Relations

Chapter 9

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Chapter Summary

- Relations and Their Properties
- Representing Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

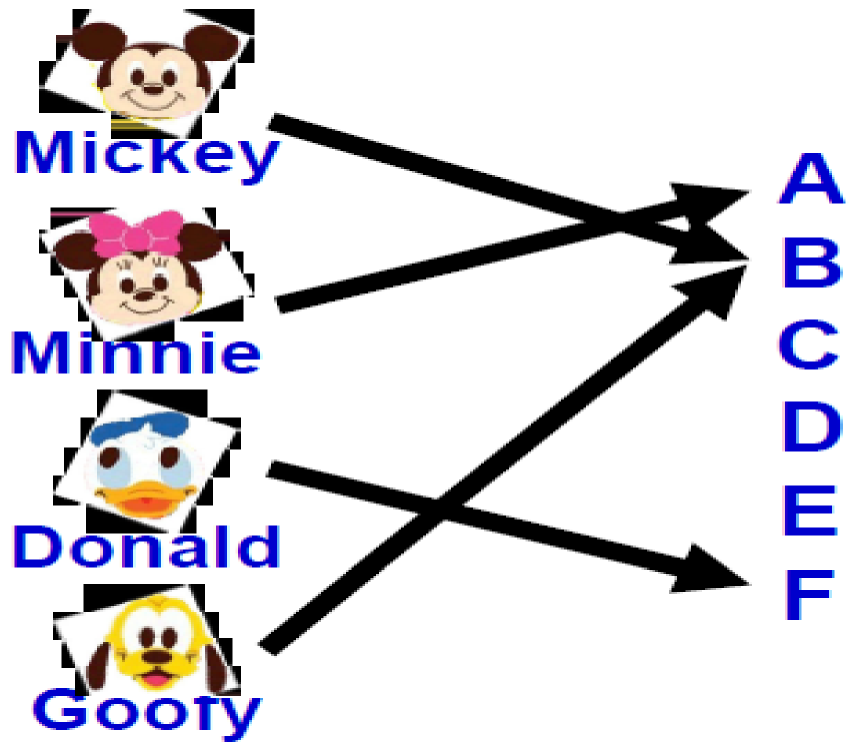
Section 9.1

Section Summary

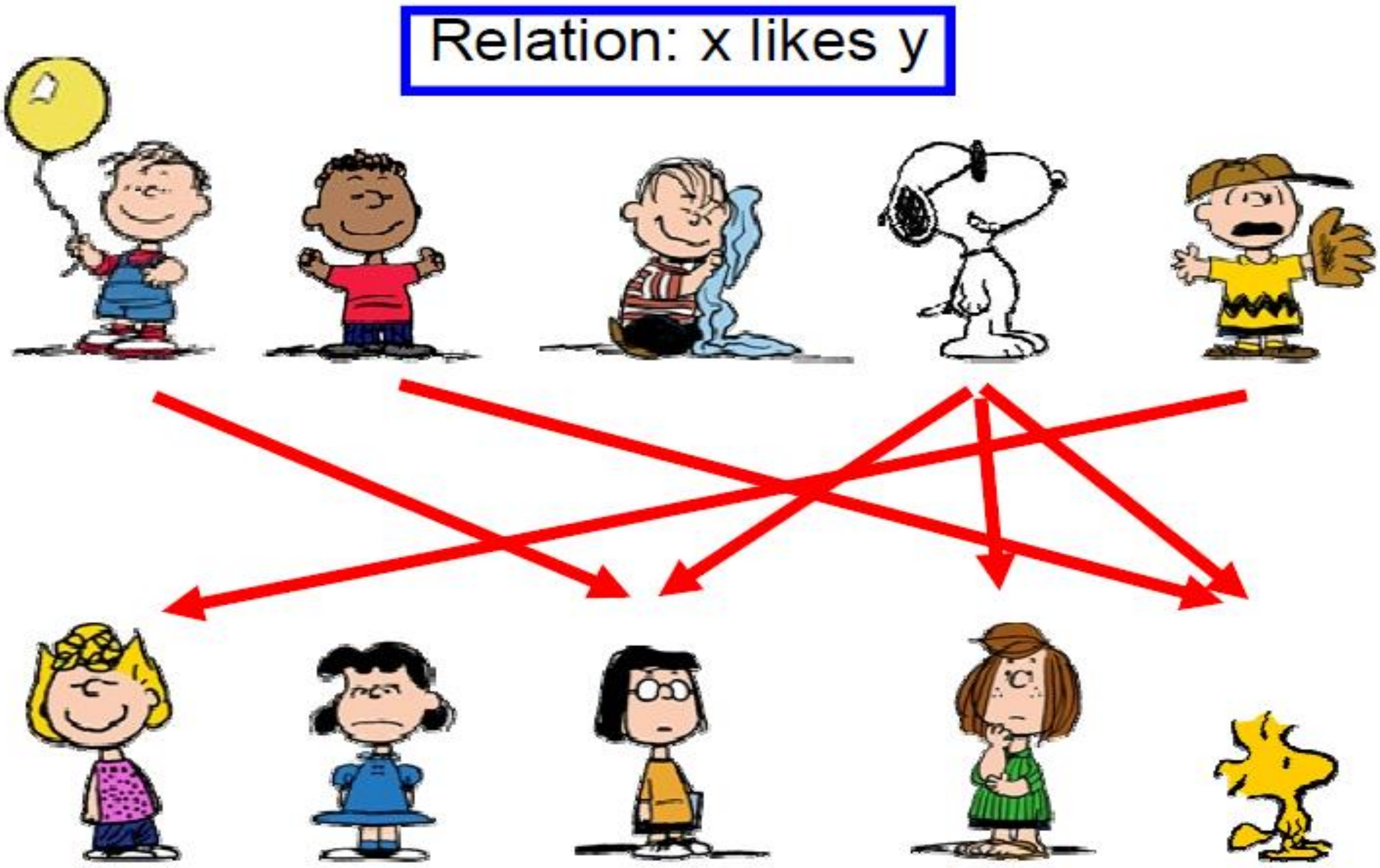
- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric Relations
  - Antisymmetric Relations
  - Transitive Relations
  - Irreflexive Relations
  - Asymmetric Relations
- Combining Relations

