

MT104 – LINEAR ALGEBRA

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MATRICES & MATRIX OPERATIONS

Matrices & Matrix Operations

- Consider the following rectangular array represents number of hours a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

- If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a “matrix”:

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Some Basic Concepts of a Matrix

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

- **Size of matrix:** (Number of Rows x Number of Columns). For e.g:

(3 x 2)

(1 x 4)

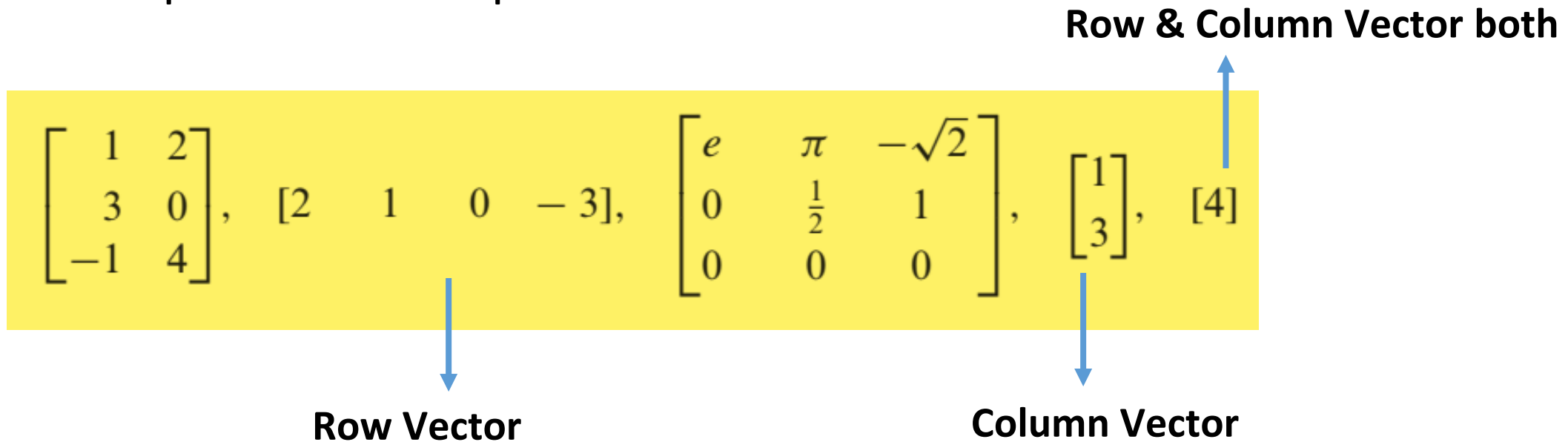
(3 x 3)

(2 x 1)

(1 x 1)

Row & Column Vector

- Consider the previous example:



How to represent Matrices?

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij} . Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$


A General (m x n) Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The **entry** in **row i** and **column j** of a matrix **A** is also commonly denoted by the symbol $(A)_{ij}$. Thus for this matrix, we have

$$(A_{ij}) = a_{ij}$$


$$[a_{ij}]_{m \times n}$$


$$[a_{ij}]$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

How to Represent Vector?

- It is common practice to denote Row or Column Vector by boldface lowercase letters rather than capital letters.
- Thus a general $\mathbf{1} \times \mathbf{n}$ row vector \mathbf{a} and a general $\mathbf{m} \times \mathbf{1}$ column vector \mathbf{b} would be written as:

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Square Matrix

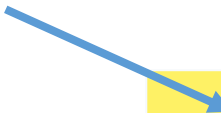
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (n \times n)$$

No. of Rows = No. of Columns

What is the order of the matrix?

Square Matrix

Main diagonal of A



$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (n \times n)$$

No. of Rows = No. of Columns

What is the order of the matrix?

Equality of a Matrix

- Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

- If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal.

Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.


Scalar Multiples

- If A is any matrix and c is any scalar, then the **product** cA is the matrix obtained by multiplying each entry of the matrix A by c . The matrix cA is said to be a **scalar multiple** of A .
- In matrix notation, if $A = [a_{ij}]$, then $(cA)_{ij} = c(A)_{ij} = ca_{ij}$
- For the matrices:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$. 

