

VECTOR SPACES

Chapter # 03 & 04: "Elementary Linear Algebra" by Howard Anton & Chris Rorres

Course Instructor:

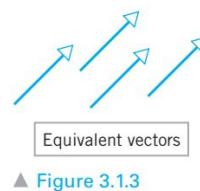
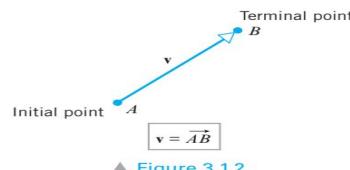
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EUCLIDEAN VECTOR SPACES

3.1: Vector in 2-Space, 3-Space & n-Space

- **Scalars** are quantities that are described by a numerical value alone. For example: temperature, length, and speed etc.
- **Vectors** are quantities that require both a number and a direction. For example: velocity, force etc.
- **Geometric Vectors:**
- **Equivalent Vectors:** Same length & direction even though they may be in different positions.



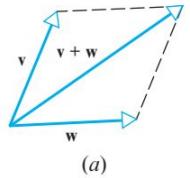
Vector Addition

Parallelogram Rule for Vector Addition If \mathbf{v} and \mathbf{w} are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the *sum* $\mathbf{v} + \mathbf{w}$ is the vector represented by the arrow from the common initial point of \mathbf{v} and \mathbf{w} to the opposite vertex of the parallelogram (Figure 3.1.4a).

Triangle Rule for Vector Addition If \mathbf{v} and \mathbf{w} are vectors in 2-space or 3-space that are positioned so the initial point of \mathbf{w} is at the terminal point of \mathbf{v} , then the *sum* $\mathbf{v} + \mathbf{w}$ is represented by the arrow from the initial point of \mathbf{v} to the terminal point of \mathbf{w} (Figure 3.1.4b).

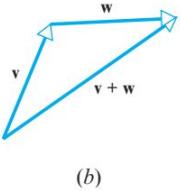
Vector Addition (Geometrical View)

Parallelogram Rule
for Vector Addition



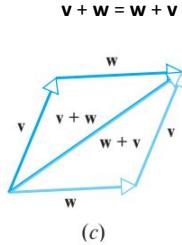
(a)

Triangle Rule for
Vector Addition



(b)

$$v + w = w + v$$



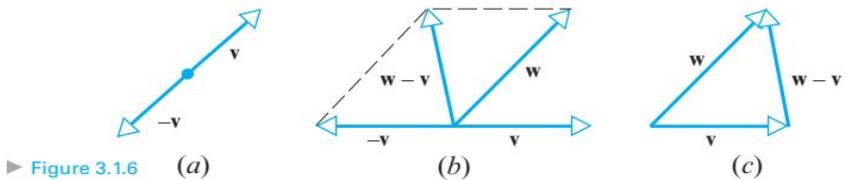
(c)

► Figure 3.14:

Vector Subtraction

Vector Subtraction The *negative* of a vector v , denoted by $-v$, is the vector that has the same length as v but is oppositely directed (Figure 3.1.6a), and the *difference* of v from w , denoted by $w - v$, is taken to be the sum

$$w - v = w + (-v) \quad (2)$$



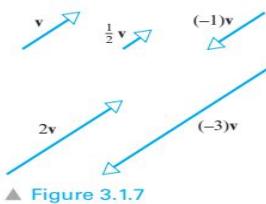
► Figure 3.1.6 (a)

(b)

(c)

Scalar Multiplication

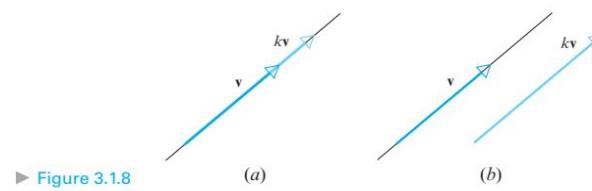
Scalar Multiplication If v is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the *scalar product of v by k* to be the vector whose length is $|k|$ times the length of v and whose direction is the same as that of v if k is positive and opposite to that of v if k is negative. If $k = 0$ or $v = \mathbf{0}$, then we define kv to be $\mathbf{0}$.



▲ Figure 3.1.7

Parallel and Collinear Vectors

- Suppose that v and w are vectors in 2-space or 3-space with a common initial point. If one of the vectors is a scalar multiple of the other, then the vectors lie on a common line, so it is reasonable to say that they are *collinear*.
- The terms *parallel* and *collinear* mean the same things when applied to vectors.



► Figure 3.1.8

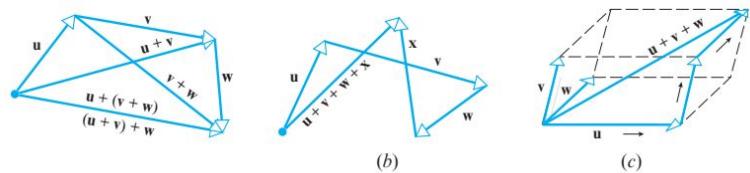
(a)

(b)

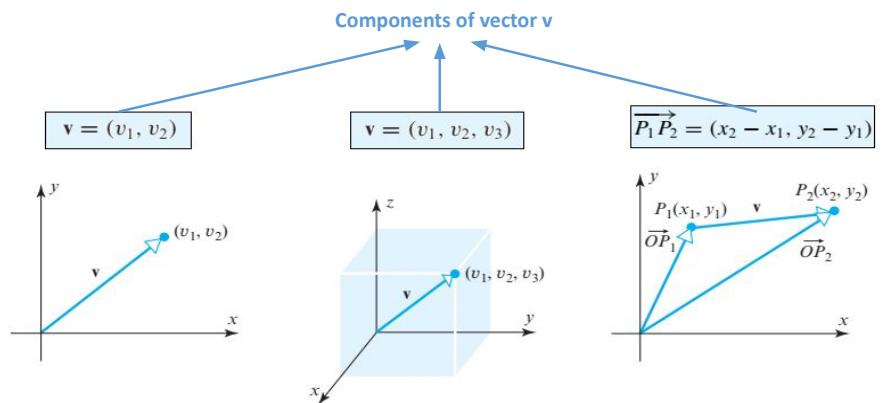
Vectors in Coordinate Systems

Sum of three or more vectors

- Vector addition satisfies the *associative law for addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- A simple way to construct $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is to place the vectors “tip to tail” in succession and then draw the vector from the initial point of \mathbf{u} to the terminal point of \mathbf{w} .



► Figure 3.1.9 (a)



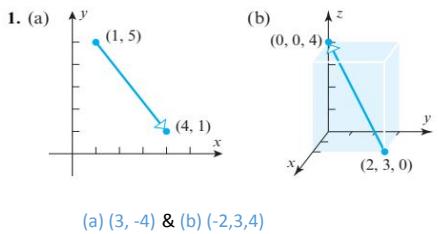
- The ordered pair (v_1, v_2) can represent a point or a vector.

► EXAMPLE 1 Finding the Components of a Vector

The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

$$\mathbf{v} = (7 - 2, 5 - (-1), -8 - 4) = (5, 6, -12)$$

► In Exercises 1–2, find the components of the vector. ◀ ► In Exercises 3–4, find the components of the vector $\overrightarrow{P_1P_2}$. ◀



3. (a) $P_1(3, 5), P_2(2, 8)$ (b) $P_1(5, -2, 1), P_2(2, 4, 2)$
 4. (a) $P_1(-6, 2), P_2(-4, -1)$ (b) $P_1(0, 0, 0), P_2(-1, 6, 1)$
- 3a. $(-1, 3)$
 3b. $(-3, 6, 1)$
 4a. $(2, -3)$
 4b. $(-1, 6, 1)$

n-Space

- Set of all real numbers can be represented as a line $\rightarrow \mathbb{R}^1$
- Set of all ordered pairs of real numbers (2-tuples) $\rightarrow \mathbb{R}^2$
- Set of all ordered triples of real numbers (3-tuples) $\rightarrow \mathbb{R}^3$
- Set of all ordered n-tuples: $\rightarrow \mathbb{R}^n$

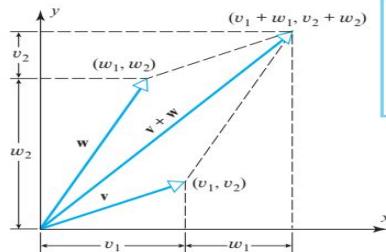
DEFINITION 1 If n is a positive integer, then an *ordered n-tuple* is a sequence of n real numbers (v_1, v_2, \dots, v_n) . The set of all ordered n -tuples is called ***n*-space** and is denoted by R^n .