Recall, Function is…

- Let **A** and **B** be nonempty sets Function ***f*** from **A** to **B** is an assignment of exactly one element of **B** to each element of **A.**
- By **defining** using a **relation**, a **function** from A to B contains **unique** ordered pair (a, b) for **every** element a ∈ A.



What is Relation?



Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

- Recall, for example:

  A = {a1, a2} and B = {b1, b2, b3}

  A x B = { (a1, b1), (a1, b2), (a1, b3),

  (a2, b1), (a2, b2), (a2, b3)}

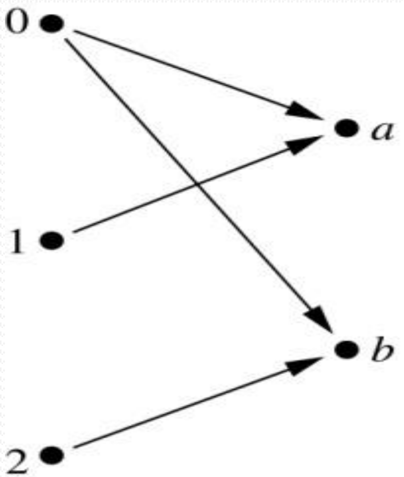
- Ordered pairs, which
  - First element comes from A
  - Second element comes from B
  - **aRb**: (a, b) ∈ R
  - **a$\not R$b** : (a, b) ∉ R

Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Binary Relations

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Binary Relations

EXAMPLE:

- Let A = {eggs, milk, corn} and B = {cows, goats, hens}

Define a relation R from A to B by (a, b) ∈R *iff* a is produced by b.

- Then R = {(eggs, hens), (milk, cows), (milk, goats)}
- Thus, with respect to this relation eggs R hens , milk R cows, etc.

Binary Relations

EXAMPLE #1:

- S = {Peter, Paul, Mary}
- C = {C++, DisMath}

- Given
  - Peter takes C++ — Peter R C++ Peter $\cancel{R}$ DisMath
  - Paul takes DisMath — Paul $\cancel{R}$ C++ Paul R DisMath
  - Mary takes none of them — Mary $\cancel{R}$ C++ Mary $\cancel{R}$ DisMath

- R = {(Peter, C++), (Paul, DisMath)}

Domain and Range of a Relation

DOMAIN OF A RELATION:

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted by Dom(R).