Product of Matrices

- If \mathbf{A} is an $(m \times r)$ matrix and B is an $(r \times n)$ matrix, then the **product** AB is the $(m \times n)$ matrix. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

2×3 3×4

- *The number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB . If this condition is not satisfied, the product is undefined.*
- The product \mathbf{AB} would be (2×4) matrix:

Product of Matrices (Example # 05)

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}_{3 \times 4} = \begin{bmatrix} \boxed{12} & \boxed{27} & \boxed{30} & \boxed{13} \\ \boxed{8} & \boxed{-4} & \boxed{26} & \boxed{12} \end{bmatrix}_{2 \times 4}$$

- $(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) = 12$
- $(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) = 27$
- $(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) = 30$
- $(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$
- $(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) = 8$
- $(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) = -4$
- $(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$
- $(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) = 12$

Determine Whether a Product Is Defined ?

Suppose that A , B , and C are matrices with the following sizes:

A	B	C
3×4	4×7	7×3

- **AB** is defined as: **3 x 7**
- **BC** is defined as: **4 x 3**
- **CA** is defined as: **7 x 4**
- **AC** is defined as: **undefined**
- **CD** is defined as: **undefined**
- **BA** is defined as **undefined**

Partitioned Matrices

- A matrix can be subdivided or **partitioned** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix A .

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Product of Matrices (Example # 06)

- Consider the following :

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

Second column of B
Second column of AB

First row of A
First row of AB

j th column vector of $AB = A[j$ th column vector of $B]$

- In general,

i th row vector of $AB = [i$ th row vector of $A]B$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

2×3
 3×4
 2×4

Matrix Products as Linear Combinations

DEFINITION 6 If A_1, A_2, \dots, A_r are matrices of the same size, and if c_1, c_2, \dots, c_r are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$$

is called a *linear combination* of A_1, A_2, \dots, A_r with *coefficients* c_1, c_2, \dots, c_r .

Matrix Products as Linear Combinations (Contd.)

- To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and \mathbf{x} an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then,

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

THEOREM 1.3.1 *If A is an $m \times n$ matrix, and if \mathbf{x} is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of \mathbf{x} .*

Example # 08

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Example # 09 (Columns of Product as Linear Combination)

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Recall, [formula 6](#) (pg. 31, see e.g. 6) and [Theorem 1.3.1](#) (pg. 32).

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Column-Row Expansion

Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

has size $m \times n$ so the sum itself is an $m \times n$ matrix. We leave it as an exercise for you to verify that the entry in row i and column j of the sum is given by the expression on the right side of Formula (5), from which it follows that

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r \tag{11}$$

We call (11) the *column-row expansion* of AB .

$$\textbf{Formula (5)} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

Example # 10

- Find the column-row expansion of the product: $AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$

- The column vectors of A and the row vectors of B are, respectively

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \quad \mathbf{r}_1 = [2 \quad 0 \quad 4], \quad \mathbf{r}_2 = [-3 \quad 5 \quad 1]$$

- the column-row expansion of AB is:

$$\begin{aligned} AB &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} [2 \quad 0 \quad 4] + \begin{bmatrix} 3 \\ -1 \end{bmatrix} [-3 \quad 5 \quad 1] \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

Summarizing Matrix Multiplication

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **product** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB , single out row i from the matrix A and column j from the matrix B . Multiply the corresponding entries from the row and column together, and then add the resulting products.

1. Entry by entry

2. Row-column method

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

3. Column by column

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

4. Row by row

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

5. Column-row expansion

$$AB = \mathbf{c}_1 \mathbf{r}_1 + \mathbf{c}_2 \mathbf{r}_2 + \cdots + \mathbf{c}_r \mathbf{r}_r$$

Matrix Form of a Linear System

Consider a system of m linear equations in n unknowns:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

Replace the m equations in this system by the single matrix equation:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times 1$ matrix on the left side of this eq. can be written as a product to give:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Replace the original system of m equations in n unknowns by the single matrix equation:

$$\mathbf{Ax} = \mathbf{b}$$

Transpose of a Matrix

DEFINITION 7 If A is any $m \times n$ matrix, then the *transpose of A* , denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A ; that is, the first column of A^T is the first row of A , the second column of A^T is the second row of A , and so forth.