DEFINITION 2 Vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ in R^n are said to be *equivalent* (also called *equal*) if

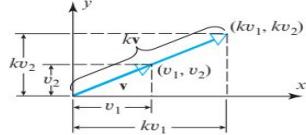
$$v_1 = w_1, v_2 = w_2, \dots, v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

Operations of Addition, Subtraction, and Scalar multiplication in R^n



$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (v_1 + w_1, v_2 + w_2) & (6) \\ k\mathbf{v} &= (kv_1, kv_2) & (7) \\ -\mathbf{v} &= (-1)\mathbf{v} = (-v_1, -v_2) & (8) \\ \mathbf{w} - \mathbf{v} &= \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2) & (9) \end{aligned}$$



DEFINITION 3 If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ are vectors in R^n , and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \quad (10)$$

$$k\mathbf{v} = (kv_1, kv_2, \dots, kv_n) \quad (11)$$

$$-\mathbf{v} = (-v_1, -v_2, \dots, -v_n) \quad (12)$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \dots, w_n - v_n) \quad (13)$$

Algebraic Operations Using Components

► EXAMPLE 3 Algebraic Operations Using Components

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \quad 2\mathbf{v} = (2, -6, 4)$$

$$-\mathbf{w} = (-4, -2, -1), \quad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1) \quad \blacktriangleleft$$

Algebraic Operations Using Components (Contd.)

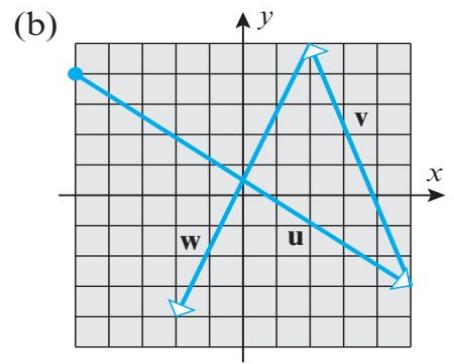
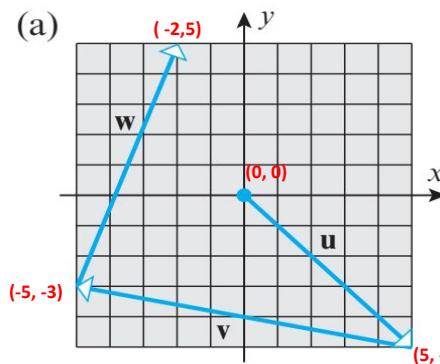
THEOREM 3.1.1 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , and if k and m are scalars, then:

- (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- (d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- (e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
- (g) $k(m\mathbf{u}) = (km)\mathbf{u}$
- (h) $1\mathbf{u} = \mathbf{u}$

THEOREM 3.1.2 If \mathbf{v} is a vector in R^n and k is a scalar, then:

- (a) $0\mathbf{v} = \mathbf{0}$
- (b) $k\mathbf{0} = \mathbf{0}$
- (c) $(-1)\mathbf{v} = -\mathbf{v}$

In each part, find the components of the vector $\mathbf{u} + \mathbf{v} + \mathbf{w}$.



$$(a) \mathbf{u} + \mathbf{v} + \mathbf{w} = (5, -5) + (-10, 2) + (3, 8) = (-2, 5)$$

$$(b) \mathbf{u} + \mathbf{v} + \mathbf{w} = (10, -7) + (-3, 8) + (-4, -9) = (3, -8)$$

Linear Combinations

Find an initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3, 0, -5)$ and such that

- (a) \mathbf{u} has the same direction as $\mathbf{v} = (4, -2, -1)$.
- (b) \mathbf{u} is oppositely directed to $\mathbf{v} = (4, -2, -1)$.

- (a) $(3 - 4, 0 - (-2), -5 - (-1))$, i.e., $(-1, 2, -4)$
- (b) $(3 - (-4), 0 - 2, -5 - 1)$, i.e., $(7, -2, -6)$

DEFINITION 4 If \mathbf{w} is a vector in R^n , then \mathbf{w} is said to be a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \quad (14)$$

where k_1, k_2, \dots, k_r are scalars. These scalars are called the *coefficients* of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

True-False Exercises

FALSE

(a) Two equivalent vectors must have the same initial point.

FALSE

(b) The vectors (a, b) and $(a, b, 0)$ are equivalent. **FALSE**

(c) If k is a scalar and \mathbf{v} is a vector, then \mathbf{v} and $k\mathbf{v}$ are parallel if and only if $k \geq 0$. **FALSE**

(d) The vectors $\mathbf{v} + (\mathbf{u} + \mathbf{w})$ and $(\mathbf{w} + \mathbf{v}) + \mathbf{u}$ are the same. **TRUE**

(e) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. **TRUE**

(f) If a and b are scalars such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then \mathbf{u} and \mathbf{v} are parallel vectors. **FALSE**

(g) Collinear vectors with the same length are equal. **FALSE**

(h) If $(a, b, c) + (x, y, z) = (x, y, z)$, then (a, b, c) must be the zero vector. **TRUE**

(i) If k and m are scalars and \mathbf{u} and \mathbf{v} are vectors, then

FALSE

$$(k+m)(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + m\mathbf{v}$$

(j) If the vectors \mathbf{v} and \mathbf{w} are given, then the vector equation

TRUE

$$3(2\mathbf{v} - \mathbf{x}) = 5\mathbf{x} - 4\mathbf{w} + \mathbf{v}$$

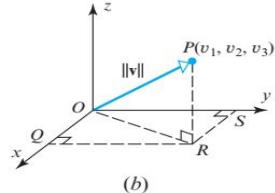
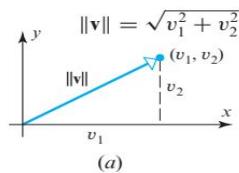
can be solved for \mathbf{x} .

(k) The linear combinations $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ and $b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ can only be equal if $a_1 = b_1$ and $a_2 = b_2$. **FALSE**

3.2: Norm, Dot Product, and Distance in n-space

Norm of a Vector

- The term “norm” is a mathematical synonym for length.



▲ Figure 3.2.1

From the theorem of Pythagoras the norm of \mathbf{v} in 2-space (a) and in 3-space (b):

$$\begin{aligned}\|\mathbf{v}\|^2 &= (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2 \\ \|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2}\end{aligned}$$

Norm of a vector in n-space

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a vector in R^n , then the **norm** of \mathbf{v} (also called the **length** of \mathbf{v} or the **magnitude** of \mathbf{v}) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad (3)$$

► EXAMPLE 1 Calculating Norms

It follows from Formula (2) that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula (3) that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39} \quad \blacktriangleleft$$

THEOREM 3.2.1 If \mathbf{v} is a vector in R^n , and if k is any scalar, then:

- (a) $\|\mathbf{v}\| \geq 0$
 - Distances are non-negative
- (b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
 - The zero vector is the only vector of length zero.
- (c) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$
 - Multiplying a vector by a scalar multiplies its length by the absolute value of that scalar.

- A vector of norm 1 is called a **unit vector**. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand.
- More generally, if \mathbf{v} is any nonzero vector in R^n , then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$

- The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called **normalizing** \mathbf{v} .