Symbolically, $\text{Dom}(R) = \{a \in A \mid (a, b) \in R\}$

RANGE OF A RELATION:

The range of a relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically, $\text{Ran}(R) = \{b \in B \mid (a, b) \in R\}$

Domain and Range of a Relation

EXERCISE:

Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

Define a binary relation R from A to B as follows:

$R = \{(a, b) \in A \times B \mid a < b\}$ Then

- a. Find the ordered pairs in R.
- b. Find the Domain and Range of R.
- c. Is 1R3, 2R2?

SOLUTION:

Given $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

- a. $R = \{(a, b) \in A \times B \mid a < b\}$

  $R = \{(1,2), (1,3), (2,3)\}$

Domain and Range of a Relation

Given $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

- b. Find the Domain and Range of R.

Solution:

$Dom(R) = \{1,2\}$ and $Ran(R) = \{2, 3\}$

- c. Is $1R3$, $2R2$?

Solution:

c. Since $(1, 3) \in R$ so $1R3$.

Since $(2, 2) \in R$ so $2R2$.

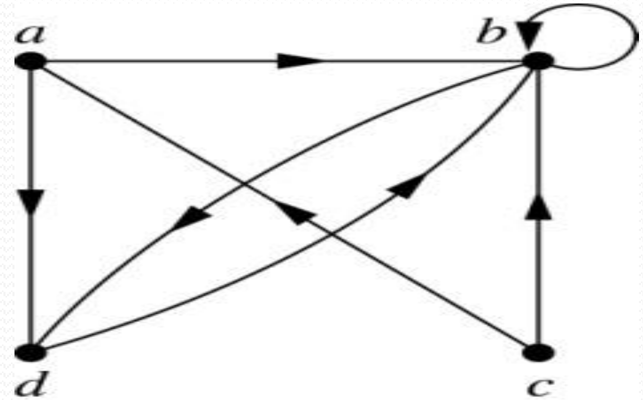
Representing Relations

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form (a,a) is called a *loop*.

Example: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

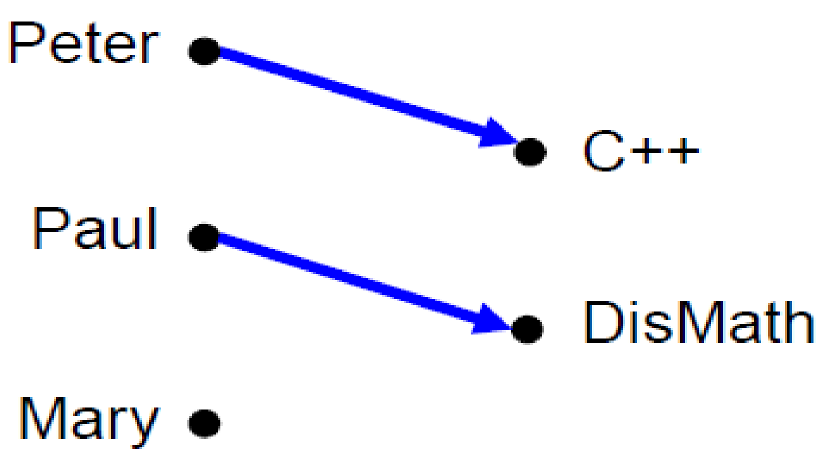
Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Binary Relations

EXAMPLE #1: (cont.)

- Peter R C++, Peter $\not R$ DisMath
  Paul $\not R$ C++, Paul R DisMath
  Mary $\not R$ C++, Mary $\not R$ DisMath



Directed Graph

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Matrix

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b),\ (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are

  $\{(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3),$ and $(4, 4)\}$.

REMARK:

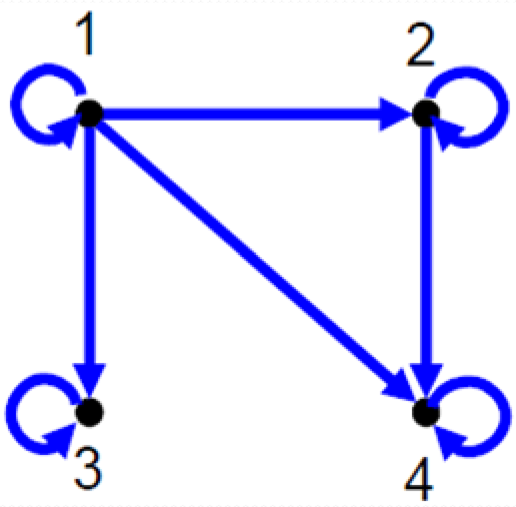
For any set A

1. A × A is known as the universal relation.
2. ∅ is known as the empty relation.

Binary Relation on a Set

- Let A be the set {1, 2, 3, 4}, which ordered pairs are in the relation R = {(a, b) | a divides b}?