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ 3 \ 5], \quad D = [4]$$
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4] \quad \blacktriangleleft$$

Trace of a Matrix

DEFINITION 8 If A is a square matrix, then the *trace of A* , denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$



TRUE-FALSE Exercises

TF. In parts (a)–(o) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal. **TRUE**

(b) An $m \times n$ matrix has m column vectors and n row vectors. **FALSE**

(c) If A and B are 2×2 matrices, then $AB = BA$. **FALSE**

(d) The i th row vector of a matrix product AB can be computed by multiplying A by the i th row vector of B . **FALSE**

(e) For every matrix A , it is true that $(A^T)^T = A$. **TRUE**

(f) If A and B are square matrices of the same order, then
 $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$ **FALSE**

(g) If A and B are square matrices of the same order, then
 $(AB)^T = A^T B^T$ **FALSE**

(h) For every square matrix A , it is true that $\text{tr}(A^T) = \text{tr}(A)$. **TRUE**

(i) If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$. **TRUE**

(j) If A is an $n \times n$ matrix and c is a scalar, then $\text{tr}(cA) = c \text{tr}(A)$. **TRUE**

(k) If A , B , and C are matrices of the same size such that $A - C = B - C$, then $A = B$. **TRUE**

(l) If A , B , and C are square matrices of the same order such that $AC = BC$, then $A = B$. **FALSE**

(m) If $AB + BA$ is defined, then A and B are square matrices of the same size. **TRUE**

(n) If B has a column of zeros, then so does AB if this product is defined. **TRUE**

(o) If B has a column of zeros, then so does BA if this product is defined. **FALSE**

Exercises 1.3

► In Exercises 1–2, suppose that A , B , C , D , and E are matrices with the following sizes:

$$\begin{array}{ccccc} A & B & C & D & E \\ (4 \times 5) & (4 \times 5) & (5 \times 2) & (4 \times 2) & (5 \times 4) \end{array}$$

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix. ◀

1. (a) BA (b) AB^T (c) $AC + D$
 (d) $E(AC)$ (e) $A - 3E^T$ (f) $E(5B + A)$
2. (a) CD^T (b) DC (c) $BC - 3D$
 (d) $D^T(BE)$ (e) $B^TD + ED$ (f) $BA^T + D$

1(a)= undefined

1(b)= 4 x 4

1(c)= 4 x 2

1d)= 5 x 2

1(e)= 4 x 5

1(f)= 5 x 5

► In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \quad \text{◀}$$

3. (a) $D + E$ (b) $D - E$ (c) $5A$
 (d) $-7C$ (e) $2B - C$ (f) $4E - 2D$
 (g) $-3(D + 2E)$ (h) $A - A$ (i) $\text{tr}(D)$
 (j) $\text{tr}(D - 3E)$ (k) $4 \text{tr}(7B)$ (l) $\text{tr}(A)$

3 (a) = Defined

3 (g) = Defined

3 (b) = Defined

3 (h) = Defined

3 (c) = Defined

3 (i) = Defined

3 (d) = Defined

3 (j) = Defined

3 (e) = Undefined

3 (k) = Defined

3 (f) = Defined

3 (l) = Undefined

► In each part of Exercises 13–14, express the matrix equation as a system of linear equations. ◀

$$13. (a) \begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$$

34. The accompanying table shows a record of May and June unit sales for a clothing store. Let M denote the 4×3 matrix of May sales and J the 4×3 matrix of June sales.

- What does the matrix $M + J$ represent?
- What does the matrix $M - J$ represent?
- Find a column vector \mathbf{x} for which $M\mathbf{x}$ provides a list of the number of shirts, jeans, suits, and raincoats sold in May.
- Find a row vector \mathbf{y} for which $\mathbf{y}M$ provides a list of the number of small, medium, and large items sold in May.
- Using the matrices \mathbf{x} and \mathbf{y} that you found in parts (c) and (d), what does $\mathbf{y}M\mathbf{x}$ represent?