Unit Vector

The standard Unit Vector

► EXAMPLE 2 Normalizing a Vector

Find the unit vector \mathbf{u} that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector \mathbf{v} has length

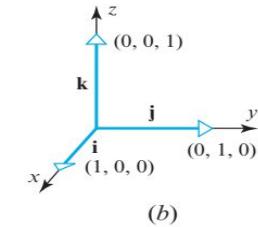
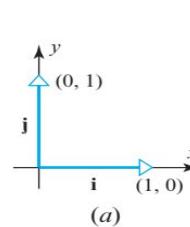
$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from (4)

$$\mathbf{u} = \frac{1}{3}(2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$. ◀

When a rectangular coordinate system is introduced in R^2 or R^3 , the unit vectors in the positive directions of the coordinate axes are called the *standard unit vectors*. In R^2 these



Moreover, we can generalize these formulas to R^n by defining the *standard unit vectors* in R^n to be

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

in which case every vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = (v_1, v_2, \dots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n$$

Distance in R^n

► EXAMPLE 3 Linear Combinations of Standard Unit Vectors

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4 \quad \blacktriangleleft$$

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance in R^2

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance in R^3

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are points in R^n , then we denote the *distance* between \mathbf{u} and \mathbf{v} by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2} \quad (11)$$

► EXAMPLE 4 Calculating Distance in R^n

If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between \mathbf{u} and \mathbf{v} is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1 - 0)^2 + (3 - 7)^2 + (-2 - 2)^2 + (7 - 2)^2} = \sqrt{58} \quad \blacktriangleleft$$

► In Exercises 1–2, find the norm of \mathbf{v} , and a unit vector that is oppositely directed to \mathbf{v} .

1. (a) $\mathbf{v} = (2, 2, 2)$ (b) $\mathbf{v} = (1, 0, 2, 1, 3)$
2. (a) $\mathbf{v} = (1, -1, 2)$ (b) $\mathbf{v} = (-2, 3, 3, -1)$

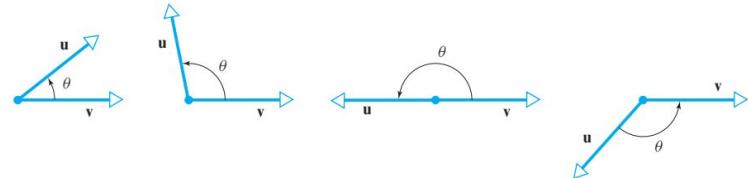
► In Exercises 3–4, evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

3. (a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$
(c) $\|-2\mathbf{u} + 2\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$
4. (a) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$ (b) $\|\mathbf{u} - \mathbf{v}\|$
(c) $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$ (d) $\|\mathbf{u}\| - \|\mathbf{v}\|$

1. (a) $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$;
 $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{2\sqrt{3}}(2, 2, 2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; $-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{2\sqrt{3}}(2, 2, 2) = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

3. (a) $\mathbf{u} + \mathbf{v} = (3, -5, 7)$; $\|\mathbf{u} + \mathbf{v}\| = \sqrt{3^2 + (-5)^2 + 7^2} = \sqrt{83}$
(b) $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 3^2} + \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{17} + \sqrt{26}$
(c) $-2\mathbf{u} + 2\mathbf{v} = (-4, 4, -6) + (2, -6, 8) = (-2, -2, 2)$;
 $\|-2\mathbf{u} + 2\mathbf{v}\| = \sqrt{(-2)^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$
(d) $3\mathbf{u} - 5\mathbf{v} + \mathbf{w} = (6, -6, 9) - (5, -15, 20) + (3, 6, -4) = (4, 15, -15)$;
 $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\| = \sqrt{4^2 + 15^2 + (-15)^2} = \sqrt{466}$

Angle between two non zero vector



The angle θ between \mathbf{u} and \mathbf{v} satisfies $0 \leq \theta \leq \pi$.

Dot Product

DEFINITION 3 If \mathbf{u} and \mathbf{v} are nonzero vectors in R^2 or R^3 , and if θ is the angle between \mathbf{u} and \mathbf{v} , then the **dot product** (also called the **Euclidean inner product**) of \mathbf{u} and \mathbf{v} is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (12)$$

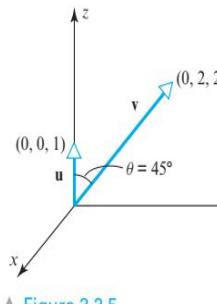
If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad (13)$$

Since $0 \leq \theta \leq \pi$, it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.



▲ Figure 3.2.5

► EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

Solution The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \text{ and } \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle θ between them is

$$\cos(45^\circ) = 1/\sqrt{2}$$

Thus, it follows from Formula (12) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$$

Components form of dot product

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in R^n , then the **dot product** (also called the **Euclidean inner product**) of \mathbf{u} and \mathbf{v} is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n \quad (17)$$

► EXAMPLE 6 Calculating Dot Products Using Components

- (a) Use Formula (15) to compute the dot product of the vectors \mathbf{u} and \mathbf{v} in Example 5.
 (b) Calculate $\mathbf{u} \cdot \mathbf{v}$ for the following vectors in R^4 :

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

Solution (a) The component forms of the vectors are $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)(0) + (0)(2) + (1)(2) = 2$$

which agrees with the result obtained geometrically in Example 5.

Solution (b)

$$\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4$$

Algebraic Properties of Dot Product

In the special case where $\mathbf{u} = \mathbf{v}$ in Definition 4, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \dots + v_n^2 = \|\mathbf{v}\|^2 \quad (18)$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad (19)$$

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , and if k is a scalar, then:

- | | |
|---|-------------------------|
| (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | [Symmetry property] |
| (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | [Distributive property] |
| (c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ | [Homogeneity property] |
| (d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ | [Positivity property] |

THEOREM 3.2.3 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , and if k is a scalar, then:

- | |
|--|
| (a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$ |
| (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ |
| (c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ |
| (d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$ |
| (e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$ |

Algebraic Properties of Dot Product (Contd.)

THEOREM 3.2.3 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , and if k is a scalar, then:

- | |
|--|
| (a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$ |
| (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ |
| (c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ |
| (d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$ |
| (e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$ |

Some other Theorems

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in R^n , then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad (22)$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \dots + u_nv_n| \leq (u_1^2 + u_2^2 + \dots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \dots + v_n^2)^{1/2} \quad (23)$$

THEOREM 3.2.5 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , then:

- | | |
|--|-------------------------------------|
| (a) $\ \mathbf{u} + \mathbf{v}\ \leq \ \mathbf{u}\ + \ \mathbf{v}\ $ | [Triangle inequality for vectors] |
| (b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ | [Triangle inequality for distances] |

THEOREM 3.2.6 Parallelogram Equation for Vectors

If \mathbf{u} and \mathbf{v} are vectors in R^n , then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) \quad (24)$$

► EXAMPLE 8 Calculating with Dot Products

$$\begin{aligned} (\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v}) \\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2 \end{aligned}$$

► In Exercises 5–6, evaluate the given expression with $\mathbf{u} = (-2, -1, 4, 5)$, $\mathbf{v} = (3, 1, -5, 7)$, and $\mathbf{w} = (-6, 2, 1, 1)$.

5. (a) $\|\mathbf{3u} - \mathbf{5v} + \mathbf{w}\|$

(b) $\|\mathbf{3u}\| - 5\|\mathbf{v}\| + \|\mathbf{w}\|$

(c) $-\|\mathbf{u}\|\|\mathbf{v}\|$

6. (a) $\|\mathbf{u}\| + \|\mathbf{-2v}\| + \|\mathbf{-3w}\|$

(b) $\|\mathbf{u} - \mathbf{v} - \mathbf{w}\|$

► In Exercises 9–10, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$.

9. (a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$

(b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

10. (a) $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$

(b) $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

5. (a) $\sqrt{2570}$

(b) $3\sqrt{46} - 10\sqrt{21} + \sqrt{42}$

(c) $\|\mathbf{-u}\|\|\mathbf{v}\| = \sqrt{46}\sqrt{84} = 2\sqrt{966}$

6. (a) $\sqrt{46} + 4\sqrt{21} + 3\sqrt{42}$

(b) $\|\mathbf{u} - \mathbf{v}\|\|\mathbf{w}\| = \sqrt{4788} = 6\sqrt{133}$

9. (a) $\mathbf{u} \cdot \mathbf{v} = -8$

$\mathbf{u} \cdot \mathbf{u} = 26$

$\mathbf{v} \cdot \mathbf{v} = 24$

11. (a) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(3-1)^2 + (3-0)^2 + (3-4)^2} = \sqrt{14}$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5}{\sqrt{51}}$; the angle is acute since $\mathbf{u} \cdot \mathbf{v} > 0$

(b) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{59}$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-4}{\sqrt{51}}$; the angle is obtuse since $\mathbf{u} \cdot \mathbf{v} < 0$

► In Exercises 1–2, determine whether \mathbf{u} and \mathbf{v} are orthogonal vectors.

