MT1004 – LINEAR ALGEBRA

Course Instructor:

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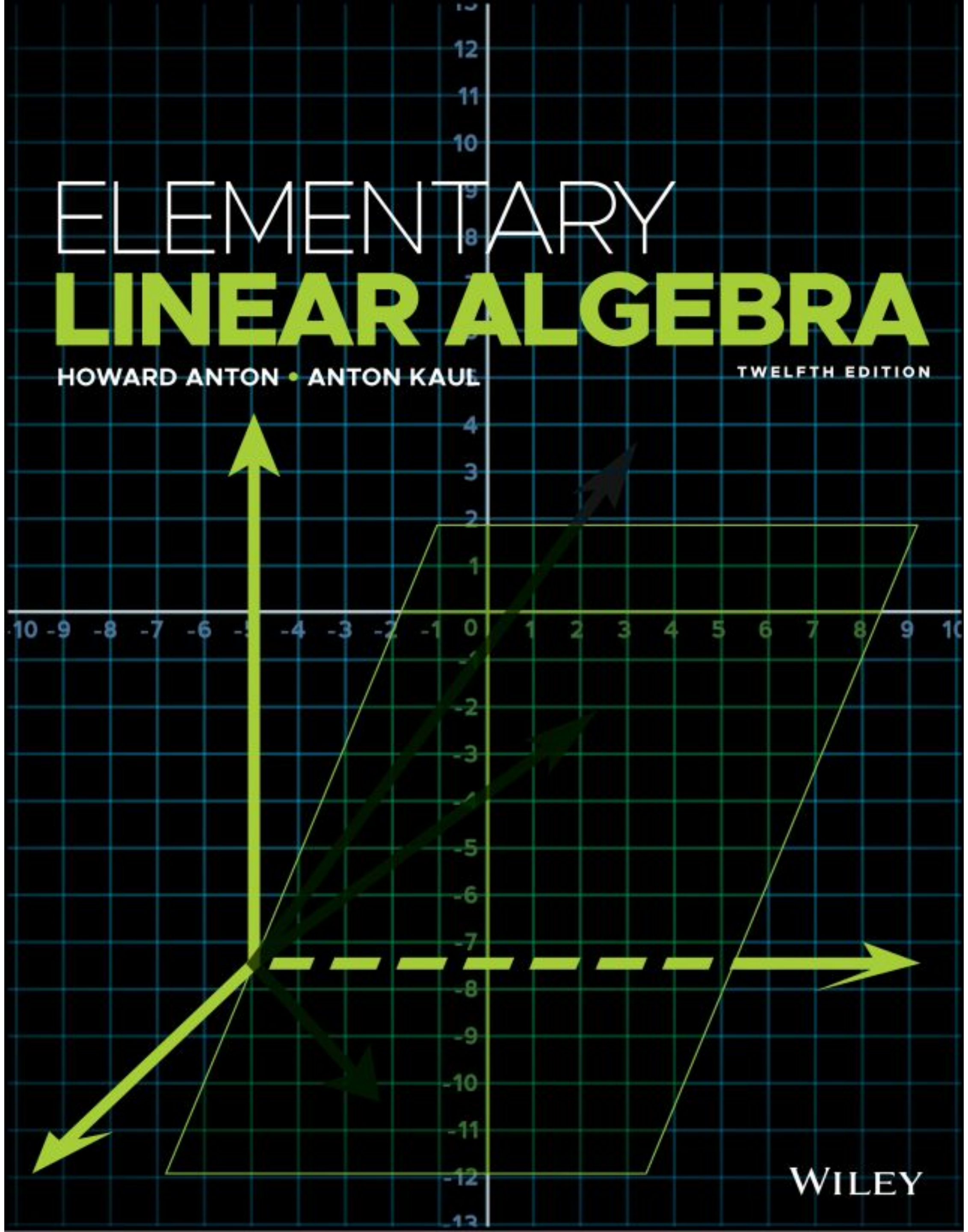
ELEMENTARY
LINEAR ALGEBRA
HOWARD ANTON • ANTON KAUL
TWELFTH EDITION
WILEY