29. A matrix B is said to be a **square root** of a matrix A if $BB = A$.

$$(a) \text{ Find two square roots of } A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

(b) How many different square roots can you find of

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}?$$

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

Table Ex-34

May Sales				June Sales			
	Small	Medium	Large		Small	Medium	Large
Shirts	45	60	75	Shirts	30	33	40
Jeans	30	30	40	Jeans	21	23	25
Suits	12	65	45	Suits	9	12	11
Raincoats	15	40	35	Raincoats	8	10	9

► In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \quad \blacktriangleleft$$

7. (a) the first row of AB (b) the third row of AB
 (c) the second column of AB (d) the first column of BA
 (e) the third row of AA (f) the third column of AA

► In Exercises 17–20, use the column-row expansion of AB to express this product as a sum of matrices. ◀

$$17. A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$18. A = \begin{bmatrix} 0 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}$$

$$19. A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$20. A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & -1 \end{bmatrix}$$

► In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \quad \blacktriangleleft$$

7. (a) the first row of AB (b) the third row of AB
(c) the second column of AB (d) the first column of BA
(e) the third row of AA (f) the third column of AA

► In Exercises 23–24, solve the matrix equation for a , b , c , and d . ◀

$$23. \begin{bmatrix} a & 3 \\ -1 & a + b \end{bmatrix} = \begin{bmatrix} 4 & d - 2c \\ d + 2c & -2 \end{bmatrix}$$

Inverses; Algebraic Properties of Matrices

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

- (a) $A + B = B + A$ [Commutative law for matrix addition]
- (b) $A + (B + C) = (A + B) + C$ [Associative law for matrix addition]
- (c) $A(BC) = (AB)C$ [Associative law for matrix multiplication]
- (d) $A(B + C) = AB + AC$ [Left distributive law]
- (e) $(B + C)A = BA + CA$ [Right distributive law]
- (f) $A(B - C) = AB - AC$
- (g) $(B - C)A = BA - CA$
- (h) $a(B + C) = aB + aC$
- (i) $a(B - C) = aB - aC$
- (j) $(a + b)C = aC + bC$
- (k) $(a - b)C = aC - bC$
- (l) $a(bC) = (ab)C$
- (m) $a(BC) = (aB)C = B(aC)$

In matrix arithmetic, the equality of AB and BA can fail for three possible reasons:

1. AB may be defined and BA may not (for example, if A is 2×3 and B is 3×4).
2. AB and BA may both be defined, but they may have different sizes (for example, if A is 2×3 and B is 3×2).
3. AB and BA may both be defined and have the same size, but the two products may be different (as illustrated in the next example).

► **EXAMPLE 2 Order Matters in Matrix Multiplication**

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$. ◀

Zero Matrix

A matrix whose entries are all zero is called a *zero matrix*. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

We will denote a zero matrix by 0 unless it is important to specify its size, in which case we will denote the $m \times n$ zero matrix by $0_{m \times n}$.

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be performed, then:

- (a) $A + 0 = 0 + A = A$
- (b) $A - 0 = A$
- (c) $A - A = A + (-A) = 0$
- (d) $0A = 0$
- (e) *If $cA = 0$, then $c = 0$ or $A = 0$.*

► **EXAMPLE 3 Failure of the Cancellation Law**

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).

► **EXAMPLE 4 A Zero Product with Nonzero Factors**

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \quad \blacktriangleleft$$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An identity matrix is denoted by the letter I . If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

if A is any $m \times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_m A = A$$

THEOREM 1.4.3 *If R is the reduced row echelon form of an $n \times n$ matrix A , then either R has a row of zeros or R is the identity matrix I_n .*

Inverse of a Matrix

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* (or *nonsingular*) and B is called an *inverse* of A . If no such matrix B can be found, then A is said to be *singular*.

Remark The relationship $AB = BA = I$ is not changed by interchanging A and B , so if A is invertible and B is an inverse of A , then it is also true that B is invertible, and A is an inverse of B . Thus, when

$$AB = BA = I$$

we say that A and B are *inverses of one another*.