- **R** = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}



$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Relations and Their Properties

Binary Relation on a Set (*cont.*)

Question: How many different relations are there on a set A with n elements?

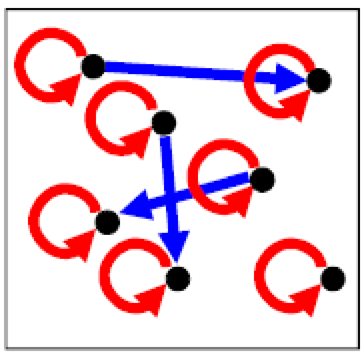
Solution:

- Suppose A has n elements
- Recall, a relation on a set A is a subset of A x A.
- A x A has n^2elements.
- If a set has m element, its has 2^m subsets.
- Therefore, the answer is 2^{n^2}.

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall a[a \in U \longrightarrow (a,a) \in R]$$



Reflexive

$\forall a\ ((a, a) \in R)$

Every node has a self-loop

$$\begin{bmatrix} 1 & & ? & \\ & 1 & & \\ ? & & 1 & \\ & & & 1 \end{bmatrix}$$

Reflexive

$\forall a\ ((a, a) \in R)$

All 1's on diagonal

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Reflexive Relations

EXAMPLE: Let $A = \{1, 2, 3, 4\}$ and determine whether relations R_1, R_2, R_3, and R_4 are Reflexive?

$$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$$
$$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$$
$$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$$
$$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$$

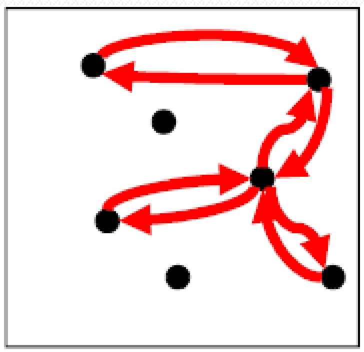
Solution:

- R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.
- R_2 is not reflexive, because $(4, 4) \notin R_2$.
- R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.
- R_4 is not reflexive, because $(1, 1) \notin R_4$, $(3, 3) \notin R_4$.

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

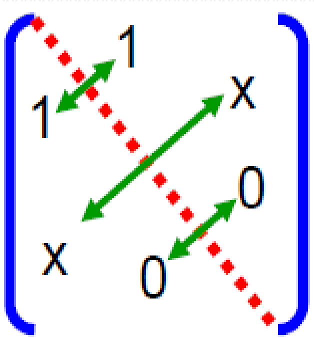
$$\forall a \forall b \, [(a,b) \in R \longrightarrow (b,a) \in R]$$



Symmetric

$\forall a \, \forall b \, (((a, b) \in R) \rightarrow ((b, a) \in R))$

Every link is bidirectional



Symmetric

$\forall a \, \forall b \, (((a, b) \in R) \rightarrow ((b, a) \in R))$

All identical across diagonal

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R. More precisely, M is a symmetric matrix
i.e. M = M^t

Symmetric Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R_1, R_2, R_3, and R_4 are Symmetric?

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

$R_3 = \{(2, 2), (2, 3), (3, 4)\}$

$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

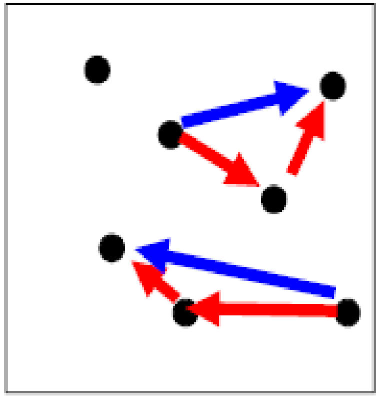
Solution:

- R_1 is Symmetric, since (a, b) and (b, a) $\in R_1$ for all (a, b) $\in$A.
- R_2 is also symmetric. We say it is vacuously true.
- R_3 is not symmetric, because $(2,3) \in R_3$ but $(3,2) \notin R_3$.
- R_4 is not symmetric because $(4,3) \in R_4$ but $(3,4) \notin R_4$.

