunction p v q is false when both p and q are false and is true otherwise.

- Note:

  inclusive or : The disjunction is true when at least one of the two propositions is true.

  - E.g. "Students who have taken calculus or computer science can take this class." – those who take one or both classes.

  exclusive or : The disjunction is true only when one of the proposition is true.

  - E.g. "Students who have taken calculus or computer science, (but not both), can take this class." – only those who take one of them.

- Definition 3 uses inclusive or.

Disjunction (OR)

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

example

- $p_1 \vee p_2$: The Moon revolves around the Earth or elephants can fly.
  $T \vee F : T$

4. Exclusive OR:

DEFINITION 4

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true, but not both.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive Or (XOR) of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Examples

1. Find the *exclusive or* of the propositions *p* and *q*, where

   p : Atif will pass the course CSC102.

   q : Atif will fail the course CSC102.

The *exclusive or* is

p@q : Atif will pass or fail the course CSC102.

The following proposition uses the (English) connective "or". Determine from the context whether "or" is intended to be used in the inclusive or exclusive sense.

1. "Nabeel has one or two brothers".

A person cannot have both one and two brothers.
Therefore, "or" is used in the exclusive sense.

Examples (OR vs XOR)

2. To register for BSC you must have passed the qualifying exam or be listed as an Math major.

Presumably, if you have passed the qualifying exam and are also listed as an Math major, you can still register for BCS. Therefore, “or” is inclusive.

5. Implication / Conditional Statements:

DEFINITION 5

Let p and q be propositions. The conditional statement $p \to q$, is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

- A conditional statement is also called an implication.
- Example: "If I am elected, then I will lower taxes." $p \to q$

implication:

	p	q	$p \to q$
elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

Example: Conditional Statements

- Example:
  - Let p be the statement “Maria learns discrete mathematics.” and q the statement “Maria will find a good job.” Express the statement $p \rightarrow q$ as a statement in English.

    Solution: Any of the following -

    “If Maria learns discrete mathematics, then she will find a good job.

    “Maria will find a good job when she learns discrete mathematics.”

    “For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

    “Maria will find a good job unless she does not learn discrete mathematics.”

Examples: Implication

Examples of implications:

> If you stand in the rain, then you'll get wet.
> If you got an A in this class, I gave you \$5.

An implication $P \Longrightarrow Q$ is false only when P is true and Q is false. For example, the first statement would be false only if you stood in the rain but didn't get wet. The second statement above would be false only if you got an "A," yet I didn't give you \$5.

Here is the truth table for $P \Longrightarrow Q$:

P	Q	$P \Longrightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

[1] P is also called the *antecedent* and Q the *consequent*.

Examples: Implication

Note that $P \Longrightarrow Q$ is always true when P is false. This means that many statements that sound nonsensical in English are true, mathematically speaking. Examples are statements like: "If pigs can fly, then horses can read" or "If 14 is odd then $1+2=18$." When an implication is stupidly true because the hypothesis is false, we say that it is **vacuously true**. Note also that $P \Longrightarrow Q$ is logically equivalent to $\neg P \vee Q$, as can be seen in the above truth table.

$P \Longrightarrow Q$ is the most common form mathematical theorems take. Here are some of the different ways of saying it:

(1) If P, then Q.
(2) Q if P.
(3) P only if Q.
(4) P is sufficient for Q.
(5) Q is necessary for P.

Some other cases of implications:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	

Implication Example

- "If I weigh over 70 kg, then I will exercise."

Implication Example

- "If I weigh over 70 kg, then I will exercise."

- p: I weigh over 70 kg.
- q: I exercise.

- when is this claim false?

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication Examples

- p_4: $3 < 8$, p_5: $3 < 14$, p_6: $3 < 2$, p_7: $8 < 6$

- $p_4 \rightarrow p_5$:
  if $3 < 8$, then $3 < 14$
  $T \rightarrow T : T$
- $p_4 \rightarrow p_6$:
  if $3 < 8$, then $3 < 2$
  $T \rightarrow F : F$
- $p_6 \rightarrow p_4$:
  if $3 < 2$, then $3 < 8$
  $F \rightarrow T : T$
- $p_6 \rightarrow p_7$:
  if $3 < 2$, then $8 < 6$
  $F \rightarrow F : T$

- Other conditional statements:
  - Converse of p → q : q → p
  - Contrapositive of p → q : ¬ q → ¬ p
  - Inverse of p → q : ¬ p → ¬ q

The contrapositive of "If you got an A in this class, I gave you \$5," is "If I did not give you \$5, you didn't get an A in this class." The converse is "If I gave you \$5 you must have received an A in this class." Does the contrapositive say the same thing as the original statement? Does the converse?

Let's look at the truth table:

P	Q	$\neg P$	$\neg Q$	$P \Longrightarrow Q$	$Q \Longrightarrow P$	$\neg Q \Longrightarrow \neg P$	$P \Longleftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

Note that the contrapositive of $P \Longrightarrow Q$ has the same truth values, while the converse does not. Many students unreasonably assume that the converse is true, but the above truth table shows that it is not necessarily the case. When two propositional forms have the same truth values, they are said to be **logically equivalent** – they mean the same thing. We'll see next time how useful this can be for proving theorems.

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
  - $q \rightarrow p$ is the converse of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$

Example: Find the converse, inverse, and contrapositive of “If it is raining then I do not go to the town.”

Solution:

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

inverse: If it is not raining, then I will go to town.

Contrapositive:

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- Any proposition and its contrapositive are logically equivalent (have the same truth table values) – Check with the truth table.
- E.g. The contrapositive of “If you get 100% in this course, you will get an A+” is “If you do not get an A+ in this course, you did not get 100%”.

Converse:

- Converse of $p \rightarrow q$ is $q \rightarrow p$
- Both are not logically equivalent.
- Ex 1: “If you get 100% in this course, you will get an A+” and “If you get an A+ in this course, you scored 100%” are not equivalent.
- Ex 2: If you won the lottery, you are rich.

Inverse:

- Inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- Both are not logically equivalent.
- Ex1 : “If you get 100% in this course, you will get an A+” and “If you didn’t 100%, then won’t have an A+ in this course.” are not equivalent.
- Ex2: You can not ride the roller coaster if you are under 4 feet. What is its inverse statement?

Example of converse (1/2)

- Find the converse of the following statement:
- R: ‘Raining tomorrow is a sufficient condition for my not going to town.’
- Step 1: Assign propositional variables to
- component propositions
- P: It will rain tomorrow
- Q: I will not go to town

Example of converse (2/2)

- Step 2: Symbolize the assertion R: $P \rightarrow Q$
- Step 3: Symbolize the converse $Q \rightarrow P$
- Step 4: Convert the symbols back into words

‘If I don’t go to town then it will rain tomorrow’ or

‘Raining tomorrow is a necessary condition for my not going to town.’ or

‘My not going to town is a sufficient condition for it raining tomorrow.’