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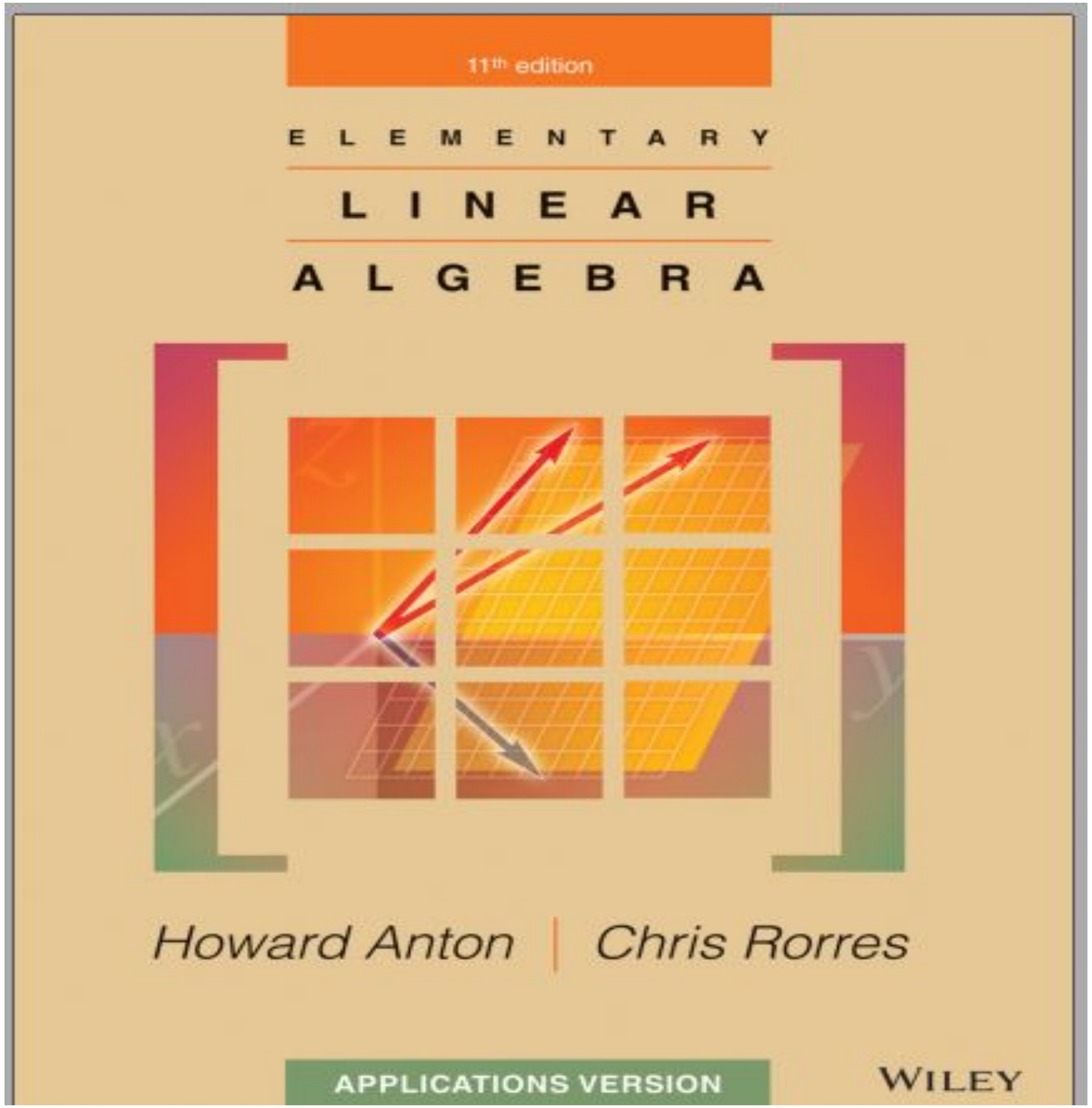
11th edition
ELEMENTARY
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ALGEBRA
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APPLICATIONS VERSION
WILEY

Tentative Course outline

Weeks	Contents/Topics	Remarks	Exercises
Week 1	Introduction, System of Linear equations, Elementary row operation		**1.1** (1-20)
Week 2	**Solving system of Linear equations**: Gaussian Elimination and Gauss Jordan methods **Matrix Operations** Elementary Matrices, Methods for finding Inverse, Invertible Matrices,	**Assignment 1**	**1.2** (1-26) **1.5** (1-6, 11-18) **1.6** (1-20)
Week 3	Diagonal, triangular, and symmetric matrices, Matrix Transformations		**1.7** (1-10, 19-28) **1.8** (1-24, 27-41) (CLO 2)
Week 4	Application no 1: Network Analysis Determinants and their properties, Minors, Cofactors, Inverse using cofactors, Cramer's Rule	**Quiz 1**	**1.10** (1-4) (CLO 3) **2.1** (1-32) **2.2** (1-23) **2.3**(1-29,31,32)
Week 5	General Vector Space Subspaces		**4.1** (1,2,9,11, 12) Example: 1-5,7 **4.2** (1-5, 19) Example: 1-6,13
Week 6	**1st Mid Term Exam**		