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall a \forall b \forall c[(a,b) \in R \land (b,c) \in \mathrm{R} \longrightarrow (a,c) \in R]$$



Transitive

$\forall a \forall b \forall c\ (\ ((a,b) \in R \land (b,c) \in R) \rightarrow ((a,c) \in R))$

Every two adjacent forms a triangle
(Not easy to observe in Graph)

Transitive

$\forall a \forall b \forall c\ (\ ((a,b) \in R \land (b,c) \in R) \rightarrow ((a,c) \in R))$

Not easy to observe in Matrix

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

Transitive Relations

EXAMPLE: Let $A = \{1, 2, 3, 4\}$ and determine whether relations R1, R2 and R3 are Transitive?

$$R1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$

$$R3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$$

Solution:

- R1 is transitive because (1, 1), (1, 2) are in R, then to be transitive relation(1,2) must be there and it belongs to R.
- R2 is not transitive since $(1,2)$ and $(2,3) \in R2$ but $(1,3) \notin R2$.
- R3 is transitive.(check by definition)

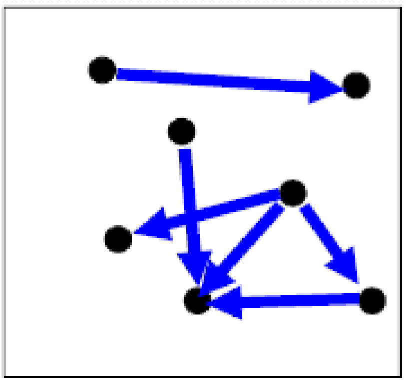
Irreflexive Relations

Definition: R is irreflexive iff for all a∈A,(a,a) ∉R. That is, R is irreflexive if no element in A is related to itself by R.

Written symbolically, *R* is irreflexive if and only if

$$\forall a\,[(a \in A) \longrightarrow (a\,,\,a) \notin R]$$

R is not irreflexive iff there is an element a∈A such that (a,a) ∈R.



Irreflexive

$\forall a\ (\ (a \in A) \rightarrow ((a,\ a) \notin R)\ )$

No node links to itself

$$\begin{bmatrix} 0 & & ? & \\ & 0 & & \\ ? & & 0 & \\ & & & 0 \end{bmatrix}$$

Irreflexive

$\forall a\ (\ (a \in A) \rightarrow ((a,\ a) \notin R)\ )$

All 0's on diagonal

Irreflexive Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R_1, R_2 and R_3 are Irreflexive?

$$R_1 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$$

$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$

$$R_3 = \{(1,2), (2,3), (3,3), (3,4)\}$$

Solution:

- R_1 is irreflexive since no element of A is related to itself in R_1. i.e. $(1,1) \notin R_1$, $(2,2) \notin R_1$, $(3,3) \notin R_1$, $(4,4) \notin R_1$.
- R_2 is not irreflexive, since all elements of A are related to themselves in R_2.
- R_3 is not irreflexive since $(3,3) \in R_3$. Note that R_3 is not reflexive.

Antisymmetric Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2, R3, and R4 are Antisymmetric?

R1 = {(1,1),(2,2),(3,3)}

R2 = {(1,2),(2,2), (2,3), (3,4), (4,1)}

R3={(1,3),(2,2), (2,4), (3,1), (4,2)}

R4={(1,3),(2,4), (3,1), (4,3)}

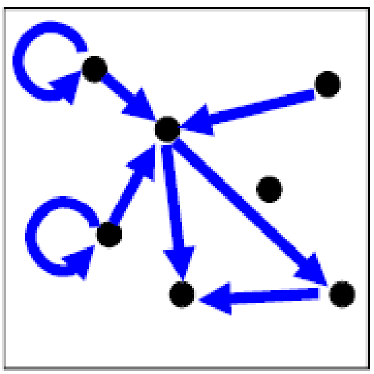
Solution:

- R1 is anti-symmetric and symmetric.
- R2 is anti-symmetric but not symmetric because $(1,2) \in R2$ but $(2,1) \notin R2$.
- R3 is not anti-symmetric since $(1,3)$ & $(3,1) \in R3$ but $1 \neq 3$. Note that R3 is symmetric.
- R4 is neither anti-symmetric because $(1,3)$ & $(3,1) \in R4$ but $1 \neq 3$ nor symmetric because $(2,4) \in R4$ but $(4,2) \notin R4$.

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if $\forall a \forall b\,[(a,b) \in R \wedge (b,a) \in R \longrightarrow a = b]$

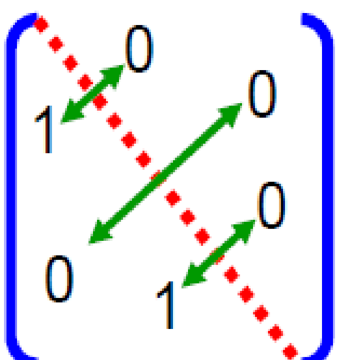
Note: (a,a) may be an element in R.