1. (a) $\mathbf{u} = (6, 1, 4)$, $\mathbf{v} = (2, 0, -3)$

Orthogonal

(b) $\mathbf{u} = (0, 0, -1)$, $\mathbf{v} = (1, 1, 1)$

Non-Orthogonal

(c) $\mathbf{u} = (3, -2, 1, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$

Non-Orthogonal

(d) $\mathbf{u} = (5, -4, 0, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$

Non-Orthogonal

► In Exercises 11–12, determine whether the given planes are perpendicular.

11. $3x - y + z - 4 = 0$, $x + 2z = -1$

Not perpendicular

12. $x - 2y + 3z = 4$, $-2x + 5y + 4z = -1$

Not perpendicular

► In Exercises 7–10, determine whether the given planes are parallel.

7. $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$

8. $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$

9. $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$

10. $(-4, 1, 2) \cdot (x, y, z) = 0$ and $(8, -2, -4) \cdot (x, y, z) = 0$

Not parallel

Parallel

Not parallel

Parallel

Orthogonality

DEFINITION 1 Two nonzero vectors \mathbf{u} and \mathbf{v} in R^n are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n .

EXAMPLE 1 Orthogonal Vectors

- (a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4 .
- (b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3 . Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = \mathbf{0}$

$$\mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = \mathbf{0}$$

$$\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1, 0, 0) \cdot (0, 0, 1) = 0$$

$$\mathbf{j} \cdot \mathbf{k} = (0, 1, 0) \cdot (0, 0, 1) = 0$$

GENERAL VECTOR SPACES (Algebraic properties of vectors in R^n as axioms)

Vector Space

Nonempty set of objects with two operations: Addition & Scalar Multiplication

If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m , then we call V a **vector space** and we call the objects in V **vectors**.

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V . → Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V , called a **zero vector** for V , such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .
5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a **negative** of \mathbf{u} , such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and \mathbf{u} is any object in V , then $k\mathbf{u}$ is in V . → Closure under Scalar Multiplication
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

► EXAMPLE 3 The Vector Space of Infinite Sequences of Real Numbers R^∞

Let V consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \dots, u_n, \dots)$$

in which $u_1, u_2, \dots, u_n, \dots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication componentwise by

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n, \dots) + (v_1, v_2, \dots, v_n, \dots) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n, \dots) \\ k\mathbf{u} &= (ku_1, ku_2, \dots, ku_n, \dots)\end{aligned}$$



► EXAMPLE 1 The Zero Vector Space

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalars k . It is easy to check that all the vector space axioms are satisfied. We call this the **zero vector space**. ◀

► EXAMPLE 2 R^n Is a Vector Space

Let $V = R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n -tuples; that is,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \\ k\mathbf{u} &= (ku_1, ku_2, \dots, ku_n)\end{aligned}$$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce n -tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1. ◀

► EXAMPLE 4 The Vector Space of 2×2 Matrices

Let V be the set of 2×2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \quad (1) \\ k\mathbf{u} &= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}\end{aligned}$$

- The set V is closed under **addition** and **scalar multiplication**.
- Axioms 2, 3, 4, 5, 8, 9, 10 satisfied by **Theorem 1.4.1**.

THEOREM 1.4.1 Properties of Matrix Arithmetic

- | | |
|---------------------------------|-----------------------------|
| (a) $A + B = B + A$ | (g) $(B - C)A = BA - CA$ |
| (b) $A + (B + C) = (A + B) + C$ | (h) $a(B + C) = aB + aC$ |
| (c) $A(BC) = (AB)C$ | (i) $a(B - C) = aB - aC$ |
| (d) $A(B + C) = AB + AC$ | (j) $(a + b)C = aC + bC$ |
| (e) $(B + C)A = BA + CA$ | (k) $(a - b)C = aC - bC$ |
| (f) $A(B - C) = AB - AC$ | (l) $a(bC) = (ab)C$ |
| | (m) $a(BC) = (aB)C = B(aC)$ |

► EXAMPLE 5 The Vector Space of $m \times n$ Matrices

- Example 4 is a special case of a more general class of vector spaces.
- The set V of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication is a vector space.
- We will denote this vector space by the symbol M_{mn} .
- The vector space in Example 4 is denoted as M_{22} .

► EXAMPLE 6 The Vector Space of Real-Valued Functions

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \quad (2)$$

$$(k\mathbf{f})(x) = kf(x) \quad (3)$$

- Think about these operations is to view the numbers $f(x)$ and $g(x)$ as “components” of \mathbf{f} and \mathbf{g} at the point x .
- The set V with these operations is denoted by the symbol $F(-\infty, \infty)$.

► EXAMPLE 7 A Set That Is Not a Vector Space

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

- The first nine vector space axioms are satisfied.
- However, Axiom 10 fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then $1\mathbf{u} = 1(u_1, u_2) = (1.u_1, 0) = (u_1, 0) \neq \mathbf{u}$
- Thus, V is not a vector space with the stated operations.

► EXAMPLE 8 An Unusual Vector Space

Let V be the set of positive real numbers, let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors (i.e., positive real numbers) in V , and let k be any scalar. Define the operations on V to be

$$u + v = uv \quad [\text{Vector addition is numerical multiplication.}]$$

$$ku = u^k \quad [\text{Scalar multiplication is numerical exponentiation.}]$$

Thus, for example, $1 + 1 = 1$ and $(2)(1) = 1^2 = 1$ —strange indeed, but nevertheless the set V with these operations satisfies the ten vector space axioms and hence is a vector space. We will confirm Axioms 4, 5, and 7, and leave the others as exercises.

- Axiom 4—The zero vector in this space is the number 1 (i.e., $\mathbf{0} = 1$) since

$$u + 1 = u \cdot 1 = u$$

- Axiom 5—The negative of a vector u is its reciprocal (i.e., $-u = 1/u$) since

$$u + \frac{1}{u} = u \left(\frac{1}{u} \right) = 1 (= \mathbf{0})$$

- Axiom 7— $k(u + v) = (uv)^k = u^k v^k = (ku) + (kv)$. ◀

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V .

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

4. There is an object $\mathbf{0}$ in V , called a *zero vector* for V , such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .

5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a *negative* of \mathbf{u} , such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

6. If k is any scalar and \mathbf{u} is any object in V , then $k\mathbf{u}$ is in V .

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

9. $k(m\mathbf{u}) = (km)(\mathbf{u})$

10. $1\mathbf{u} = \mathbf{u}$

$$\boxed{\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{uv} \\ k\mathbf{u} &= \mathbf{u}^k \end{aligned}}$$

THEOREM 4.1.1 Let V be a vector space, \mathbf{u} a vector in V , and k a scalar; then:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

TRUE-FALSE

(a) A vector is any element of a vector space.

TRUE

(b) A vector space must contain at least two vectors.

FALSE

(c) If \mathbf{u} is a vector and k is a scalar such that $k\mathbf{u} = \mathbf{0}$, then it must be true that $k = 0$.

FALSE

(d) The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers.

FALSE

(e) In every vector space the vectors $(-1)\mathbf{u}$ and $-\mathbf{u}$ are the same.

TRUE

(f) In the vector space $F(-\infty, \infty)$ any function whose graph passes through the origin is a zero vector.

FALSE

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$. $\mathbf{u} + \mathbf{v} = (2, 6); \quad k\mathbf{u} = (0, 6)$

(b) In words, explain why V is closed under addition and scalar multiplication.

(c) Since addition on V is the standard addition operation on R^2 , certain vector space axioms hold for V because they are known to hold for R^2 . Which axioms are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations. $1(u_1, u_2) = (0, u_2)$ does not generally equal (u_1, u_2) .

- For any \mathbf{u} and \mathbf{v} in V , $\mathbf{u} + \mathbf{v}$ is an ordered pair of real numbers, therefore $\mathbf{u} + \mathbf{v}$ is in V .
- For any \mathbf{u} in V and for any scalar k , $k\mathbf{u} = (0, ku_2)$ is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V .

Axioms 1-5 hold for V because they are known to hold for R^2 .

