

Lecture # 3

6. Bi-implications:

DEFINITION 6

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition “ p if and only if q .” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
 - “if and only if” can be expressed by “iff”
 - Example:
 - Let p be the statement “You can take the flight” and let q be the statement “You buy a ticket.” Then $p \leftrightarrow q$ is the statement “You can take the flight if and only if you buy a ticket.”
- Implication:
- If you buy a ticket you can take the flight.
- If you don't buy a ticket you cannot take the flight.

Bi-implications:

If p denotes “I am at home.” and q denotes “It is raining.” then
 $p \leftrightarrow q$ denotes “I am at home if and only if it is raining.”

If both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then we say “ P if and only if Q ” (abbreviated P iff Q). Formally, we write $P \Leftrightarrow Q$. P if and only if Q is true only when P and Q have the same truth values.

For example, if we let P be “3 is odd,” Q be “4 is odd,” and R be “6 is even,” then $P \Rightarrow R$, $Q \Rightarrow P$ (vacuously), and $R \Rightarrow P$. Because $P \Rightarrow R$ and $R \Rightarrow P$, P if and only if R .

Given an implication $P \Rightarrow Q$, we can also define its

(a) **Contrapositive:** $\neg Q \Rightarrow \neg P$

(b) **Converse:** $Q \Rightarrow P$

The contrapositive of “If you got an A in this class, I gave you \$5,” is “If I did not give you \$5, you didn’t get an A in this class.” The converse is “If I gave you \$5 you must have received an A in this class.” Does the contrapositive say the same thing as the original statement? Does the converse?

The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Expressing the Biconditional

- Some alternative ways “p if and only if q” is expressed in English:
 - p is necessary and sufficient for q
 - if p then q , and conversely
 - p iff q

Without changing their meanings, convert each of the following sentences into a sentence having the form
"p iff q"

For a matrix to be invertible, it is necessary and sufficient that its determinant is not zero.

Answer: A matrix is invertible if and only if its determinant is not zero.

If $xy = 0$ then $x = 0$ or $y = 0$, and conversely.

Answer: $xy = 0$ if and only if $x = 0$ or $y = 0$

For an occurrence to become an adventure, it is necessary and sufficient for one to recount it.

Answer: An occurrence becomes an adventure if and only if one recounts it.

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T



Problem:

- How many rows are there in a truth table with n propositional variables?

Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.


Precedence of Logical Operators.	
Operator	Precedence
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$$p \wedge q \vee r = (p \wedge q) \vee r$$

$$p \vee q \wedge r = p \vee (q \wedge r)$$

Translating English Sentences

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- (1) If P , then Q .
 - (2) Q if P .
 - (3) P only if Q .
 - (4) P is sufficient for Q .
 - (5) Q is necessary for P .

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let q , r , and s represent “You can ride the roller coaster,”

“You are under 4 feet tall,” and “You are older than 16 years old.” The sentence can be translated into:

$$(r \wedge \neg s) \rightarrow \neg q.$$

You cannot ride the coaster if You are under 4 feet tall and you are not older than 16 Years old.

Translating English Sentences

- (1) If P , then Q .
- (2) Q if P .
- (3) P only if Q .
- (4) P is sufficient for Q .
- (5) Q is necessary for P .

- Steps to convert an English sentence to a statement in propositional logic
 - Identify atomic propositions and represent using propositional variables.
 - Determine appropriate logical connectives
- “If I go to Harry’s or to the country, I will not go shopping.”
 - p : I go to Harry’s
 - q : I go to the country.
 - r : I will go shopping.

If p or q then not r .

$$(p \vee q) \rightarrow \neg r$$

Translating English Sentences

- Example: How can this English sentence be translated into a logical expression?

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution: Let a , c , and f represent “You can access the Internet from campus,” “You are a computer science major,” and “You are a freshman.” The sentence can be translated into:

$$a \rightarrow (c \vee \neg f).$$

- (1) If P , then Q .
- (2) Q if P .
- (3) P only if Q .
- (4) P is sufficient for Q .
- (5) Q is necessary for P .

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

“The automated reply cannot be sent when the file system is full”

Solution: One possible solution: Let p denote “The automated reply can be sent” and q denote “The file system is full.”

$$q \rightarrow \neg p$$

1.1 Propositional Logic

Logic and Bit Operations

- Computers represent information using bits.
- A bit is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators OR, AND, and XOR.				
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

1.1 Propositional Logic

DEFINITION 7

A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

- Example: Find the bitwise OR, bitwise AND, and bitwise XOR of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110	
11 0001 1101	

