Sequences and Series(Sums)

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations

Introduction

- Sequences are ordered lists of elements.
- EXAMPLES:
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, ...
  - 1, 2, 3, 4, 5, ...
  - 4, 8, 12, 16, 20,...
  - 2, 4, 8, 16, 32, ...
  - 1, 1/2, 1/3, 1/4, 1/5, ...
  - 1, 4, 9, 16, 25, ...
  - 1, -1, 1, -1, 1, -1, ...
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

SEQUENCES IN COMPUTER PROGRAMMING:

- An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.

EXAMPLE:

- The names of k students in a class may be represented by an array of k elements “name” as:

  name [0], name[1], name[2], …, name[k-1]

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, ...} or {1, 2, 3, 4, ...}) to a set S.

- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,.....} to S. We call a_n a *term* of the sequence.

OR

A sequence is just a list of elements usually written in a row.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Sequences

EXAMPLE:

Write the first four terms of the sequence defined by the formula: $b_j = 1 + 2^j$, for all integers $j \geq 0$

SOLUTION:

- $b_0 = 1 + 2^0 = 1 + 1 = 2$
- $b_1 = 1 + 2^1 = 1 + 2 = 3$
- $b_2 = 1 + 2^2 = 1 + 4 = 5$
- $b_3 = 1 + 2^3 = 1 + 8 = 9$

REMARK:

The formula $b_j = 1 + 2^j$, for all integers $j \geq 0$ defines an infinite sequence having infinite number of values.

Sequences

- **EXERCISE:**

Compute the first six terms of the sequence defined by the formula

- Cn = 1+ $(-1)^n$ for all integers n ≥ 0

SOLUTION :

- C0 = 1 + $(-1)^0$ = 1 + 1 = 2
- C1 = 1 + $(-1)^1$ = 1 + (-1) = 0
- C2 = 1 + $(-1)^2$ = 1 + 1 = 2
- C3 = 1 + $(-1)^3$ = 1 + (-1) = 0
- C4 = 1 + $(-1)^4$ = 1 + 1 = 2
- C5 = 1 + $(-1)^5$ = 1 + (-1) = 0

REMARK:

1) If n is even, then Cn = 2 and if n is odd, then Cn = 0. Hence, the sequence oscillates endlessly between 2 and 0.

2) An infinite sequence may have only a finite number of

Sequences

EXAMPLE:

Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \quad \text{for all integers } n \geq 1$$

SOLUTION:

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$

$$\textit{And fourth term is} C_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

REMARK: A sequence whose terms alternate in sign is called an alternating sequence.

Sequences

Find explicit formulas for sequences with the initial terms given:

1) 0, 1, -2, 3, -4, 5, …

SOLUTION:

$a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

2) $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \cdots$

SOLUTION:

$$b_k = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all integers } n \geq 1$$

Sequences

3) 2, 6, 12, 20, 30, 42, 56, …

SOLUTION:

$C_n = n\,(n + 1)$ for all integers $n \geq 1$

4) 1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …

SOLUTION:

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers} \quad i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers} \quad j \geq 0$$

Arithmetic Progression OR Sequences

- A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)
- The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by “d”.

EXAMPLES:

1. 5, 9, 13, 17, … (common difference = 4)
2. 0, -5, -10, -15, … (common difference = -5)
3. x + a, x + 3a, x + 5a, … (common difference = 2a)

Arithmetic Sequences

GENERAL TERM OF AN ARITHMETIC SEQUENCE:

Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is:

$$a, a+d, a+2d, a+3d, \ldots$$

If a_i, for $i \geq 1$, represents the terms of the sequence then

a_1 = first term = a = a + (1-1) d

a_2 = second term = a + d = a + (2-1) d

a_3 = third term = a + 2d = a + (3 -1) d

By symmetry

a_n = nth term = a + (n - 1)d for all integers n ≥1.

Arithmetic Sequences

Examples:

1. Let a = −1 and d = 4:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let a = 7 and d = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let a = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Arithmetic Sequences

EXAMPLE:

Find the 20th term of the arithmetic sequence

3, 9, 15, 21, …

SOLUTION:

- Here a = first term = 3
- d = common difference = 9 - 3 = 6
- n = term number = 20
- a_{20} = value of 20th term = ?
- Since $a_n = a + (n - 1) d$; $n \geq 1$

$\therefore a_{20} = 3 + (20 - 1)\, 6$

$= 3 + 114$

$= 117$

Arithmetic Sequences

EXERCISE:

Find the 36th term of the arithmetic sequence whose 3rd term is 7 and 8th term is 17.

SOLUTION:

Let **a** be the first term and **d** be the common difference of the arithmetic sequence.