Antisymmetric

$\forall a\, \forall b\, (\, ((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)\,)$

No link is bidirectional



Antisymmetric

$\forall a\, \forall b\, (\, ((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)\,)$

All 1's are across from 0's

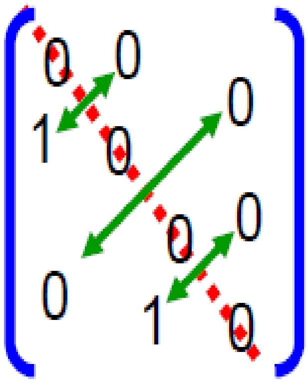
Let R be an anti-symmetric relation on a set A = {a1, a2, …, an}. Then if (ai, aj) ∈R for i ≠ j then (ai, aj) ∉R. Thus in the matrix representation of R there is a 1 in the ith row and jth column iff the jth row and ith column contains 0 vice versa.

Asymmetric Relations

Definition: R is Asymmetric iff for all $(a,b) \in R$ *than* $(b,a) \notin R$.
Written symbolically, R is Asymmetric if and only if

$$\forall a \forall b[((a,b) \in R)) \longrightarrow ((b\,,\,a) \notin R)]$$

Note:(a,a) cannot be an element in R.

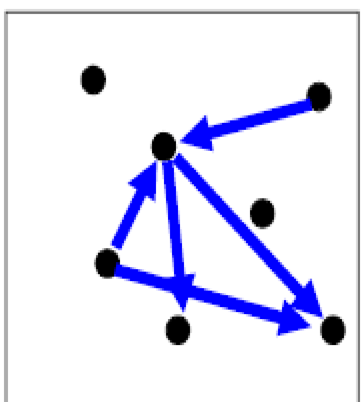


Asymmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \notin R)\ )$

All 1's are across from 0's (Antisymmetric)
All 0's on diagonal (Irreflexive)

Asymmetry = Antisymmetry + Irreflexivity



Asymmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \notin R)\ )$

No link is bidirectional (Antisymmetric)
No node links to itself (Irreflexive)

Asymmetric Relations

- **EXAMPLE**: Let A = {1, 2, 3, 4} and determine whether relations R1, R2 and R3 are Asymmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

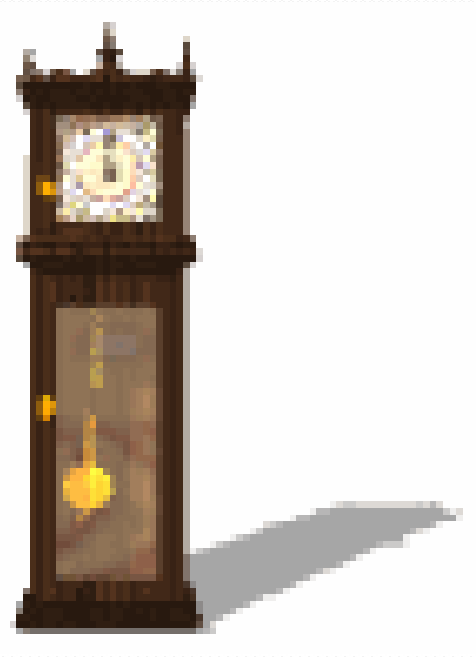
$R_2 = \{(1,2),(2,3),(3,4)\}$

$R_3 = \{(2,3),(3,3),(3,4)\}$

Solution:

- R1 is not Asymmetric since R1 is neither Antisymmetric nor Irreflexive.
- R2 is Asymmetric since R2 is both Antisymmetric and Irreflexive.
- R3 is not Asymmetric since it is Antisymmetric but not irreflexive.

Activity Time



Consider the following relations on {1, 2, 3, 4}:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
$R2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
$R6 = \{(3, 4)\}$.

Determine which of these relation are Reflexive, Symmetric, Transitive, Antisymmetric, Irreflexive and Asymmetric.

Combining Relations

As R is a subsets of A x B, the set operations can be applied

- Union (U)
- Intersection (∩)
- Difference (-)
- Symmetric Complement (⊕)

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ and $R_1 \oplus R_2$.