- (d) Axiom 7: $k((u_1, u_2) + (v_1, v_2)) = k(u_1 + v_1, u_2 + v_2) = (0, k(u_2 + v_2)) = (0, ku_2) + (0, kv_2)$
 $= k(u_1, u_2) + k(v_1, v_2)$ for all real k , u_1, u_2, v_1 , and v_2 ;
- Axiom 8: $(k + m)(u_1, u_2) = (0, (k + m)u_2) = (0, ku_2 + mu_2) = (0, ku_2) + (0, mu_2)$
 $= k(u_1, u_2) + m(u_1, u_2)$ for all real k , m , u_1 , and u_2 ;
- Axiom 9: $k(m(u_1, u_2)) = k(0, mu_2) = (0, kmu_2) = (km)(u_1, u_2)$ for all real k , m , u_1 , and u_2 .

Vector Space Assumptions

If the following axioms are satisfied by all objects \mathbf{u} , \mathbf{v} , \mathbf{w} in V and all scalars k and m , then we call V a **vector space** and we call the objects in V **vectors**.

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V , called a **zero vector** for V , such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .
5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a **negative** of \mathbf{u} , such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and \mathbf{u} is any object in V , then $k\mathbf{u}$ is in V .
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Determine whether each set equipped with the given operations is a vector space.

3. The set of all real numbers with the standard operations of addition and multiplication. **Vector Space**
4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2 . **Vector Space**
5. The set of all pairs of real numbers of the form (x, y) , where $x \geq 0$, with the standard operations on R^2 . **Axioms 6 failed**
6. The set of all n -tuples of real numbers that have the form (x, x, \dots, x) with the standard operations on R^n . **Vector Space**
7. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by $k(x, y, z) = (k^2x, k^2y, k^2z)$ **Axioms 8 failed**
8. The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication. **Axioms 6 failed**

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \text{Vector Space}$$

with the standard matrix addition and scalar multiplication.

10. The set of all real-valued functions f defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6. **Vector Space**
11. The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

12. The set of polynomials of the form $a_0 + a_1x$ with the operations

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x$$

and $k(a_0 + a_1x) = (ka_0) + (ka_1)x$

Subspaces

DEFINITION 1 A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V .

THEOREM 4.2.1 If W is a set of one or more vectors in a vector space V , then W is a subspace of V if and only if the following conditions are satisfied.

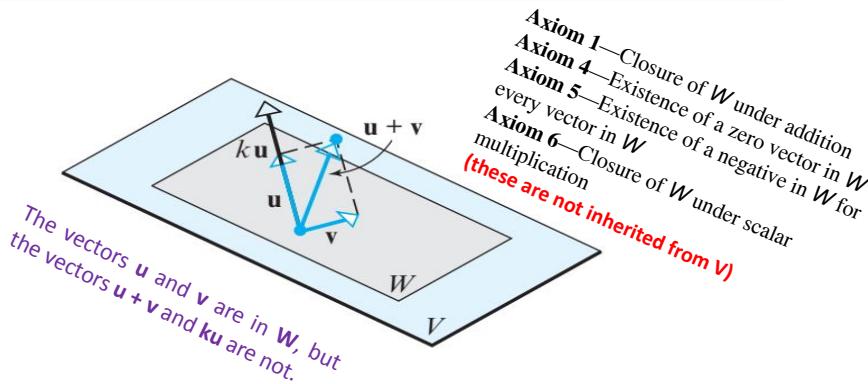
- If \mathbf{u} and \mathbf{v} are vectors in W , then $\mathbf{u} + \mathbf{v}$ is in W .
- If k is a scalar and \mathbf{u} is a vector in W , then $k\mathbf{u}$ is in W .

► EXAMPLE 1 The Zero Subspace

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

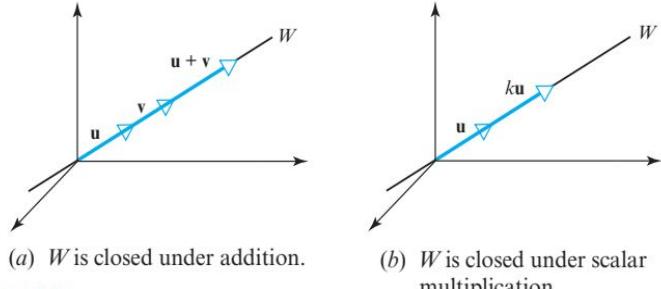
$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for any scalar k . We call W the **zero subspace** of V .



► **EXAMPLE 2 Lines Through the Origin Are Subspaces of R^2 and of R^3**

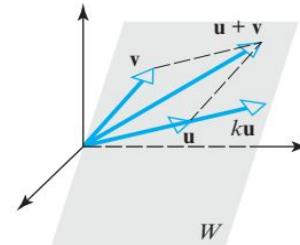
If W is a line through the origin of either R^2 or R^3 , then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication (see Figure 4.2.2 for an illustration in R^3).



► **Figure 4.2.2**

► **EXAMPLE 3 Planes Through the Origin Are Subspaces of R^3**

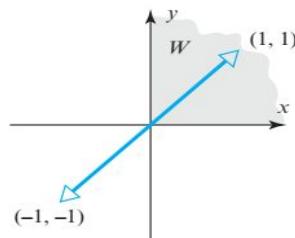
If \mathbf{u} and \mathbf{v} are vectors in a plane W through the origin of R^3 , then it is evident geometrically that $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ also lie in the same plane W for any scalar k (Figure 4.2.3). Thus W is closed under addition and scalar multiplication. ◀



▲ **Figure 4.2.3** The vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as \mathbf{u} and \mathbf{v} .

► **EXAMPLE 4 A Subset of R^2 That Is Not a Subspace**

- Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$.



- This set is not a subspace of R^2 because it is not closed under scalar multiplication.
- For example, $\mathbf{v} = (1, 1)$ is a vector in W , but $(-1)\mathbf{v} = (-1, -1)$ is not.

► **EXAMPLE 5 Subspaces of M_{nn}**

- Let W be the set of symmetric matrices.
- Then W is a subspace of V because the sum of two symmetric matrices and the scalar multiplication of symmetric matrices are in W .
- Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn} .

► **EXAMPLE 6 A Subset of M_{nn} That Is Not a Subspace**

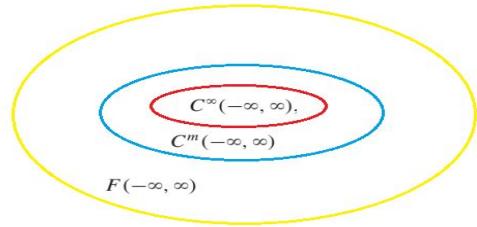
- Let W be the set of invertible $n \times n$ matrices. W is not a subspace because the zero matrix is not in W .
- Consider the matrices: $U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$
- The matrix OU is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

► EXAMPLE 7 The Subspace $C(-\infty, \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous. Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C(-\infty, \infty)$.

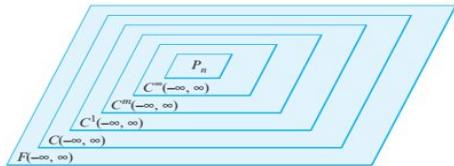
► EXAMPLE 8 Functions with Continuous Derivatives

- The functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$.
- The sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable.



The Hierarchy of Function Spaces

- It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$.
- Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$.
- Last 4 examples are “nested” one inside other as illustrated in the figure:



THEOREM 4.2.2 If W_1, W_2, \dots, W_r are subspaces of a vector space V , then the intersection of these subspaces is also a subspace of V .

► EXAMPLE 9 The Subspace of All Polynomials

Recall that a **polynomial** is a function that can be expressed in the form

$$p(x) = a_0 + a_1x + \dots + a_nx^n \quad (1)$$

where a_0, a_1, \dots, a_n are constants. It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial. Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞ .

► EXAMPLE 10 The Subspace of Polynomials of Degree $\leq n$

Recall that the **degree** of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n . It is *not* true that the set W of polynomials with positive degree n is a subspace of $F(-\infty, \infty)$ because that set is not closed under addition. For example, the polynomials

$$1 + 2x + 3x^2 \text{ and } 5 + 7x - 3x^2$$

both have degree 2, but their sum has degree 1. What *is* true, however, is that for each nonnegative integer n the polynomials of degree n or less form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n . ◀

Spanning Set

DEFINITION 2 If w is a vector in a vector space V , then w is said to be a **linear combination** of the vectors v_1, v_2, \dots, v_r in V if w can be expressed in the form

$$w = k_1v_1 + k_2v_2 + \dots + k_rv_r \quad (2)$$

where k_1, k_2, \dots, k_r are scalars. These scalars are called the **coefficients** of the linear combination.