Week 7	Spanning Sets Linear Independence		**4.3** (1-20) **4.4** (1-15)
Week 8	Coordinates and Bases Dimensions Change of basis	**Quiz 2**	**4.5** (1-22) **4.6** (1-8,10,12-13,15-20) **4.7** (1-19)
Week 9	Bases for row, column, and null spaces, Rank and Nullity	**Assignment 2**	**4.8** (1-19,21-30) **4.9** (1-14,19-36)
Week 10	Eigenvalues and Eigenvectors Diagonalization		**5.1** (1-16) **5.2** (1-20)
Week 11	**2nd Mid Term Exam**		

Week 12	Inner product spaces, Orthogonal and orthonormal bases, Gram-Schmidt Process;	**Assignment 3**	**6.1** (1-26) **6.2** (1-12, 17-19)
Week 13	QR-Decomposition. Orthogonal Matrices		**6.3** (1-14, 27-31, 44-49) **7.1** (1-6) (CLO 1)
Week 14	Orthogonal Diagonalization, Quadratic Forms	**Quiz 3**	**7.2** (1-18) (CLO 1) **7.3** (1-8(CLO 1)
Week 15	Application no 2: Single Value Decomposition Markov Chains	**Presentation**	**5.5** **10.19**
Week 16	Revision		

Marks Distribution

Particulars	% Marks
1. Quizzes and Assignments / Presentations	20
2. First Mid Exam	15
3. Second Mid Exam	15
4. Final Exam	50
Total:	**100**

Classroom & Course Protocols

- Be in Classroom on time.
- Student who arrive more than **10 minutes** late will be marked **LATE** & after **20 minutes** as **ABSENT**.
- Keep remember to t**urn off your Cell phone before entering the class.**
- **Avoid conversation** during lecture.
- Submit your Assignment on time.
- Try to code your mathematical learnings into any suitable programming language and analyze the output by changing the input parameters.

Introduction to Systems of Linear Equations

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called “matrices” (plural of “matrix”).

Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well.

Example: Sales data

The accompanying table shows a record of May and June unit sales for a clothing store. Let M denote the 4 × 3 matrix of May sales and J the 4 × 3 matrix of June sales.

May Sales

	Small	Medium	Large
Shirts	45	60	75
Jeans	30	30	40
Suits	12	65	45
Raincoats	15	40	35

$$\begin{pmatrix} 45 & 60 & 75 \\ 30 & 30 & 40 \\ 12 & 65 & 45 \\ 15 & 40 & 35 \end{pmatrix}$$

Example: System of equation

- All of the information required to solve a system of equations such as:

$$5x + y = 3$$
$$2x - y = 4$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

Example: LINEAR MODELS IN ECONOMICS

- It was late summer in 1949. Harvard **Professor Wassily Leontief** was carefully feeding the last of his punched cards into the university's **Mark II computer**.

- The cards contained information about the U.S. economy and represented a summary of more than **250,000 pieces of information** produced by the U.S. Bureau of Labor Statistics after two years of intensive work.

- Leontief had divided the U.S. economy into **500 "sectors**," such as the coal industry, the automotive industry, communications, and so on.

- For **each sector**, he had written a **linear equation** that described how the sector distributed its output to the other sectors of the economy.

- Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns.
- Leontief had distilled the problem into a system of 42 equations in 42 unknowns.
- Programming the Mark II computer for Leontief's 42 equations had required several months of effort.
- The Mark II hummed and blinked for 56 hours before finally producing a solution

This is particularly important in developing **computer programs** for solving systems of equations because computers are well suited for manipulating **arrays** of numerical information.

The **study of matrices** and related topics that forms the mathematical field that we call “**linear algebra.**”

Linear Equations

- The linear equation for **n** variables is defined as:

$$\mathbf{a_1x_2 + a_2x_2 + \ \ldots + a_nx_n = b} \qquad \text{(a's are not all zero)} \qquad \mathbf{(1)}$$

Where a_1, a_2, …, a_n and b are constants

- The above equation is called ***homogeneous linear equation*** if b = 0

- For n = 2 and n = 3, we define linear equations as:

$$\mathbf{ax + by = c} \qquad \text{(a, b not both zero)} \qquad \mathbf{(2)}$$

$$\mathbf{ax + by + cz = d} \qquad \text{(a, b, c not all zero)} \qquad \mathbf{(3)}$$

Example # 01

- Following are examples of linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$
$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

- Following are examples of non-linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$
$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

- A linear equation **does not involve any products or roots of variables**. .
- **All variables occur only to the first power** and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions.

LINEAR SYSTEM

- A finite set of linear equations is called a ***system of linear equations*** or, a ***linear system***. The variables are called ***unknowns***.

- For example, system (5) that follows has two unknowns x & y and system (6) has three unknowns x_1, x_2, and x_3.