Combining Relations

Given, $A = \{1,2,3\}$, $B = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$,

$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

- $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 - R_2 = \{(2,2),(3,3)\}$
- $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$
- $R_1 \oplus R_2 = \{(1,2),(1,3),(1,4),(2,2),(3,3)\}$

Composition of Relations

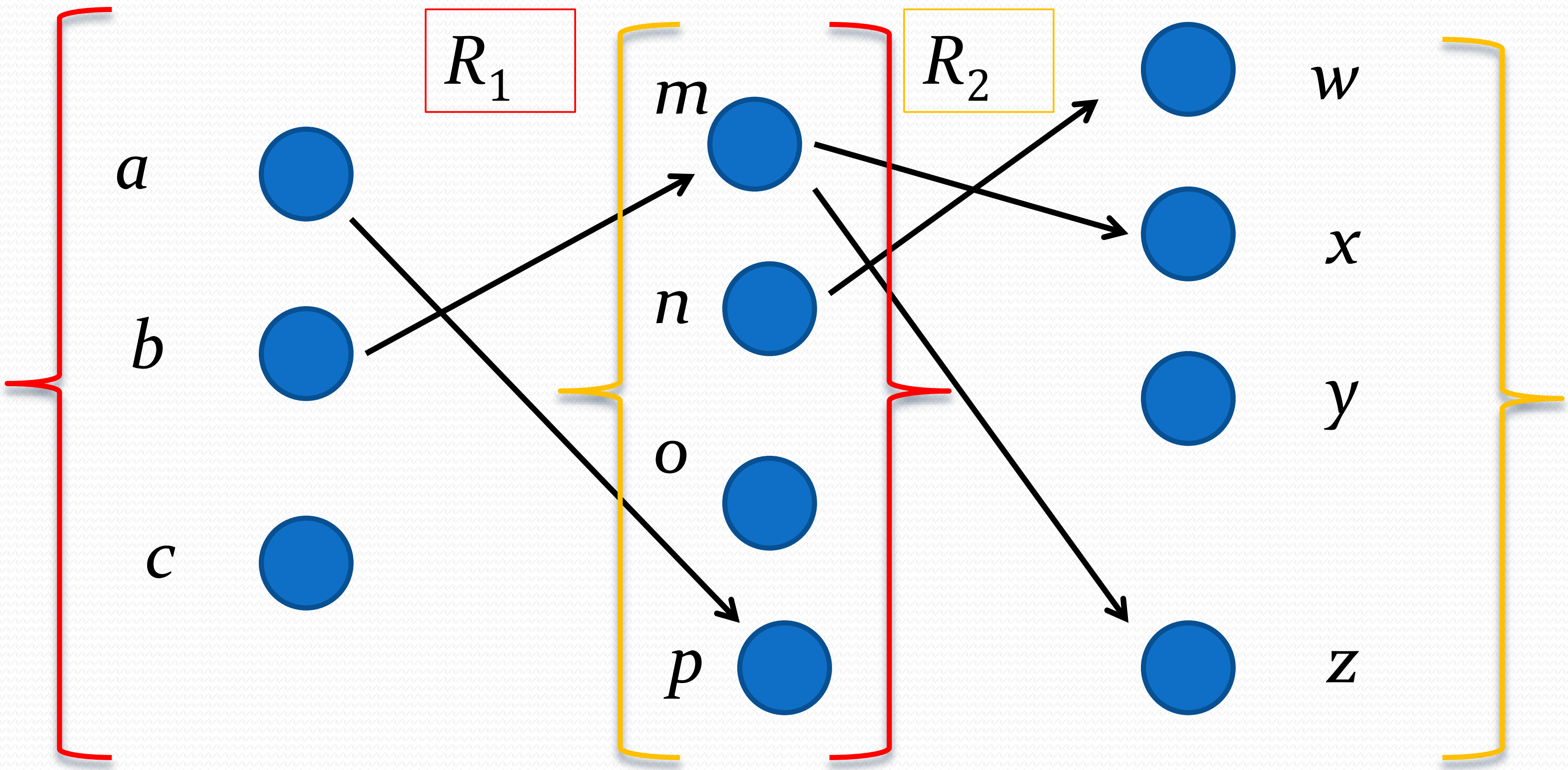
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_1 \circ R_2 = \{(b,D),(b,B)\}$$