DEFINITION 3 If $S = \{w_1, w_2, \dots, w_r\}$ is a nonempty set of vectors in a vector space V , then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V **generated** by S , and we say that the vectors w_1, w_2, \dots, w_r **span** W . We denote this subspace as

$$W = \text{span}\{w_1, w_2, \dots, w_r\} \text{ or } W = \text{span}(S)$$

- vectors (v_1, v_2, \dots, v_r) span a space means:** The space consists of all combinations of those vectors.
- Span (v_1, v_2, \dots, v_r) → set of all possible linear combination of v_1, v_2, \dots, v_r .**

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \left\{ \sum_{i=1}^n c_i \vec{v}_i : c_i \in \mathbb{R} \right\}$$

- When we say that a vector \vec{w} is in $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$

- we mean that \vec{w} can be written as a linear combination of the \vec{v}_i .
- We say that a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a **spanning set**

for V if $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

Let W be a subset of a vector space V .

- If $W = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\text{Span}(W)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

- If W is an infinite set then $\text{Span}(W)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in W$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ ($k \geq 1$).

- If W is the empty set then $\text{Span}(W) = \{\mathbf{0}\}$.

► EXAMPLE 11 The Standard Unit Vectors Span R^n

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Thus, for example, the vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

span R^3 since every vector $\mathbf{v} = (a, b, c)$ in this space can be expressed as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Unit Vector

- A vector of norm 1 is called a **unit vector**. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand.

- More generally, if \mathbf{v} is any nonzero vector in R^n , then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$

- The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called **normalizing** \mathbf{v} .

$$\bullet \text{ norm} \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

► EXAMPLE 1 Calculating Norms

It follows from Formula (2) that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula (3) that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39} \quad \blacktriangleleft$$

THEOREM 4.2.1 If W is a set of one or more vectors in a vector space V , then W is a subspace of V if and only if the following conditions are satisfied.

- (a) If \mathbf{u} and \mathbf{v} are vectors in W , then $\mathbf{u} + \mathbf{v}$ is in W .
- (b) If k is a scalar and \mathbf{u} is a vector in W , then $k\mathbf{u}$ is in W .

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3 .

- (a) All vectors of the form $(a, 0, 0)$. W is a subspace of R^3 .
- (b) All vectors of the form $(a, 1, 1)$. W is not a subspace of R^3 .
- (c) All vectors of the form (a, b, c) , where $b = a + c$. W is a subspace of R^3 .
- (d) All vectors of the form (a, b, c) , where $b = a + c + 1$. W is not a subspace of R^3 .
- (e) All vectors of the form $(a, b, 0)$. W is a subspace of R^3 .

2. Use Theorem 4.2.1 to determine which of the following are subspaces of M_{nn} .

- (a) The set of all diagonal $n \times n$ matrices. W is a subspace of M_{nn} .
- (b) The set of all $n \times n$ matrices A such that $\det(A) = 0$. W is not a subspace of M_{nn} .
- (c) The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$. W is a subspace of M_{nn} .
- (d) The set of all symmetric $n \times n$ matrices. W is a subspace of M_{nn} .
- (e) The set of all $n \times n$ matrices A such that $A^T = -A$. W is a subspace of M_{nn} .
- (f) The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. W is not a subspace of M_{nn} .

► EXAMPLE 14 Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3 . Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of \mathbf{u} and \mathbf{v} and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of \mathbf{u} and \mathbf{v} .

- $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$
- $(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$

$k_1 + 6k_2 = 9$ • Solving this system using Gaussian elimination
 $2k_1 + 4k_2 = 2$ yields $k_1 = -3$, $k_2 = 2$, so $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$
 $-k_1 + 2k_2 = 7$

Similarly, $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$k_1 + 6k_2 = 4$ This system of equations is inconsistent (verify),
 $2k_1 + 4k_2 = -1$ so no such scalars k_1 and k_2 exist.
 $-k_1 + 2k_2 = 8$

► EXAMPLE 15 Testing for Spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3 .

- $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$
- $(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$

- $$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 &\quad + k_3 = b_2 \longrightarrow A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$
- since $\det(A) = 0$ (verify), so \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 do not span R^3 .

Which of the following are linear combinations of

$\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

- (a) $(2, 2, 2)$ (b) $(0, 4, 5)$ (c) $(0, 0, 0)$

- $a(0, -2, 2) + b(1, 3, -1) = (2, 2, 2)$ (b) $a(0, -2, 2) + b(1, 3, -1) = (0, 4, 5)$
- $\begin{array}{rcl} 0a & + & 1b = 2 \\ -2a & + & 3b = 2 \\ 2a & - & 1b = 2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$
- The linear system is consistent.
- The system is inconsistent, $(0, 4, 5)$ is not a linear combination of \mathbf{u} and \mathbf{v} .

- (c) The zero vector $(0, 0, 0)$ is a lc of \mathbf{u} and \mathbf{v} .

- $0(0, -2, 2) + 0(1, 3, -1) = (0, 0, 0)$

In each part, determine whether the vectors span R^3 .

- (a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$
 (b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

(a) • The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination $(b_1, b_2, b_3) = k_1(2, 2, 2) + k_2(0, 0, 3) + k_3(0, 1, 1)$

$$\begin{array}{rcl} 2k_1 + 0k_2 + 0k_3 & = & b_1 \\ 2k_1 + 0k_2 + 1k_3 & = & b_2 \\ 2k_1 + 3k_2 + 1k_3 & = & b_3 \end{array}$$

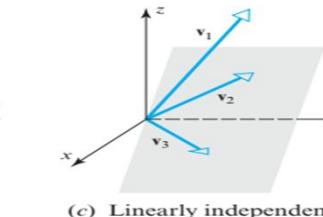
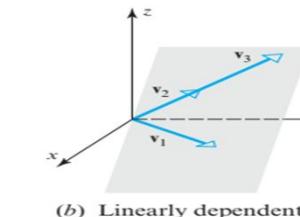
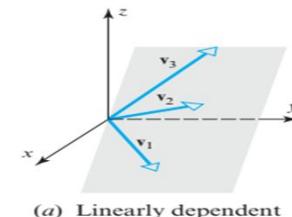
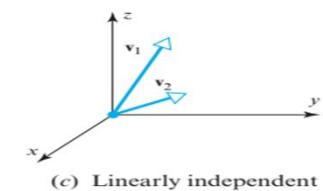
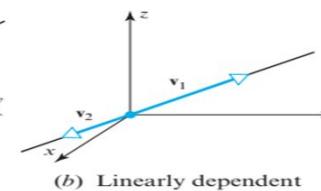
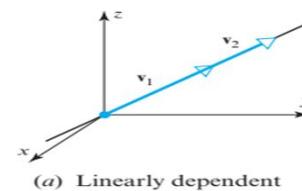
• We conclude that \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 span R^3 .

(b) • The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination $(b_1, b_2, b_3) = k_1(2, -1, 3) + k_2(4, 1, 2) + k_3(8, -1, 8)$

$$\begin{array}{rcl} 2k_1 + 4k_2 + 8k_3 & = & b_1 \\ -1k_1 + 1k_2 - 1k_3 & = & b_2 \\ 3k_1 + 2k_2 + 8k_3 & = & b_3 \end{array} \rightarrow \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0,$$

• Theorem 2.3.8, the system cannot be consistent for all right hand side vectors \mathbf{b} .
 We conclude that \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 do not span R^3 .

Linear Independence



Linear Independence (Contd.)

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V , then S is said to be a *linearly independent set* if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be *linearly dependent*.

THEOREM 4.3.1 A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \dots, k_r = 0$.

Linear Independence in \mathbb{R}^3

Determine whether the vectors: $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$, $\mathbf{v}_3 = (3, 2, 1)$ are linearly independent or linearly dependent in \mathbb{R}^3 .

Solution The linear independence or dependence of these vectors is determined by whether the vector equation :

- $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$
- $k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

$$\bullet k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$$

- This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

Linear Independence in \mathbb{R}^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in \mathbb{R}^4 are linearly dependent or linearly independent.