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \textbf{(5)}$$

$$\begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \qquad \textbf{(6)}$$

General Linear System Of m Equations In The n Unknowns

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad & \quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{7}$$

- A solution of linear system corresponding to unknown **x's** can be written as :

$(s_1, s_2, \ldots, s_n)$ → also called Ordered n - tuple

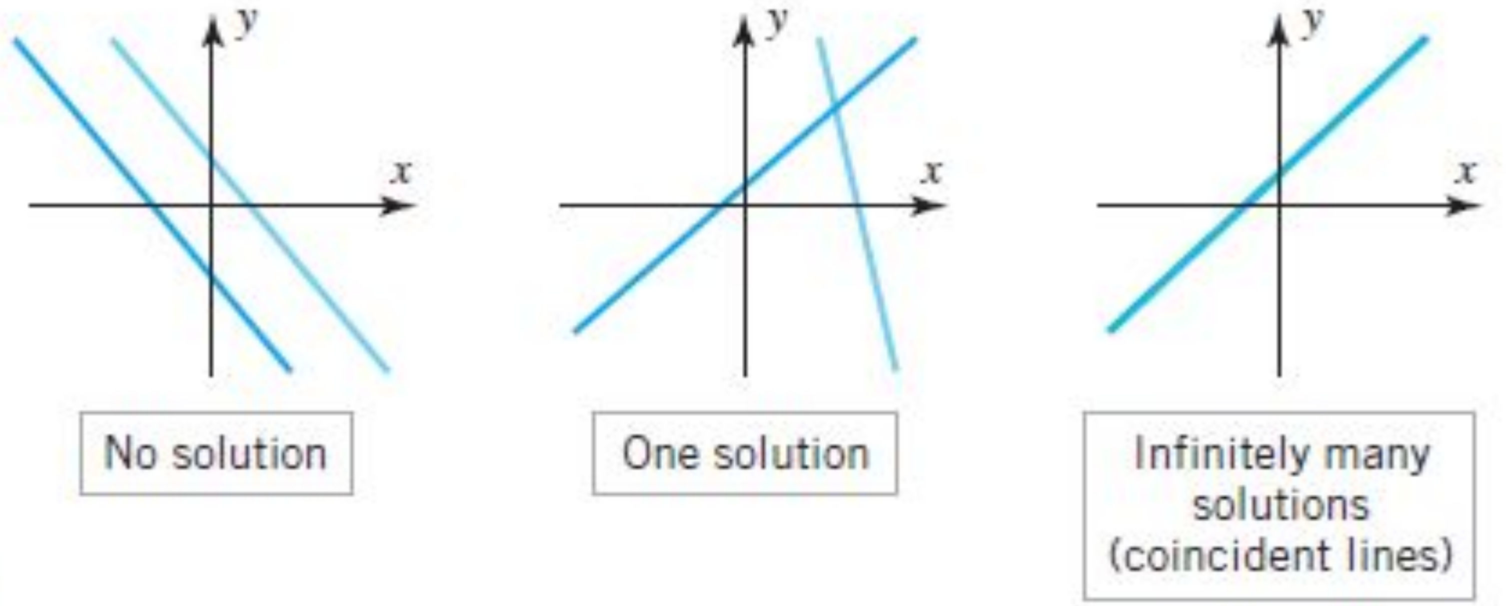
- For example system (5) has the solution $x = 1, \quad y = -2$ and the system (6) has solution $x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$