11 1011 1111	bitwise OR
01 0001 0100	bitwise AND
10 1010 1011	bitwise XOR

Propositional Equivalences

DEFINITION 1

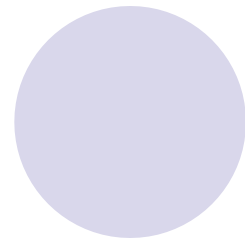
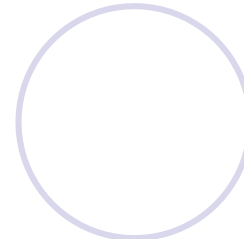
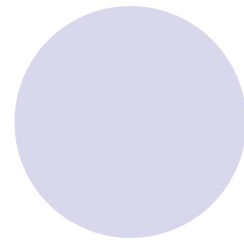
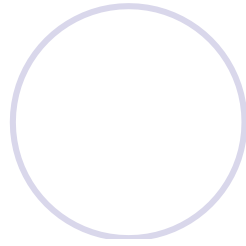
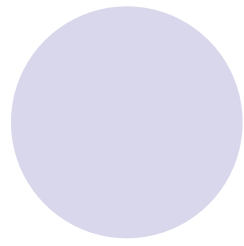
A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a **tautology**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology or a contradiction is called a **contingency**.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F



Tautologies and Contradictions

- Tautology is a statement that is always true regardless of the truth values of the individual logical variables
- Examples:
- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$



Tautologies and Contradictions

- A Contradiction is a statement that is always false regardless of the truth values of the individual logical variables

Examples

- The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

Propositional Equivalences

DEFINITION 2

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositional Equivalences

- In general, 2^n rows are required if a compound proposition involves n propositional variables in order to get the combination of all truth values.
- Prove that $\neg(\neg p) \equiv p$

Solution

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

As you can see the corresponding truth values of p and $\neg(\neg p)$ are same, hence **equivalence** is justified.



Applications: Boolean Searches

- Logical connectives are used extensively in searches of large collections of information.
 - Example: indexes of Web pages.
- AND - used to match records that contain both of two search terms.
- OR - used to match one or both of two search terms.
- NOT - used to exclude a particular search term.
- Read about: Web Page Searching

Applications: Logic Puzzles

- Puzzles (**important job interview question**) that can be solved using logical reasoning
- [Sm78] Smullyan: An island that has two kinds of inhabitants.
 - knights, who always tell the truth.
 - knaves, who always lie.
- You encounter two people A and B.
- What are the types of A and B?:
 - A says “B is a knight” and
 - B says “The two of us are opposite types.”

Example 1:

- p : A is a knight
- q : B is a knight

- $\neg p$: A is a knave
- $\neg q$: B is a knave

- Consider the possibility that A is a knight;
 - So, p is true. And he is telling truth.
 - & q is true. So, A and B are the same type.
- However, if B is a knight, then B's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

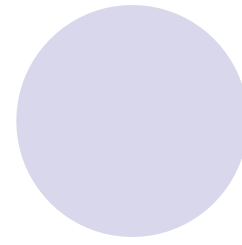
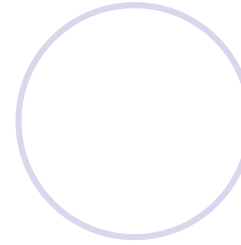
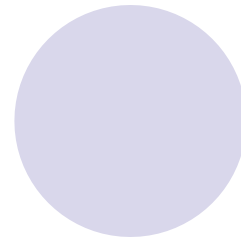
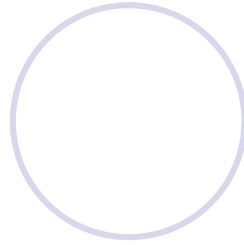
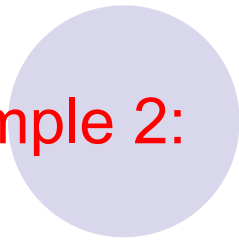
Example 1: Solution (cont..)

- Consider the possibility that A is a knave,
 - everything a knave says is false; q is true, is a lie.
 - So, q is false. B is also a knave.
 - B 's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves.
- We can conclude that both A and B are knaves.

Example 2:

- A father tells his two children, a boy and a girl, to play in their backyard without getting dirty.
- However, while playing, both children get mud on their foreheads. When the children stop playing, the father says “At least one of you has a muddy forehead,” and then asks the children to answer “Yes” or “No” to the question: “Do you know whether you have a muddy forehead?”
- The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead?
- Assume that both children are honest and that the children answer each question simultaneously.

Example 2:



Solution: S denotes "Son has a muddy forehead" and D denotes " Daughter has a muddy forehead". The father states that $S \vee D$ is True. Boy can know D is True but can't know S . Girl can know D is True but can't know S . So no for the first time. After that they can conclude that both D and S are True. Since one of them will say yes for the first time if one of D and S is not True.

Example 3:

6. Determine whether these system specifications are consistent:
- "the diagnostic message is stored in the buffer or it is retransmitted"
 - "the diagnostic message is not stored in the buffer"
 - "if the diagnostic message is stored in the buffer, then it is retransmitted"

Solution: p denotes "the diagnostic message is stored in the buffer", q denotes "the diagnostic message is retransmitted". Then the specifications can be written as $p \vee q$, $\neg p$ and $p \rightarrow q$. $\neg p$ is True, so p is False. $p \vee q$ is True, so q is True. So $p \rightarrow q$ is True. They are consistent.