Solution

- $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$
- $k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$
$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$
- $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$
- Linearily independent vectors.

THEOREM 4.3.2

- A finite set that contains $\mathbf{0}$ is linearly dependent.
- A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.
- A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

THEOREM 4.3.3 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ be a set of vectors in \mathbb{R}^n . If $r > n$, then S is linearly dependent.

Explain why the following form linearly dependent sets of vectors.

- (a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3
- (b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2
- (c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2
- (d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

TRUE-FALSE

- (a) A set containing a single vector is linearly independent. FALSE
- (b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k . TRUE
- (c) Every linearly dependent set contains the zero vector. FALSE
- (d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k . TRUE
- (e) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector \mathbf{v}_k is a unique linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$. TRUE
- (f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22} . FALSE
- (g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent. TRUE
- (h) The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2 . FALSE

BASIS FOR A VECTOR SPACE

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of vectors in a vector space V , then S is called a **basis** for V if:

- (a) S spans V .
- (b) S is linearly independent.

• Recall standard unit vectors:

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

• Span R^3 & linearly independent,

• Thus, they form a basis for R^3 that we call the **standard basis for R^3** .

• In particular, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ is the standard basis for R^3 .

► EXAMPLE

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3 .

Solution We must show that these vectors are linearly independent and span R^3 .

- $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ (to prove linear independence)
- $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ (vector span R^3)

where, $\mathbf{b} = (b_1, b_2, b_3)$

$$\begin{array}{lcl} c_1 + 2c_2 + 3c_3 = 0 & & c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = 0 & \text{and} & 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 + 4c_3 = 0 & & c_1 + 4c_3 = b_3 \end{array} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

• $\det(A) = -1$, which proves that the vector $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 form a basis for R^3 .

• The homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of b_1, b_2 , and b_3 .

► EXAMPLE The Standard Basis for M_{mn}

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

- $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \mathbf{0}$ (to prove linear independence)
- $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$ (to prove matrices span M_{22} .
where, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$)
- $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$$

the matrices span M_{22} . This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22} .

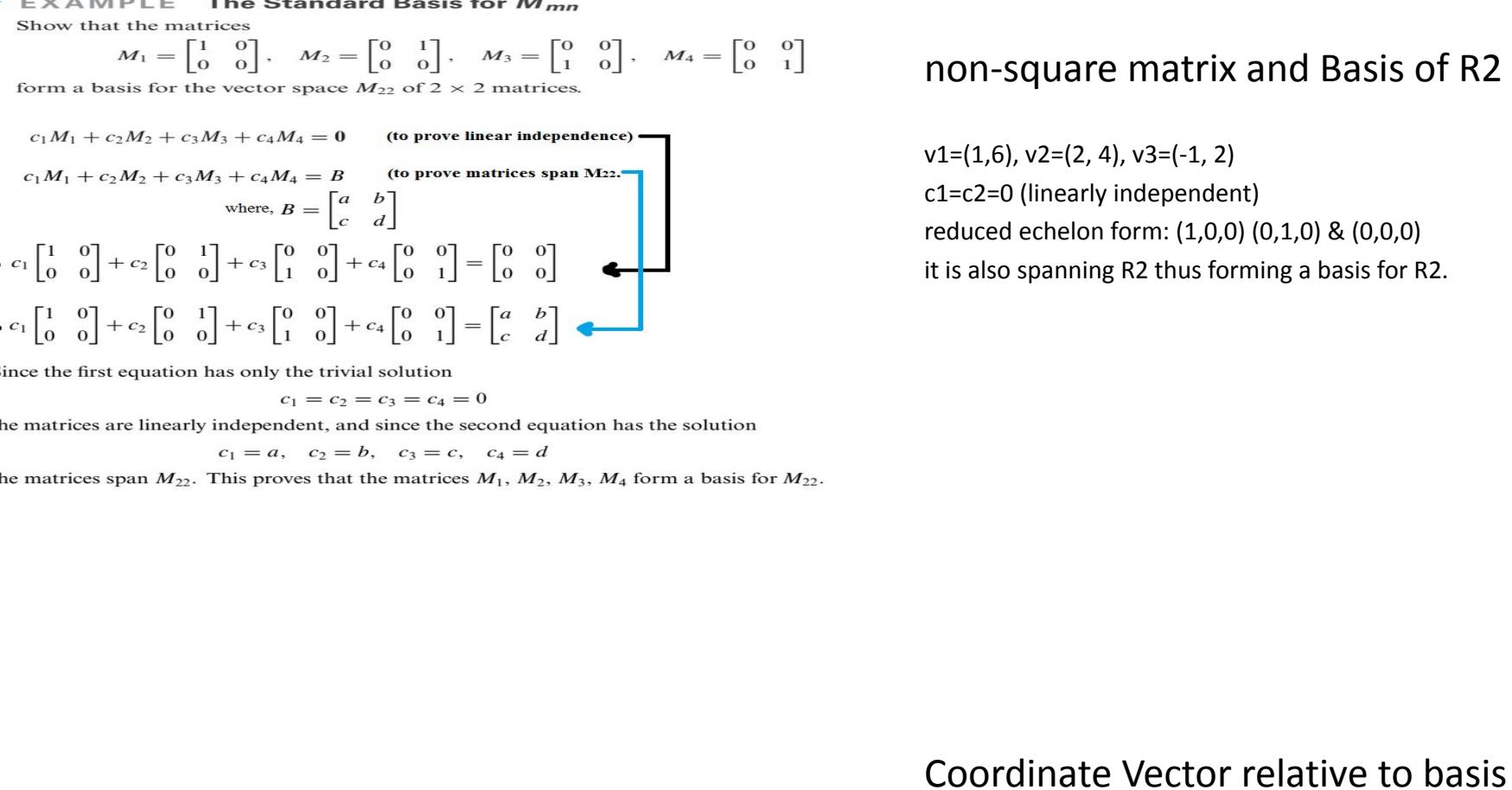
non-square matrix and Basis of R^2

$$v1=(1,6), v2=(2, 4), v3=(-1, 2)$$

$c_1=c_2=0$ (linearly independent)

reduced echelon form: $(1,0,0) (0,1,0) \& (0,0,0)$

it is also spanning R^2 thus forming a basis for R^2 .



Coordinate Vector relative to basis

let $B = \{ b_1, b_2, b_3 \}$ is a base for a vector space V . if another vector v is added to B (belongs to V) then B will be a set of **linearly dependent variables**.

DEFINITION 2 If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V , and

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

is the expression for a vector v in terms of the basis S , then the scalars c_1, c_2, \dots, c_n are called the *coordinates* of v relative to the basis S . The vector (c_1, c_2, \dots, c_n) in R^n constructed from these coordinates is called the *coordinate vector of v relative to S* ; it is denoted by

$$(v)_S = (c_1, c_2, \dots, c_n) \quad (6)$$

► **EXAMPLE 8 Coordinate Vectors Relative to Standard Bases**

- (a) Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n .

- (b) Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22} .

Solution (a) The given formula for $\mathbf{p}(x)$ expresses this polynomial as a linear combination of the standard basis vectors $S = \{1, x, x^2, \dots, x^n\}$. Thus, the coordinate vector for \mathbf{p} relative to S is

$$(\mathbf{p})_S = (c_0, c_1, c_2, \dots, c_n)$$

Solution (b) We showed in Example 4 that the representation of a vector

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a linear combination of the standard basis vectors is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is

$$(B)_S = (a, b, c, d)$$

it follows from

► **EXAMPLE 9 Coordinates in R^3**

- (a) We showed in Example 3 that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3 . Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

- (b) Find the vector \mathbf{v} in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution (a) To find $(\mathbf{v})_S$ we must first express \mathbf{v} as a linear combination of the vectors in S ; that is, we must find values of c_1, c_2 , and c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or, in terms of components,

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1, c_2 = -1, c_3 = 2$ (verify). Therefore,

$$(\mathbf{v})_S = (1, -1, 2)$$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7) \end{aligned}$$

TRUE-FALSE

- (a) If $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V . FALSE

- (b) Every linearly independent subset of a vector space V is a basis for V . FALSE

- (c) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. TRUE

- (d) The coordinate vector of a vector \mathbf{x} in R^n relative to the standard basis for R^n is \mathbf{x} . TRUE

- (e) Every basis of P_4 contains at least one polynomial of degree 3 or less. FALSE

1. show that the following set of vectors forms a basis for R^3

$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

2. show that the set of vectors is not a basis for R^3 .

$$\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$$

3. Find the coordinate vector of \mathbf{v} relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3 .

(a) $\mathbf{v} = (2, -1, 3); \mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (2, 2, 0), \mathbf{v}_3 = (3, 3, 3)$

(b) $\mathbf{v} = (5, -12, 3); \mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (-4, 5, 6), \mathbf{v}_3 = (7, -8, 9)$

Dimension

DEFINITION 1 The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V . In addition, the zero vector space is defined to have dimension zero.