Ordered pair

Ordered triple

- These solutions can be written as: $(1, -2)$ and $(1, 2, -1)$

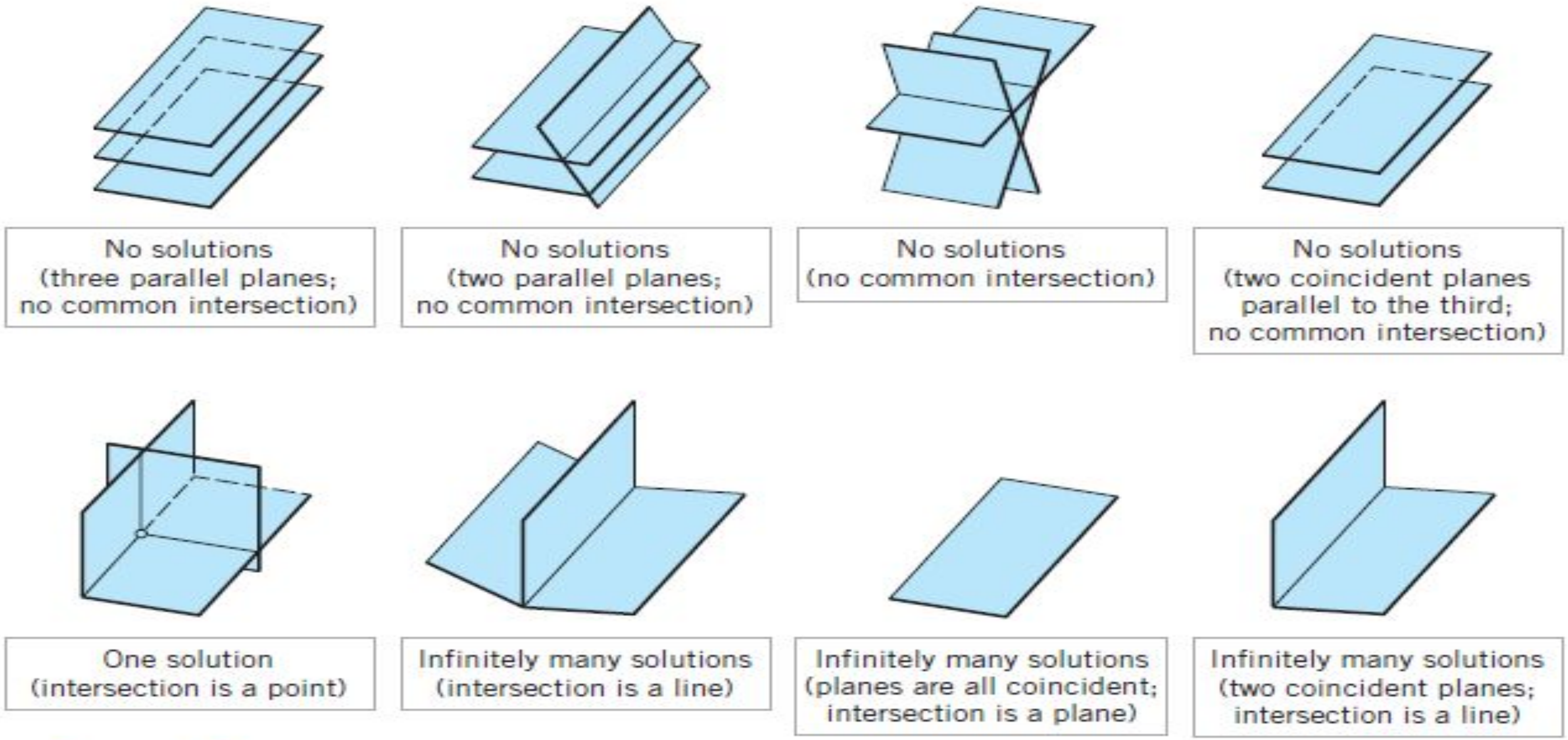
Linear Systems in Two Unknowns



► Figure 1.1.1

- *A linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions.*

Linear Systems in Three Unknowns



▲ Figure 1.1.2

Example # 02 (A Linear System with One Solution)

Solving following linear system:

$x - y = 1$ **(1)**

$2x + y = 6$ **(2)**

eq(1)*2 – eq(2)

$2x - 2y = 2$

$2x + y = 6$

$\mathbf{y} = 3/4$

$\mathbf{x} = 7/3$

Geometrically, point of intersection is (7/3 , 3/4)

Example # 03 (A linear system with no solution)

- Solve the following linear system:

$$x + y = 4 \quad \textbf{(1)}$$

$$3x + 3y = 6 \quad \textbf{(2)}$$

Example # 03 (A linear system with no solution)

- Solve the following linear system:

$$x + y = 4 \quad \textbf{(1)}$$

$$3x + 3y = 6 \quad \textbf{(2)}$$

eq(2) - **eq(1)*3**

$$3x - 3y = 6$$

$$3x + 3y = 12$$

$$0 = -6$$

The given system has no solution.

Example # 04 (A linear system with infinitely many solution)

- Solve the following linear system:

$$4x - 2y = 1 \qquad (1)$$

$$16x - 8y = 4 \qquad (2)$$

Example # 04 (A linear system with infinitely many solution)

- Solve the following linear system:

$$4x - 2y = 1 \qquad (1)$$

$$16x - 8y = 4 \qquad (2)$$

eq(2) – eq(1)*4

$$\begin{array}{r} 16x - 8y = 4 \\ \underline{16x - 8y = 4} \\ 0 = 0 \end{array}$$