► **EXAMPLE 5 An Invertible Matrix**

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

► **EXAMPLE 6 A Class of Singular Matrices**

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that $AB = BA = I$. For this purpose let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{0}$ be the column vectors of A . Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \text{ [Formula (6) of Section 1.3]}$$

The column of zeros shows that $BA \neq I$ and hence that A is singular. ◀

Properties of Inverses

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (2)$$

- The quantity **$ad - bc$** in above theorem is called the **determinant** of the 2×2 matrix A and is denoted by: $\det(A) = ad - bc$
- If A is invertible, then its inverse will be denoted by the symbol A^{-1} .
Thus,

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

► **EXAMPLE 7 Calculating the Inverse of a 2×2 Matrix**

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$(a) A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad (b) A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

► EXAMPLE 8 Solution of a Linear System by Matrix Inversion

Consider the following pair of equations:

$$\begin{aligned}u &= ax + by \\ v &= cx + dy\end{aligned}$$

Let us replace the two equations by the single matrix equation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If we assume that the 2×2 matrix is invertible (i.e., $ad - bc \neq 0$), then we can multiply through on the left by the inverse and rewrite the equation as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From theorem 1.4.5

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

from which we obtain

$$x = \frac{du - bv}{ad - bc}, \quad y = \frac{av - cu}{ad - bc} \quad \blacktriangleleft$$

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$. ◀

- This result can be extended to three or more factors.

► **EXAMPLE 9 The Inverse of a Product**

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix},$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6. ◀

Powers of a Matrix

If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA \cdots A \quad [n \text{ factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1} \quad [n \text{ factors}]$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$A^r A^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

In addition, we have the following properties of negative exponents.

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

- (a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- (b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
- (c) kA is invertible for any nonzero scalar k , and $(kA)^{-1} = k^{-1}A^{-1}$.

► **EXAMPLE 10 Properties of Exponents**

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(b),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

► **EXAMPLE 11 The Square of a Matrix Sum**

In real arithmetic, where we have a commutative law for multiplication, we can write

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

However, in matrix arithmetic, where we have no commutative law for multiplication, the best we can do is to write

$$(A + B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B *commute* (i.e., $AB = BA$) that we can go a step further and write

$$(A + B)^2 = A^2 + 2AB + B^2 \quad \blacktriangleleft$$

Matrix Polynomials

If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m \tag{3}$$

where I is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting A for x and replacing the constant term a_0 by the matrix a_0I . An expression of form (3) is called a *matrix polynomial in A* .

► **EXAMPLE 12 A Matrix Polynomial**

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$. ◀

Properties of the Transpose

The following theorem lists the main properties of the transpose.

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

- (a) $(A^T)^T = A$
- (b) $(A + B)^T = A^T + B^T$
- (c) $(A - B)^T = A^T - B^T$
- (d) $(kA)^T = kA^T$
- (e) $(AB)^T = B^T A^T$

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

► **EXAMPLE 13 Inverse of a Transpose**

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since A is invertible, its determinant $ad - bc$ is nonzero. But the determinant of A^T is also $ad - bc$ (verify), so A^T is also invertible. It follows from Theorem 1.4.5 that

$$(A^T)^{-1} = \begin{bmatrix} \frac{d}{ad - bc} & -\frac{c}{ad - bc} \\ -\frac{b}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

which is the same matrix that results if A^{-1} is transposed (verify). Thus,

$$(A^T)^{-1} = (A^{-1})^T$$

as guaranteed by Theorem 1.4.9. ◀

TRUE – FALSE Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B , are inverses of one another if and only if $AB = BA = 0$. **FALSE**

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$. **FALSE**

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$. **FALSE**

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$. **FALSE**

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^TB^T$. **FALSE**

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{TRUE}$$

is invertible if and only if $ad - bc \neq 0$.

(g) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$. **TRUE**

(h) If A is an invertible matrix, then so is A^T . **TRUE**

(i) If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$. **FALSE**

(j) A square matrix containing a row or column of zeros cannot be invertible. **TRUE**

(k) The sum of two invertible matrices of the same size must be invertible. **FALSE**