► EXAMPLE 1 Dimensions of Some Familiar Vector Spaces

$$\dim(R^n) = n \quad [\text{The standard basis has } n \text{ vectors.}]$$

$$\dim(P_n) = n + 1 \quad [\text{The standard basis has } n + 1 \text{ vectors.}]$$

$$\dim(M_{mn}) = mn \quad [\text{The standard basis has } mn \text{ vectors.}]$$

For exercise set 4.5 (q1)

THEOREM 4.3.2

- (a) A finite set that contains $\mathbf{0}$ is linearly dependent.
- (b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.
- (c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Dimension of Span (S)

$$\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}] = r$$

the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

Dimension of a solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}x_1 + 3x_2 - 2x_3 &+ 2x_5 = 0 \\2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\5x_3 + 10x_4 &+ 15x_6 = 0 \\2x_1 + 6x_2 &+ 8x_4 + 4x_5 + 18x_6 = 0\end{aligned}$$

Solution

- $x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$
- $(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$
- $\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$
- These vector span the solution space & are linearly independent.
- Thus solution space has dimension 3.

Bases by inspection

Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2 .

Solution Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2 , and hence they form a basis by Theorem 4.5.4.

- find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

$$\begin{aligned} \mathbf{1.} \quad & x_1 + x_2 - x_3 = 0 \\ & -2x_1 - x_2 + 2x_3 = 0 \\ & -x_1 + x_3 = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{2.} \quad & 3x_1 + x_2 + x_3 + x_4 = 0 \\ & 5x_1 - x_2 + x_3 - x_4 = 0 \end{aligned}$$

- We conclude that \mathbf{v}_1 forms a basis for the solution space and that the dimension of the solution space is 1.

- We conclude that \mathbf{v}_1 and \mathbf{v}_2 form a basis for the solution space and that the dimension of the solution space is 2.

7. In each part, find a basis for the given subspace of R^3 , and state its dimension.
- The plane $3x - 2y + 5z = 0$.
 - The plane $x - y = 0$.
 - The line $x = 2t, y = -t, z = 4t$.
 - All vectors of the form (a, b, c) , where $b = a + c$.

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4 .

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4 .

Row Space, Column Space & Null Space

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the *row space* of A , and the subspace of R^m spanned by the column vectors of A is called the *column space* of A . The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n , is called the *null space* of A .

THEOREM 4.7.1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is in the column space of A .

THEOREM 4.7.3 Elementary row operations do not change the null space of a matrix.

THEOREM 4.7.4 Elementary row operations do not change the row space of a matrix.

• An elementary row operation can alter its column space.

► EXAMPLE 2 A Vector \mathbf{b} in the Column Space of A

Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that \mathbf{b} is in the column space of A by expressing it as a linear combination of the column vectors of A .

Solution Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix} \quad \blacktriangleleft$$

► **EXAMPLE 4** Finding a Basis for the Null Space of a Matrix

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Solution The null space of A is the solution space of the homogeneous linear system $Ax = \mathbf{0}$, which has the basis

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



THEOREM 4.7.5 If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R , and the column vectors with the leading 1's of the row vectors form a basis for the column space of R .

► **EXAMPLE 5** Bases for the Row and Column Spaces of a Matrix in Row Echelon Form

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution Since the matrix R is in row echelon form, it follows from Theorem 4.7.5 that the vectors

$$\mathbf{r}_1 = [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r}_2 = [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 1 \quad 0]$$

form a basis for the row space of R , and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the column space of R .

► **EXAMPLE 7** Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of A .

Solution

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THEOREM 4.7.6 If A and B are row equivalent matrices, then:

- (a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
- (b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B .

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by $\text{rank}(A)$; the dimension of the null space of A is called the **nullity** of A and is denoted by $\text{nullity}(A)$.

► EXAMPLE 1 Rank and Nullity of a 4×6 Matrix

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since this matrix has two leading 1's, its row and column spaces are two dimensional and $\text{rank}(A) = 2$.

Example # 01 (Contd.)

- To find the nullity of A , we must find the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$.
- This system can be solved by reducing its augmented matrix to reduced row echelon form.

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$

$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

from which we obtain the general solution

$$x_1 = 4r + 28s + 37t - 13u$$

$$x_2 = 2r + 12s + 16t - 5u$$

$$x_3 = r$$

$$x_4 = s$$

$$x_5 = t$$

$$x_6 = u$$

or in column vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (3)$$

Because the four vectors on the right side of (3) form a basis for the solution space, $\text{nullity}(A) = 4$.

► EXAMPLE 2 Maximum Value for Rank

What is the maximum possible rank of an $m \times n$ matrix A that is not square?

Solution Since the row vectors of A lie in R^n and the column vectors in R^m , the row space of A is at most n -dimensional and the column space is at most m -dimensional. Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n . We denote this by writing

$$\text{rank}(A) \leq \min(m, n)$$

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n \quad (4)$$

► EXAMPLE 3 The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\text{rank}(A) + \text{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4 \quad \blacktriangleleft$$

THEOREM 4.8.3

If A is an $m \times n$ matrix, then

- $\text{rank}(A) =$ the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.
- $\text{nullity}(A) =$ the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

QUIZ

- (1) Use the adjoint method to find the inverse of the given matrix, if it exists:

$$\begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$$

- (2) Let $\mathbf{u} = (-2, 0, 4)$, $\mathbf{v} = (3, -1, 6)$, and $\mathbf{w} = (2, -5, -5)$. Compute

- (3) determine which of the following are subspaces of M_{nn} .

- (a) The set of all $n \times n$ matrices A such that $\det(A) = 0$.
 - (b) The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$.
 - (c) The set of all symmetric $n \times n$ matrices.

$$\begin{aligned}
 1. \quad & \left| \begin{array}{cccc} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{array} \right| = 3 \left| \begin{array}{ccc} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{array} \right| - 6 \left| \begin{array}{ccc} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{array} \right| + 0 - 1 \left| \begin{array}{ccc} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{array} \right| \\
 & = 3(-0 + (-1) \left| \begin{array}{cc} 3 & 4 \\ 2 & 2 \end{array} \right| - 1 \left| \begin{array}{cc} 1 & 4 \\ -2 & 2 \end{array} \right|) - 6(-1 \left| \begin{array}{cc} -1 & 4 \\ -2 & 2 \end{array} \right| + (-1) \left| \begin{array}{cc} -2 & 4 \\ -9 & 2 \end{array} \right| - 1 \left| \begin{array}{cc} -2 & 1 \\ -9 & -2 \end{array} \right|) + 0 \\
 & \quad - 1(-1 \left| \begin{array}{cc} 3 & 1 \\ 2 & -2 \end{array} \right| + 0 - (-1) \left| \begin{array}{cc} -2 & 3 \\ -9 & 2 \end{array} \right|) \\
 & = 3(0 - 1(-2) - 1(-8)) - 6(-1(10) - 1(32) - 1(13)) + 0 - 1(-1(-8) + 0 + 1(23)) \\
 & = 3(10) - 6(-55) + 0 - 1(31) \\
 & = 329
 \end{aligned}$$

2. (a) $3\mathbf{v} - 2\mathbf{u} = (9, -3, 18) - (-4, 0, 8) = (13, -3, 10)$
 (b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (3, -6, 5); \|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{3^2 + (-6)^2 + 5^2} = \sqrt{70}$

$$d(-3\mathbf{u}, \mathbf{v} + 5\mathbf{w}) = \| -3\mathbf{u} - (\mathbf{v} + 5\mathbf{w}) \| = \sqrt{(-7)^2 + 26^2 + 7^2} = \sqrt{774} = 3\sqrt{86}$$