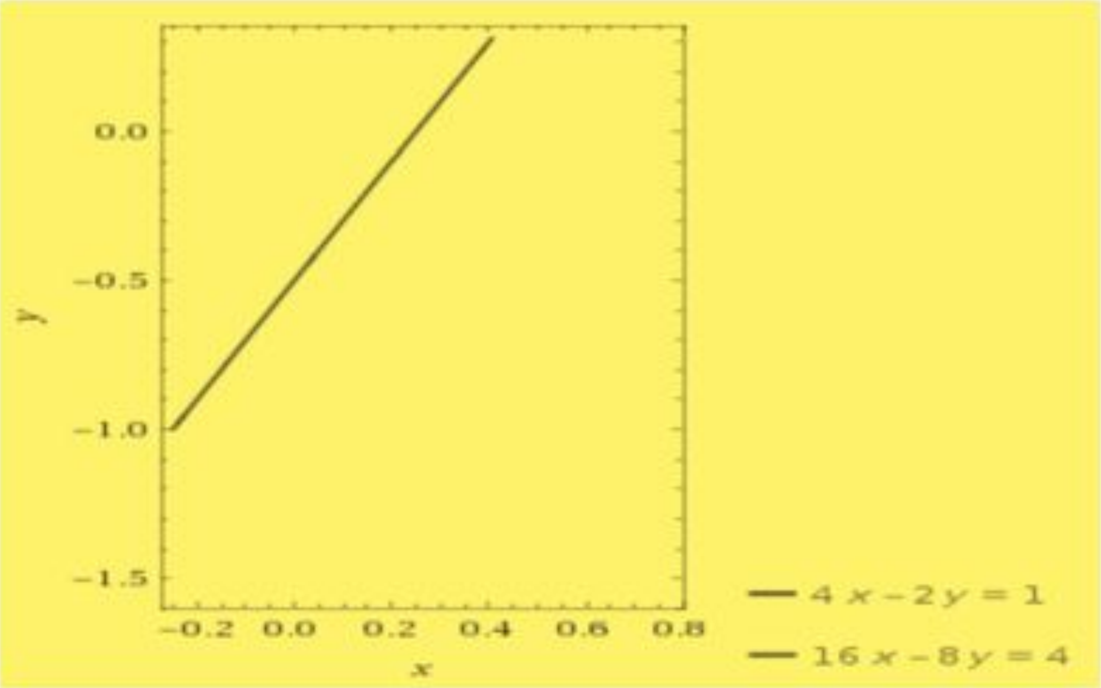
- ***The solutions of the system are those values of x and y that satisfy the single equation:*** $4x - 2y = 1$
- ***Geometrically, the two equations coincide.***

Example # 04 (A linear system with infinitely many solution)

- Solve the following linear system:

$$4x - 2y = 1 \qquad (1)$$

$$16x - 8y = 4 \qquad (2)$$



eq(2) – eq(1)*4

$$\begin{array}{r} 16x - 8y = 4 \\ \underline{16x - 8y = 4} \\ 0 = 0 \end{array}$$

- ***The solutions of the system are those values of x and y that satisfy the single equation:*** $4x - 2y = 1$

- ***Geometrically, the two equations coincide.***

Example # 04 (A linear system with infinitely many solutions)

- One way to describe the solution set is to solve eq. (1) for *x* in terms of *y* to obtain $x = \frac{1}{4} + \frac{1}{2}y$ then,
- Express the solution by the parametric equations:

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t$$

- Substituting t = 0, t = 1 and t = -1 yield the solutions:

$\left(\frac{1}{4}, 0\right)$, $\left(\frac{3}{4}, 1\right)$ and $\left(-\frac{1}{4}, -1\right)$ respectively

- You can solve for other values of t.

Example # 05 (A linear system with infinitely many solutions)

- Consider the following linear system:

$$x - y + 2z = 5 \qquad \textbf{(1)}$$

$$2x - 2y + 4z = 10 \qquad \textbf{(2)}$$

$$3x - 3y + 6z = 15 \qquad \textbf{(3)}$$

Example # 05 (A linear system with infinitely many solutions)

- Consider the following linear system:

$$x - y + 2z = 5 \qquad \textbf{(1)}$$

$$2x - 2y + 4z = 10 \qquad \textbf{(2)}$$

$$3x - 3y + 6z = 15 \qquad \textbf{(3)}$$

- Notice that, second and third equations are multiples of the first, this means that the three planes coincide. Thus, it is sufficient to find solutions of **(1).**
- Express the solution by three parametric equations:

$$x = 5 + r - 2s, \qquad y = r, \qquad z = s$$

- Now select arbitrary values of r and s to obtain specific solutions. For example taking $r = 1$ and $s = 0$ yields the solution (6, 1, 0).

Augmented Matrices

- Recall General Linear System:

$$
\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m
\end{array}
$$

Augmented Matrices

- Express the system in the rectangular array, called Augmented Matrix.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Augmented Matrices

- For example, the augmented for the system of equations is:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Elementary Row Operations

1. Multiply a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

Example # 06

Consider the following system with its augmented matrix and apply elementary row operations to obtain the solution.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Example # 06 (Contd.)

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

Add -2 times the first row to the second to obtain

Example # 06 (Contd.)

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

Add -3 times the first row to the third to obtain

Example # 06 (Contd.)