Composition of Relations

What is the composite of the relations R and S, where

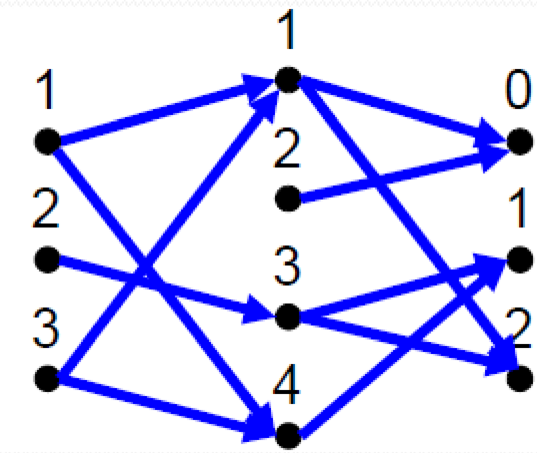
- R is the relation from **{1,2,3}** to **{1,2,3,4}** with

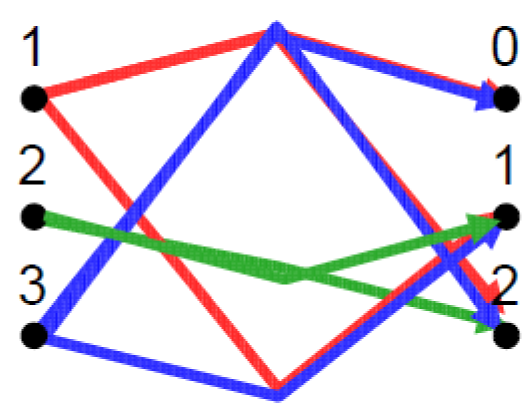
  R = {(1,1),(1,4),(2,3),(3,1),(3,4)}

- S is the relation from **{1,2,3,4}** to **{0,1,2}** with

  S = {(1,0),(1,2),(2,0),(3,1),(3,2),(4,1)}?

- S o R = {(1,0),(1,2),(1,1), (2,2),((2,1), 3,0),(3,2),(3,1)}





INVERSE OF A RELATION

Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as:

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

More simply, the inverse relation R^{-1} of R is obtained by interchanging the elements of all the ordered pairs in R.

- **Example**

X = {a, b, c} and Y={1, 2}

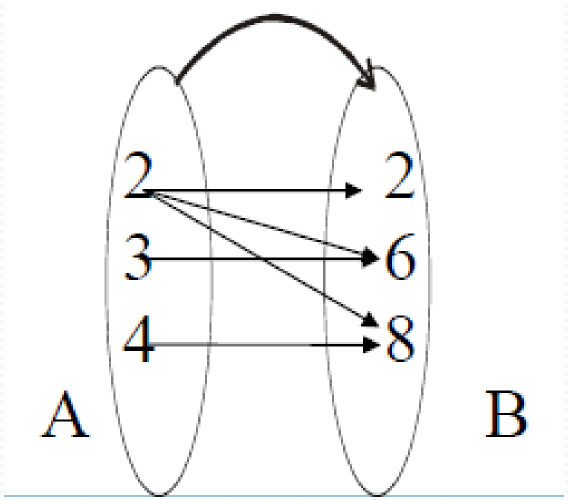
$$R = \{(a, 1), (b, 2), (c, 1)\}$$

- $R^{-1} = \{(1, a), (2, b), (1, c)\}$

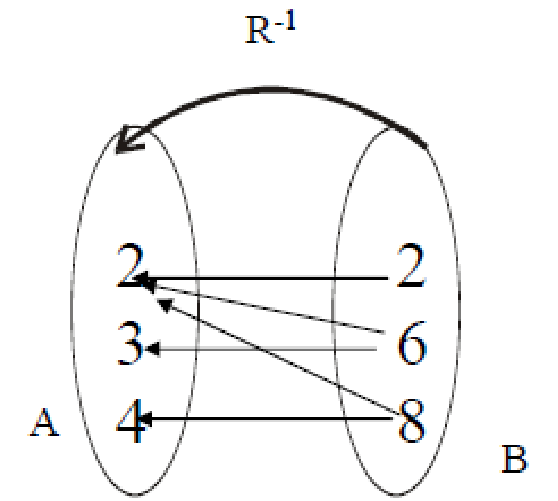
INVERSE OF A RELATION

The relation

R = {(2,2), (2,6), (2,8), (3,6), (4,8)} is represented by the arrow diagram.



Then inverse of the above relation can be obtained simply changing the directions of the arrows and hence the diagram is



Equivalence Relations

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example:

Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$, and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

  Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Partial Orderings

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive.

A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation ($|$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\mathbf{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.