Lecture# 05

Predicates and Quantifiers

Section 1.4

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers
- Translating English to Logic
- Logic Programming (optional)

Propositional Logic Not Enough

- If we have:

  "All men are mortal."

  "Socrates is a man."
- Does it follow that "Socrates is mortal?"
- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- Later we'll see how to draw inferences.

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: P(x), M(x)
  - Quantifiers (to be covered in a few slides):
- Propositional functions are a generalization of propositions.
  - They contain variables and a predicate, e.g., P(x)
  - Variables can be replaced by elements from their domain.

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain (or bound by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  $P(-3)$ is false.
  $P(0)$ is false.
  $P(3)$ is true.
- Often the domain is denoted by U. So in this example U is the integers.

Examples of Propositional Functions

- Let “x + y = z” be denoted by R(x, y, z) and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let “x - y = z” be denoted by Q(x, y, z), with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If P(x) denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ Solution: T
  $P(3) \wedge P(-1)$ Solution: F
  $P(3) \to P(-1)$ Solution: F
  $P(3) \to P(1)$ Solution: T

- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$
  $P(x) \to P(y)$

- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

- We need quantifiers to express the meaning of English words including all and some:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - Universal Quantifier, "For all," symbol: $\forall$
  - Existential Quantifier, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the domain.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the domain.
- The quantifiers are said to bind the variable x in these expressions.

Quantifiers

- Universal Quantifier, “For all,”   symbol: $\forall$
- Existential Quantifier, “There exists,” symbol: $\exists$

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Universal Quantifier

$\forall x\, P(x)$ is read as " For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Uniqueness Quantifier (optional)

- $\exists!x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists!x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists!x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that P(x) can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples:**
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

For example,
we may be interested in a wider group of people than only those in this class. If we change the domain to consist of all people, we will need to express our statement as
“For every person x, if person x is a student in this class then x has studied Java.”
If S(x) represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow J(x))$.
[Caution! Our statement cannot be expressed as $\forall x(S(x) \land J(x))$ because this statement says that all people are students in this class and have studied Java!]

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 1: But if U is all people, then translate as

$\exists x\, (S(x) \wedge J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

if we are interested in people other than those in this class, we look at the statement a little differently. Our statement can be expressed as "There is a person x having the properties that x is a student in this class and x has studied Java."

In this case, the domain for the variable x consists of all people. We introduce $S(x)$ to represent

"x is a student in this class." Our solution becomes $\exists x(S(x) \wedge J(x))$ because the statement is that there is a person x who is a student in this class and who has studied Java.

[Caution! Our statement cannot be expressed as $\exists x(S(x) \rightarrow J(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow J(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.]

Returning to the Socrates Example

- Introduce the propositional functions Man(x) denoting "x is a man" and Mortal(x) denoting "x is mortal." Specify the domain as all people.
- The two premises are: $\forall x Man(x) \rightarrow Mortal(x)$

  $Man(Socrates)$
- The conclusion is: $Mortal(Socrates)$
- Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- Statements involving predicates and only quantifiers are logically equivalent if and if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- Example: $\forall x\, \neg\neg S(x) \equiv \forall x\, S(x)$