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Example # 06 (Contd.)

Multiply the second equation by $\frac{1}{2}$ to obtain

Multiply the second row by $\frac{1}{2}$ to obtain

Add -3 times the second equation to the third to obtain

Add -3 times the second row to the third to obtain

Multiply the third equation by -2 to obtain

Multiply the third row by -2 to obtain

Example # 06 (Contd.)

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the second equation to the third to obtain

Multiply the third equation by -2 to obtain

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second row to the third to obtain

Multiply the third row by -2 to obtain

Example # 06 (Contd.)

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

Multiply the third row by -2 to obtain

Example # 06 (Contd.)

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example # 06 (Contd.)

Add -1 times the second equation to the first to obtain

Add -1 times the second row to the first to obtain

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

Example # 06 (Contd.)

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \qquad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

Example # 06 (Contd.)

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \qquad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1$, $y = 2$, $z = 3$ is now evident. ◀

(The system has a unique solution)

Example # 07: Linear systems in three unknowns

In each part, suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $0x + 0y + 0z = 1$

the system is inconsistent.

(b) infinite many solutions.

- $0x + 0y + 0z = 0$
  - $x = -1 - 3z$

    $y = 2 + 4z$
- $x = -1 - 3t, \quad y = 2 + 4t, \quad z = t$

(c) infinite many solution

$x - 5y + z = 4$

$x = 4 + 5y - z$

$x = 4 + 5s - t, \quad y = s, \quad z = t$

Exercise Set 1.1 (TRUE-FALSE)

(a) A linear system whose equations are all homogeneous must be consistent. TRUE

(b) Multiplying a row of an augmented matrix through by zero is an acceptable elementary row operation. FALSE

(c) The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

TRUE

cannot have a unique solution, regardless of the value of k.

(d) A single linear equation with two or more unknowns must have infinitely many solutions. TRUE

(e) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent. FALSE

(f) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c. FALSE

(g) Elementary row operations permit one row of an augmented matrix to be subtracted from another. TRUE

(h) The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent. FALSE

Exercise Set 1.1 (Contd.)

2. In each part, determine whether the equation is linear in x and y.

(a) $2^{1/3}x + \sqrt{3}y = 1$ LINEAR

(b) $2x^{1/3} + 3\sqrt{y} = 1$ NON-LINEAR

(c) $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ LINEAR

(d) $\frac{\pi}{7}\cos x - 4y = 0$ NON-LINEAR

(e) $xy = 1$ NON-LINEAR

(f) $y + 7 = x$ LINEAR

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

(a) $(3, 1, 1)$ (b) $(3, -1, 1)$ (c) $(13, 5, 2)$

(d) $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ (e) $(17, 7, 5)$

- **a, d, and e are solution.**
- **b and c are not solutions to this system**

	X_1	X_2	X_3	b
1	1	0	-7/2	-1/2
2	0	1	-3/2	-1/2
3	0	0	0	0

ECHELON FORM

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

ROW ECHELON FORM

REDUCCED ROW ECHELON FORM

Examples of Reduced Row Echelon Form

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Examples of Reduced Row Echelon Form

REDUCED ROW ECHELON FROM

Property 1: *Leading 1*

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Examples of Reduced Row Echelon Form

REDUCED ROW ECHELON FROM

Property 1: *Leading 1*

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Property 2: *Rows containing zeros are at the bottom of the matrix*

Examples of Reduced Row Echelon Form

REDUCED ROW ECHELON FROM

Property 1: *Leading 1*

Property 3: *the leading 1 in the lower row occurs farther to the right.*

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Property 2: *Rows containing zeros are at the bottom of the matrix*

Examples of Reduced Row Echelon Form

REDUCED ROW ECHELON FROM

Property 1: *Leading 1*

Property 3: *the leading 1 in the lower row occurs farther to the right.*

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Property 2: *Rows containing zeros are at the bottom of the matrix*

Property 4: *Each column that contains a leading 1 has zeros everywhere else in that column.*

Examples of Reduced Row Echelon Form

- These are also examples of reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

*** *Substituted for any real numbers***

Examples of Row Echelon Form

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Examples of Row Echelon Form

Property 1: Leading 1

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Examples of Row Echelon Form

Property 1: Leading 1

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Property 2: *Rows containing zeros are at the bottom of the matrix; but for this case,* ***there are no rows that consists entirely of zeros.***

Examples of Row Echelon Form

Property 1: Leading 1

Property 3: *the leading 1 in the lower row occurs farther to the right.*

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Property 2: *Rows containing zeros are at the bottom of the matrix; but for this case,* ***there are no rows that consists entirely of zeros.***

Examples of Row Echelon Form

- These are also examples of row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

*** *Substituted for any real numbers***

EXAMPLE 3 Unique Solution

Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

This matrix is in reduced row echelon form and corresponds to the equations

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -1 \\ x_3 &= 0 \\ x_4 &= 5 \end{aligned}$$

Thus, the system has a unique solution, namely, $x_1 = 3$, $x_2 = -1$, $x_3 = 0$, $x_4 = 5$.