Then $a_n = a + (n - 1)d \quad n \geq 1$

$\Rightarrow a_3 = a + (3 - 1)\, d$ and $a_8 = a + (8 - 1)\, d$

Given that $a_3 = 7$ and $a_8 = 17$. Therefore

$7 = a + 2d$……………………(1) and $17 = a + 7d$……………………..(2)

Subtracting (1) from (2), we get,

$10 = 5d \qquad \Rightarrow d = 2$

Substituting $d = 2$ in (1) we have

$7 = a + 2(2)$ which gives $a = 3$

Arithmetic Sequences

Thus, $a_n = a + (n - 1)\, d$

$a_n = 3 + (n - 1)\, 2$ (using values of a and d)

Hence the value of 36th term is

$$a_{36} = 3 + (36 - 1)\, 2$$

$$= 3 + 70$$

$$a_{36} = 73$$

Geometric Progression OR Sequence

- A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)
- The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by "r".
- **EXAMPLE:**

1. 1, 2, 4, 8, 16, … (common ratio = 2)
2. 3, - 3/2, 3/4, - 3/8, … (common ratio = - 1/2)
3. 0.1, 0.01, 0.001, 0.0001, … (common ratio = 0.1 = 1/10)

GENERAL TERM OF A GEOMETRIC SEQUENCE:

Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is a, ar, ar^2, ar^3, …

If a_i, for $i \geq 1$ represent the terms of the sequence, then

a_1 = first term = a = ar^{1-1}

a_2 = second term = ar = ar^{2-1}

a_3 = third term = ar^2 = ar^{3-1}

………………

………………

a_n = nth term = ar^{n-1}; for all integers $n \geq 1$

Geometric Progression

Examples:

1. Let a = 1 and r = −1. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let a = 2 and r = 5. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let a = 6 and r = 1/3. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Geometric Sequence

EXAMPLE:

Find the 8th term of the following geometric sequence

$$4, 12, 36, 108, \ldots$$

SOLUTION:

Here a = first term = 4

r = common ratio = $\dfrac{12}{4} = 3$

n = term number = 8

a_8 = value of 8th term = ?

Since $a_n = ar^{n-1}; \quad n \geq 1$

$\Rightarrow \quad a_8 = (4)(3)^{8-1}$

$= 4\,(2187)$

$= 8748$

Geometric Sequence

EXERCISE:

Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.

SOLUTION:

Let a be the first term and r be the common ratio of the geometric sequence. Then

$a_n = ar^{n-1} \qquad n \geq 1$

Now $\quad a_2 = ar^{2-1}$

$\Rightarrow \quad 9 = ar \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Also $\quad a_4 = ar^{4-1}$

$1 = ar^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Dividing (2) by (1), we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \quad \frac{1}{9} = r^2$$

$$\Rightarrow \quad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

Substituting r = 1/3 in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad a = 9 \times 3 = 27$$

Hence the geometric sequence is

27, 9, 3, 1, 1/3, 1/9, …

Useful Sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

SERIES

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATIONS

SUMMATION NOTATION:

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation.

where k varies from 1 to n represents the sum given in expanded form by

$= a_1 + a_2 + a_3 + \ldots + a_n$

More generally if m and n are integers and $m \leq n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

SUMMATIONS

COMPUTING SUMMATIONS:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.Compute each of the summations:

$(a)\quad \sum_{i=0}^{4} a_i \qquad (b)\quad \sum_{j=0}^{2} a_{2j} \qquad (c)\quad \sum_{k=1}^{1} a_k$

SOLUTION:

(a) $$\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4 = 2 + 3 + (-2) + 1 + 0 = 4$$

(b) $$\sum_{j=0}^{2} a_{2j} = a_0 + a_2 + a_4 = 2 + (-2) + 0 = 0$$

(c) $$\sum_{k=1}^{1} a_k = a_1 = 3$$

ARITHMETIC SERIES:

The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S). For example

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: $a + (a+d) + (a+2d) + \ldots$

SUM OF n TERMS OF AN ARITHMETIC SERIES:

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \quad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\,d]$$
$$= a + (a+d) + (a + 2d) + \ldots + a_n$$
$$= a + (a+d) + (a + 2d) + \ldots + (a_n - d) + a_n \;\ldots\ldots\ldots(1)$$

where $a_n = a + (n - 1)\,d$

Rewriting the terms in the series in reverse order,

$$Sn = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a \;\ldots\ldots\ldots\ldots(2)$$

Adding (1) and (2) term by term, gives

$$2\,S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n) \qquad (n \text{ terms})$$
$$2\,S_n = n\,(a + a_n)$$
$$\Rightarrow \qquad S_n = n(a + a_n)/2$$
$$S_n = n(a + l)/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $\qquad l = an = a + (n - 1)d$

Therefore

$$S_n = n/2\,[a + a + (n - 1)\,d]$$
$$S_n \;\; n/2\,[2\,a + (n - 1)\,d] \ldots\ldots\ldots(4)$$

ARITHMETIC SERIES:

EXERCISE:

Find the sum of first n natural numbers.

SOLUTION:

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right hand side forms an arithmetic series with

$a = 1, \quad d = 2 - 1 = 1 \qquad$ and $\; n = n$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2(1) + (n-1)(1)]$$

$$= \frac{n}{2}[2 + n - 1]$$

$$= \frac{n(n+1)}{2}$$

GEOMETRIC SERIES:

The sum of the terms of a geometric sequence forms a geometric series (G.S.). For example

$$1 + 2 + 4 + 8 + 16 + \ldots$$

is geometric series.

In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: $a + ar + ar^2 + ar^3 + \ldots$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_o = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_o + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Solving Recurrence Relations

- Finding a formula for the nth term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,....$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

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$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)