Algorithm to reduce any Matrix to Reduced Row Echelon Form

- Consider the following matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \longleftarrow$$

The first row of the preceding matrix was multiplied by $\frac{1}{2}$.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \longleftarrow$$

-2 times the first row of the preceding matrix was added to the third row.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

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$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

-5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again to Step 1.

Leftmost nonzero column in the new submatrix

The entire matrix is now in row echelon form as:

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

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$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\frac{7}{2}$ times the third row of the preceding matrix was added to the second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

The last matrix is in reduced row echelon form.

Gaussian & Gauss-Jordan Elimination

- The algorithm just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***.
- This algorithm consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and a ***backward phase*** in which zeros are introduced above the leading 1's.
- If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination*** ***(See Step 5)***.

Example 5: Gauss – Jordan Elimination

Solve by Gauss–Jordan elimination.

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 5 \\
2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

Solution The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Adding -2 times the first row to the second and fourth rows

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Multiplying the second row by -1 and then adding -5 times the new second row to the third row and -4 times the new second row to the fourth row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by 1/6 gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Solving for the leading variables, we obtain

$$x_1 = -3x_2 - 4x_4 - 2x_5 \qquad x_3 = -2x_4 \qquad x_6 = \tfrac{1}{3}$$

Finally, we express the system parametrically,

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

Example 6: Homogeneous System

- Consider the augmented matrix for the system in previous example, except for zeros in the last column.

Theorem 1.2.2:
A homogeneous linear system with more unknowns than equations has infinitely many solutions.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

Theorem 1.2.1:
*If a homogeneous linear system has **n** unknowns, and if the reduced row echelon form of its augmented matrix has **r** nonzero rows, then the system has **n - r** free variables.*

- Thus, the reduced row echelon form of this matrix will be the same as that of the augmented matrix in Example 5, except for the last column.

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 0$$

n = 6 & r = 3

free variables = (x_2, x_4, x_5)

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ignore the row of zeros

- ***Elementary row operations do not alter columns of zeros in a matrix***

Example # 08

- **Discuss the existence and uniqueness of solutions to the corresponding linear systems:**

(a) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(a) the system is inconsistent. Why?

(b) The system must have infinitely many solutions. Why?

(c) The system has unique solution. Why?

Some Facts About Echelon Forms

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss–Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.*
2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.
3. Although row echelon forms are not unique, the reduced row echelon form and all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions. Those are called the ***pivot positions*** of A. A column that contains a pivot position is called a ***pivot column*** of A.

Example # 09

Augmented Matrix

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Pivot Position:
Leading 1's are at same positions.

$$\begin{bmatrix} \underline{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \underline{1} & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

$$\begin{bmatrix} \underline{1} & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & \underline{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form

Exercise Set 1.2

(True – False Exercises)

- **In parts (a)–(i) determine whether the statement is true or false, and justify your answer**

(a) If a matrix is in reduced row echelon form, then it is also in row echelon form. TRUE

(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form. FALSE

(c) Every matrix has a unique row echelon form. FALSE

(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables. TRUE

(e) All leading 1's in a matrix in row echelon form must occur in different columns. TRUE

(f) If every column of a matrix in row echelon form has a leading 1, then all entries that are not leading 1's are zero. FALSE

(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution. TRUE

(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions. FALSE

(i) If a linear system has more unknowns than equations, then it must have infinitely many solutions. FALSE

Exercise Set 1.2

- In Exercises **13–14**, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

13.
$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

From theorem 1.2.2, this system has infinitely many solutions. Those include the trivial solution and infinitely many nontrivial solutions.

14.
$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

The system does not have nontrivial solutions. (see Back-substitution)

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

Exercise Set 1.2

In Exercises 1–2, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

1. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise Set 1.2

In each part of Exercises 23–24, the augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

23. (a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$ (b) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$

24. (a) $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$ (d) $\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 5–8, solve the linear system by Gaussian elimination.

5. $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

6. $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

7. $\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$

8. $\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$

In Exercises 9–12, solve the linear system by Gauss–Jordan elimination.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

Exercise Set 1.2

In Exercises 27–28, what condition, if any, must a, b, and c satisfy for the linear system to be consistent?

27. $$\begin{aligned} x + 3y - z &= a \\ x + y + 2z &= b \\ 2y - 3z &= c \end{aligned}$$

28. $$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

35. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

In Exercises 25–26, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

25. $$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

26. $$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.